Math 120
1.8 Inverse Functions

Objectives

1. Verify inverse functions.
2. Find the inverse of a function.
3. Use the graph of a one-to-one function to graph its inverse function.
4. Find the inverse of a function and graph both functions on the same axes.

Topic #1: The Inverse of a Function

In an inverse relationship, the domain and range
Trade places

However, inverting a function **does not guarantee** the inverse relation is also a function.

One definition of an inverse is that all members in the domain interchange with their associated member in the range. In other words, **each ordered pair (x, y) in the function becomes** (y, x) **in the inverse.**

Consider the two functions:

Suppose **function $\boldsymbol{f}$** consists of the ordered pairs (this is a function since no x-values repeat):

f $$(1,0), (2,1), (5,2), (10,3), (17,4)$$

The inverse of function f consists of the ordered pairs:

f^{-1} $$(0,1), (1,2), (2,5), (3,10), (4,17)$$

Is the inverse of function f a function? Why or why not?

Yes. No x-values repeat

It is worth noting that function f has no repeated x-values ***and*** no repeated y-values. This is the definition of One to One____________ (abbreviated as 1:1), which tells us the function MUST have an inverse function.

Suppose **function *g*** consists of the ordered pairs (this is also a function since no x-values repeat):

g $$(-1,4), (0,1), (1,0), (2,1), (3,4)$$

The inverse of function g consists of the ordered pairs:

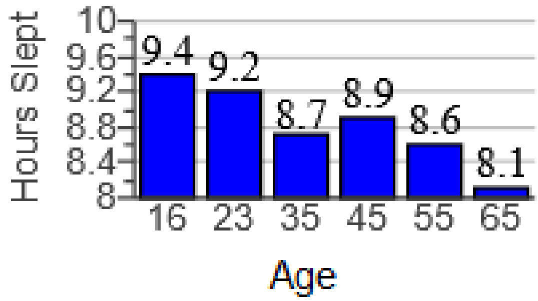
g^{-1} $$(4,-1), (1,0), (0,1), (1,2), (4,3)$$

Is the inverse of function g a function? Why or why not?

No. X value of 1 repeats.

Notice function g has repeated y-values, which results in the inverse relation for function g having repeated x values! (Because the inverse interchanges the x's and y's). This tells us that function *g* ___does NOT___ have an inverse function.

Example #1 – Determine if the Function has an Inverse: the graph below shows the average hours slept for select age groups:



Let x be: Age (years)

Let f(x) be: Hours slept

a) Write the ordered pairs of the function (this is a function because no x's repeat).

The ordered pairs are:

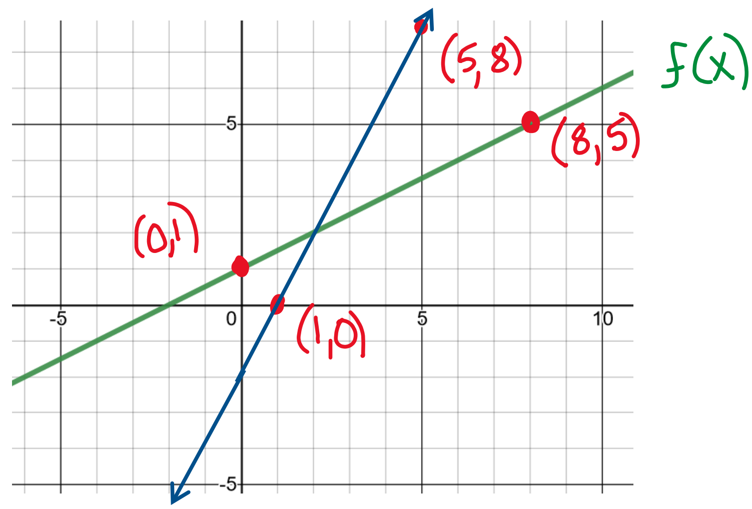
$(16,9.4), (23, 9.2), (35, 8.7)$, (45, 8.9), (55, 8.6), (65, 8.1)

b) Does the function have an inverse that is also a function? Explain

By definition of inverse, interchange all members of the domain with each member in the range:

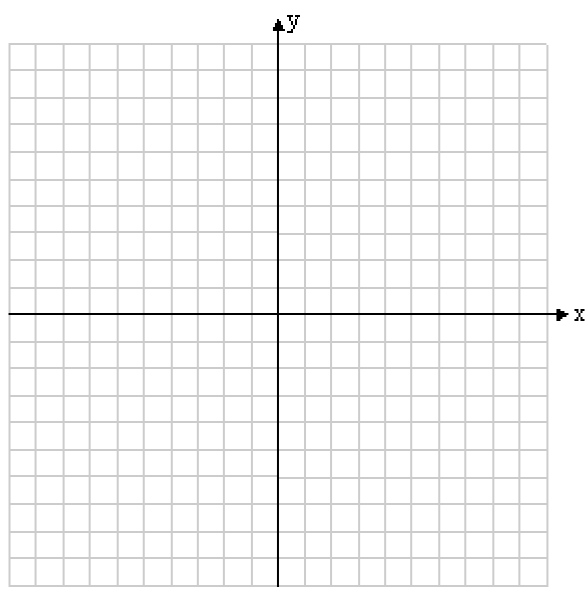
Example #2 – Graph the Inverse for the Function

A graph of f is given, graph its inverse.



Note that if you are given the graph of **any** 1:1 function, you can interchange (x,y) to sketch the graph of the inverse function.

This is linear, so its inverse must also be a line. All we need are two points We can pick any 2 points and interchange them to sketch the graph of the inverse function:



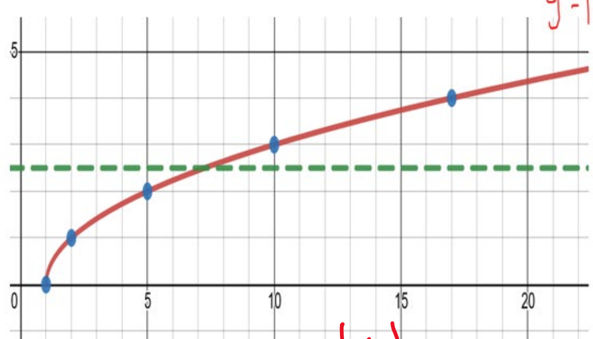
Note the symmetry across the line y=x!

Topic #2: One-to-One Functions and the Horizontal Line Test

Functions that are one-to-one (abbreviated as 1:1) have an inverse. Functions that are not 1:1 do not have an inverse.

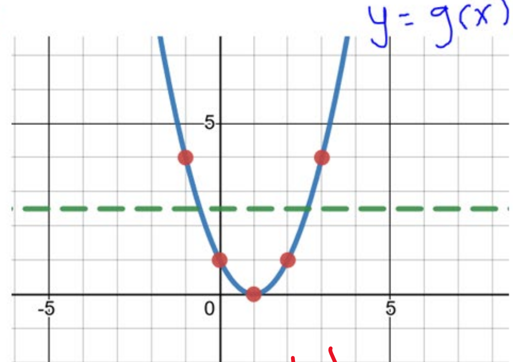
A graph of a function tells us if it is 1:1 or not. If a horizontal line does not intersect the graph more than once, then it is 1:1 since y does not repeat. Consider the graphs of two functions:

$y=f(x)$



1:1

$y=g(x)$



NOT 1:1

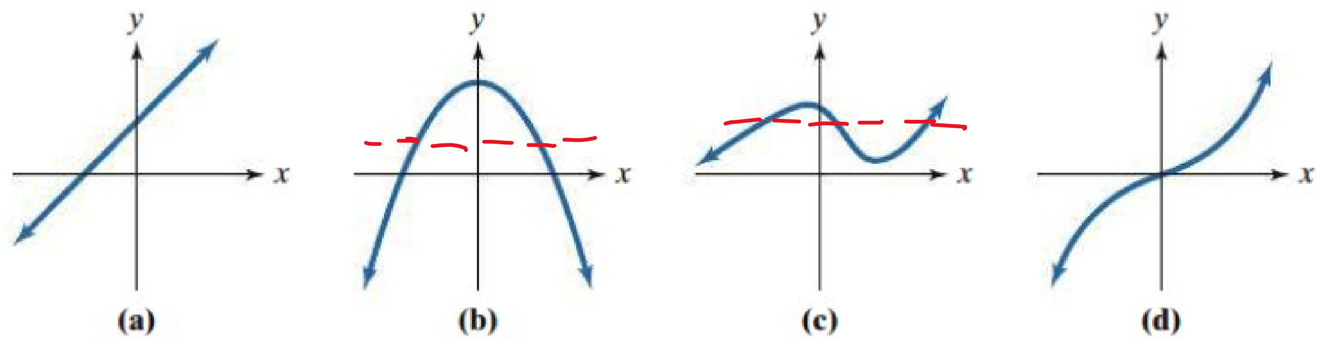
Function f passes the Horizontal Line Test and is 1:1. As a result, function f has an inverse

