Math 120
4.4 Exponential and Logarithmic Equations

Objectives:

1. Use like bases to solve exponential equations.
2. Use logarithms to solve exponential equations.
3. Use the definition of a logarithm to solve logarithmic equations.
4. Use the one-to-one property of logarithms to solve logarithmic equations.
5. Solve applied problems involving exponential and logarithmic equations.

Topic #1: Exponential Equations

Exponential equations contain a variable in an exponent. There are different techniques to solve them.

Like Base Property

In some cases, exponential equations can be solved by using the **Like Base Property of Exponents**:

If $b^M = b^N$, then $M = N$

$3^x = 3^2$

$x = 2$

Consider the equation:

$$2^{3x-8} = 16$$

$2^4 = 16$

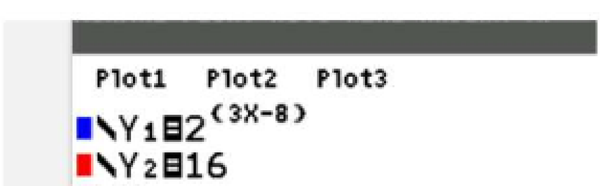
The right side is a base 2 number $(16 = 2^4)$ and can be rewritten as such. The bases cancel, leaving an equation we know how to solve:

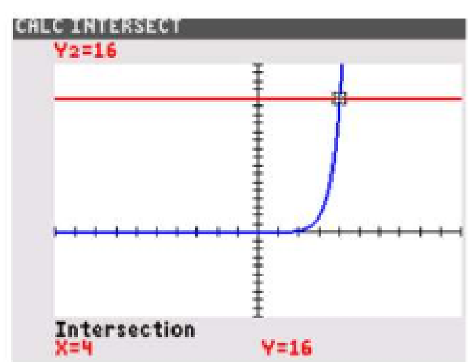
$$2^{3x-8} = 2^4$$

$$3x-8=4$$

$$\boxed{x=4}$$

Feel free to graph:





Example #1 – Solve the Exponential Equation

a) $4^{x+3} = 16$

The right side is a base 4 number $(16 = 4^2)$ and can be rewritten to apply the property:

$$4^{x+3} = 4^2$$

$$x+3 = 2$$

$$\boxed{x = -1}$$

b) $8^x = \sqrt{2}$

The right side is a base 2 number $(8 = 2^3)$, so is the left side $(\sqrt{2} = 2^{1/2})$ – rewrite both accordingly and apply the property:

$$8^x = 2^{\frac{1}{2}}$$

$$(2^3)^x = 2^{\frac{1}{2}}$$

$$2^{3x} = 2^{\frac{1}{2}}$$

$$3x = \frac{1}{2}$$

$$\boxed{x = \frac{1}{6}}$$

c) $16^x = 64$

The right side is a base 4 number $(16 = 4^2)$, so is the left side $(64 = 4^3)$ – rewrite both accordingly and apply the property:

$$(4^2)^x = 4^3$$

$$4^{2x} = 4^3$$

$$2x = 3$$

$$\boxed{x = \frac{3}{2}}$$

GRAPH to confirm the results!

Solving with logarithms

Consider the equation:

$$4^x = 15$$

The number 15 cannot be "nicely" rewritten as a base 4 number (the actual value is what we are solving for and it is not a rational number) and the like base property does not apply.

We can solve using the definition of logarithm:

$$4^x = 15 \qquad \boxed{x = \log_4 15}$$

$$b^y = x \longleftrightarrow y = \log_b x$$

Although this is correct, we can use the change of base formula for a decimal approximation:

$$x = \log_4 15 = \frac{\log 15}{\log 4} \approx 195$$

We could also use the power rule of logarithms by introducing a logarithm to both sides:

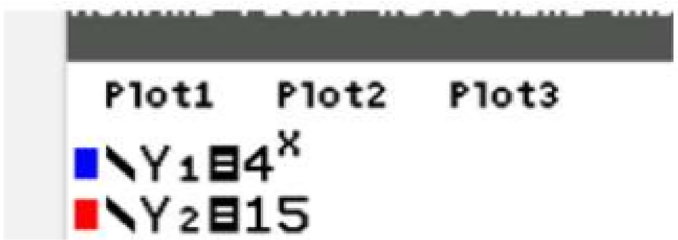
$$4^x = 15$$

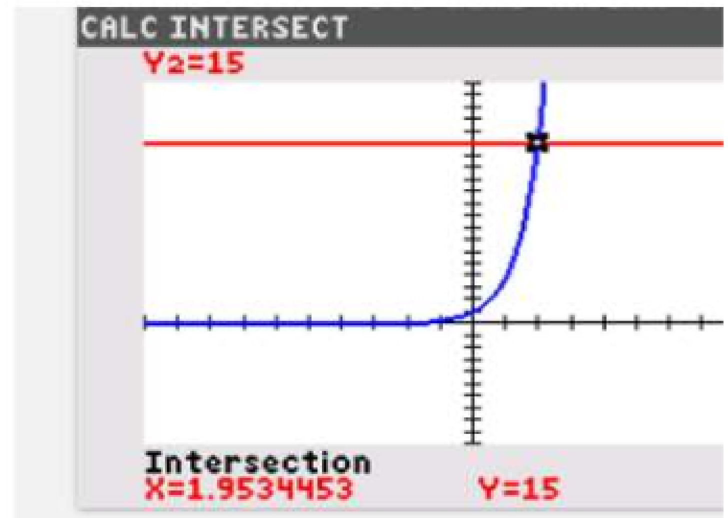
$$\log 4^x = \log 15$$

$$\frac{x \log 4}{\log 4} = \frac{\log 15}{\log 4}$$

$$x = \frac{\log 15}{\log 4} \approx 1.95$$

A graph confirms the decimal approximation:





The second approach is more common. We could have introduced a common log (base 10) instead and still have the correct answer. For consistency, the natural log will be used throughout.

Example #2 – Solve the Exponential Equation

a) $e^x = 12.75$

Introduce the natural log to both sides (this undoes base e on the left side):

$$e^x = 12.75$$
$$\ln_e e^x = \ln 12.75$$
$$x = \ln 12.75 \approx 255$$

b) $\frac{2e^{2x}}{2} = \frac{180}{2}$

Divide by 2 to Isolate the base, introduce the natural log to both sides:

$$e^{2x} = 90$$
$$\ln e^{2x} = \ln 90$$
$$\frac{2x}{2} = \frac{\ln 90}{2}$$
$$x = \frac{\ln 90}{2} \approx 2.25$$

c) $17^x = 47$

Introduce the natural log to both sides and solve:

$$\ln 17^x = \ln 47$$
$$x \ln(17) = \ln(47)$$
$$x = \frac{\ln 47}{\ln 17} \approx 136$$

Topic #2: Logarithmic Equations

Logarithmic equations contain a variable inside the logarithm. When solving these equations, we must check to see if the proposed solutions are in the domain (recall that logarithms only accept POSITIVE inputs). There are two basic techniques to solve them.

$x>0$

$\log_b x$ $\quad x \neq 1$

Definition of Logarithm

Consider the equation:

$$\log_4(x+3) = 2$$

>0

Before solving, we identify the domain:

$x+3>0$

$x>-3$

Now we can apply the definition of logarithm:

$y=\log_b x \longleftrightarrow x=b^y$

$2=\log_4(x+3)$

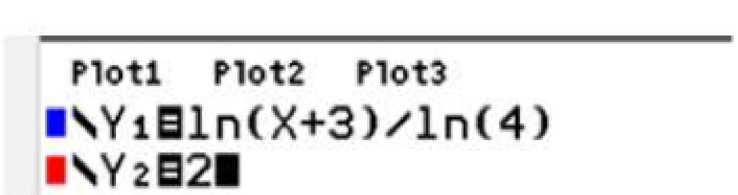
$x+3=4^2$

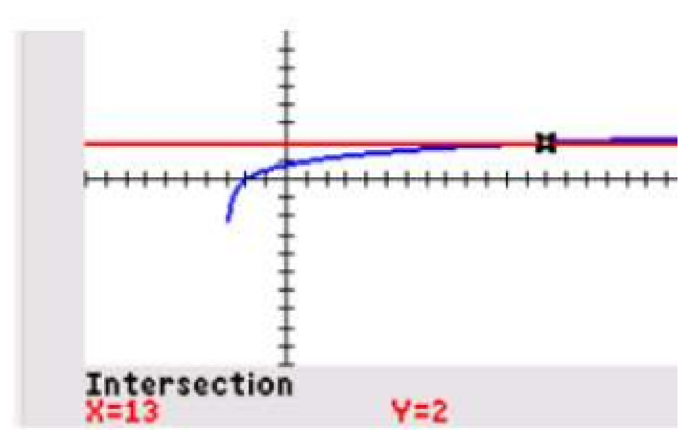
$x+3=16$

$\boxed{x=13}$

The proposed solution is in the domain and is accepted!

A graph confirms:





Example #1 – Solve the Logarithmic Equation

a) $\log_2(x + 19) = 8$

The domain is $x + 19 > 0$ $\boxed{x > -19}$

apply the definition of logarithm:

$y = \log_b x \longleftrightarrow x = b^y$

$8 = \log_2 (x+19)$ $\quad x + 19 = 2^8$

$\boxed{x = 237}$

The solution is in the domain and is accepted!

b) $\ln(6x) = 10$

The domain is $\frac{\cancel{6}x}{\cancel{6}} > \frac{0}{6}$ $\boxed{x > 0}$

apply the definition of logarithm (this is base e):

$y = \log_b x \longleftrightarrow x = b^y$

$10 = \ln_e 6x$ $\quad \frac{6x}{6} = \frac{e^{10}}{6}$

$x \approx 3671\ 1$

The solution is in the domain and is accepted!

c) $8 + 4\ln(x) = 6$

The domain is $\boxed{x > 0}$

apply the definition:

$y = \log_b x \longleftrightarrow x = b^y$

$-\frac{1}{2} = \ln x$ $\quad x = e^{-\frac{1}{2}}$

$\boxed{x \approx 0\ 61}$

$8 + 4\ln x = 6$
$-8 \qquad\qquad -8$

$\frac{4\ln x}{4} = \frac{-2}{4}$

$\ln x = -\frac{1}{2}$

Like Base Property

In some cases, logarithmic equations can be solved by using the **Like Base Property of Logarithms:**

Consider the equation:

$$\ln(x+2) + \ln(x) = \ln 8$$

Before solving, identify the domain. The first term indicates a domain _____________and the second term indicates a domain _____________ we pick the MORE restrictive of the two:

The property does not apply yet, the left side can be combined into a single logarithm by the product rule:

The bases now cancel, leaving the quadratic equation:

We cannot accept the solution ____________it is out of the domain! The only solution is ________________

__Example #2__ – Solve the Logarithmic Equation

a) $3\log x = \log 125$

The domain is ____________________The property does not apply yet; apply the power rule to the left side:

The solution is in the domain and is accepted!

b) $2\log_6 x - \log_6 5 = \log_6 405$

The domain is _____________________The property does not apply yet; apply the power and quotient rule to the left side:

We cannot accept the solution __________________it is out of the domain! The only solution is ________________

c) $\ln(8x-7)=\ln(x+3)+\ln 9$

The more restrictive domain is ____________________The property does not apply yet; apply the product rule to the right side:

We cannot accept the proposed solution, it is out of the domain! There is _________________________

Topic #3: Application of Logarithmic and Exponential Equations

Both types of functions are used to model real life situations

Example #1 – Exponential Model

The population of a small country over time is modeled by the function

$$y = 18.1e^{0.0133t}$$

where y is the population in millions and t is the number of years after 2010.

a) What was the population in 2010?

b) When will the population reach 24.3 million? Round to the nearest year.

c) When will the population double? Round to the nearest year.

