2021 AIME I Problem & Solution

2021 AIME I Problem 1

Zou and Chou are practicing their 100-meter sprints by running 6 races against each other. Zou wins the first race, and after that, the probability that one of them wins a race is $\frac{2}{3}$ if they won the previous race but only $\frac{1}{3}$ if they lost the previous race. The probability that Zou will win exactly 5 of the 6 races is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

小邹和小周两人进行了6场100米短跑比赛．小邹在第一场比赛中获胜, 此后的每场比赛, 在前一场比赛中获胜的人获胜的概率为$\frac{2}{3}$, 但在前一场比赛中失利的人获胜概率只有$\frac{1}{3}$．小邹在6场比赛中恰好赢了5场的概率是$\frac{m}{n}$, 其中m和n是互质的正整数．求$m + n$.

Solution 1 (Casework)

For the next five races, Zou wins four and loses one. There are five possible outcome sequences, and we will proceed by casework:

Case (1): Zou does not lose the last race.

The probability that Zou loses a race is $\frac{1}{3}$, and the probability that he wins the following race is $\frac{1}{3}$. For each of the three other races, the probability that he wins is $\frac{2}{3}$.

There are four such outcome sequences. The probability of one such sequence is $\left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^3$.

Case (2): Zou loses the last race.

The probability that Zou loses a race is $\frac{1}{3}$. For each of the four other races, the probability that he wins is $\frac{2}{3}$.

There is one such outcome sequence. The probability is $\left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^4$.

Answer

The requested probability is $4 \times \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^3 + \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^4 = \frac{32}{243} + \frac{16}{243} = \frac{48}{243} = \frac{16}{81}$, and the answer is $16 + 81 = \boxed{097}$.

2021 AIME I Problem 2

In the diagram below, $ABCD$ is a rectangle with side lengths $AB = 3$ and $BC = 11$, and $AECF$ is a rectangle with side lengths $AF = 7$ and $FC = 9$, as shown. The area of the shaded region common to the interiors of both rectangles is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

在下图中，$ABCD$是边长为$AB = 3$和$BC = 11$的矩形，$AECF$是边长为$AF = 7$和$FC = 9$的矩形．两个矩形内部的公共区域如阴影部分所示，面积为$\frac{m}{n}$，其中m和n为互质的正整数．求$m + n$.
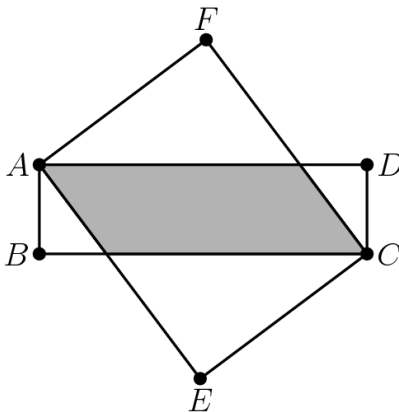
Solution 1 (Similar Triangles) Let G be the intersection of AD and FC. From vertical angles, we know that $\angle FGA = \angle DGC$. Also, given that $ABCD$ and $AFCE$ are

rectangles, we know that $\angle AFG = \angle CDG = 90°$. Therefore, by AA similarity, we know that triangles AFG and CDG are similar.

Let $AG = x$. Then, we have $DG = 11 - x$. By similar triangles, we know that $FG = \frac{7}{3}(11 - x)$ and $CG = \frac{3}{7}x$. We have $\frac{7}{3}(11 - x) + \frac{3}{7}x = FC = 9$.

Solving for x, we have $x = \frac{35}{4}$. The area of the shaded region is just $3 \cdot \frac{35}{4} = \frac{105}{4}$.

Thus, the answer is $105 + 4 = \boxed{109}$.

2021 AIME I Problem 3

Find the number of positive integers less than 1000 that can be expressed as the difference of two integral powers of 2.

求可以表示成为两个2的整数方幂之差的小于1000的正整数的个数.

Solution 1

We want to find the number of positive integers $n < 1000$ which can be written in the form $n = 2^a - 2^b$ for some non-negative integers $a > b \geq 0$ (note that if $a = b$, then $2^a - 2^b = 0$). We first observe a must be at most 10; if $a \geq 11$, then $2^a - 2^b \geq 2^{10} > 1000$. As $2^{10} = 1024 \approx 1000$, we can first choose two different numbers $a > b$ from the set $\{0, 1, 2, \ldots, 10\}$ in $\binom{11}{2} = 55$ ways. This includes $(a, b) = (10, 0)$, $(10, 1)$, $(10, 2)$, $(10, 3)$, $(10, 4)$ which are invalid as $2^a - 2^b > 1000$ in this case. For all other choices a and b, the value of $2^a - 2^b$ is less than 1000.

We claim that for all other choices of a and b, the values of $2^a - 2^b$ are pairwise distinct. More specifically, if $(a_1, b_1) \neq (a_2, b_2)$ where $10 \geq a_1 > b_1 \geq 0$ and $10 \geq a_2 > b_2 \geq 0$, we must show that $2^{a_1} - 2^{b_1} \neq 2^{a_2} - 2^{b_2}$. Suppose otherwise for sake of contradiction; rearranging yields $2^{a_1} + 2^{b_2} = 2^{a_2} + 2^{b_1}$. We use the fact that every positive integer has a unique binary representation:

If $a_1 \neq b_2$ then $\{a_1, b_2\} = \{a_2, b_1\}$; from here we can deduce either $a_1 = a_2$ and $b_1 = b_2$ (contradicting the assumption that $(a_1, b_1) \neq (a_2, b_2)$, or $a_1 = b_1$ and $a_2 = b_2$ (contradicting the assumption $a_1 > b_1$ and $a_2 > b_2$).

If $a_1 = b_2$ then $2^{a_1} + 2^{b_2} = 2 \times 2^{a_1}$, and it follows that $a_1 = a_2 = b_1 = b_2$, also contradicting the assumption $(a_1, b_1) \neq (a_2, b_2)$. Hence we obtain contradiction.*

Then there are $\binom{11}{2} - 5$ choices for (a, b) for which $2^a - 2^b$ is a positive integer less than 1000; by the above claim, each choice of (a, b) results in a different positive integer n. Then there are $55 - 5 = \boxed{050}$ integers which can be expressed as a difference of two powers of 2.

Note: The uniqueness of binary representation could be rather easily proven, but if you cannot convince yourself on the spot that this is the case, consider the following alternative proof. Let $(a_1, b_1) \neq (a_2, b_2)$ where $10 \geq a_1 > b_1 \geq 0$ and $10 \geq a_2 > b_2 \geq 0$ and $2^{a_1} - 2^{b_1} = 2^{a_2} - 2^{b_2}$, for the sake of contradiction. Therefore $\deg_2(2^{a_1} - 2^{b_1}) = \deg_2(2^{a_2} - 2^{b_2})$, or $b_1 = b_2$. Plugging in, we see that $2^{a_1} = 2^{a_2}$, or $a_1 = a_2$, contradiction. Note by Ross Gao

Solution 2 (Casework)

Case 1: When our answer is in the form $2^n - 2^i$, where i is an integer such that $0 \leq i \leq 4$.

We start with the subcase where it is $2^n - 2^0$, for some integer n where $n > 0$ (this is because the case where $n = 0$ yields $2^0 - 2^0 = 0$, which doesn't work because it must be a positive integer.) Note that $2^{10} = 1024$, and $2^9 = 512$. Our answer needs to be less than 1000, so the maximum possible result (in this case) is $2^9 - 2^0$. Our lowest result is $2^1 - 2^0$. All the positive powers of two less than 1024 work, so we have 9 possibilities for this subcase. For subcases $i = 1, i = 2, i = 3$, and $i = 4$, we have 8,7,6, and 5 possibilities, respectively.

Case 2: When our answer is in the form of $2^n - 2^j$, where j is an integer such that $5 \leq j \leq 9$.

We can start with the subcase where $j = 5$. We notice that $2^5 = 32$, and $2^{10} - 2^5 = 992$ which is less than 1000, so the greatest result in this subcase is actually $2^{10} - 2^5$, and the lowest is $2^6 - 2^5$. Thus, we have 5 possibilities. For the other four subcases, we have 4,3,2, and 1 possibilities, respectively.

Answer: We note that these are our only cases, as numbers in the form of $2^n - 2^{10}$ and beyond are greater than 1000.

Thus, our result is $9 + 8 + 7 + 6 + 5 + 5 + 4 + 3 + 2 + 1 = (9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1) + 5 = \boxed{50}$. ~jehu26

Solution 3

First, you need to notice that it is impossible to have overlapping, making the problem easier.

Case 1 : 2^0 There are 9 ways here, from 2^1 to 2^9

It is easy to see here that this continues all the way down to one. However, when the case gets to 32, there are 5 ways instead of 4 because $2^{10} - 2^5$ is smaller than 1000.

Thus, $9 + 8 + 7 + 6 + 5 + 5 + 4 + 3 + 2 + 1 = 50$, So the answer is $\boxed{050}$

2021 AIME I Problem 4

Find the number of ways 66 identical coins can be separated into three nonempty piles so that there are fewer coins in the first pile than in the second pile and fewer coins in the second pile than in the third pile.