Function g fails the Horizontal Line Test and is not 1:1. As a result, function g has no inverse

<u>YOU TRY #1</u> – Determine if the Function has an Inverse

The graphs of 4 functions follow.

Which functions have an inverse? Which functions do not have an inverse? Explain your reasoning.



Yes   No   NO   Yes

Formal definition of Inverse function (using function composition):

Let *f* and *g* be two functions such that

$f(g(x)) = x$ for every x in the domain of g and

$g(f(x)) = x$ for every x in the domain of f.

The function, *g*, is the **inverse of the function *f*** and is denoted by $f^{-1}(x)$

(read as "f inverse"; the -1 is NOT an exponent!).

Thus, $f(f^{-1}(x))$ = X and $f^{-1}(f(x)) = X$.

The domain of *f* is equal to the Range of f^{-1}, and vice versa.

__Example #1__ - $f(x) = x - 300$ and $g(x) = x + 300$,

SHOW they are inverse functions using the formal definition above: $f(f^{-1}(x)) = X$

So if $f(g(x)) = X$ then $f(x)$ and $g(x)$ must be inverses.

$f(g(x)) = f(X+300) = (X+300) - 300 = X$

$f(x)$ and $g(x)$ are inverses

__Example #2__ - $f(x)=\sqrt[3]{x-5}$ and $g(x)=x^3+5$,

SHOW they are inverse functions using the formal definition above: $f(f^{-1}(x))=x$

$$f(g(x))=f(x^3+5)=\sqrt[3]{(x^3+5)-5}=\sqrt[3]{x^3+5-5}$$
$$=\sqrt[3]{x^3}$$
$$=x$$

f(x) and g(x) are inverses.

__YOU TRY #2__ -

$f(x)=3x+2$ and $g(x)=\dfrac{x-2}{3}$; $f(f^{-1}(x))=x$

SHOW they are inverse functions:

$$f(g(x))=f\left(\frac{x-2}{3}\right)=3\left(\frac{x-2}{3}\right)+2=\cancel{3}\left(\frac{x-2}{\cancel{3}}\right)+2$$
$$=x-2+2$$
$$=x$$

Topic #3: Finding the Inverse of a 1:1 Function

When a function is 1:1 it has an inverse. If function f is 1:1 and contains the points (x, y), then its inverse f^{-1} contains the points (y, x).

In other words, the **inputs** trade places with the Outputs______.

The domain and the range are interchanged; **the domain for f is the** Range__________ **for f^{-1} and the range for f is the** Domain______________ **for f^{-1}**

To find the inverse for a function:

1. Replace $f(x)$ with y and Switch__________ the variables x and y;
2. Solve for y______ and the result is the inverse for the original function (and the original function is the inverse for the new function);
3. Last, we replace y in the equation for the inverse function with function notation $f^{-1}(x)$__________

Example #1 - Consider the function $f(x) = 7x - 5$.

To find the inverse function we rewrite without function notation, replacing $f(x)$ with y: $y = 7x - 5$