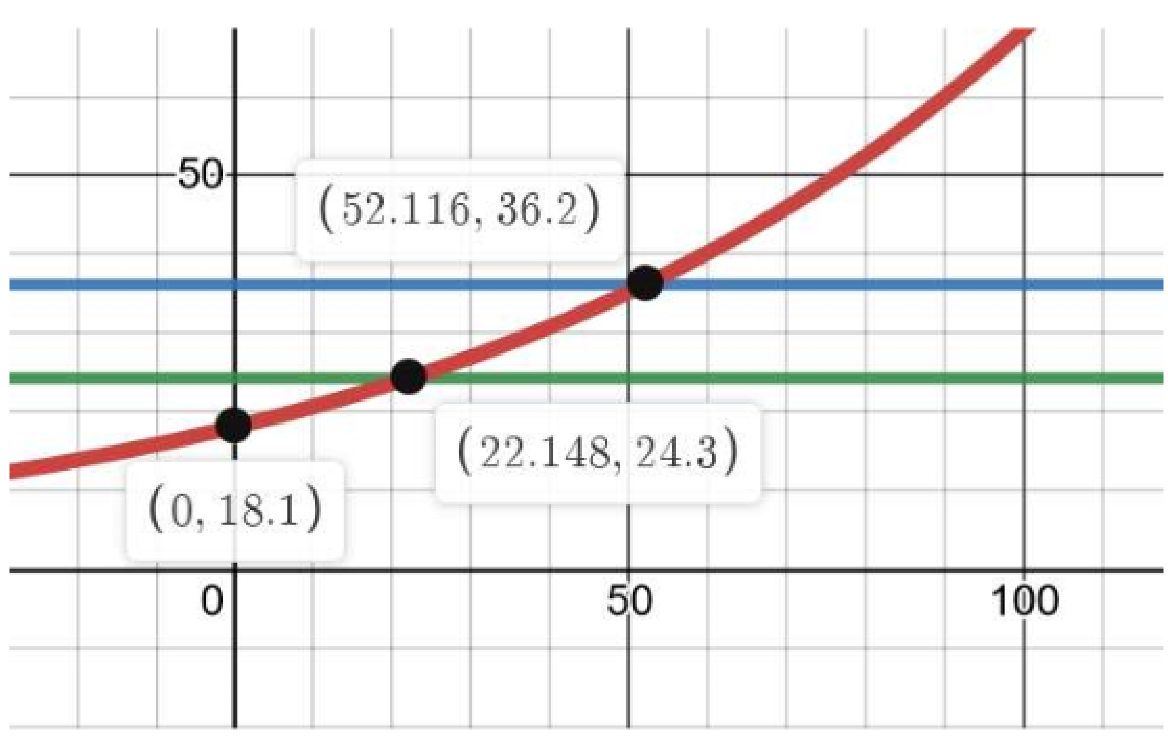
The initial population is $y_o = 18.1$, double the initial population gives $y = 2 * 18.1 \rightarrow y = 36.2$ and gives the equation:

When we isolate the base, we get the equation:

The model tells us the population will double **in**

____

We can also graph to solve either of the equations above:



<u>Example #1</u> – Logarithmic Model

The percentage of students in a class who can recall important features of a classroom lecture over time is modeled by the function

$$y = 95 - 30\log_2 x\,;\text{for } 1 \le x \le 9$$

where y is the percentage of students and x is the number of days after the lecture.

a) What percentage of the students will recall important features 1 day after a lecture?

b) After how many days will only half the students recall important information?

Half of the class is $y = 50\%$, giving the equation:

We can also graph to see the point of intersection that solves the equation: