FE-620A Pricing and Hedging
Hedging

STEVENS INSTITUTE OF TECHNOLOGY

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Hedging

- The main purpose of hedging is to minimize the ***Risk.***
- ***Black-Scholes option price formula*** eliminates the stochasticity of the underlying asset in the “static” market. In reality the market is dynamic: it evolves in time. As a result – option price is a function not just the underlying asset price and time but also a function of other market parameters: interest rate, volatility: $V(S, r, \sigma, t)$
- Sources of risk in dynamic market:
  - Underlying asset price;
  - Volatility;
  - Interest rate;
- We’ve derived Black-Scholes Greeks to account for the risks caused by dynamic changes in the market:

Hedging

- Black-Scholes Delta:

$$\Delta = \frac{\partial V(S,t)}{\partial S}$$

- Vega

$$\Upsilon = \frac{\partial V(S,t)}{\partial \sigma}$$

- Rho

$$\rho = \frac{\partial V(S,t)}{\partial r}$$

- Theta

$$\Theta = \frac{\partial V(S,t)}{\partial t}$$

Hedging

- Greeks are also subject to dynamic changes in the market so it'd make sense to consider *second* order factors (second/mixed derivatives with respect to the market parameters)
- The most "non-linear" ones are underlying asset price and volatility.
- In addition to the first order Delta one considers also Gamma

$$\Gamma = \frac{\partial^2 V(S,t)}{\partial S^2}$$

- Vanna

$$Vanna = \frac{\partial \Upsilon(S,t)}{\partial S} = \frac{\partial^2 V(S,t)}{\partial \sigma \partial S}$$

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- Volga

$$Volga = \frac{\partial \Upsilon(S,t)}{\partial \sigma} = \frac{\partial^2 V(S,t)}{\partial \sigma^2}$$

- When market "moves" the change in portfolio $\Delta\Pi$ is

$$\Delta\Pi = \frac{\partial \Pi}{\partial S}\Delta S + \frac{\partial \Pi}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 \Pi}{\partial S^2}(\Delta S)^2 + \frac{\partial \Pi}{\partial r}\Delta r + \frac{\partial \Pi}{\partial \sigma}\Delta \sigma + \cdots$$

- Profit-and-Loss (P&L) is the difference between left hand and right hand sides. Ideally it should be made zero but in practice it's not possible to achieve
- We restrict P&L to the following formula

Hedging

$$P\&L = \frac{\partial V}{\partial S}\Delta S + \frac{\partial V}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(\Delta S)^2 + \frac{\partial V}{\partial r}\Delta r + \frac{\partial V}{\partial \sigma}\Delta\sigma + Residual\ P\&L$$

- Or verbally

*P&L = Delta * Spot Variation + Theta * Time Variation + ½ * Gamma * (Spot Variation)^2 + Rho * Rate Variation + Vega * Volatility Variation + Residual P&L*

- *Residual P&L* is also called “Unexplained P&L” and accounts for the higher order Greeks and anything that cannot be explained by assumed approach

Implied Volatility

- Implied volatility is the Black-Scholes volatility σ such that the corresponding Black-Scholes option price equals to the market option price with the same expiry
- Properties of the Implied Volatility:
  - In general (in most of the cases) it's different from the realized volatility of the underlying asset
  - Can be calculated for any option type
  - In reality is not a constant but depends on the strike (volatility "skew"/"smile") and expiry (volatility term structure): $\sigma_{imp}(K, T)$
  - When considered in dynamic markets, implied volatility is also a function of asset price S and time t: $\sigma_{imp}(S, t, K, T)$
  - Implied volatility doesn't exist in real world; this is just a "wrong number that you put into wrong equation to obtain the correct answer"
  - Implied volatility is the alternative "language" to talk about options and prices

