
Math 120

1.8 Inverse Functions

Objectives

1. Verify inverse functions.
2. Find the inverse of a function.
3. Use the graph of a one-to-one function to graph its inverse function.
4. Find the inverse of a function and graph both functions on the same axes.

Topic #1: The Inverse of a Function

In an inverse relationship, the domain and range

Trade places

Places

However, inverting a function does not guarantee the inverse relation is also a function.

One definition of an inverse is that all members in the domain interchange with their associated member in the range. In other words, each ordered pair (x, y) in the function becomes (y, x) in the inverse.

Consider the two functions:

Suppose **function** f consists of the ordered pairs (this is a function since no x-values repeat):

$(1,0), (2,1), (5,2), (10,3), (17,4)$

f^{-1} The inverse of function f consists of the ordered pairs:
 $(0,1), (1,2), (2,5), (3,10), (4,17)$

Is the inverse of function f a function? Why or why not?

Yes No repeating x-values
(originally y-values)

It is worth noting that function f has no repeated x-values and no repeated y-values. This is the definition of a one to one (abbreviated as 1:1), which tells us the function **MUST** have an inverse function.

Suppose **function g** consists of the ordered pairs (this is also a function since no x-values repeat):

$$(-1, 4), (0, 1), (1, 0), (2, 1), (3, 4)$$

The inverse of function g consists of the ordered pairs:

g^{-1}

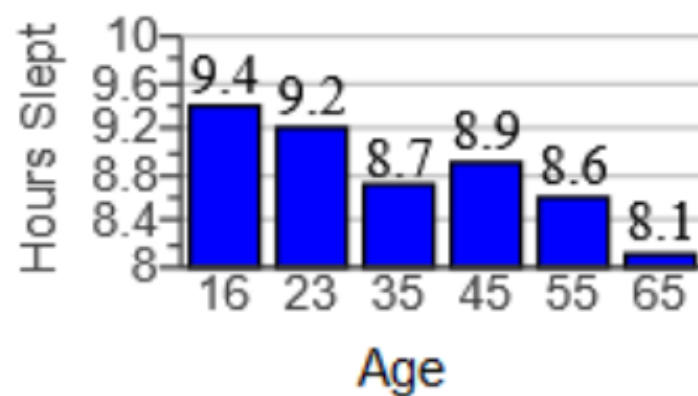
$$(4, -1), (1, 0), (0, 1), (1, 2), (4, 3)$$

Is the inverse of function g a function? Why or why not?

Not a function, x's repeat
(originally y's repeat)

Notice function g has repeated y-values, which results in the inverse relation for function g having repeated x values! (Because the inverse interchanges the x's and y's). This tells us that function g D~~oes~~ NOT have an inverse function.

Example #1 – Determine if the Function has an Inverse:
the graph below shows the average hours slept for select
age groups:



a) Write the ordered pairs of the function (this is a
function because no x's repeat).

The ordered pairs are:

$(16, 9.4), (23, 9.2), (35, 8.7), (45, 8.9), (55, 8.6), (65, 8.1)$

b) Does the function have an inverse that is also a
function? Explain

Yes y values don't repeat

By definition of inverse, interchange all members of the
domain with each member in the range:

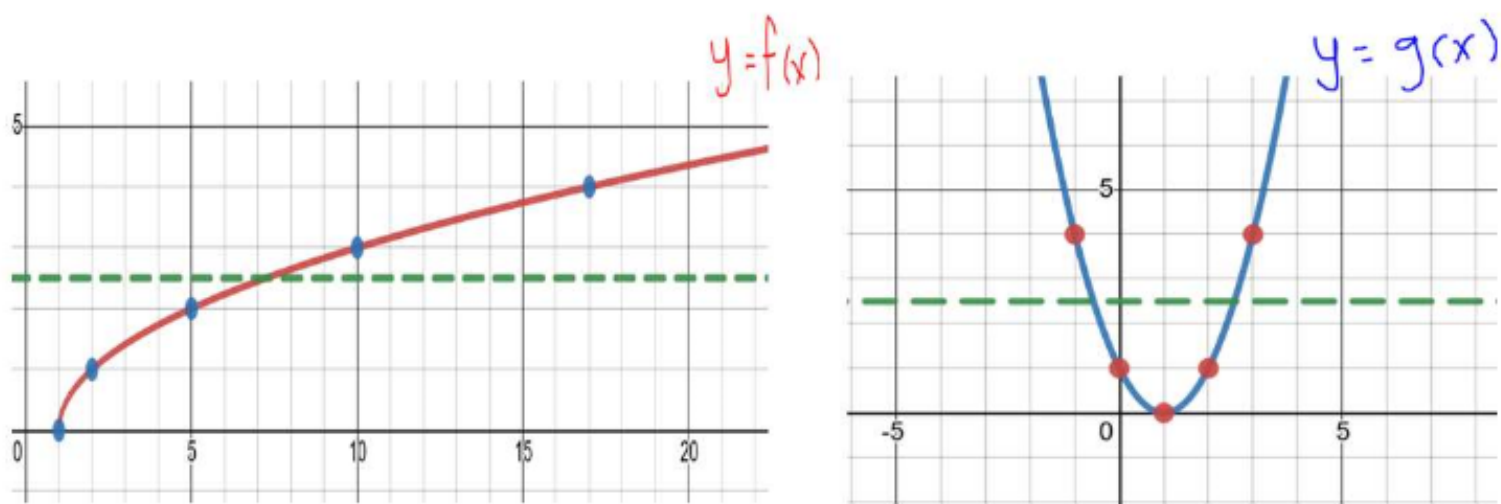
Is this inverse relation a function, why or why not?

Yes.

Topic #2: One-to-One Functions and the Horizontal Line Test

Functions that are one-to-one (abbreviated as 1:1) have an inverse. Functions that are not 1:1 do not have an inverse.

A graph of a function tells us if it is 1:1 or not. If a horizontal line does not intersect the graph more than once, then it is 1:1 since y does not repeat. Consider the graphs of two functions:



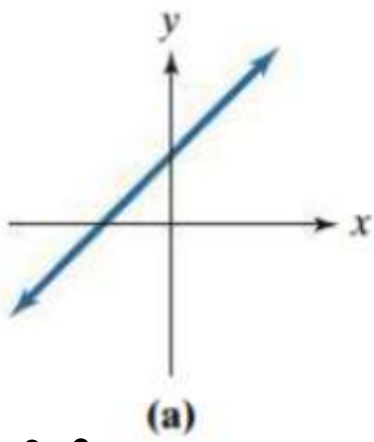
Function f passes the Horizontal Line Test and is 1:1. As a result, function f has an inverse

Function g fails the Horizontal Line Test and is not 1:1. As a result, function g has no inverse

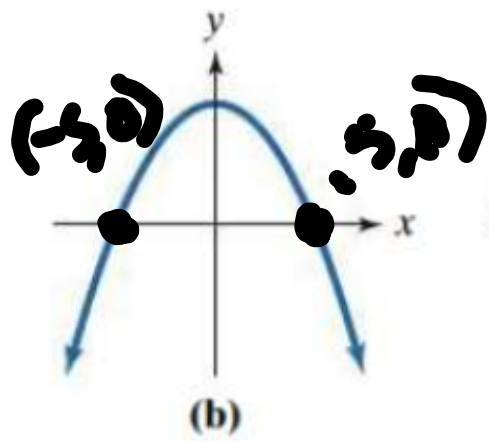
YOU TRY #1 – Determine if the Function has an Inverse

The graphs of 4 functions follow.

