Math 120
2.1 Linear Equations and Rational Equations

Objectives

1. Solve linear equations in 1 variable.
2. Solve linear equations containing fractions.
3. Recognize identities, conditional equations, and inconsistent equations.
4. Solve rational equations with variables in the denominator.
5. Solve applied problems using linear functions.

Topic #1: Solving Linear Equations in One Variable – By "Hand"

Highest power is 1

Linear functions can be written in the form
$f(x) = mx + b$.

Consider the linear functions $f(x) = 2(x-3) - 17$ and $g(x) = 13 - 3(x+2)$. Suppose we want to find where $f(x) = g(x)$. It may not be obvious that the functions are linear, but the resulting equation is:

$$f(x) = g(x)$$
$$2(x-3) - 17 = 13 - 3(x+2)$$

$2x - 6 - 17 = 13 - 3x - 6$

$2x - 23 = -3x + 7$

$-7 \qquad -7$

$2x - 30 = -3x$

$-2x \qquad -2x$

$-30 = -5x$

$\frac{-30}{-5} = \frac{-5x}{-5}$

$\boxed{6 = x}$

The result is the solution set to the equation, we can plug into the original equation to verify:

$$2(6-3)-17 = 13-3(6+2)$$
$$2(3)-17 = 13-3(8)$$
$$6-17 = 13-24$$
$$-11 = -11$$

Moreover, the original functions are shown to be equivalent at $x = 6$:

$$f(6) = 2(6-3)-17 = -11$$

$$g(6) = 13-3(6+2) = -11$$

Thinking of linear equations as 2 lines provides the basis for the steps to solving a linear equation:

1. Simplify both sides of the equation as much as possible (get both sides in $y = mx + b$ form). Clear __Fractions__ if necessary (LCM).
   Least Common Multiple
2. Collect all variable terms to one side of the equation and the constant terms __other side__
3. __Isolate__ the variable to solve.
4. Check the proposed solution in the original equation.

Example #1 – Solve the Linear Equation

a. $3(x-2)+7=2(x+5)$

$3x-6+7=2x+10$

$3x+1=2x+10$

$-2x \qquad -2x$

$x+1=10$

$-1 \qquad -1$

$\boxed{x=9}$

Check:

$3(9-2)+7=2(9+5)$

$3(7)+7=2(14)$

$21+7=28$

$28=28$ ✓

b. Suppose $y=5[x-(2-x)]-9(x+1)$. **Find the zeros of the equation. In other words, find all x values such that $y=0$.**

$0=5[x-(2-x)]-9(x+1)$

$0=5[x-2+x]-9x-9$

$0=5[2x-2]-9x-9$

$0=10x-10-9x-9$

$0=x-19$

$+19 \qquad +19$

$\boxed{19=x}$

Check:

$0=5[19-(2-19)]-9(19+1)$

$0=5[19-(-17)]-9(20)$

$0=5(36)-180$

$0=180-180$

$0=0$ ✓

c. $\dfrac{x}{11} = \dfrac{x}{12} - 3$

$LCD = 11 \cdot 12 = 132$

$\dfrac{132}{1} \cdot \dfrac{x}{11} = \dfrac{132}{1} \cdot \dfrac{x}{12} - \dfrac{132}{1} \cdot \dfrac{3}{1}$

$\cancel{\dfrac{132}{1}} \cdot \cancel{\dfrac{x}{11}} = \cancel{\dfrac{132}{1}} \cdot \cancel{\dfrac{x}{12}} - \dfrac{132}{1} \cdot \dfrac{3}{1}$

$12x = 11x - 396$

$-11x \quad -11x$

$\boxed{x = -396}$

Check:

$\dfrac{-396}{11} = \dfrac{-396}{12} - 3$

$-36 = -33 - 3$

$-36 = -36$ ✓

d. $\dfrac{x+5}{4} = 1 - \dfrac{x+6}{9}$

$LCD = 4 \cdot 9 = 36$

$\dfrac{36}{1} \cdot \dfrac{(x+5)}{4} = \dfrac{36}{1} \cdot 1 - \dfrac{36}{1} \cdot \dfrac{(x+6)}{9}$

