第三章 矩阵分析

§1 矩阵序列的极限

例 矩阵

$$A = \begin{pmatrix} 0.2 & 0.1 & 0.2 \\ 0.5 & 0.5 & 0.4 \\ 0.1 & 0.3 & 0.2 \end{pmatrix}$$

是否为收敛矩阵? 为什么?

解 因为 $\|A\|_1 = 0.9 < 1$, (或 $\|A\|_E = \sqrt{0.89} \approx 0.943 < 1$) 所以4是收敛矩阵。

(可求得 $\|A\|_{m_1} = 2.5$, $\|A\|_{m_{\infty}} = 3 \times 0.5 = 1.5$, $\|A\|_{\infty} = 1.4$)



所以收敛半径为 $r = \frac{1}{1} = 6$ 。可求得A的特征值为 $\lambda_1 = 5, \quad \lambda_2 = -3$

即 $\rho(A) = 5 < 6$,故矩阵幂级数绝对收敛。

法2. 取幂级数
$$\sum_{k=0}^{+\infty} kx^k$$
,则 $A = \frac{1}{6} \begin{pmatrix} 1 & -8 \\ -2 & 1 \end{pmatrix}$ 。

可求得 r=1, A的特征值为 $\lambda_1=\frac{5}{6}$, $\lambda_2=-\frac{1}{2}$

于是 $\rho(A) = \frac{5}{6} < 1$, 故矩阵幂级数绝对收敛。



例 矩阵 $A = \begin{pmatrix} \frac{1}{6} & -\frac{4}{3} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$ 是否为收敛矩阵? 为什么?

 $\det(\lambda \mathbf{I} - \mathbf{A}) = \begin{vmatrix} \lambda - \frac{1}{6} & \frac{4}{3} \\ \frac{1}{3} & \lambda - \frac{1}{6} \end{vmatrix} = \lambda^2 - \frac{1}{3}\lambda - \frac{15}{36} = (\lambda - \frac{5}{6})(\lambda + \frac{3}{6})$

得A的特征值为 $\lambda_1 = \frac{5}{6}$, $\lambda_2 = -\frac{1}{2}$ 从而 $\rho(A) = \frac{5}{6} < 1$,故A是收敛矩阵。

 $A = \begin{bmatrix} 0.5 & 0.5 & 0.4 \end{bmatrix}$

判断 $\sum A^k$ 的敛散性。若收敛,求其和。

解 因为 $\|A\|_1 = 0.9 < 1$,所以 $\sum_{k=0}^{\infty} A^k$ 收敛,且

 $\sum_{k=0}^{+\infty} \boldsymbol{A}^k = (\boldsymbol{I} - \boldsymbol{A})^{-1} = \begin{pmatrix} 0.8 & -0.1 & -0.2 \\ -0.5 & 0.5 & -0.4 \\ -0.1 & -0.3 & 0.8 \end{pmatrix}^{-1} = \frac{1}{14} \begin{pmatrix} 28 & 14 & 14 \\ 44 & 62 & 42 \\ 20 & 25 & 35 \end{pmatrix}$

例 已知 $A = \begin{pmatrix} \frac{1}{6} & -\frac{4}{3} \\ -\frac{1}{2} & \frac{1}{4} \end{pmatrix}$, 则 $\sum_{k=0}^{+\infty} A^k$ 收敛的原因是

§ 2 矩阵级数

例 判断矩阵幂级数

$$\sum_{k=0}^{+\infty} \frac{k}{6^k} \begin{pmatrix} 1 & -8 \\ -2 & 1 \end{pmatrix}^k$$

的敛散性。

解 法1. 令 $A = \begin{pmatrix} 1 & -8 \\ -2 & 1 \end{pmatrix}$, 取幂级数 $\sum_{k=0}^{+\infty} \frac{k}{6^k} x^k$ 。

因为

 $\rho = \lim_{k \to +\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \to +\infty} \frac{\frac{k+1}{6^{k+1}}}{\frac{k}{k}} = \lim_{k \to +\infty} \frac{1}{6} \cdot \frac{k+1}{k} = \frac{1}{6}$

分析 可求得A的特征值为

 $\rho(A) = \frac{5}{6} < 1$,且其和为 $\begin{pmatrix} \frac{10}{3} & -\frac{16}{3} \\ -\frac{4}{3} & \frac{10}{3} \end{pmatrix}$ 。

