4.10. (i) $X_t = \Delta_t S_t + \Gamma_t M_t$ (4.10.16)

$dX_t = \Delta_t dS_t + r(X_t - \Delta_t S_t)dt$ (4.10.9)

Proof: $S_t d\Delta_t + dS_t d\Delta_t + M_t d\Gamma_t + dM_t d\Gamma_t = 0$ (4.10.15)

4.10.15. 4.10.9 is ~~a~~ Itô differential equations.

Using Itô's Lemma into 4.10.16 (Train rule)

$$dX_t = \Delta_t dS_t + S_t d\Delta_t + dS_t d\Delta_t + \Gamma_t dM_t + M_t d\Gamma_t + d\Gamma_t dM_t \quad ①$$

Substitude 4.10.16 into 4.10.19

$$dX_t = \Delta_t dS_t + r\Gamma_t M_t \, dt \quad ②$$

Combine ① and ② $\Delta_t dS_t$ can be canceled.

$$S_t d\Delta_t + dS_t d\Delta_t + \underline{\Gamma_t dM_t} + M_t d\Gamma_t + d\Gamma_t dM_t - \underline{r\Gamma_t M_t dt} =$$

Moreover $M_t = e^{rt}$ $dM_t = rM_t dt$,

We have proofed 4.10.15.

(ii). The correct version of the value that long a call and short Δ_t sh

of underlying asset is

$$N_t = c(t, S_t) - \Delta_t S_t$$

$$dN_t = dc - \Delta_t dS_t - S_t d\Delta_t - d\Delta_t dS_t \quad ③$$

$$\Gamma_t = N_t / M_t$$

$$d\Gamma_t = d(e^{-rt} N_t) = -r \frac{N_t}{M_t} dt + e^{-rt} dN_t \quad ④$$

$$\Rightarrow M_t d\Gamma_t = -rN_t dt + dN_t \quad ⑤$$

Substitude ~~these~~ ④⑤ into 4.10.15

We got :

$$S_t d\Delta_t + dS_t d\Delta_t = rN_t dt + dN_t + \frac{dM_t}{M_t}\overbrace{\left(-rN_t dt + dN_t\right)}^{=0} \quad ⑥$$

Moreover we have dN_t as the formed in ③

we have $dM_t = rM_t dt$ ⑦

Then, $\cancel{S_t d\Delta_t} + \cancel{dS_t d\Delta_t} - rN_t dt + dC - \Delta_t dS_t - \cancel{S_t d\Delta_t} - \cancel{d\Delta_t dS_t} +$
$r dt\left(-rN_t dt + dC - \Delta_t dS_t - S_t d\Delta_t - d\Delta_t dS_t\right) = 0$

$$-rN_t dt + dC - \Delta_t dS_t + o(dt) = 0$$

Ignore the higher order items, we get the equation

$rN_t dt = dC - \Delta_t dS_t$, the same as previous ~~stat~~ argum

Similarly, to ~~cancel the stochastic item, we have~~ expand dc as Itô Lemma :

$$rN_t dt = C_x dS + \frac{1}{2} C_{xx} dS dS + C_t dt - \Delta_t dS_t$$

$$\left(\cancel{C_x \mu S_t} + \frac{1}{2} C_{xx} \sigma^2 S_t^2 + C_t\right) dt + \left(C_x \cancel{\sigma S_t} - \Delta_t\right) dS$$

a) $\Delta_t = C_x$

b) $rN_t = \frac{1}{2} C_{xx} \sigma^2 S_t^2 + C_t$

or $C_t + \frac{1}{2} C_{xx} \sigma^2 S_t^2 = r\left(C(t,S_t) - C_x S_t\right)$

$$C_t + \frac{1}{2} C_{xx} \sigma^2 S_t^2 + rC_x S_t = rC(t,S_t) \quad \ldots\ldots (4.$$

4.11 $dX_t = dc(t,S_t) - c_x(t,S_t)dS_t + r[X_t - c(t,S_t) + S_t c_x]dt$
$- \frac{1}{2}(\sigma_2^2 - \sigma_1^2)S^2 c_{xx}\, dt$

