Math 120
4.5 Exponential Growth and Decay

Objectives:

1. Model exponential growth and decay.
2. Use logistic growth models.
3. Choose an appropriate model for data.
4. Express an exponential model in base *e*.

Topic #1: Exponential Growth Models

Recall the formula for continuous interest:

$$A = Pe^{rT}$$

Where A is the future value, P is the initial value, e is the natural base, r is the relative growth rate, and t is time.

Although it is unlikely that money in an account will grow per continuous interest, many quantities that grow over time are modeled by continuous growth.

Here is the general formula for Exponential Growth:

$$A(t) = A_0 e^{kT} \qquad k > 0$$

Where $A(t)$ is the future value, A_0 is the initial value, e is the natural base, k is the relative **growth** rate, and t is time. Notice it is the SAME formula, just with some differences in the notation.

Example #1 – Evaluate an Exponential Growth Model

The population of Canada is modeled with the exponential growth model

$A(t) = 33.1e^{0.009t}$

Where A is the population in millions and t is the number of years after 2006.

$A(t)$

A = pop in millions

t = years after 2006

a) What is the relative growth rate of the population per year?

$k = 0.009 = 0.9\%$

b) What was the population in 2006?

$t=0$

$A(0) = 33.1e^{.009(0)}$

$= 33.1$ million people

c) Based on the model, what is the projected population in 2029?

$t=23$

$A(23) = 33.1e^{.009(23)}$

$A(23) \approx 40.7$

In 2029, the population will be 40.7 million.

d) When is the population expected to reach 70 million?

This gives $A(t) = 70$ plug in and solve the equation: $70 = 33.1e^{.009T}$

We can solve by isolating the base and introducing the natural log or by graphing.

We can solve the general equation for t using the same techniques discussed in the last section:

$$\frac{70}{33.1} = \frac{33.1e^{.009T}}{33.1}$$

$$\frac{70}{33.1} = e^{.009T}$$

$$\ln\left(\frac{70}{33.1}\right) = \cancel{\ln} e^{.009T}$$

$$\frac{\ln\left(\frac{70}{33.1}\right)}{.009} = \frac{.009 \cdot T}{.009}$$

$$\frac{\ln\left(\frac{70}{33.1}\right)}{.009} = T$$

2089

$83 \text{ years} = T$

Example #2 – Build and Evaluate an Exponential Growth Model

The population of Columbia in 2010 was 44.2 million; the projected population in 2050 is expected to be 62.9 million. Assume the projection will hold true and that the population will grow based on an exponential model.

$A(t) =$ pop in Millions

a) Find the exponential growth model that describes the population t years after 2010.

$A(t) = A_0 e^{KT}$

$A(t) = 44.2e^{KT}$

$\frac{62.9}{442} = \frac{44.2e^{k \cdot 40}}{44.2}$

We need to solve for k. Feel free to graph or isolate the base and introduce logarithms. We can solve the general equation for k using the same techniques discussed in the last section:

$\frac{629}{44.2} = e^{40k}$

$\ln\left(\frac{62.9}{442}\right) = \ln e^{40k}$

$\frac{\ln\left(\frac{62.9}{44.2}\right)}{40} = \frac{40k}{40}$

$K \approx .0088$

This gives the exponential growth model:

$A(t) = 442e^{0.0088T}$

b) Based on the model what is the projected population in 2035?

$A(25) \approx 551$

c) When is the population expected to reach 65 million?

Feel free to set up the equation OR use the "shortcut"

$65 = 44.2e^{.0088T}$

2054

$T = \frac{\ln\left(\frac{65}{44.2}\right)}{.0088} \approx 44$

d) When is the population expected to double the size of the population in 2010? 44.2

Double the initial value is 88.4

feel free to plug into the model and solve. However, when we divide a number twice as big as the initial value, it is always the number 2. This gives the formula for doubling time:

$\frac{88.4}{44.2} = \frac{44.2e^{.0088T}}{44.2}$

$2 = e^{.0088T}$

$\ln_e 2 = \ln_e e^{.0088T}$

$\frac{\ln 2}{.0088} = \frac{.0088T}{.0088}$

≈ 79 years

2089

Topic #2: Exponential Decay Models

Some quantities grow exponentially/continuously over time; others decay.

Here is the general formula for Exponential $\boxed{\text{Decay:}}$

$$A(t) = A_0 e^{kt} \qquad k < 0$$

Where $A(t)$ is the future value, A_0 is the initial value, e is the natural base, k is the relative **decay** rate, and t is time.

Notice it is ~~the~~ ALMOST the SAME formula for growth, but the rate is Negative

<u>Example #1</u> – Build and Evaluate an Exponential Decay Model

The population of Japan in 2010 was 127.3 million; the projected population in 2050 is expected to be 100.1 million.

a) Find the exponential decay model that describes the population t years after 2010.