 $\lambda_1 = \frac{5}{6}$, $\lambda_2 = -\frac{1}{2}$ 从而 $\rho(A) = \frac{5}{6} < 1$, $\sum_{k=0}^{+\infty} A^k = (I - A)^{-1} = \begin{pmatrix} \frac{10}{3} & -\frac{16}{3} \\ -\frac{4}{3} & \frac{10}{3} \end{pmatrix}$







§3 矩阵函数

例 已知
$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
, 试求 e^A , e^{At} , $\sin A$, $\cos At$ 。

解 因为 $\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 \\ 1 & \lambda \end{vmatrix} = \lambda^2 + 1$

所以
$$A^2 + I = 0$$
,即 $A^2 = -I$ 。从而

$$A^{3} = -A$$
, $A^{4} = I$, $A^{5} = A$, $A^{6} = -I$, $A^{7} = -A$, $A^{8} = I$, ...

可知
$$A^{2k} = (-1)^k I$$
, $A^{2k+1} = (-1)^k A$ $(k = 1, 2, \cdots)$ 故

例 设
$$A \in \mathbb{C}^{n \times n}$$
满足 $A^2 = A$,试求 e^A , e^{At} , $\sin A$,

解由于
$$A^k = A (k = 2,3,\cdots)$$
,所以
 $e^A = I + \frac{1}{1!}A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \frac{1}{4!}A^4 + \frac{1}{5!}A^5 + \frac{1}{6!}A^6 + \cdots$
 $= I + A(\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots)$
 $= I + A[(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots) - 1] = I + (e-1)A$
 $e^{At} = I + \frac{1}{1!}At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \frac{1}{4!}A^4t^4 + \cdots$
 $= I + (\frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots)A = I + (e^t - 1)A$
 $\sin A = A - \frac{1}{3!}A^3 + \frac{1}{5!}A^5 - \frac{1}{7!}A^7 + \frac{1}{9!}A^9 - \cdots$

$$= A - \frac{1}{3!}A + \frac{1}{5!}A - \frac{1}{7!}A + \cdots = A \sin 1$$

$$e^{A} = I + \frac{1}{1!}A + \frac{1}{2!}A^{2} + \frac{1}{3!}A^{3} + \frac{1}{4!}A^{4} + \frac{1}{5!}A^{5} + \frac{1}{6!}A^{6} + \cdots$$

$$= I + \frac{1}{1!}A - \frac{1}{2!}I - \frac{1}{3!}A + \frac{1}{4!}I + \frac{1}{5!}A - \frac{1}{6!}I - \frac{1}{7!}A + \frac{1}{8!}I - \cdots$$

$$= (1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!} - \cdots)I + (1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots)A$$

$$= (\cos 1)I + (\sin 1)A = \begin{pmatrix} \cos 1 & \sin 1 \\ -\sin 1 & \cos 1 \end{pmatrix}$$

$$e^{At} = I + \frac{1}{1!}At + \frac{1}{2!}A^{2}t^{2} + \frac{1}{3!}A^{3}t^{3} + \frac{1}{4!}A^{4}t^{4} + \cdots$$

$$= (1 - \frac{t^{2}}{2!} + \frac{t^{4}}{4!} - \frac{t^{6}}{6!} + \cdots)I + (t - \frac{t^{3}}{3!} + \frac{t^{5}}{5!} - \cdots)A$$

$$= (\cos t)I + (\sin t)A = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

$$\cos At = I - \frac{1}{2!}A^{2}t^{2} + \frac{1}{4!}A^{4}t^{4} - \frac{1}{6!}A^{6}t^{6} + \cdots$$

$$= I + A(-\frac{1}{2!}t^{2} + \frac{1}{4!}t^{4} + \cdots)$$

$$= I + (\cos t - 1)A$$

例 已知 $\Lambda = \text{diag}(1, -2, 0, 4)$, 求 e^{Λ} , $e^{\Lambda t}$, $\sin \Lambda t$, $\cos \Lambda_0$

