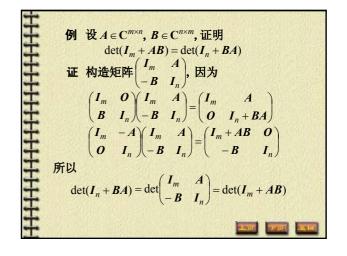
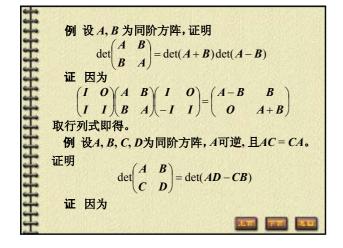
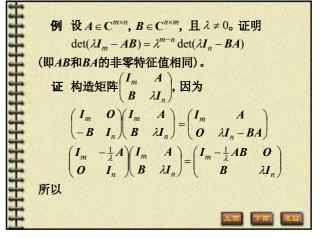


例 求正定矩阵
$$A = \begin{pmatrix} 5 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$
 的Cholesky分解。

解
$$\frac{\sqrt{5}}{-\frac{2}{\sqrt{5}}} \sqrt{\frac{11}{5}}$$
所以 $A = \begin{pmatrix} \sqrt{5} & \sqrt{\frac{11}{5}} & \sqrt{\frac{6}{11}} \\ 0 & -\sqrt{\frac{5}{11}} & \sqrt{\frac{6}{11}} \end{pmatrix}$







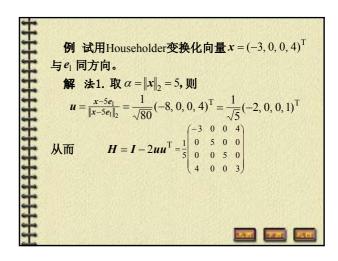
$$\det(\lambda \mathbf{I}_n - \mathbf{B}\mathbf{A}) = \det\begin{pmatrix} \mathbf{I}_m & \mathbf{A} \\ \mathbf{B} & \lambda \mathbf{I}_n \end{pmatrix}$$

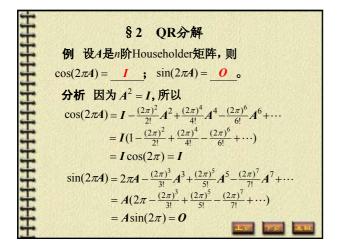
$$= \det\begin{pmatrix} \mathbf{I}_m - \frac{1}{\lambda} \mathbf{A} \mathbf{B} & \mathbf{O} \\ \mathbf{B} & \lambda \mathbf{I}_n \end{pmatrix}$$

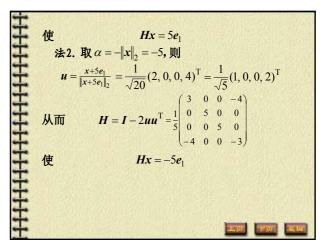
$$= \det(\mathbf{I}_m - \frac{1}{\lambda} \mathbf{A} \mathbf{B}) \det(\lambda \mathbf{I}_n)$$

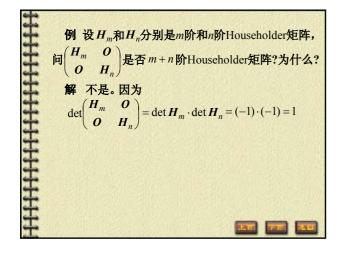
$$= \frac{1}{\lambda^m} \det(\lambda \mathbf{I}_m - \mathbf{A} \mathbf{B}) \lambda^n$$

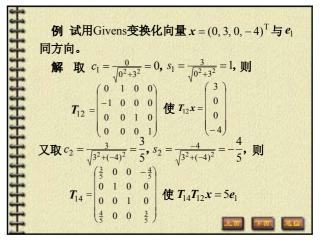
$$= \lambda^{n-m} \det(\lambda \mathbf{I}_m - \mathbf{A} \mathbf{B})$$











例 试求矩阵
$$A = \begin{pmatrix} 0 & 3 & 1 \\ 0 & 4 & -2 \\ 2 & 1 & 2 \end{pmatrix}$$
 的QR分解。