将66枚相同的硬币分成三堆, 每堆都有硬币, 并且第一堆的硬币比第二堆的少, 第二堆的硬币比第三堆的少. 问这样的分法共有多少种?

Solution 1

Suppose we have 1 coin in the first pile. Then $(1,2,63), (1,3,62), \ldots, (1,32,33)$ all work for a total of 31 piles. Suppose we have 2 coins in the first pile, then $(2,3,61), (2,4,60), \ldots, (2,31,33)$ all work, for a total of 29. Continuing this pattern

until 21 coins in the first pile, we have the sum $31 + 29 + 28 + 26 + 25 + \cdots + 4 + 2 + 1 = (31 + 28 + 25 + 22 + \cdots + 1) + (29 + 26 + 23 + \cdots + 2) = 176 + 155 = \boxed{331}$.

(Minor edit to make everything fit in the page made by KingRavi)

Solution 2

Let the three piles have a, b, c coins respectively. If we disregard order, then we just need to divide by $3! = 6$ at the end.

We know $a + b + c = 66$. Since a, b, c are positive integers, there are $\binom{65}{2}$ ways from Stars and Bars.

However, we must discard the cases where $a = b$ or $a = c$ or $b = c$. The three cases are symmetric, so we just take the first case and multiply by 3. We have $2a + c = 66 \implies a = 1, 2, \ldots 32$ for 32 solutions. Multiplying by 3, we will subtract 96 from our total.

But we undercounted where $a = b = c = 22$. This is first counted 1 time, then we subtract it 3 times, so we add it back twice. There is clearly only 1 way, for a total of 2.

Hence, the answer is $\dfrac{\binom{65}{2} - 96 + 2}{6} = \boxed{331}$.

Solution 3

Let the piles have a, b and c coins, with $0 < a < b < c$. Then, let $b = a + k_1$, and $c = b + k_2$, such that each $k_i \geq 1$. The sum is then $a + a + k_1 + a + k_1 + k_2 = 66 \implies 3a + 2k_1 + k_2 = 66$. This is simply the number of positive solutions to the equation $3x + 2y + z = 66$. Now, we take cases on a.

If $a = 1$, then $2k_1 + k_2 = 63 \implies 1 \leq k_1 \leq 31$. Each value of k_1 corresponds to a unique value of k_2, so there are 31 solutions in this case. Similarly, if $a = 2$, then $2k_1 + k_2 = 60 \implies 1 \leq k_1 \leq 29$, for a total of 29 solutions in this case. If $a = 3$, then $2k_1 + k_1 = 57 \implies 1 \leq k_1 \leq 28$, for a total of 28 solutions. In general, the number of

solutions is just all the numbers that aren't a multiple of 3, that are less than or equal to 31.

We then add our cases to get $1 + 2 + 4 + 5 + \cdots + 31 = 1 + 2 + 3 + \cdots + 31 - 3(1 + 2 + 3 + \cdots + 10) = \frac{31(32)}{2} - 3(55) = 31 \cdot 16 - 165 = 496 - 165 = \boxed{331}$ as our answer.

2021 AIME I Problem 5

Call a three-term strictly increasing arithmetic sequence of integers special if the sum of the squares of the three terms equals the product of the middle term and the square of the common difference. Find the sum of the third terms of all special sequences.

对于一个由三个整数组成的严格递增的等差数列, 如果三项的平方和等于中间项与公差平方的乘积, 那么它称为特殊的. 求所有特殊数列的第三项之和.

Solution 1

Let the terms be $a - b, a$, and $a + b$. Then we want $(a - b)^2 + a^2 + (a + b)^2 = ab^2$, or $3a^2 + 2b^2 = ab^2$. Rearranging, we get $b^2 = \frac{3a^2}{a-2}$. Simplifying further, $b^2 = 3a + 6 + \frac{12}{a-2}$. Looking at this second equation, since the right side must be an integer, $a - 2$ must equal $\pm 1,2,3,4,6,12$. Looking at the first equation, we see $a > 2$ since b^2 is positive. This means we must test $a = 3,4,5,6,8,14$. After testing these, we see that only $a = 5$ and $a = 14$ work which give $b = 5$ and $b = 7$ respectively. Thus the answer is $10 + 21 = \boxed{031}$. ~JHawk0224

Solution 2

Let the common difference be d and let the middle term be x. Then, we have that the sequence is $x - d, x, x + d$. This means that the sum of the squares of the 3 terms of the sequence is $(x - d)^2 + x^2 + (x + d)^2 = x^2 - 2xd + d^2 + x^2 + x^2 + 2xd + d^2 = 3x^2 + 2d^2$. We know that this must be equal to xd^2, so we can write that $3x^2 + 2d^2 = xd^2$, and it follows that $3x^2 - xd^2 + 2d^2 = 3x^2 - (d^2)x + 2d^2 = 0$. Now, we can

treat d as a constant and use the quadratic formula to get $x = \frac{d^2 \pm \sqrt{d^4 - 4(3)(2d^2)}}{6}$. We can factor pull d^2 out of the square root to get $x = \frac{d^2 \pm d\sqrt{d^2 - 24}}{6}$. Here, it is easy to test values of d. We find that $d = 5$ and $d = 7$ are the only positive integer values of d that make $\sqrt{d^2 - 24}$ a positive integer. (To prove this, let $\sqrt{d^2 - 24} = k$, then $d^2 - k^2 = 24$ which is $(d + k)(d - k) = 24$, then remembering that d and k are integers see if you can figure it out. -PureSwag) $d = 5$ gives $x = 5$ and $x = \frac{10}{3}$, but we can ignore the latter. $d = 7$ gives $x = 14$, as well as a fraction which we can ignore.

Since $d = 5$, $x = 5$ and $d = 7, x = 14$ are the only two solutions and we want the sum of the third terms, our answer is $(5 + 5) + (7 + 14) = 10 + 21 = \boxed{031}$. - BorealBear, minor edit by Kinglogic

Solution 3

Proceed as in solution 2, until we reach $3x^2 + 2d^2 = xd^2$, Write

$d^2 = \frac{3x^2}{x-2}$, it follows that $x - 2 = 3k^2$ for some (positive) integer k and $k \mid x$.

Taking both sides modulo k, $-2 \equiv 0 \mod k$, so $k \mid 2 \rightarrow k = 1, 2$.

When $k = 1$, we have $x = 5$ and $d = 5$. When $k = 2$, we have $x = 14$ and $d = 7$.

Summing the two cases, we have $10 + 21 = \boxed{031}$. ~Ross Gao

Solution 4 (Combining Solution 1 and Solution 3)

As in Solution 1, write the three integers in the sequence as $a - d$, a, and $a + d$.

Then the sum of the squares of the three integers is $(a - d)^2 + a^2 + (a + d)^2 = 3a^2 + 2d^2$.

Setting this equal to the middle term times the common difference squared, which is ad^2,

and solving for d^2 we get:

$$3a^2 + 2d^2 = ad^2 \implies ad^2 - 2d^2 = 3a^2 \implies d^2(a - 2) = 3a^2 \implies d^2 = \frac{3a^2}{a - 2}$$

The numerator has to be positive, so the denominator has to be positive too for the sequence

to be strictly increasing; that is, $a > 2$.

For $\frac{3a^2}{a-2}$ to be a perfect square, $\frac{3}{a-2}$ must be a perfect square as well.

This means that $a - 2$ is divisible by 3, and whatever left over is a perfect square.

We can express this as an equation: let the perfect square left over be n^2. Then:

$3n^2 = a - 2$. Now when you divide the numerator and denominator by 3, you are left with

$d^2 = \frac{a^2}{n^2} \implies d = \frac{a}{n}$. Because the sequence is of integers, d must also be an

integer, which means that n must divide a.

Taking the above equation we can solve for a: $3n^2 = a - 2 \implies a = 3n^2 + 2$.

This means that $3n^2 + 2$ is divisible by n. $3n^2$ is automatically divisible by n, so

2 must be divisible by n. Then n must be either of $\{1,2\}$. Plugging back into the equation,

$n = 1 \implies a = 5 \implies d = 5$, so $a + d = 5 + 5 = 10$.

$n = 2 \implies a = 14 \implies d = 7$, so $a + d = 14 + 7 = 21$.

Finally, $10 + 21 = \boxed{031}$

2021 AIME I Problem 6

Segments $\overline{AB}, \overline{AC}$, and $\overline{AD}$ are edges of a cube and $\overline{AG}$ is a diagonal through the center of the cube. Point P satisfies $PB = 60\sqrt{10}, PC = 60\sqrt{5}, PD = 120\sqrt{2}$, and $PG = 36\sqrt{7}$. What is PA?

线段AB, AC和AD是一个立方体的边, 而线段AG是通过立方体中心的对角线. 点P满足$BP = 60\sqrt{10}, CP = 60\sqrt{5}, DP = 120\sqrt{2}$, 并且$GP = 36\sqrt{7}$. 求$AP$的长度.

