Fixed Income Derivatives (introduction)

FE-620A Pricing and Hedging

Stevens Institute of Technology

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Fixed Income Derivatives (Rates)

- There are different types of interest rates. The first distinction can be made between interbank and government rates.
- *Government* rates are usually deduces from the bonds issued by government.
- *Interbank* rates are the rates at which deposits are exchanged between banks and at which swap transactions between banks occur.
- The most important interbank rate is the LIBOR (London InterBank Offered Rate) rate, fixing daily in London. LIBOR rate is fixed for the particular tenor (1-month, 3-month, 6-month etc.) at particular time but changes on daily basis (so, in general, LIBOR rate has stochastic nature).
- Zero-Coupon Bond (ZCB). *Definition*: *A T-maturity zero-coupon bond is a contract that guarantees its holder the payment of one unit of currency at time T, with no intermediate payments*.

Fixed Income Derivatives (Zero-Coupon Bond)

- Consider interest rate $r(t)$ as a known deterministic function of time. The change in the value of ZCB, $P(t,T)$, in a time step dt (from t to $t+dt$) is $\frac{dP}{dt}dt$. Arbitrage considerations (risk neutrality) impose the equality

$$\frac{dP}{dt} = r(t)P$$

with obvious boundary condition $P(T,T) = 1$. Solution to the above equation is given by

$$P(t,T) = e^{-\int_t^T r(z)dz}$$

- By differentiating both sides by T we obtain the relation

$$r(T) = -\frac{1}{P(t,T)}\frac{\partial P(t,T)}{\partial T}$$

- If we have three time points $t \leq T \leq S$ then

$$P(t,S) = P(t,T)P(T,S)$$

Fixed Income Derivatives (Coupon-bearing Bond)

- In case Bond pays coupons $C(t)$ the differential equation becomes

$$\frac{dP}{dt} + C(t) = r(t)P$$

and the terminal condition remains the same $P(T,T) = 1$. The solution of the equation with specified terminal condition takes the form

$$P(t,T) = e^{-\int_t^T r(y)dy}\left(1 + \int_t^T C(x)e^{\int_x^T r(y)dy}dx\right)$$

- In case of discrete coupons paid at times $T_i, 0 \leq i \leq N$ we have

$$P(T_i^-, T) = P(T_i^+, T) + C(T_i)$$

and

$$\frac{dP}{dt} + \sum_{i=0}^{N} C(T_i)\delta(t - T_i) = r(t)P$$

so

$$P(t,T) = e^{-\int_t^T r(y)dy}\left(1 + \sum_{i=0}^{N} C(T_i)\mathrm{H}(t - T_i)\, e^{\int_{T_i}^T r(y)dy}\right)$$

Fixed Income Derivatives (Zero-Coupon Bond)

- Day-count convention (day-count basis). *We denote by* $\tau(t, T)$ *the chosen time measure between t and T, which is usually referred to as* year fraction *between the dates t and T.*
- There are different day-count conventions:
  - Actual/365

$$\tau(D_1, D_2) = \frac{\Delta D}{365}$$

  - Actual/360

$$\tau(D_1, D_2) = \frac{\Delta D}{360}$$

  - 30/360

$$\tau(D_1, D_2) = \frac{\Delta D^*}{360}$$

Here ΔD – is the difference in days between dates D_2 and D_1 assuming actual number of days in each month and ΔD^* - is the difference in days between dates D_2 and D_1 assuming 30 days in each month

Fixed Income Derivatives (rate compounding)

- **Continuously-compounded spot interest rate**. *The continuously-compounded spot interest rate at time t for the maturity* T *is denoted by* $R(t,T)$ *and is the constant rate at which an investment of* $P(t,T)$ *units of currency at time t accrues continuously to yield a unit amount of currency at maturity* T:

$$R(t,T) = -\frac{lnP(t,T)}{\tau(t,T)} \text{ or } P(t,T) = e^{-R(t,T)\tau(t,T)}$$

- **Simply-compounded spot interest rate**. *The simply compounded spot interest rate at time t for the maturity* T *is denoted by* $L(t,T)$ *and is the constant rate at which an investment has to be made to produce an amount of one unit of currency at maturity, starting from* $P(t,T)$ *units of currency at time t, when accruing occurs proportionally to the investment time:*

$$L(t,T) = \frac{1-P(t,T)}{\tau(t,T)P(t,T)} \text{ or } P(t,T) = \frac{1}{1+L(t,T)\tau(t,T)}$$