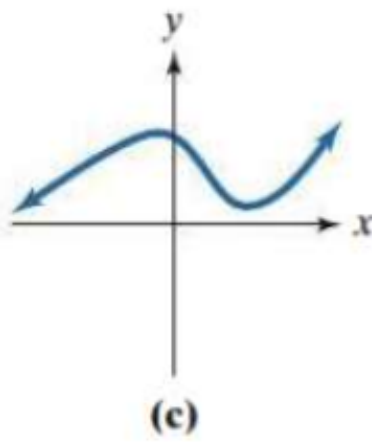
Which functions have an inverse? Which functions do not have an inverse? Explain your reasoning.



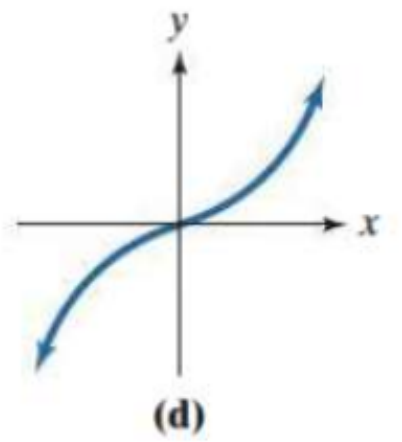
yes



NO



NO



yes

Formal definition of Inverse function (using function composition):

Let f and g be two functions such that

$$f(g(x)) = x \text{ for every } x \text{ in the domain of } g \text{ and}$$

$$g(f(x)) = x \text{ for every } x \text{ in the domain of } f.$$

The function, g , is the **inverse of the function f** and is denoted by f^{-1}

(read as "f inverse"; the -1 is NOT an exponent!).

Thus, $f(f^{-1}(x)) = \underline{x}$ and $\underline{f^{-1}(f(x)) = x}$.

x -values

y -values

The domain of f is equal to the range of f^{-1} , and vice versa.

Example #1 - $f(x) = x - 300$ and $g(x) = x + 300$,

SHOW they are inverse functions using the formal definition above:

$$\boxed{f(g(x)) = x}$$

Should

$$\begin{aligned} f(x+300) &= (x+300) - 300 \\ &= x \end{aligned}$$

Example #2 - $f(x) = \sqrt[3]{x-5}$ and $g(x) = x^3 + 5$,

SHOW they are inverse functions using the formal definition above:

$$\begin{aligned}
 f(g(x)) &= x \\
 \text{Should} & \\
 f(x^3 + 5) &= \sqrt[3]{(x^3 + 5) - 5} \\
 &= \sqrt[3]{x^3} \\
 &= x
 \end{aligned}
 \quad (x^3)^3$$

YOU TRY #2 -

$$f(x) = 3x + 2 \text{ and } g(x) = \frac{x-2}{3};$$

SHOW they are inverse functions:

$$f(g(x)) = x$$

Should

$$\begin{aligned}
 f\left(\frac{x-2}{3}\right) &= 3\left(\frac{x-2}{3}\right) + 2 \\
 &= x - 2 + 2 \\
 &= x
 \end{aligned}$$

Topic #3: Finding the Inverse of a 1:1 Function

When a function is 1:1 it has an inverse. If function f is 1:1 and contains the points (x, y) , then its inverse f^{-1} contains the points (y, x) .

In other words, the **inputs** trade places with the **Outputs**.

The domain and the range are interchanged; the domain for f is the **RANGE** for f^{-1} and the range for f is the **DOMAIN** for f^{-1} .

To find the inverse for a function:

1. Replace $f(x)$ with y and **Switch** the variables x and y ;
2. Solve for **y** and the result is the inverse for the original function (and the original function is the inverse for the new function);
3. Last, we replace y in the equation for the inverse function with function notation **$f(x)^{-1}$**

Example #1 - Consider the function $f(x) = 7x - 5$.

To find the inverse function we rewrite without function notation, replacing $f(x)$ with y $y = 7x - 5$

Step 1: we interchange x and y : $x = 7y - 5$

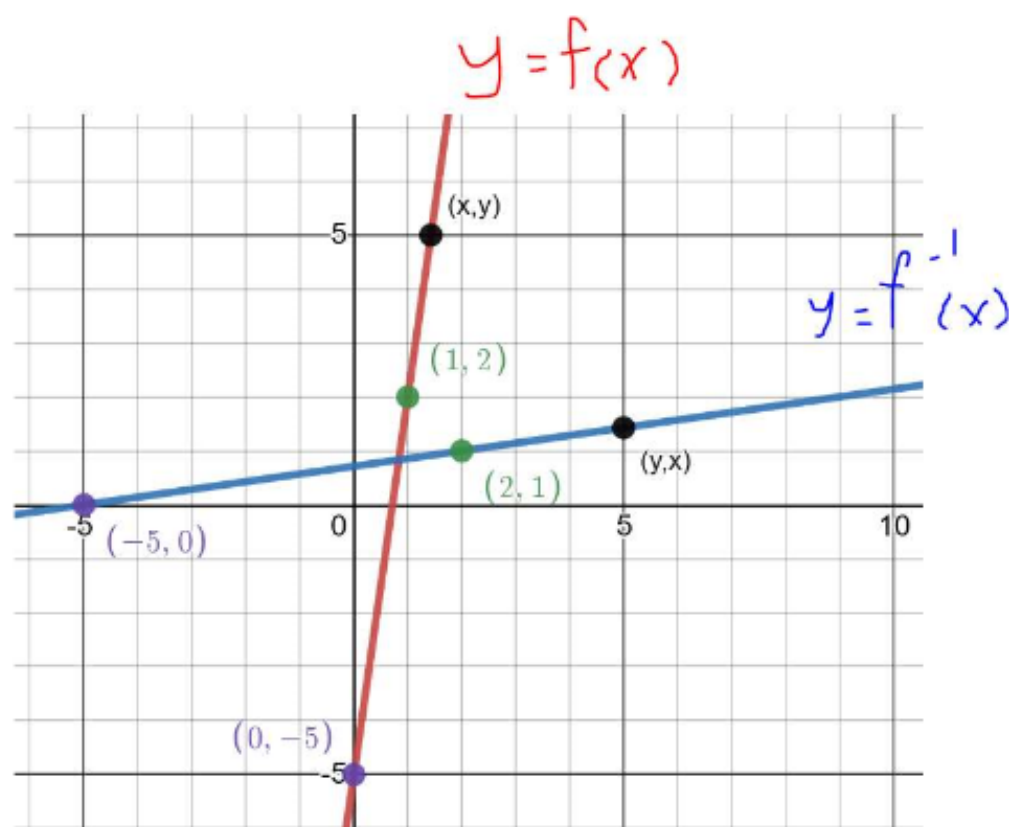
Step 2: we solve for y (this will create a new function where y is a function of x).

$$\begin{array}{l} x = 7y - 5 \\ +5 \quad +5 \\ \hline x + 5 = 7y \\ \frac{x+5}{7} = y \end{array} \quad \frac{x+5}{7} = y$$

Step 3: Last, we identify the inverse with the notation:

$$f^{-1}(x) = \frac{x+5}{7}$$

A graph shows, the 2 functions have all x and y values interchanged.



Another way to look at this and check visually is that ***the graph of the inverse is the reflection of the graph of f over the line $y=x$***

This is true for every function and its inverse function.