$dc(t,S_t) = (c_t + \frac{1}{2}c_{xx}\sigma_2^2 S_t^2)dt + c_x dS_t$

Then $dX_t = dt\left[c_t + \frac{1}{2}c_{xx}\sigma_2^2 S_t^2 + r[X_t - c(t,S_t) + S_t c_x] - \frac{1}{2}(\sigma_2^2-\sigma_1^2)S^2 c_{xx}\right]$

$= dt\left\{c_t + \frac{1}{2}c_{xx}\sigma_1^2 S_t^2 + r[X_t - c + S_t c_x]\right\}$ ①

Moreover

From Black-Scholes formula, $c(t,x)$ statisfies :

$$\left(\frac{\partial}{\partial t} + rx\frac{\partial}{\partial x} + \frac{1}{2}x^2\sigma_1^2\frac{\partial^2}{\partial x^2} - r\right)c(t,x) = 0$$

$$c_t + rxc_x + \frac{1}{2}x^2\sigma_1^2 c_{xx} - rc = 0 \quad ②$$

Substitude to ① $\Rightarrow$ $dX_t = rX_t dt$

Then, $X_t = \mathcal{E}\left(\int r\,dt\right) = \exp\left(\int_0^t r\,ds - 0 - \frac{1}{2}\cdot 0 + c\right)$
$= X_0 e^{rt}$

or $d(e^{-rt}X_t) = 0$

4.12 (i) the value of a forward contract

$$f(t,x) = x - e^{-r(T-t)}K \qquad 4.5.26$$

put-call parity

$$f(t,x) = c(t,x) - p(t,x) \qquad 4.5.29.$$

$$p(t,x) = c(t,x) - f(t,x) = c(t,x) - x + e^{-r\tau}K$$

Since $p_x(t,x) = \frac{\partial}{\partial x} p(t,x) = \frac{\partial}{\partial x}\left(c(t,x) - x + e^{-r\tau}K\right)$

$$= c_x(t,x) - 1 + 0$$
$$= N(d_+) - 1$$
$$= -N(-d_+)$$

$\cancel{d_+ = d_- + \sqrt{\tau}\sigma}$

$$p_{xx}(t,x) = \frac{\partial^2}{\partial x^2} p(t,x) = \frac{\partial^2}{\partial x^2}\left(c(t,x) - x + e^{-r\tau}K\right)$$
$$= \frac{\partial^2}{\partial x^2} c(t,x) = c_{xx} = \frac{N'(d_+)}{\sigma x\sqrt{\tau}}$$

$$p_t(t,x) = \frac{\partial}{\partial t} p(t,x) = \frac{\partial}{\partial t}\left(c(t,x) - x + e^{-r\tau}K\right)$$
$$= c_t(t,x) + \cancel{r}e^{-r\tau}K$$
$$= [1 - N(d_-)]re^{-r\tau}K - \frac{\sigma x}{2\sqrt{\tau}} N'(d_+)$$
$$= +N(-d_-)re^{-r\tau}K - \frac{\sigma x}{2\sqrt{\tau}} N'(d_+)$$

(ii) Let X_t be a portfolio with short position in put, underlying ~~and money account~~

$$dX_t = -dp(t,x) + \Delta dS_t \cancel{+ r(X_t - p - \Delta S_t)dt}$$
$$= -\left(\cancel{rx\frac{\partial}{\partial x}} + \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}\right) p\, dt + -\frac{\partial}{\partial x} p\, dx + \Delta dS_t$$
$$\cancel{+ r(X_t - p - \Delta S_t)dt}$$
$$= \cancel{\frac{\partial}{\partial t}} -\left(p_t + \frac{1}{2}\sigma^2 x^2 p_{xx}\right)dt + (-p_x + \Delta)dx$$

In order to cancel risk here. we should make Stochastic item equals to zero. such that:

$$(-P_x + \Delta)dx = 0 \qquad \Delta = +P_x = -N(-d_+) < 0$$

So, the agent should short underlying asset.