Solution 1

First scale down the whole cube by 12. Let point P have coordinates (x, y, z), A have coordinates $(0,0,0)$, and s be the side length. Then we have the equations$(s - x)^2 + y^2 + z^2 = (5\sqrt{10})^2, x^2 + (s - y)^2 + z^2 = (5\sqrt{5})^2, x^2 + y^2 + (s - z)^2 = (10\sqrt{2})^2, (s - x)^2 + (s - y)^2 + (s - z)^2 = (3\sqrt{7})^2$. These simplify into $s^2 + x^2 + y^2 + z^2 - 2sx = 250, s^2 + x^2 + y^2 + z^2 - 2sy = 125, s^2 + x^2 + y^2 + z^2 - 2sz = 200, 3s^2 - 2s(x + y + z) + x^2 + y^2 + z^2 = 63$.

Adding the first three equations together, we get $3s^2 - 2s(x + y + z) + 3(x^2 + y^2 + z^2) = 575$. Subtracting this from the fourth equation, we get $2(x^2 + y^2 + z^2) = 512$, so $x^2 + y^2 + z^2 = 256$. This means $PA = 16$. However, we scaled down everything by 12 so our answer is $16 * 12 = \boxed{192}$. ~JHawk0224

Solution 2 (Solution 1 with slight simplification)

Once the equations for the distance between point P and the vertices of the cube have been written. We can add the first, second, and third to receive, $2(x^2 + y^2 + z^2) + (s - x)^2 + (s - y)^2 + (s - z)^2 = 250 + 125 + 200$. Subtracting the fourth equation gives,$2(x^2 + y^2 + z^2) = 575 - 63x^2 + y^2 + z^2 = 256\sqrt{x^2 + y^2 + z^2} = 16$. Since point $A = (0,0,0), PA = 16$, and since we scaled the answer is $16 \cdot 12 = \boxed{192}$ ~Aaryabhatta1

Solution 3

Let E be the vertex of the cube such that ABED is a square. By the British Flag Theorem, we can easily we can show that$PA^2 + PE^2 = PB^2 + PD^2andPA^2 + PG^2 = PC^2 + PE^2$Hence, adding the two equations together, we get $2PA^2 + PG^2 = PB^2 + PC^2 + PD^2$. Substituting in the values we know, we get $2PA^2 + 7 \cdot 36^2 = 10 \cdot 60^2 + 5 \cdot 60^2 + 2 \cdot 120^2$.

Thus, we can solve for PA, which ends up being $\boxed{192}$.

2021 AIME I Problem 7

Find the number of pairs (m, n) of positive integers with $1 \le m < n \le 30$ such that there exists a real number x satisfying $\sin(mx) + \sin(nx) = 2$.

求具有如下性质的正整数对(m, n)的数目: $1 \le m < n \le 30$, 并且存在实数x满足 $\sin(mx) + \sin(mx) = 2$.

Solution 1

The maximum value of $\sin\theta$ is 1, which is achieved at $\theta = \frac{\pi}{2} + 2k\pi$ for some integer k. This is left as an exercise to the reader.

This implies that $\sin(mx) = \sin(nx) = 1$, and that $mx = \frac{\pi}{2} + 2a\pi$ and $nx = \frac{\pi}{2} + 2b\pi$, for integers a, b.

Taking their ratio, we have $\frac{mx}{nx} = \frac{\frac{\pi}{2}+2a\pi}{\frac{\pi}{2}+2b\pi} \implies \frac{m}{n} = \frac{4a+1}{4b+1} \implies \frac{m}{4a+1} = \frac{n}{4b+1} = k$. It remains to find all m, n that satisfy this equation.

If $k = 1$, then $m \equiv n \equiv 1 \mod 4$. This corresponds to choosing two elements from the set $\{1,5,9,13,17,21,25,29\}$. There are $\binom{8}{2}$ ways to do so.

If $k < 1$, by multiplying m and n by the same constant $c = \frac{1}{k}$, we have that $mc \equiv nc \equiv 1 \mod 4$. Then either $m \equiv n \equiv 1 \mod 4$, or $m \equiv n \equiv 3 \mod 4$. But the first case was already counted, so we don't need to consider that case. The other case corresponds to choosing two numbers from the set $\{3,7,11,15,19,23,27\}$. There are $\binom{7}{2}$ ways here.

Finally, if $k > 1$, note that k must be an integer. This means that m, n belong to the set $\{k, 5k, 9k, \ldots\}$, or $\{3k, 7k, 11k, \ldots\}$. Taking casework on k, we get the sets $\{2,10,18,26\}, \{6,14,22,30\}, \{4,20\}, \{12,28\}$. Some sets have been omitted; this is

because they were counted in the other cases already. This sums to $\binom{4}{2} + \binom{4}{2} +$

$\binom{2}{2} + \binom{2}{2}$.

In total, there are $\binom{8}{2} + \binom{7}{2} + \binom{4}{2} + \binom{4}{2} + \binom{2}{2} + \binom{2}{2} = \boxed{63}$ pairs of (m, n).

This solution was brought to you by Leonard_my_dude

Solution 2

In order for $\sin(mx) + \sin(nx) = 2$, $\sin(mx) = \sin(nx) = 1$.

This happens when $mx \equiv nx \equiv \frac{\pi}{2} \pmod{2\pi}$.

This means that $mx = \frac{\pi}{2} + 2\pi\alpha$ and $nx = \frac{\pi}{2} + 2\pi\beta$ for any integers α and β.

As in Solution 1, take the ratio of the two equations: $\frac{mx}{nx} = \frac{\frac{\pi}{2}+2\pi\alpha}{\frac{\pi}{2}+2\pi\beta} \implies \frac{m}{n} = \frac{\frac{1}{2}+2\alpha}{\frac{1}{2}+2\beta} \implies \frac{m}{n} =$

$\frac{4\alpha+1}{4\beta+1}$ Now notice that the numerator and denominator of $\frac{4\alpha+1}{4\beta+1}$ are both odd, which

means that m and n have the same power of two (the powers of 2 cancel out).

Let the common power be p: then $m = 2^p \cdot a$, and $n = 2^p \cdot b$ where a and b are

integers between 1 and 30.

We can now rewrite the equation: $\frac{2^p \cdot a}{2^p \cdot b} = \frac{4\alpha+1}{4\beta+1} \implies \frac{a}{b} = \frac{4\alpha+1}{4\beta+1}$ Now it is easy to tell that

$a \equiv 1 \pmod 4$ and $b \equiv 1 \pmod 4$. However, there is another case: that

$a \equiv 3 \pmod 4$ and $b \equiv 3 \pmod 4$. This is because multiplying both $4\alpha + 1$ and $4\beta + 1$ by -1 will not change the fraction, but each congruence will be changed to $-1 \pmod 4 \equiv 3 \pmod 4$.

From the first set of congruences, we find that a and b can be two of $\{1,5,9,\cdots,29\}$.

From the second set of congruences, we find that a and b can be two of

$\{3,7,11,\cdots,27\}$.

Now all we have to do is multiply by 2^p to get back to m and n. Let's organize the solutions in order of increasing values of p, keeping in mind that m and n are bounded between 1 and 30.

For $p = 0$ we get $\{1,5,9,\cdots,29\}, \{3,7,11,\cdots,27\}$.

For $p = 1$ we get $\{2,10,18,26\}, \{6,14,22,30\}$

For $p = 2$ we get $\{4,20\}, \{12,28\}$

If we increase the value of p more, there will be less than two integers in our sets, so we are done there.

There are 8 numbers in the first set, 7 in the second, 4 in the third, 4 in the fourth, 2 in the fifth, and 2 in the sixth.

In each of these sets we can choose 2 numbers to be m and n and then assign them in increasing order. Thus there are:

$\binom{8}{2} + \binom{7}{2} + \binom{4}{2} + \binom{4}{2} + \binom{2}{2} + \binom{2}{2} = 28 + 21 + 6 + 6 + 1 + 1 = \boxed{63}$ possible pairings of m, n that satisfy the conditions.

-KingRavi

Solution 3

We know that the range of sin is between -1 and 1.

Thus, the only way for the sum to be 2 is for sin of mx and nx to both be 1.

The sin of $(90 + 360k)$ is equal to 1.

Assuming mx and nm are both positive, m and n could be 1,5,9,13,17,21,25,29. There are 8 ways, so $\binom{8}{2}$.

If bother are negative, m and n could be 3,7,11,15,19,23,27. There are 7 ways, so $\binom{7}{2}$.

However, the pair (1,5) could also be (2,10) and so on. The same goes for some other pairs.

In total there are 14 of these extra pairs.

The answer is $28 + 21 + 14 = \boxed{063}$

2021 AIME I Problem 8

Find the number of integers c such that the equation $||20|x| - x^2| - c| = 21$ has 12 distinct real solutions.

求使得方程 $||20|x| - x^2| - c| = 21$ 有12个不同的实数解的整数 c 的个数.

Solution 1

Let $y = |x|$. Then the equation becomes $||20y - y^2| - c| = 21$, or $|y^2 - 20y| = c \pm 21$. Note that since $y = |x|$, y is nonnegative, so we only care about nonnegative solutions in y. Notice that each positive solution in y gives two solutions in x ($x = \pm y$), whereas if $y = 0$ is a solution, this only gives one solution in x, $x = 0$. Since the total number of solutions in x is even, $y = 0$ must not be a solution. Hence, we require that $|y^2 - 20y| = c \pm 21$ has exactly 6 positive solutions and is not solved by $y = 0$.

If $c < 21$, then $c - 21$ is negative, and therefore cannot be the absolute value of $y^2 - 20y$. This means the equation's only solutions are in $|y^2 - 20y| = c + 21$. There is no way for this equation to have 6 solutions, since the quadratic $y^2 - 20y$ can only take on each of the two values $\pm(c + 21)$ at most twice, yielding at most 4 solutions. Hence, $c \geq 21$. c also can't equal 21, since this would mean $y = 0$ would solve the equation. Hence, $c > 21$.

At this point, the equation $y^2 - 20y = c \pm 21$ will always have exactly 2 positive solutions, since $y^2 - 20y$ takes on each positive value exactly once when y is restricted to positive values (graph it to see this), and $c \pm 21$ are both positive. Therefore, we just need $y^2 - 20y = -(c \pm 21)$ to have the remaining 4 solutions exactly. This means the horizontal lines at $-(c \pm 21)$ each intersect the parabola $y^2 - 20y$ in two places. This occurs when the two lines are above the parabola's vertex $(10, -100)$. Hence we have: $-(c + 21) > -100$ $c + 21 < 100$ $c < 79$ Hence,

the integers c satisfying the conditions are those satisfying $21 < c < 79$. There are $\boxed{057}$ such integers. Note: Be careful of counting at the end, you may mess up and get 59.

Solution 2 (also graphing)

Graph $y = |20|x| - x^2|$ (If you are having trouble, look at the description in the next two lines and/or the diagram in solution 3). Notice that we want this to be equal to $c - 21$ and $c + 21$.

We see that from left to right, the graph first dips from very positive to 0 at $x = -20$, then rebounds up to 100 at $x = -10$, then falls back down to 0 at $x = 0$.

The positive x are symmetric, so the graph re-ascends to 100 at $x = 10$, falls back to 0 at $x = 10$, and rises to arbitrarily large values afterwards.

Now we analyze the y (varied by c) values. At $y = k < 0$, we will have no solutions, as the line $y = k$ will have no intersections with our graph.

At $y = 0$, we will have exactly 3 solutions for the three zeroes.

At $y = n$ for any n strictly between 0 and 100, we will have exactly 6 solutions.

At $y = 100$, we will have 4 solutions, because local maxima are reached at $x = \pm 10$.

At $y = m > 100$, we will have exactly 2 solutions.

To get 12 distinct solutions for $y = |20|x| - x^2| = c \pm 21$, both $c + 21$ and $c - 21$ must produce 6 solutions.

Thus $0 < c - 21$ and $c + 21 < 100$, so $c \in \{22, 23, \ldots, 77, 78\}$ is required.

It is easy to verify that all of these choices of c produce 12 distinct solutions (none overlap), so our answer is $\boxed{057}$.

Solution 3 (Piecewise Functions: Analyses and Graphs)

We take cases for the outermost absolute value, then rearrange: $|20|x| - x^2| - c = \pm 21$.

Let $f(x) = |20|x| - x^2|$. We will rewrite $f(x)$ as a piecewise function without using any absolute value:

$$f(x) = \begin{cases} |-20x - x^2|, & \text{if } x \le 0 \begin{cases} 20x + x^2 & \text{if } x \le -20 \\ -20x - x^2 & \text{if } -20 < x \le 0 \end{cases} \\ |20x - x^2| & \text{if } x > 0 \begin{cases} 20x - x^2 & \text{if } 0 < x \le 20 \\ -20x + x^2 & \text{if } x > 20 \end{cases} \end{cases}.$$

We graph $f(x)$ as shown below, with some key points labeled. The fact that $f(x)$ is an even function ($f(x) = f(-x)$ holds for all real numbers x, from which the graph of $f(x)$ is symmetric about the y-axis) should facilitate the process of graphing.

Since $f(x) - c = \pm 21$ has 12 distinct real solutions, it is clear that each case has 6 distinct real solutions geometrically. We shift the graph of $f(x)$ down by c units, where $c > 0$:

For $f(x) - c = 21$ to have 6 distinct real solutions, we need $0 < c < 79$.

For $f(x) - c = -21$ to have 6 distinct real solutions, we need $21 < c < 121$.

Taking the intersection of these two cases gives $21 < c < 79$, from which there are $79 - 21 - 1 = \boxed{057}$ such integers c. ~MRENTHUSIASM

Solution 4

Removing the absolute value bars from the equation successively, we get
$||20|x| - x^2| - c| = 21, |20|x| - x^2| = c \pm 21, 20|x| - x^2 = \pm c \pm 21, x^2 \pm 20x \pm c \pm 21 = 0$.

The discriminant of this equation is $\sqrt{400 - 4(\pm c \pm 21)}$

Equating the discriminant to 0, we see that there will be two distinct solutions to each of the possible quadratics above only in the interval $-79 < c < 79$. However, the number of zeros the equation $ax^2 + b|x| + k$ has is determined by where $ax^2 + bx + k$ and $ax^2 - bx + k$ intersect, namely at $(0, k)$. When $k < 0, a > 0, ax^2 +$

$b|x| + k$ will have only 2 solutions, and when $k > 0, a > 0$, then there will be 4 real solutions, if they exist at all. In order to have 12 solutions here, we thus need to ensure $-c + 21 < 0$, so that exactly 2 out of the 4 possible equations of the form $ax^2 + b|x| + k$ given above have y-intercepts below 0 and only 2 real solutions, while the remaining 2 equations have 4 solutions. This occurs when $c > 21$, so our final bounds are $21 < c < 79$, giving us $\boxed{057}$ valid values of c.

2021 AIME I Problem 9

Let $ABCD$ be an isosceles trapezoid with $AD = BC$ and $AB < CD$. Suppose that the distances from A to the lines BC, CD, and BD are 15,18, and 10, respectively. Let K be the area of $ABCD$. Find $\sqrt{2} \cdot K$.

设$ABCD$为等腰梯形, $AD = BC$, 并且$AB < CD$. 假设从A到直线BC, CD和BD的距离分别为15,18和10. 设K为$ABCD$的面积, 求$\sqrt{2} \cdot K$的值.

Solution 1

Construct your isosceles trapezoid. Let, for simplicity, $AB = a$, $AD = BC = b$, and $CD = c$. Extend the sides BC and AD mark the intersection as P. Following what the question states, drop a perpendicular from A to BC labeling the foot as G. Drop another perpendicular from A to CD, calling the foot E. Lastly, drop a perpendicular from A to BD, labeling it F. In addition, drop a perpendicular from B to AC calling its foot F'.

–DIAGRAM COMING SOON–

Start out by constructing a triangle ADH congruent to $\triangle ABC$ with its side of length a on line DE. This works because all isosceles triangles are cyclic and as a result, $\angle ADC + \angle ABC = 180°$.

Notice that $\triangle AGC \sim \triangle BF'C$ by AA similarity. We are given that $AG = 15$ and by symmetry we can deduce that $F'B = 10$. As a result, $\frac{BF}{AG} = \frac{BC}{AC} = \frac{3}{2}$. This gives us that $AC = BD = \frac{3}{2}b$.

The question asks us along the lines of finding the area, K, of the trapezoid $ABCD$. We look at the area of ABC and notice that it can be represented as $\frac{1}{2} \cdot AC \cdot 10 = \frac{1}{2} \cdot a \cdot 18$. Substituting $AC = \frac{3}{2}b$, we solve for a, getting $a = \frac{5}{6}b$.

Now let us focus on isosceles triangle ACH, where $AH = AC = \frac{3}{2}b$. Since, AE is an altitude from A to CH of an isosceles triangle, HE must be equal to EC. Since $DH = a$ and $DC = c$, we can solve to get that $DE = \frac{c-a}{2}$ and $EC = \frac{a+c}{2}$.

We must then set up equations using the Pythagorean Theorem, writing everything in terms of a, b, and c. Looking at right triangle AEC we get $324 + \frac{(a+c)^2}{4} = \frac{9}{4}b^2$ Looking at right triangle AED we get $b^2 - 324 = \frac{(c-a)^2}{4}$ Now rearranging and solving, we get two equation $a + c = 3\sqrt{b^2 - 144}$ $c - a = 2\sqrt{b^2 - 324}$ Those are convenient equations as $c + a - (c - a) = 2a = \frac{5}{3}b$ which gives us $3\sqrt{b^2 - 324} - 2\sqrt{b^2 - 324} = \frac{5}{3}b$ After some "smart" calculation, we get that $b = \frac{27}{\sqrt{2}}$.

Notice that the question asks for $K\sqrt{2}$, and $K = \frac{1}{2} \cdot 18 \cdot (a + c)$ by applying the trapezoid area formula. Fortunately, this is just $27\sqrt{b^2 - 144}$, and plugging in the value of $b = \frac{27}{\sqrt{2}}$, we get that $K\sqrt{2} = \boxed{567}$. ~Math_Genius_164

Solution 2 (LOC and Trig)

Call AD and BC a. Draw diagonal AC and call the foot of the perpendicular from B to AC G. Call the foot of the perpendicular from A to line BC F, and call the foot of the perpindicular from A to DC H. Triangles CBG and CAF are similar, and we get that $\frac{10}{15} = \frac{a}{AC}$ Therefore, $AC = 1.5a$. It then follows that triangles ABF and ADH are similar. Using similar triangles, we can then find that $AB = \frac{5}{6}a$. Using the Law of Cosine on ABC, We can find that the cosine of angle ABC is $-\frac{1}{3}$. Since angles ABF and ADH are equivalent and supplementary to angle ABC, we know that the cosine of angle ADH

is 1/3. It then follows that $a = \frac{27\sqrt{2}}{2}$. Then it can be found that the area K is $\frac{567\sqrt{2}}{2}$.

Multiplying this by $\sqrt{2}$, the answer is $\boxed{567}$. -happykeeper

Solution 3 (Similarity)

Let the foot of the altitude from A to BC be P, to CD be Q, and to BD be R.

Note that all isosceles trapezoids are cyclic quadrilaterals; thus, A is on the circumcircle of $\triangle BCD$ and we have that PRQ is the Simson Line from A. As $\angle QAB = 90°$, we have that $\angle QAR = 90° - \angle RAB = \angle ABR = \angle APR = \angle APQ$, with the last equality coming from cyclic quadrilateral $APBR$. Thus, $\triangle QAR \sim \triangle QPA$ and we have that $\frac{AQ}{AR} = \frac{PQ}{PA}$ or that $\frac{18}{10} = \frac{QP}{15}$, which we can see gives us that $QP = 27$. Further ratios using the same similar triangles gives that $QR = \frac{25}{3}$ and $RP = \frac{56}{3}$.

We also see that quadrilaterals $APBR$ and $ARDQ$ are both cyclic, with diameters of the circumcircles being AB and AQ respectively. The intersection of the circumcircles are the points A and R, and we know DRB and QRP are both line segments passing through an intersection of the two circles with one endpoint on each circle. By Fact 5, we know then that there exists a spiral similarity with center A taking $\triangle APQ$ to $\triangle APD$. Because we know a lot about $\triangle APQ$ but very little about $\triangle APD$ and we would like to know more, we wish to find the ratio of similitude between the two triangles.

To do this, we use the one number we have for $\triangle APD$: we know that the altitude from A to BD has length 10. As the two triangles are similar, if we can find the height from A to PQ, we can take the ratio of the two heights as the ratio of similitude. To do this, we once again note that $QP = 27$. Using this, we can drop the altitude from A to QP and let it intersect QP at H. Then, let $QH = x$ and thus $HP = 27 - x$. We then have by the Pythagorean Theorem on $\triangle AQH$ and $\triangle APH$: $15^2 - x^2 = 18^2 - (27 - x)^2 225 - x^2 = 324 - (x^2 - 54x + 729) 54x = 630 x = \frac{35}{3}$ Then, $RH = QH - QR = \frac{35}{3} - \frac{25}{3} = \frac{10}{3}$. This gives us then from right triangle $\triangle ARH$ that $AH = \frac{20\sqrt{2}}{3}$ and thus

the ratio of $\triangle APQ$ to $\triangle ABD$ is $\frac{3\sqrt{2}}{4}$. From this, we see then that $AB = AP * \frac{3\sqrt{2}}{4} = 15 *$

$\frac{3\sqrt{2}}{4} = \frac{45\sqrt{2}}{4}$ and $AD = AQ * \frac{3\sqrt{2}}{4} = 18 * \frac{3\sqrt{2}}{4} = \frac{27\sqrt{2}}{2}$ The Pythagorean Theorem on $\triangle AQD$

then gives that $QD = \sqrt{AD^2 - AQ^2} = \sqrt{(\frac{27\sqrt{2}}{2})^2 - 18^2} = \sqrt{\frac{81}{2}} = \frac{9\sqrt{2}}{2}$ Then, we have the

height of trapezoid $ABCD$ is $AQ = 18$, the top base is $AB = \frac{45\sqrt{2}}{4}$, and the bottom

base is $CD = \frac{45\sqrt{2}}{4} + 2 * \frac{9\sqrt{2}}{2}$. From the equation of a trapezoid, $K = \frac{b_1 + b_2}{2} h = \frac{63\sqrt{2}}{4} *$

$18 = \frac{567\sqrt{2}}{2}$, so the answer is $K\sqrt{2} = \boxed{567}$. ~lvmath

Solution 4 (Cool Solution by advanture)

First, draw the diagram. Then, notice that since $ABCD$ is isosceles, $\triangle ABD \cong \triangle BAC$,
and the length of the altitude from B to AC is also 10. Let the foot of this altitude be
F, and let the foot of the altitude from A to BC be denoted as E. Then, $\triangle BCF \sim$
$\triangle ACE$. So, $\frac{BC}{AC} = \frac{BF}{AE} = \frac{2}{3}$. Now, notice that $[ABC] = \frac{10 \times AC}{2} = \frac{AB \times 18}{2} \Longrightarrow AC = \frac{9 \times AB}{5}$,
where $[ABC]$ denotes the area of triangle ABC. Letting $AB = x$, this equality
becomes $AC = \frac{9x}{5}$. Also, from $\frac{BC}{AC} = \frac{2}{3}$, we have $BC = \frac{6x}{5}$. Now, by the Pythagorean
theorem on triangles ABF and CBF, we have $AF = \sqrt{x^2 - 100}$ and $CF =$
$\sqrt{\left(\frac{6x}{5}\right)^2 - 100}$. Notice that $AC = AF + CF$, so $\frac{9x}{5} = \sqrt{x^2 - 100} + \sqrt{\left(\frac{6x}{5}\right)^2 - 100}$.

Squaring both sides of the equation once, moving $x^2 - 100$ and $\left(\frac{6x}{5}\right)^2 - 100$ to the
right, dividing both sides by 2, and squaring the equation once more, we are left
with $\frac{32x^4}{25} = 324x^2$. Dividing both sides by x^2 (since we know x is positive), we are
left with $\frac{32x^2}{25} = 324$. Solving for x gives us $x = \frac{45}{2\sqrt{2}}$.

Now, let the foot of the perpendicular from A to CD be G. Then let $DG = y$. Let the
foot of the perpendicular from B to CD be H. Then, CH is also equal to y. Notice that
$ABHG$ is a rectangle, so $GH = x$. Now, we have $CG = GH + CH = x + y$. By the

Pythagorean theorem applied to $\triangle AGC$, we have $(x + y)^2 + 18^2 = \left(\frac{9x}{5}\right)^2$. We know

that $\frac{9x}{5} = \frac{9}{5} \cdot \frac{45}{2\sqrt{2}} = \frac{81}{2\sqrt{2}}$, so we can plug this into this equation. Solving for $x + y$, we get $x + y = \frac{63}{2\sqrt{2}}$.

Finally, to find $[ABCD]$, we use the formula for the area of a trapezoid: $K = [ABCD] = \frac{b_1 + b_2}{2} \cdot h = \frac{AB + CD}{2} \cdot 18 = \frac{x + (CG + DG)}{2} \cdot 18 = \frac{2x + 2y}{2} \cdot 18 = (x + y) \cdot 18 = \frac{63}{2\sqrt{2}} \cdot 18 = \frac{567}{\sqrt{2}}$. The problem asks us for $K \cdot \sqrt{2}$, which comes out to be $\boxed{567}$. ~advanture

Solution 5 (Compact similarity solution)

Let E, F, and G be the feet of the altitudes from A to BC, CD, and DB, respectively.

Claim: We have 2 pairs of similar right triangles: $\triangle AEB \sim \triangle AFD$ and $\triangle AGD \sim \triangle AEC$.

Proof: Note that $ABCD$ is cyclic. We need one more angle, and we get this from this cyclic quad: $\angle ABE = 180° - \angle ABC = \angle ADC = \angle ADG \angle ADG = \angle ADB = \angle ACB = \angle ACE$ ▪ Let $AD = a$. We obtain from the similarities $AB = \frac{5a}{6}$ and $AC = BD = \frac{3a}{2}$.

By Ptolemy, $\left(\frac{3a}{2}\right)^2 = a^2 + \frac{5a}{6} \cdot CD$, so $\frac{5a^2}{4} = \frac{5a}{6} \cdot CD$.

We obtain $CD = \frac{3a}{2}$, so $DF = \frac{CD - AB}{2} = \frac{a}{3}$.

Applying the Pythagorean theorem on $\triangle ADF$, we get $324 = a^2 - \frac{a^2}{9} = \frac{8a^2}{9}$.

Thus, $a = \frac{27}{\sqrt{2}}$, and $[ABCD] = \frac{AB + CD}{2} \cdot 18 = \frac{\frac{5a}{6} + \frac{9a}{6}}{2} \cdot 18 = 18 \cdot \frac{7}{6} \cdot \frac{27}{\sqrt{2}} = \frac{567}{\sqrt{2}}$, yielding $\boxed{567}$.

Solution 6 (Similar Triangles, Two Variables, Two Equations)

Let $\overleftrightarrow{AE}, \overleftrightarrow{AF}$, and $\overleftrightarrow{AG}$ be the perpendiculars from A to $\overleftrightarrow{BC}, \overleftrightarrow{CD}$, and $\overleftrightarrow{BD}$, respectively. Next, let H be the intersection of $\overline{AF}$ and $\overline{BD}$.

We set $AB = x$ and $AH = y$, as shown below.

From here, we obtain $HF = 18 - y$ by segment subtraction, and $BG = \sqrt{x^2 - 10^2}$ and $HG = \sqrt{y^2 - 10^2}$ by the Pythagorean Theorem.

Since $\angle ABG$ and $\angle HAG$ are both complementary to $\angle AHB$, we have $\angle ABG = \angle HAG$, from which $\triangle ABG \sim \triangle HAG$ by AA. It follows that $\frac{BG}{AG} = \frac{AG}{HG} \Rightarrow BG \cdot HG = AG^2$, or $\sqrt{x^2 - 10^2} \cdot \sqrt{y^2 - 10^2} = 10^2$. (1)

Since $\angle AHB = \angle FHD$ by vertical angles, we have $\triangle AHB \sim \triangle FHD$ by AA, with ratio of similitude $\frac{AH}{FH} = \frac{y}{18-y}$. It follows that $DF = x \cdot \frac{18-y}{y}$.

Since $\angle EBA = \angle ECD = \angle FDA$ by angle chasing, we have $\triangle EBA \sim \triangle FDA$ by AA, with ratio of similitude $\frac{EA}{FA} = \frac{15}{18} = \frac{5}{6}$. It follows that $DA = \frac{6}{5}x$.

By the Pythagorean Theorem on right $\triangle ADF$, we have $DF^2 + AF^2 = AD^2$, or $\left(x \cdot \frac{18-y}{y}\right)^2 + 18^2 = \left(\frac{6}{5}x\right)^2$. (2)

Solving this system of equations ((1) and (2)), we get $x = \frac{45\sqrt{2}}{4}$ and $y = \frac{90}{7}$, from which $AB = x = \frac{45\sqrt{2}}{4}$ and $CD = AB + 2DF = x + 2\left(x \cdot \frac{18-y}{y}\right) = \frac{81\sqrt{2}}{4}$. Finally, the area of $ABCD$ is $K = \frac{AB+CD}{2} \cdot AF = \frac{567\sqrt{2}}{2}$, from which $\sqrt{2} \cdot K = \boxed{567}$.

2021 AIME I Problem 10

Consider the sequence $(a_k)_{k \geq 1}$ of positive rational numbers defined by $a_1 = \frac{2020}{2021}$ and for $k \geq 1$, if $a_k = \frac{m}{n}$ for relatively prime positive integers m and n, then $a_{k+1} = \frac{m+18}{n+19}$. Determine the sum of all positive integers j such that the rational number a_j can be written in the form $\frac{t}{t+1}$ for some positive integer t.

由有理数组成的数列$\{a_k\}$按如下方式定义: $a_1 = \frac{2020}{2021}$, 并且对于$k \geq 1$, 如果$a_k = \frac{m}{n}$对于互质的正整数m和n成立, 那么 $a_{k+1} = \frac{m+18}{n+19}$. 求能够找到正整数$t$, 使得有理数$a_j$i可以写成 $\frac{t}{t+1}$ 形式的所有正整数j的总和.

Solution

We know that $a_1 = \frac{t}{t+1}$ when $t = 2020$ so 1 is a possible value of j.

Note also that $a_2 = \frac{2038}{2040} = \frac{1019}{1020} = \frac{t}{t+1}$ for $t = 1019$.

Then $a_{2+q} = \frac{1019+18q}{1020+19q}$ unless $1019 + 18q$ and $1020 + 19q$ are not relatively prime which happens when $q + 1$ divides $18q + 1019$ or $q + 1$ divides 1001, so the least value of q is 6 and $j = 2 + 6 = 8$.

We know $a_8 = \frac{1019+108}{1020+114} = \frac{1127}{1134} = \frac{161}{162}$. Now $a_{8+q} = \frac{161+18q}{162+19q}$ unless $18q + 161$ and $19q + 162$ are not relatively prime which happens the first time $q + 1$ divides $18q + 161$ or $q + 1$ divides 143 or $q = 10$, and $j = 8 + 10 = 18$.

We have $a_{18} = \frac{161+180}{162+190} = \frac{341}{352} = \frac{31}{32}$. Now $a_{18+q} = \frac{31+18q}{32+19q}$ unless $18q + 31$ and $19q + 32$ are not relatively prime.

This happens the first time $q + 1$ divides $18q + 31$ implying $q + 1$ divides 13, which is prime so $q = 12$ and $j = 18 + 12 = 30$. We have $a_{30} = \frac{31+216}{32+228} = \frac{247}{260} = \frac{19}{20}$.

We have $a_{30+q} = \frac{18q+19}{19q+20}$, which is always reduced by EA, so the sum of all j is $1 + 2 + 8 + 18 + 30 = \boxed{059}$.

2021 AIME I Problem 11

Let $ABCD$ be a cyclic quadrilateral with $AB = 4, BC = 5, CD = 6$, and $DA = 7$. Let A_1 and C_1 be the feet of the perpendiculars from A and C, respectively, to line BD, and let B_1 and D_1 be the feet of the perpendiculars from B and D, respectively, to line AC. The perimeter of $A_1B_1C_1D_1$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

设$ABCD$为圆内接四边形, $AB = 4, BC = 5, CD = 6$, 并且$DA = 7$. 设A_1和C_1分别是A和C到直线BD的垂线的垂足, 设B_1和D_1分别是B和D到直线AC的垂线的垂足. $A_1B_1C_1D_1$周长为$\frac{m}{n}$, 其中m和n为互质的正整数, 求$m + n$.

Solution
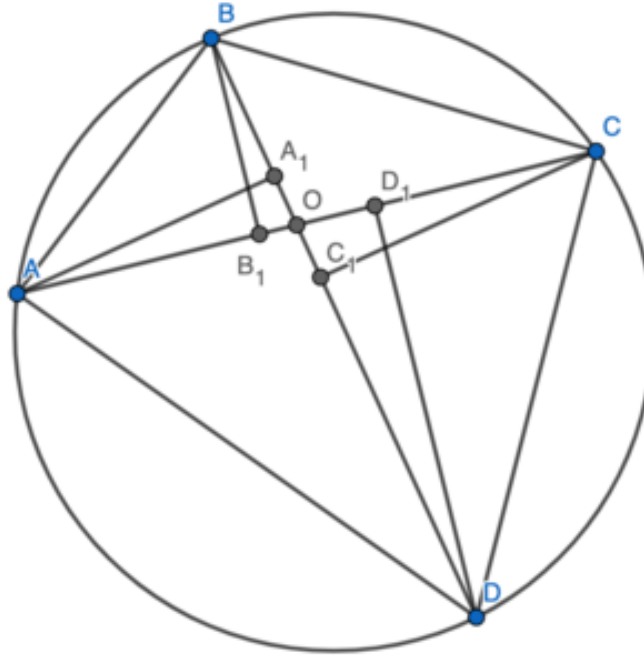
Let O be the intersection of AC and BD. Let $\theta = \angle AOB$.

Firstly, since $\angle AA_1D = \angle AD_1D = 90°$, we deduce that AA_1D_1D is cyclic. This implies that $\triangle A_1OD_1 \sim \triangle AOD$, with a ratio of $\frac{A_1O}{AO} = \cos\angle A_1OA = \cos\theta$. This means that $\frac{A_1D_1}{AD} = \cos\theta$. Similarly, $\frac{A_1B_1}{AB} = \frac{B_1C_1}{BC} = \frac{C_1D_1}{CD} = \cos\theta$. Hence $A_1B_1 + B_1C_1 + C_1D_1 + D_1A_1 = (AB + BC + CD + DA)\cos\theta$ It therefore only remains to find $\cos\theta$.

From Ptolemy's theorem, we have that $(BD)(AC) = 4 \times 6 + 5 \times 7 = 59$. From Brahmagupta's Formula, $[ABCD] = \sqrt{(11-4)(11-5)(11-6)(11-7)} = 2\sqrt{210}$. But the area is also $\frac{1}{2}(BD)(AC)\sin\theta = \frac{59}{2}\sin\theta$, so $\sin\theta = \frac{4\sqrt{210}}{59} \implies \cos\theta = \frac{11}{59}$. Then the desired fraction is $(4 + 5 + 6 + 7)\cos\theta = \frac{242}{59}$ for an answer of $\boxed{301}$.

Finding $\cos x$ 2

The angle θ between diagonals satisfies $\tan\frac{\theta}{2} = \sqrt{\frac{(s-b)(s-d)}{(s-a)(s-c)}}$(see

https://en.wikipedia.org/wiki/Cyclic_quadrilateral#Angle_formulas).

Thus,$\tan\frac{\theta}{2} = \sqrt{\frac{(11-4)(11-6)}{(11-5)(11-7)}}or\tan\frac{\theta}{2} = \sqrt{\frac{(11-5)(11-7)}{(11-4)(11-6)}}$. That is, $\tan^2\frac{\theta}{2} = \frac{1-\cos^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}} = \frac{24}{35}$ or $\frac{35}{24}$

Thus, $\cos^2\frac{\theta}{2} = \frac{35}{59}$ or $\frac{24}{59}$ $\cos\theta = 2\cos^2\frac{\theta}{2} - 1 = \frac{\pm11}{59}$In this context, $\cos\theta > 0$.

Thus, $\cos\theta = \frac{11}{59}, Ans = 22 * \cos\theta = 22 * \frac{11}{59} = \frac{242}{59} = \frac{m}{n}, m+n = 242 + 59 = \boxed{301}$

~y.grace.yu

Solution 3 (Pythagorean Theorem)

We assume that the two quadrilateral mentioned in the problem are similar (due to both of them being cyclic). Note that by Ptolemy's, one of the diagonals has length $\sqrt{4\cdot6 + 5\cdot7} = \sqrt{59}$. WLOG we focus on diagonal BD. To find the diagonal of the inner quadrilateral, we drop the altitude from A and C and calculate the length of A_1C_1. Let x be A_1D (Thus $A_1B = \sqrt{59} - x$. By Pythagorean theorem, we have$49 - x^2 = 16 - (\sqrt{59} - x)^2 \Longrightarrow 92 = 2\sqrt{59}x \Longrightarrow x = \frac{46}{\sqrt{59}} = \frac{46\sqrt{59}}{59}$.Now let y be C_1D. (thus making $C_1B = \sqrt{59} - y$). Similarly, we have$36 - y^2 = 25 - (\sqrt{59} - y)^2 \Longrightarrow 70 = 2\sqrt{59}y \Longrightarrow y = \frac{35}{\sqrt{59}} = \frac{35\sqrt{59}}{59}$.We see that A_1C_1, the scaled down diagonal is just $x - y = \frac{11\sqrt{59}}{59}$, which is $\frac{\frac{11\sqrt{59}}{59}}{\sqrt{59}} = \frac{11}{59}$ times our original diagonal BD, implying a scale factor of $\frac{11}{59}$. Thus, due to perimeters scaling linearly, the perimeter of the new quadrilateral is simply $\frac{11}{59} \cdot 22 = \frac{242}{59}$, making our answer $242 + 59 = \boxed{301}$. -fidgetboss_4000

2021 AIME I Problem 12

Let $A_1A_2A_3...A_{12}$ be a dodecagon (12-gon). Three frogs initially sit at A_4, A_8, and A_{12}. At the end of each minute, simultaneously, each of the three frogs jumps to one of the two vertices adjacent to its current position, chosen randomly and

independently with both choices being equally likely. All three frogs stop jumping as soon as two frogs arrive at the same vertex at the same time. The expected number of minutes until the frogs stop jumping is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

设$A_1 A_2 A_3 \ldots A_{12}$是一个十二边形, 三只青蛙分别坐在$A_4, A_8$和$A_{12}$处. 每分钟结束时, 三只青蛙同时分别跳到与其当前位置相邻的两个顶点中的一个, 这两个顶点随机独立的选择, 可能性相同. 只要有两只青蛙同时到达同一个顶点, 那么所有三只青蛙就会停止跳跃. 从开始到青蛙停止跳跃所需的分钟数的期望为$\frac{m}{n}$, 其中m和n为互质的正整数, 求$m + n$.

Solution

The expected number of steps depends on the distance between the frogs, not on the order in which these distances appear. Let $E(a, b, c)$ where $a + b + c = 9$ denote the expected number of steps that it takes for two frogs to meet if traversing in clockwise or counterclockwise order, the frogs are a, b and c vertices apart. Then

$E(3,3,3) = 1 + \frac{1}{4}E(3,3,3) + \frac{3}{4}E(1,3,5)$, giving $E(3,3,3) = \frac{4}{3} + E(1,3,5)$; (1)

$E(1,3,5) = 1 + \frac{1}{8}E(1,1,7) + \frac{1}{2}E(1,3,5) + \frac{1}{8}E(3,3,3)$, giving $E(1,3,5) = 2 + \frac{1}{4}E(1,1,7) + \frac{1}{4}E(3,3,3)$; (2)

$E(1,1,7) = 1 + \frac{1}{4}E(1,1,7) + \frac{1}{4}E(1,3,5)$, giving $E(1,1,7) = \frac{4}{3} + \frac{1}{3}E(1,3,5)$; (3)

Plug in (1) and (3) into (2), we see that $E(1,3,5) = 4$. $E(3,3,3) = \frac{4}{3} + 4 = \frac{16}{3}$.

Each step is one minute. The answer is $16 + 3 = \boxed{19}$.

2021 AIME I Problem 13

Circles ω_1 and ω_2 with radii 961 and 625, respectively, intersect at distinct points A and B. A third circle ω is externally tangent to both ω_1 and ω_2. Suppose line AB

intersects ω at two points P and Q such that the measure of minor arc $\widehat{PQ}$ is 120°. Find the distance between the centers of ω_1 and ω_2.

圆ω_1和ω_2的半径分别为961和625, 两圆交于不同的点A和B. 第三个圆ω与ω_1和ω_2相外切. 假设直线AB与ω相交于两点P和Q, 使得劣弧$\widehat{PQ}$是120°, 求$_{1}1$ 和$_{2}$的圆心之间的距离.

Solution

Let O_i and r_i be the center and radius of ω_i, and let O and r be the center and radius of ω.

Since $\overline{AB}$ extends to an arc with arc 120°, the distance from O to $\overline{AB}$ is $r/2$. Let $X = \overline{AB} \cap \overline{O_1O_2}$. Consider $\triangle OO_1O_2$. The line $\overline{AB}$ is perpendicular to $\overline{O_1O_2}$ and passes through X. Let H be the foot from O to $\overline{O_1O_2}$; so $HX = r/2$. We have by tangency $OO_1 = r + r_1$ and $OO_2 = r + r_2$. Let $O_1O_2 = d$.
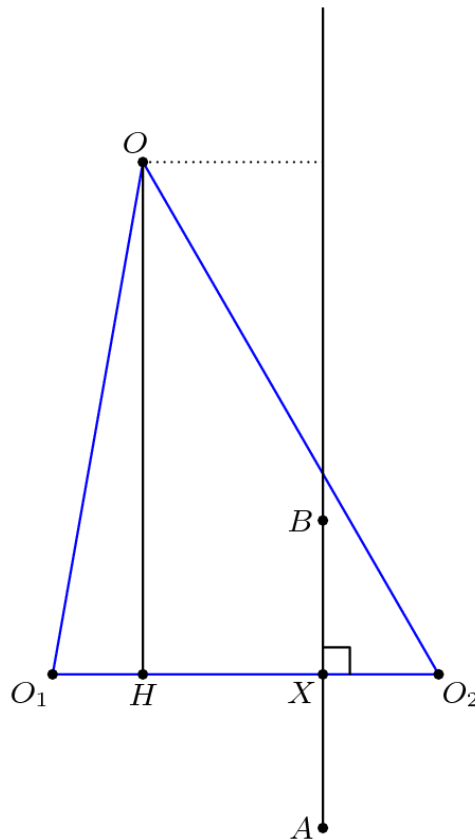
Since X is on the radical axis of ω_1 and ω_2, it has equal power with respect to both circles, so

$$O_1X^2 - r_1^2 = O_2X^2 - r_2^2 \Longrightarrow O_1X - O_2X = \frac{r_1^2 - r_2^2}{d}$$

since $O_1X + O_2X = d$.

Now we can solve for O_1X and O_2X, and in particular,

$$O_1H = O_1X - HX = \frac{d + \dfrac{r_1^2 - r_2^2}{d}}{2} - \frac{r}{2},$$

$$O_2H = O_2X + HX = \frac{d - \dfrac{r_1^2 - r_2^2}{d}}{2} + \frac{r}{2}.$$

We want to solve for d. By the Pythagorean Theorem (twice):

$$-OH^2 = O_2H^2 - (r + r_2)^2 = O_1H^2 - (r + r_1)^2 \Longrightarrow \left(d + r - \frac{r_1^2 - r_2^2}{d}\right)^2 - 4(r + r_2)^2$$

$$= \left(d - r + \frac{r_1^2 - r_2^2}{d}\right)^2 - 4(r + r_1)^2 \Longrightarrow 2dr - 2(r_1^2 - r_2^2)^2 - 8rr_2 - 4r_2^2$$

$$= -2dr + 2(r_1^2 - r_2^2) - 8rr_1 - 4r_1^2 \Longrightarrow 4dr = 8rr_2 - 8rr_1 \Longrightarrow d = 2r_2 - 2r_1$$

Therefore, $d = 2(r_2 - r_1) = 2(961 - 625) = \boxed{672}$.

