Math 120
4.2 Logarithmic Functions

Objectives:

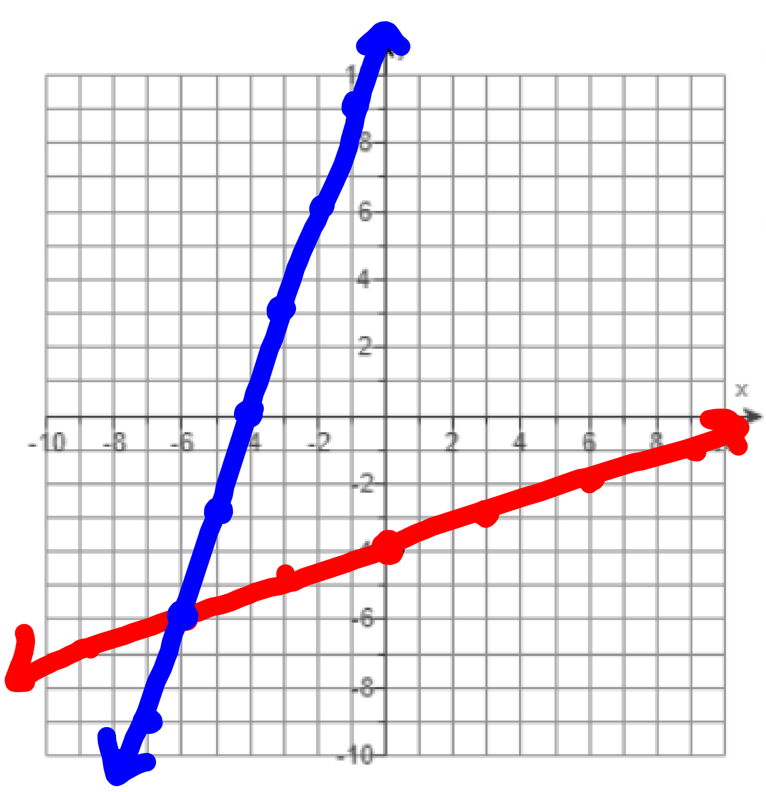
1. Change from logarithmic to exponential form.
2. Change from exponential to logarithmic form.
3. Evaluate logarithms.
4. Use basic logarithmic properties.
5. Graph logarithmic functions.
6. Find the domain of a logarithmic function.
7. Use common logarithms.
8. Use natural logarithms.

Topic #1: Definition of Logarithmic Functions

REMINDER:

Recall that inverse functions "switch" the x and the y-values on the graph and in the equation.

Example: $y = \frac{1}{3}x - 4$



x	y
-9	-7
-6	-6
-3	-5
0	-4
3	-3
6	-2
9	-1

Equation:

$x = \frac{1}{3}y - 4$

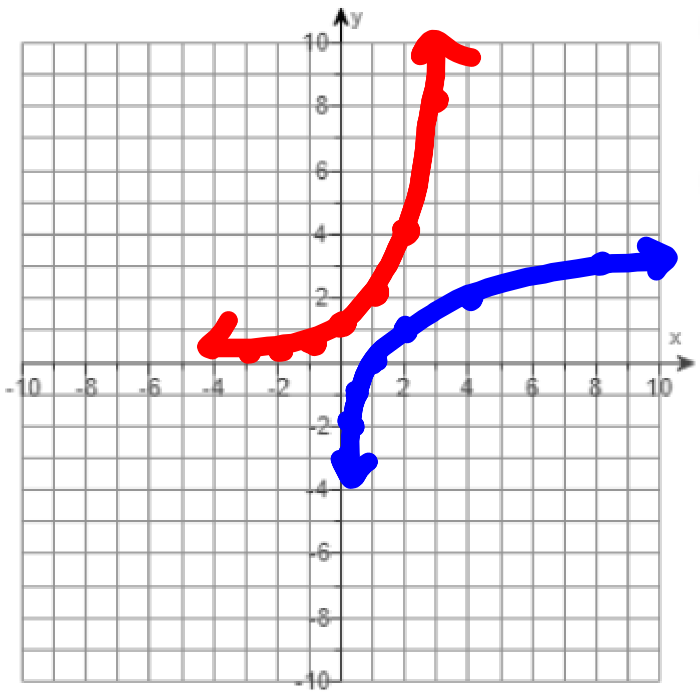
$+4 \qquad +4$

$3(x+4 = \frac{1}{3}y)3$

$\boxed{3x+12=y}$

x	y
-7	-9
-6	-6
-5	-3
-4	0
-3	3
-2	6
-1	9

$y = 2^x$ $\qquad$ $x = 2^y$ $\qquad$ Equation:



x	y
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8

x	y
$\frac{1}{8}$	-3
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2
8	3

When taking the inverse of exponential functions, we run into a problem actually solving the equation. But we know exactly how this inverse function should behave. So, mathematicians defined a new type of function that acts as the inverse of an exponential function.

This is called a Logarithmic Function.

Consider the equivalent statements that show the inverse relationship between exponents and logarithms:

$$x = b^y \longleftrightarrow y = \log_b x$$

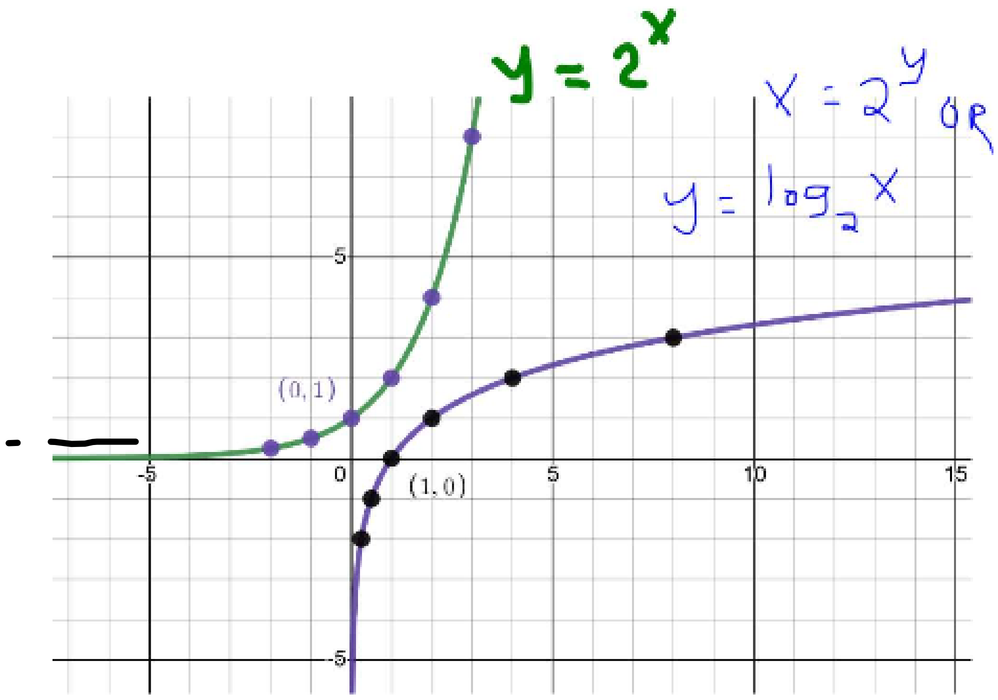
This provides the definition of a logarithmic function:

$$f(x) = \log_b x$$

where b is the base of the logarithmic function.

The function is read "y equals log base b of x". As with exponential functions $b > 0$ and $b \neq 1$. Unlike exponential functions, the domain is restricted to $x > 0$.

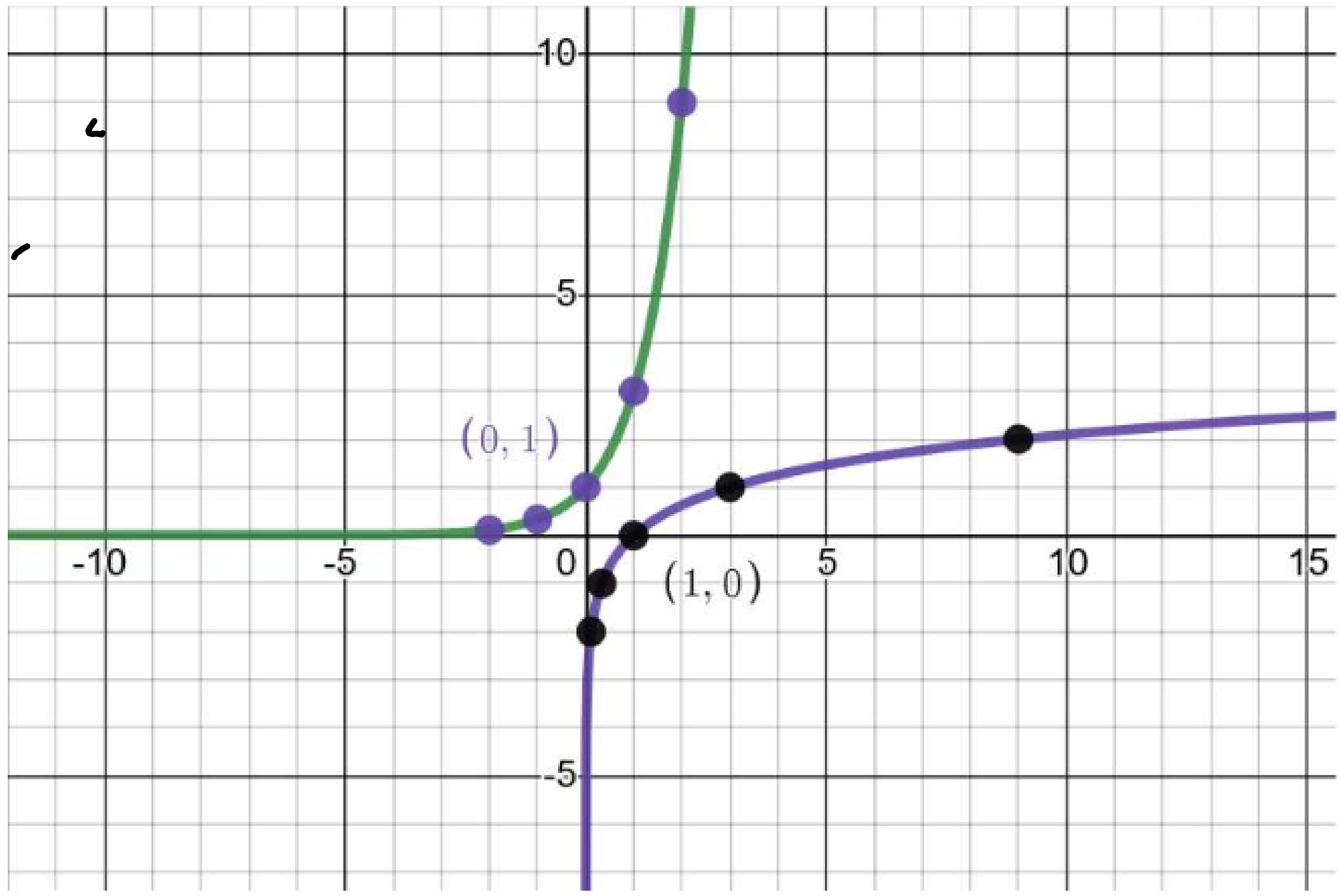
Consider the graph of the functions $f(x) = 2^x$ and $f(x) = \log_2 x$



Notice the points are reversed, showing the two functions are inverses! The exponential function has an horizontal asymptote at $y = 0$ ____________________

and the logarithmic function has a vertical asymptote at $x = 0$ ____________________

A graph confirms:



Example #2 – Use the Definition to Rewrite the Logarithm Equation in Exponential Form

a) $2 = \log_5 x$ $\longrightarrow$ $x = 5^2$

$y = \log_b x$

Apply the definition: $y = \log_b x \leftrightarrow x = b^y$

$y = \log_b x \longleftrightarrow x = b^y$

b) $3 = \log_b 64$ $\quad 64 = b^3$

Apply the definition:

$\log_b x = y \longleftrightarrow x = b^y$

c) $\log_3 7 = y$ $\quad 7 = 3^y$

Apply the definition:

Example #3 – Use the Definition to Rewrite the Exponential Equation in Logarithm Form

a) $10^2 = x$ $\quad x = 10^2 \leftrightarrow 2 = \log_{10} X$

Apply the definition: $x = b^y \leftrightarrow y = \log_b x$

$b^y = X \longleftrightarrow Y = \log_b X$

b) $b^3 = 8$ $\quad 3 = \log_b 8$

Apply the definition:

$b^y = X \longleftrightarrow y = \log_b X$

c) $e^y = 9$ $\quad Y = \log_e 9$

Apply the definition:

Example #3 – Use the Definition to Rewrite; Radicals Involved

$3 = 9^{\frac{1}{2}}$ $\quad 9^{\frac{1}{2}} = \sqrt[2]{9}$

a) $\log_9 3 = 1/2$

$\log_b x = y \longleftrightarrow x = b^y$

$\log_b x = y \longleftrightarrow x = b^y$

b) $\log_{32} 2 = 1/5$ $\quad 2 = 32^{\frac{1}{5}}$ $\quad 32^{\frac{1}{5}} = \sqrt[5]{32}$

$2^5 = 32$

$b^y = x \longleftrightarrow y = \log_b x$

c) $\sqrt[3]{27} = 3$

$27^{\frac{1}{3}} = 3$ $\quad \frac{1}{3} = \log_{27} 3$

$b^y = x \longrightarrow y = \log_b x$

d) $\sqrt[4]{81} = 3$ $\quad \frac{1}{4} = \log_{81} 3$

$81^{\frac{1}{4}} = 3$

Topic #2: Evaluating Logarithmic Expressions with Definitions and Properties

It is worth stating multiple times – here is the definition of logarithm:

$$y = \log_b x \longleftrightarrow x = b^y$$

Certain logarithmic expressions can be evaluated without a calculator by using what we know about their exponent counterparts. For example, we can evaluate:

$$\log_2 32 = y$$

Then we can rewrite as an exponential equation using the above property:

$$2^y = 32$$

Finally, we can use trial and error OR what we know about base 2 numbers:

$$\boxed{y = 5}$$

Example #1 – Use the Definition of Logarithm to Evaluate the Expression

a) $\log_3 27 = y$

Write as an equation, then write as an exponent:

$3^y = 27$ $\boxed{y=3}$

b) $\log_{27} 3 = y$

Write as an equation, then write as an exponent:

$27^y = 3$ $\boxed{y=\frac{1}{3}}$

$27^{\frac{1}{3}} = \sqrt[3]{27}$

c) $\log_5 125$

Write as an equation, then write as an exponent:

$5^y = 125$ $y=3$

d) $\log_5 1/25$

Write as an equation, then write as an exponent:

$1/25 = 5^y$ $y=-2$

$\frac{1}{5^2} =$

$5^{-2} = 5^y$

e) $\log_9 81$

Write as an equation, then write as an exponent:

$9^y = 81$

$y = 2$

f) $\log_3 81$

Write as an equation, then write as an exponent:

$3^y = 81$ $y = 4$

g) $\log_7 1/7$

Write as an equation, then write as an exponent:

$7^y = 1/7$ $y = -1$

$\frac{1}{7} = 7^{-1}$

$7^y = 7^{-1}$

Note: Perhaps not all the answers are obvious and might require some trial and error. However, the more important part is to understand that **logarithms are** exponents in reverse

Topic #3: Basic Logarithmic Properties

The definition of logarithm:

$y = \log_b x \quad \leftrightarrow \quad x = b^y$

Can be used to develop the 4 basic properties:

1. $\log_b 1 = 0$ $\qquad b^0 = 1$
2. $\log_b b^1 = 1$ $\qquad b^1 = b$
3. $\log_b b^x = x$
4. $b^{\log_b x} = x$

For example, the first property must be true since any base b to the power ZERO is ONE. Rewriting the exponential fact as a logarithm shows the property:

The second property must be true since any base b to the power of ONE is ITSELF. Rewriting this fact as a logarithm shows the property:

The third and fourth properties can be reasoned with similarly, but they can also be explained by the inverse relationship between logarithms and exponents. The third property states that a ________________________, the fourth states that an ______________________________.

Example #1 – Use a Basic Logarithmic Property to Evaluate the Expression

a) $\log_5 5 = 1$

Use property two:

b) $\log_5 1 = 0$

Use property one:

c) $\log_6 \sqrt{6} \longrightarrow \log_6 6^{\frac{1}{2}} = \frac{1}{2}$

Rewrite the radical as an exponent and use property three:

d) $\log_4 4^7 = 7$

Use property three:

$\frac{1}{2}$

e) $\log_2(1/2)$ $\longrightarrow$ $\log_2 2^{-1} = -1$

Rewrite the fraction as an exponent and use property three:

f) $3^{\log_3 27}$ $= 27$

Use property four:

Topic #4: Common and Natural Logarithms

Recall the general logarithmic function:

$$\boldsymbol{f(x) = \log_b x}$$

Where b is any base such that $b > 0$ and $b \neq 1$ and the domain is $x > 0$

The base can be any number that is positive and not ONE. Two widely used bases in the family of logarithmic functions are:

1. The Common Log, which uses the base 10

$$\boldsymbol{f(x) = \log_{10} x}$$

$f(x) = \log x$

Which we can rewrite as:

$$\boldsymbol{f(x) = \log x}$$

2. The Natural Log, which uses base e

$$\boldsymbol{f(x) = \log_e x}$$

$f(x) = \ln x$

Which we can rewrite as:

$$\boldsymbol{f(x) = \ln x}$$

Example #1 – Evaluate the Expression

a) $\log 10$ $\longrightarrow$ $\log_{10} 10 = y$

This a common log, the implied base is 10:

$10^y = 10$ $\boxed{y=1}$

b) $\log(1/100)$ $\rightarrow$ $\log_{10} \frac{1}{10^2}$ $\rightarrow$ $\log_{10} 10^{-2} = -2$

This is a common/base ten log; rewrite the fraction as an exponent:

$\cancel{3^{\log_3}}{}^{7} = 7$

c) $10^{\log 10^2}$ $\rightarrow$ $\cancel{10^{\log_{10}}}\overset{x}{10^2} = 10^2$

An exponent of base 10 undoes a common log:

d) $\ln e$ $\rightarrow$ $\log_e e = 1$

This is a natural log, the implied base is e:

e) $\ln(1/e^7)$ $\longrightarrow$ $\log_e e^{-7} \longrightarrow -7$

This is a natural log; rewrite the exponent:

.

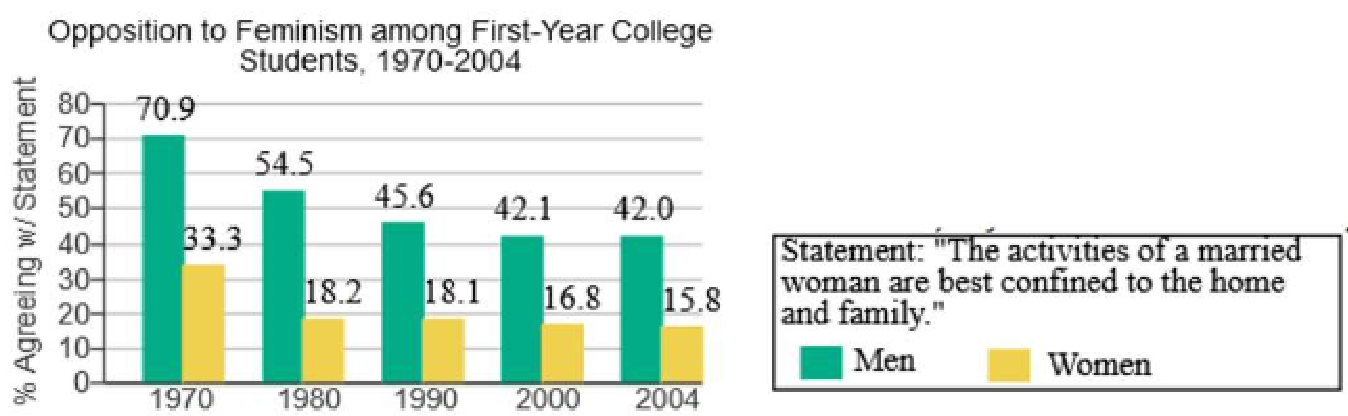
f) $e^{\ln 16}$ $\longrightarrow$ $e^{\log_e 16} = 16$

An exponent of base e undoes a natural log:

Topic #4: Applications of Logarithms

Example #1 – Modeling Data with a Natural Logarithmic Function

The graph shows the percentage of first-year college students who agree with the statement “The activities of a married woman are best confined to the home and family” for select years.



