Math 120
1.7 Combinations of Functions and Composite Functions

Objectives:

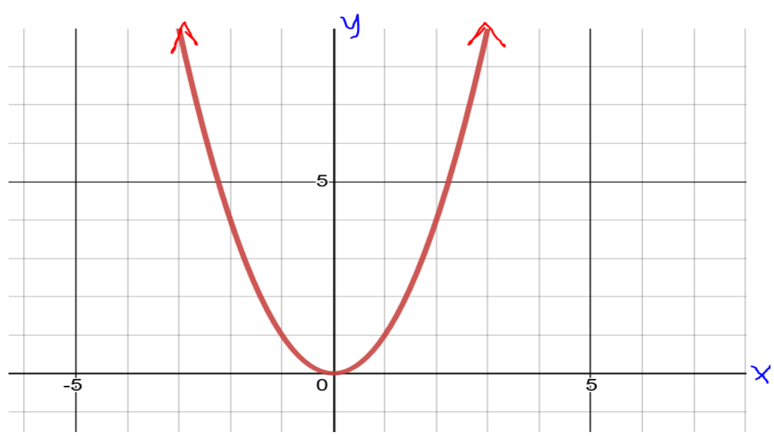
1. Find the domain of a function.
2. Combine functions using the algebra of functions, specifying domains.
3. Form composite functions.
4. Determine domains for composite functions (minimally)
5. Write functions as compositions.

Topic #1: The Domain of a Function

Recall that the domain of a function is the set of all x-values that prove an output *y*. Some functions do not have domain restrictions, some functions do have domain restrictions.

A. Domain is unrestricted:

Consider the function $f(x) = x^2$.



D: $(-\infty, \infty)$

Any real number can be squared, so the domain is **all real numbers**.

Moreover, the graph shows all x values from left to right have an output on the graph. To write as an interval, the domain is $(-\infty, \infty)$ or in words, the domain is all real numbers.

B. Domain is Restricted – Division by 0

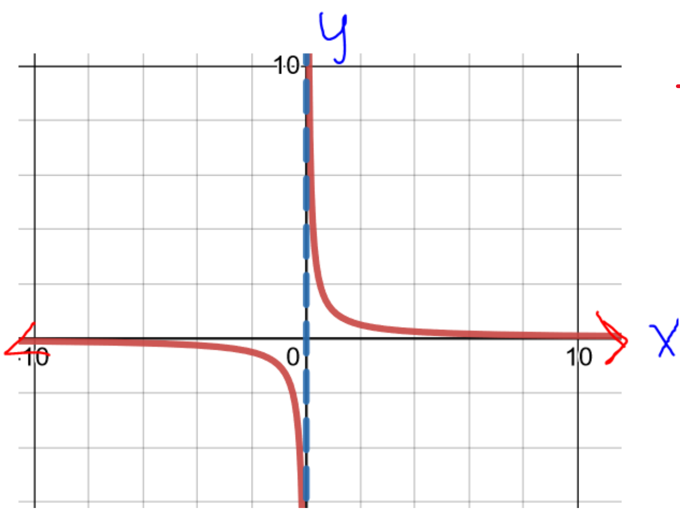
Consider the function $f(x) = \frac{1}{x}$

$\frac{0}{K} = 0$

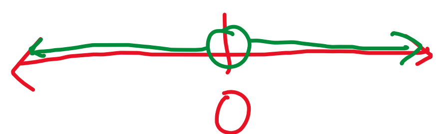
$\frac{N}{0} = \text{undefined}$

$f(0) = \frac{1}{0}$

$= \text{undefined}$



$(-\infty, 0) \cup (0, \infty)$



Since division by zero is undefined, the domain is all real numbers except 0.

Moreover, the graph shows that when $x = 0$ there is no output on the graph.