Fixed Income Derivatives (rate compounding)

- **Annually-compounded spot interest rate**. *The annually-compounded spot interest rate at time t for the maturity T is denoted by $Y(t,T)$ and is the constant rate at which an investment has to be made to produce an amount of one unit of currency at maturity, starting from $P(t,T)$ units of currency at time t, when reinvesting the obtained amounts once a year:*

$$Y(t,T) = \frac{1}{[P(t,T)]^{1/\tau(t,T)}} - 1 \text{ or } P(t,T) = \frac{1}{(1+Y(t,T))^{\tau(t,T)}}$$

When reinvestment happens k time a year then

$$Y(k,t,T) = \frac{k}{[P(t,T)]^{1/(k\tau(t,T))}} - k \text{ and } P(t,T) = \frac{1}{\left(1+\frac{Y(k,t,T)}{k}\right)^{k\tau(t,T)}}$$

When $k \to \infty$, k-times-per-year compounded rates converge to the continuously compounded rates:

$$\lim_{k\to\infty} \frac{k}{[P(t,T)]^{1/(k\tau(t,T))}} - k = -\frac{ln(P(t,T))}{\tau(t,T)} = R(t,T)$$

(Prove this!)

Fixed Income Derivatives (Forward Rates)

- *Forward rates.* Forward rates are characterized by three time instants:
  - time t at which the rate is considered;
  - time T at which the parties are entered into the contract (fixing time);
  - time S – contract maturity;

  with $\text{Now} \equiv 0 \leq t \leq T \leq S$.
- Forward rates are interest rates that can be locked in today $(t = 0)$ for an investment in the future time period $[T, S]$, and are set consistently with the current term structure of discount factors.
- Example: Forward Rate Agreement (FRA) gives its holder an interest rate payment for the period $[T, S]$. At the maturity S, fixed payment based on the fixed rate K (strike) is exchanged against floating payment based on the spot rate $L(T, S)$ resetting at T with maturity S. At time S one receives $\tau(T, S)K$ units of currency and pays the amount $\tau(T, S)L(T, S)$. The value of the contract is $\tau(T, S)(K - L(T, S))$. Substituting expression for $L(T, S)$ we get $\tau(T, S)K - \frac{1}{P(T,S)} + 1$. Value of the contract at time t is
  $$P(t, S)\left(\tau(T, S)K - \frac{1}{P(T,S)} + 1\right) = P(t, S)\tau(T, S)K - P(t, T) + P(t, S)$$

Fixed Income Derivatives (Forward Rates)

- From the example above, there exists only one value for the strike K that values FRA contract at zero (fair contract) at time t:

$$F(t;T,S) = \frac{1}{\tau(T,S)}\left(\frac{P(t,T)}{P(t,S)} - 1\right)$$

- Simply-compounded Forward Interest Rate.

- The value of the FRA contract can be rewritten as

$$FRA(t,T,S,K) = P(t,S)\tau(T,S)\big(K - F(t;T,S)\big)$$

- In the limit $S \to T^+$ we obtain the <u>*Instantaneous Forward Rate*</u> $f(t,T)$*:*

$$f(t,T) \equiv \lim_{S\to T^+} F(t;T,S) = \lim_{S\to T^+} -\frac{1}{P(t,S)}\frac{P(t,S) - P(t,T)}{S - T} = -\frac{1}{P(t,T)}\frac{\partial P(t,T)}{\partial T}$$

Or

$$f(t,T) = -\frac{\partial lnP(t,T)}{\partial T}$$

and

$$P(t,T) = exp\left(-\int_t^T f(t,z)dz\right)$$

Fixed Income Derivatives (IR Swaps)

- *Interest Rate Swap (IRS)* is the generalization of a FRA: a contract that exchanges payments between two legs (fixed and floating) starting from the future time instants $T_i, 0 \leq i \leq N$. Fixed leg pays the amount $\tau_i K$, and floating leg pays the amount $\tau_i L(T_{i-1}, T_i)$, where $\tau_i = T_i - T_{i-1}$. When the fixed leg is paid and floating leg is received the swap is called *Payer IRS (PIRS)*, in the other case – *Receiver IRS (RIRS)*.
- Discounted payoff at time $t < T_0$ of PIRS is

$$\sum_{i=1}^{N} D(t, T_i)\tau_i(L(T_{i-1}, T_i) - K)$$

whereas the same of RIRS is

$$\sum_{i=1}^{N} D(t, T_i)\tau_i(K - L(T_{i-1}, T_i))$$

In terms of FRAs

$$RIRS = \sum_{i=1}^{N} \tau_i P(t, T_i)(K - F(t; T_{i-1}, T_i)) = -P(t, T_0) + P(t, T_N) + K \sum_{i=1}^{N} \tau_i P(t, T_i)$$

