Math 120
4.1 Exponential Functions

Quadratic → $ax^2+bx+c = f(x)$
Linear → $f(x)=mx+b$
Rational → $f(x)=\frac{q(x)}{p(x)}$
polynomials → $f(x)=ax^N,\ldots+a$

Objectives:
1. Evaluate exponential functions.
2. Graph exponential functions.
3. Evaluate functions with base *e*.
4. Use compound interest formulas.

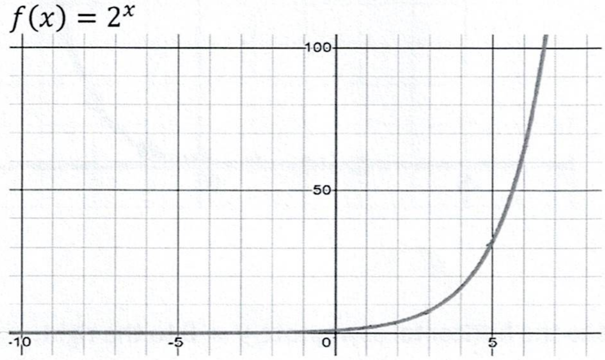
<u>Topic #1: Definition of an Exponential Function and Their Graphs</u>

$b>0$

Exponential functions occur when the independent variable is a power to some positive base: $f(x)=b^x$
Where b is the ___base___ and is a positive constant (other than the number 1) and x is any real number.
The y-intercept for any exponential function in this form is the point $f(0)=b^0=1$ $(0,1)$
$X=0$

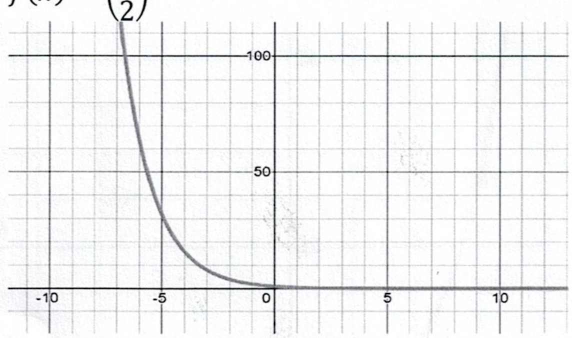
X^x

Consider the graph of the base "doubling" exponential function:
$f(x)=2^x$



The base is $b=2$ and the curve is smooth, so the domain is all real numbers $(-\infty,\infty)$.
Moving to the right, the curve grows rapidly and without bound. Moving to the left, the curve flattens to the horizontal asymptote of $y=0$. This gives the range as $(0,\infty)$
Any base where $b>1$has a similar curve; as the base gets larger the exponential function grows faster.

Consider the graph of the base "halving" exponential function:
$f(x)=\left(\frac{1}{2}\right)^x$



The base is $b=1/2$ and the curve is smooth, so the domain is all real numbers $(-\infty,\infty)$

Moving to the right, the curve flattens to the horizontal asymptote $y=0$. Moving to the left, the curve grows rapidly and without bound. This gives the range as $(0,\infty)$
Any base where $0<b<1$has a similar curve; as the base gets closer to zero the exponential function decreases faster.

Example #1 – Sketch a Graph of the Exponential Function

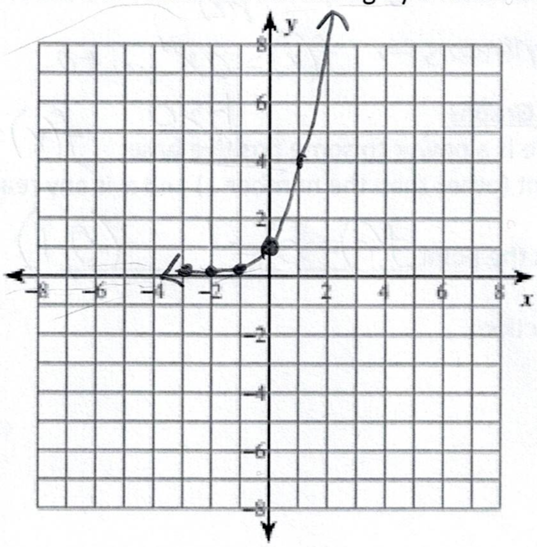
a) $f(x) = 4^x$

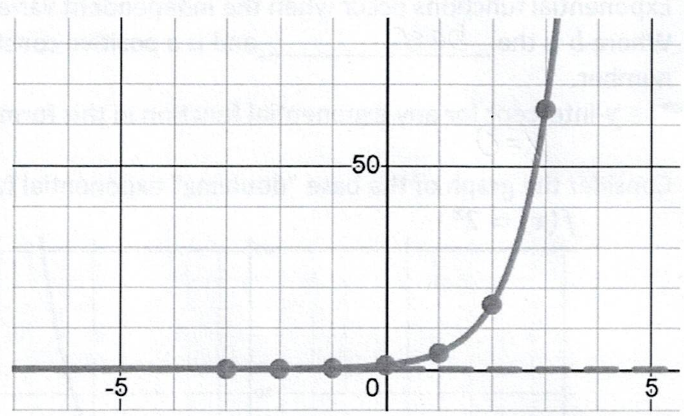
Since $b > 1$ the function grows without bound and moves to the horizontal asymptote $y = 0$ to the left.