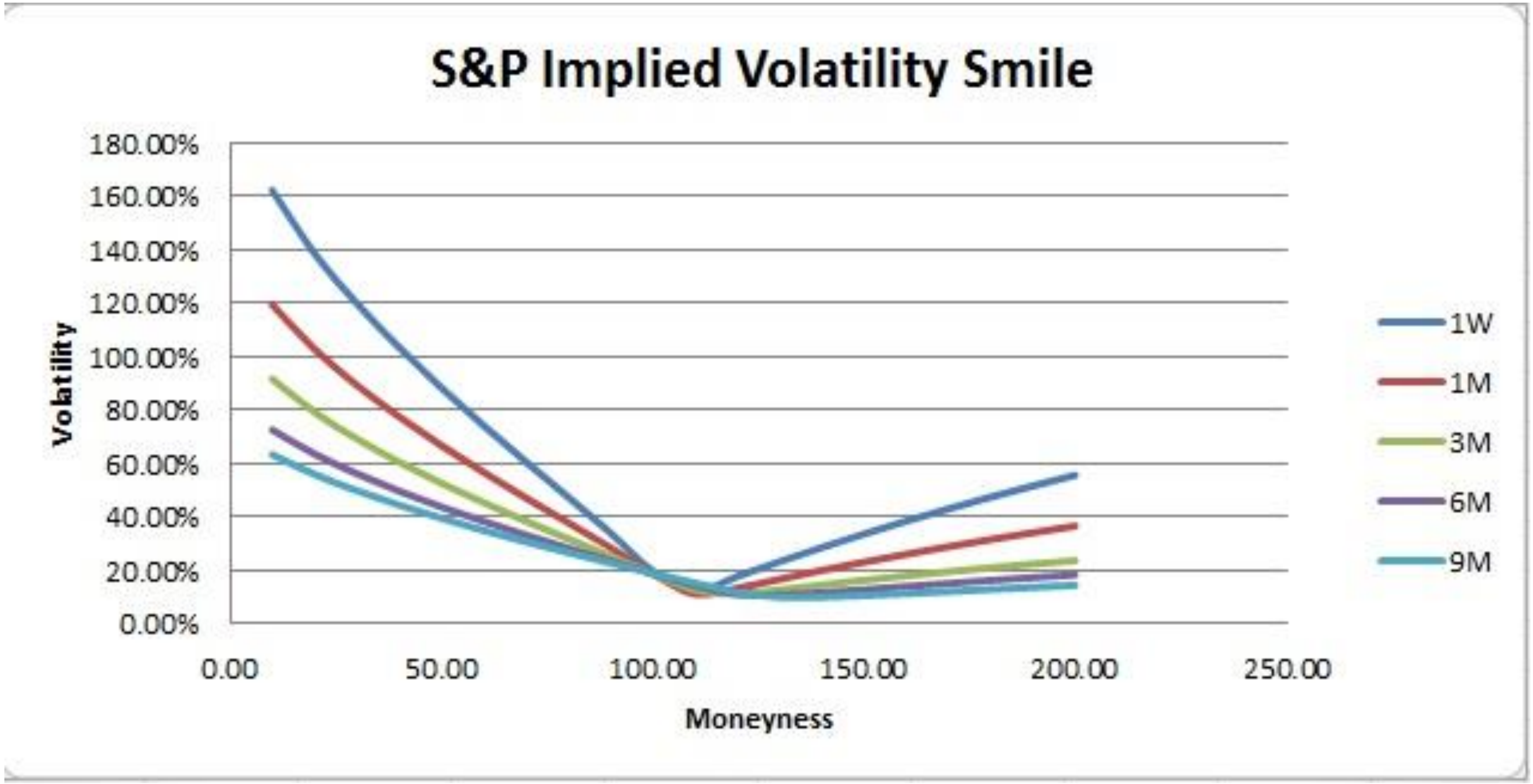
Implied volatility - Calculation

- Implied volatility is the solution of the inverse problem: given the strike K, expiry T, option price $V(K,T)$, what is the volatility σ_{BS} that equates $V(K,T)$ and $V_{BS}(\sigma_{BS}, K, T)$?
- There is no an explicit (analytical) solution to this problem so numerical calculation based on optimization is used:

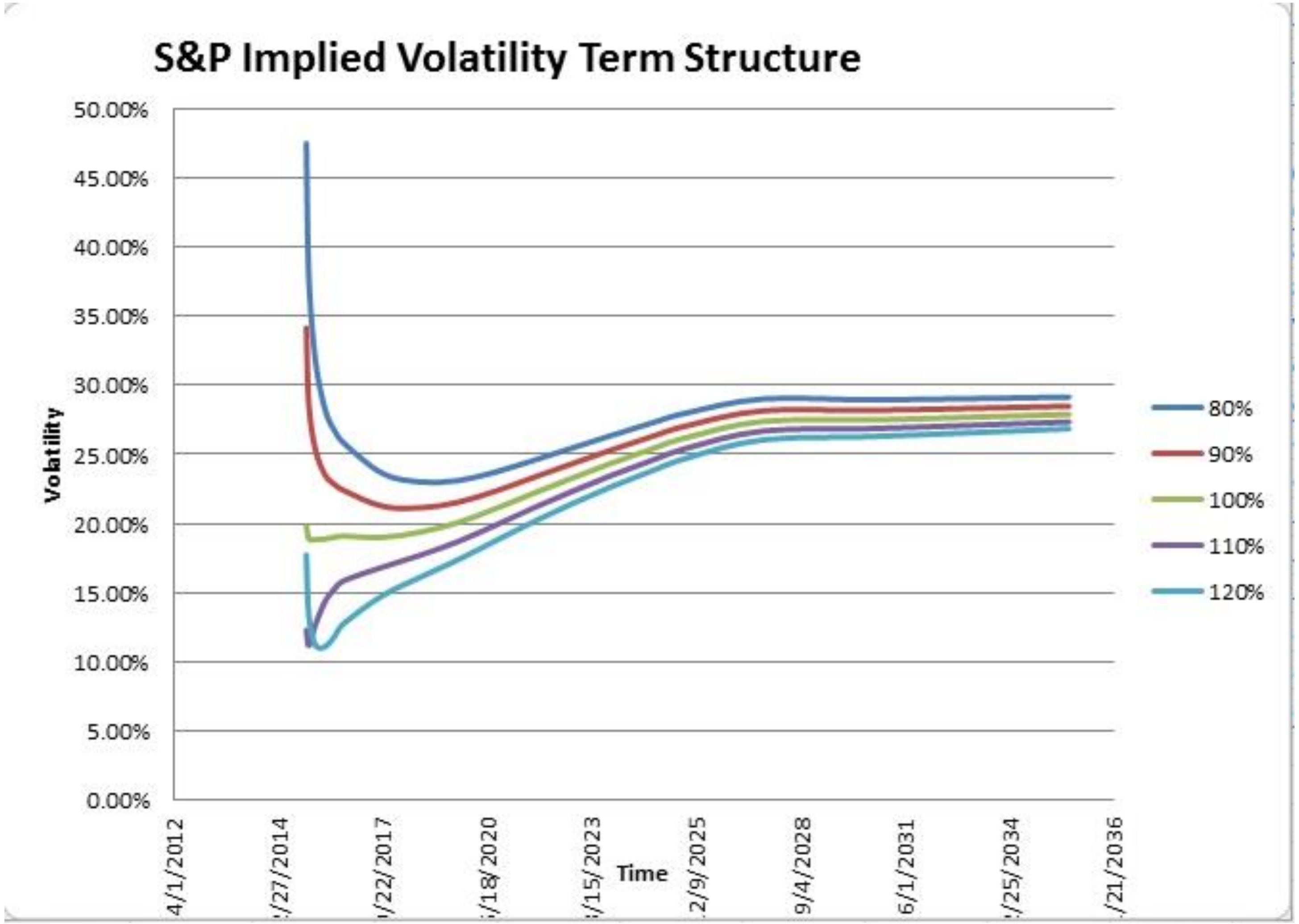
$$\min_{\sigma} |V(K,T) - V_{BS}(\sigma, K, T)|$$

- By definition, σ_{BS} is constant, so implied volatility is seen as piece-wise constant approximation of the volatility in the neighborhood of strikes $K_i, 1 \leq i \leq N$ and expiries $T_j, 1 \leq j \leq M$ (*Implied Volatility Grid*). If we connect obtained points by smooth lines in strike and expiry dimensions, - we will have an *Implied Volatility Surface.*

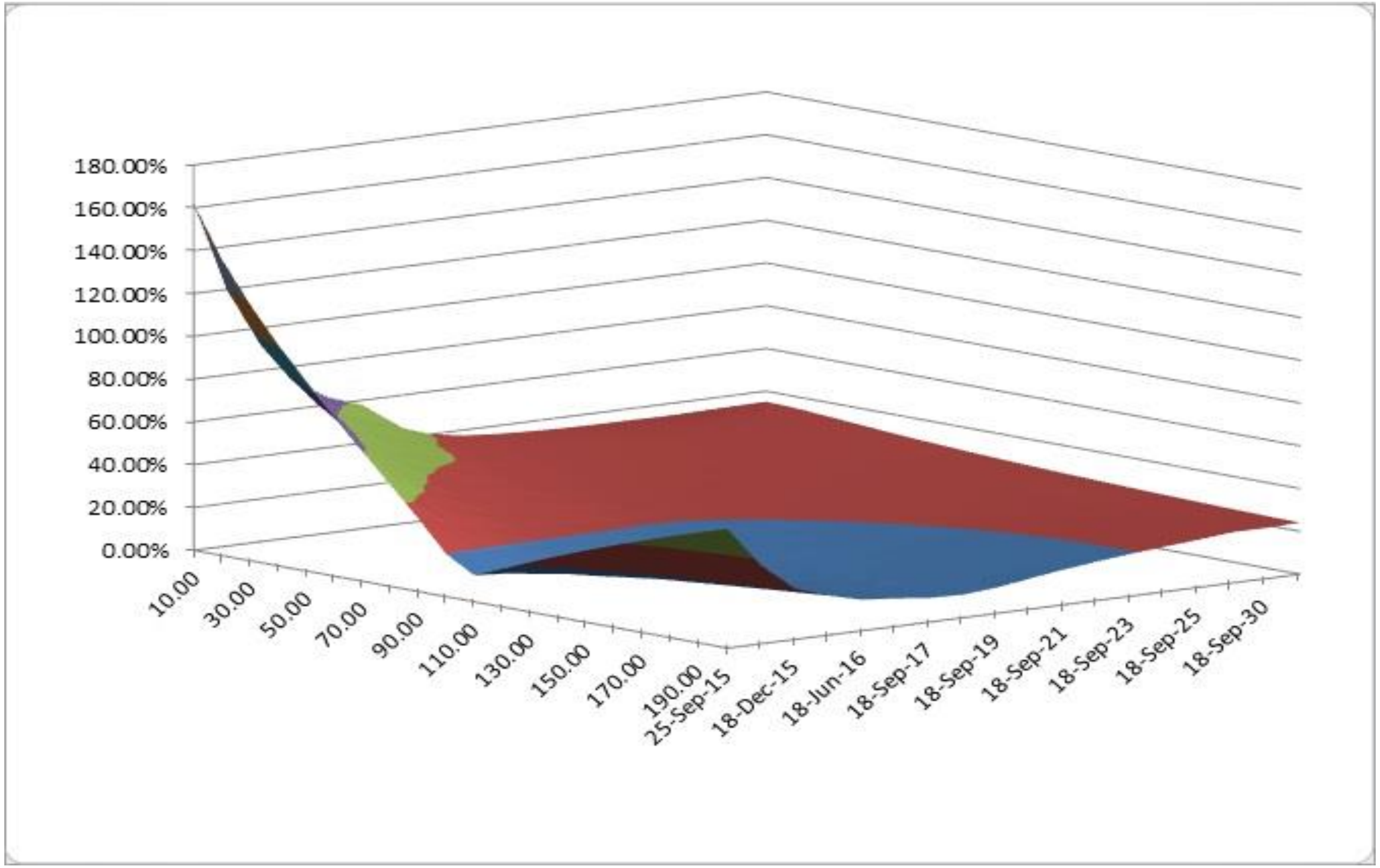
Implied Volatility - Example



Implied Volatility - Example



Implied Volatility - Example



Implied Volatility - Parameterization

- What would happen if we wanted to price an option with the same expiry T but with different strike K? Suppose, $K_1 \leq K \leq K_2$, where K_1 and K_2 belong to the IV Grid. We'd need to interpolate the implied volatility between values σ_{K_1} and σ_{K_2}. It is more convenient to come up with the *functional representation* of the dependence $\sigma(K)$ assuming it is continuous. Another words, we need to *parameterize* the function $\sigma(K)$.
- Requirements:
  - Parameterization must be flexible enough to reflect the dynamics of the implied volatility in changing markets;
  - Number of the parameters shouldn't be big;
  - Parameterization shouldn't violate no-arbitrage condition.

Implied Volatility - Parameterization

- There is a number of different parameterizations suggested. Most of them – heuristic (do not reflect the true nature of the implied volatility)
- One of them is Jim Gatheral's SVI (Stochastic Volatility Inspired) parameterization:

$$\text{Var}(K;\, a, b, \sigma, \rho, m) = a + b\left\{\rho(K-m) + \sqrt{(K-m)^2+\sigma^2}\right\}$$

- Properties:
  - It's a hyperbola with
    - ➢ Left asymptote $\text{Var}_L(K;\, a, b, \sigma, \rho, m) = a - b\,(1-\rho)(K-m)$
    - ➢ Right asymptote $\text{Var}_R(K;\, a, b, \sigma, \rho, m) = a + b\,(1+\rho)(K-m)$

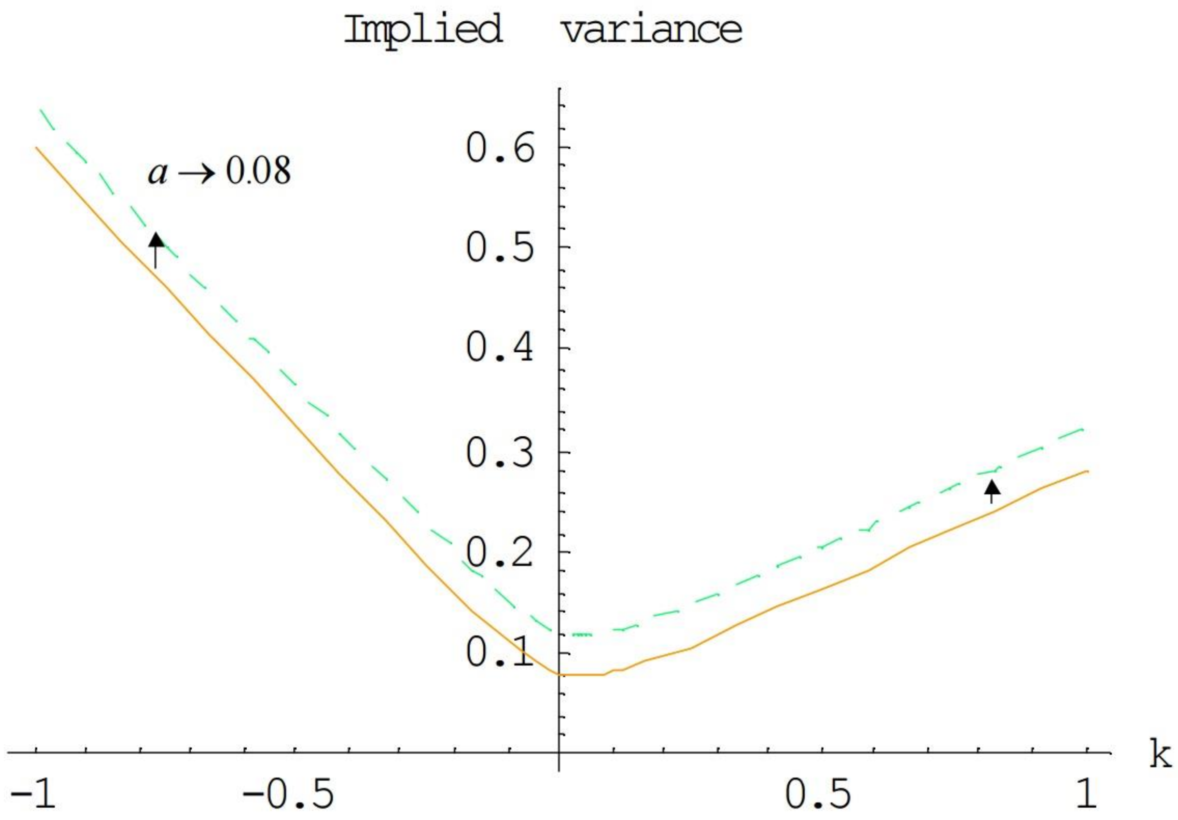
Implied Volatility - Parameterization

- Variance is always positive
- Variance increases linearly with |K| for k very positive or very negative K
- a gives the overall level of variance
- b gives the angle between the left and right asymptotes
- σ determines how smooth the vertex is
- ρ determines the orientation of the graph
- m translates the graph