$\cancel{\dfrac{36}{1}}^{9} \cdot \cancel{\dfrac{(x+5)}{4}} = \dfrac{36}{1} \cdot 1 - \cancel{\dfrac{36}{1}}^{4} \cdot \cancel{\dfrac{(x+6)}{9}}$

$9(x+5) = 36 - 4(x+6)$

$9x + 45 = 36 - 4x - 24$

$9x + 45 = -4x + 12$

$+4x \quad +4x$

$13x + 45 = 12$

$-45 \quad -45$

$\dfrac{13x}{13} = \dfrac{-33}{13}$

$\boxed{x = -\dfrac{33}{13}}$

Check:

$\dfrac{\left(-\frac{33}{13} + 5\right)}{4} = 1 - \dfrac{\left(-\frac{33}{13} + 6\right)}{9}$

use calculator

$-\dfrac{8}{13} = -\dfrac{8}{13}$

Topic #2: Solving Linear Equations in One Variable – With Technology

We can use a graphing calculator to solve equations.

Recall the equation:

$$2(x-3)-17=13-3(x+2)$$

We can input the left side of the equation in the calculator as $y_1 = 2(x-3)-17$ and the right side of as $y_2 = 13-3(x+2)$

Look at a table to see what x values makes them equal (in other words $y_1 = y_2$).

Based on the table below, we can see that both sides are equivalent when X=6

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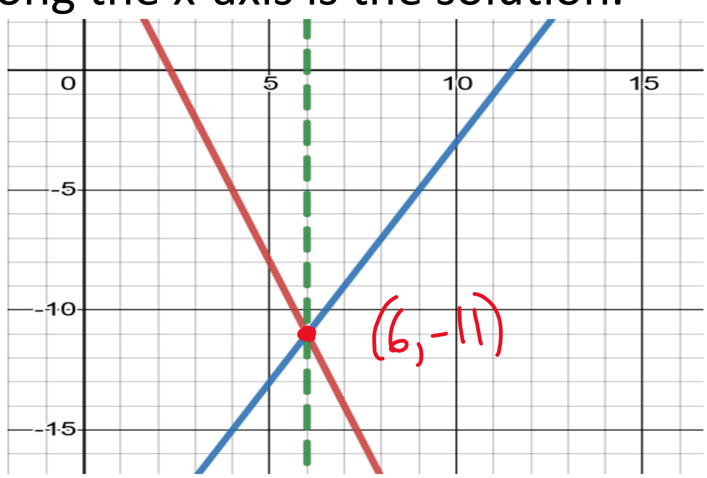
X	Y_1	Y_2
-2	-27	13
-1	-25	10
0	-23	7
1	-21	4
2	-19	1
3	-17	-2
4	-15	-5
5	-13	-8
6	-11	-11
7	-9	-14
8	-7	-17

X=6

We can also look at the graph of the two sides of the equation to see where they meet. This point along the x-axis is the solution.



Example #1 – Use the Table to Write and Solve the Linear Equation

Plot1 Plot2 Plot3
\Y1 = −5(X − 3)
\Y2 = 7(2 − X)
\Y3 =
\Y4 =
\Y5 =
\Y6 =
\Y7 =

X	Y1	Y2
−2	25	28
−1.5	22.5	24.5
−1	20	21
−0.5	17.5	17.5
0	15	14
0.5	12.5	10.5
1	10	7
X = −2		

Here $y_1 =$ -5(x-3) and $y_2 =$ 7(2-x).

The resulting equation is

$y_1 = y_2$ OR $\boldsymbol{-5(x-3) = 7(2-x)}$.

The solution to the equation is at $\boldsymbol{x} =$ -0.5

Technology is useful, feel free to use it as you see fit!

Topic #3: Solution Types to Linear Equations

There are 3 types possible solution sets when solving linear equations:

1) **Conditional** – there is a Finite solution set. The equation is only True for certain values.

Recall that the equation:
$2(x-3)-17=13-3(x+2)$ has a solution when $\boxed{x=6}$.

This equation is **only** true when $x=6$, so the equation is Conditional

2) **Identity** – there is an infinite solution set. The equation is always true and x is a real number.

