Congruences and Partition Cranks

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In this post, I will develop the concept of partition cranks from their original motivation in Ramanujan congruences. I came across this topic for the first time recently and thought it would be nice to share this bit of information. Along the way, I'll mention a couple other interesting questions regarding partitions and some general topics of interest in modern number theory. I've also adopted Evan Chen's Infinite Napkin latex template, which you can find on his github.

1 Introduction

Recall the definition of a partition:

Definition 1.1 (Partition) — Consider $n \in \mathbb{N}$. We call $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_m) \vdash n$ a partition of n if and only if

$$\sum_{i=1}^{m} \lambda_i = n \text{ and } \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m > 0$$

Each λ_i is called a **part** and we denote $\ell(\lambda) = m$ as the **length** of λ and $|\lambda| = n$ as the size of λ.

To walk through a simple example, valid partitions of 4 include (4), $(3, 1)$, $(2, 2)$, $(2, 1, 1)$, and $(1, 1, 1, 1)$. Notice that the definition accounts for partitions that have the same parts but are ordered differently from one another through the requirement that each λ_i must be greater than or equal to the the subsequent one. Otherwise, $(1, 1, 2)$ and $(2, 1, 1)$ would be distinct from one another.

Naturally, the next question to ask is how many ways are there to partition a given number. To study this, we will consider the partition function. The partition function is defined by $p(n)$ = the number of partitions of n and we can take it a step further by defining $p(n, m)$ as the number of partitions of n with length m. For the sake of clarity, $p(0)$ is defined to be 1, and the partition λ of 0 is called the empty partition. There is no known closed form expression for the partition function, so instead we study it using generating functions. If you're unfamiliar with generating functions, I suggest reading this article to brush up on the definition and other details that will help with understanding its applications.

Theorem 1.2 — The generation function for the partition numbers with given length is

$$P(z,q) := \sum_{\lambda\in\mathcal{P}} z^{\ell(\lambda)}q^{|\lambda|} = \sum p(n,m)z^m q^n = \prod_{n=1}^{\infty} \frac{1}{1-zq^n}$$

where $\mathcal{P}$ is the set of all partitions of n.

Proof. Note that both z and q are formal variables so $|zq| < 1$, yielding the following geometric series on the right hand side:

$$\frac{1}{1-zq^n} = \sum_{k=0}^{\infty} z^k q^{nk}$$

Then, the right side becomes

$$\prod_{n=1}^{\infty} \frac{1}{1-zq^n} = \prod_{n=1}^{\infty}\sum_{k=0}^{\infty} z^k q^{nk} = (1+zq+z^2q^2+\cdots)(1+zq^2+z^2q^4+\cdots)(1+zq^3+z^2q^6+\cdots$$

The i'th sum in this product describes the number of times i appears in our partition. We have a multitude of choices to construct q^n term. We could take the q^n term from the first sum and multiply it by 1 in all the other sums, which would yield a length n partition full of 1's. Our z^n attached to q^n is also in accordance with $\ell(\lambda) = n$ so this is a valid partition of n. Alternately, we could take zq^2 from the second sum and $z^{n-2}q^{n-2}$ from the first sum yielding the partition $(2, 1, \ldots, 1)$. We have constructed a q^n term and z^{n-1} term attached to it which is in accordance with the length of the partition $n-1$. Continuing in this fashion, we can observe that there are $p(n)$ ways to construct a q^n term. □

Corollary 1.3 — The generating function for the partition numbers with no restriction on length (i.e. the normal partition function) is

$$P(q) := \sum_{\lambda\in\mathcal{P}} q^{|\lambda|} = \sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}$$

Proof. This follows from our proof of Theorem 1.2. □

There are other, special types of partition functions, including partitions into odd/even parts, partitions into distinct parts, and colored partition. Generating functions are an extremely useful tools in statistics and number theory. And although understanding the partition generating function is not necessary to the theory of partition cranks, it is important philosophical background for understanding why the crank function was developed.

2 Ramanujan's Congruences

With a way to get exact values of the partition function, the next area of interest is whether the partition function has any notable division properties. In a 1919 paper and 1921 posthumous manuscript, Ramanujan was able to identify some of these properties, specifically for partitions of numbers related to primes.