- Example: a = 0.04, b = 0.4, σ = 0.1, ρ = -0.4, m = 0

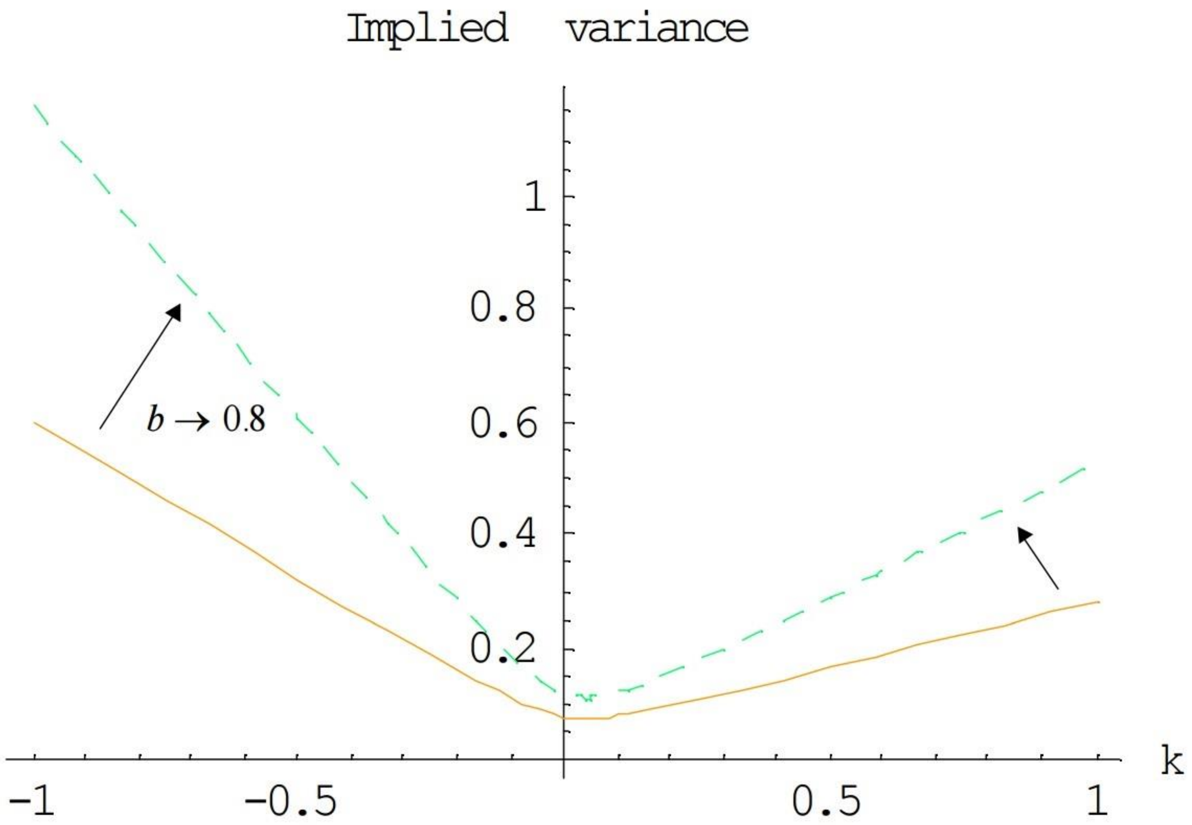
Implied Volatility - Parameterization

Changing a



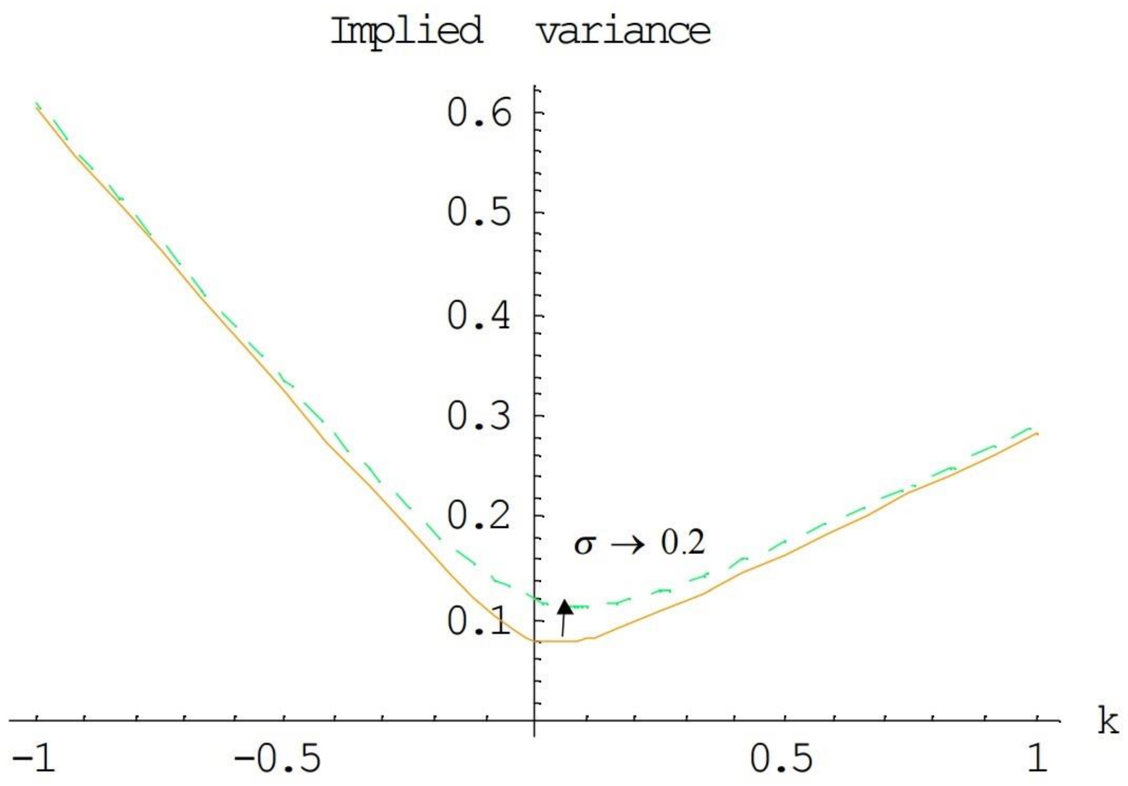
Implied Volatility - Parameterization

Changing b



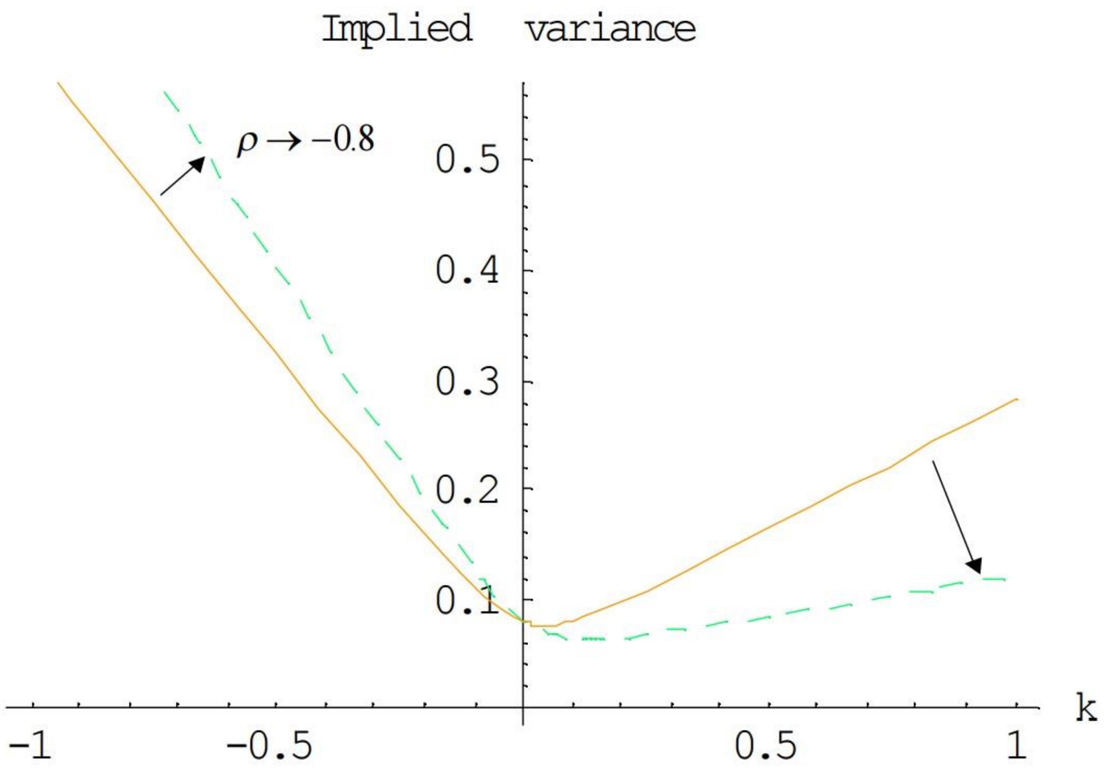
Implied Volatility - Parameterization

Changing σ

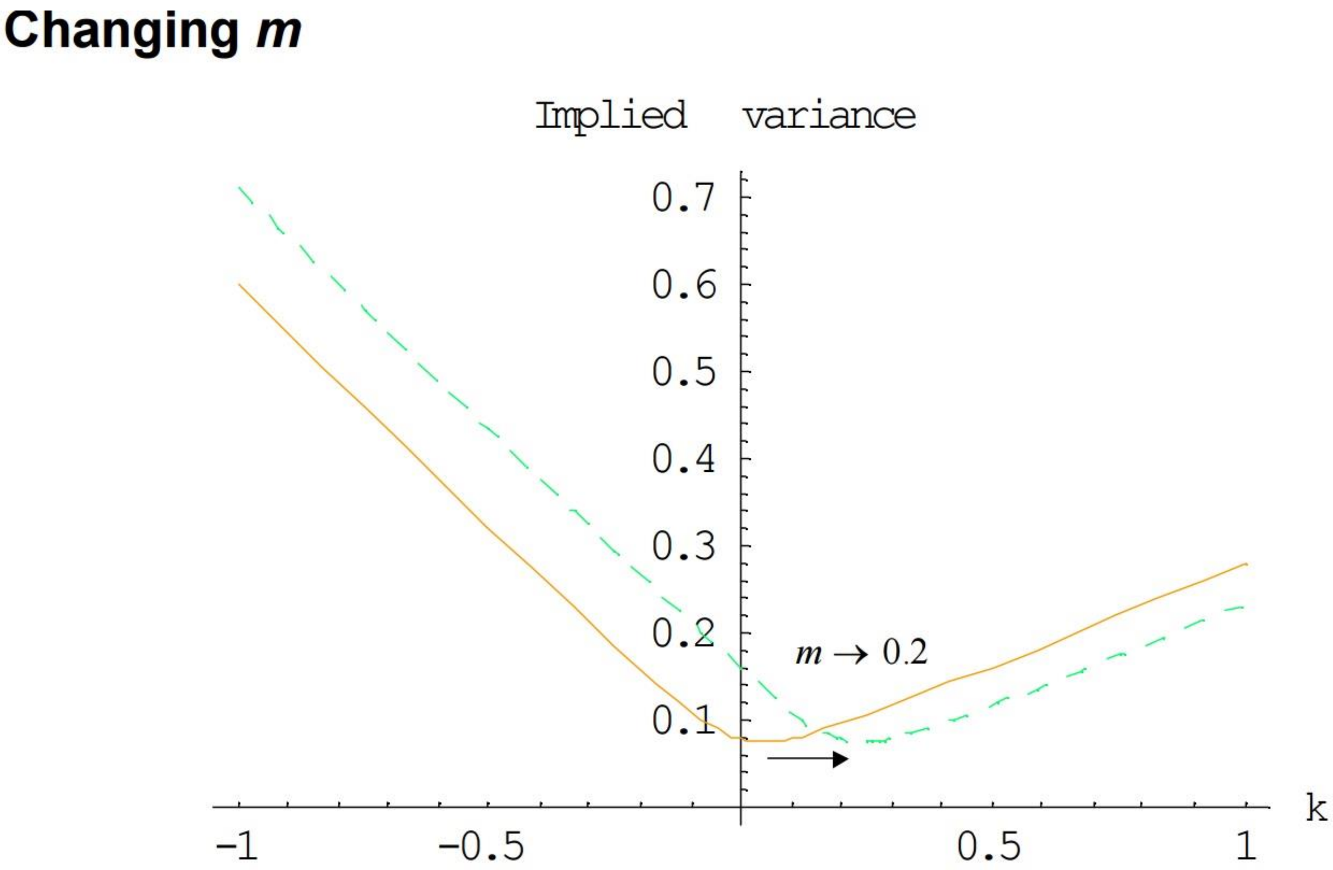


Implied Volatility - Parameterization

Changing ρ



Implied Volatility - Parameterization



Implied Volatility - Parameterization

- Solving the inverse problem of option pricing one obtains the points V_i of implied volatility corresponding to the market strikes K_i
- Parameterization process relies on “fitting” of the two sets of points $\{K_i, V_i\}$ into the functional form $V(K)$. Usually the number of pairs $\{K_i, V_i\}$ is much bigger then the number of parameters forcing the process to use the numerical procedure
- Consider the parameter set $\Omega = \{a, b, \sigma, \rho, m\}$ for SVI model. Then the problem is

$$\min_{\Omega} \sum_{i=1}^{N} [V_i - V(K_i)]^2$$

- For this kind of problems Levenberg-Marquardt algorithm is the most suitable one