Pick a few values and plot the points: $(-1, \frac{1}{4})$ $(1, 4)$

x	-3	-2	-1	0	1	2	3
f(x)	$4^{-3} = \frac{1}{4^3} = \frac{1}{64}$	$4^{-2} = \frac{1}{4^2} = \frac{1}{16}$	$4^{-1} = \frac{1}{4^1}$	$4^0 = 1$	$4^1 = 4$	$4^2 = 16$	$4^3 = 64$

Notice the values are increasing by a factor of 4, that is because $b = 4$.





b) $y = \left(\frac{1}{3}\right)^x$

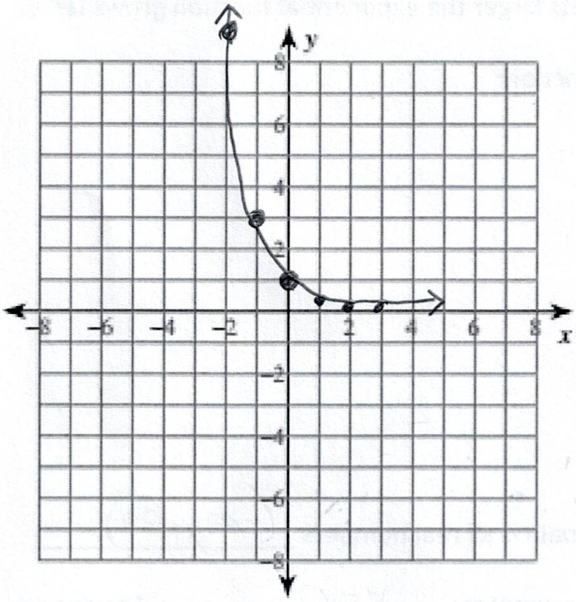
Since $b < 1$ the function decreases and moves to the horizontal asymptote $y = 0$ to the right.

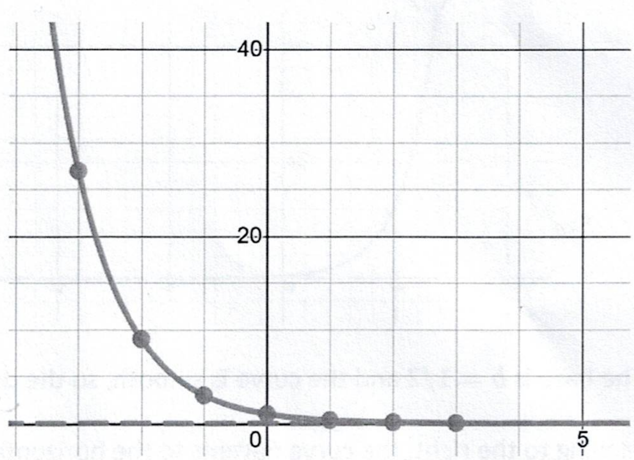
Pick a few values and plot the points:

x	-3	-2	-1	0	1	2	3
f(x)	$(\frac{1}{3})^{-3} = 3^3 = 27$	$(\frac{1}{3})^{-2} = 3^2 = 9$	$(\frac{1}{3})^{-1} = \frac{3}{1}$	1	$(\frac{1}{3})^1 = \frac{1}{3}$	$(\frac{1}{3})^2 = \frac{1}{9}$	$(\frac{1}{3})^3 = \frac{1}{27}$

Notice the values are decreasing by a factor of 1/3, that is because $b = 1/3$.

$\left(\frac{1}{3}\right)^2 = \frac{1^2}{3^2} = \frac{1}{9}$





Notice negative inputs DO NOT give negative outputs. It is due to the property of exponents:

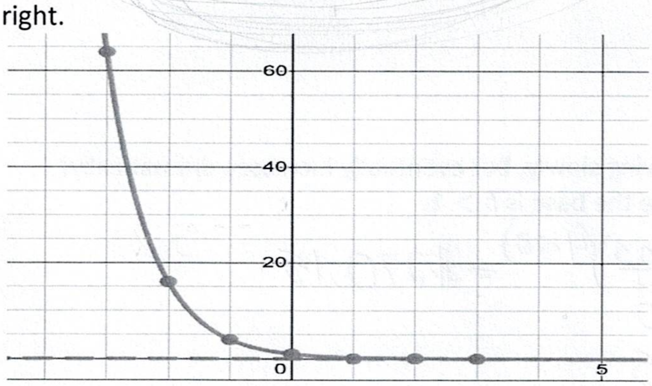
c) $y = 4^{-x}$

The exponent is negative, so we need to be careful since it is not in the base form definition. Alternatively, we could use the property of exponents to rewrite in base form:

In this form, we see the exponential function has a base $b < 1$ which is confirmed by the graph above.
Plot some points to get a sense of what happens left to right:

x	-3	-2	-1	0	1	2	3
f(x)	64	16	4	1	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{64}$

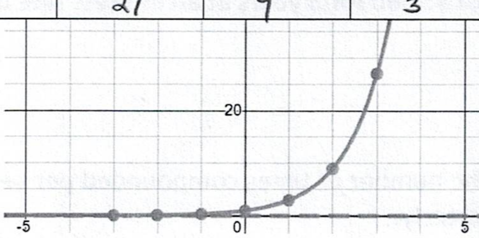
Plotting the points shows the function decreases and moves to the horizontal asymptote $y = 0$ to the right.



d) $y = \left(\frac{1}{3}\right)^{-x}$

The exponent is negative, so we need to be careful since it is not in the base form definition.
Again, we could rewrite:
Plot some points to get a sense of what happens left to right:

x	-3	-2	-1	0	1	2	3
f(x)	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27



$100
$10
$110
$11
$121

Topic #2: Compound Interest

Money in a savings account earns interest over time. When interest is compounded, the amount in the account earns interest, then the interest is added to the balance, and the new value becomes the amount in the account. The process of accumulating interest repeats and eventually the initial value invested grows significantly.

Here is the formula for compound interest: $A = P\left(1 + \frac{R}{N}\right)^{(N \cdot T)}$

Where
A is the Accumulated Amount
P is the initial Amount
r is the rate
n is the # compounds per year
t is the Time in years

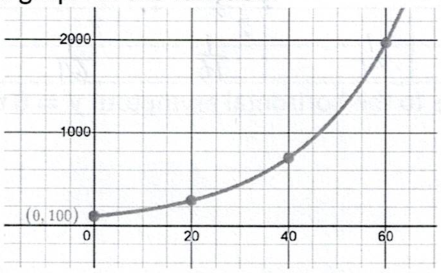
The most common ways interest is compounded per year are:

Name	Annual	Semiannual	Quarterly	Monthly
Times per year	$n = 1$	$n = 2$	$n = 4$	$n = 12$

D $\leftrightarrow$ P

Suppose a person has an initial investment of \$100 in an account that is compounded quarterly at a 5% interest rate. This tells us $P = 100, r = 0.05$, and $n = 4$. Using these values in the compound interest formula, we get the function: $A(t) = 100\left(1+\frac{.05}{4}\right)^{4T}$

The only variables are time in years, t, and the new value, A, over time.
Here is a graph of the function:



Notice the new amount in the account starts growing slowly, but eventually increases dramatically! That is the result of an exponential function where the base is $b > 1$.
To the nearest cent:

when $t = 20$, $A(20) = 100\left(1+\frac{.05}{4}\right)^{(4\cdot 20)} = \$270.15$
when $t = 40$, $A(40) = \$729.80$
when $t = 60$, $A(60) = \$1971.55$

Example #1 – Calculate the Accumulated Value

a) Find the accumulated value of an investment of \$5000 for 5 years at an interest rate of 5.5% compounded semiannually. $A=$
$P = \$5000$
$R = .055$
$N = 2$
$T = 5$

$A = 5000\left(1+\frac{.055}{2}\right)^{2\cdot 5} = \$6558.26$

b) Find the accumulated value of an investment of \$5000 for 5 years at an interest rate of 5.5% compounded monthly.

Note: The only difference in the two examples above is the number of times compounded per year. The monthly compound earned about \$20 more than the quarterly.

c) A person has \$2000 to invest for 4 years. One bank offers an interest rate of 8.25% compounded quarterly, another bank offers an interest rate of 8.5% compounded annually. Which offer will earn a yield a higher return on the investment?

Topic #3: The Natural Base e and Continuous Growth

Introduction to the number e:

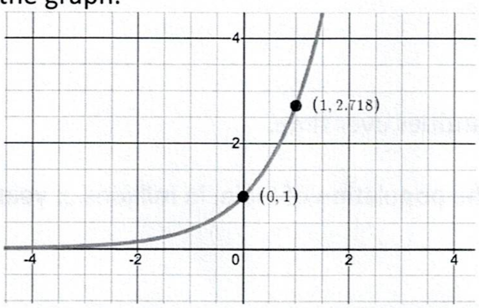
Suppose a person deposits \$1 into an account for 1 year and the account has an interest rate of 100%. This gives $P = 1, r = 1, t = 1$. Using these fixed values in the compound interest formula gives the function: $A = 1\left(1+\frac{1}{N}\right)^{N\cdot 1} \rightarrow A = \left(1+\frac{1}{N}\right)^{N}$

The only variables are how many times per year the interest is compounded, n, and the accumulated value, A, over the number of times the interest is compounded.
What happens when n increases? Through intuition, the higher n gets should produce a higher value for A.

However, consider the table of values when the interest is compounded: once, quarterly, monthly, daily, by the hour, by the minute, and by the second: hr min sec

n	1	4	12	365	8760	525600	31536000
A	2	2.4414	2.613	2.7146	2.7181	2.7183	2.7183

Notice that when n gets large, that A looks to flatten out to the value 2.7183
We can break one year up into as many tiny little pieces as we want (which is what n does). This introduces a very useful number, e. The number e is irrational and is approximately:
2.718281828459045
The function $f(x) = e^x$ is called the Natural base function since base e models continuous change (meaning, there are no breaks in how something changes over time, which is a **natural** occurrence).
Here is the graph:



Here is a table of select values for the natural exponential function (approximated)

x	-3	-2	-1	0	1	2	3
f(x)	0.04979	0.13534	0.36788	1	2.7183	7.3891	20.086

Continuous Interest:

When the number of times that interest is compounded per year is continuous, the formula for compound interest no longer applies (since n becomes infinite). Here is the formula for continuous interest:

$$A = Pe^{rT}$$

Where
A is the Accumulated balance
P is the initial Amount
r is the rate
t is the time

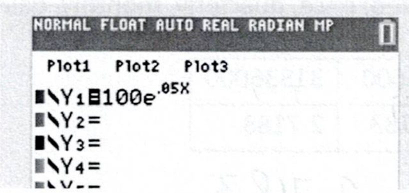
Although it is highly unlikely that a bank would compound interest continuously, the formula for continuous interest is useful since it introduces the concept of continuous change. Time never stops in nature and base e is the perfect number to model how things change when the "clock never stops".

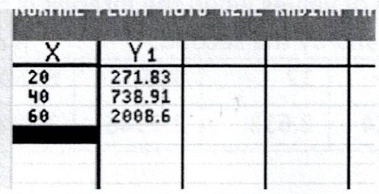
$A = Pe^{rT}$

Example #1 – Calculate the Future Value

a) Suppose a person wants to invest \$100 in an account that is compounded continuously at a 5% interest rate. How much will be in the account after 20 years? 40 years? 60 years?

We can also use the formula:
when $t = 20$, $A =$ $100e^{.05 \cdot 20}$ = \$271.83
when $t = 40$, $A =$ = \$738.91
when $t = 60$, $A =$ = \$2008.55
We can plug this function into the calculator to help evaluate at the indicated times.





b) A person has \$2000 to invest for 4 years. One bank offers an interest rate of 7% compounded monthly, another bank offers an interest rate of 6.85% compounded continuously. Which offer will earn a yield a higher return on the investment?

Use the appropriate model and inputs to compare:

Option #1: $A =$

Option #2: $A =$

<u>Topic #3: Other Applications of Exponential Functions</u>

Exponential functions are useful to model changes in quantities over time.

<u>Example #1</u> – Exponential Growth of a Population

The exponential function $f(x) = 574(1.026)^x$ models the population of India, in millions, x years after 1974.

a) Find $f(0)$ and interpret the meaning.

b) According to the model, what was the estimated population in 2001? Round to the nearest whole number (which gives the population to the nearest million).

c) According to the model, what is the projected population in the years 2028 and 2055? Round to the nearest whole number and include units.

d) What appears to be happening to India's population every 27 years?

The years above are in 27 year increments: $0, 27, 54, 81$

The corresponding populations (in millions) are:

The model suggests **the population will** ____________________________

<u>Example #2</u> – Exponential Decay of Learned Information

The exponential function $f(x) = 80e^{-0.5x} + 20$ describes the percentage of information that the average person remembers x weeks after learning the information.

a) What percentage of information is initially retained when it is first learned?

b) What percentage of information is retained after 1 week? After 4 weeks?

Evaluate:

c) Graph the function. What happens to retention as the time after learning the information increases?

d) Use the graph to estimate how long it takes retention to drop to 50%. Round to the nearest week.