Math 116
Homework 5
Due Thursday, February 25, 2016

- Do the following problems:
  1. Let R be a ring. Prove that the following are equivalent statements about R:
     (1) For all $a, b \in R$, if $ab = 0$ then $a = 0$ or $b = 0$.
     (2) For all $a, b, c \in R$, if $ab = ac$ and $a \neq 0$ then $b = c$.

     In other words, prove that if (1) is true for R, then so is (2), *and* that if (2) is true for R, then so is (1).

     Note that (1) is the same as saying that R has no zero divisors, and is sometimes called the "zero product property". (2) is the "multiplicative cancellation law". So what you have proved is that R has no zero divisors iff the zero product property is true in R iff the cancellation law works in R.
  2. Let F be a field, and let $P(X) \in F[X]$ (that is, $P(X)$ is a polynomial with coefficients in F). Recall that in class we defined, for $A(X), B(X) \in F[X]$,
     $$A(X) \equiv B(X) \pmod{P(X)} \quad \text{if } P(X) \mid (A(X) - B(X)).$$
     Prove that this is an equivalence relation. (Remember that this means you must show that it is reflexive, symmetric, and transitive.)
  3. Again let F be a field, and let $P(X) \in F[X]$. Assume $A_1(X) \equiv A_2(X) \pmod{P(X)}$ and $B_1(X) \equiv B_2(X) \pmod{P(X)}$. Show that
     $$A_1(X) + B_1(X) \equiv A_2(X) + B_2(X) \pmod{P(X)} \qquad \text{and}$$
     $$A_1(X) \cdot B_1(X) \equiv A_2(X) \cdot B_2(X) \pmod{P(X)}$$
  4. Use the Euclidean algorithm for polynomials to find the greatest common divisor of $4X^3 - 4X^2 - 3X + 2$ and $8X^4 - 12X^3 + 8X - 3$ in $\mathbb{R}[X]$.
  5. Let $A(X) = X^3 + 2X + 2$ and $B(X) = X^2 + 3X + 4$ in $\mathbb{F}_5[X]$. Use the extended Euclidean algorithm for polynomials to find polynomials $P(X), Q(X) \in \mathbb{F}_5[X]$ such that
     $$A(X) \cdot P(X) + B(X) \cdot Q(X) = 1.$$
- Do Problems 21, 22, 33 and 34 at the end of chapter 3 (section 3.13).
- ~~Do Problems 1, 3, 5, 6, 7 and 8 at the end of chapter 7 (section 7.6).~~ Postponed until next week.