To write as an interval, the domain is $(-\infty, 0) \cup (0, \infty)$

To write as a set; $x \neq 0$. $\{x \mid x \neq 0\}$

C. Domain is Restricted – Negative even roots

Consider the function: $f(x) = \sqrt{x}$

Example:

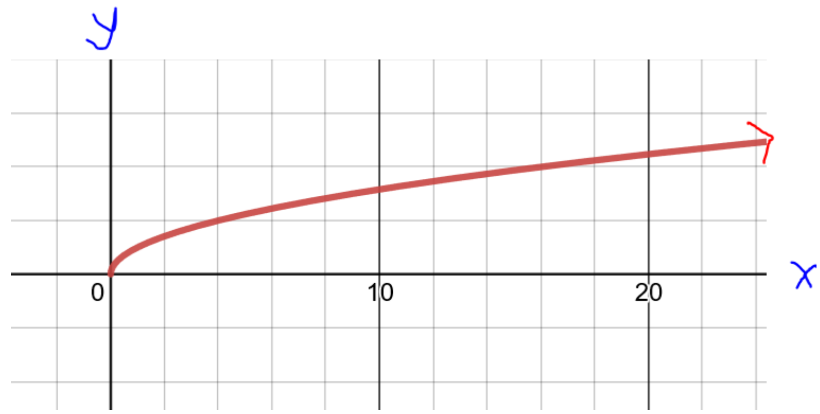
$\sqrt{4} = 2$

because $2^2 = 4$

$\sqrt{-4}$ = Does Not exist

$(\ )^2 = -4$

nothing



Since negative square roots are undefined with real numbers, the domain is ALL non-negative numbers $x \geq 0$

Moreover, the graph shows that all negative x values have no output on the graph.

To write as an interval, the domain is $[0, \infty)$

CONCLUSION: When determining the DOMAIN for a given function, we need to check if there are restrictions on the domain due to Division of 0

OR Negative even roots

Example #1 – Find the Domain of the Function, Write in Interval Notation

Denominator $\neq 0$

a) $g(x) = \dfrac{-3x}{2x+1}$

$2x+1 \neq 0$

The function contains **division** and is undefined when the denominator is zero (later we will define this type of function as a RATIONAL FUNCTION).

Set the denominator equal to 0 and solve:

why? $g(-\frac{1}{2}) = \dfrac{-3(-\frac{1}{2})}{2(-\frac{1}{2})+1} = \dfrac{\frac{3}{2}}{0}$

$\frac{N}{0}$ = undefined

$2x + 1 \neq 0 \rightarrow$ $x \neq -\frac{1}{2}$

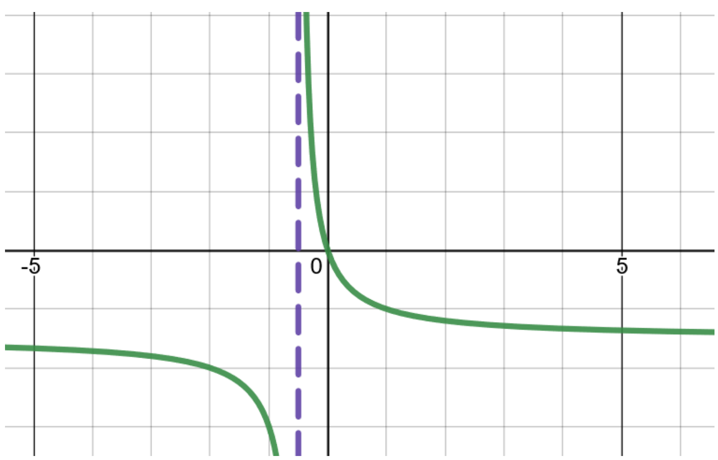
$-1 \quad -1$

$\frac{2x}{2} \neq \frac{-1}{2}$

A graph shows the same restriction;

notice when $x = -1/2$ there is no output on the graph:

$g(x) = \dfrac{-3x}{2x+1}$



x can't be $-\frac{1}{2}$ because it causes division by 0

$x = -\frac{1}{2}$

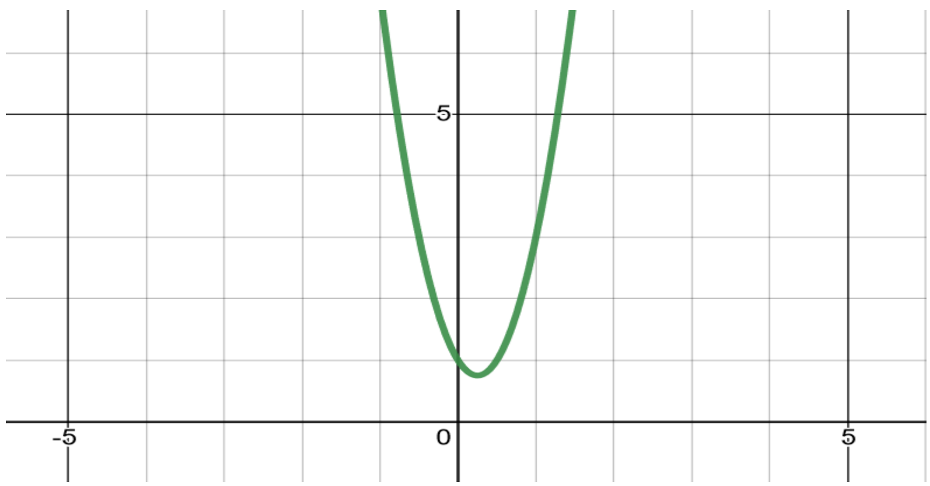
b) $f(t) = 4t^2 - 2t + 1$

The function does not have Denominator and does not have radical (later we will define this type of function as a polynomial).

The function is DEFINED for all real numbers; note the inputs have been assigned the variable t.

As an interval, the domain is $(-\infty, \infty)$

A graph shows that there are no restrictions:



X can be anything.

No restrictions.

c) $h(x) = \sqrt{3x - 2}$

The function contains a **square root** and is restricted when the radicand (the value under the square root) is negative.

radicand ≥ 0

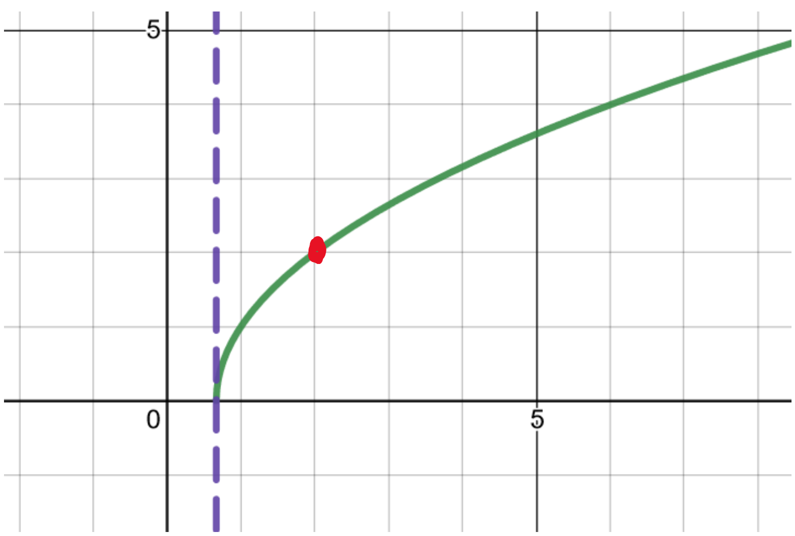
Set the radicand greater or equal to zero (which means non-negative) and solve:

$$3x - 2 \geq 0 \rightarrow$$

$$\frac{3x}{3} \geq \frac{2}{3}$$

$x \geq \frac{2}{3}$ $\left[\frac{2}{3}, \infty\right)$

A graph shows that all values for x less than 2/3 have no output on the graph:



x can be anything greater or equal to $\frac{2}{3}$

$f(2) = \sqrt{3(2)-2} = \sqrt{4} = 2$

$f(0) = \sqrt{3(0)-2} = \sqrt{-2}$ Does Not exist

__YOU TRY #1__ – Find the Domain of each function. State in interval notation.

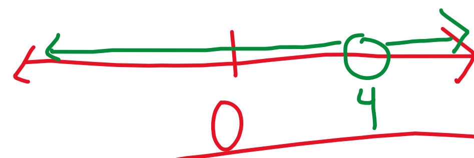
a. $f(x) = \dfrac{2}{x-4}$

Denominator $\neq 0$

$x - 4 \neq 0$

$\underline{\quad +4 \quad +4}$

$x \neq 4$



$\boxed{(-\infty, 4) \cup (4, \infty)}$

b. $g(x) = 2x - 3$

Linear

NO denominator

NO radical

NO restrictions

$\boxed{(-\infty, \infty)}$

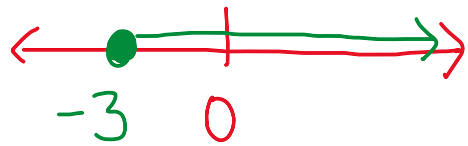
c. $h(x) = \sqrt{x+3}$

radicand ≥ 0

$x + 3 \geq 0$

$\underline{\quad -3 \quad -3}$

$x \geq -3$



$\boxed{[-3, \infty)}$

2 restrictions

d. $f(x) = \dfrac{1}{\sqrt{x-4}}$

radicand ≥ 0 AND Denominator $\neq 0$

$x - 4 \geq 0$

$x \geq 4$

$\sqrt{x-4} \neq 0$

$(\sqrt{x-4})^2 \neq 0^2$

$x - 4 \neq 0$

$x \neq 4$

$x > 4$

$\boxed{\begin{array}{c}(4, \infty) \\ \{x \mid x > 4\}\end{array}}$

Topic #2: Combining Functions

Functions possess algebraic properties. Two functions can be combined through addition, subtraction, multiplication, and division (to name a few possibilities) to create a new function.