Theorem 2.1 (Ramanujan's Congruences) — Let $n \in \mathbb{N}_{\geq 0}$. Then we have the following congruences

1. $p(5n+4) \equiv 0 \pmod 5$
2. $p(7n+5) \equiv 0 \pmod 7$
3. $p(11n+6) \equiv 0 \pmod{11}$

Proof. For the proofs mod 5 and mod 7 I will give a basic outline of the technique used. Introducing some new notation,

$$\sum_{n\geq 0} p(n)q^n = \prod_{n\geq 1} \frac{1}{1-q^n} =: \frac{1}{E(q)} \text{ where } E(q) = \prod_{n\geq 1} 1-q^n$$

It can be shown that $E(q) = 1 - q - q^2 + q^5 + q^7 - \cdots$ implying $p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + \cdots$.
Using Jacobi's identity

$$E(q)^3 = \sum_{n\geq 0} (-1)^n (2n+1) q^{\frac{n^2+n}{2}}$$

we have the equations $E(q) = J_0 + J_1 \pmod 5$ and $E(q) \equiv J_0 + J_1 + J_3 \pmod 7$. Here, J_i represents the terms for which the power of q is congruence to i mod 5 and 7 respectively. Extracting the terms for which the power of q is congruent to 4 (mod 5) and 5 (mod 7), we have

$$\sum_{n\geq 0} p(5n+4)q^{5n+4} \equiv 0 \pmod 5 \text{ and } \sum_{n\geq 0} p(7n+5)q^{7n+5} \equiv 0 \pmod 7$$

which yield our desired result. The proof for the mod 11 congruence is much more elusive and was originally proven using Eisenstein series and modular forms, which you can learn about here. It is also possible to use a similar approach as the mod 5 and 7 cases, and I leave it as a challenge to you to find such a proof. ☐

This result is quite fascinating as we've found that numbers that are 4, 5, and 6 greater than multiples of 5, 7, and 11 respectively have a total number of partitions that is divisible by the prime it was constructed by. We can illustrate these congruences with a short example:

Example 2.2

Consider $n = 0$. Listing the partitions of $5n+4$ gives us (4), $(3,1)$, $(2,2)$, $(2,1,1)$, and $(1,1,1,1)$. Then $p(5n+4) = 5 \equiv 0 \pmod 5$.

For $n = 1$, the partitions of $5n+4$ are (9), $(8,1)$, $(7,2)$, $(7,1,1)$, $(6,3)$, $(6,2,1)$, $(6,1,1), \ldots$ which give us $p(5n+4) = 30 \equiv 0 \pmod 5$. These examples focus on the $p(5n+4)$ case but similar results can be observed mod 7 and mod 11, completing Ramanujan's congruences.

There are other congruences that can be observed for the partition function, but none as simple and eloquent as these. In fact, the 3 congruences proven by Ramanujan are the only ones of the form $p(\ell n+m) \equiv 0 \pmod{\ell}$ where ℓ is prime. For example, there does a exist a partition congruence mod 13, namely $p(11^3 * 13n + 237) \equiv 0 \pmod{13}$, but it is not nearly as eloquent as any of the mod 5, 7, or 11 congruences.

3 Ranks and Cranks of Partitions

Unfortunately, Ramanujan's results on partition functions and modular congruences were done through generating functions and do not provide much combinatorial insight on the behavior of the partition function. We know that certain numbers have a total number of partitions that is divisible by either 5, 7, or 11 and can therefore be split up among groups of equal size, but we don't know why this occurs. In 1944, Freeman Dyson attempted to resolve this through defining a rank function and making several rank conjectures.

Definition 3.1 (Partition Rank) — Let $n \in \mathbb{N}$ and λ be a partition of n. Then the rank of λ, denoted $r(\lambda)$, is equal to the largest part of λ minus the number of parts of λ. That is, $r(\lambda) = \lambda_1 - \ell(\lambda)$. In addition, the number of partitions of n with rank m be denoted $N(m, n)$ and the number of partitions of n that are congruent to m modulo q be denoted by $N(m, q, n)$.

