3.2 Proof $\mathbb{E}[W^2(t) - t \mid \mathcal{F}(s)] = W^2(s) - s$

Properties

As $W^2(t) = [W(t) - W(s)]^2 - W^2(s) + 2W(t)W(s)$

Linearity

$\mathbb{E}[[W(t) - W(s)]^2 - W^2(s) + 2W(s)W(t) - t \mid \mathcal{F}(s)]$

Taking out what is known $= \mathbb{E}[(W(t) - W(s))^2] + 2W(s)\mathbb{E}[W(t)|\mathcal{F}(s)] - W^2(s) - t$

Independence $= (t-s) + 2W^2(s) - W^2(s) - t$

Varience $= W^2(s) - s$

Proof Done! $W(t)-t$ is a Martingale

3.3 $\varphi^{(3)}(u) = \frac{d}{du}\varphi''(u) = \frac{d}{du}\left[(\sigma^2 + \sigma^4 u^2)e^{\frac{1}{2}\sigma^2u^2}\right]$

$= \sigma^4 \cdot 2u\, e^{\frac{1}{2}\sigma^2u^2} + (\sigma^2 + \sigma^4u^2)\cdot \frac{1}{2}\sigma^2 \cdot 2u\, e^{\frac{1}{2}\sigma^2u^2}$

$= (2u\cdot\sigma^4 + \frac{1}{2}\sigma^4\cdot 2u + \sigma^6\cdot u^3)\, e^{\frac{1}{2}\sigma^2u^2}$

$= (3u\sigma^4 + u^3\sigma^6)\, e^{\frac{1}{2}\sigma^2u^2}$

$\mathbb{E}[(X-\mu)^3 e^{u(X-\mu)}] = \varphi^{(3)}(u) = (3u\sigma^4 + u^3\sigma^6)\exp(\frac{1}{2}\sigma^2u^2)$

$\varphi^{(4)}(u) = \frac{d}{du}(\varphi^{(3)}(u)) = (3\sigma^4 + 3u^2\sigma^6)\exp(\frac{1}{2}\sigma^2u^2) + (3u\sigma^4 + u^3\sigma^6)\frac{d}{du}e^{\frac{1}{2}\sigma^2u^2}$

$\Rightarrow \mathbb{E}[(X-\mu)^4 e^{u(X-\mu)}] = (3\sigma^4 + 3u^2\sigma^6 + 3u^2\sigma^6 + u^4\sigma^8)\exp(\frac{1}{2}\sigma^2u^2)$

$= (3\sigma^4 + 6u^2\sigma^6 + u^4\sigma^8)\exp(\frac{1}{2}\sigma^2u^2)$

Substitute $u=0$

$\Rightarrow \mathbb{E}[(X-\mu)^4] = 3\sigma^4$

3.4 i) $\sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)| \to \infty$

To proof that, we have $\sum_{j=0}^{n-1}(W(t_{j+1}) - W(t_j))^2 \le \max|W(t_{j+1}) - W(t_j)|\,\Sigma|W(t_{j+1}) - W(t_j)|$

The second term on the right hand of equation is what we ~~also~~ want to proof.

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Equivalantly, $\sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)| \geq \dfrac{\sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2}{\max |W(t_{j+1}) - W(t_j)|}$

As $n \to \infty$ and more precisely speaking

$\|\Pi\| = \text{sub} |t_{j+1} - t_j|$ approaches zero, $\sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2 = T$

Continuity Function:

$\max |W(t_{j+1}) - W(t_j)| = \cancel{\max | \Sigma X}$ approaches zero

More precisely convergent to 0 on the right side

So $\lim_{\|\Pi\| \to 0} \dfrac{\sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2}{\max |W(t_{j+1}) - W(t_j)|} = +\infty$

Hence $\lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)| = +\infty$ which is the first variation

iii) Cubic variation of Brownian Motion $\overset{\text{def}}{=} \lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|^3$

As $\sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|^3 \leq \max |W(t_{j+1}) - W(t_j)| \sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2$

When $\text{sub} |t_{j+1} - t_j| \to 0$ $\max |W(t_{j+1}) - W(t_j)| \to 0$, Continuity

$\sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|^2 = T$, quadra variation

so, the cubic variation ≤ 0

Moreover, $\sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|^3 \geq 0$ as every item are a absolute value

Consequentially the cubic variation ~~appro~~ approaches zero.

3.6 i) $\mathbb{E}\left[f(X(t)) \mid \mathcal{F}(s) \right] = \mathbb{E}\left[f(\mu t + W(t) - W(s) + W(s)) \mid \mathcal{F}(s) \right]$

$$= \frac{1}{\sqrt{2\pi \tau}} \int_{-\infty}^{+\infty} f(x + \mu t + W(s))\, e^{-\frac{x^2}{2\tau}}\, dx$$

let $y = x + \mu t + W(s)$ $\quad x = y - \mu t - W(s)$ $\quad dx = dy$

more over $\cancel{x + \mu}$ $X(s) = \mu s + W(s)$

$$= \frac{1}{\sqrt{2\pi \tau}} \int_{-\infty}^{+\infty} f(y) \exp\left\{ - \frac{y - \mu t - W(s)}{2\tau} \right\} dy$$

$$= \frac{1}{\sqrt{2\pi \tau}} \int_{-\infty}^{+\infty} f(y) \exp\left\{ - \frac{y - \overbrace{(\mu s + W(s))}^{X(s)} - \mu(t-s)}{2\tau} \right\} dy$$

So. $$g(X(s)) = \frac{1}{\sqrt{2\pi \tau}} \int_{-\infty}^{+\infty} f(y) \exp\left\{ - \frac{y - X(s) - \mu(t-s)}{2\tau} \right\} dy$$

Then, $$p(\tau, x, y) = \frac{1}{\sqrt{2\pi \tau}} \exp\left\{ - \frac{y - x - \mu\tau}{2\tau} \right\} \cancel{dy}$$

ii) By i) $\mathbb{E}[f(X(t)) \mid \mathcal{F}(s)] = g(X(s))$

$X(t) = \mu t + W(t)$

$$g(x) = \frac{1}{\sqrt{2\pi\tau}} \int_{-\infty}^{+\infty} f(y) \exp\left\{-\frac{[y-x-\mu(t-s)]^2}{2\tau}\right\} dy$$

$$g(X(s)) = g(x)\Big|_{x=X(s)}$$

$$S(t) = S(0) \cdot \exp(\sigma W(t) + \nu t) = S(0) \exp[\sigma(W(t)+\mu t)]\Big|_{\mu\sigma=\nu}$$

$$= S(0) \exp[\sigma X(t)]\Big|_{\mu=\frac{\nu}{\sigma}}$$

suppose $= \cancel{f_1(x) = f(S(0)e^{\sigma x})}$. $f_1(x) = f(S(0)e^{\sigma x})$ or $f_1(S(t)) = f(e^{\sigma X(t)})$

$$\text{So, } \mathbb{E}[f_1(S(t)) \mid \mathcal{F}(s)] = \frac{1}{\sqrt{2\pi\tau}} \int_{-\infty}^{+\infty} f(y) \exp\left\{-\frac{(y-X(s)-\mu\tau)^2}{2\tau}\right\} dy$$

$$\Rightarrow \mathbb{E}[f_1(S(0)e^{\sigma X(t)}) \mid \mathcal{F}(s)] = \frac{1}{\sqrt{2\pi\tau}} \int_{-\infty}^{+\infty} f_1(S(0)e^{\sigma y}) \exp\left\{-\frac{(y-X(s)-\mu\tau)^2}{2\tau}\right\} dy$$

$$= \frac{1}{\sqrt{2\pi\tau}} \int_{-\infty}^{+\infty} f_1(S(0)e^{\sigma y}) \exp\left\{-\frac{(y-X(s)-\mu\tau)^2}{2\tau}\right\} dy$$

$$S(0)e^{\sigma X(s)} = S(s) \Rightarrow X(s) = \frac{1}{\sigma}\ln\left(\frac{S(s)}{S(0)}\right)$$

Suppose $S(0)e^{\sigma y} = z \Rightarrow y = \frac{1}{\sigma}\ln\frac{z}{S(0)}$, $dy = \frac{dz}{\sigma z}$

Substitute to last equation $\Rightarrow$

$$\mathbb{E}[f_1(S(t)) \mid \mathcal{F}(s)] = \frac{1}{\sqrt{2\pi\tau}} \int_{-\infty}^{+\infty} f_1(z) \exp\left\{-\frac{\left[\frac{1}{\sigma}\ln\left(\frac{z}{S(s)}\right) - \frac{\nu}{\sigma}\tau\right]^2}{2\tau}\right\} \frac{dz}{\sigma z}$$

Replace z with y, Proof done!

3.7 i) $Z(t) = \exp\{\sigma X(t) - (\sigma\mu + \frac{1}{2}\sigma^2)t\}$ ①

$X(t) = \mu t + W(t)$ ②

Combine ①②

$Z(t) = \exp\{\sigma\mu t + \sigma W(t) - \sigma\mu t - \frac{1}{2}\sigma^2 t\}$

$= \exp\{\sigma W(t) - \frac{1}{2}\sigma^2 t\}$ is a martingale

Method I

$\mathbb{E}[Z(t) \mid \mathcal{F}(s)] = \mathbb{E}[\exp\{\sigma W(t) - \sigma W(s) + \sigma W(s) - \frac{1}{2}\sigma^2 t\} \mid \mathcal{F}(s)]$

Taking out what is known $= \mathbb{E}[\exp\{\sigma(W(t) - W(s))\} \mid \mathcal{F}(s)] \exp(\sigma W(s) - \frac{1}{2}\sigma^2 t)$

Independence $= \mathbb{E}[\exp\{\sigma(W(t) - W(s))\}] \exp(\sigma W(s) - \frac{1}{2}\sigma^2 t)$

Normal ↓ Moment Generation $= \exp\{\underbrace{\cancel{\sigma\cdot\mu}}_{=0} + \frac{1}{2}\sigma^2 \cdot (t-s)\} \exp(\sigma W(s) - \frac{1}{2}\sigma^2 t)$

$= Z(s)$

Method 2. $Z_x = \sigma Z \quad Z_t = -\frac{1}{2}\sigma^2 Z \quad Z_{xx} = \sigma^2 Z$

$dZ(t) = (\sigma\, dW - \frac{1}{2}\sigma^2 dt)\, Z(t) + \frac{1}{2}\sigma^2 Z(t)\, dt$

$Z(t) = \int Z(t)\sigma\, dW(t) - \cancel{\frac{1}{2} Z(t)\sigma^2 dt}$ which is a Itô-Integral

So, it is a Martingale!

ii) $Z(t)$ is a Martingale, so $\mathbb{E}[Z(t)] = Z(0)$

$Z(0) = \exp\{\sigma W(0) - \frac{1}{2}\sigma^2 \cdot 0\} = \exp(0) = 1$

So, $\mathbb{E}[Z(t)] = 1$

Moreover $\mathbb{E}[Z(t \wedge \tau_m)] = 1$ Proof done.

As For $\exp\{\sigma X(t\wedge \tau_m) - (\sigma\mu + \frac{1}{2}\sigma^2)(t\wedge \tau_m)\}$

iii) When $\tau_m = \infty$

We have $\mathbb{E}[\exp\{\sigma X(t) - (\sigma\mu + \frac{1}{2}\sigma^2)t\}] \cancel{=1}$

When $\tau_m < \infty$, and t is big enough

We have $\mathbb{E}[\exp\{\sigma m - (\sigma\mu + \frac{1}{2}\sigma^2)\tau_m\}] \cancel{=1}$

So, $\lim_{t\to\infty} \exp\{\sigma X(t\wedge\tau_m) - (\sigma\mu + \frac{1}{2}\sigma^2)(t\wedge\tau_m)\} = \mathbb{I}_{\{\tau_m<\infty\}} \exp\{\sigma m - (\sigma\mu + \frac{1}{2}\sigma^2)\tau_m\}$

$\lim_{t\to\infty} \mathbb{E}[\exp\{\sigma X(t\wedge\tau_m) - (\sigma\mu + \frac{1}{2}\sigma^2)(t\wedge\tau_m)\}]$

$= \mathbb{E}[\mathbb{I}_{\{\tau_m<\infty\}} \exp\{\sigma m - (\sigma\mu + \frac{1}{2}\sigma^2)\tau_m\}] = 1$

When $\sigma \downarrow 0$, we deduced that $\mathbb{E}[\mathbb{I}_{\{\tau_m<\infty\}}] = 1$

$P(\tau_m < \infty) = 1$ $\quad \tau_m < \infty$ is almost surely!

Hence, $\mathbb{E}[\exp\{\sigma m - (\sigma\mu + \frac{1}{2}\sigma^2)\tau_m\}] = 1$

$$\mathbb{E}[\exp\{-(\sigma\mu + \frac{1}{2}\sigma^2)\tau_m\}] = \exp(-\sigma m)$$

let $\alpha = \sigma\mu + \frac{1}{2}\sigma^2$ and $\alpha > 0$

$\Rightarrow \sigma = -\mu \pm \sqrt{\mu^2 + 2\alpha}$ and because $\sigma > 0$

$\sigma = \sqrt{\mu^2 + 2\alpha} - \mu$

$\therefore \mathbb{E}[\exp(-\alpha\tau_m)] = \exp\{(\mu - \sqrt{\mu^2+2\alpha})m\}$

iv) by iii) $\mathbb{E}(e^{-\alpha \tau_m}) = e^{m\mu - m\sqrt{2\alpha+\mu^2}}$

Differentiate the formula with respect to α

$\Rightarrow -\mathbb{E}(\tau_m e^{-\alpha \tau_m}) = \frac{d}{d\alpha} \exp\{m\mu - m(2\alpha+\mu^2)^{\frac{1}{2}}\}$

$= -m \cdot \frac{1}{2}(2\alpha+\mu^2)^{-\frac{1}{2}} \cdot 2 \exp\{m\mu - m\sqrt{2\alpha+\mu^2}\}$

$= -\frac{m}{\sqrt{2\alpha+\mu^2}} \exp\{m\mu - m\sqrt{2\alpha+\mu^2}\}$

Let $\alpha = 0$, substitude $\mu > 0$

$\mathbb{E}(\tau_m) = + \frac{m}{\mu} \exp\{m\mu - m\mu\}$

$= \frac{m}{\mu} < \infty$

v) $\sigma > 2\mu$, when $\tau = \infty$

We have $\exp\{\sigma X(t) - (\sigma\mu + \frac{1}{2}\sigma^2)t\}$ and $(\sigma\mu + \frac{1}{2}\sigma^2) = \frac{1}{2}\sigma(2\mu+\sigma) > 0$

so it is the same as what we proofed in iii)

$\mathbb{E}[\exp\{\sigma m - (\sigma\mu + \frac{1}{2}\sigma^2)\tau_m\} \mathbb{I}_{\{\tau_m < \infty\}}] = 1$

$\mathbb{E}[\exp[-(\sigma\mu + \frac{1}{2}\sigma^2)\tau_m] \mathbb{I}_{\{\tau_m<\infty\}}] = \exp[-\sigma m]$

Let $\sigma \downarrow -2\mu$

$P(\tau_m < \infty) = \mathbb{E}[\mathbb{I}_{\{\tau_m<\infty\}}] = \exp(2\mu m) = e^{-2|\mu| m}$

$\mathbb{E}\{e^{-\alpha\tau_m}\} = \mathbb{E}\{e^{-\alpha\tau_m} \mathbb{I}_{\tau_m<\infty}\} = e^{-\sigma m}$

$\Rightarrow$ As $\alpha = \sigma\mu + \frac{1}{2}\sigma^2$

We have $\mathbb{E} e^{-\alpha\tau_m} = e^{m\mu - m\sqrt{2\alpha+\mu^2}}$

Judgement Questions.

a) False

$$\mathbb{E}X(t) = \frac{1}{2}\mathbb{E}W_t + \frac{1}{2}\mathbb{E}B_t = \frac{W+B}{2} \qquad \text{Martingale}$$

$$[X,X](t) = \frac{1}{4}\{[W,W](t) + [W,B](t) + [B,B](t) + [B,W](t)\}$$

$$= \frac{1}{4}\{t + t + [W,B](t) + [B,W](t)\}$$

Moreover: Because of Independence

$$[B,W](t) = [W,B](t) = \sum_{j=0}^{N-1}(W(t_{j+1}) - W(t_j))(B(t_{j+1}) - B(t_j)) \quad \|\Pi\| \to 0$$

$$\leq \max|W(t_{j+1}) - W(t_j)| \sum |B(t_{j+1}) - B(t_j)| \quad \|\Pi\| \to 0$$

$$= 0$$

So, $[X,X](t) = \frac{1}{2}t \neq t$

b) True

$$\mathbb{E}[Z_t \mid \mathcal{F}_s] = \mathbb{E}[\tfrac{1}{2}(X_t + Y_t) \mid \mathcal{F}_s] = \frac{1}{2}\mathbb{E}(X_t \mid \mathcal{F}_s) + \mathbb{E}(Y_t \mid \mathcal{F}_s)$$

$$= \frac{1}{2}X_s + \frac{1}{2}Y_s = \frac{1}{2}(X_s + Y_s) = Z_s$$

c) False! Only if it is CONTINUOUS!

Proof in Continuous Process

Brownian Motion W has a finit none-zero quadratic variation. t

But $FV(W) = \lim_{\|\Pi\| \to 0} \sum_{j=0}^{N-1} |W(t_{j+1}) - W(t_j)| = \infty$

$$[X,X] = \lim_{\|\Pi\| \to 0} \sum_{j=0}^{N-1} |X(t_{j+1}) - X(t_j)|^2$$

$$\leq \lim_{\|\Pi\| \to 0} \max|X(t_{j+1}) - X(t_j)| \sum_{j=0}^{N-1} |X(t_{j+1}) - X(t_j)|$$

Equivalently: $FV(X) \cdot \lim_{\|\Pi\| \to 0} \max|X(t_{j+1}) - X(t_j)| \geq$ quadratic variation > 0

If X is continuous, $FV(X) = \infty$

However, let's see X is discontinous, and has a jump.

For example $X(t) = \begin{cases} 0 & 0<t<1 \\ 1 & t \geq 1 \end{cases}$

$[X,X](t) = 1 \qquad FV(X) = 1$

Bonus Problem

I.2 i) ~~$f(\sigma) = pe^{\sigma} + qe^{-\sigma} \geq 2\sqrt{pq\, e^{\sigma-\sigma}}$~~

~~geometry inequation~~

~~$2\sqrt{p(1-p)} >$~~

i) $f(\sigma) = pe^{\sigma} + qe^{-\sigma} = pe^{\sigma} + (1-p)e^{-\sigma}$

suppose $e^{\sigma} = y \quad e^{-\sigma} = \frac{1}{y}$

$f(\sigma) = py + \frac{q}{y}$

$\frac{\partial f}{\partial y} = p + -\frac{q}{y^2} = p - \frac{q}{y^2}$

$\sigma > 0 \Rightarrow y > 1$, moreover $p > q$ so $\frac{\partial f}{\partial y} > 0$

$f(\sigma) > f(\sigma)\big|_{e^{\sigma}=1} = p + q = 1$

ii) ~~$E[S_n \mid \mathcal{F}_s]$~~ $= E\left[e^{\sigma M_n} \left(\frac{1}{f(\sigma)}\right)^n \mid \mathcal{F}_s \right]$

$E\left[\frac{S_{n+1}}{S_n}\right] = E\left[e^{\sigma(M_{n+1} - M_n)} \frac{1}{f(\sigma)} \right]$

$= p \cdot \frac{e^{\sigma}}{f(\sigma)} + q \cdot \frac{e^{-\sigma}}{f(\sigma)}$

$= \frac{pe^{\sigma} + qe^{-\sigma}}{f(\sigma)} = 1$

S_n is a Martingale

iii) $S_n = e^{\sigma M_n} \left(\frac{1}{f(\sigma)}\right)^n$ is a Martingale

$$\mathbb{E}\left(S_{\tau_1 \wedge n}\right) = \mathbb{E}\left(e^{\sigma M_{\tau_1 \wedge n}} \left(\frac{1}{f(\sigma)}\right)^{\tau_1 \wedge n}\right) = S_0 = 1$$

if $\tau_m = \infty$, $S_{\tau_1 \wedge n} = S_n$

if $\tau_m < \infty$, $S_{\tau_1 \wedge n} = S_{\tau_1}$

So, $\lim\limits_{n \to \infty} S_{\tau_1 \wedge n} = \chi_{\{\tau_1 < \infty\}} \, e^{\sigma M_{\tau_1}} \left(\frac{1}{f(\sigma)}\right)^{\tau_1}$

$$\mathbb{E}\left[\lim_{n\to\infty} S_{n \wedge \tau_1}\right] = \mathbb{E}\left[\chi_{\{\tau_1<\infty\}} e^{\sigma \cdot 1} \left(\frac{1}{f(\sigma)}\right)^{\tau_1}\right] = 1$$

$$e^{-\sigma} = \mathbb{E}\left[\chi_{\{\tau_1<\infty\}} \left(\frac{1}{f(\sigma)}\right)^{\tau_1}\right]$$

$$\sigma \to \infty \Rightarrow 1 = \mathbb{E}\left[\chi_{\{\tau_1<\infty\}}\right] = P\{\tau_1 < \infty\}$$

iv) As $\tau_1 < \infty$ Almost surely, cancel the χ function.

We have $e^{-\sigma} = \mathbb{E}\left[\left(\frac{1}{f(\sigma)}\right)^{\tau_1}\right]$

let $\alpha = \frac{1}{f(\sigma)}$ $\quad \alpha \in (0,1)$ $\quad \sigma = f^{-1}\left(\frac{1}{\alpha}\right)$

so, $\mathbb{E}(\alpha^{\tau_1}) = e^{-f^{-1}\left(\frac{1}{\alpha}\right)} = \frac{1-\sqrt{1-4pq\alpha^2}}{2q\alpha}$

v) $\mathbb{E}(\tau_1 \alpha^{\tau_1}) = \mathbb{E}\left(\frac{\partial}{\partial \tau_1} \alpha^{\tau_1}\right) = \frac{\partial}{\partial \tau_1} e^{-f^{-1}\left(\frac{1}{\alpha}\right)}$

$$= \frac{1}{2q}\left[-\frac{1}{2}\left(1-4pq\alpha^2\right)^{-\frac{1}{2}}\left(-4pq\,2\alpha\right)\alpha^{-1} + \left(1-\sqrt{1-4pq\alpha^2}\right)\alpha^{-2}\right)$$

let $\alpha \uparrow 1$ $\quad \mathbb{E}(\tau_1) = \frac{1}{2p-1}$