Consider the equation: $6x+2=2(3x+1)$
Which simplifies to: $6x+2=6x+2$ Same

$$\begin{array}{r} 6x+2=6x+2 \\ -6x \quad\quad -6x \\ \hline 2=2 \end{array}$$

All real #'s

Collecting the variable terms and constant terms to either side gives

This equation is **always** true, so the equation is an identity

3) **Inconsistent** – there is NO solution.
The equation is Never true, which we can convey with the null symbol: $\varnothing$

Consider the equation: $6x + 2 = 2(3x + 2)$
Which simplifies to: $6x+2 = 6x + 4$

$2 = 4$

Collecting the variable terms and constant terms to either side gives: $2=4$

This is **never** true, so the equation is inconsistent

Example #1 – Solve the Linear Equation and Categorize the Solution Type

a. $3(x + 2) = 7 + 3x$

$3x + 6 = 7 + 3x$

$-3x \qquad -3x$

$6 = 7$

$\boxed{\varnothing \text{ or No Solution}}$

Inconsistent

b. $5x + 9 = 9(x + 1) - 4x$

$5x+9 = 9x+9-4x$

$5x+9 = 5x+9$

$\boxed{\text{All real \#'s}}$

identity

c. $10x + 3 = 8x + 3$

$$\begin{array}{rcl} 10x + 3 & = & 8x + 3 \\ -8x & & -8x \\ \hline 2x+3 & = & 3 \\ -3 & & -3 \\ \hline \end{array}$$

$\frac{2x}{2} = \frac{0}{2}$

$\boxed{x=0}$

Conditional

YOU TRY #1 **–** Solve the Linear Equation and Categorize the Solution Type

a. $3(x-4)-4(x-3)=x+3-(x-2)$

$3x-12-4x+12=x+3-x+2$

$\frac{-x}{-1} = \frac{5}{-1}$

$\boxed{x=-5}$

Conditional

b. $4x+7=7(x+1)-3x$

$4x+7=7x+7-3x$

$4x+7=4x+7$

$\boxed{\text{All real \#'s}}$

identity

c. $3(x+1) = 7 + 3x$

$$\begin{aligned} 3x + 3 &= 7 + 3x \\ -3x \quad & \quad\ \ -3x \\ \hline 3 &= 7 \end{aligned}$$

$\boxed{\text{NO Solution}}$

inconsistent

d. $\dfrac{3x}{5} - x = \dfrac{x}{10} - \dfrac{5}{2}$

LCD = 10

$$\frac{10}{1}\cdot\frac{3x}{5} - \frac{10}{1}\cdot x = \frac{10}{1}\cdot\frac{x}{10} - \frac{10}{1}\cdot\frac{5}{2}$$

$$\frac{\cancel{10}^{2}}{1}\cdot\frac{3x}{\cancel{5}} - \frac{10}{1}\cdot x = \frac{\cancel{10}}{1}\cdot\frac{x}{\cancel{10}} - \frac{\cancel{10}^{5}}{1}\cdot\frac{5}{\cancel{2}}$$

$$6x - 10x = x - 25$$

$$\begin{aligned} -4x &= x - 25 \\ -x \quad & \quad -x \\ \hline \frac{-5x}{-5} &= \frac{-25}{-5} \end{aligned}$$

$\boxed{x = 5}$ conditional

Topic #4: Solving Rational Equations

Fraction

Rational equations include a variable in the denominator.

Since division by zero is undefined, we cannot accept any solution that makes the denominator 0.

We clear out denominators with the LCD

From there, the equation will become "linear" – we just have to make sure none of the solutions make the denominator zero. Otherwise, we throw it out of the solution set.