Fixed Income Derivatives (IR Swaps)

- The two legs of IRS can be interpreted as two separate contracts:
  - Fixed leg represents a coupon-bearing bond;
  - Floating leg can be thought of a floating-rate note.
- Recall the definition of coupon-bearing bond (CB): it is a contract that ensures payments at future times $T_i, 0 \leq i \leq N$ of the deterministic amounts of currency (cash-flows) $C(T_i) = \tau_i K, 0 \leq i \leq N-1, C(T_N) = \tau_N K + 1$.
- The current value of the bond is

$$CB(t) = \sum_{i=0}^{N} C(T_i) P(t, T_i)$$

- *Floating-rate Note* – is a contract ensuring the payment at future times $T_i, 1 \leq i \leq N$ of the LIBOR rates that reset at the previous instants T_{i-1}. The note pays the last cash-flow consisting of the notional value of the note at final time T_N.
- Value of the note is obtained by changing sign to the above value of RIRS with $K = 0$ (no fixed leg) and by adding to it the present value of the notional at time T_N. Thus

$$-RIRS + P(t, T_N) = P(t, T_0)$$

Fixed Income Derivatives (IR Swaps)

- Definition: The *<u>Forward Swap Rate</u>* $S(t)$ at time t for the set of times T_i and year fractions $\tau_i, 1 \leq i \leq N$ is the rate in the fixed leg of the above IRS that makes the IRS a fair contract at the present time, i.e. it is the fixed rate K for which $RIRS = 0$.
- We easily obtain

$$S(t) = \frac{P(t,T_0) - P(t,T_N)}{\sum_{i=1}^{N} \tau_i P(t,T_i)}$$

- Using the relations

$$\frac{P(t,T_i)}{P(t,T_0)} = \prod_{j=1}^{i} \frac{P(t,T_j)}{P(t,T_{j-1})} = \prod_{j=1}^{i} \frac{1}{1 + \tau_j F_j(t)}, i > 0$$

Forward swap rate $S(t)$ can be written in terms of forward rates

$$S(t) = \frac{1 - \prod_{j=1}^{N} \left(1 + \tau_j F_j(t)\right)^{-1}}{\sum_{i=1}^{N} \tau_i \prod_{j=1}^{i} \left(1 + \tau_j F_j(t)\right)^{-1}}$$

Fixed Income Derivatives (stochastic interest rates)

- The theory of interest-rate modeling was originally based on the assumption of specific one-dimensional dynamics for the instantaneous spot rate process $r(\mathrm{t})$. Modeling directly such dynamics is very convenient since all fundamental quantities (rates and bonds) are readily defined, by no-arbitrage arguments, as the expectation of a functional of the process $r(\mathrm{t})$.
- The existence of a risk-neutral measure implies that the arbitrage-free price at time t of a contingent claim with payoff Π_T at time T is given by
$$E_t\{D(t,T)\Pi_T\} = E_t\left\{e^{-\int_t^T r(y)dy}\Pi_T\right\}$$
- The ZCB price at time t for the maturity T is characterized by the unit amount of currency at time T, so that $\Pi_T = 1$ and we obtain
$$P(t,T) = E_t\left\{e^{-\int_t^T r(y)dy}\right\}$$
- From bond prices all kind of rates are available, so that indeed the whole zero-coupon curve is characterized in terms of distributional properties of $r(\mathrm{t})$.