A
$$e^{\mathbf{A}} = diag(e, e^{-2}, 1, e^4)$$

 $e^{\mathbf{A}t} = diag(e^t, e^{-2t}, 1, e^{4t})$
 $\sin \mathbf{A}t = diag(\sin t, -\sin 2t, 0, \sin 4t)$
 $\cos \mathbf{A} = diag(\cos 1, \cos 2, 1, \cos 4)$

$$\sin A = A - \frac{1}{3!}A^{3} + \frac{1}{5!}A^{5} - \frac{1}{7!}A^{7} + \frac{1}{9!}A^{9} - \cdots$$

$$= A + \frac{1}{3!}A + \frac{1}{5!}A + \frac{1}{7!}A + \cdots$$

$$= A[\frac{1}{2}(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots) - \frac{1}{2}(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots)]$$

$$= A[\frac{e - e^{-1}}{2}] = A \operatorname{sh} 1 = \begin{pmatrix} 0 & \operatorname{sh} 1 \\ -\operatorname{sh} 1 & 0 \end{pmatrix}$$

$$\cos At = I - \frac{1}{2!}A^{2}t^{2} + \frac{1}{4!}A^{4}t^{4} - \frac{1}{6!}A^{6}t^{6} + \cdots$$

$$= I(1 + \frac{1}{2!}t^{2} + \frac{1}{4!}t^{4} + \cdots)$$

$$= I[\frac{e^{t} + e^{-t}}{2}] = I \operatorname{ch} t = \begin{pmatrix} \operatorname{ch} t & 0 \\ 0 & \operatorname{ch} t \end{pmatrix}$$

例 已知
$$A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & 1 & 2 \end{pmatrix}$$
, 试求 e^A , e^{At} , $\sin At_o$

解 可求得 $\det(\lambda I - A) = (\lambda - 1)(\lambda - 2)(\lambda - 3)$ A的特征值为 $\lambda_1 = 1, \ \lambda_2 = 2, \ \lambda_3 = 3$

对应的特征向量分别为

 $p_1 = (-1, 0, 1)^T$, $p_2 = (-1, 1, 1)^T$, $p_3 = (-1, 1, 0)^T$ 故相似变换阵

$$\mathbf{P} = \begin{pmatrix} -1 & -1 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \notin \mathbf{P}^{-1} \mathbf{A} \mathbf{P} = \begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \end{pmatrix}$$

从而





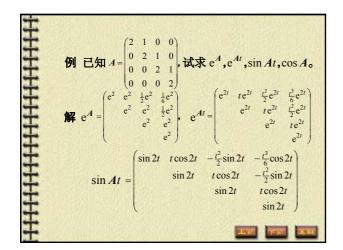
$$e^{A} = P \begin{pmatrix} e \\ e^{2} \\ e^{3} \end{pmatrix} P^{-1} = \begin{pmatrix} e - e^{2} + e^{3} & e - e^{2} & -e^{2} + e^{3} \\ e^{2} - e^{3} & e^{2} & e^{2} - e^{3} \\ -e + e^{2} & -e + e^{2} & e^{2} \end{pmatrix}$$

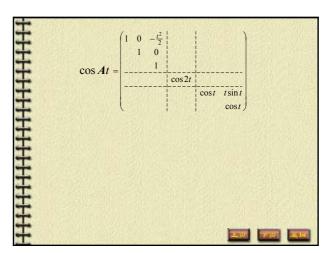
$$e^{At} = P \begin{pmatrix} e^{t} \\ e^{2t} \\ e^{3t} \end{pmatrix} P^{-1} = \begin{pmatrix} e^{t} - e^{2t} + e^{3t} & e^{t} - e^{2t} & -e^{2t} + e^{3t} \\ e^{2t} - e^{3t} & e^{2t} & e^{2t} - e^{3t} \\ -e^{t} + e^{2t} & -e^{t} + e^{2t} & e^{2t} \end{pmatrix}$$

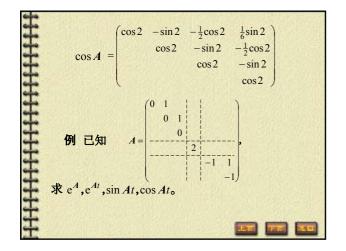
$$\sin At = P \begin{pmatrix} \sin t \\ \sin 2t \\ \sin 3t \end{pmatrix} P^{-1}$$

