Math 120
2.3 Complex Numbers

Objectives:

1. Add and subtract complex numbers
2. Multiply complex numbers
3. Divide complex numbers
4. Perform operations with square roots of negative numbers

Topic #1: The Imaginary Unit and Complex Numbers

Recall that the domain for the square root function $f(x) = \sqrt{x}$ must be a non-negative number; $x \geq 0$

However, the domain is a discussion about **Real Numbers**. Here we will look at the imaginary unit and complex #'s

Consider the expression $\sqrt{-1}$; it is not defined by the real number system. However, expressions like this show up in mathematics and a new number system helps us deal with them.

The Imaginary Unit

The imaginary unit is defined as $\sqrt{-1}$

For example, we can use the unit to rewrite undefined numbers: $\sqrt{-25} = \sqrt{-1} \cdot \sqrt{25} = i\sqrt{25} = 5i$

The result is a pure imaginary number.
The imaginary unit $\boldsymbol{i}$ is the basis for the complex number system and also the solution to the equation $x^2 = -1$. In other words, $\boldsymbol{i^2 = -1}$.

The Complex Number System

Complex numbers have two parts – a real part and an imaginary part.
They are in the form ____a + bi____
where a is the real part and b is the imaginary part.

The conjugate of $\boldsymbol{a + bi}$ is $\boldsymbol{a - bi}$.

__Example #1__ – Identify the Real and Imaginary Part of the Complex Number

a. $3 - 6i$ Real = 3 I = -6

b. $16i$ Real = 0 I = 16

c. -7 Real = -7 I = 0

Topic #2: Operations on Complex Numbers

Addition and Subtraction

Adding or subtracting two complex numbers produces a new complex number.

Consider the complex numbers: $5 - 11i$ and $7 + 4i$

They can be added by combining like parts:

$(5-11i)+(7+4i)=$ $5+7$ and $-11i+4i$ $= \boxed{12-7i}$

They can be subtracted by distributing and combining like parts: $(5-11i)-(7+4i) = 5-11i-7-4i = \boxed{-2-15i}$

In both cases, a new complex number results!

Multiplication (FOIL)

Multiplying has the same effect; two complex numbers multiply to create a new one.

Consider the complex numbers: $5 - 11i$ and $7 + 4i$

They can be multiplied by using the distributive property/FOIL:

$(5-11i)(7+4i) = 35+20i-77i-44i^2$

$i^2=-1$

$35-57i-44(-1) = 35-57i+44$

$\boxed{79-57i}$

YOU TRY #1 – Perform the Operation and Write Result in Standard Form $a + bi$

a. $(7 + 2i) + (1 - 4i)$ $= 7 + 1$ and $2i + -4i$

$\boxed{8 - 2i}$

b. $(7 + 2i) - (1 - 4i)$ $= 7 + 2i - 1 + 4i$

$\boxed{6 + 6i}$

c. $(7 + 2i)(1 - 4i)$

FOIL, rewrite i^2, and combine like parts:

$7 - 28i + 2i - 8i^2$ $\qquad i^2 = -1$

$7 - 26i - 8(-1)$

$7 - 26i + 8 \longrightarrow \boxed{15 - 26i}$

d. $(-3+8i)(-3-8i)$ Conjugates

FOIL, rewrite i^2, and combine like parts:

$9+24i-24i-64i^2$

$9-64(-1)$

$9+64$

73

Notice the imaginary parts cancelled, this ALWAYS happens when multiplying conjugates: $a+bi$ and $a+bi$. **The useful formula for multiplying conjugates is** $\boxed{(\boldsymbol{a}+\boldsymbol{bi})(\boldsymbol{a}-\boldsymbol{bi})=\boldsymbol{a}^2+\boldsymbol{b}^2}$

e. $(2-5i)^2 = (2-5i)(2-5i)$

$4-10i-10i+25i^2$

$4-20i+25(-1)$

$\boxed{-21-20i}$

f. $(7+2i)(7-2i)$ Conjugates

$49-14i+14i-4i^2$

$49-4(-1)$

$49+4$

$\boxed{53}$

OR

$(a+bi)(a-bi)=a^2+b^2$

$(7+2i)(7-2i)=7^2+2^2$

$=49+4$

$=\boxed{53}$

Topic #3: Operations on Complex Numbers – Division

Besides addition, subtraction, and multiplication, we can also divide one complex number into another. The result gives a new complex number.

Consider the complex numbers: $5 - 11i$ and $7 + 4i$

Suppose we want to divide the first number by the second:

$$\frac{5-11i}{7+4i}$$

To do this, we can multiply both the numerator and denominator by the conjugate of the denominator;

we do this to create a real number divisor:

$$\frac{(5-11i)(7-4i)}{(7+4i)(7-4i)}$$

This will make the imaginary part of the complex number in the denominator go away

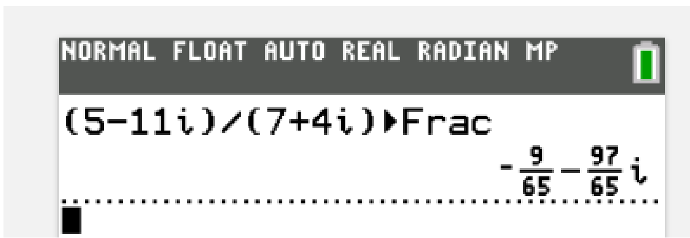
Since $(a+bi)(a-bi) = \boxed{a^2 + b^2}$

$$\frac{(5-11i)}{(7+4i)} \cdot \frac{(7-4i)}{(7-4i)} = \frac{35-20i-77i+44i^2}{7^2+4^2} = \frac{35-97i+44(-1)}{49+16}$$

$$= \frac{-9-97i}{55}$$

Technically this is not in $a + bi$ form, so split up the fraction: $\dfrac{-9-97i}{55} = \dfrac{-9}{55} - \dfrac{97i}{55}$

It is good to understand the process of introducing a conjugate to do the division, but feel free to use a graphing calculator to double check:



Example #1 – Perform the Division and Write Result in Standard Form $a + bi$

a. $\dfrac{7+4i}{5-11i}$

Multiply both the numerator and denominator by the conjugate of the denominator: $\dfrac{(7+4i)}{(5-11i)} \cdot \dfrac{(5+11i)}{(5+11i)}$

FOIL, combine like terms (feel free to use the conjugate formula for the denominator), and split up the fraction:

$$\frac{(7+4i)}{(5-11i)} \cdot \frac{(5+11i)}{(5+11i)} = \frac{35+77i+20i+44i^2}{5^2+11^2} = \frac{35+97i+44(-1)}{25+121}$$

$$= \frac{-9+97i}{146}$$

$$= \boxed{-\frac{9}{146} + \frac{97i}{146}}$$

b. $\dfrac{9i}{8-3i}$

Multiply both the numerator and denominator by the conjugate of the denominator: $\dfrac{9i}{(8-3i)} \cdot \dfrac{(8+3i)}{(8+3i)}$

Distribute, combine like terms (feel free to use the conjugate formula for the denominator), and split up the fraction: $\dfrac{72i+27i^2}{8^2+3^2} = \dfrac{-27+72i}{73}$

$$= -\frac{27}{73} + \frac{72i}{73}$$

Feel free to check answers with the calculator. The most useful idea here is that multiplying two conjugate complex numbers will remove the imaginary part!

__YOU TRY #2__ - Perform the Division and Write Result in Standard Form $a + bi$

a. $\dfrac{9i}{8+3i} \cdot \dfrac{(8-3i)}{(8-3i)}$

$$\frac{72i-27i^2}{8^2+3^2} = \frac{72i-27(-1)}{64+9}$$

$$= \frac{27+72i}{73}$$

$$= \boxed{\frac{27}{73} + \frac{72i}{73}}$$

b. $\dfrac{3+2i}{5-4i} \cdot \dfrac{(5+4i)}{(5+4i)}$

$$\frac{15+10i+12i+8i^2}{5^2+4^2} = \frac{15+22i+8(-1)}{25+16}$$

$$= \frac{7+22i}{41}$$

$$= \boxed{\frac{7}{41} + \frac{22i}{41}}$$

Topic #4: Operations on Square Roots with Negative Numbers

Review Pefect Squares 1, 4, 9, 16, 25 . . .

As stated before, square roots with negative numbers are undefined with the real number system. However, they are defined with the complex number system.

To operate, the undefined real numbers are converted to defined complex numbers and we proceed with the rules established earlier.

The principal square root of a negative number is defined as:

$\sqrt{-b} = \sqrt{-1} \cdot \sqrt{b} = i\sqrt{b}$, where $b > 0$ and $i = \sqrt{-1}$

Example #1 – Convert, Perform the Operation, and Write Result in Standard Form $a + bi$

a. $\sqrt{-4} + \sqrt{-36}$

$\sqrt{-1} \cdot \sqrt{4} + \sqrt{-1} \cdot \sqrt{36}$

$i \cdot 2 + i \cdot 6$

$2i + 6i$

$\boxed{8i}$

b. $3\sqrt{-4} - 7\sqrt{-64}$

$3 \cdot \sqrt{-1} \cdot \sqrt{4} - 7 \cdot \sqrt{-1} \cdot \sqrt{64}$

$3 \cdot i \cdot 2 - 7 \cdot i \cdot 8$

$6i - 56i$

$\boxed{-50i}$

c. $\sqrt{-25} * \sqrt{-225}$

Must deal with negative Square root BEFORE multiplying

$\sqrt{-1} \cdot \sqrt{25} \cdot \sqrt{-1} \cdot \sqrt{225}$

$i \cdot 5 \cdot i \cdot 15$

$75i^2 = 75(-1) = \boxed{-75}$

d. $\sqrt{-18} + \sqrt{-8}$

$\sqrt{-1} \cdot \sqrt{18} + \sqrt{-1} \cdot \sqrt{8}$

Not perfect squares so seperate into perfect squares

$i \cdot \sqrt{9} \cdot \sqrt{2} + i\sqrt{4} \cdot \sqrt{2}$

$i \cdot 3 \cdot \sqrt{2} + i \cdot 2 \cdot \sqrt{2}$

$3i\sqrt{2} + 2i\sqrt{2}$

Like terms

$\boxed{5i\sqrt{2}}$

e. $\sqrt{-18} * \sqrt{-8}$

$\sqrt{-1} \cdot \sqrt{9} \cdot \sqrt{2} \cdot \sqrt{-1} \cdot \sqrt{4} \cdot \sqrt{2}$

$i \cdot 3 \cdot \sqrt{2} \cdot i \cdot 2 \cdot \sqrt{2}$

$\sqrt{2} \cdot \sqrt{2} = (\sqrt{2})^2 = 2$

$i^2 \cdot 6 \cdot 2$

$-1 \cdot 6 \cdot 2 = \boxed{-12}$

f. $\left(-1+\sqrt{-5}\right)^2 = \left(-1+\sqrt{-5}\right)\left(-1+\sqrt{-5}\right)$

$= 1 - \sqrt{-5} - \sqrt{-5} + \sqrt{-5}\cdot\sqrt{-5}$

$= 1 - 2\sqrt{-5} + i\sqrt{5}\cdot i\sqrt{5}$

$= 1 - 2i\sqrt{5} + i^2\cdot\left(\sqrt{5}\right)^2$

$= 1 - 2i\sqrt{5} + -1\cdot 5$

$= \boxed{-4 - 2i\sqrt{5}}$

g. $\dfrac{-25+\sqrt{-50}}{15} = -\dfrac{25}{15} + \dfrac{\sqrt{-50}}{15}$

$\sqrt{-50}$

$\sqrt{-1}\cdot\sqrt{25}\cdot\sqrt{2}$

$i\cdot 5\cdot\sqrt{2}$

$= -\dfrac{5}{3} + \dfrac{5i\sqrt{2}}{15}$

$= \boxed{-\dfrac{5}{3} + \dfrac{i\sqrt{2}}{3}}$

YOU TRY #3 - Convert, Perform the Operation, and Write Result in Standard Form $a + bi$

a. $5\sqrt{-16} + 3\sqrt{-81}$

$5 \cdot \sqrt{-1} \cdot \sqrt{16} + 3 \cdot \sqrt{-1} \cdot \sqrt{81}$

$5 \cdot i \cdot 4 + 3 \cdot i \cdot 9$

$20i + 27i$

$\boxed{47i}$

b. $\left(-2 + \sqrt{-3}\right)^2$

$(-2+\sqrt{-3})(-2+\sqrt{-3})$

$4 - 2\sqrt{-3} - 2\sqrt{-3} + \sqrt{-3} \cdot \sqrt{-3}$

$4 - 4\sqrt{-3} + i\sqrt{3} \cdot i\sqrt{3}$

$4 - 4 \cdot \sqrt{-1} \cdot \sqrt{3} + i^2 (\sqrt{3})^2$

$4 - 4i\sqrt{3} + 3(-1)$

$\boxed{1 - 4i\sqrt{3}}$

c. $\dfrac{-14+\sqrt{-12}}{2}$

$\dfrac{-14}{2} + \dfrac{\sqrt{-12}}{2}$

$-7 + \dfrac{\sqrt{-1}\sqrt{4}\sqrt{3}}{2}$

$-7 + \dfrac{2i\sqrt{3}}{2}$

$\boxed{-7 + i\sqrt{3}}$