Fixed Income Derivatives (short rate models)

- In the same way we dealt with the model for the asset price as a log-normal random walk, let us suppose that the interest rate $r(t)$ is driven by a stochastic differential equation of the form

$$dr = \mu(r,t)dt + \sigma(r,t)dW_t$$

- Pricing a bond is technically harder that pricing an option, because there is no underlying asset to hedge with: one cannot ‘buy’ an interest rate on the Market. The solution would be to hedge with bonds of different maturities.
- We setup a portfolio Π of two bonds with maturities T_1 and T_2. The corresponding prices are P_1 and P_2:

$$\Pi = P_1 - \Delta P_2$$

- For the change in the portfolio over time dt, by using Ito’s lemma we have

$$d\Pi = \frac{\partial P_1}{\partial t}dt + \frac{\partial P_1}{\partial r}dr + \frac{1}{2}\sigma^2\frac{\partial^2 P_1}{\partial r^2}dt - \Delta\left(\frac{\partial P_2}{\partial t}dt + \frac{\partial P_2}{\partial r}dr + \frac{1}{2}\sigma^2\frac{\partial^2 P_2}{\partial r^2}\right)$$

The choice of

$$\Delta = \frac{\partial P_1}{\partial r} / \frac{\partial P_2}{\partial r}$$

Fixed Income Derivatives (short rate models)

eliminates the random component in $d\Pi$ so we get

$$d\Pi = \left(\frac{\partial P_1}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 P_1}{\partial r^2} - \frac{\partial P_1/\partial r}{\partial P_2/\partial r}\left(\frac{\partial P_2}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 P_2}{\partial r^2}\right)\right) dt = r\left(P_1 - \frac{\partial P_1/\partial r}{\partial P_2/\partial r} P_2\right) dt$$

$$= r\Pi \mathrm{dt}$$

Combining terms with P_1 and P_2 we find that

$$\frac{\left(\frac{\partial P_1}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 P_1}{\partial r^2} - rP_1\right)}{\frac{\partial P_1}{\partial r}} = \frac{\left(\frac{\partial P_2}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 P_2}{\partial r^2} - rP_2\right)}{\frac{\partial P_2}{\partial r}}$$

The left-hand side of the equation is a function of T_1 only and the right-hand side is the function of T_2 only. This is possible when both sides do not depend on maturity. Thus,

$$\frac{\left(\frac{\partial P_i}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 P_i}{\partial r^2} - rP_i\right)}{\frac{\partial P_i}{\partial r}} = a(r,t), i = 1,2$$

where $a(r,t)$ - is some function.

Fixed Income Derivatives (short rate models)

- We choose the following representation for the function $a(r,t)$:

$$a(r,t) = \sigma(r,t)\lambda(r,t) - \mu(r,t)$$

where $\lambda(r,t)$ - (Market Price of Risk).

- As in the case of Heston model, function $\lambda(r,t)$ can take various forms depending on Market conditions and model tolerance to those conditions. For example, one of the choices is $\lambda(r,t) = \lambda\sqrt{r(t)}$.
- The ZCB pricing equation is therefore

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 P}{\partial r^2} + (\mu - \lambda\sigma)\frac{\partial P}{\partial r} - rP = 0$$

with terminal condition $P(r,T) = 1$. Boundary conditions depend on the concrete expressions for $\mu(r,t)$ and $\sigma(r,t)$ and will be discussed later.

- Assume that $\mu(r,t)$ and $\sigma(r,t)$ have the form

$$\sigma(r,t) = \sqrt{\alpha(t)r(t) - \beta(t)}$$

$$\mu(r,t) = -\gamma(t)r(t) + \eta(t) + \lambda(r,t)\sqrt{\alpha(t)r(t) - \beta(t)}$$

- This choice allows us to ensure that
  - We can avoid negative rates;
  - We can make the short rate mean reverting;
  - We can impose boundaries for the dynamics of the rate.

Fixed Income Derivatives (short rate models)

- There are many interest rate models that fit into the developed framework:
  - Vasicek model ($\alpha(r,t) = 0,$ no time dependence in other parameters)
  - Cox, Ingersoll & Ross model (CIR) ($\beta(r,t) = 0$, no time dependence in other parameters)
  - Hull & White model (HW) (either $\alpha(r,t) = 0$ or $\beta(r,t) = 0$ but all other parameters are time dependent).
- Boundary conditions are $P(r,t) \to 0$ as $r \to \infty$ and while $r = \beta/\alpha$, $P(r,t)$ remains finite.
- Under these assumptions the solution $P(r,t)$ takes a simple special form:
$$P(r,t) = e^{A(t,T) - r(t)B(t,T)}$$
- After substituting this expression into bond equation we obtain
$$\frac{\partial A}{\partial t} - r\frac{\partial B}{\partial t} + \frac{1}{2}\sigma^2 B^2 - (\mu - \lambda\sigma)B - r = 0$$