A general graph showing the inverse relationship is shown below

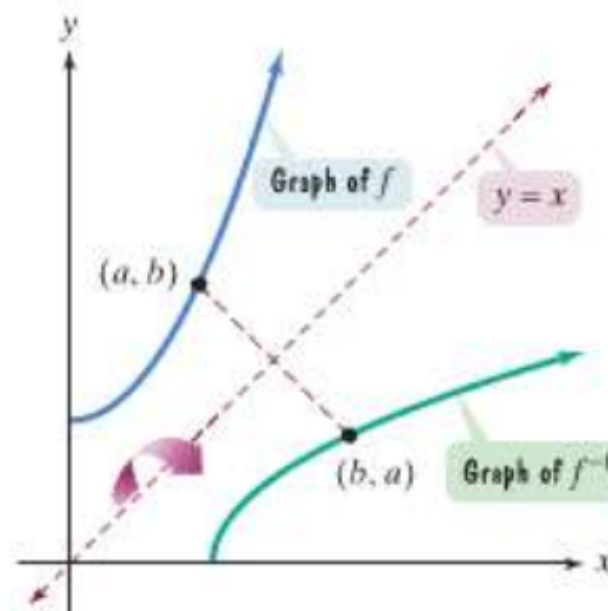


FIGURE 1.67 The graph of f^{-1} is a reflection of the graph of f about _____

It is worth pointing out that the operations on the 2 functions in the above example are completely **OPPOSITE** and **IN REVERSE**:

$f(x) = 7x - 5$ starts by **multiplying** by 7, then **subtracting** 5, and $f^{-1}(x) = \frac{x+5}{7}$ starts by **adding** 5, then **dividing** by 7.

$$f(f^{-1}(x)) = x$$

11

$$\begin{aligned} f\left(\frac{x+5}{7}\right) &= 7\left(\frac{x+5}{7}\right) - 5 \\ &= x+5-5 \\ &= x \end{aligned}$$

Example #2 – Find the Inverse for the Function

a) $f(x) = 2x - 1$

First, rewrite the function using y in place of $f(x)$:

$$y = 2x - 1$$

Next, interchange x and y (which means to apply the definition of inverse): $x = 2y - 1$

Solve for y (this is the hardest step):

$$x = 2y - 1$$

$$\frac{x+1}{2} = y$$

The last step is to indicate with notation that the above equation is the inverse of f : $f^{-1}(x) = \frac{x+1}{2}$

Notice the operations are OPPOSITES and REVERSED.

The domain for the original function is ~~$(-\infty, \infty)$~~ and the range is ~~$(-\infty, \infty)$~~ . The inverse function uses the range of the original function for its domain ~~$(-\infty, \infty)$~~ and uses the domain for its range ~~$(-\infty, \infty)$~~ .

$$b) f(x) = 2x^3 + 3$$

First, rewrite the function using y in place of $f(x)$:

$$y = 2x^3 + 3$$

Next, interchange x and y (apply the definition of inverse):

$$\frac{x-3}{-3} = \frac{2y^3}{-3}$$

Solve for y :

$$\sqrt[3]{\frac{x-3}{2}} = y$$

The last step is to indicate with notation that the above equation is the inverse of f :

$$f^{-1}(x) = \sqrt[3]{\frac{x-3}{2}}$$

Notice the operations are OPPOSITES and REVERSED.

The domain for the original function is $(-\infty, \infty)$ and the range is $(-\infty, \infty)$. The inverse function uses the range of the original function for its domain $(-\infty, \infty)$ and uses the domain for its range $(-\infty, \infty)$.

$$c) f(x) = \sqrt{x-1}$$

Note the domain is $[1, \infty)$ and the range is $[0, \infty)$

If you aren't sure, graph it in your calculator.

First, rewrite the function using y in place of $f(x)$:

$$y = \sqrt{x-1}$$

Next, interchange x and y (apply the definition of inverse):

$$x = \sqrt{y-1}$$

Solve for y :

$$\begin{aligned} x^2 &= y-1 \\ x^2 + 1 &= y \end{aligned}$$

$$f^{-1}(x) = x^2 + 1$$

The new function has a domain $[0, \infty)$ and a range $[1, \infty)$.

Had the domain of $f(x)$ not been restricted, then the new function would not be 1:1 and would not be an inverse!

YOU TRY #3 - $f(x) = x^2 + 1$, for $x \geq 0$

This function is restricted to make it 1:1.

The restricted domain is $x \geq 0$ which makes the restricted range $y \geq 1$.