Step 1: we interchange x and y: $x = 7y - 5$

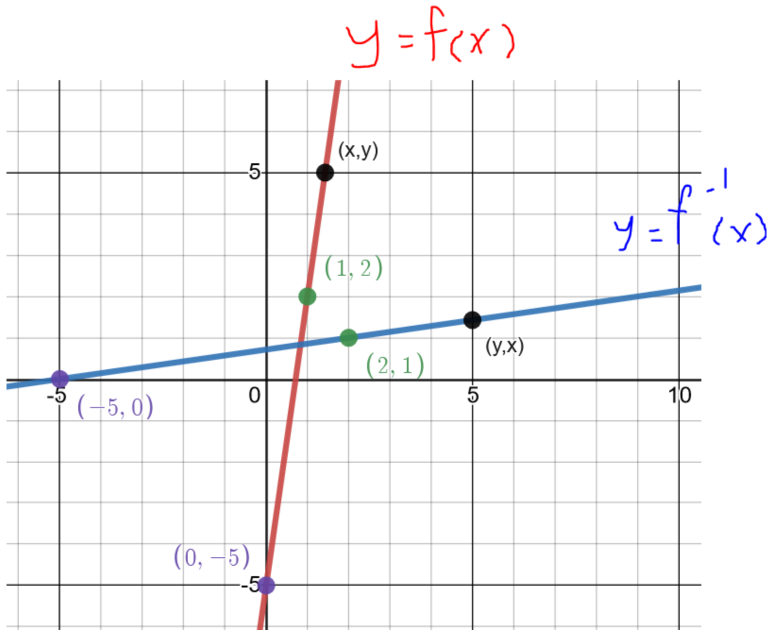
Step 2: we solve for y (this will create a new function where y is a function of x).

$$\begin{array}{r} x = 7y - 5 \\ +5 \quad\quad +5 \\ \hline x + 5 = 7y \end{array} \qquad \frac{x+5}{7} = \frac{7y}{7} \qquad \boxed{\frac{x+5}{7} = y}$$

Step 3: Last, we identify the inverse with the notation:

$$f^{-1}(x) = \frac{x+5}{7}$$

A graph shows, the 2 functions have all x and y values interchanged.



Another way to look at this and check visually is that ***the graph of the inverse is the*** reflection ***of the graph of f over the line y=x***

This is true for every function and its inverse function.

A general graph showing the inverse relationship is shown below

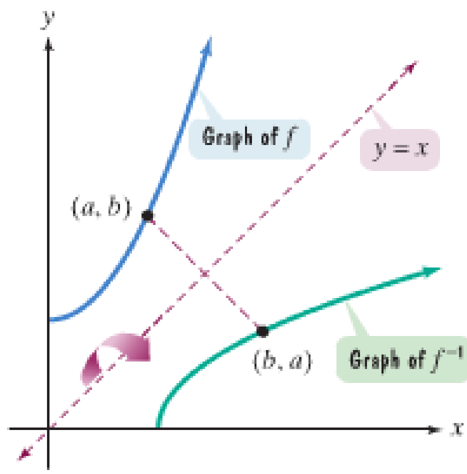


FIGURE 1.67 The graph of f^{-1} is a reflection of the graph of f about ________

It is worth pointing out that the operations on the 2 functions in the above example are completely OPPOSITE and IN REVERSE:

$f(x) = 7x - 5$ starts by **multiplying** by 7, then **subtracting** 5, and $f^{-1}(x) = \frac{x+5}{7}$ starts by **adding** 5, then **dividing** by 7.

Example #2 – Find the Inverse for the Function

a) $f(x) = 2x - 1$

First, rewrite the function using y in place of *f(x)*:

$y = 2x - 1$

Next, interchange x and y (which means to apply the definition of inverse): $x = 2y - 1$

Solve for y (this is the hardest step):

$$x = 2y - 1$$
$$\underline{+1 \qquad\quad +1}$$
$$\frac{x+1}{2} = \frac{2y}{2}$$

$$\boxed{\frac{x+1}{2} = y}$$

The last step is to indicate with notation that the above equation is the inverse of f: $f^{-1}(x) = \frac{x+1}{2}$

Notice the operations are OPPOSITES and REVERSED.

The domain for the original function is $(-\infty, \infty)$ and the range is $(-\infty, \infty)$ The inverse function uses the range of the original function for its domain $(-\infty, \infty)$ and uses the domain for its range $(-\infty, \infty)$

b) $f(x) = 2x^3 + 3$

First, rewrite the function using y in place of *f(x)*:

$$y = 2x^3 + 3$$

Next, interchange x and y (apply the definition of inverse):

$$x = 2y^3 + 3$$

Solve for y:

$$\begin{array}{rcl} x & = & 2y^3 + 3 \\ -3 & & \quad -3 \\ \hline \dfrac{x-3}{2} & = & \dfrac{2y^3}{2} \end{array}$$

$$\frac{x-3}{2} = y^3$$

$$\sqrt[3]{\frac{x-3}{2}} = \sqrt[3]{y^3}$$

$$\boxed{\sqrt[3]{\frac{x-3}{2}} = y}$$

The last step is to indicate with notation that the above equation is the inverse of f:

$$f^{-1}(x) = \sqrt[3]{\frac{x-3}{2}}$$

Notice the operations are OPPOSITES and REVERSED.