The function $f(x) = -7.58 \ln x + 70$ models the percentage of men who agree with the statement, x years after 1969.

a) According to the model, what percentage of men agree with the statement in the year 2004?

$f(35) = 43.1$ In 2004, 43.1% of men agree w/statement

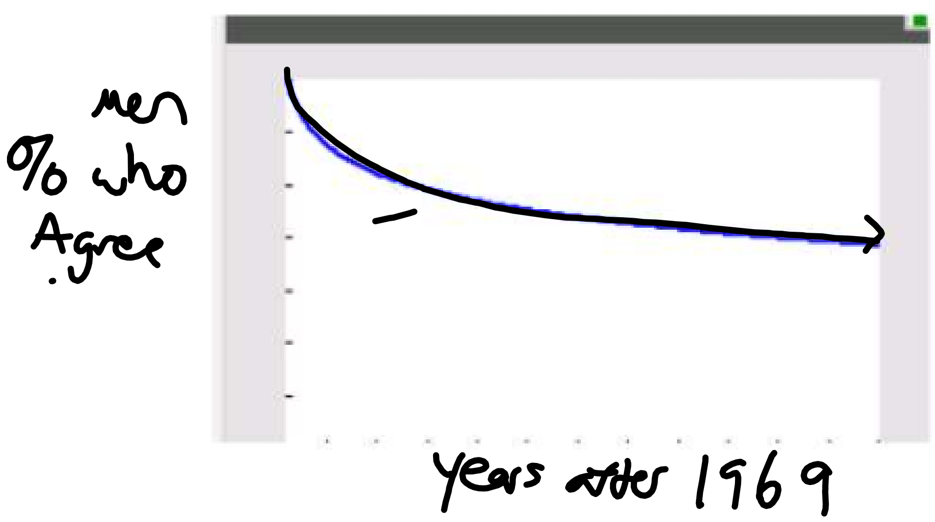
b) How does this estimate compare with the actual percentage?

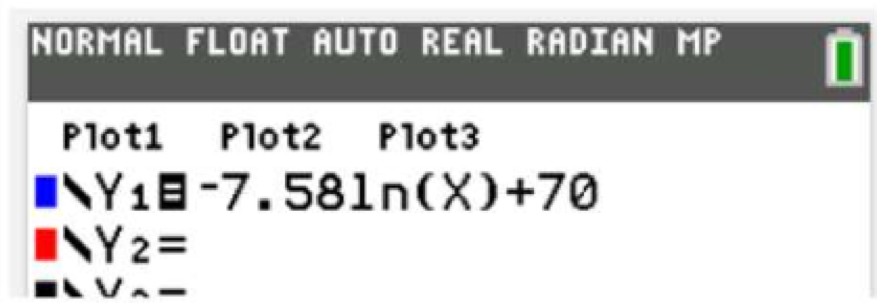
c) Use the model to predict the percentage of men who will agree with the statement in the year 2025.

$f(56) = 39.5\%$

d) Graph the function and interpret.

Use a graphing device:





Example #2 – Modeling Height with a Common Logarithmic Function

The percentage of adult height attained by a boy who is x years old is modeled by the function:

$$f(x) = 29 + 48.8\log(x+1)\,; \text{ for } 5 \le x \le 17$$

where x represents the boy's age (from 5 to 17) and $f(x)$ represents the percentage of his adult height.

a) What percentage of his adult height will the boy attain at age 8?

$f(8) = 75.6$

At 8 years old, the boy will have 75.6% of his adult height

b) What percentage of his adult height will the boy attain at 15?

$f(15) = 87.8\%$

Y_1 Y_2

$80 = 29 + 48.8 \log(x+1)$

c) When will the boy reach 80% of his adult height?

This gives an output $f(x) = 80$ and the equation:

$$29 + 48.8\log(x+1) = 80$$

Use technology to solve the equation; on a graphing calculator, let the left side equal y1 and the right side equal y2. Find the intersection: