Math 307 (Section 201) Midterm Examination

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Name SOLUTIONS

Student Number _____

1		21
2		14
3		12
4		13
Total		60

- Be sure that this examination has 10 pages. Write your name on top of each page.
- No notes, no calculators allowed.
- **Show your work and make your reasoning clear.**

Problem 1. [21] **Short Answer.** The questions (a)-(g) below are worth 3 marks each and require a short explanation (unless otherwise stated).

(a) Given the vector $x = [2, 1, 4]$, write a MATLAB code with input x that would produce the matrix

$$\begin{bmatrix} 4 & 2 & 1 \\ 1 & 1 & 1 \\ 16 & 4 & 1 \end{bmatrix}.$$

You may assume that x is already stored in MATLAB.

```
>> vander(x)
```

OR various ad hoc ways such as

```
>> [x'.^2 x' ones(3,1)]
```

(b) Which of the following sets are linear subspaces? **(Each correct answer is worth 1 mark. No explanation required, no partial marks.)**

(i) S_1: the set of all points $[x_1, x_2, x_3]^T$ in $\mathbb{R}^3$ such that $x_1 + x_2 = 0$.

(ii) S_2: the set of all polynomials of degree at most 2.

(iii) S_3: the set of all 2 x 2 matrices that are not invertible.

S_1 and S_2 are linear subspaces.

S_3 is not a linear subspace.

(c) Let $x = [x_1, x_2, \ldots, x_n]^T$. Does the quantity

$$|||x||| = \min_i |x_i|$$

define a norm on $\mathbb{R}^n$?

No. For example set $x = [0, 1, 1, \ldots, 1]^T \neq \vec{0}$ for $n > 1$. However, $|||x||| = 0$. This contradicts with the requirement: $|||x|||$ is a norm $\Rightarrow$ $|||x||| = 0 \Leftrightarrow x = \vec{0}$.

In the next 3 questions, let

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 5/4 \\ 0 & 0 & 1 & 3/4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

be the reduced row echelon form of a matrix A.

(d) What is the rank of A?

$\text{rank}(A) = \#\text{pivots} = 3$

(e) What is the dimension of $\mathcal{N}(A^T)$ (the nullspace of A^T)?

$\dim(N(A^T)) = 4 - \text{rank}(A) = 1.$

(f) Is the vector $[1, 2, 4, 5]^T$ in the range of A^T?

YES. As $R(A^T) = R(U^T)$, we check if $U^T y = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 5 \end{bmatrix}$ has a solution. It is easy to observe that $y = [1, 2, 4, 0]^T$ indeed solves this system.

Alternative justification: Recall $R(A^T) = N(A)^\perp$. Also, $N(A) = N(U) = \text{span}\{[1/2, -5/4, -3/4, 1]^T\}$. Then $\langle y, [1/2, -5/4, -3/4, 1] \rangle = 0 \Rightarrow y \in N(A)^\perp = R(A^T)$.

(g) The chemical reaction $C_{10}H_8 + 12O_2 = 10CO_2 + 4H_2O$ can be written as $An = 0$ for some integer vector n and matrix A. Write down A and n.

$$A = \begin{bmatrix} 10 & 0 & 1 & 0 \\ 8 & 0 & 0 & 2 \\ 0 & 2 & 2 & 1 \end{bmatrix} ; \quad \vec{n} = \begin{bmatrix} 1 \\ 12 \\ -10 \\ -4 \end{bmatrix}.$$

Problem 2. [14]

(a) Give the definitions of the matrix norm and the condition number of a matrix.

Let A be a matrix. Then

- $\|A\| = \max_{x \neq \vec{0}} \frac{\|Ax\|_2}{\|x\|_2} = \max_{\|x\|=1} \|Ax\|$

- $\text{cond}(A) = \|A\| \cdot \|A^{-1}\|$

(b) Find the Frobenius norm, the matrix norm, and the condition number of the matrix

$$A = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

$\|A\|_{FR} = \sqrt{\frac{1}{4} + 9 + 4} = \sqrt{53/4}$;

$\|A\| = 3$; $\text{Cond}(A) = \frac{3}{1/2} = 6$

(c) Let A be a 2×3 matrix with $||A|| = 3$. Suppose you are given that

$$A \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} b \\ 3 \end{bmatrix}.$$

List **all integer values** that b can take.

Since $\|A\| = 3$, $\left\| A \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\| = \left\| \begin{bmatrix} b \\ 3 \end{bmatrix} \right\| \leq 3 \left\| \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\|$

That is

$\sqrt{b^2+9} \leq 3 \cdot \sqrt{1+1+4} \iff b^2+9 \leq 9 \cdot 6 = 54$

$\iff b^2 \leq 45.$

Then the integer values b can take are

$\boxed{0, \pm 1, \pm 2, \ldots, \pm 6.}$

(d) Let $B = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ be the 2×2 matrix that rotates a vector in $\mathbb{R}^2$ counterclockwise by an angle of $\pi/4$. Find the matrix norm and condition number of B.

Notice that $\|Bx\| = \|x\|$ for all x. This implies that $\boxed{\|B\| = 1}$ and $\|B^{-1}\| = 1$. Accordingly, $\boxed{\text{Cond}(B) = 1.}$

(e) The norm and condition number of a 2×2 matrix C are $||C|| = 2$ and $\text{cond}(C) = 3$. For x with $\|x\| = 1$, what is the minimum norm of Cx?

$$\text{Cond}(C) = \frac{\text{max stretch ratio}}{\text{min stretch ratio}}. \text{ So,}$$

$$3 = \frac{\max_{\|x\|=1} \|Cx\|}{\min_{\|x\|=1} \|Cx\|} = \frac{\|C\|}{\min_{\|x\|=1} \|Cx\|} = \frac{2}{\min_{\|x\|=1} \|Cx\|}$$

$$\Rightarrow \boxed{\min_{\|x\|=1} \|Cx\| = 2/3.}$$

Problem 3. [12] Consider the differential equation

$$f''(x) + 2^x f(x) = 1; f'(-1) = 3, f(5) = 10$$

on the interval $-1 \leq x \leq 5$. Let $F = [F_0, F_1, F_2, F_3]^T$ be the finite difference approximation to $f(x)$ when $N = 3$.

(a) Write down the matrix equation that F must solve (note that the first boundary condition involves the derivative $f'(1)$).

$N=3$; so $\Delta x = \dfrac{5-(-1)}{3} = 2$; which gives $x_0 = -1$, $x_1 = 1$, $x_2 = 3$, $x_3 = 5$.

Noting that $f'(-1) \cong \dfrac{F_1 - F_0}{\Delta x} = \dfrac{F_1 - F_0}{2}$,

$f(5) \cong F_3$,

set

$$L = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2^3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$b = \begin{bmatrix} 3 \cdot \Delta x \\ 1 \cdot \Delta x^2 \\ 1 \cdot \Delta x^2 \\ 10 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 4 \\ 10 \end{bmatrix}.$$

The eqn is:

$$(L + \Delta x^2 Q) F = b.$$

<u>Explicitly:</u>

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 6 & 1 & 0 \\ 0 & 1 & 30 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 4 \\ 10 \end{bmatrix}$$

(b) Write a MATLAB code that will compute the vector F and plot the result.

```
>> A=[-1,1,0,0;1,6,1,0;0,1,30,1;0,0,0,1];
>> b=[6;4;4;10];
>> F=A\b;
>> plot([-1,1,3,5],F)
```

Problem 4. [13] Consider the $(N-1) \times (N+1)$ matrix $A = D_{N-1}D_N$ which we use when we compute the discrete approximation to the second derivative of a function on a given interval. That is,

$$A = \begin{bmatrix} 1 & -2 & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & -2 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & -2 & 1 & 0 \\ 0 & \dots & 0 & 0 & 1 & -2 & 1 \end{bmatrix}.$$

(a) Are the rows of A linearly independent?

Yes. Every row of A is a pivot row, so they are linearly independent.

(b) Are the columns of A linearly independent?

No. There are N+1 columns, each in $\mathbb{R}^{N-1}$, so they cannot be linearly independent.

(c) What is $\dim(R(A))$?

$\dim(R(A)) = \text{rank}(A) = N-1.$

(d) What is $\dim(N(A))$?

$$\dim(N(A)) = N+1 - \text{rank}(A) = 2$$

(e) What is the range of A?

Since $R(A) \subseteq \mathbb{R}^{N-1}$ and $\text{rank}(A) = \dim(R(A)) = N-1$ we conclude that $R(A) = \mathbb{R}^{N-1}$.

(f) What is the null space of A? (Hint: Recall that A computes the "discrete second derviative" of a vector.)

Need to find two linearly indep. vectors in $N(A)$ (as we showed above $\dim(N(A)) = 2$).

<u>Claim:</u> $n_1 = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$; $n_2 = \begin{bmatrix} 1 \\ 2 \\ \vdots \\ n+1 \end{bmatrix}$ are in $N(A)$.

Why? Because the "discrete derivative of n_1 given by $D_N n_1$ is 0. Also, the discrete derivative of n_2, i.e., $D_N n_2 = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^N$,

so $D_{N-1}(D_N n_2) = 0$.

Finally, noting that n_1 and n_2 are lin. indep., we have

$$N(A) = \text{Span}\{n_1, n_2\}.$$

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<u>Alternatively:</u> Solve the homog. eqn

$$\begin{bmatrix} 1 & -2 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & -2 & 1 & \cdots & 0 \\ 0 & \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & \ddots \\ 0 & \cdots & & 0 & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \\ u_{N+1} \end{bmatrix} = \vec{0}$$

<u>Two free variables:</u> $u_{N+1} = s$; $u_N = t$

Last row: $u_{N-1} = 2t - s$

$$u_{N-2} = 2u_{N-1} - u_N = 3t - 2s$$

$$u_{N-3} = 4t - 3s$$

$$\vdots$$

$$u_1 = Nt + (N-1)s$$

$$\Rightarrow u = t \underbrace{\begin{bmatrix} N \\ N-1 \\ \vdots \\ 1 \\ 0 \end{bmatrix}}_{n_3} + s \underbrace{\begin{bmatrix} N-1 \\ N-2 \\ \vdots \\ 1 \\ 0 \\ -1 \end{bmatrix}}_{n_4} \Rightarrow N(A) = \text{Span}\{n_3, n_4\}.$$