Consider the equation:

$$\frac{1}{x} = \frac{1}{5} + \frac{3}{2x}$$

Denominator $\neq 0$

$x \neq 0$

$2x \neq 0$

What is the restricted value? $x \neq 0$

$$\frac{1}{x} = \frac{1}{5} + \frac{3}{2x}$$

1. Identify the LCM.
   The unique factors are $x, 5, 2$
   (the factor x shows up twice, but is only unique once),
   which gives an LCM of $x \cdot 5 \cdot 2 = 10x$

2. Multiply each term by the LCM $10x$ to clear out the denominators:

$$\frac{10x}{1} \cdot \frac{1}{x} = \frac{10x}{1} \cdot \frac{1}{5} + \frac{10x}{1} \cdot \frac{3}{2x}$$

$$\frac{10x}{x} = \frac{10x}{5} + \frac{30x}{2x}$$

$$\frac{10\cancel{x}}{\cancel{x}} = 2\frac{\cancel{10}x}{\cancel{5}} + 15\frac{\cancel{30}\cancel{x}}{\cancel{2}\cancel{x}}$$

3. This simplifies to a linear equation:

$$10 = 2x + 15$$

4. Solve the equation with techniques discussed earlier and check the restricted values: $\rightarrow x \neq 0$

$$10 = 2x + 15$$
$$-15 \qquad\qquad -15$$

$$\frac{-5}{2} = \frac{2x}{2}$$

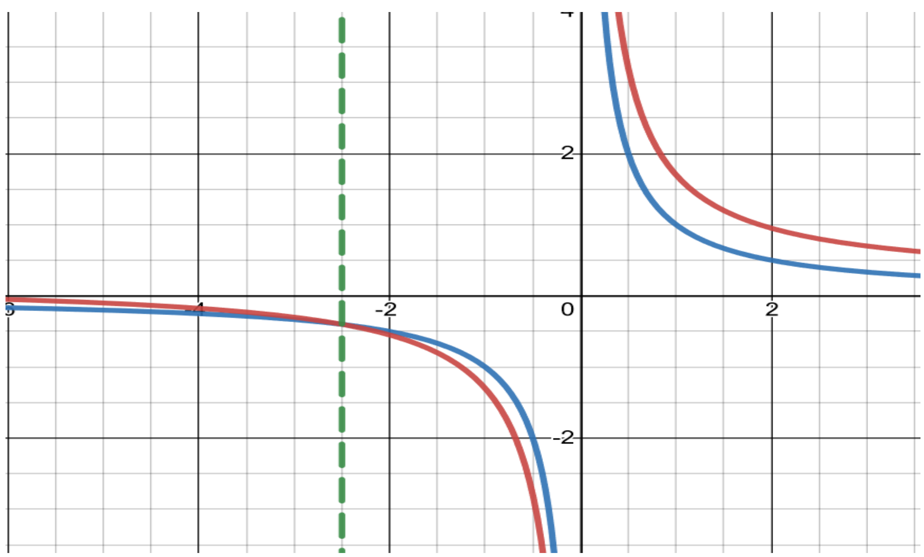
$$\boxed{-\frac{5}{2} = x}$$

Since this is not the excluded value $x \neq 0$, the solution works.

A graph also confirms the solution, where:

$y_1 =$ $\frac{1}{x}$ and $y_2 =$ $\frac{1}{5} + \frac{3}{2x}$

Notice the two sides meet when x = $-\frac{5}{2}$



Example #1 – State the Excluded Values and Solve the Rational Equation

a. $\frac{4}{5x+25} = \frac{6}{x+5} - \frac{2}{5}$

Factor! $5(x+5)$

1. Identify restricted values: Denominator $\neq 0$

$5x+25 \neq 0$ $\quad x+5 \neq 0$ $\quad \boxed{x \neq 5}$

2. Identify LCM Factors: $5, x+5$

$5(x+5)$

3. Multiply every term by LCM

$$\frac{5(x+5)}{1} \cdot \frac{4}{5(x+5)} = \frac{5(x+5)}{1} \cdot \frac{6}{(x+5)} - \frac{5(x+5)}{1} \cdot \frac{2}{5}$$