解 由于 $a_1 = (0,0,2)^T$,取 $\alpha_1 = \|a_1\|_2 = 2$,则

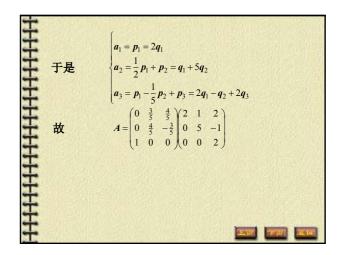
$$u_1 = \frac{a_1 - 2e_1}{\|a_1 - 2e_1\|_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$
于是 $H_1 = I - 2u_1u_1^T = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ 使 $H_1A = \begin{pmatrix} 2 & 1 & 2 \\ 0 & 4 & -2 \\ 0 & 3 & 1 \end{pmatrix}$

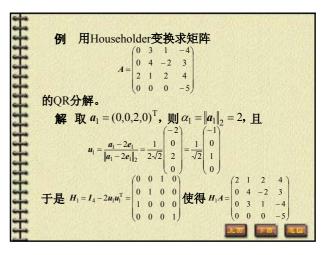
例 试求矩阵
$$A = \begin{pmatrix} 0 & 3 & 1 \\ 0 & 4 & -2 \\ 2 & 1 & 2 \end{pmatrix}$$
 的QR分解。

解 将列向量 $a_1 = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$, $a_2 = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$, $a_3 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ 正交化得

$$p_1 = a_1 = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$
, $p_2 = a_2 - \frac{2}{4}p_1 = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$, $p_3 = a_3 - \frac{4}{4}p_1 - \frac{-5}{25}p_2 = \begin{pmatrix} \frac{8}{5} \\ -\frac{6}{5} \\ 0 \end{pmatrix}$
单位化得
$$q_1 = \frac{1}{2}p_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$
, $q_2 = \frac{1}{5}p_2 = \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \\ 0 \end{pmatrix}$, $q_3 = \frac{1}{2}p_3 = \begin{pmatrix} \frac{4}{5} \\ -\frac{3}{5} \\ 0 \end{pmatrix}$

又由于
$$b_2 = (4,3)^T$$
,取 $\alpha_2 = \|b_2\|_2 = 5$,则
$$u_2 = \frac{b_2 - 5\widetilde{e_1}}{\|b_2 - 5\widetilde{e_1}\|_2} = \frac{1}{\sqrt{10}} \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$
于是 $\widetilde{H}_2 = I - 2u_2u_2^T = \frac{1}{5} \begin{pmatrix} 4 & 3 \\ 3 & -4 \end{pmatrix}$,会
$$H_2 = \begin{pmatrix} 1 \\ \widetilde{H}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{4}{5} & \frac{3}{5} \\ 0 & \frac{3}{5} & -\frac{4}{5} \end{pmatrix}$$
 则 $H_2(H_1A) = \begin{pmatrix} 2 & 1 & 2 \\ 0 & 5 & -1 \\ 0 & 0 & -2 \end{pmatrix}$ 故
$$A = (H_1H_2)R = \begin{pmatrix} 0 & \frac{3}{5} & -\frac{4}{5} \\ 0 & \frac{4}{5} & \frac{3}{5} \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 & 2 \\ 0 & 5 & -1 \\ 1 & 0 & 0 \end{pmatrix}$$