Let f and g be two functions. The sum $f + g$, the difference $f - g$, the product fg, and the quotient $\frac{f}{g}$ are functions whose domains are the set of all real numbers common to the domains of f and g : $(D_f \cap D_g)$, defined as follows:

1. Sum: $(f+g)(x) = f(x) + g(x)$
2. Difference: $(f-g)(x) = f(x) - g(x)$
3. Product: $(fg)(x) = f(x) \cdot g(x)$
4. Quotient: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)},\ g(x) \neq 0.$

Example #1 – Combine the Functions

Use the following functions to find the given combination of functions. State any domain restrictions as needed.:

$f(x) = x^2 + 2x - 1$ and $g(x) = x - 1$

a) $(f+g)(x) = f(x) + g(x)$

$= (x^2+2x-1) + (x-1) = \boxed{x^2+3x-2}$

b) $(f-g)(x) = f(x) - g(x)$

$= (x^2+2x-1) - (x-1) = \boxed{x^2+x}$

c) $(fg)(x) = f(x) \cdot g(x)$

$= (x^2+2x-1) \cdot (x-1) = x^3+2x^2-x-x^2-2x+1$

$= \boxed{x^3+x^2-3x+1}$

d) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{(x^2+2x-1)}{(x-1)}$

$= \boxed{\frac{x^2+2x-1}{x-1}}$

Denominator $\neq 0$

$x \neq 1$

__YOU TRY #2__ - Use the following functions to find the given combination of functions. State any domain restrictions as needed.:

$f(x) = 2x - 1$ and $g(x) = 3x + 2$

$(f+g)(x) = (2x-1) + (3x+2) = \boxed{5x+1}$

$(f-g)(x) = (2x-1) - (3x+2) = 2x-1-3x-2$

$= \boxed{-x-3}$

$(fg)(x) = (2x-1)\cdot(3x+2)$

$= 6x^2+4x-3x+1 = \boxed{6x^2+x+1}$

$\left(\frac{f}{g}\right)(x) = \boxed{\frac{2x-1}{3x+2}}$

Denominator $\neq 0$

$3x+2 \neq 0$

$-2 \quad -2$

$\frac{3x}{3} \neq \frac{-2}{3}$

$x \neq -\frac{2}{3}$

Topic #3: Composite Functions

Functions have the algebraic property that two functions can be combined by composition.

This means one function can be ***plugged into*** another to create a new function.

We can plug function g into function f to get a new function, the notation is:

$$(f \circ g)(x) = f\big(g(x)\big)$$

$f(x)$

and is read as "f of g".

The inputs/domain/x values go into function g first, those outputs then become the new inputs for function f.

We can also plug function f into function g to get a new function, the notation is:

$g(x)$

$$(g \circ f)(x) = g\big(f(x)\big)$$

and is read as "g of f". The inputs/domain/x values go into function f first, those outputs then become the new inputs for function g.

Example #1 -

Consider the functions:

$f(x) = 5x - 1$ and $g(x) = 7x$

We can plug function g into function f:

$$(f \circ g)(x) = f\big(g(x)\big) = f(g(x)) = f(7x)$$

$$= 5(7x) - 1$$

$$= \boxed{35x - 1}$$

We can plug function f into function g:

$$(g \circ f)(x) = g\big(f(x)\big) = g(f(x)) = g(5x - 1)$$

$$= 7(5x - 1)$$

$$= \boxed{35x - 7}$$

In both cases, we get a new function. To check the domain, look at the "inner" function first and the new "composite" function last. Both new functions above have no restrictions for the "inner" function and no final restrictions for the "composite/new" function.