4. Simplify to linear equation.

$$\frac{\cancel{5(x+5)}}{1} \cdot \frac{4}{\cancel{5(x+5)}} = \frac{5\cancel{(x+5)}}{1} \cdot \frac{6}{\cancel{(x+5)}} - \frac{\cancel{5}(x+5)}{1} \cdot \frac{2}{\cancel{5}}$$

$$4 = 30 - 2(x+5)$$

5. Solve and check the restricted values. $\rightarrow x \neq 5$

$4 = 30 - 2x - 10$

$4 = -2x + 20$

$-20 \qquad -20$

$\frac{-16}{-2} = \frac{-2x}{-2}$ $\quad \boxed{8 = x}$

b. $\dfrac{2}{x+4} + \dfrac{3}{x-4} = \dfrac{24}{(x+4)(x-4)}$

Denominator $\neq 0$

1. Identify restricted values: $x+4 \neq 0$ $\quad x-4 \neq 0$

$x \neq -4 \quad x \neq 4$

2. Identify LCM $(x+4)\cdot(x-4)$

3. Multiply every term by LCM

$$\frac{(x+4)(x-4)}{1}\cdot\frac{2}{(x+4)} + \frac{(x+4)(x-4)}{1}\cdot\frac{3}{(x-4)} = \frac{(x+4)(x-4)}{1}\cdot\frac{24}{(x+4)(x-4)}$$

4. Simplify to linear equation.

$$\frac{\cancel{(x+4)}(x-4)}{1}\cdot\frac{2}{\cancel{(x+4)}} + \frac{(x+4)\cancel{(x-4)}}{1}\cdot\frac{3}{\cancel{(x-4)}} = \frac{\cancel{(x+4)}\cancel{(x-4)}}{1}\cdot\frac{24}{\cancel{(x+4)}\cancel{(x-4)}}$$

$$2(x-4) + 3(x+4) = 24$$

5. Solve and check the restricted values. $\rightarrow x \neq 4, -4$

$$2x - 8 + 3x + 12 = 24$$

$$5x + 4 = 24$$

$$-4 \qquad -4$$

$$\frac{5x}{5} = \frac{20}{5}$$

$$x = 4$$

$\boxed{\text{NO Solution}}$

c. $\dfrac{2}{x+1} - \dfrac{1}{x-1} = \dfrac{2x}{x^2-1}$ $\qquad \dfrac{2}{x+1} - \dfrac{1}{x-1} = \dfrac{2x}{(x+1)(x-1)}$

Factor

1. Identify restricted values: $x \neq 1, -1$

2. Identify LCM $(x+1)(x-1)$

3. Multiply every term by LCM

$$\frac{(x+1)(x-1)}{1} \cdot \frac{2}{(x+1)} - \frac{(x+1)(x-1)}{1} \cdot \frac{1}{(x-1)} = \frac{(x+1)(x-1)}{1} \cdot \frac{2x}{(x+1)(x-1)}$$

4. Simplify to linear equation.

$$\frac{\cancel{(x+1)}(x-1)}{1} \cdot \frac{2}{\cancel{(x+1)}} - \frac{(x+1)\cancel{(x-1)}}{1} \cdot \frac{1}{\cancel{(x-1)}} = \frac{\cancel{(x+1)}\cancel{(x-1)}}{1} \cdot \frac{2x}{\cancel{(x+1)}\cancel{(x-1)}}$$

$$2(x-1) - (x+1) = 2x$$

5. Solve and check the restricted values. $\rightarrow$ $x \neq 1, -1$

$$2x - 2 - x - 1 = 2x$$

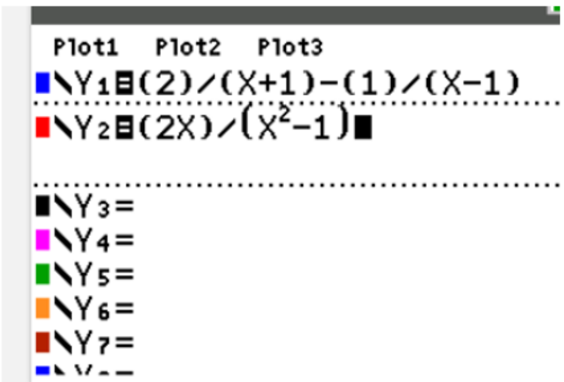
$$x - 3 = 2x$$

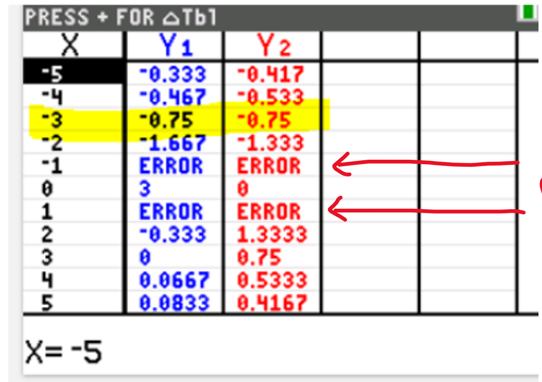
$$-x \qquad -x$$

$$\boxed{-3 = x}$$

A table generated by a graphing calculator can also solve the equation (notice the parenthesis), where:

$$y_1 = \frac{2}{x+1} - \frac{1}{x-1} \text{ AND } y_2 = \frac{2x}{x^2-1}$$

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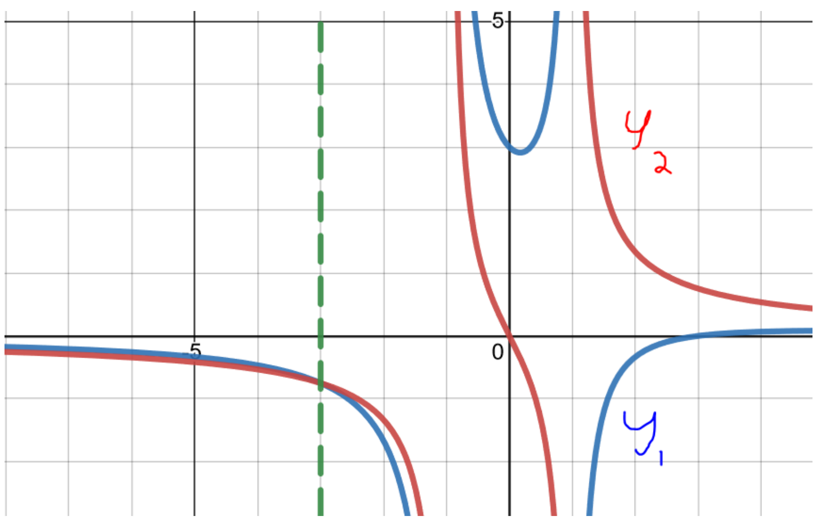
X	Y1	Y2
-5	-0.333	-0.417
-4	-0.467	-0.533
-3	-0.75	-0.75
-2	-1.667	-1.333
-1	ERROR	ERROR
0	3	0
1	ERROR	ERROR
2	-0.333	1.3333
3	0	0.75
4	0.0667	0.5333
5	0.0833	0.4167

X= -5

restricted values

A graph also works!



YOU TRY #2 – Solve the following equations. Make sure to include any restricted values if there are any.

a. $\frac{2}{x}+\frac{1}{2}=\frac{3}{4}$

$x \neq 0$

$LCD = 4x$

$$\frac{4x}{1}\cdot\frac{2}{x}+\frac{4x}{1}\cdot\frac{1}{2}=\frac{4x}{1}\cdot\frac{3}{4}$$

$$\frac{4\cancel{x}}{1}\cdot\frac{2}{\cancel{x}}+\frac{\overset{2}{\cancel{4}}x}{1}\cdot\frac{1}{\cancel{2}}=\frac{\cancel{4}x}{1}\cdot\frac{3}{\cancel{4}}$$

$$8 + 2x = 3x$$
$$-2x \quad\quad -2x$$
$$\boxed{8 = x}$$

b. $\frac{1}{x-2}+\frac{3}{x+5}=\frac{7}{(x+5)(x-2)}$

$x \neq -5, 2$

$LCD = (x-2)(x+5)$

$$\frac{(x+5)(x-2)}{1}\cdot\frac{1}{(x-2)}+\frac{(x+5)(x-2)}{1}\cdot\frac{3}{(x+5)}=\frac{(x+5)(x-2)}{1}\cdot\frac{7}{(x+5)(x-2)}$$

$$\frac{(x+5)\cancel{(x-2)}}{1}\cdot\frac{1}{\cancel{(x-2)}}+\frac{\cancel{(x+5)}(x-2)}{1}\cdot\frac{3}{\cancel{(x+5)}}=\frac{\cancel{(x+5)}\cancel{(x-2)}}{1}\cdot\frac{7}{\cancel{(x+5)}\cancel{(x-2)}}$$

$$(x+5) + 3(x-2) = 7$$
$$x+5+3x-6 = 7$$
$$4x-1 = 7$$
$$+1 \quad\quad +1$$
$$\frac{4x}{4} = \frac{8}{4}$$