Differentiating with respect to r gives
$$-\frac{\partial B}{\partial t} + \frac{1}{2}B^2\frac{\partial \sigma^2}{\partial r} - B\frac{\partial(\mu - \lambda\sigma)}{\partial r} = 1$$

Fixed Income Derivatives (short rate models)

- Differentiating one more time and then dividing by B we get

$$\frac{1}{2}B\frac{\partial^2\sigma^2}{\partial r^2} - \frac{\partial^2(\mu - \lambda\sigma)}{\partial r^2} = 0$$

B is a function of T, so we must have $\frac{\partial^2\sigma^2}{\partial r^2} = 0$ and therefore $\frac{\partial^2(\mu-\lambda\sigma)}{\partial r^2} = 0$ which is consistent with the choice of $\mu(r,t)$ and $\sigma(r,t)$.

- Substituting the expressions for $\mu(r,t)$ and $\sigma(r,t)$ into the differential equations for $A(t,T)$ and $\mathrm{B}(t;T)$ yields:

$$\frac{\partial A}{\partial t} = \eta(t)B + \frac{1}{2}\beta(t)B^2$$

and

$$\frac{\partial B}{\partial t} = \frac{1}{2}\alpha(t)B^2 + \gamma(t)B - 1$$

with terminal conditions $A(T;T) = 0$ and $\mathrm{B}(T;T) = 0$.

Fixed Income Derivatives (short rate models)

- Fitting parameters. The general stochastic process developed above involves four time-dependent parameters, $\alpha(t), \beta(t), \gamma(t)$ and $\eta(t)$. If these parameters are assumed to be constant then explicit forms for $A(t;T), B(t;T)$ and hence the bond prices are easily obtained. However, in reality the market's expectations about future interest rates are time-varying.
- The fitting is performed to the historical data of the rate and the Yield Curve.
- Yield Curve Y in terms of ZCB price is defined as

$$Y(t;T) = -\frac{\ln\big(P(r,t)\big)}{T-t} = -\frac{A - rB}{T-t}$$

- For the simplicity we assume that α, β, γ are constant and η is time dependent.
- When $T - t \to 0$ function $Y(t;T)$ can be expanded into the Taylor series

$$Y \sim r - \frac{1}{2}(T-t)\big(\gamma r - \eta(0)\big) + \cdots$$

Fixed Income Derivatives (short rate models)

- Fitting process boils down to the following:
  - Based on historical data of, say, 10-year bond we find the lower bound of the rate which in terms of our parameters is β/α;
  - The spot rate volatility is $\sqrt{\alpha r - \beta}$ - so we find α and β;
  - The slope of the Yield Curve at $T = t$ is given by
$$s = \frac{1}{2}(\eta(0) - \gamma r)$$
    from which it follows that
$$ds = -\frac{1}{2}\gamma dr$$
    So the analysis of correlation between YC slope and the rate gives the value for γ;
  - Having the Yield Curve, $Y^*(t^*; T)$, from the market data at time t^* we can find $\eta(t)$ such that $Y(t; T)$ would fit the whole current market's Yield Curve.

Fixed Income Derivatives (short rate models)

- Example: Assume $\alpha(t) = 0; \lambda(t) = 0; -\beta(t) = \sigma^2 = const; \gamma(t) = \gamma = const$ and let's see how the function $\eta(t)$ can be fitted to match the whole Yield Curve $Y(t^*, T)$ at particular time t^*. Solving the equation
$$\frac{\partial B}{\partial t} = \gamma B - 1$$

with terminal condition $B(T,T) = 0$ we obtain $B(t,T) = \frac{1}{\gamma}\left(1 - e^{-\gamma(T-t)}\right)$ and

$$A(t,T) = -\frac{1}{\gamma}\int_t^T \eta(x)\left(1 - e^{-\gamma(T-x)}\right)dx + \frac{\sigma^2}{2\gamma^2}\left[T - t - \frac{2}{\gamma}\left(1 - e^{-\gamma(T-t)}\right) + \frac{1}{2\gamma}\left(1 - e^{-2\gamma(T-t)}\right)\right]$$

