Math 151B Homework 1

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Part 2

Here we have the initial value problem (IVP).

$$\begin{cases} y'(t) = r(1-\frac{y}{K})y, 0 \le t \le 50 \\ y(0) = y_0 \end{cases}$$

To get the error bound specified by the equation from Theorem 5.9

$$|y(t_i) - w_i| \le \frac{hM}{2L}(e^{L(t_i-a)} - 1)$$

we need to know the constant L and constant M.
Since $y'(t) = r(1-\frac{y}{K})y$, we have

$$y''(t) = r(1-\frac{2y}{K})y'(t) = r(1-\frac{2y}{K})r(1-\frac{y}{K})y$$

Let $x = \frac{y}{K} \Rightarrow x \in [0.25, 1)$ since $1000 \le y < 4000$ as we obtained from the plot.
Then when $x = 0.25$ or 0.75, $g(x) = |(1-2x)(1-x)4000x| = 4000\left|2x^3 - 3x^2 + x\right|$ reaches its maximum $\frac{3}{32} \times 4000 = 375$. Then we have

$$\left|y''(t)\right| \le r^2 \cdot 375 = 0.2^2 \times 375 = 15 = M$$

Then we want to find its Lipschitz constant.

$$\begin{aligned} |f(t,y_1) - f(t,y_2)| &= \left|r(1-\frac{y_1}{K})y_1 - r(1-\frac{y_2}{K})y_2\right| = \left|r(y_1 - \frac{y_1^2}{K} - y_2 + \frac{y_2^2}{K})\right| \\ &= |r|\left|(y_1-y_2) - \frac{1}{K}(y_1-y_2)(y_1+y_2)\right| = |r|\left|1 - \frac{y_1+y_2}{K}\right||y_1-y_2| \\ &\le 0.2 \cdot \left|1 - \frac{4000+4000}{4000}\right||y_1-y_2| = 0.2\,|y_1-y_2| = L\,|y_1-y_2| \\ \Rightarrow L = 0.2 \end{aligned}$$

Thus we have the error bound

$$|y(t_i) - w_i| \le \frac{75h}{2}(e^{0.2t_i} - 1)$$

Here are the outputs from MATLAB:

```
step size: h = 10.000000
Maximum error: 583.340025, time is: 20.000000, error bound: 20099.306262
step size: h = 1.000000
Maximum error: 29.974716, time is: 6.000000, error bound: 87.004385
step size: h = 0.100000
Maximum error: 2.890099, time is: 5.500000, error bound: 7.515623
```

As we can see, actual errors are much smaller than theoretical bounds. Also, we can think of the error as a function of h where the error increases as h increases and decreases as h decreases. If h is too large, the error will probably exponentially grow and the results will be unusable. It also breaks the assumption of Euler's method, which is to have small step size h. Plot is shown in Figure 1.

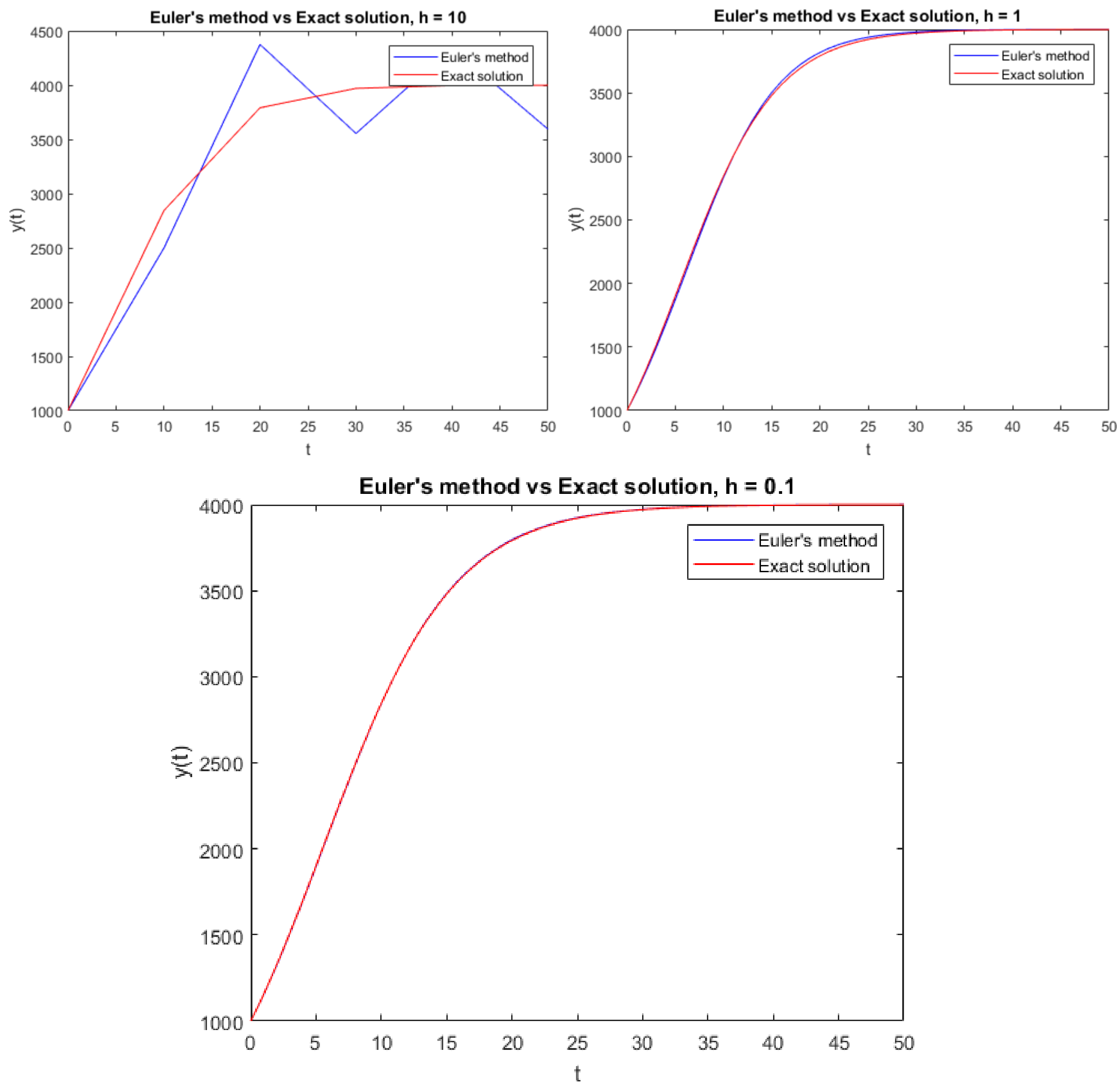

Figure 1: Plot of computed and exact solutions w.r.t. different step size h