$x=2$ restricted

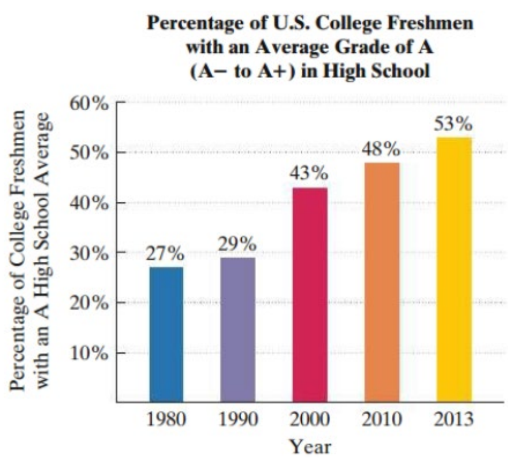
$\boxed{\text{No Solution}}$

Topic #5: Applications of Linear and Rational Equations

Linear and rational equations are often are used to model data.

Example #1 – Application of a Linear Equation

The bar graph shows the percentage of US college freshmen with an average grade of A in high school over time.



One model for the data is given by the equation:

$$p - \frac{4x}{5} = 25$$

Where p is the percentage with an average grade of A and x are years after 1980.

Let x be: years after 1980

Let p be: % of freshman with high school average of A

a. According to the model, what percentage of US college freshmen have an average grade of A in 2010? How far off is the model from the actual data? $X = 30$

2010 is 30 years after 1980, which gives x = 30

$p - \frac{4 \cdot 30}{5} = 25$ $p = 49\%$ off by 1%

$p - 24 = 25$

$+24 \quad +24$

b. Based on the model, when will the percentage of US college freshmen with an average grade of A be 61%?

The projected percentage is given, $\underline{p = 61}$. Plug into the equation and solve for x, which is the number of years after 1980:

$$61 - \frac{4x}{5} = 25$$

LCD = 5

$$\frac{5}{1} \cdot 61 - \frac{5}{1} \cdot \frac{4x}{5} = \frac{5}{1} \cdot 25$$

$$305 - 4x = 125$$
$$-305 \qquad\qquad -305$$

$$\frac{-4x}{-4} = \frac{-180}{-4}$$

$$x = 45$$

years after 1980

In 2025, 61% of college freshman should have a high school average grade of A.

Example #2 – Application of a Rational Equation

A learning curve is a math model that estimates the proportion of correct responses on a test/task in terms of the number of trials/attempts. As the number of trials/attempts increase, the proportion of correct responses increase.

Consider the model

$$P(x) = \frac{0.9x - 0.4}{0.9x + 0.1}$$

where P is the proportion correct and x is the number of trials.

Let x be: # of trials

Let P(x) be: proportion correct

a. Find $P(1)$ and interpret its meaning.

$$P(1) = \frac{0.9(1) - 0.4}{0.9(1) + 0.1} = 0.5$$

After 1 trial, the proportion of correct responses is 50%.

b. How many trials are necessary to get 95% correct responses? $P(x) = .95$

$$0.95 = \frac{0.9x - 0.4}{0.9x + 0.1}$$

$$(0.9x + 0.1) \cdot 0.95 = \frac{(0.9x - 0.4)}{(0.9x + 0.1)} \cdot (0.9x + 0.1)$$

$$(0.9x + 0.1) \cdot 0.95 = \frac{(0.9x - 0.4)}{\cancel{(0.9x + 0.1)}} \cdot \cancel{(0.9x + 0.1)}$$

$$0.95(0.9x + 0.1) = 0.9x - 0.4$$

$$0.855x + 0.095 = 0.9x - 0.4$$

$$-0.855x \qquad -.855x$$

$$0.095 = .045x - 0.4$$

$$+0.4 \qquad +0.4$$

$$\frac{0.495}{0.045} = \frac{0.045x}{0.045}$$

$$11 = x$$

11 trials