At particular time t^* function $A(t^*, T)$ can be expressed in terms of $Y(t^*, T)$ as

$$A(t^*, T) = -Y(t^*, T) + r(t^*)B(t^*, T)$$

Suppose we also have calibrated values for γ and σ then the equation for $\eta(t)$ becomes

$$\int_{t^*}^T \eta(x)\left(1 - e^{-\gamma(T-x)}\right)dx$$
$$= \gamma\left(Y(t^*, T) + r(t^*)B(t^*, T)\right) + \frac{\sigma^2}{2\gamma}\left[T - t^* - \frac{2}{\gamma}\left(1 - e^{-\gamma(T-t^*)}\right) + \frac{1}{2\gamma}\left(1 - e^{-2\gamma(T-t^*)}\right)\right]$$

Fixed Income Derivatives (short rate models)

We rewrite this equation as

$$\int_{t^*}^{T} \eta(x)\left(1 - e^{-\gamma(T-x)}\right)dx = F(T)$$

where

$F(T) = \gamma\big(Y(t^*, T) + r(t^*)B(t^*, T)\big) + \frac{\sigma^2}{2\gamma}\left[T - t^* - \frac{2}{\gamma}\left(1 - e^{-\gamma(T-t^*)}\right) + \frac{1}{2\gamma}\left(1 - e^{-2\gamma(T-t^*)}\right)\right]$ - is known function. As functions of T we differentiate both sides of equation by T

$$\gamma \int_{t^*}^{T} \eta(x)\, e^{-\gamma(T-x)}dx = F'\,(T)$$

The second differentiation gives

$$\eta(T) - \gamma \int_{t^*}^{T} \eta(x)\, e^{-\gamma(T-x)}dx = F''\,(T)/\gamma$$

After adding these two we find $\eta(T) = \frac{1}{\gamma}F''\,(T) + F'\,(T)$. We can now find the function $A(t, T)$ and ZCB price $P(t, T)$.

Fixed Income Derivatives (short rate models)

- SDE for the short rate $r(t)$

$$dr(t) = [\eta(t) - \gamma r(t)]dt + \sigma dW_t$$

can be integrated to yield

$$r(t) = r(s)e^{-\gamma(t-s)} + \int_s^t e^{-\gamma(t-x)}\eta(x)\,dx + \sigma\int_s^t e^{-\gamma(t-x)}\,dW_x$$

Therefore, $r(t)$ is normally distributed with mean

$$E[r(t)|\mathcal{F}_s] = r(s)e^{-\gamma(t-s)} + \int_s^t e^{-\gamma(t-x)}\eta(x)\,dx$$

and variance

$$Var[r(t)|\mathcal{F}_s] = \frac{\sigma^2}{2\gamma}\left[1 - e^{-2\gamma(t-s)}\right]$$

Fixed Income Derivatives (Caps/Floors and Swaptions)

- A ***Cap*** is a contract that can be viewed as a payer IRS where each exchange payment is executed only if it has positive value. The Cap discounted payoff is given by

$$\sum_{i=1}^{N} D(t,T_i)\tau_i(L(T_{i-1},T_i) - K)^+$$

Each term under the sum is called ***Caplet***

- A ***Floor*** is a contract that can be viewed as a receiver IRS where each exchange payment is executed only if it has positive value. The Floor discounted payoff is given by

$$\sum_{i=1}^{N} D(t,T_i)\tau_i\big(K - L(T_{i-1},T_i)\big)^+$$

Each term under the sum is called ***Floorlet***

Fixed Income Derivatives (Caps/Floors and Swaptions)

- Caps/Floors can be priced using Black-Scholes formula:

$$Cap(K) = \sum_{i=1}^{N} P(0, T_i)\tau_i B_c(K, F(0, T_{i-1}, T_i), \nu_i\ )$$

where

$$B_c(K, F(0, T_{i-1}, T_i), \nu_i\ ) = F\Phi(d_1(K, F_i, \nu_i)) - K\Phi(d_2(K, F_i, \nu_i))$$

$$d_{1,2}(K, F, \nu) = \frac{\ln(\frac{F}{K}) \pm \nu^2/2}{\nu}, \nu_i^2 = \frac{\sigma^2}{2\gamma}[1 - e^{-2\gamma\tau_i}]B^2\ (T_{i-1}, T_i); F_i = \frac{P(0, T_i)}{P(0, T_{i-1})}$$