In order to be arbitrage free, there should statisfy:

$$dX_t = rX_t dt = r(-P + \Delta \cancel{X})dt$$

$$-\left(P_t(t,x) + \frac{1}{2}\sigma^2 x^2 P_{xx}(t,x)\right) = r(-P(t,x) + \Delta x)$$

$$= r(-P(t,x) + P_x x)$$

$$\text{or} \quad \left(rx\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t} - r\right) P(t,x) = 0$$

Since

$$\cancel{X_t = (-P} \quad X_t = -P + \Delta_t S_t = -P + P_x S_t < 0$$

So, the agent can long ($P - P_x S_t > 0$) position in money market

(iii)

See (ii). It is the same SDE as call option

4.13 $\begin{cases} B_1(t) = W_1(t) \\ B_2(t) = \int_0^t \rho(s)\, dW_1(s) + \int_0^t \sqrt{1-\rho(s)^2}\, dW_2(s) \end{cases}$

Let $B(t) = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}(t)$ $\quad W(t) = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}(t)$

$$dB(t) = \begin{pmatrix} 1 & 0 \\ \rho(t) & \sqrt{1-\rho(t)^2} \end{pmatrix} dW(t) = A\, dW(t) \quad (A \text{ is inversible})$$

$$A^{-1} = \begin{pmatrix} 1 & 0 \\ -\frac{\rho}{\sqrt{1-\rho^2}} & \frac{1}{\sqrt{1-\rho^2}} \end{pmatrix} \qquad A^{-1} dB_t = dW_t$$

$$dW_1 = dB_1$$
$$dW_2 = -\frac{\rho}{\sqrt{1-\rho^2}} dB_1 + \frac{1}{\sqrt{1-\rho^2}} dB_2$$
$$d[W_1, W_2] = -\frac{\rho}{\sqrt{1-\rho^2}} dB_1^2 + \frac{dB_1 dB_2}{\sqrt{1-\rho^2}} = 0 \Rightarrow \text{Independence.}$$

4.14 (i) Define $Z_j = f''(W(t_j))\left[(W(t_{j+1}) - W(t_j))^2 - (t_{j+1} - t_j)\right]$

$W(t_j)$ t_j is $\mathcal{F}(t_j)$-measurable, it is ~~surely~~ clearly $\mathcal{F}(t_{j+1})$-measurable

$W(t_{j+1})$ t_{j+1} is $\mathcal{F}(t_{j+1})$-measurable

So, Z_j is $\mathcal{F}(t_{j+1})$-measurable

$$\mathbb{E}[Z_j \mid \mathcal{F}(t_j)] \xlongequal[\text{what is known}]{\text{taking out}} f''(W(t_j))\left(\mathbb{E}\left[(W(t_{j+1}) - W(t_j))^2 \mid \mathcal{F}(t_j)\right] - (t_{j+1} - t_j)\right)$$

$$= f''(W(t_j))\left[\mathbb{E}[W^2_{(t_{j+1}-t_j)}] - (t_{j+1} - t_j)\right] = 0$$

with $\mathbb{E}[W^2_{(t_{j+1}-t_j)}] = t_{j+1} - t_j$ written above the term; note marks 1 and 2 refer to the lines below.

1: Marcov property for Itô diffusion $\quad E^x[f(X_{t+h}) \mid \mathcal{F}_t^{(m)}](\omega) = E^{X_t(\omega)}[f(X_h)]$