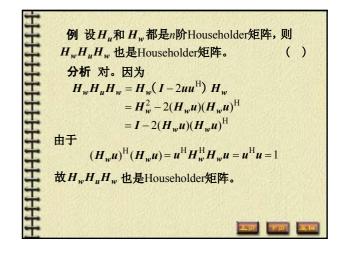


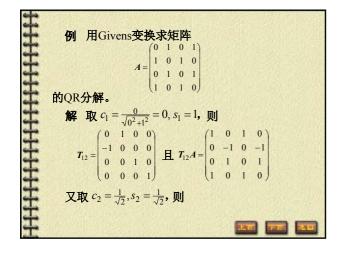


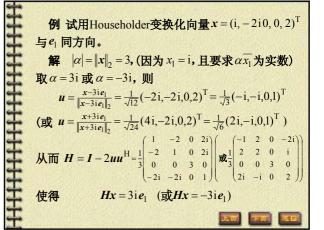
又取
$$b_2 = (4,3,0)^{\mathrm{T}}$$
,则 $\alpha_2 = \|b_2\|_2 = 5$,且
$$u_2 = \frac{b_2 - 5\tilde{e}_1}{\|b_2 - 5\tilde{e}_1\|_2} = \frac{1}{\sqrt{10}} \begin{pmatrix} -1\\3\\0 \end{pmatrix}$$
于是 $\tilde{H}_2 = I_3 - 2u_2u_1^{\mathrm{T}} = \begin{pmatrix} \frac{4}{5} & \frac{3}{5} & 0\\ \frac{3}{5} & -\frac{4}{5} & 0\\0 & 0 & 1 \end{pmatrix}$
令 $H_2 = \begin{pmatrix} 1 & \mathbf{0}^{\mathrm{T}}\\\mathbf{0} & \tilde{H}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0\\0 & \frac{4}{3} & \frac{3}{5} & 0\\0 & 0 & \frac{3}{5} & -\frac{4}{5} & 0\\0 & 0 & 0 & 1 \end{pmatrix}$
则 $H_2(H_1A) = \begin{pmatrix} 2 & 1 & 2 & 4\\0 & 5 & -1 & 0\\0 & 0 & -2 & 5\\0 & 0 & 0 & -5 \end{pmatrix} = R$

$$T_{14} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \mathbf{H} \quad T_{14}(T_{12}A) = \begin{pmatrix} \sqrt{2} & 0 & \sqrt{2} & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
再取 $c_3 = -\frac{1}{\sqrt{2}}, s_3 = \frac{1}{\sqrt{2}}$,则
$$T_{23} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mathbf{H} \quad T_{23}(T_{14}T_{12}A) = \begin{pmatrix} \sqrt{2} & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathbf{R}$$
于是
$$\mathbf{A} = \mathbf{Q}\mathbf{R} = \mathbf{T}_{12}^{\mathrm{T}}\mathbf{T}_{14}^{\mathrm{T}}\mathbf{T}_{23}^{\mathrm{T}}\mathbf{R} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

故
$$A = QR = (H_1 H_2) = \begin{pmatrix} 0 & \frac{3}{5} & -\frac{4}{5} & 0 \\ 0 & \frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 2 & 4 \\ 0 & 5 & -1 & 0 \\ 0 & 0 & -2 & 5 \\ 0 & 0 & 0 & -5 \end{pmatrix}$$







例 设
$$T$$
是 n 阶 Given矩阵, H_u 是 n 阶 Householder 矩阵,则 TH_uT^{-1} 也是 Householder矩阵。 () 分析 对。因为
$$TH_uT^{-1} = T (I-2uu^H) T^H = TT^H - 2(Tu)(Tu)^H = I-2(Tu)(Tu)^H$$
 由于
$$(Tu)^H(Tu) = u^HT^HTu = u^Hu = 1$$
 故 TH_uT^{-1} 也是 Householder矩阵。

则
$$H_1AH_1 = \begin{pmatrix} 1 & \frac{4}{5} & -\frac{3}{5} \\ 5 & 1 & -2 \\ 0 & 2 & 1 \end{pmatrix}$$
 法2. 利用 Givens变换 取 $c = \frac{3}{5}, s = \frac{4}{5}$,则 $T_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & \frac{4}{5} \\ 0 & -\frac{4}{5} & \frac{3}{5} \end{pmatrix}$ 且 $T_{23}AT_{23}^T = \begin{pmatrix} 1 & \frac{4}{5} & \frac{3}{5} \\ 5 & 1 & 2 \\ 0 & -2 & 1 \end{pmatrix}$