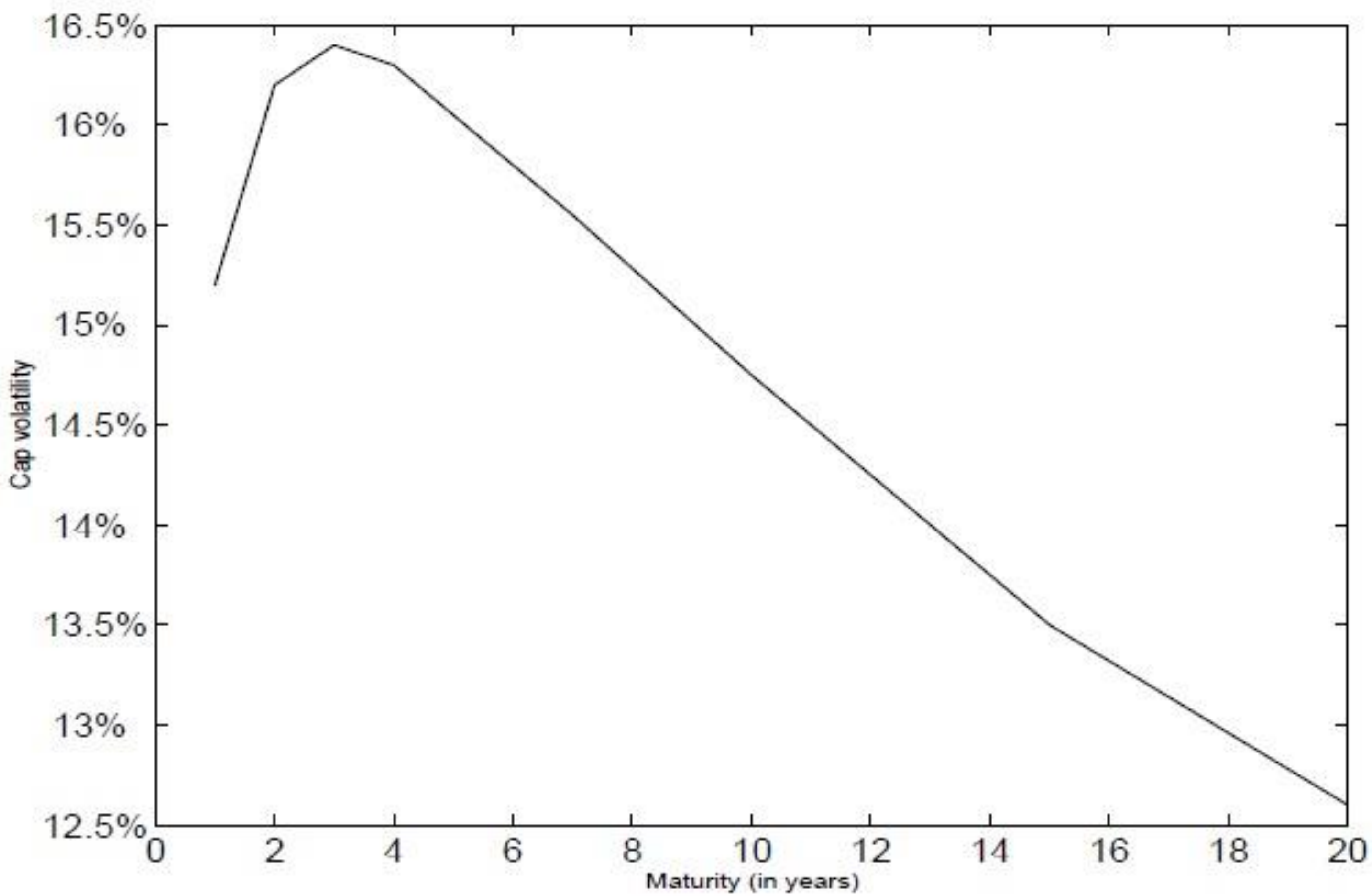
$$Floor(K) = \sum_{i=1}^{N} P(0, T_i)\tau_i B_f(K, F(0, T_{i-1}, T_i), \nu_i\ )$$

$$B_f(K, F(0, T_{i-1}, T_i), \nu_i\ ) = -\mathrm{F}\Phi(-d_1(K, F_i, \nu_i)) + K\Phi(-d_2(K, F_i, \nu_i))$$

- Cap is said to be at-the-money if and only if

$$K = S(0) = \frac{P(0, T_0) - P(0, T_N)}{\sum_{i=1}^{N} \tau_i P(0, T_i)}$$

Fixed Income Derivatives (Caps/Floors and Swaptions)



