Fixed Income Derivative (Short Rate Model)

1. Interest Rate Model = (Marcov with respect to (r, t))

PDE: $dr = \mu(r,t)\,dt + \sigma(r,t)\,dW$

$\mu(r,t) = -\gamma(t)\,r(t) + \eta(t)$

$\sigma(r,t) = \sqrt{\alpha(t)\,r(t) - \beta(t)}$

2. Zero coupon bond: (Marcov with respect to (r, t))

No arbitrage condition
Risk-neutral measure

$$\underbrace{D(t)}_{\text{Discounted factor}}\;\underbrace{B(t,T)}_{\text{Zero coupon bond}} = \tilde{E}[\,D(T)\cdot 1\,]$$

Since ZCB is a Marcov with respect to r and t,

Let $B(t,T) = f(t,r)$

Because $D(t)f(r,t)$ is a Martingale and Marcov

$d(D(t)f(r,t))$ only contains diffusion term

3. PDE for zero coupon bond

$$d(D(t)f(r,t)) = D(t)\,df(r,t) + f(r,t)\,dD(t) + \underbrace{d[f, D]_t}_{\approx 0}$$

FV process → semi-martingale

$$= D(t)\left(\frac{\partial}{\partial t} + \mu\frac{\partial}{\partial r} + \frac{\sigma^2}{2}\frac{\partial^2}{\partial r^2}\right)f(r,t)\,dt + \sigma D(t)\frac{\partial}{\partial r}f(r,t)\,dW_t$$

$$+ \left(-R(t)D(t)f(r,t)\right)dt + 0 \qquad \text{(bears no drift)}$$

$$\Rightarrow \text{PDE: } \left(\frac{\partial}{\partial t} + \mu(r,t)\frac{\partial}{\partial r} + \frac{\sigma^2(r,t)}{2}\frac{\partial^2}{\partial r^2} - r\right)f(r,t) = 0$$

Same as Feynman-Kac Equation states

4. Solution

Assum the solution is of the form

$$f(r,t) = \exp(-r\,C(t,T) - A(t,T))$$

Substitude this expression into bond equation we obtain:

$$(-rC' - A')f + \mu(-C)f + \frac{\sigma^2}{2}C^2 f - rf = 0$$

$$\Rightarrow A' = -rC' - \mu C + \frac{\sigma^2}{2}C^2 - r$$

Differentiating with respect to r gives

$$\frac{\partial C}{\partial t} = C' = -\frac{\partial \mu}{\partial r}C + \frac{C^2}{2}\frac{\partial}{\partial r}(\sigma^2) - 1$$

$$= \gamma(t)C + \frac{\alpha(t)}{2}C^2 - 1$$

Moreover, $\frac{\partial A}{\partial t} = \eta(t)C + \frac{1}{2}\beta(t)C^2$

Boundary condition $A(T,T) = 0$ $B(T,T) = 0$