* The inverse function must therefore have a domain $x \geq 1$ and a range $y \geq 0$.

First, rewrite the function using y in place of $f(x)$:

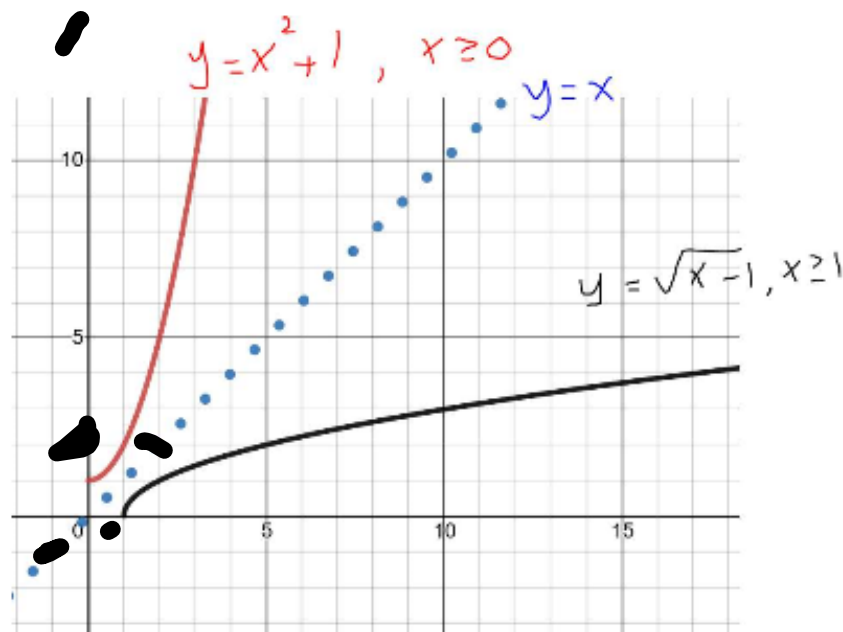
Next, switch the input with the output (apply the definition of inverse):

Solve for y :

$$f^{-1}(x) = \sqrt{x-1}$$

Our new function has a domain $x \geq 1$ and a range $y \geq 0$, just like we anticipated!

A graph of the original functions from example c) and d) show that if function 1 is the inverse of function 2, then function 2 must be the inverse of function 1. Also, notice the symmetry about the line $y = x$.



YOU TRY #4 – Find the equation for $f^{-1}(x)$

Use interval notation to give the domain and range of f and $f^{-1}(x)$.

$$f(x) = x^2 - 4 \text{ for } x \geq 0$$

$$D: [0, \infty)$$

$$R: [-4, \infty)$$

$$\begin{aligned} y &= x^2 - 4 \\ x &= y^2 - 4 \\ +4 & \quad +4 \\ \hline \sqrt{x+4} &= \sqrt{y^2} \\ \sqrt{x+4} &= y \end{aligned}$$