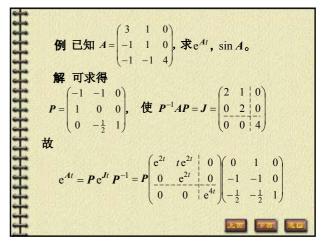
$$= \begin{pmatrix} \sin t - \sin 2t + \sin 3t & \sin t - \sin 2t & -\sin 2t + \sin 3t \\ \sin 2t - \sin 3t & \sin 2t & \sin 2t - \sin 3t \\ -\sin t + \sin 2t & -\sin t + \sin 2t & \sin 2t \end{pmatrix}$$

$$e^{A} = \begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 1 & 1 & 1 \\ & & & \\ & &$$









$$= \begin{pmatrix} te^{t} + e^{2t} & te^{2t} & 0\\ -te^{2t} & e^{2t} - te^{2t} & 0\\ \frac{1}{2}e^{2t} - \frac{1}{2}e^{4t} & \frac{1}{2}e^{2t} - \frac{1}{2}e^{4t} & e^{4t} \end{pmatrix}$$

$$\sin A = P(\sin J)P^{-1} = P\begin{pmatrix} \sin 2 & \cos 2 & 0\\ 0 & \sin 2 & 0\\ 0 & 0 & \sin 4 \end{pmatrix}$$

$$= \begin{pmatrix} \sin 2 + \cos 2 & \cos 2 & 0\\ -\cos 2 & \sin 2 - \cos 2 & 0\\ \frac{1}{2}\sin 2 - \frac{1}{2}\sin 4 & \frac{1}{2}\sin 2 - \frac{1}{2}\sin 4 & \sin 4 \end{pmatrix}$$

例 已知
$$A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 3 \end{pmatrix}$$
, 试计算 e^A , e^{At} , $\sin At$, $\cos A_o$ 解 法1. $\det(\lambda I - A) = (\lambda - 2)^3$
 A 的特征值为 $\lambda_1 = \lambda_2 = \lambda_3 = 2$ (三重) 设 $r(\lambda) = b_0 + b_1 \lambda + b_2 \lambda^2$
列方程组:
1) 求 e^A

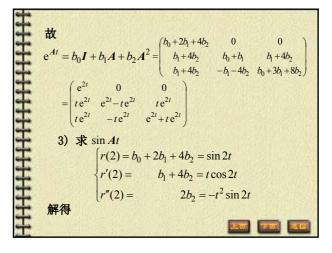
$$\begin{cases} r(2) = b_0 + 2b_1 + 4b_2 = e^2 \\ r'(2) = b_1 + 4b_2 = e^2 \end{cases}$$
 解得 $\begin{cases} b_0 = e^2 \\ b_1 = -e^2 \\ b_2 = \frac{1}{2}e^2 \end{cases}$

例 已知
$$A = \begin{pmatrix} -1 & -2 & 6 \\ -1 & 0 & 3 \\ -1 & -1 & 4 \end{pmatrix}$$
, 求 e^{At} , $\sin A_o$
解 可求得相似变换阵
$$P = \begin{pmatrix} -1 & 2 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$
, 使 $P^{-1}AP = J = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 \end{pmatrix}$
故
$$e^{At} = Pe^{Jt}P^{-1} = P\begin{pmatrix} e^t & e^t & te^t \\ e^t & e^t \end{pmatrix}$$

$$= \begin{pmatrix} (1-2t)e^{t} & -2te^{t} & 6te^{t} \\ -te^{t} & (1-t)e^{t} & 3te^{t} \\ -te^{t} & -te^{t} & (1+3t)e^{t} \end{pmatrix}$$

$$\sin A = P(\sin J)P^{-1} = P\begin{pmatrix} \frac{\sin 1}{\sin 1} & \cos 1 \\ \sin 1 & \sin 1 \end{pmatrix} P^{-1}$$

$$= \begin{pmatrix} \sin 1 - 2\cos 1 & -2\cos 1 & 6\cos 1 \\ -\cos 1 & \sin 1 - \cos 1 & 3\cos 1 \\ -\cos 1 & -\cos 1 & \sin 1 + 3\cos 1 \end{pmatrix}$$

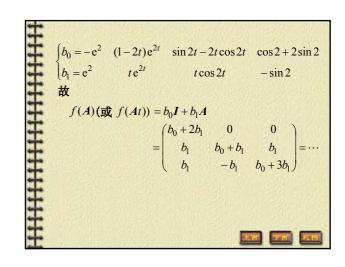


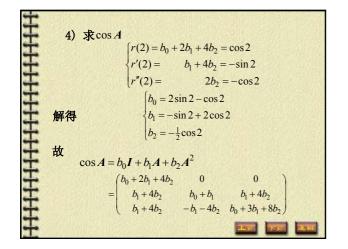
$$\begin{cases} b_0 = \sin 2t - 2t\cos 2t - 2t^2 \sin 2t \\ b_1 = t\cos 2t + 2t^2 \sin 2t \\ b_2 = -\frac{1}{2}t^2 \sin 2t \end{cases}$$

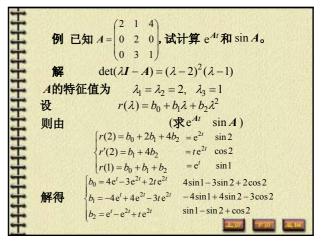
$$\Rightarrow \sin At = b_0 I + b_1 A + b_2 A^2$$

$$= \begin{pmatrix} b_0 + 2b_1 + 4b_2 & 0 & 0 \\ b_1 + 4b_2 & b_0 + b_1 & b_1 + 4b_2 \\ b_1 + 4b_2 & -b_1 - 4b_2 & b_0 + 3b_1 + 8b_2 \end{pmatrix}$$

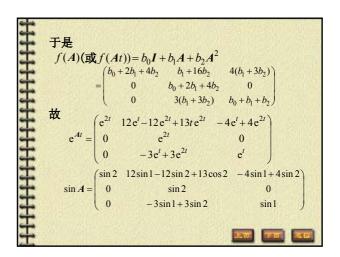
$$= \begin{pmatrix} \sin 2t & 0 & 0 \\ t\cos 2t & \sin 2t - t\cos 2t & t\cos 2t \\ t\cos 2t & -t\cos 2t & \sin 2t + t\cos 2t \end{pmatrix}$$

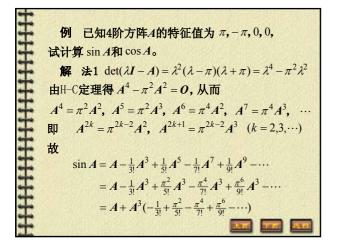


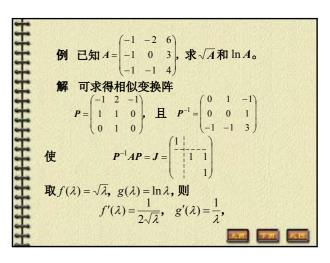


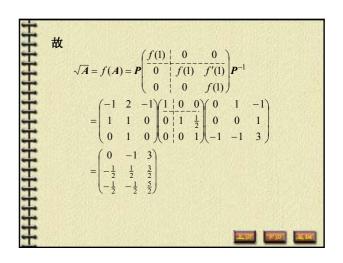


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cos 2
                          0
              -\sin 2 \sin 2 + \cos 2
                                     -\sin 2
                                -\sin 2 + \cos 2
             -\sin 2
                        sin 2
法2. 对应特征值2有2个线性无关的特征向量,
于是m_A(\lambda) = (\lambda - 2)^2是A的最小多项式。设
                  r(\lambda) = b_0 + b_1 \lambda
由
              (求eA
                      e^{At}
                             \sin At
                                     \cos A)
                       e^{2t}
(r(2) = b_0 + 2b_1 = e^2
                             \sin 2t
                                       cos 2
          b_1 = e^2 t e^{2t}
 r'(2) =
                             t\cos 2t
                                        -\sin 2
解得
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則由
$$(求 \sin A \cos A)$$

$$r(\pi) = b_0 + \pi b_1 + \pi^2 b_2 + \pi^3 b_3 = \sin \pi = 0 = \cos \pi = -1$$

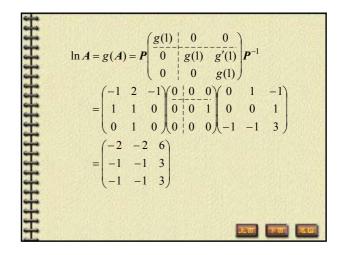
$$r(-\pi) = b_0 - \pi b_1 + \pi^2 b_2 - \pi^3 b_3 = \sin(-\pi) = 0 = \cos(-\pi) - 1$$

$$r(0) = b_0 = \sin 0 = 0 = \cos 0 = 1$$

$$r'(0) = b_1 = \cos 0 = 1 = -\sin 0 = 0$$

$$\begin{cases} b_0 = 0 & b_1 = 1 \\ b_1 = 1 & b_1 = 0 \\ b_3 = -\frac{1}{\pi^2} & b_3 = 0 \end{cases}$$

$$\Rightarrow \sin A = A - \frac{1}{\pi^2} A^3, \cos A = I - \frac{2}{\pi^2} A^2$$



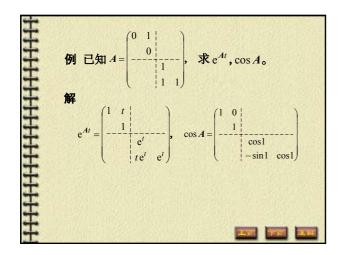
例 已知
$$A = \begin{pmatrix} -2 \\ 1 & -2 \\ 1 & -2 \end{pmatrix}$$
,求 $f(A)$ 。

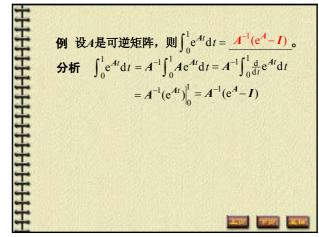
解
$$A^{T} = \begin{pmatrix} -2 & 1 \\ -2 & 1 \\ -2 \end{pmatrix}, f(A^{T}) = \begin{pmatrix} f(-2) & f'(-2) & \frac{1}{2}f''(-2) \\ f(-2) & f'(-2) \\ f(-2) \end{pmatrix}$$
故 $f(A) = (f(A^{T}))^{T} = \begin{pmatrix} f(-2) \\ f'(-2) & f(-2) \\ \frac{1}{2}f''(-2) & f'(-2) \end{pmatrix}$

所以
$$\frac{d}{dt}A^{-1}(t) = \begin{pmatrix} 0 & \frac{1}{2}e^{-t} \\ 0 & -\frac{1+2t}{2t^2}e^{-2t} \end{pmatrix} (由定义)$$
法2.
$$\frac{d}{dt}A^{-1}(t) = -A^{-1}(t)(\frac{d}{dt}A(t))A^{-1}(t)$$

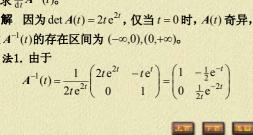
$$= -\begin{pmatrix} 1 & -\frac{1}{2}e^{-t} \\ 0 & \frac{1}{2t}e^{-2t} \end{pmatrix}\begin{pmatrix} 0 & (1+t)e^{t} \\ 0 & 2(1+2t)e^{2t} \end{pmatrix}\begin{pmatrix} 1 & -\frac{1}{2}e^{-t} \\ 0 & \frac{1}{2t}e^{-2t} \end{pmatrix}$$

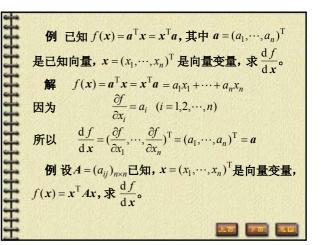
$$= \begin{pmatrix} 0 & \frac{1}{2}e^{-t} \\ 0 & -\frac{1+2t}{2t^2}e^{-2t} \end{pmatrix}$$





§ 4 矩阵微积分 **例** 设 $A(t) = \begin{pmatrix} 1 & t e^t \\ 0 & 2t e^{2t} \end{pmatrix}$, 求 $A^{-1}(t)$ 的存在区间,并求 $\frac{d}{dt}A^{-1}(t)$ 。 **解** 因为 det $A(t) = 2t e^{2t}$,仅当 t = 0 时,A(t) 奇异,故 $A^{-1}(t)$ 的存在区间为 $(-\infty,0),(0,+\infty)$ 。





解
$$f(x) = x^{T} A x = \sum_{s=1}^{n} \sum_{t=1}^{n} a_{st} x_{s} x_{t}$$

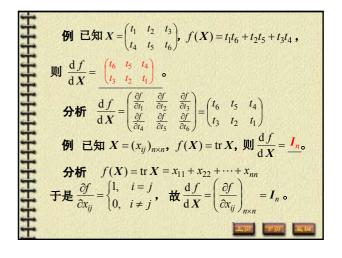
$$= x_{1} \sum_{t=1}^{n} a_{1t} x_{t} + x_{2} \sum_{t=1}^{n} a_{2t} x_{t} + \dots + x_{t} \sum_{t=1}^{n} a_{it} x_{t} + \dots + x_{n} \sum_{t=1}^{n} a_{nt} x_{t}$$
因为
$$\frac{\partial f}{\partial x_{i}} = a_{1t} x_{1} + \dots + a_{i-1,i} x_{i-1} + (a_{it} x_{i} + \sum_{t=1}^{n} a_{it} x_{t}) + a_{i+1,i} x_{i+1} + \dots + a_{nt} x_{n}$$