Implied Volatility – No-arbitrage conditions

- The price of an European Call option with strike K is given by

$$V(x,t,K,T) = e^{-rt}\int_{-\infty}^{\infty} \max(y-K,0)\, p(y,x,t)dy$$

where $p(y,x,t)$ - is the T*ransition Probability Density* of the Markov process. It follows that the price function must be a decreasing and convex function in the option's strike and

$$-e^{-rt} \leq \frac{\partial\, V(x,t,K,T)}{\partial K} \leq 0$$

Implied Volatility – No-arbitrage conditions

- Differentiating $V(x,t,K,T)$ twice by K we obtain

$$\frac{\partial^2 V(x,t,K,T)}{\partial K^2} = e^{-rt} p(x,K,t) \geq 0$$

- Convex ("butterfly spread") restriction: for $K_1 < K_2 < K_3$

$$\frac{V(K_3) - V(K_2)}{K_3 - K_2} - \frac{V(K_2) - V(K_1)}{K_2 - K_1} \geq 0$$

- Calendar spread restriction: let $\omega(K,t) = \sigma(K,t)^2 t$ then

$$\frac{\partial \omega(K,t)}{\partial t} \geq 0$$

Local Volatility Model

- If we want to price a path dependent option (option whose value depends on the events that happen during the whole lifetime of the option) then implied volatility obtained by the method described above won’t work. We’d need different model.
- One of the alternative models for the volatility is ***Local Volatility Model***. It assumes building a function $\sigma(S,t)$ such that market option prices $V_{Mkt}\left(K_i,T_j\right), 1 \le i \le N, 1 \le j \le M$ match model option prices $V_{Mod}\left(\sigma(S,t),K_i,T_j\right), 1 \le i \le N, 1 \le j \le M$.
- Dupire formulas give a connection between option prices and local volatility as well as between implied and local volatilities

Local Volatility Model

- Local variance in terms of option prices can be computed as

$$\sigma(S,t)^2 = \frac{\dfrac{\partial V(S,t,K,T)}{\partial T} - (r-q)\left(V(S,t,K,T) - K\dfrac{\partial V(S,t,K,T)}{\partial K}\right)}{\dfrac{1}{2}K^2\dfrac{\partial^2 V(S,t,K,T)}{\partial K^2}}$$

Local variance $\sigma(S,t)^2$ in terms of implied volatility $\Sigma(\mathrm{K},\mathrm{T})$

$$\sigma(S,t)^2 = \frac{\Sigma^2 + 2\Sigma T\left[\dfrac{\partial\Sigma}{\partial t} + (r-q)K\dfrac{\partial\Sigma}{\partial K}\right]}{\left(1 - \dfrac{Kd}{\Sigma}\dfrac{\partial\Sigma}{\partial K}\right)^2 + K\Sigma T\left(\dfrac{\partial\Sigma}{\partial K} - \dfrac{1}{2}K\Sigma T\left(\dfrac{\partial\Sigma}{\partial K}\right)^2 + K\dfrac{\partial^2\Sigma}{\partial K^2}\right)}$$

$$d = ln\left(\frac{K}{S_0}\right) - (r-q)T$$

Local Volatility - Interpretation

- Consider for simplicity the case $r = q = 0$ so

$$\sigma(S,t)^2 = \frac{2\,\frac{\partial V(S,t,K,T)}{\partial T}}{K^2\,\frac{\partial^2 V(S,t,K,T)}{\partial K^2}}$$

  - The numerator term

$$\frac{\partial V}{\partial T} = \frac{V(S,t,K,T+dT) - V(S,t,K,T)}{dT}$$

is proportional to an infinitesimal calendar spread for the call option with the strike K : long a call with expiry $T + dT$ and short a call with expiry T

Local Volatility - Interpretation

- The denominator term

$$\frac{\partial^2 V}{\partial K^2} = \frac{V(S,t,K+dK,T) - 2V(S,t,K,T) + V(S,t,K-dK,T)}{(dK)^2}$$

is proportional to an infinitesimal butterfly spread for the calls with strikes $K - dK$, K, $K + dK$. So

$$\sigma(S,t)^2 \sim \frac{calendar\ spread}{butterfly\ spread}$$

and no-arbitrage condition in terms of local volatility follows:

$$anumerator \sim calendar\ spread \geq 0,$$

$$denominator \sim butterfly\ spread \geq 0.$$