FE-620A Pricing and Hedging Other pricing models

STEVENS INSTITUTE OF TECHNOLOGY

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American type option pricing

- American call (put) option is a contract which gives the right to buy (sell) an underlying asset at the agreed price K (strike price) at or before the specified time T (maturity time). Therefore, an American option can be exercised at any time τ , up until maturity T, where 0 < τ ≤ T.
- For the pricing of an American option we can use binomial or trinomial tree with slightly modified logic: the pay-off has to be defined at every node of the tree

$$Payoff_{Call} = \max(S_{ij} - K, 0)$$

$$Payoff_{Put} = \max(K - S_{ij}, 0)$$

where i and j are time and space indices respectively.

American type option pricing

- The modified backward recursion takes the following form

$V_{ij} = \max(Payoff, e^{-r\Delta t}\left[p_u V_{i+1,j+1} + p_0 V_{i+1,j} + p_d V_{i+1,j-1}\right])$

Compare to the European type option price

$$V_{ij} = e^{-r\Delta t}\left[p_u V_{i+1,j+1} + p_0 V_{i+1,j} + p_d V_{i+1,j-1}\right]$$

In other words, on each time step we have to compare the “intrinsic value” of an option (payoff) to the current value of the price.

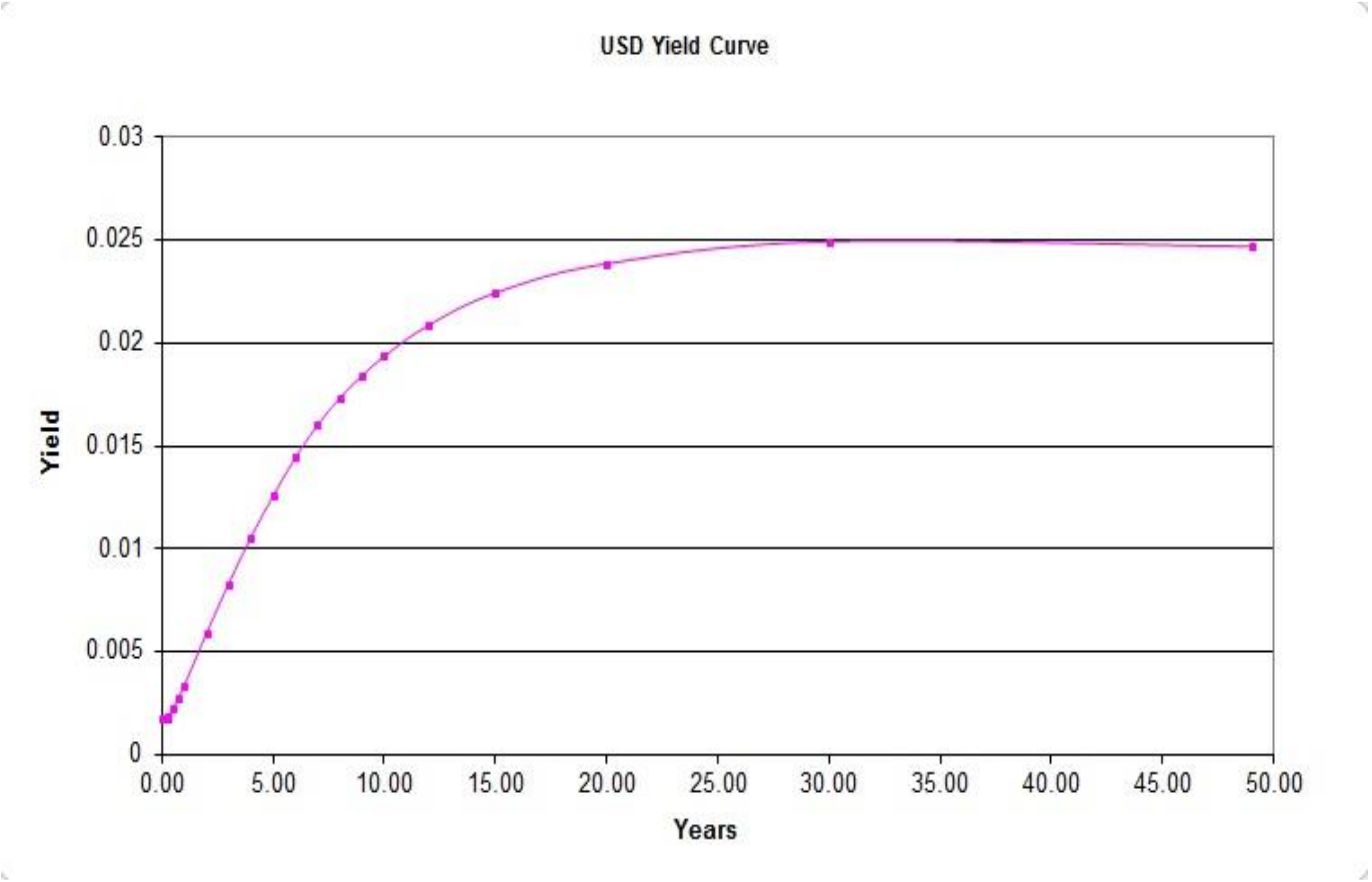
American type option pricing

- The algorithm for the computing of the price using trees is as follows:
- **Algorithm 1: American Option Pricing**
  - 1: Declare and initialize *S(0)*
  - 2: Calculate the jump sizes *u; d*
  - 3: Calculate the *transition probabilities* p_u, p_d, p_0
  - 4: Build the *price tree*
  - 5: Calculate the payoff of the option at maturity (node *N*)
  - 6: *for* (int i = *N; i > 0; --i) do*
    - 7: Calculate option price at node i using formula max(payoff, current price):
      $V_{ij} = \max(Payoff, e^{-r\Delta t}[p_u V_{i+1,j+1} + p_0 V_{i+1,j} + p_d V_{i+1,j-1}])$
  - 7: *end for*
  - 8: Output option price
- Pricing American type Puts and Calls

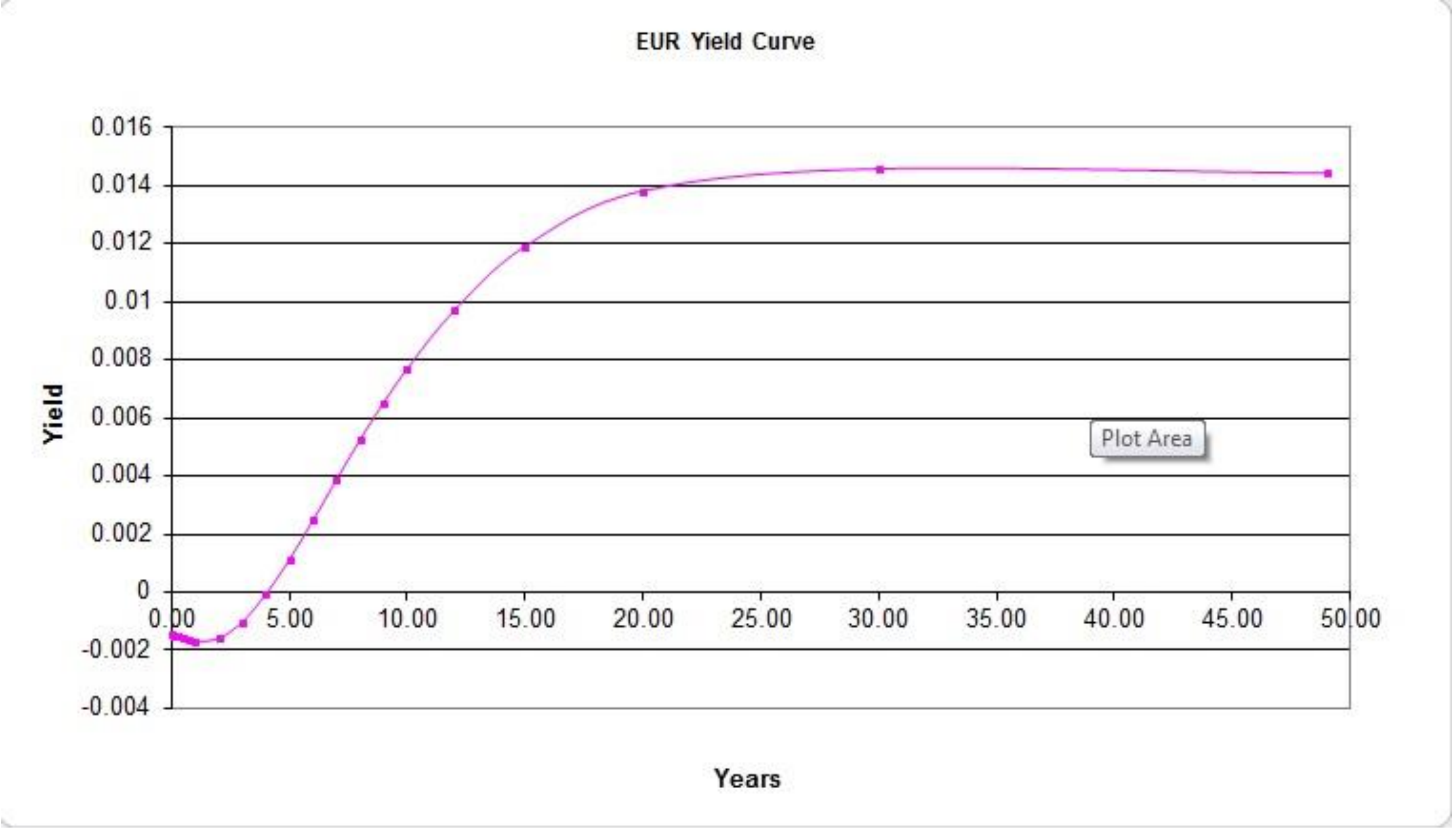
Interest rate term structure

- Up until now we were considering only *constant interest rate.* In reality it’s not constant: it has some term structure represented by the *Yield Curve*.
- *Yield curve* is constructed using Market prices of Fixed Income related instruments (Money Market, Forwards (FRAs), Futures, Swaps)
- Examples: shapes of the USD and EUR Yield Curves

Interest rate term structure



Interest rate term structure



Interest rate term structure

- When pricing, *risk free rate* component $e^{r\Delta t}$ needs to be modified to $e^{r_i \Delta t}$ where r_i represents time dependent *forward rate.*
- The relation between *yields* r_1 and r_2 corresponding to the times T_1 and T_2 $(T_2 > T_1)$ and *forward rate* r_{12} corresponding to the time interval $[T_1, T_2]$ is as follows

$$r_{12} = \frac{r_2 T_2 - r_1 T_1}{T_2 - T_1}$$

- In case of constant rate $r_1 = r_2 = r$ we have $r_{12} = r$.
- Usually Yield Curve is given as the map $terms \rightarrow yields$ where *terms* are some standard dates (times) and *yields* are corresponding yields.

Interest rate term structure

- In order to adapt the Yield Curve to your particular time grid you'd need to perform yield interpolation. Common interpolation schemes are
  - Linear-in-yield
  - Linear-in-term-yield
- Example: given yields r_1, r_2 corresponding to times T_1, T_2; find yield r^* corresponding to time T^* where $T_1 \leq T^* \leq T_2$ using "linear-in-yield" method. Solution:

$$r^* = \frac{T^* - T_1}{T_2 - T_1}(r_2 - r_1) + r_1$$

Dividends

- A ***dividend*** is a distribution of a portion of a company's earnings, decided by the board of directors, to a class of its shareholders. Dividends can be issued as cash payments, as shares of stock, or other property.
- The ***dividend rate*** may be quoted in terms of the dollar amount each share receives (***dividends per share (DPS)***), or It can also be quoted in terms of a percent of the current market price, which is referred to as the ***dividend yield.***
- Dividend yield is represented as a percentage and can be calculated as

$$\frac{Annual\ Dividends\ per\ Share}{Price\ of\ an\ Asset\ per\ Share}$$

- Properties of the dividends:
  - Are dependent on the company's earnings in the future and therefore are unpredictable in general (random amount at random time);
  - Dividend yield is a function of asset price and therefore is stochastic in general;
  - It’s a “drag down” to the asset price at particular time by particular amount;
  - One has to use some dividend forecasting scheme in order to price an option on an asset that pays dividends.

Dividends

- Simple example: let a represent the annualized percentage rate of the dividend paid N times a year. Then annualized *constant continuous dividend rate* q is:

$$q = Nln\left(1 + \frac{a}{N}\right)$$

which is just a conversion from simple compounding to continuous one.

- Consequences for the pricing: if q is known then the risk free rate r needs to be substituted by the difference $(r - q)$.
- Handling dividends in general is non-trivial problem

Stochastic volatility models

- Problems with local volatility models:
  - When time increases local volatility “flattens out”;
  - Unable to price forward starting options;
  - Unable to price cliquet options
    - **Cliquet option** or **ratchet option** is an exotic option consisting of a series of consecutive forward start options. The first is active immediately. The second becomes active when the first expires, etc. Each option is struck at-the-money when it becomes active;
    - Typical payoff of the cliquet option (global floor, local cap):

$$Payoff = max\left(\sum_{i=1}^{N} max\left(0, min\left(Cap, \frac{S_i - S_{i-1}}{S_{i-1}}\right)\right), Floor\right)$$

  - Systematically misprice OTM and ITM options

Stochastic volatility models

- One of the alternatives to LVMs that still is able to handle volatility "smile/skew" and solve some of the LVM's problems is *Stochastic Volatility Model*
- Consider a system of stochastic differential equations

$$dS = S(r - q)dt + S\sqrt{v}dW_1$$
$$dv = k(\theta - v)dt + \sigma\sqrt{v}dW_2$$

where $Corr(W_1, W_2) = \rho, \sigma$ – is the volatility-of-volatility.

- Volatility is not a "material" market instrument so one cannot trade/hedge volatility directly

Stochastic volatility models

- Consider portfolio Π:

$$\Pi = V + \phi_1 S + \phi_2 Q$$

where V- is the option and Q - is some volatility dependent instrument (may be another option) which is used to hedge volatility. The change in portfolio value $d\Pi$ is

$$d\Pi = dV + \phi_1 dS + \phi_2 dQ.$$

Using Ito's Lemma we can write

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial \upsilon}d\upsilon + \frac{1}{2}\upsilon S^2\frac{\partial^2 V}{\partial S^2}dt + \frac{1}{2}\upsilon\sigma^2\frac{\partial^2 V}{\partial \upsilon^2}dt + \upsilon\sigma\rho S\frac{\partial^2 V}{\partial \upsilon \partial S}dt + o(dt^2)$$

$$dQ = \frac{\partial Q}{\partial t}dt + \frac{\partial Q}{\partial S}dS + \frac{\partial Q}{\partial \upsilon}d\upsilon + \frac{1}{2}\upsilon S^2\frac{\partial^2 Q}{\partial S^2}dt + \frac{1}{2}\upsilon\sigma^2\frac{\partial^2 Q}{\partial \upsilon^2}dt + \upsilon\sigma\rho S\frac{\partial^2 Q}{\partial \upsilon \partial S}dt + o(dt^2)$$

Stochastic volatility models

- Ito's Lemma in this case (two variable model) is based on two variable Taylor expansion for the function $f(x, y)$ around point (a, b) :

$$f(x,y) = f(a,b) + \frac{\partial f(a,b)}{\partial x}(x-a) + \frac{\partial f(a,b)}{\partial y}(y-b)$$

$$+\frac{1}{2!}\left[\frac{\partial^2 f(a,b)}{\partial x^2}(x-a)^2 + 2\frac{\partial^2 f(a,b)}{\partial x \partial y}(x-a)(y-b) + \frac{\partial^2 f(a,b)}{\partial y^2}(y-b)^2\right] + \cdots$$

Stochastic volatility models

- Combining these two expressions we obtain

$$d\Pi = \left\{\frac{\partial V}{\partial t} + \frac{1}{2}\upsilon S^2 \frac{\partial^2 V}{\partial S^2} + \upsilon\sigma\rho S \frac{\partial^2 V}{\partial \upsilon \partial S} + \frac{1}{2}\upsilon\sigma^2 \frac{\partial^2 V}{\partial \upsilon^2}\right\} dt +$$
$$+\phi_2 \left\{\frac{\partial Q}{\partial t} + \frac{1}{2}\upsilon S^2 \frac{\partial^2 Q}{\partial S^2} + \upsilon\sigma\rho S \frac{\partial^2 Q}{\partial \upsilon \partial S} + \frac{1}{2}\upsilon\sigma^2 \frac{\partial^2 Q}{\partial \upsilon^2}\right\} dt +$$
$$+\left\{\frac{\partial V}{\partial S} + \phi_2 \frac{\partial Q}{\partial S} + \phi_1\right\} dS + \left\{\frac{\partial V}{\partial \upsilon} + \phi_2 \frac{\partial Q}{\partial \upsilon}\right\} d\upsilon + o(dt^2)$$

To eliminate the "stochasticity" we must require

$$\frac{\partial V}{\partial S} + \phi_2 \frac{\partial Q}{\partial S} + \phi_1 = 0 \text{ and } \frac{\partial V}{\partial \upsilon} + \phi_2 \frac{\partial Q}{\partial \upsilon} = 0 \text{ or}$$

$$\phi_2 = -\frac{\frac{\partial V}{\partial \upsilon}}{\frac{\partial Q}{\partial \upsilon}} \text{ and } \phi_1 = -\frac{\partial V}{\partial S} + \frac{\frac{\partial V}{\partial \upsilon}}{\frac{\partial Q}{\partial \upsilon}} \frac{\partial Q}{\partial S}$$

Stochastic volatility models

- Taking into account risk neutrality of our valuation we get

$$d\Pi = r\Pi \mathrm{dt}$$

which leads to the representation

$$d\Pi = (A + \phi_2 B)dt$$

or

$$A + \phi_2 B = r(V + \phi_1 S + \phi_2 Q)$$

Substituting values for ϕ_1 and ϕ_2 and rearranging

$$\frac{A - rV + rS\frac{\partial V}{\partial S}}{\frac{\partial V}{\partial \nu}} = \frac{B - rQ + rS\frac{\partial Q}{\partial S}}{\frac{\partial Q}{\partial \nu}}$$

Stochastic volatility models

- Left hand side of the previous equation is a function of V only and the right hand side is the function of Q only. This is possible only in the case when both LHS and RHS are represented as some function of $S, \nu \; and \; t$: $f(S, \nu,$ t). This function is called *price of volatility risk.*
- After substitution and rearrangement finally we obtain

$$\frac{\partial V}{\partial t} + \frac{1}{2}\nu S^2 \frac{\partial^2 V}{\partial S^2} + \rho\vartheta\nu S \frac{\partial^2 V}{\partial \nu \partial S} + \frac{1}{2}\vartheta^2 \nu \frac{\partial^2 V}{\partial \nu^2} - rV + rS\frac{\partial V}{\partial S} + f(S, \nu, t)\frac{\partial V}{\partial \nu} = 0$$

- This is Heston stochastic volatility model.
- There are different representations (assumptions) for the price of volatility risk (function $f(S, \nu,$ t)). For example, Heston suggested $f(S, \nu, t) = \kappa(\theta - \nu) - \lambda\nu$.

Stochastic volatility models

- Boundary conditions:
  - For the call option at expiry *T* the following “terminal” condition is satisfied
$$V(S,\nu,T) = \max(S-K,0)$$
  - When the underlying asset price is zero the call is worthless:
$$S=0 \rightarrow V(0,\nu,\mathrm{t}) = 0$$
  - As the underlying asset price increases, delta approaches one:
$$\frac{\partial V}{\partial S}(\infty,\nu,\mathrm{t}) = 1$$
  - When the volatility increases, the call option becomes equal to the underlying asset price:
$$V(S,\infty,t) = S$$

Volatility Derivatives

- Consider undiscounted Vanilla option written on underlying described by the process $S(t)$. The price of the option can be calculated by

$$V(x,t) = \int_{-\infty}^{\infty} f(y)p(x,y,t)dy$$

where $f(y)$ - is the payoff, $p(x,y,t)$ - is the transition probability density of the process $S(t)$. Apply this formula to the Call option with the payoff

$$f(y) = \max(y - K, 0) = \begin{cases} 0, & y \le K \\ y - K, & y > K \end{cases}$$

Then

$$\frac{\partial V_C\,(x,t)}{\partial K} = -\int_{K}^{\infty} h(y-K)\,p(x,y,t)dy$$

where $h(y)$ – is Heaviside step function

Volatility Derivatives

Differentiating second time and using properties of Delta function we get

$$\frac{\partial^2 V_C\ (x,t)}{\partial K^2} = p(x,K,t)$$

It is assumed that we have continuous set of strikes and corresponding option prices. Same is true for the Put (check!):

$$\frac{\partial^2 V_P\ (x,t)}{\partial K^2} = p(x,K,t)$$

The value of the claim with a generalized payoff $g(x)$ at time T is given by

$$E[g(S_T|S_t)] = \int_0^\infty g(K)p(S_t,K,t)dK = \int_0^F \frac{\partial^2 V_P\ (x,t)}{\partial K^2} g(K)dK + \int_F^\infty \frac{\partial^2 V_C\ (x,t)}{\partial K^2} g(K)dK$$

Volatility derivatives

Where F represents the time T forward price of the underlying asset. Integrating by parts and using the call-put parity $V_C - V_P = F - K$ give

$$E[g(S_T|S_t)] = \frac{\partial V_P}{\partial K} g(K)\bigg|_0^F - \int_0^F g'(k)\frac{\partial V_P}{\partial K}dK + \frac{\partial V_C}{\partial K} g(K)\bigg|_F^\infty -$$

$$- \int_F^\infty g'(K)\frac{\partial V_C}{\partial K}dK = g(F) - \int_0^F g'(k)\frac{\partial V_P}{\partial K}dK - \int_F^\infty g'(k)\frac{\partial V_C}{\partial K}dK =$$

$$= g(F) - V_P(K)g'(K)|_0^F + \int_0^F V_P(K)g''(K)dK - V_C(K)g'(K)|_F^\infty$$

$$+ \int_F^\infty V_C(K)g''(K)dK = g(F) + \int_0^F V_P(K)g''(K)dK + \int_F^\infty V_C(K)g''(K)dK$$

Volatility derivatives

- By letting $t \to T$ one can see that any Vanilla twice-differentiable payoff may be replicated in a portfolio of puts and calls with strikes 0 to ∞ with weight of each option equal to the second derivative of the payoff at the strike price of the option.

- Example: consider a contract with the payoff $Payoff = \log\left(\frac{S_T}{F}\right)$.

  Then $g''(K) = -\frac{1}{K^2}$ and it follows that

$$E\left[log\left(\frac{S_T}{F}\right)\right] = -\int_0^F \frac{dK}{K^2} V_P(K,t) - \int_F^\infty \frac{dK}{K^2} V_C(K,t)$$

Volatility derivatives

- Assume $r = q = 0$. In this case $F = S_0$. Applying Ito's Lemma we get

$$\log\left(\frac{S_T}{F}\right) = \log\left(\frac{S_T}{S_0}\right) = \int_0^T d\, log(S_t) = \int_0^T \frac{dS_t}{S_t} - \int_0^T \frac{\sigma^2(S_t)}{2} dt$$

The term $\int_0^T \frac{\sigma^2(S_t)}{2} dt$ represents a total variance of the underlying price over the time interval $[0, T]$. The term $\int_0^T \frac{dS_t}{S_t}$ - is the payoff of the hedging strategy. Taking into account that the log payoff can be replicated in a portfolio of puts and calls, it follows that the total variance can be replicated in a model independent way by a portfolio of Vanilla options.