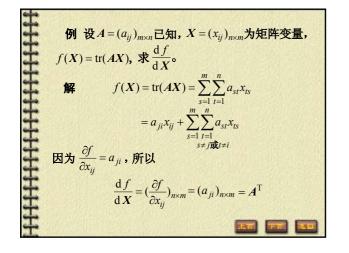
$$= \sum_{s=1}^{n} a_{si} x_{s} + \sum_{t=1}^{n} a_{it} x_{t}$$

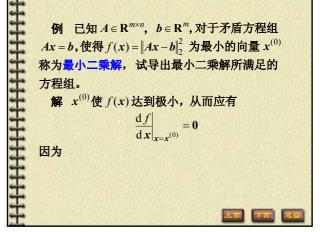
例 设
$$X = (x_{ij})_{n \times n}$$
 是矩阵变量, $f(X) = \det X$,
试求 $\frac{\mathrm{d} f}{\mathrm{d} X}$ 。
解 设 X_{ij} 是 $\det X$ 中元素 x_{ij} 的代数余子式,则
 $f(X) = \det X = x_{i1}X_{i1} + \dots + x_{ij}X_{ij} + \dots + x_{in}X_{in}$
因为 $\frac{\partial f}{\partial x_{ij}} = X_{ij}$,所以
 $\frac{\mathrm{d} f}{\mathrm{d} X} = (\frac{\partial f}{\partial x_{ij}})_{n \times n} = (X_{ij})_{n \times n} = (X^*)^{\mathrm{T}}$
当 X 可逆时,
 $\frac{\mathrm{d} f}{\mathrm{d} X} = ((\det X)X^{-1})^{\mathrm{T}} = (\det X)X^{-\mathrm{T}}$

所以
$$\frac{\mathrm{d}f}{\mathrm{d}x} = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix} = \begin{pmatrix} \sum_{s=1}^n a_{s1} x_s + \sum_{t=1}^n a_{t1} x_t \\ \vdots \\ \sum_{s=1}^n a_{sn} x_s + \sum_{t=1}^n a_{nt} x_t \end{pmatrix}$$

$$= A^{\mathrm{T}} x + A x = (A^{\mathrm{T}} + A) x$$
特例,当 $A^{\mathrm{T}} = A$ 时,即A对称时, $\frac{\mathrm{d}f}{\mathrm{d}x} = 2Ax$ 。

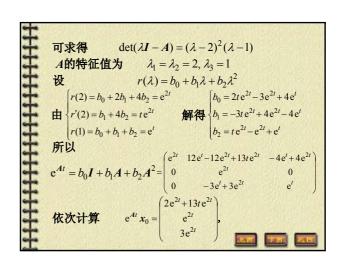




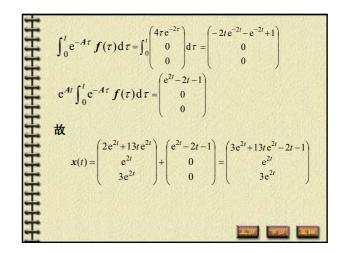


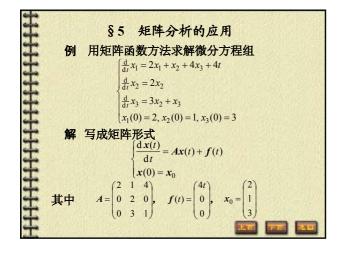
$$f(x) = \|Ax - b\|_{2}^{2} = (Ax - b)^{T} (Ax - b)$$

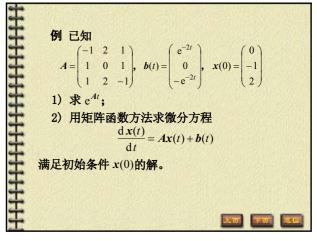
$$= x^{T} A^{T} Ax - x^{T} A^{T} b - b^{T} Ax + b^{T} b$$
由前几例得
$$\frac{d f}{d x} = 2A^{T} Ax - 2A^{T} b$$
于是
$$\frac{d f}{d x}\Big|_{x = x^{(0)}} = 2A^{T} Ax^{(0)} - 2A^{T} b = 0$$
即
$$A^{T} Ax^{(0)} = A^{T} b$$
称 $A^{T} Ax = A^{T} b$ 为法方程组,它是最小二乘解所满足的方程组。