Fixed Income Derivatives (Caps/Floors and Swaptions)

- A European ***Payer Swaption*** is an option giving the right (and no obligation) to enter a payer IRS at a given future time (swaption maturity).
- Usually the swaption maturity coincides with the first reset date of the underlying IRS.
- The underlying IRS length T_N is called the *tenor* of the swaption.
- Swaption payoff:

$$PayerSwaption = \left(\sum_{i=1}^{N} P(0, T_i) \tau_i \left(F(T_{i-1}, T_i) - K \right) \right)^+$$

- Contrary to the Cap case, this payoff cannot be decomposed in more elementary products and this is a fundamental difference between the two derivatives.
- Swaption can be viewed as an option on a basket of forwards.
- Black-Scholes formula can also be used to price swaptions.

Fixed Income Derivatives (pricing)

- We can use our interest rate model to price swaps, caps/floors.
- *Swaps*: suppose party A pays the interest on an mount N to party B at a fixed rate r^* and B pays interest to A at the floating rate r. These payments continue until time T. Let's denote the value of the swap to A by $NS(r,t)$.
- At time step dt party A receives $(r - r^*)Ndt$. This is similar to the coupon payment on a simple bond so we find that

$$\frac{\partial S}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 S}{\partial r^2} + (\mu - \lambda\sigma)\frac{\partial S}{\partial r} - rS + (r - r^*) = 0$$

with terminal condition $S(r,T) = 0$.

- *Caps*: cap is a loan at the floating interest rate but the interest rate charged is guarantied not to exceed K. The loan of N is to be paid back at time T. It's straight forward that the value of a capped loan, $NC(r,t)$, satisfies

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 C}{\partial r^2} + (\mu - \lambda\sigma)\frac{\partial C}{\partial r} - rC + min(r,K) = 0$$

with terminal condition $C(r,T) = 1$.

Fixed Income Derivatives (pricing)

- *<u>Floors</u>*: is similar to the cap except that the interest rate does not go below K. So the pricing equation takes the form

$$\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial r^2} + (\mu - \lambda\sigma)\frac{\partial F}{\partial r} - rF + max(r, K) = 0$$

with terminal condition $F(r, T) = 1$.

- Having valued swaps, caps and floors it's easy to value options on these instruments (swaptions, captions and floortions).
- Suppose that our swap (cap, floor) which expires at time T has value $S(r, t)$. An option to buy this swap (call swaption) for an amount K at time $t < T$ has value $\hat{S}(r, t)$ where

$$\frac{\partial \hat{S}}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 \hat{S}}{\partial r^2} + (\mu - \lambda\sigma)\frac{\partial \hat{S}}{\partial r} - r\hat{S} = 0$$

with

$$\hat{S}(r, t) = max(S(r, t) - K, 0)$$

Thus we solve for the value of the swap first and than use this value as the terminal condition for the swaption.

- Captions and floortions are treated similarly.

Fixed Income Derivatives (volatility cube)

- Market implied volatilities are organized by:
  - Option maturity
  - Tenor of the underlying instrument
  - Strike of the option
- These three-dimensional objects are called *volatility cubes*.
- Market quotes ATM swaption volatilities for certain standard maturities and underlyings

mat\tenor	0.25	1	2	3	4	5	7	10	15	20
0.25	6.7	13.3	15.5	15.7	15.6	15.5	15.0	14.2	13.5	13.1
0.5	11.9	14.8	16.2	16.2	16.1	15.9	15.3	14.5	13.8	13.3
1	16.7	17.1	17.2	17.0	16.8	16.6	16.0	15.2	14.4	13.9
2	18.5	18.2	17.90	17.7	17.4	17.2	16.7	15.9	15.0	14.5
3	18.9	18.4	18.2	18.0	17.7	17.5	17.0	16.3	15.3	14.8
4	18.9	18.3	18.1	17.9	17.6	17.5	16.9	16.2	15.2	14.7
5	18.8	18.1	17.9	17.6	17.4	17.3	16.7	16.0	15.0	14.5
7	18.0	17.4	17.1	16.8	16.6	16.4	15.9	15.3	14.2	13.8
10	16.2	16.1	15.8	15.6	15.4	15.2	14.8	14.2	13.0	12.6