Ramnujan's proof provides that $p(5n+4) \equiv 0 \pmod{5}$, or that the partitions of $5n+4$ can be split into 5 categories of equal size. The rank function is an initial attempt to define these categories by using modular congruences. To illustrate this, consider the following example:

Example 3.2

Let $n = 0$ and consider the mod 5 case. $5n+4 = 4$ has partitions (4), $(3,1)$, $(2,2)$, $(2,1,1)$, and $(1,1,1,1)$ whose ranks are:

$(4) \to 4-1 = 3 \equiv 3 \pmod{5}$
$(3,1) \to 3-2 = 1 \equiv 1 \pmod{5}$
$(2,2) \to 2-2 = 0 \equiv 0 \pmod{5}$
$(2,1,1) \to 2-3 = -1 \equiv 4 \pmod{5}$
$(1,1,1,1) \to 1-4 = -3 \equiv 2 \pmod{5}$

Grouping the partitions by their ranks modulo 5, we can see that there are an equal number of partitions in each member of the congruence class of 5.

The next problem to consider is whether the rank function can be applied in this manner to all numbers of the form $5n+4$ and whether it applies to the mod 7 and mod 11 cases. Note that the rank can also be illustrated through the use of Young/Ferrer diagrams, which show that the rank can be used to characterize symmetry in a partition, an interesting topic I recommend the reader look into. Additionally, a second definition of the partition rank exists, although it is based on Durfee squares and primarily used in combinatorics and statistics.

Theorem 3.3 (Dyson's Rank Conjecture) — The following are conjectured to be true regarding the rank function:

1. $N(0,5,5n+4) = N(1,5,5n+4) = N(2,5,5n+4) = N(3,5,5n+4) = N(0,5,5n+4)$
2. $N(0,7,7n+5) = N(1,7,7n+5) = N(2,7,7n+5) = N(3,7,7n+5) = N(4,7,7n+5) = N(5,7,7n+5) = N(6,7,7n+5)$

This conjecture states that the phenomenon observed in Example 3.2 applies to all integers of the form $5n+4$ and $7n+5$. That is, the partitions of these numbers can be grouped into equally sized categories through their ranks modulo 5 and 7 respectively. This explains the mod 5 and mod 7 congruences but leaves the mod 11 case unaccounted for. Applying the rank function to the integer 6, which has the form $11n+6$ proves problematic as $N(4,11,11n+6) = 0 = N(7,11,11n+6)$ implying that the rank modulo 11 cannot split up the partitions of a $11n+6$ number equally. Because of this, the rank function cannot be used to explain all three congruences simultaneously, which Dyson addressed by introducing the crank.

Theorem 3.4 (Dyson's Crank Conjecture) — There exists an arithmetical coefficient similar to, but more recondite than the rank of a partition that will be referred to as the crank. Suppose $M(m,q,n)$ denotes the number of partitions of n with crank congruent to m modulo p. Then the following properties are satisfied:

1. $M(m,q,n) = M(q-m,q,n)$
2. $M(0,11,11n+6) = M(1,11,11n+6) = M(2,11,11n+6) = M(3,11,11n+6) = M(4,11,11n+6) = M(5,11,11n+6) = M(6,11,11n+6)$

The crank function was defined explicitly in 1988 by Andrews and Garvan, resolving the mod 11 congruence and introducing a new branch of interest for research in partition functions.

Definition 3.5 (Crank of a Partition) — Introducing some new notation, let $n \in \mathbb{N}$ and $\lambda \vdash n$. Then $\omega(\lambda) :=$ the number of 1's in λ and $\mu(\lambda) :=$ the number of parts of λ that are greater than $\omega(\lambda)$. Then, the crank of λ, denoted $c(\lambda)$ is defined by

$$c(\lambda) = \begin{cases} \ell(\lambda) & \text{if } \omega(\lambda) = 0 \\ \mu(\lambda) - \omega(\lambda) & \text{if } \omega(\lambda) > 0 \end{cases}$$

The definition of the crank is a bit more complex than that of the rank function, but still resembles it in some form. The crank function is proven to satisfy both of the properties from Dyson's conjecture and accounts for the mod 5 and mod 7 congruences as well. As a result, the crank simultaneously explains all three of Ramnujan's congruences and is a topic of interest for mathematicians studying the partition function.

Example 3.6

Consider $n = 0$ and the congruence $p(11n + 6) \equiv 0 \pmod{11}$. The partitions of 11 and their associated ranks are below:

$(6) \to 6 \equiv 6 \pmod{11}$
$(5, 1) \to 1 - 1 = 0 \equiv 0 \pmod{11}$
$(4, 2) \to 4 \equiv 4 \pmod{11}$
$(4, 1, 1) \to 1 - 2 = -1 \equiv 10 \pmod{11}$
$(3, 3) \to 3 \equiv 3 \pmod{11}$
$(3, 2, 1) \to 2 - 1 = 1 \pmod{11}$
$(3, 1, 1, 1) \to 0 - 3 = -3 \equiv 8 \pmod{11}$
$(2, 2, 2) \to 2 \equiv 2 \pmod{11}$
$(2, 2, 1, 1) \to 0 - 2 = -2 \equiv 9 \pmod{11}$
$(2, 1, 1, 1, 1) \to 0 - 4 = -4 \equiv 7 \pmod{11}$
$(1, 1, 1, 1, 1, 1) \to 0 - 6 = -6 \equiv 5 \pmod{11}$

Grouping the partitions together by their crank modulo 11, one can see that we can obtain 11 equally sized categories of partitions, which "explains" why $p(11n + 6)$ is divisible by 11. Further investigation will show that this holds for all integers of the form $11n + 6$ and a similar method can be applied for mod 5 and mod 7 as well.

Taking a step back, it's interesting to see that these divisibility properties of partition functions exist and they they seem to be particular (pun intended) to the numbers 5, 7, and 11. It makes us wonder whether these three numbers are special in some way and whether other analogs of the ordinary partition function can yield similar result. I mentioned the timeline of the development of the rank and cranks of partitions to emphasize the passage of time and how long it takes to come up with these types of concepts. Something as simple as the rank took almost 30 years to construct and another 10 prove and keeping this in mind helps illustrate that the development of mathematics is ongoing!

As a final exercise, I suggest looking into an sort of relationship that may exist between the rank and crank. Both of them can be used to combinatorially prove the mod 5 and mod 7 congruence and it's interesting to compare how the rank and crank distribute the partitions of a number. You will find that the rank of a certain partition $\pmod{\ell}$ is different from that of the crank $\pmod{\ell}$ but they do seem to agree on specific types of partitions.

4 Further Results on Partitions

This section will serve as a brief summary of some of the other results that have been proved about partitions that I think are extremely cool. Although I didn't mention it earlier, the partition function has asymptotic behavior, specifically that

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)$$

I thought that this was especially interesting as it gives us a better understanding of the growth of the partition function and has a form that can be compared to the prime counting function. The proof of this was the origin of the circle method, which is now a general tool used in analytical number theory to prove some pretty surprising properties of partitions and generation functions.

I've focusd mainly on the ordinary partition as the most well known results have all been about the regular partition, but there are various other types of partitions. Specifically, partitions into distinct parts, and partitions into odd and even parts (as the names suggest, these are functions that generate partitions where each part is distinct, odd, or even). We can also place restrictions on the smallest and largest part, for example, the generating function for partitions with largest part less than or equal to k is

$$P_{\leq k}(q) = \sum_{n\geq 0} p_{\leq k}(n)q^n = \prod_{n=1}^{k} \frac{1}{1-q^n}$$

This is simply the generating function for the ordinary partition function but with the product capped at k instead of infinity. I personally am interested in colored partitions, which are partitions in the parts are assigned a specific color and each part with a color different from another is a distinct part. All of these partitions are interesting to look into and they don't require a lot of mathematical background to dive into. That being said, thanks for taking the time to tune in and I hope this article was an enjoyable read.For more content similar to this check out my blog!