We need to solve for k. Feel free to graph or isolate the base and introduce logarithms. We can solve this general equation for k using the same techniques discussed in the last section:

b) Based on the model what is the projected population in 2035?

c) When is the population expected to be half the size of the population in 2010?
Half the initial value is ____________________________
feel free to plug into the model and solve. However, when we divide a number half as big as the initial value, it is always the number $1/2$. This gives the formula for half-time:

Exponential Decay and Carbon-14 Dating

The amount of Carbon-14 in an artifact or a fossil decays exponentially over time. The age of an artifact or a fossil can be determined using the formula:

$$A(t) = A_0 e^{-0.000121t}$$

Where $A(t)$ is the amount of Carbon-14 present (in grams), A_0 is the original amount of Carbon-14 (in grams), e is the natural base, the rate of decay is about -0.000121 grams per year, and t is time in years.

<u>Example #2</u> – Exponential Decay Model and Carbon Dating

a) An artifact originally has 16 grams of Carbon-14. How many grams will be present in 800 years?

Use the model for Carbon-14, where $A_0 =$ 16g and $t =$ 800

$A(800) = 16g\, e^{-.000121(800)}$

$\approx 14.5g$

b) An artifact originally has 80 grams of Carbon-14. When will the artifact have 20 grams of Carbon-14?

Use the model for Carbon-14, where $A_0 =$ 80g and $A(t) =$ 20g

$20g = 80g\, e^{-.000121T}$

Solve for t with any desired technique:

$T \approx 11{,}457$ years

c) A fossil is found during an excavation project. It is determined that the fossil contains 15% of its original Carbon-14. How old is the fossil?

We are asked to solve for time, but we do not have a specific original and new amount of Carbon-14. All we know is the new amount is 15% of the original. Feel free to pick some original value, for example if $A_0 = 100$ then $A(t) =$ $(.15)(100) = 15$ Plug these values into the model and solve for t.

However; if the new quantity is 15% of the original quantity, when we divide the new quantity by the original it is always 15% OR 0.15, regardless of the original amount.

$$A(t) = A_0 e^{-.000121T}$$

$$15 = 100 \cdot e^{-.000121T}$$

$$.15 = e^{-.000121T}$$

$$\ln(.15) = \ln e^{-.000121T}$$

$$\frac{\ln(.15)}{-.000121} = \frac{-.000121T}{-.000121}$$

$$15{,}679 \text{ years} \approx T$$

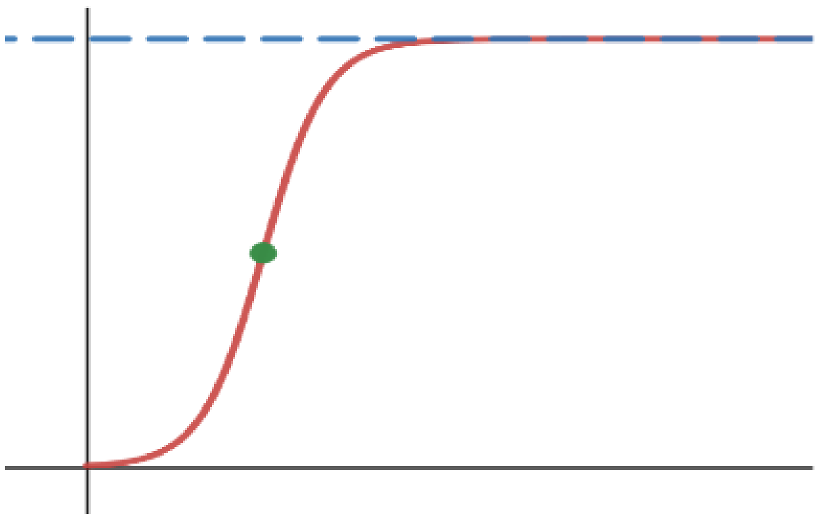
Topic #2: Logistical Growth Models

Exponential growth models have limitations when time increases. The models increase without bound, but nothing in nature grows exponentially indefinitely.

Growth is limited, to model such behavior consider the Logistical Growth Model:

$$f(t) = \frac{c}{1+ae^{-bt}}$$

Where $f(t)$ is the new quantity, a, b, c are positive constants, e is the natural base, and t is time.



The graph creates an "S" curve. The quantity grows quickly to start, but slows down and flattens out to a horizontal asymptote. The equation of the asymptote is $y = c$ and is the limiting capacity of growth.

Suppose that a disease is spreading though a population and the number of people infected is modeled by the logistical model:

$$f(t) = \frac{30000}{1 + 20e^{-1.5t}}$$

where $f(t)$ are the number of people infected t weeks after the initial outbreak.