The domain for the original function is $(-\infty, \infty)$ and the range is $(-\infty, \infty)$. The inverse function uses the range of the original function for its domain $(-\infty, \infty)$ and uses the domain for its range $(-\infty, \infty)$

c) $f(x) = \sqrt{x-1}$

Note the domain is $[1, \infty)$ and the range is $[0, \infty)$

If you aren't sure, graph it in your calculator.

First, rewrite the function using y in place of *f(x)*:

$y = \sqrt{x-1}$

Next, interchange x and y (apply the definition of inverse):

$x = \sqrt{y-1}$

Solve for y:

$x = \sqrt{y-1}$

$(x)^2 = \left(\sqrt{y-1}\right)^2$

$x^2 = y - 1$

$+1 \quad +1$

$\boxed{x^2+1 = y}$

$f^{-1}(x) = x^2 + 1$

The new function has a domain $[1, \infty)$ and a range $[0, \infty)$.

Had the domain of $f(x)$ not been restricted, then the new function would not be 1:1 and would not be an inverse!

d) $f(x) = x^2 + 1$, for $x \geq 0$

This function is restricted to make it 1:1.

Note the restricted domain is $\underline{[0, \infty)}$ and the restricted range is $\underline{[1, \infty)}$

If you aren't sure, graph it in your calculator.

First, rewrite the function using y in place of *f(x)*:

$$y = x^2 + 1$$

Next, interchange x and y (apply the definition of inverse):

$$x = y^2 + 1$$

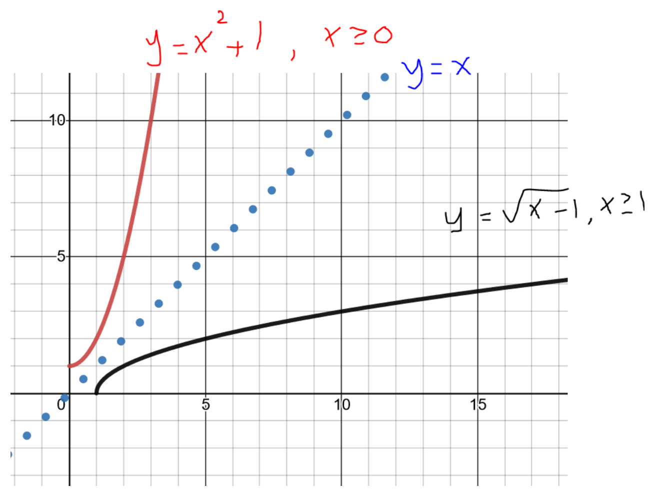
Solve for y:

$$x = y^2 + 1$$
$$-1 \qquad -1$$
$$x - 1 = y^2$$
$$\sqrt{x-1} = \sqrt{y^2}$$
$$\boxed{\sqrt{x-1} = y}$$

$$f^{-1}(x) = \sqrt{x-1}$$

The new function has a domain $\underline{[1, \infty)}$ and a range $\underline{[0, \infty)}$.

A graph of the original functions from example c) and d) show that if function 1 is the inverse of function 2, then function 2 must be the inverse of function 1. Also, notice the symmetry about the line $y = x$.



<u>*YOU TRY #3*</u> – Find the equation for $f^{-1}(x)$

Use interval notation to give the domain and range of *f* and *f*$^{-1}$*(x)*.

$f(x) = x^2 - 4 \;\; for \; x \geq 0$

$f(x)$ Domain : $[0, \infty)$

Range : $[-4, \infty)$

$y = x^2 - 4$

$x = y^2 - 4$

$+4 \qquad +4$

$x + 4 = y^2$

$\sqrt{x+4} = \sqrt{y^2}$

$\sqrt{x+4} = y$

$f^{-1}(x) = \sqrt{x+4}$

Domain: $[-4, \infty)$

Range: $[0, \infty)$