$$f^{-1}(x) = \sqrt{x+4}$$

$$x+4 \geq 0$$

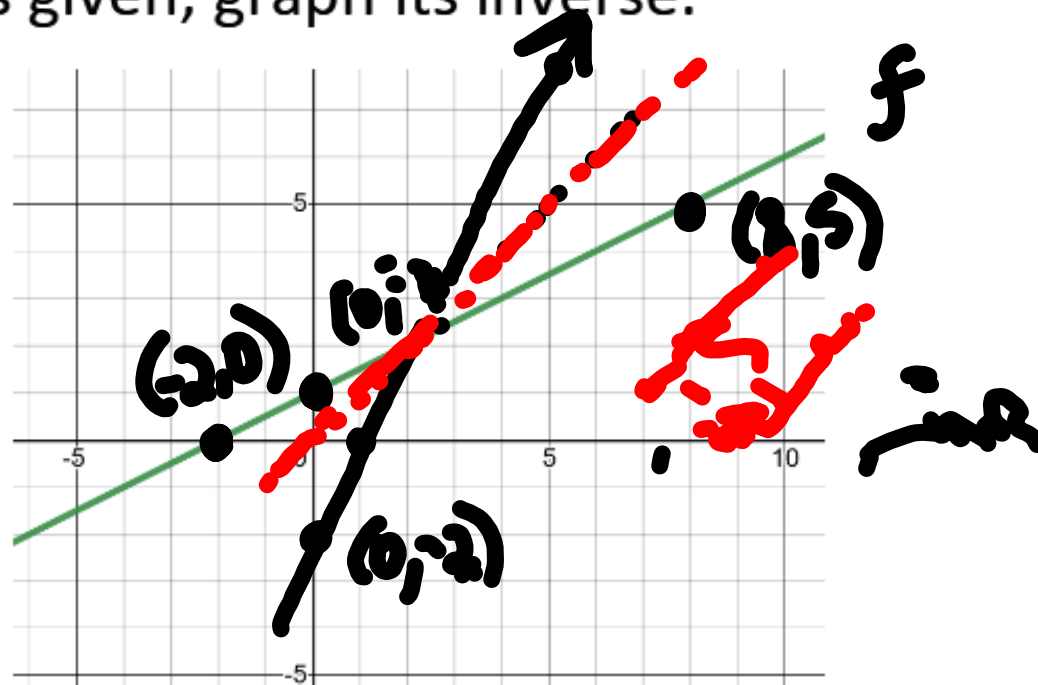
$$x \geq -4$$

$$D: [-4, \infty)$$

$$R: [0, \infty)$$

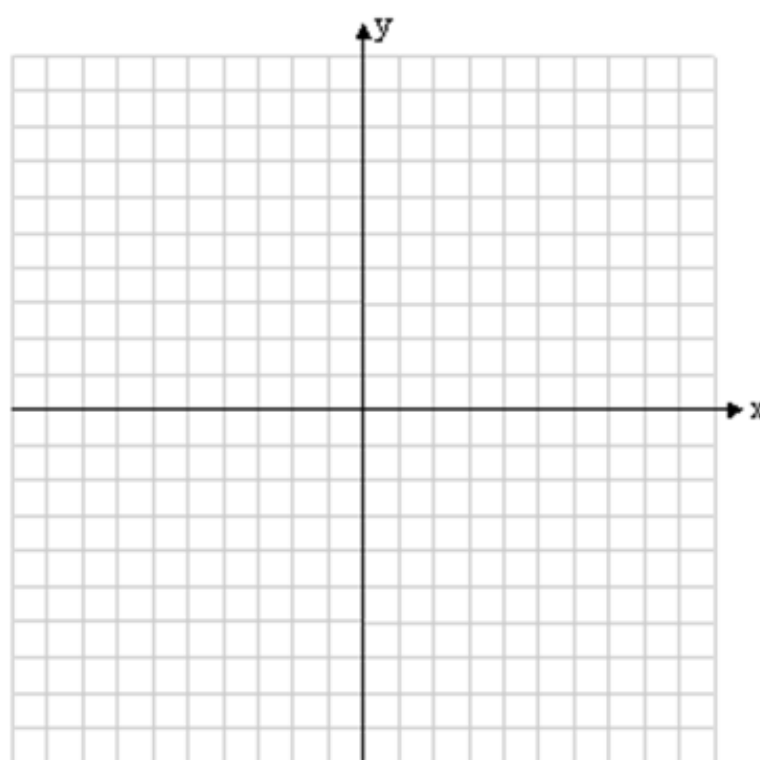
Example #3 – Graph the Inverse for the Function

A graph of f is given, graph its inverse.



Note that if you are given the graph of **any** 1:1 function, you can _____ to sketch the graph of the inverse function.

This is linear, so its inverse must also be a line. All we need are _____ We can pick any 2 points and interchange them to sketch the graph of the inverse function:



Note the symmetry across the line $y=x$!