例 试用Givens变换化向量
$$x = (i, -2i, 0, 2)^T$$
与 e_1 同方向。 解 取 $c_1 = \frac{i}{\sqrt{5}}$, $s_1 = -\frac{2i}{\sqrt{5}}$, 则
$$T_{12} = \begin{pmatrix} -\frac{i}{\sqrt{5}} & \frac{2i}{\sqrt{5}} & 0 & 0 \\ \frac{2i}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
使 $T_{12}x = \begin{pmatrix} \sqrt{5} \\ 0 \\ 0 \\ 2 \end{pmatrix}$ 又取 $c_2 = \frac{\sqrt{5}}{3}$, $s_2 = \frac{2}{3}$,则
$$T_{14} = \begin{pmatrix} \frac{\sqrt{5}}{3} & 0 & 0 & \frac{2}{3} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{2}{3} & 0 & 0 & \frac{\sqrt{5}}{3} \end{pmatrix}$$
 使 $T_{14}T_{12}x = 3e_1$

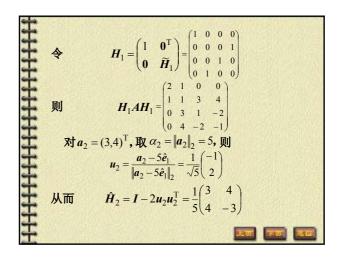
例 化矩阵
$$A = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & -1 & -2 & 4 \\ 0 & -2 & 1 & 3 \\ 1 & 4 & 3 & 1 \end{pmatrix}$$
 正交相似于三对角阵。
解 法1. 用 Householder变换
对 $a_3 = (0,0,1)^T$,取 $\alpha_1 = \|a_3\|_2 = 1$,则
$$u_1 = \frac{a_3 - \tilde{e}_1}{\|a_3 - \tilde{e}_1\|_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$
于是 $\tilde{H}_1 = I - 2u_1u_1^T = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

例 化矩阵
$$A = \begin{pmatrix} 1 & 0 & 1 \ 3 & 1 & 2 \ 4 & -2 & 1 \end{pmatrix}$$
 正交相似于Hessenberg阵。解 法1. 利用Householder变换

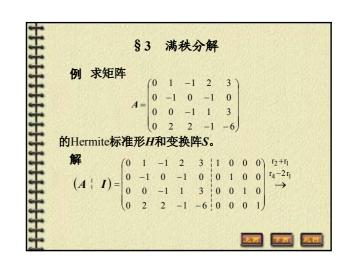
对 $a_2 = (3,4)^T$,取 $\alpha_1 = \|a_2\|_2 = 5$,则

$$u_1 = \frac{a_2 - 5\tilde{e}_1}{\|a_2 - 5\tilde{e}_1\|_2} = \frac{1}{\sqrt{5}} \begin{pmatrix} -1 \ 2 \end{pmatrix}$$
于是 $\widetilde{H}_1 = I - 2u_1u_1^T = \frac{1}{5} \begin{pmatrix} 3 & 4 \ 4 & -3 \end{pmatrix}$,令

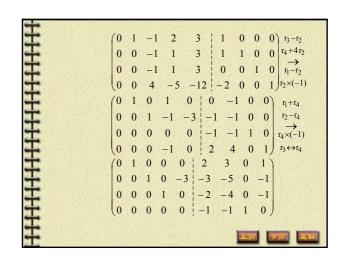
$$H_1 = \begin{pmatrix} 1 & \mathbf{0}^T \ \mathbf{0} & \widetilde{H}_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \ 0 & \frac{3}{5} & \frac{4}{5} \ 0 & \frac{4}{5} & -\frac{3}{5} \end{pmatrix}$$

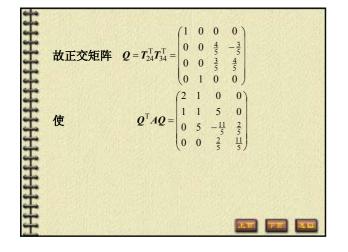


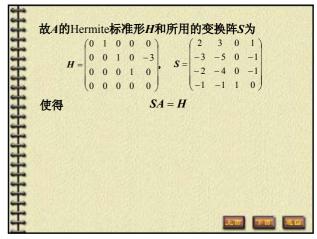
令
$$H_2 = \begin{pmatrix} I_2 & O \\ O & \hat{H}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{3}{5} & \frac{4}{5} \\ 0 & 0 & \frac{4}{5} & -\frac{3}{5} \end{pmatrix}$$
则 $H_2H_1AH_1H_2 = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 1 & 5 & 0 \\ 0 & 5 & -\frac{11}{5} & \frac{2}{5} \\ 0 & 0 & \frac{2}{5} & \frac{11}{5} \end{pmatrix} = T$
故正交矩阵 $Q = H_1H_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{4}{5} & -\frac{3}{5} \\ 0 & 0 & \frac{3}{5} & \frac{4}{5} \\ 0 & 0 & \frac{3}{5} & \frac{4}{5} \\ 0 & 1 & 0 & 0 \end{pmatrix}$
使 $Q^TAQ = T$

