Math 120
4.3 Properties of Logarithms

Objectives:

1. Use the product rule.
2. Use the quotient rule
3. Use the power rule.
4. Expand logarithmic expressions.
5. Condense logarithmic expressions.
6. Use the change-of-base property.

Topic #1: Properties of Logarithms – The Product Rule

Since logarithms are exponents in reverse, certain properties of exponents apply to logarithms too. Recall the definition of logarithm:

$$y = \log_b x \longleftrightarrow x = b^y$$

The product rule for exponents states:

$$b^M \cdot b^N = b^{M+N} \qquad x^2 \cdot x^3 = x^5$$

This indicates that to **multiply** like base exponents means to **add** the powers.

The Product Rule for logarithms is similar:

$$\log_b (M \cdot N) = \log_b M + \log_b N$$

This also indicates that multiplication becomes addition.

For example, we can expand a logarithmic expression that involves multiplication: $\ln 7x = \ln 7 + \ln x$

Which gives an equivalent statement:

Example #1 – Expand the Expression and Simplify when Possible

a) $\log_6(7 * 3)$ $= \log_6 7 + \log_6 3$

The product of 7 and 3 is an input to the operation "log base 6" and the product rule applies:

b) $\log(10000x) =$ $\log 10000 + \log x$

The product of 10000 and x is an input to the operation "log base 10" and the product rule applies:

$\log_{10} 10^4 + \log x$

$4 + \log x$

The first term of the expansion can be simplified since $10^4 = 10000$:

c) $\ln(10e^2)$ $= \ln 10 + \ln_e e^2$

The product of 10 and e^2 is an input to the operation "log base e" and the product rule applies

The second term of the expansion can be simplified since a natural log (base e) undoes a base e exponent:

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ln(10e²)
                 4.302585093
ln(10)+2
                 4.302585093
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$\ln 10 + 2$

Topic #2: Properties of Logarithms – The Quotient Rule

The quotient rule for exponents states:

$$\frac{b^M}{b^N} = b^{M-N}$$

This indicates that to **divide** like base exponents means to **subtract** the powers (in order).

The corresponding Quotient Rule for logarithms is similar: $\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$

This also indicates that division becomes subtraction.

For example, we can expand a logarithmic expression that involves division:

$$\log_2\left(\frac{7}{x}\right) = \log_2 7 - \log_2 x$$

Which gives an equivalent statement:

Example #1 – Expand the Expression and Simplify when Possible

a) $\log_9\left(\frac{9}{y}\right) = \log_9 9^1 - \log_9 Y$

The quotient of 9 and y is an input to the operation "log base 9" and the quotient rule applies. The first term of the expansion can be simplified since $9^1 = 9$:

$$1 - \log_9 Y$$

b) $\log\left(\frac{z}{100}\right) = \log Z - \log_{10} 10^2$

The quotient of z and 100 is an input to the operation to the common log (base 9) and the quotient rule applies. The second term of the expansion can be simplified since $10^2 = 100$: $\log Z - 2$

c) $\ln\left(\frac{e^6}{8}\right) = \ln_e e^6 - \ln 8$

The quotient of e^6 and 8 is an input to the operation to the natural log (base e) and the quotient rule applies. The first term of the expansion can be simplified since the natural log undoes base e: $6 - \ln 8$

ln(e⁶/8)	3.920558458
6-ln(8)	3.920558458

<u>**Topic #3: Properties of Logarithms – The Power Rule**</u>

The power rule for exponents states:

$$(b^M)^N = b^{MN}$$

This indicates that to raise a **power to a power** is to **multiply** the powers.

The Power Rule for logarithms is similar:

$$\log_b(M^N) = N \cdot \log_b M$$

This also indicates that raising a power becomes multiplication.

For example, we can rewrite a logarithmic expression with a power to become one with multiplication:

$$\ln x^2 = 2\ln x$$

Which gives the equivalent statement:

Example #1 – Expand the Expression

a) $\log_b x^4$

The base x to the power of 4 is an input to the operation "log base b" (which can be any suitable base); the power rule applies and the power of 4 becomes a multiplier/coefficient: $4 \cdot \log_b X$

b) $\ln\sqrt{z} \longrightarrow \ln z^{\frac{1}{2}} = \frac{1}{2}\ln(z)$

The fractional power of square root is "one half". The base z to the power of $1/2$ is an input to the operation of the natural log; the power rule applies and the power of $1/2$ becomes a multiplier/coefficient:

c) $\log(t^{-5}) = -5 \log t$

The base x to the power of -5 is an input to the operation of the common log; the power rule applies and the power of -5 becomes a multiplier/coefficient:

Topic #4: Properties of Logarithms – Using Multiple Rules

Some logarithmic expressions can be expanded by using multiple rules. Here are the three fundamental properties again:

Product Rule: $\log_b(MN) = \log_b M + \log_b N$

Quotient Rule: $\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$

Power Rule: $\log_b(M^N) = N \log_b M$

Example #1 – Expand the Expression and Simplify when Possible

a) $\log(x^2\sqrt{y}) = \log x^2 \cdot y^{\frac{1}{2}}$

Rewrite the radical as a fractional power; there is a product rule and two power rules:

$\log x^2 + \log y^{\frac{1}{2}}$

$$\boxed{\log(x^2\sqrt{y}) = 2\log x + \frac{1}{2}\log y}$$

b) $\log_6\left(\frac{\sqrt[3]{y}}{36x^3}\right) = \log_6 \frac{y^{\frac{1}{3}}}{36x^3}$

Rewrite the radical as a fractional power; there is a quotient rule, a product rule in the denominator, and two power rules:

$\log_6 y^{\frac{1}{3}} - \log_6 36x^3$

$\log_6 6^2$

$\log_6 y^{\frac{1}{3}} - (\log_6 36 + \log_6 x^3)$

$$\boxed{\frac{1}{3}\log_6 y - 2 - 3\log_6 x}$$

c) $\ln\left(\frac{x^2 z}{\sqrt{y+1}}\right)$

Rewrite the radical as a fractional power; there is a quotient rule, a product rule in the numerator, and two power rules:

$\ln x^2 \cdot z - \ln (y+1)^{\frac{1}{2}}$

$\ln x^2 + \ln z - \ln (y+1)^{\frac{1}{2}}$

$\boxed{2\ln x + \ln z - \frac{1}{2}\ln(y+1)}$

Topic #5: Properties of Logarithms in Reverse – Condensing a Logarithmic Expression

The rules established above are a two-way street. We can expand a logarithmic expression into two or more terms, but we can also condense two or more terms into one expression. Here are the rules (again) in reverse:

$$\log_b M + \log_b N = \log_b(M * N)$$

$$\log_b M - \log_b N = \log_b(M/N)$$

$$N \cdot \log_b M = \log_b M^N$$

Consider the expression

$$2\log x + \frac{1}{2}\log y$$

We have two terms; the coefficients become powers (the half power can be written as a square root) and addition becomes multiplication:

$$\log x^2 + \log \sqrt{y} + \log_3 x$$

$$\boxed{\log x^2\sqrt{y}}$$

Example #1 – Condense the Expression

a) $\frac{1}{3}\ln y + 2\ln z - 3\ln x$

Rewrite the coefficients as powers; positive terms make up the numerator and negative terms make up the denominator: $\ln y^{\frac{1}{3}}$

$$\ln \sqrt[3]{y} + \ln z^2 - \ln x^3$$

$$\ln\left(\frac{z^2\sqrt[3]{y}}{x^3}\right)$$

b) $6\ln x - \frac{1}{8}\ln z + \ln y$

Rewrite the coefficients as powers (the coefficient for the third term is 1, so we do not need to do anything); positive terms make up the numerator and negative terms make up the denominator:

$$\ln x^6 - \ln z^{\frac{1}{8}} + \ln y$$

$$\ln \frac{x^6}{\sqrt[8]{z}} + \ln y$$

$$\boxed{\ln \frac{x^6 y}{\sqrt[8]{z}}}$$

c) $\log 8 - \frac{1}{2}\log x - 3\log y$

Rewrite the coefficients as powers (the coefficient for the first term is 1, so we do not need to do anything); positive terms make up the numerator and negative terms make up the denominator:

$\log 8 - \log x^{\frac{1}{2}} - \log y^3$

$\log\left(\frac{8}{y^3\sqrt{x}}\right)$

Topic #6: Properties of Logarithms – Change of Base

Not all values of logarithmic expressions are rational numbers. For example $\log 15$ is not a rational number, it is somewhere between 1 and 2 since $\log 10 = 1$ and $\log 100 = 2$.

Scientific and graphing calculators are programmed to give values for common logs (base 10):

$\log_{10} 15 \approx 1.18$

Scientific and graphing calculators also evaluate natural logs (base e). Using a calculator, we can approximate values such as: $\ln 15 \approx 2.71$

Most calculators do not directly evaluate other bases. For example $\log_2 15$ is not a rational number, it is somewhere between 3 and 4 since $\log_2 8 = 3$ and $\log_2 16 = 4$. However, most calculators do not give this value directly.

The Change of Base Property is the way around this issue: $\log_b M = \dfrac{\log_a M}{\log_a b}$

The property tell us we can express the logarithm of base b in terms of another base a. The two most convenient and useful bases are base 10 and base e:

$$\log_b M = \frac{\log M}{\log b} = \frac{\ln M}{\ln b}$$

We can use either common logs or natural logs to evaluate other bases. For example,

$$\log_2 15 = \frac{\log 15}{\log 2} \approx 3.91 \qquad \log_2 15 = \frac{\ln 15}{\ln 2} \approx 3.91$$

We can also graph logarithmic functions other bases with this property. Suppose we want to graph $\boxed{f(x) = \log_3 x.}$ Using change of base (pick either a common or natural base): $f(x) = \log_3 x \rightarrow f(x) = \dfrac{\ln x}{\ln 3} = \dfrac{\log x}{\log 3}$