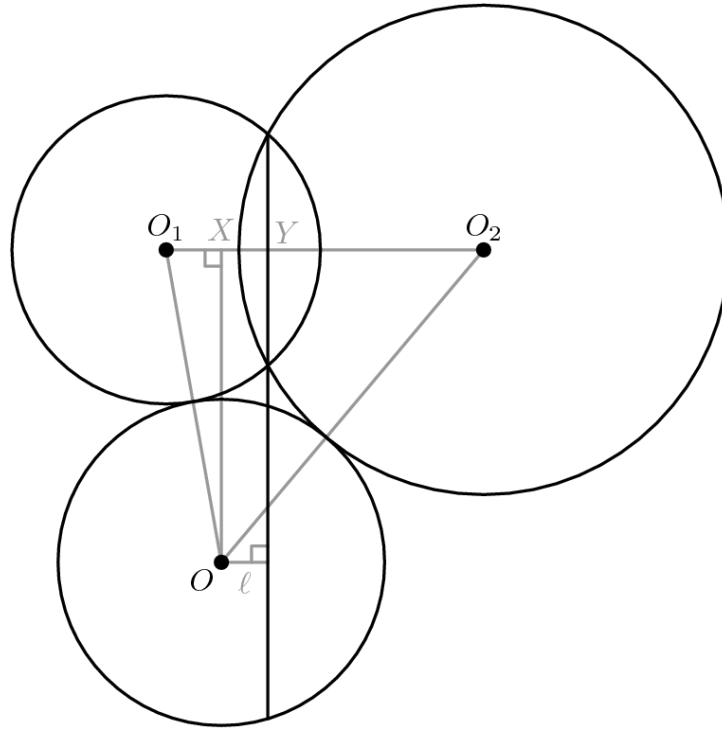Solution 2 (Official MAA, Unedited)

Denote by O_1, O_2, and O the centers of ω_1, ω_2, and ω, respectively. Let $R_1 = 961$ and $R_2 = 625$ denote the radii of ω_1 and ω_2 respectively, r be the radius of ω, and ℓ the distance from O to the line AB. We claim that

$$\frac{\ell}{r} = \frac{R_2 - R_1}{d},$$

where $d = O_1O_2$.

This solves the problem, for then the $\widehat{PQ} = 120°$ condition implies $\frac{\ell}{r} = \cos 60° = \frac{1}{2}$, and then we can solve to get $d = \boxed{672}$.

Denote by O_1 and O_2 the centers of ω_1 and ω_2 respectively. Set X as the projection of O onto $O_1 O_2$, and denote by Y the intersection of AB with $O_1 O_2$. Note that $\ell = XY$.
Now recall that

$$d(O_2 Y - O_1 Y) = O_2 Y^2 - O_1 Y^2 = R_2^2 - R_1^2.$$

Furthermore, note that $d(O_2 X - O_1 X) = O_2 X^2 - O_1 X^2 = O_2 O^2 - O_1 O^2 = (R_2 + r)^2 - (R_1 + r)^2 = (R_2^2 - R_1^2) + 2r(R_2 - R_1)$.

Substituting the first equality into the second one and subtracting yields $2r(R_2 - R_1) = d(O_2 Y - O_2 X) - d(O_2 X - O_1 X) = 2dXY$, which rearranges to the desired.

2021 AIME I Problem 14

For any positive integer a, $\sigma(a)$ denotes the sum of the positive integer divisors of a. Let n be the least positive integer such that $\sigma(a^n) - 1$ is divisible by 2021 for all positive integers a. Find the sum of the prime factors in the prime factorization of n.

对于任何正整数a, 用$\sigma(a)$表示a的正整数因数之和. 令n是最小的正整数, 使得对于所有的正整数a, $\sigma(a^n) - 1$可以被2021整除. 求n的质因数分解中所有质因数的和.

Solution 1

We require that $\prod_{p^e \| a} \frac{p^{en+1}-1}{p-1} \equiv 1 \pmod{2021}$ for all a, so it is necessary and sufficient to ensure $\frac{p^{en+1}-1}{p-1} \equiv 1 \pmod{2021}$ for all p, e.

We solve the problem for primes:

Claim: A prime q always divides $\frac{p^{en+1}-1}{p-1} - 1$ if and only if $q(q-1)$ divides n.

Proof. For choices of $p \not\equiv 1 \pmod q$, we need $p^{en+1} - 1 \equiv p - 1 \pmod q$, i.e. $p^{en} \equiv 1 \pmod q$. This holds for all p, e if and only if $q - 1 \mid n$ by Fermat's little theorem.

If $q^k \| p - 1$, we need $v_q(p^{en+1} - 1) = v_q(p - 1)$. By lifting the exponent, this implies $v_q(en+1) = 0$. It follows that we must have $q \mid n$, since otherwise some choice of e would force $en + 1 \equiv 0 \pmod q$. We will verify $q \mid n$ is sufficient for $p^{en+1} - 1 \equiv p - 1 \pmod{q^{k+1}}$ to hold: indeed, $p^{en} \equiv (\text{const} \cdot q^k + 1)^{en} \equiv en \cdot q^k + 1 \equiv 1 \pmod{q^{k+1}}$.

Therefore, the smallest n is $\text{lcm}(43 \cdot 42, 47 \cdot 46) = 2 \cdot 3 \cdot 7 \cdot 23 \cdot 43 \cdot 47$, and the requested sum is $2 + 3 + 7 + 23 + 43 + 47 = 125$.

Solution 2

You get that (I'll just do the 43 case, but 47 is the same)

$\frac{p^{n+1}-1}{p-1} \equiv 1 \bmod 43$. If $p \not\equiv 1 \bmod 43$, we have that $p^{n+1} \equiv p \bmod 43$. The minimum value of n s.t. $p^n \equiv 1 \bmod 43$ is $\lambda(43) = \phi(43) = 42$ where $\lambda(n)$ is the carmichael function.

When $p \equiv 1 \bmod 43$, we require that $1 + p + p^2 + \cdots p^n \equiv 1 \bmod 43, n + 1 \equiv 1 \bmod 43, n \equiv 0 \bmod 43$.

Symmetrically we also require that $n \equiv 0 \mod 46, 47$. So our answer is

lcm$(42,43,46,47) \Longrightarrow \boxed{125}$

2021 AIME I Problem 15

Let S be the set of positive integers k such that the two parabolas $y = x^2 - k$ and $x = 2(y - 20)^2 - k$ intersect in four distinct points, and these four points lie on a circle with radius at most 21. Find the sum of the least element of S and the greatest element of S.

使得两条抛物线$y = x^2 - k$和$x = 2(v - 20)^2 - k$相交于四个不同点, 并且这四个点在某个半径不超过21的圆上的正整数k组成的集合记 为S, 求S中最小元素与S中最大元素的和.

Solution

Make the translation $y \to y + 20$ to obtain $20 + y = x^2 - k, x = 2y^2 - k$. Multiply the first equation by 2 and sum, we see that $2(x^2 + y^2) = 3k + 40 + 2y + x$. Completing the square gives us $(y - \frac{1}{2})^2 + (x - \frac{1}{4})^2 = \frac{325 + 24k}{16}$; this explains why the two parabolas intersect at four points that lie on a circle*. For the upper bound, observe that $LHS \leq 21^2 = 441 \to 24k \leq 6731$, so $k \leq 280$.

For the lower bound, we need to ensure there are 4 intersections to begin with. (Here I'm using the un-translated coordinates.) Draw up a graph, and realize that two intersections are guaranteed, on the so called "right branch" of $y = x^2 - k$. As we increase the value of k, two more intersections appear on the "left branch."

$k = 4$ does not work because the "leftmost" point of $x = 2(y - 20)^2 - 4$ is $(-4, 20)$ which lies to the right of $(-\sqrt{24}, 20)$, which is on the graph $y = x^2 - 4$. While technically speaking this doesn't prove that there are no intersections (why?), drawing the graph should convince you that this is the case. Clearly, no k less than 4 works either.

$k = 5$ does work because the two graphs intersect at $(-5, 20)$, and by drawing the graph, you realize this is not a tangent point and there is in fact another intersection nearby, due to slope. Therefore, the answer is $5 + 280 = 285$.

In general, (Assuming four intersections exist) when two conics intersect, if one conic can be written as $ax^2 + by^2 = f(x, y)$ and the other as $cx^2 + dy^2 = g(x, y)$ for f,g polynomials of degree at most 1, whenever $(a, b), (c, d)$ are linearly independent, we can combine the two equations and then complete the square to achieve $(x - p)^2 + (y - q)^2 = r^2$. We can also combine these two equations to form a parabola, or a hyperbola, or an ellipse. When $(a, b), (c, d)$ are not L.I., the intersection points instead lie on a line, which is a circle of radius infinity. When the two conics only have 3,2 or 1 intersection points, the statement that all these points lie on a circle is trivially true. ~Ross Gao