2: Using the truth that the transition probability of Brownian motion

$p(t, x, y) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{|x-y|^2}{2t}\right)$, variance equals to time spent t

$$\mathbb{E}[Z_j^2 \mid \mathcal{F}(t_j)] = \mathbb{E}\Big[f''^2(W(t_j)) \big[(W(t_{j+1}) - W(t_j))^4 + (t_{j+1} - t_j)^2 - 2(t_{j+1} - t_j)(W(t_{j+1}) - W(t_j)) \big] \Big| \mathcal{F}(t_j) \Big]$$

$$= f''^2(W(t_j)) \Big(\mathbb{E}[(W(t_{j+1}) - W(t_j))^4] + (t_{j+1} - t_j)^2 - 2\mathbb{E}[(t_{j+1} - t_j)(W(t_{j+1}) - W(t_j))^2] \Big)$$

$$= f''^2(W(t_j)) \ \cancel{3(t_{j+1}-t_j)^2} \Big(\mathbb{E}[W^4_{t_{j+1}-t_j}] + (t_{j+1} - t_j)^2 - 2(t_{j+1} - t_j) \cdot \mathbb{E}[W^2_{(t_{j+1}-t_j)}] \Big)$$

Independence increment ~~Marcov~~ property

$$= f''^2(W(t_j)) \, (3 + 1 - 2)(t_{j+1} - t_j)^2 = 2 f''^2(W(t_j)) (t_{j+1} - t_j)^2$$

Where $\mathbb{E}[W^4] = 3\mathbb{E}^2[W^2]$ as W is a Gaussian process

(ii) $$\mathbb{E} \sum_{j=0}^{n-1} Z_j = \mathbb{E}[Z_0 \mid \mathcal{F}(t_0)] + \cdots + \mathbb{E}[Z_{n-1} \mid \mathcal{F}(t_{n-1})] = 0$$

(iii) $$\mathrm{Var}\Big[\sum_{j=0}^{n-1} Z_j\Big] = \mathbb{E}\Big[\Big(\sum_{j=0}^{n-1} Z_j\Big)^2\Big] - \mathbb{E}^2\Big(\sum_{j=0}^{n-1} Z_j\Big)$$

$$= \mathbb{E}\Big[\Big(\sum_{j=0}^{n-1} Z_j\Big)^2\Big] = \mathbb{E}\Big[\sum_{i,j} Z_i Z_j\Big] = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \mathbb{E}[Z_i Z_j]$$

$$= \mathbb{E}\Big[\sum_{j=0}^{n-1} Z_j^2 + 2\sum_{0 \le i < j \le n-1} Z_i Z_j\Big]$$

$$= \sum_{j=0}^{n-1} 2 f''^2(W(t_j))(t_{j+1} - t_j)^2 + 2\sum_{0 \le i < j \le n-1} \mathbb{E}\big[Z_i \, \mathbb{E}[Z_j \mid \mathcal{F}(t_j)] \big]$$

$$= \sum_{j=0}^{n-1} 2 f''^2(W(t_j))(t_{j+1} - t_j)^2$$

So, $$\lim_{\|\Pi\| \to 0} \mathrm{Var}\Big[\sum_{j=0}^{n-1} Z_j\Big] = \lim_{\|\Pi\| \to 0} 2\sum_{j=0}^{n-1} f''^2(W(t_j))(t_{j+1} - t_j)^2 = 0$$

4.15 (i) $B_i = \sum_{j=1}^{d} \int_0^t \frac{\sigma_{ij}(u)}{\sigma_i(u)} dW_j(u)$

~~a) Continuity~~ : B_i is the sum of integral with respect to a

a) Continuity and martingale — continuous and Martingale function, so, B_i is continuous and a martingale

b) $B_i(0) = 0$

c) Since B_i is continue process, it's quadratic variation

$$[B_i]_t = \int_0^t d[B_i]_s = \int_0^t \left(\sum_{j=1}^{d} \frac{\sigma_{ij}^2(s)}{\sigma_i^2(s)} \right) ds$$

$$= \int_0^t 1 \cdot ds = t$$

Using Levy's characterization, it's a Brownian motion.