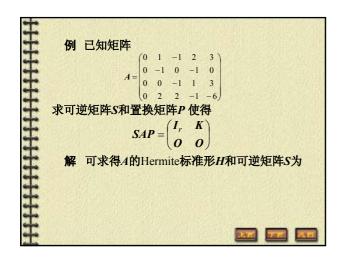


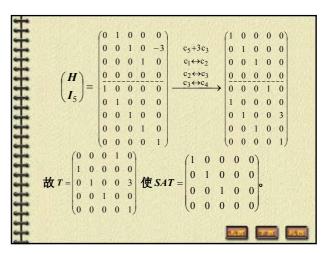
法2. 利用 Givens 变换
 取
$$c_1 = 0$$
, $s_1 = 1$, 则
 $T_{24} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$ 且 $T_{24}AT_{24}^{\mathrm{T}} = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 1 & 3 & -4 \\ 0 & 3 & 1 & 2 \\ 0 & -4 & 2 & 1 \end{pmatrix}$
又取 $c_2 = \frac{3}{5}$, $s_2 = -\frac{4}{5}$, 则
 $T_{34} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{3}{5} & -\frac{4}{5} \\ 0 & 0 & \frac{4}{5} & \frac{3}{5} \end{pmatrix}$ 且 $T_{34}T_{24}T_{34}^{\mathrm{T}} = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 1 & 5 & 0 \\ 0 & 5 & -\frac{11}{5} & \frac{2}{5} \\ 0 & 0 & \frac{2}{5} & \frac{11}{5} \end{pmatrix}$

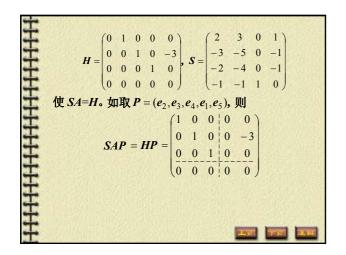


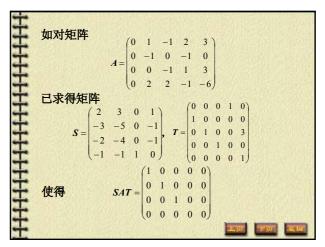


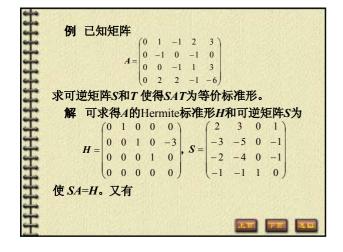


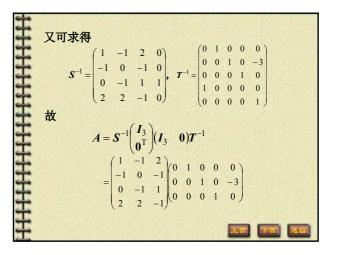












例 求矩阵
$$A = \begin{pmatrix} 0 & 1 & -1 & 2 & 3 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & 3 \\ 0 & 2 & 2 & -1 & -6 \end{pmatrix}$$
 的满秩分解。
$$\mathbf{A} \xrightarrow{\substack{\mathbf{r}_2 + \mathbf{r}_1 \\ \mathbf{r}_4 - 2\mathbf{r}_1}} \begin{pmatrix} 0 & 1 & -1 & 2 & 3 \\ 0 & 0 & -1 & 1 & 3 \\ 0 & 0 & -1 & 1 & 3 \\ 0 & 0 & 4 & -5 & -12 \end{pmatrix} \xrightarrow{\substack{\mathbf{r}_1 - \mathbf{r}_2 \\ \mathbf{r}_4 + 4\mathbf{r}_2 \\ \mathbf{r}_2 \times (-1)}}$$