例 已知
$$A = (a_{ij})_{m \times n}$$
, $x = (x_1, \dots, x_m)^T$, 且
$$F(x) = x^T A$$
, 求 $\frac{\mathrm{d} F}{\mathrm{d} x} \circ$
解 $F(x) = x^T A = (\sum_{k=1}^m x_k a_{k1}, \sum_{k=1}^m x_k a_{k2}, \dots, \sum_{k=1}^m x_k a_{kn})$
因为 $\frac{\partial F}{\partial x_i} = (a_{i1}, a_{i2}, \dots, a_{in})$
所以 $\frac{\mathrm{d} F}{\mathrm{d} x} = \begin{pmatrix} \frac{\partial F}{\partial x_1} \\ \vdots \\ \frac{\partial F}{\partial x_m} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} = A$







解 1)
$$\det(\lambda I - A) = (\lambda + 2)^{2}(\lambda - 2)$$

法1. 可求得
$$P = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}, \quad \notin P^{-1}AP = \begin{pmatrix} -2 \\ & -2 \\ & 2 \end{pmatrix}$$
所以
$$e^{At} = P \begin{pmatrix} e^{-2t} \\ e^{-2t} \\ e^{2t} \end{pmatrix} P^{-1}$$

$$= \frac{1}{4} \begin{pmatrix} 3e^{-2t} + e^{2t} & -2e^{-2t} + 2e^{2t} & -e^{-2t} + e^{2t} \\ -e^{-2t} + e^{2t} & -2e^{-2t} + 2e^{2t} & -e^{-2t} + e^{2t} \\ -e^{-2t} + e^{2t} & -2e^{-2t} + 2e^{2t} & 3e^{-2t} + e^{2t} \end{pmatrix}$$

法2. 可求得
$$m_A = (\lambda + 2)(\lambda - 2)_{\text{o}}$$
 设
$$r(\lambda) = b_0 + b_1 \lambda$$
 由
$$\begin{cases} r(-2) = b_0 - 2b_1 = \mathrm{e}^{-2t} \\ r(2) = b_0 + 2b_1 = \mathrm{e}^{2t} \end{cases}$$
 解得
$$\begin{cases} b_0 = \frac{1}{2}(\mathrm{e}^{-2t} + \mathrm{e}^{2t}) \\ b_1 = \frac{1}{4}(\mathrm{e}^{2t} - \mathrm{e}^{-2t}) \end{cases}$$
 于是
$$\mathrm{e}^{At} = b_0 \mathbf{I} + b_1 \mathbf{A} = \begin{pmatrix} b_0 - b_1 & 2b_1 & b_1 \\ b_1 & b_0 & b_1 \\ b_1 & 2b_1 & b_0 - b_1 \end{pmatrix} = \cdots$$
 法3. 设 $r(\lambda) = b_0 + b_1 \lambda + b_2 \lambda^2_{\text{o}}$ 由
$$\begin{cases} r(-2) = b_0 - 2b_1 + 4b_2 = \mathrm{e}^{-2t} \\ r'(-2) = b_1 - 4b_2 = t\mathrm{e}^{-2t} \end{cases}$$
 解得
$$\begin{cases} b_0 = t\mathrm{e}^{-2t} + \frac{1}{4}\mathrm{e}^{2t} + \frac{3}{4}\mathrm{e}^{-2t} \\ b_1 = \frac{1}{4}(\mathrm{e}^{2t} - \mathrm{e}^{-2t}) \\ b_2 = -\frac{1}{4}t\mathrm{e}^{-2t} + \frac{1}{16}(\mathrm{e}^{2t} - \mathrm{e}^{-2t}) \end{cases}$$

故
$$e^{At} = b_0 I + b_1 A + b_2 A^2$$

$$= \begin{pmatrix} b_0 - b_1 + 4b_2 & 2b_1 & b_1 \\ b_1 & b_0 + 4b_2 & b_1 \\ b_1 & 2b_1 & b_0 - b_1 + 4b_2 \end{pmatrix} = \cdots$$
2) 计算 $e^{-A\tau} b(\tau) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \int_0^t e^{-A\tau} b(\tau) d\tau = \begin{pmatrix} t \\ 0 \\ -t \end{pmatrix},$

$$x(t) = e^{At} x(0) + e^{At} \int_0^t e^{-A\tau} b(\tau) d\tau$$

$$= \begin{pmatrix} 0 \\ -e^{-2t} \\ 2e^{-2t} \end{pmatrix} + \begin{pmatrix} te^{-2t} \\ 0 \\ -te^{-2t} \end{pmatrix} = e^{-2t} \begin{pmatrix} t \\ -1 \\ 2-t \end{pmatrix}$$