(ii) $dB_i(u) = \sum_{j=1}^{d} \frac{\sigma_{ij}(u)}{\sigma_i(u)} dW_j(u)$

$$dB_i(t)\, dB_j(t) = \sum_{k=1}^{d} \frac{\sigma_{ik}(t)}{\sigma_i(t)} \sum_{l=1}^{d} \frac{\sigma_{jl}(t)}{\sigma_j(t)} dW_k(t)\, dW_l(t)$$

$$= \sum_{k=1}^{d} \frac{\sigma_{ik}(t)\sigma_{jk}(t)}{\sigma_i(t)\,\sigma_j(t)} dt + o(dt)$$

$$= \rho_{ij}(t)\, dt$$

where $\rho_{ij}(t) = \frac{\sum_{k=1}^{d} \sigma_{ik}(t)\, \sigma_{jk}(t)}{\sigma_i(t)\, \sigma_j(t)}$

5.1 (i) $dX_t = \sigma_t dW_t + (\alpha_t - R_t - \frac{1}{2}\sigma_t^2)dt$

$f: x \mapsto S(0)e^x \quad \frac{\partial}{\partial x} f = f \quad \frac{\partial^2}{\partial x^2} f = f$

$df(x) = \frac{\partial f}{\partial x}(x)dx + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x)dx^2$

$df(X_t) = f(X_t)\left(\sigma_t dW_t + (\alpha_t - R_t - \frac{1}{2}\sigma_t^2)dt\right) + \frac{1}{2}f(X_t)\sigma_t^2 dt$

$= f(X_t)\left[(\alpha_t - R_t)dt + \sigma_t dW_t\right]$

Because $f(X_t) = D_t S_t \quad df(X_t) = D_t S_t \sigma_t [\Theta dt + dW_t]$

(ii) $dS_t = \alpha_t S_t dt + \sigma_t S_t dW_t$

$dD_t = -R_t D_t dt \Rightarrow \quad D_t = e^{-\int_0^t R_s ds}$

$d[D,S]_t = 0$

$d(D_t S_t) = D_t dS_t + S_t dD_t + dD_t dS_t$

$\cancel{= D_t dS_t + (S_t - R_t D_t dt)}$

$dD_t dS_t = d[D,S]_t$ since D_t is a FV process, S is a semimartingale, $d[D,S]_t \equiv 0$

So, $d(D_t S_t) = D_t dS_t + S_t dD_t$

$= D_t(\alpha_t S_t dt + \sigma_t S_t dW_t) - R_t D_t S_t dt$

$= (\alpha_t - R_t) D_t S_t dt + D_t \sigma_t S_t dW_t$

$= D_t\left[(\alpha_t - R_t)S_t dt + \sigma_t S_t dW_t\right]$

$= \sigma_t D_t S_t [\Theta_t dt + dW_t]$

5.2 risk-neutral pricing formula

$$D_t V_t = \tilde{E}[D_T V_T \mid \mathcal{F}_t]$$

Preliminary

1) $\tilde{E}[\cdot] = E[Z \cdot]$

2) For any $\mathcal{F}_t$-measurable variable Y $\tilde{E}[Y] = E[Z_t Y]$

3) Let Y be an $\mathcal{F}_t$-measurable random variable

$$\tilde{E}[Y \mid \mathcal{F}_s] = \frac{1}{Z_s} E[Y Z_t \mid \mathcal{F}_s] = E\left[Y \frac{d\tilde{P}|_{\mathcal{F}_s}}{dP|_{\mathcal{F}_s}} \mid \mathcal{F}_s\right]$$

In this case $D_T V_T$ is $\mathcal{F}_T$-measurable, using Preliminary 3)

$$\tilde{E}[D_T V_T \mid \mathcal{F}_t] = E[D_T V_T Z_T \mid \mathcal{F}_t] \frac{1}{Z_t}$$

So, $D_t V_t Z_t = E[D_T V_T Z_T \mid \mathcal{F}_t]$

5.3. (i) $c(t,x) = \tilde{E}[e^{-r\tau} (S_T - k)^+]$

$$\frac{\partial c}{\partial x} = \tilde{E}\left[e^{-r\tau} \frac{\partial}{\partial x}(S_T - k)^+\right]$$

In risk neutral measure: ~~$= \tilde{E}[e^{-r\tau} \mathbb{1}_{(x>k)}]$~~ $dS_T = r S_T dT + \sigma S_T d\tilde{W}_T$

$$\Rightarrow S_T = S_0 \exp\left(\int r dT + \int \sigma d\tilde{W}_T - \frac{1}{2}\int \sigma^2 dT\right)$$

$$= S_0 \exp\left(\int_0^T (r - \tfrac{1}{2}\sigma^2) ds + \int_0^T \sigma d\tilde{W}_s\right)$$

$$= x \exp\left((r - \tfrac{1}{2}\sigma^2)T + \sigma \tilde{W}_T\right)$$

$$\frac{\partial}{\partial x} c(t,x) = \tilde{E}[e^{-r\tau} (S_T - k)' \mathbb{1}_{S_T > k}]$$

~~Then $\frac{\partial c(t,x)}{\partial x} = \tilde{E}\left[e^{-r\tau} \frac{\partial}{\partial x}\left(x \exp((r - \frac{1}{2}\sigma^2)T + \sigma \tilde{W}_T - k\right) \mathbb{1}_{S_T > k}\right]$~~

$$\frac{\partial}{\partial x}(S_T - K)^+ = \mathbb{1}_{S_T>K} \frac{\partial S_T}{\partial x} = \mathbb{1}_{S_T>K} \frac{\partial}{\partial x}\left[x \exp\left((r-\tfrac{1}{2}\sigma^2)T + \sigma \tilde{W}_T\right)\right]$$

$$= \mathbb{1}_{S_T>K} \exp\left[(r-\tfrac{1}{2}\sigma^2)T + \sigma \tilde{W}_T\right]$$

$$\frac{\partial c(0,x)}{\partial x} = \tilde{E}\left[e^{-r\tau}\ \mathbb{1}_{S_T>K} \exp\left[(r-\tfrac{1}{2}\sigma^2)T + \sigma\tilde{W}_T\right]\ \right]$$

$$= \tilde{E}\left[\cancel{e^{rt}}\ \mathbb{1}_{S_T>K} \exp\left[\cancel{\phi} - \tfrac{1}{2}\sigma^2 T + \sigma\tilde{W}_T\right]\ \right]$$

$$= \cancel{e^{rt}}\ e^{-\frac{\sigma^2}{2}T}\ \tilde{E}\left[\mathbb{1}_{S_T>K} \exp(\sigma\tilde{W}_T)\ \right]$$

Where $S_T = x\exp\left((r-\tfrac{1}{2}\sigma^2)T + \sigma\tilde{W}_T\right) > K$

$$\Leftrightarrow \quad (r-\tfrac{1}{2}\sigma^2)T + \sigma\tilde{W}_T > \ln\frac{K}{x}$$

$$-\frac{\tilde{W}_T}{\sqrt{T}} < \frac{\ln\frac{x}{K} + (r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

Define random variable $Z = -\dfrac{\tilde{W}_T}{\sqrt{T}} \sim N(0,1)$, $d_- = \dfrac{\ln\frac{x}{K} + (r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$

$$\frac{\partial c(0,x)}{\partial x} = e^{-\frac{\sigma^2}{2}T} \int_0^{d_-} \exp(-\sigma\sqrt{T}Z)\frac{e^{-\frac{Z^2}{2}}}{\sqrt{2\pi}}\,dZ$$

$$= e^{-\frac{\sigma^2}{2}T} \int_0^{d_-} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{Z^2 + 2\sigma\sqrt{T}Z + \sigma^2 T}{2}}\, e^{\frac{\sigma^2 T}{2}}\, dZ$$

$$= \int_0^{d_-} \frac{1}{\sqrt{2\pi}} e^{-\frac{(Z+\sigma\sqrt{T})^2}{2}}\,dZ$$

$d_+ = d_- + \sigma\sqrt{T}$

$$= \int_0^{d_+} \frac{1}{\sqrt{2\pi}} e^{-\frac{(Z+\sigma\sqrt{T})^2}{2}}\,d(Z+\sigma\sqrt{T})$$

$$= N(d_+)$$

(ii) $\hat{W}_t = \tilde{W}_t - \sigma t$

$[X_t, \hat{W}_t] = \sigma t$, let $X_t = \sigma \tilde{W}_t$

Girsanov's Theorem: $\left.\frac{d\hat{P}}{d\tilde{P}}\right|_{\mathcal{F}_t} = \mathcal{E}(X_t) = \exp\left(X_t - X_0 - \frac{1}{2}[X]_t \right)$

$$= \exp\left(\sigma \tilde{W}_t - \frac{1}{2}\sigma^2 t \right)$$

In measure $\hat{P}$, $\hat{W}$ is a Brownian motion

Moreover, $\frac{\partial}{\partial x} c(0,x) = \tilde{E}[\, 1_{S_T > K} \exp(\sigma \tilde{W}_T) \,] e^{-\frac{\sigma^2}{2}T}$

$$\cancel{= \hat{E}\left[\left.\frac{d\tilde{P}}{d\hat{P}}\right|_{\mathcal{F}_T} 1_{S_T > K} \exp(\sigma \tilde{W}_T) \right.}$$

$$\hat{P}(S_T > K) = \hat{E}[\, 1_{S_T > K} \,]$$

$$= \tilde{E}\left[1_{S_T > K} \left.\frac{d\hat{P}}{d\tilde{P}}\right|_{\mathcal{F}_T} \right] = \tilde{E}\left[1_{S_T > K} \exp\left(\sigma \tilde{W}_T - \frac{1}{2}\sigma^2 T\right) \right]$$

$$= \frac{\partial c}{\partial x}(0, x)$$

(iii) $S_T = x \exp\left((r - \frac{1}{2}\sigma^2)\cancel{T} + \sigma \tilde{W}_T \right)$

$$= x \exp\left((r + \frac{1}{2}\sigma^2) T + \sigma \hat{W}_T \right)$$

$S_T > K \Rightarrow (\frac{\sigma^2}{2} + r)T + \sigma \hat{W}_T > \ln\frac{K}{x} \Rightarrow -\frac{\hat{W}_T}{\sqrt{T}} < \frac{\ln\frac{x}{K} + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} = c$

$\hat{P}(S_T > K) = \hat{E}[\, 1_{S_T > K} \,] =$

$$\int_0^{d_+} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = N(d_+)$$

SPE

5.4 $\begin{cases} \frac{dS_t}{S_t} = r_t dS_t + \sigma_t d\tilde{W}_t \\ dD_t = -r_t D_t dt \end{cases}$

$c(0,x) = \tilde{E}[\, D_T (S_T - K)^+ \,]$

(i) $dS_t = S_t (r_t dS_t + \sigma_t d\tilde{W}_t) = S_t \, dX_t$

Let $dX_t = r_t dS_t + \sigma_t d\tilde{W}_t$

Then, using Doleans exponential

$$S_t = \exp\left(X_t - X_0 - \frac{1}{2}[X]_t \right)$$
$$= S_0 \exp\left(\int_0^t r_u dS_u + \int_0^t \sigma_u d\tilde{W}_u - \frac{1}{2}\int_0^t \sigma_u^2 du \right)$$
$$= S_0 \exp\left(\int_0^t \left(r_u - \frac{\sigma_u^2}{2}\right) du + \int_0^t \sigma_u d\tilde{W}_u \right)$$

Let $X_t = \int_0^t \left(r_u - \frac{\sigma_u^2}{2}\right) du + \int_0^t \sigma_u d\tilde{W}_u$

The first term is a constant

The second term is a Gaussian random variable whose distribution satisfies: $N(0, \int_0^t \sigma_u^2 du)$

Then, $\cancel{S_t}$ is a normal random variable

$$\cancel{S_t} \sim N\left(\int_0^t \left(r_u - \frac{\sigma_u^2}{2}\right) du , \int_0^t \sigma_u^2 du \right)$$

(ii) In the case, r_t, σ_t is not constant but deterministic

Risk-neutral pricing formula

$$c(0,x) = \tilde{E}[D_T(S_T-K)^+]$$

$$= \int D_T(S_T-K) I_{S_T>K} \, d\cancel{S}_T \, p(\cancel{S}_T)$$

$$S_T > K \Rightarrow S_0 \exp\left(\int_0^T (r_u - \frac{\sigma_u^2}{2}) du + \int_0^T \sigma_u d\tilde{W}_u\right) > K$$

$$X_T = \int_0^T (r_u - \frac{\sigma_u^2}{2}) du + \int_0^T \sigma_u d\tilde{W}_u > \ln\frac{K}{S_0}$$

$$X_T \sim N\left(\int_0^T (r_u - \frac{\sigma_u^2}{2}) du, \int_0^T \sigma_u^2 du\right)$$

$$\cancel{c(0,x) = \int_{\ln\frac{K}{S_0}}^{+\infty} D_T \exp(S_0 e^{X_T} - K) \frac{e^{-\frac{(X_T - \int_0^T (r_u - \frac{\sigma_u^2}{2})du)^2}{2}}}{\sqrt{2\pi \int_0^T \sigma_u^2 du}} dX_T}$$

Let $\bar{r} = \frac{1}{T}\int_0^T r_t dt$

$$\bar{\sigma} = \sqrt{\frac{1}{T}\int_0^T \sigma_u^2 du}$$

$$Z = -\frac{X_T - \int_0^T (r_u - \frac{\sigma_u^2}{2}) du}{\sqrt{\int_0^T \sigma_u^2 du}} \sim N(0,1) \qquad Z = -\frac{X_T - T\bar{r} + \int_0^T \frac{\sigma_u^2}{2} du}{\sqrt{T}\,\bar{\sigma}}$$

$$\cancel{c(0,x) =} \; S_0 e^{X_T} > K \Rightarrow -\sqrt{T}\bar{\sigma} Z + T\bar{r} - \int_0^T \frac{\sigma_u^2}{2} du > \ln\frac{K}{S_0}$$

$$\Rightarrow Z < \frac{\ln\frac{S_0}{K} + T\bar{r} - \int_0^T \frac{\sigma_u^2}{2} du}{\sqrt{T}\,\bar{\sigma}}$$

$$D(T) = e^{-\int_0^T r_u du} = e^{-\bar{r}}$$

$$c(0,x) = \tilde{E}\left[D(T)(S_T - K)^+\right]$$

$$= \tilde{E}\left[D(T)(S_0 e^{X_T} - K)^+\right] \qquad -\sqrt{T}\bar{\sigma}Z + \left(\bar{r} - \frac{\bar{\sigma}^2}{2}\right)T$$

$$= \tilde{E}\left[D(T)\left\{S_0 \exp\left(-\sqrt{T}\bar{\sigma}Z + T\bar{r} - \int_0^T \frac{\sigma_u^2}{2}du\right) - K\right\}^+\right]$$

$$= \cancel{D(T)}\, e^{-\bar{r}} \int_{-\infty}^{\frac{1}{\sqrt{T}\bar{\sigma}}\left[\ln\frac{S_0}{K} + \left(r - \frac{\bar{\sigma}^2}{2}\right)T\right]} \left(S_0 \exp\left(-\sqrt{T}\bar{\sigma}Z + \left(\bar{r} - \frac{\bar{\sigma}^2}{2}\right)T\right) - K\right) \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz$$

Comparing with BSM Formula with constant rate r and volatility σ.

$$c(0,x) = e^{-r} \int_{-\infty}^{\frac{1}{\sqrt{T}\sigma}\left[\ln\frac{S_0}{K} + \left(r - \frac{\sigma^2}{2}\right)T\right]} \left(S_0 \exp\left(-\sqrt{T}\sigma Z + \left(r - \frac{\sigma^2}{2}\right)T\right) - K\right) \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz$$

Only replace r with $\bar{r} = \frac{1}{T}\int_0^T r_u du$

replace σ with $\bar{\sigma} = \sqrt{\frac{1}{T}\int_0^T \sigma_u^2 du}$