$t=0$

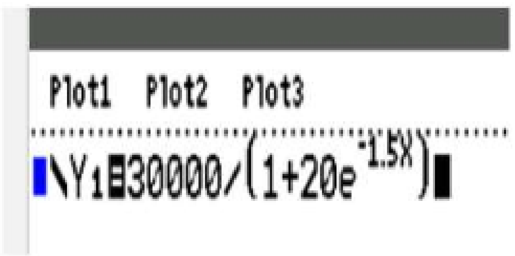
According to the model, the initial number of people infected at the onset of the epidemic is:

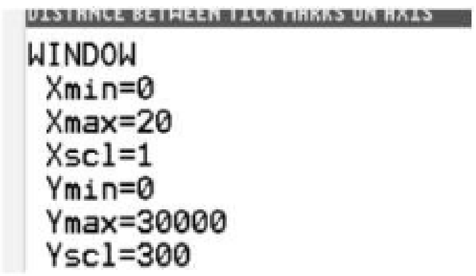
$$f_i(0) = \frac{30000}{1+20e^{-1.5(0)}} = \frac{30000}{1+20} = \frac{30000}{21}$$

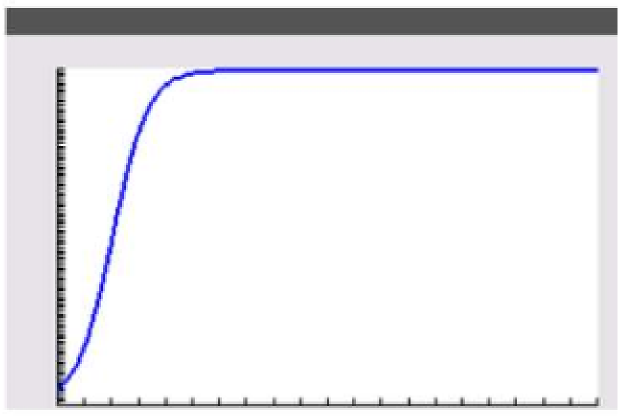
≈ 1429 infected

The disease does not spread without bound, eventually the number of people infected flattens out to $y = 30000$ OR $30{,}000$ people infected.

Here is a graph of the model:







Example #1 – Evaluate a Logistical Growth Model

The world population over time is modeled by the logistical function

$$f(t) = \frac{12.57}{1 + 4.11e^{-0.026t}}$$

where $f(t)$ is the world population in billions and t is the number of years after 1949.

a) According to the model, what was the world population in 1949? $t=0$

$$f(0) = \frac{12.57}{1+4.11e^{-.026(0)}} = \frac{12.57}{1+4.11} \approx 2.5 \text{ billion}$$

b) According to the model, what was the world population in 2010? $t=61$

$$f(61) = \frac{12.57}{1+4.11e^{-.026(61)}} \approx 6.8 \text{ billion}$$

c) Based on actual census data, the population in 2010 was 6.9 billion. How well does the model work with the estimation above?

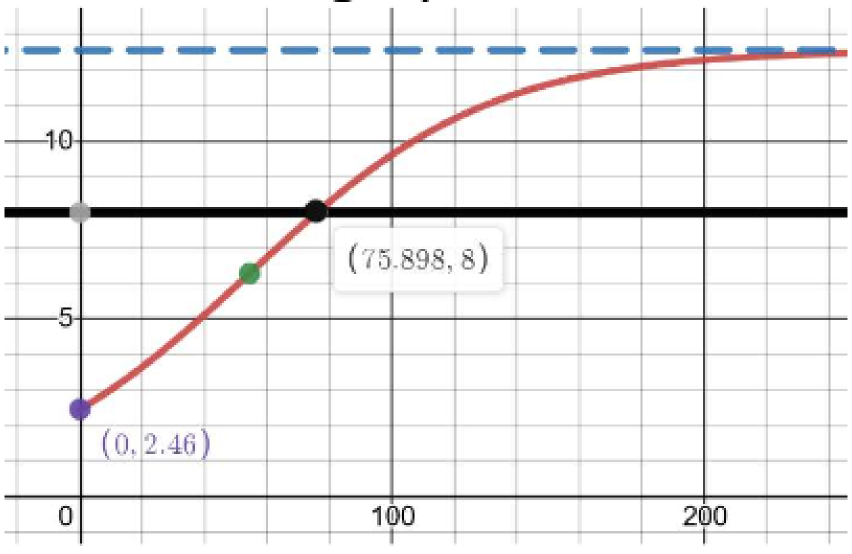
d) When will the population reach 8 billion?

This gives $f(t) = 8$ and the equation:

$$\underbrace{\frac{12.57}{1+4.11e^{-.026t}}}_{Y_1} = \underbrace{8}_{Y_2}$$

We can cross-multiply, isolate the base, and introduce a natural log.

Here is the graphical solution:



Based on the graph, the population will reach 8 billion in about 76 years after 1949 which is in the year 2025.

e) According to the model, what is the limiting size of the world population?