Example #2-

Consider the functions:

$f(x) = 2x - 1$ and $g(x) = 3x + 2$

We can plug function g into function f:

$$(f \circ g)(x) = f(g(x)) = f(3x+2) = 2(3x+2)-1$$

$3x+2$

$$= 6x + 4 - 1$$

$$= \boxed{6x+3}$$

We can plug function f into function g:

$$(g \circ f)(x) = g(f(x)) = g(2x-1) = 3(2x-1)+2$$

$2x-1$

$$= 6x-3+2$$

$$= \boxed{6x-1}$$

In both cases, we get a new function. To check the domain, look at the "inner" function first and the new "composite" function last. Both new functions above have no restrictions for the "inner" function and no final restrictions for the "composite/new" function.

Example #3 – Compose the Functions, NO restrictions

Let $f(x) = 3x - 4$ and $g(x) = x^2 - 2x + 6$

a) Find $(f \circ g)(x)$

Plug g into f: $(f \circ g)(x) = f(g(x)) = f(x^2 - 2x + 6)$

Replace all inputs for f with function g, distribute and combine like terms: $f(x) = 3x - 4$

$$f(x^2 - 2x + 6) = 3(x^2 - 2x + 6) - 4$$

$$f(g(x)) = 3x^2 - 6x + 18 - 4$$

$$\boxed{f(g(x)) = 3x^2 - 6x + 14}$$

b) Find $(g \circ f)(x)$

Plug f into g: $(g \circ f)(x) = g(f(x)) = g(3x - 4)$

Replace all inputs for g with function f, distribute and combine like terms: $g(x) = x^2 - 2x + 6$

$$g(3x-4) = (3x-4)^2 - 2(3x-4) + 6$$

$$= 9x^2 - 12x - 12x + 16 - 6x + 8 + 6$$

$$\boxed{g(f(x)) = 9x^2 - 30x + 30}$$

c) Evaluate $(g \circ f)(1) = g(f(1)) = 9(1)^2 - 30(1) + 30 = \boxed{9}$

You can also find $(g \circ f)(1)$ by first finding $f(1)$, and then plugging this output into g: $f(1) = 3(1) - 4 = -1$

$$g(-1) = (-1)^2 - 2(-1) + 6 = \boxed{9}$$

Example #4 – Compose the Functions, w/restrictions

Let $f(x) = \sqrt{x}$ and $g(x) = x - 7$

a) Find $(f \circ g)(x)$ and state its domain.

Plug g into f: $(f \circ g)(x) = f(g(x)) =$ $f(x-7)$

At this point, there are no restrictions for the "inner" function. Now replace all inputs for f with g:

$f(x) = \sqrt{x}$
$f(x-7) = \sqrt{x-7}$

$f(g(x)) = \sqrt{x-7}$

The new function has a square root and cannot be negative so there is a restriction on the domain: radicand ≥ 0

$x-7 \geq 0$
$x \geq 7$

$[7, \infty)$

b) Find $(g \circ f)(x)$ and state its domain.

Plug f into g: $(g \circ f)(x) = g(f(x)) =$ $g(\sqrt{x})$

The "inner" function is a square root and what is the domain restriction? radicand ≥ 0

$x \geq 0$

$[0, \infty)$

Now replace the input of g with f: $g(x) = x - 7$

$g(\sqrt{x}) = \sqrt{x} - 7$

Are there any further domain restrictions for the new function? What do we need to do to find them?

$[0, \infty)$

c) Find $(f \circ g)(10)$ $f(g(x)) = \sqrt{x-7}$

You can use the composite function from part a):

$(f \circ g)(10) = f(g(10)) = \sqrt{10-7} = \boxed{\sqrt{3}}$

You can also find $(f \circ g)(10)$ by first finding g(10), and then plugging this output into f: $g(10) = 10-7 = 3$

$f(3) = \boxed{\sqrt{3}}$

YOU TRY #3-

Let $f(x) = 5x + 6$ and $g(x) = x^2 - x - 1$

a) Find $(f \circ g)(x)$ and state its domain.

$f(g(x)) = f(x^2-x-1) = 5(x^2-x-1)+6 = 5x^2-5x-5+6$

$= \boxed{5x^2-5x+1}$

NO restrictions

$(-\infty, \infty)$

b) Find $(g \circ f)(x)$ and state its domain.

$g(f(x)) = \underbrace{(5x+6)^2}_{(5x+6)(5x+6)} - (5x+6) - 1$

$= 25x^2+30x+30x+36-5x-6-1$

$= \boxed{25x^2+55x+29}$

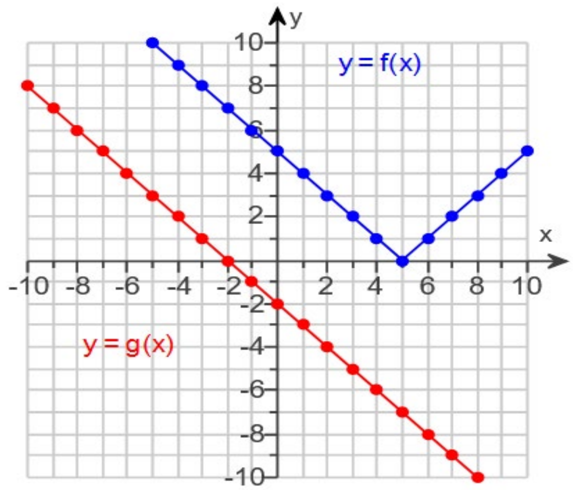
c) Find $(f \circ g)(-1) = f(g(-1)) = 5(-1)^2 - 5(-1) + 1 = \boxed{11}$

OR

$g(-1) = (-1)^2 - (-1) - 1 = 1$ $f(1) = 5(1)+6 = \boxed{11}$

Example #5 – Evaluate the Composite Function with a Graph

The graphs of functions f and g follow, use the graphs to evaluate the composite functions.



a) $(f \circ g)(-2) = 5$

The equations are not given, but we still start with the definition: $(f \circ g)(-2) = f(g(-2))$

red

Use the graph to find the "inner" value: $g(-2) = 0$

We input this into the "outer" function: $f(0) = 5$

Blue

b) $(g \circ f)(-2) = -9$

$f(-2) = 7$

$g(7) = -9$

c) $(g \circ f)(3) = -4$

$f(3) = 2$

$g(2) = -4$

d) $(f \circ g)(3) = 10$

$g(3) = -5$

$f(-5) = 10$

Topic #4: Decomposition of Functions

Two functions can be composed into one function, and one function can also be decomposed into two functions.

Consider the function $h(x) = (3x - 1)^5$

Looking at function h we see the "big picture" is 5^{th} power

However, before that operation happens we multiply by 3 and subtract 1 inside the parentheses.

This demonstrates function h can be decomposed into an "inner" and "outer" function:

$$h(x) = (f \circ g)(x)$$

where $f(x) = x^5$ and $g(x) = 3x - 1$

Evaluate the composite function to confirm by inputting g(x) for x in f:

$$(f \circ g)(x) = f(g(x)) = f(3x-1) = (3x-1)^5$$

$\uparrow$

back where we started

$$h(x) = f(g(x))$$

Example #1 – Decompose the Function

Find functions f and g such that $h(x) = (f \circ g)(x)$

a) $h(x) = (x^3 - 5)^{10}$

The big picture is 10th power; that is the outer function f.

Before that operation, $x^3 - 5$; that is the inner function g:

$f(x) = x^{10}$ $g(x) = x^3 - 5$

g(x) must be the inner function

As a check, you can evaluate the composite functions to confirm that $h(x) = (f \circ g)(x)$ for each example above.

CHECK: $h(x) = f(x^3 - 5) = (x^3 - 5)^{10}$

b) $h(x) = \sqrt[3]{5 - x^8}$

The big picture is Cubed root; that is the outer function f.

Before that operation, $5 - x^8$; that is the inner function g:

$f(x) = \sqrt[3]{x}$ $g(x) = 5 - x^8$

CHECK: $h(x) = f(g(x)) = f(5 - x^8) = \sqrt[3]{5 - x^8}$

c) $h(x) = \frac{1}{5x+6}$

The big picture is ___ $\frac{1}{x}$ ___; that is the outer function f.

Before that operation, ___ $5x+6$ ___; that is the inner function g:

$f(x) = \frac{1}{x}$ $\qquad$ $g(x) = 5x+6$

CHECK: $h(x) = f(g(x)) = f(5x+6) = \frac{1}{5x+6}$

<u>*YOU TRY #4-*</u> Express h(x) as a composition of two functions.

$$h(x) = \sqrt{x^2+5}$$

"Outer" $f(x)$ is square root

"inner" $g(x)$ is x^2+5

$f(x) = \sqrt{x}$ $\qquad$ $g(x) = x^2+5$

CHECK: $h(x) = f(g(x)) = f(x^2+5) = \sqrt{x^2+5}$