$dS_t =$

4.1 As $\Delta(t)$ is a simple process adapted to $\mathcal{F}(t)$

It is equivalent to define a new partition that $t = t_{k+1}$

So, rewrite the stochastic integral on simple process:

The increment of Martingale is expected to be zero ✓

The increment of Martingale is independent ×

$$I(t) = I(t_k) = \sum_{j=0}^{k} \Delta(t_j)\,(M(t_{j+1}) - M(t_j))$$

$$\mathbb{E}[I(t)\,|\,\mathcal{F}(s)] = \mathbb{E}\{[I(t) - I(s) + I(s)]\,|\,\mathcal{F}(s)\}$$

Like what we define t_{k+1} before, naturally, we can assume $s = t_{\ell+1}$

$$\begin{cases} I(s) = \sum_{j=0}^{\ell} \Delta(t_j)\,(M(t_{j+1}) - M(t_j)) & \text{is } \mathcal{F}(s) \text{ adapted} \\ I(t) - I(s) = \sum_{j=\ell+1}^{k} \Delta(t_j)\,\underline{(M(t_{j+1}) - M(t_j))} & \text{is } \cancel{\text{independent of}}\ \mathcal{F}(s) \end{cases}$$

increment of Martingale has no property ?

Therefore, $\mathbb{E}[I(t)\,|\,\mathcal{F}(s)] = \mathbb{E}[I(t) - I(s)\,|\,\mathcal{F}(s)] + \mathbb{E}[I(s)\,|\,\mathcal{F}(s)]$

$= I(s) + \mathbb{E}[\sum \Delta(t_j)(M(t_{j+1}) - M(t_j))\,|\,\mathcal{F}(s)]$

$= \mathbb{E}[\sum \Delta(t_j)\,\mathbb{E}[M(t_{j+1}) - M(t_j)\,|\,\mathcal{F}(t_j)]\,|\,\mathcal{F}(s)]$

So, the stochastic integral is a Martingale

4.2 i) Just as simplification, insert $s = t_\ell$ and $t = t_k$.

$$I(t) - I(s) = \sum_{j=\ell}^{k-1} \Delta(t_j)\,[W(t_{j+1}) - W(t_j)]$$

None random ⇒ Deterministic, independent

In which, $\Delta(t_j)$ is $\mathcal{F}(t_j)$ ~~adapted~~ measurable $\mathcal{F}(t_j) \supset \mathcal{F}(s)$

$W(t_{j+1})$ is $\mathcal{F}(t_{j+1})$-measurable $\mathcal{F}(t_{j+1}) \supset \mathcal{F}(s)$ $= 0$

$W(t_j)$ is $\mathcal{F}(t_j)$-measurable $\mathcal{F}(t_j) \supset \mathcal{F}(s)$

~~Therefore, $I(t) - I(s)$ is independent of $\mathcal{F}(s)$ me~~

The increment of Brownian Motion is independent of ~~backward~~ forward Filtration $W(t_{j+1}) - W(t_j)$ is independent of $\mathcal{F}(t_j)$, so, it is independent of $\mathcal{F}(s)$ as $\mathcal{F}(s) \subset \mathcal{F}(t_j)$

~~$\Delta(t_j)$ is independent of $\mathcal{F}(t_{j-1})$~~
~~$\Delta(t_l)$ is independent of $\mathcal{F}(t_l)$~~

So, As every item in $I(t) - I(s)$ are independent of $\mathcal{F}(s)$
$I(t) - I(s)$ is independent of $\mathcal{F}(s)$

ii) $I(t) - I(s) = \sum_{j=l}^{k-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)]$

As Increment $W(t_{j+1}) - W(t_j)$ is Normal distributed as $N(0, t_{j+1} - t_j)$
$I(t) - I(s)$ is the linearal combination of normal distribution
is surely normal distributed with mean zero and
variance $\sum_{j=l}^{k-1} \Delta^2(t_j)(t_{j+1} - t_j)$ which is equivalent to
$\int_s^t \Delta^2(u)\,du$

iii) $\mathbb{E}[I(t) | \mathcal{F}(s)] = \mathbb{E}[I(t) - I(s) + I(s) | \mathcal{F}(s)]$
$= \mathbb{E}[I(t) - I(s) | \mathcal{F}(s)] + \mathbb{E}[I(s) | \mathcal{F}(s)]$
$= \mathbb{E}[I(t) - I(s)] + I(s)$
$= \mathbb{E}(I(0) - I(0)) + I(s) = I(s)$

iv) $\mathbb{E}\left[I^2(t) - \int_0^t \Delta^2 du \,\middle|\, \mathcal{F}(s)\right] = \mathbb{E}\left[(I(t) - I(s))^2 - I^2(s) + 2I(t)I(s) - \int_0^t \Delta^2 du \,\middle|\, \mathcal{F}(s)\right]$
$= \mathbb{E}(I(t) - I(s))^2 \cancel{- \mathbb{E}\{I(s)[I(s) - 2I(t)] | \mathcal{F}(s)\}} - \mathbb{E}\int_0^t \Delta^2 du \,|\, \mathcal{F}(s)$
$+ \mathbb{E}[I^2(s) + 2I(s)(I(t) - I(s)) | \mathcal{F}(s)]$
$= \int_s^t \Delta^2(u)\,du + \int_0^s \Delta^2(u)\,du + 0 - \int_0^t \Delta^2 du \,|\, \mathcal{F}(s)$
$= 0$

~~$\mathbb{E}[I^2(t) - \int_0^t \Delta^2(u) du \,|\, \mathcal{F}(s)] = \mathbb{E}$~~

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iv) Proof:

$$\mathbb{E}\left[\left(I^2(t) - \int_0^t \Delta^2(u)\,du\right) - \left(I^2(s) - \int_0^s \Delta^2(u)\,du\right) \middle| \mathcal{F}(s)\right]$$

$$= \mathbb{E}\left[I^2(t) - I^2(s) - \int_s^t \Delta^2(u)\,du \middle| \mathcal{F}_s\right]$$

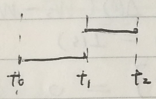
$$= \mathbb{E}\left[(I(t)-I(s))(I(t)+I(s)) - \int_s^t \Delta^2(u)\,du \middle| \mathcal{F}(s)\right]$$

$$= \mathbb{E}\left[(I(t)-I(s))^2 - 2I^2(s) + 2I(t)I(s) - \int_s^t \Delta^2(u)\,du \middle| \mathcal{F}(s)\right]$$

$$= \mathbb{E}\left[(I(t)-I(s))^2 \middle| \mathcal{F}_s\right] + 2\mathbb{E}\left[I(s)(I(t)-I(s)) \middle| \mathcal{F}_s\right] - \mathbb{E}\left[\int_s^t \Delta^2(u)\,du\right]$$

$$= \int_s^t \Delta^2(u)\,du + 0 - \int_s^t \Delta^2(u)\,du = 0$$

4.3 i)



$$I(t) = \overbrace{\Delta(t_0)[W(t_1) - W(t_0)]}^{I(s)} + \Delta(\overset{s}{\overset{\|}{t_1}})[W(t_2) - W(t_1)]$$

$$I(t) - I(s) = \Delta(s)[W(t) - W(s)]$$

$$= W(s)[W(t) - W(s)]$$

i) False $\quad I(t) - I(s) = \underbrace{W(s)}_{\mathcal{F}_s\text{ - measurable}}\ \underbrace{[W(t) - W(s)]}_{\text{independent}}$

False

ii) $\mathbb{E}(I(t) - I(s)) = \mathbb{E}\,W(s)\ \mathbb{E}(W(t) - W(s)) = 0$

$\mathbb{E}(I(t) - I(s))^2 = \mathbb{E}\,W^2(s)\ \mathbb{E}(W(t) - W(s))^2 = s\cdot(t-s)$

$\mathbb{E}(I(t) - I(s))^3 = \mathbb{E}\,W^3(s)\ \mathbb{E}(W(t) - W(s))^3 = 0$

$\mathbb{E}(I(t) - I(s))^4 = \mathbb{E}\,W^4(s)\ \mathbb{E}(W(t) - W(s))^4$

$$d(W^4) = 4W^3\,dW + \frac{1}{2}\cdot 12W^2\,dt$$

$$\Rightarrow \mathbb{E}\,W^4 = \mathbb{E}\left(W^4(0) + \int 4W^3\,dW + \int 6W^2\,dt\right) = \cancel{6W^2\frac{t^2}{2}}\ 6\int \mathbb{E}W^2\,dt$$

$$= 6\int t\,dt = \frac{t^2}{2}\cdot 6 = 3t^2$$

$$\mathbb{E}(W(t) - W(s))^4 = \mathbb{E}\left[4\int (W_t - W_s)^3\,dW_t + \frac{1}{2}\cdot 4\cdot 3\cdot\int (W_t - W_s)^2\,d(t-s)\right]$$

$$= 6\cdot \cancel{t-s}\int \mathbb{E}(W_t - W_s)^2\,dt$$

$$= 6\int (t-s)\,d(t-s) = \frac{(t-s)^2}{2}\cdot 6 = 3(t-s)^2$$

$\mathbb{E}(I(t)-I(s))^4 = 3s^2 \cdot 3(t-s)^2 = 9[s(t-s)]^2.$

$\mathbb{E}(I(t)-I(s))^2 = s(t-s)$

Since $\mathbb{E}(I(t)-I(s))^4 = \cancel{3}\,\mathbb{E}^2(I(t)-I(s))^2 \neq 3\,\mathbb{E}(I(t)-I(s))^2$

$I(t)-I(s)$ is not normal distributed.

iii) $\mathbb{E}[I(t) \mid \mathcal{F}_s] = \mathbb{E}[I(t)-I(s)+I(s) \mid \mathcal{F}_s]$

True

$= \mathbb{E}(I(t)-I(s) \mid \mathcal{F}_s) + \mathbb{E}[I(s) \mid \mathcal{F}_s]$

$= \cancel{0 + I(t)}\ \mathbb{E}[W_s(W_t-W_s) \mid \mathcal{F}_s] + \mathbb{E}[\Delta(0)(W_s-W_0) \mid \mathcal{F}_s]$

$= I(s) = W_s\,\mathbb{E}(W_t-W_s) + \Delta(0)(W_s-W_0)$

$= 0 + I(s) = I(s)$

iv) Proof $\mathbb{E}\left[I^2(t) - I^2(s) - \int_s^t \Delta^2 du \mid \mathcal{F}_s\right] = 0$

True

$\cancel{= \mathbb{E}\left[(I(t)-I(s))^2 + I^2(s) + 2(I(t)-I(s))I(s) - \int_s^t \Delta^2 du \mid \mathcal{F}_s\right]}$

Adaptivity $\cancel{= \mathbb{E}(I(t)-I(s))^2 + I^2(s) + 2I_s\,\mathbb{E}(I_t-I_s \mid \mathcal{F}_s) - \mathbb{E}\left(\int_s^t \Delta^2 du \mid \mathcal{F}_s\right)}$

$\cancel{= \int_s^t \Delta^2 du - \int_s^t \Delta^2 du = 0}$

$\cancel{= \mathbb{E}(W_s^2(W_t-W_s)^2 \mid \mathcal{F}_s) + I_s^2 + 2I_s\,\mathbb{E}[W_s(W_t-W_s) \mid \mathcal{F}_s] - \int_s^t W_s^2 du}$

As $I_t = I_s + W_s(W_t-W_s)$

$I_s = \Delta(0)(W_s-0) \qquad \int_s^t \Delta^2 du = W_s^2(t-s)$

$\mathbb{E}[I_t^2 - I_s^2 - \int_s^t \Delta du \mid \mathcal{F}_s] = \mathbb{E}[\cancel{I_s^2} + W_s^2(W_t-W_s)^2 + 2I_s W_s(W_t-W_s) - \int_s^t \Delta^2 du \mid \mathcal{F}_s]$

$= s\cdot(t-s) + 2\Delta(0)\,s\cdot 0 - \mathbb{E}(W_s^2(t-s) \mid \mathcal{F}_s)$

$= s(t-s) - s(t-s) = 0$

4.5 i) $d \log S_t = \frac{dS_t}{S_t} + \frac{1}{2}\left(-\frac{1}{S_t^2}\right) \cdot dS_t dS_t$

$d[S_t S_t](t) = \sigma^2 S(t)^2 dt$, substitute to equation

$dS_t = \alpha S_t dt + \sigma_t S_t dW_t$

$d \log S_t = \frac{1}{S_t} \cdot (\alpha S_t dt + \sigma_t S_t dW_t) - \frac{1}{2S_t^2} \sigma^2 S_t^2 dt$

$= \frac{\alpha S_t}{S_t} dt = \alpha dt$ $\quad (\alpha - \frac{\sigma^2}{2}) dt + \sigma_t dW_t$

ii) $\int (d\log S_t) = \int (\alpha - \frac{\sigma^2}{2}) dt + \int \sigma_t dW_t$

$\Rightarrow S_t = \exp\left\{ \int (\alpha - \frac{\sigma^2}{2}) dt + \int \sigma dW_t \right\}$

4. ~~(iii)~~ $d(S_t^p) = p S^{p-1} dS_t + \frac{1}{2} p(p-1) S^{p-2} dS dS$

As $dS = \alpha S dt + \sigma dW$

$dS dS = \sigma^2 dt$ substitute to ~~the~~ formula ~~equation~~

$d(S(t)) = pS^{p-1}(\alpha S dt + \sigma dW) + \frac{1}{2} p(p-1) S^{p-2} \sigma^2 dt$

$= p\alpha S^p dt + pS^{p-1}\sigma dW + \frac{1}{2} p(p-1) S^{p-2} \sigma^2 dt$

$= pS^{p-2}(\alpha p^2 + \frac{1}{2}(p-1)\sigma^2) dt + pS^{p-1}\sigma dW$

4.7 i) $dW^4 = 4W^3 dW + 4 \cdot 3 \cdot W^2 dW dW$

$W^4(t) - W^4(0) = 4\int_0^t W_u^3 dW(u) + \frac{1}{2} 12 \cdot \int_0^t W_u^2 du$

ii) $\mathbb{E} W^4(t) = 4\,\mathbb{E}\int_0^t W_u^3 dW_u + \frac{1}{2} 12 \int_0^t \mathbb{E} W_u^2 du$

$= 0 + 12\int_0^t u du = \frac{1}{2} \frac{u^2}{2} \cdot 12 \Big|_0^t = 3t^2$ $\quad$ 5

iii) $d(W^6) = 6W^5 dW + \frac{1}{2} 30 W^4 dt \Rightarrow \mathbb{E}W^6 = 0 + 15\int_0^t \mathbb{E}W^4 du = 15\int_0^t 3u^2 du$

$$= 5 \cdot \frac{u^3}{3} \cdot 3 \Big|_0^t = 15t^3$$

So $\mathbb{E}\, W^6(T) = 15\,T^3$

4.8 $d(e^{\beta t}R(t))$, Define a function $f(t,x) = e^{\beta t}x$

Then, $f_x = e^{\beta t}$ $f_{xx} = 0$ $f_t = \beta e^{\beta t} x$

So $df = f_x dx + \frac{1}{2} f_{xx}\, dx dx + f_t\, dt$

$$= e^{\beta t} dx + x \beta e^{\beta t} dt$$

Substitute $dX = dR_t = (\alpha - \beta R_t)\,dt + \sigma dW_t$

$$d(R_t e^{\beta t}) = e^{\beta t}(\alpha - \beta R_t)dt + e^{\beta t}\sigma dW_t + \beta e^{\beta t} R_t\, dt$$

$$= e^{\beta t}\alpha\, dt + e^{\beta t}\sigma dW_t$$

$$R(T)e^{\beta T} - R(0) = e^{\beta t} \cdot \frac{\alpha}{\beta} - \frac{\alpha}{\beta} + \int e^{\beta t}\sigma dW_t$$

$$= \frac{\alpha}{\beta}(e^{\beta T} - 1) + \int_0^T e^{\beta t}\sigma dW_t$$

Therefore $R(t) = e^{-\beta t}R(0) + \frac{\alpha}{\beta}(1 - e^{-\beta t}) + \int_0^t \sigma dW_t$

Additional Question :

$$dX_t = \Delta_t \, dW_t + \theta_t \, dt$$

a) $$dX_t = d(e^{tW_t}) = W_t e^{tW_t} dt + t e^{tW_t} dW_t + \frac{1}{2} t^2 e^{tW_t} dt$$

$$\left(f(t,x) = e^{tx} \quad f_x = t e^{tx} \quad f_{xx} = t^2 e^{tx} \quad f_t = x e^{tx} \right)$$

b) $$d[f(W_t)] \overset{\text{Itô}}{=} f'(W_t) \, dW_t + \frac{1}{2} f''(W_t) \, dt$$

$$= f'(W_t) \, dW_t + \frac{1}{2} \lambda f(W_t) \, dt$$

$$\Rightarrow \mathbb{E}[f(W_t)] = f(W(0)) + \mathbb{E}\int_0^T f'(W_t) \, dW_t + \frac{1}{2}\lambda \int_0^T \mathbb{E}[f(W_t)] \, dt$$

$$\mathbb{E} f(W(T)) - 1 = 0 + \frac{1}{2}\lambda \int_0^T \mathbb{E}[f(W_t)] \, dt.$$

Add $\frac{d}{dT}$ in both sides of last equation, we can deduce that =

$$\cancel{\frac{d}{dT} \mathbb{E} f(W(T)) = \frac{1}{2}\lambda \, \mathbb{E} f(W_T)}$$

→ No derivative. Only differential

$$d\,\mathbb{E} f(W_t) = \frac{1}{2}\lambda \, \mathbb{E} f(W_t) \, dt$$

$$\mathbb{E} f(W_T) = \frac{1}{2} e^{\frac{\lambda T}{2}}$$

Cannot deduce further!

If any, please tell me

Method 2: Assume $X_t = e^{\gamma t} f(W_t)$

$$dX_t = e^{\gamma t} f'(W_t) \, dW_t + \frac{1}{2} e^{\gamma t} f''(W_t) \, dW_t dW_t + \gamma e^{\gamma t} f(W_t)$$

$$= e^{\gamma t} f'(W_t) \, dW_t + \frac{\lambda}{2} e^{\gamma t} f(W_t) \, dt + \gamma e^{\gamma t} f(W_t)$$

Let γ equals to $\left(-\frac{\lambda}{2}\right)$ ~~Any γ, cannot cancel the dt~~

$$\mathbb{E} \quad d\left[e^{-\frac{\lambda}{2}t} f(W_t)\right] = dX_t = e^{-\frac{\lambda}{2}t} f'(W_t) \, dW_t$$

$$\cancel{\mathbb{E}\left[e^{\gamma t} f(W_t)\right]}$$

So $$\mathbb{E} \, e^{-\frac{\lambda}{2}t} f(W_t) = e^{-\frac{\lambda}{2} \cdot 0} f(W_0) = 1$$

$$\mathbb{E} f(W_t) = e^{\frac{\lambda}{2}t}$$

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