5.6 $\tilde{W}(t) = W(t) - [W(t), X(t)] = W(t) - \int_0^t \Theta(u)\,du$

$X(t) = \int_0^t \Theta(u)\cdot dW(u)$

$Z(t) = \mathcal{E}(X)_t = \exp\left(X(t) - \frac{1}{2}[X]_t\right)$

$dZ(t) = Z(t)\,dX(t)$

$dX(t) = \Theta(u)\cdot dW(u)$

2-dimensional Lévy's theorem:

1. $M_1(t)$ $M_2(t)$ are $\mathcal{F}(t)$-measurable martingale
2. $M_1(t)$ $M_2(t)$ have continuous path
3. $M_1(0) = M_2(0) = 0$
4. $[M_i, M_j](t) = t\,\delta_{ij}$

1. $\tilde{E}[\tilde{W}_i(t)|\mathcal{F}_s] = \frac{1}{Z_s}E[\tilde{W}_i(t)\,Z(t)\,\cancel{Z(s)}\,|\mathcal{F}_s] = \tilde{W}_i(s)$

Proof:

$d(\tilde{W}_i(t)\,Z(t))$ has no drift

Using Itô-product rule:

$= \tilde{W}_i(t)\,dZ(t) + Z(t)\,d\tilde{W}_i(t) + d\tilde{W}_i(t)\,dZ(t)$

$= \left(W_i(t) - \int_0^t \Theta_i(u)\,du\right) Z(t)\,\Theta(u)\cdot dW(u) + Z(t)\left(dW_i(t) - \Theta_i(t)\,dt\right)$

$+ \cancel{\left(W_i(t) - \int_0^t \Theta_i(u)\,d\right)}\ Z(t)\left(\Theta_1(t)\,dW_1(t) + \Theta_2(t)\,dW_2(t)\right)\left(dW_i(t) - \Theta_i(t)\,dt\right)$

$= \left(W_i(t) - \int_0^t \Theta_i(u)\,du\right) Z(t)\left(\Theta_1(t)\,dW_1(t) + \Theta_2(t)\,dW_2(t)\right) + Z(t)\left(dW_i(t) - \Theta_i(t)\,dt\right)$

$+ Z(t)\,\Theta_i(t)\,dW_i(t)\,dW_i(t)$

$= \left(W_i(t) - \int_0^t \Theta_i(u)\,du\right) Z(t)\left(\Theta(t)\cdot dW(t)\right) + Z(t)\,dW_i(t)$

Shows that $Z(t)\tilde{W}(t)$ is a $\mathcal{F}_t$-Martingale under P

Equivalently, $\tilde{W}(t)$ is a $\mathcal{F}_t$-Martingale under $\tilde{P}$ 1

2. $[\tilde{W}_i, \tilde{W}_j](t) = t\delta_{ij}$

Proof: $\tilde{W}_i = W_i - [X, W_i]$

$\tilde{W}_i(t) = W_i(t) - \int_0^t \theta(u)\,du$

$W_i(t)$ is path-continuous, $\int_0^t \theta(u)du$ is continuous

Then, $\tilde{W}_i(t)$ is path-continuous ②

Moreover $\tilde{W}_i(0) = W_i(0) - 0 = 0$ ③

Then, $[X, W_i](t) = \int_0^t d[X, W_i](u)$

$= \int_0^t \theta(u)\,du$ is a FV-process.

$[\tilde{W}_i, \tilde{W}_j](t) = \cancel{\int_0^t d[\tilde{W}_i, \tilde{W}_j]}$

$= \cancel{\int_0^t} [W_i - [X, W_i], W_j - [X, W_j]]$

$= [W_i, W_j] - [W_i, [X, W_j]] - [[X, W_i], W_j]$

$+ [[X, W_i], [X, W_j]]$

Since W_i, W_j is semi-martingale

$[X, W_i]$, $[X, W_j]$ is FV process

We have $[M, F] = 0$ where $\begin{cases} M \text{ is semi-martingale} \\ F \text{ is FV process} \end{cases}$

Consequently $[\tilde{W}_i, \tilde{W}_j] = [W_i, W_j] = t\delta_{ij}$ ④

Combining ①②③④, this proves the two-dimentional Girsanov's Theorem. where after transformation $\tilde{W}(t)$ is a 2-dimensional Brownian motion under probability measure $\tilde{P}$

5.10 Chooser Option

(i) at time t_0, to maximize the profit, one has chooser option would choose the option type with higher value

So, the value of choose option $= \max\{C(t_0), P(t_0)\}$

$$= \max\{C(t_0), C(t_0) - F(t_0)\} = \max\{C(t_0), C(t_0) - F(t_0)\}$$

$$= C(t_0) + \max\{0, -F(t_0)\} = C(t_0) + (e^{-r(T-t_0)}K - S(t_0))^+$$

(ii) Using risk-neutral ~~no-arbitrage~~ pricing $\quad D(t) = e^{-\int_0^t r_s ds}$

in this case $= e^{-tr}$

$$V(0) = \tilde{E}[V(t)D(t)] = \tilde{E}[V(t_0)D(t_0)]$$

$$= \tilde{E}[e^{-rt_0} C(t_0) + e^{-rt_0}(e^{-r(T-t_0)}K - S(t_0))^+]$$

$$= C(0) + \tilde{E}[e^{-rt_0}(e^{-r(T-t_0)}K - S(t_0))^+]$$

The first term ~~=~~ is the value of a call option expiring at time T with strike K.

The second term : is the value of a put option expiring at time t_0 with strike price $e^{-r(T-T_0)}K$

6.1 (i) $Z(t) = \exp\left\{ \int_t^t \sigma(v)dW(v) + \int_t^t \left(b(v) - \frac{1}{2}\sigma^2(v)\right) dv \right\} = e^0 = 1$

let $X(u) = \int_t^u \sigma(v)dW(v) + \int_t^u \left(b(v) - \frac{1}{2}\sigma^2(v)\right) dv$

$dX(u) = \sigma(u)\,dW(u) + \left(b(u) - \frac{1}{2}\sigma^2(u)\right)du \quad u \geq t$

$f(x) = e^x \quad f_x(x) = f(x) = e^x \quad f_{xx}(x) = e^x$

Itô's Lemma :

$dZ(u) = df(X_u) = Z(u)\,dX(u) + \frac{1}{2} Z(u)\,\sigma^2(u)\,du$

$\Rightarrow dZ(u) = b(u)\,Z(u)\,du + \sigma(u)\,Z(u)\,dW(u) \quad u \geq t$

(ii) $X(u) = Y(u)\,Z(u)$

Using Itô's product rule :

$dX(u) = Y(u)\,dZ(u) + Z(u)\,dY(u) + dY(u)\,dZ(u)$

$= Y(u)\,Z(u)\left(b(u)\,du + \sigma(u)\,dW(u)\right) + \left(a(u) - \cancel{\sigma(u)\gamma(u)}\right)du$

$+ \gamma(u)dW(u) + \cancel{\sigma(u)\gamma(u)}\,du$

$= \left(X(u)\,b(u) + a(u)\right)du + \left(\sigma(u)\,X(u) + \gamma(u)\right)dW(u)$

$X(t) = Y(t)\,Z(t) = x \cdot 1 = x$

6.2 (i) $X(t) = \Delta_1(t)\,B(t, T_1) + \Delta_2(t)\,B(t, T_2)$

$= \Delta_1(t)\,f(t, R(t), T_1) + \Delta_2(t)\,f(t, R(t), T_2)$

Self-financing :

$dX(t) = \Delta_1(t)\; df(t, R(t), T_1) + \Delta_2(t)\,df(t, R(t), T_2)$

To simplify the symbol , define $f(t, R(t), T_1) = f_1(t, R(t))$

$f(t, R(t), T_2) = f_2(t, R(t))$

$$d\big(D(t)X(t)\big) = D(t)\,dX(t) + X(t)\,dD(t) + d[\overset{0}{\overset{\|}{X}}, D](t)$$

$$= D(t)\big(\Delta_1 df_1 + \Delta_2 df_2\big) - R(t)D(t)\big(\Delta_1 f_1 + \Delta_2 f_2\big)\,dt$$

$$= \Delta_1(t)D(t)\big[-R(t)f_1(t, R(t))\,dt + df_1(t, R(t))\big]$$
$$+ \Delta_2(t)D(t)\big[-R(t)f_2(t, R(t))\,dt + df_2(t, R(t))\big] \quad ①$$

Since $dR(t) = \alpha(t, R(t))\,dt + \gamma(t, R(t))\,dW(t)$

$$df_1(t, R(t)) = f_t(t, R(t), T_1) + \alpha(t, R(t)) f_r(t, R(t), T_1) + \frac{\gamma^2(t, R(t))}{2} f_{rr}(\ldots)\,dt + \gamma(t, R(t)) f_r(t, R(t), T_1)\,dW(t) \quad ②$$

Using (6.9.3): $$f_t(t, R(t), T_1) + \frac{\gamma^2(t, R(t))}{2} f_{rr}(t, R(t), T_1)$$
$$= R(t) f(t, R(t), T_1) - \beta(t, R(t), T_1) f_r(t, R(t), T_1) \quad ③$$

substitute ②③ to ①

$$d\big(D(t)X(t)\big) = \Delta_1(t)D(t)\big[\alpha(t, R(t)) - \beta(t, R(t), T_1)\big] f_r(t, R(t), T_1)\,dt$$
$$+ \Delta_2(t)D(t)\big[\alpha(t, R(t)) - \beta(t, R(t), T_2)\big] f_r(t, R(t), T_2)\,dt$$
$$+ D(t)\gamma(t, R(t))\big[\Delta_1(t) f_r(t, R(t), T_1) + \Delta_2 f_r(t, R(t), T_2)\big]\,dW(t)$$

(ii) $\Delta_1(t) = S(t) f_r(t, R(t), T_2) \qquad \Delta_2(t) = S(t) f_r(t, R(t), T_1)$

Then, $$d\big(D(t)X(t)\big) = S(t)D(t) f_r(t, R(t), T_1) f_r(t, R(t), T_2)\big[\cancel{\alpha(t, R(t))} - \beta(t, R(t), T_1) + \beta(t, R(t), T_2)\big]\,dt +$$
$$\cancel{2D(t)\gamma(t, R(t)) S(t) f_r(t, R(t), T_1) f_r(t, R(t), T_2)\,dW(t)}$$
$$= D(t)\,\big|\, f_r(t, R(t), T_1) f_r(t, R(t), T_2)\big[\beta(t, R(t), T_2) - \beta(t, R(t), T_1)\big]\big|\,dt$$

If. $\beta(t,r,T)$ depend on T, $(\beta(t,r,T_1) \neq \beta(t,r,T_2))$
Then, ~~there~~ is an arbitrage ~~chance~~. opportunity.

(iii) The portfolio which invests only in the bond of maturity T
Self-financing only in bond and money market with interest rate $R(t)$, the portfolio's value satisfies:

$$dX(t) = \Delta(t)\,dB(t) + (X(t) - \Delta(t)B(t))R(t)\,dt$$

semi martingale FV

$$d\,D(t)X(t) = D(t)\,dX(t) + X(t)\,dD(t) + d[X(t), D(t)] = 0$$
$$= \Delta(t)D(t)\,dB(t) + D(t)(X(t) - \Delta(t)B(t))R(t)\,dt$$
$$+ \cancel{\Delta(t)B(t)} + D(t)R(t)X(t)\,dt$$
$$= \Delta(t)D(t)\,dB(t) - \Delta(t)D(t)B(t)R(t)\,dt$$

Since $B(t) = f(t, R(t), T)$
Itô's Lemma: $dB(t) = df(t, R(t), T) = (f_t + \alpha f_r + \frac{\gamma^2}{2} f_{rr})dt + \gamma f_r\,dW$
Then, substitude to former equation:

$$d(D(t)X(t)) = \Delta(t)D(t)\Big[f_t(t,R(t),T) + \alpha(t,R(t))f_r(t,R(t),T) + \frac{\gamma^2(t,R(t))}{2} f_{rr}(t,R(t),T)$$
$$- R(t)f(t,R(t),T)\Big]dt + \Delta(t)D(t)\gamma(t,R(t))f_r(t,R(t),T)\,dW(t)$$

In this case
If $f_r(t, R(t), T) \equiv 0$ we can invest

$$\Delta(t) = \operatorname{sign}\left\{ f_t(t,R(t),T) + \frac{\gamma^2(t,R(t))}{2} f_{rr}(t,R(t),T) - R(t)f(t,R(t),T) \right\}$$

that $d(D(t)X(t)) = D(t)\left| f_t + \frac{\gamma^2}{2} f_{rr} - R(t)f \right| dt$
the discounted value beats no risk and has a nonnegative drift

It's an arbitrage opportunity unless

$$f_t^{(t,r,T)} + \frac{\gamma^2}{2} f_{rr}(t,r,T) = r f(t,r,T)$$

So, in another word, if $f_r(t,r,T)=0$, then (6.9.3) must hold regardless of $\beta(t,r,T)$ we choosed.

Change to the risk-neutral measure

1) No arbitrage PDE:

$$f_t(t,r,T) + \beta(t,r) f_r(t,r,T) + \frac{1}{2}\gamma^2(t,r) f_{rr}(t,r,T) = r f(t,r,T)$$

2) ~~Assuming~~ Suppose the interest rate is given by a stochastic differential equation

$$dR(t) = \alpha(t, R(t))dt + \gamma(t, R(t))\, dW(t)$$

3) Risk neutral SDE:

$$dR(t) = \beta(t, R(t))\, dt + \gamma(t, R(t))\, d\tilde{W}(t)$$

4) Change the measure.

$$d\tilde{W}(t) = dW(t) + \frac{\beta(t, R(t)) - \alpha(t, R(t))}{\gamma(t, R(t))} dt$$

Girsanov's theorem:

Suppose $X(t) = -\int_0^t \frac{\beta(u, R(u)) - \alpha(u, R(u))}{\gamma(u, R(u))} dW(t)$

$d\tilde{W}(t) = dW(t) - d[X, W](t)$

Radon-Nycodym derivative $\left.\frac{d\tilde{P}}{dP}\right|_{\mathcal{F}_t} = \mathcal{E}(X)_t = \exp\Big(\int_0^t \frac{\alpha(u, R(u)) - \beta(u, R(u))}{\gamma(u, R(u))} dW(t)$

$$- \frac{1}{2}\int_0^t \left(\frac{\beta(u, R(u)) - \alpha(u, R(u))}{\gamma(u, R(u))}\right)^2 du \Big)$$

conclusion = PDE

$$f_t(t,r) + \beta(t,r) f_r(t,r) + \frac{1}{2}\gamma^2(t,r) f_{rr}(t,r) = r f(t,r)$$

can be derivated by no-arbitrage considerations.

6.3. Hull-White interest rate model

$$dR(t) = \left(a(t) - b(t)R(t)\right)dt + \sigma(t)\, d\tilde{W}(t)$$

Based on the risk-neutral pricing formula

$D(t)B(t,T) = \tilde{E}\left[D(T)B(T,T) \mid \mathcal{F}_t\right] \Rightarrow$ the discounted price of ZCB is a martingale under the risk-neutral probability.

That means $D(t)f(t,R(t))$ bears no drift (dt term $=0$)

$$d\left(D(t)f(t,R(t))\right) = D(t)\,df(t,R(t)) + f(t,R(t))\,dD(t)$$

$$\cancel{= D_t\, f_t + \beta(t,R(t))} = D(t)\left(\beta f_r + f_t + \frac{\gamma^2}{2} f_{rr} - Rf\right)dt + D(t)\gamma f_r\, d\tilde{W}$$

SDE:

$$f_t + \left(a(t) - b(t)R(t)\right)f_r + \frac{\sigma^2(t)}{2} f_{rr} - R(t) f = 0$$

PDE:

$$f_t(t,r) + \left(a(t) - b(t)r\right)f_r(t,r) + \frac{\sigma^2(t)}{2} f_{rr}(t,r) - r f(t,r) = 0$$

Guess the solution is of the form $f(t,r) = e^{-rC(t,T) - A(t,T)}$

$f_t = (-r\,C'(t,T) - A'(t,T))\,f(t,r)$

$f_r = -C(t,T)\,f(t,r)$

$f_{rr} = C^2(t,T)\,f(t,r)$

Substitution into the PDE gives:

$$-rC'(t,T) - A'(t,T) + (a(t) - rb(t))(-C(t,T)) + \frac{\sigma^2(t)}{2}C^2(t,T) - r = 0$$

$f(t,r) \neq 0$

$$r(C'(t,T) + 1 - b(t)C(t,T)) = -A'(t,T) - a(t)C(t,T) + \frac{\sigma^2(t)}{2}C^2(t,T)$$

derivative with respect to r in both sides:

$$\begin{cases} C'(t,T) + 1 - b(t)C(t,T) = 0 & (6.5.8) \\ \text{Substitution into the PDE given:} & \\ A'(t,T) + a(t)C(t,T) = \frac{\sigma^2(t)}{2}C^2(t,T) & (6.5.9) \end{cases}$$

(ii)

$$\frac{d}{ds}C(s,T) + 1 - b(s)C(s,T) = 0$$

$$\frac{d}{ds}\left(e^{-\int_0^s b(v)dv}C(s,T)\right) = -b(s)C(s,T)e^{-\int_0^s b(v)dv} + e^{-\int_0^s b(v)dv}C'(s,T)$$

$$= e^{-\int_0^s b(v)dv}\left[-b(s)C(s,T) + b(s)C(s,T) - 1\right] = \exp\left(-\int_0^s b(v)dv\right)$$

(iii) $f(T,r) = e^{-r C(T,T) - A(T,T)} = 1 \Rightarrow C(T,T) = A(T,T) = 0$

$$\int_t^T d\left(\exp\left(+\int_s^T b(v)dv\right)C(s,T)\right) = \int_t^T \exp\left(\int_s^T b(v)dv\right)ds$$

$$\exp\int_t^T b(v)dv\; C(t,T) = \int_t^T \exp\left(\int_s^T b(v)dv\right)ds$$

$$C(t,T) = \left(\int_t^T e^{\int_s^T b(v)dv} ds \right) \exp\left(-\int_t^T b(v)dv\right)$$

$$e^{\int_s^T b(v)dv} = e^{-\int_t^s b(v)dv} e^{\int_t^T b(v)dv}$$

$$C(t,T) = \cancel{\exp\left(-\int_t^T b(v)dv\right)} \int_t^T e^{-\int_t^s b(v)dv} ds \quad (6.5.10)$$

(iii) $$\frac{d}{dt} A(t,T) = -a(t)C(t,T) + \frac{\sigma^2(t)}{2} C^2(t,T)$$

$$A(T,T) - A(t,T) = \int_t^T -a(s)C(s,T) + \frac{\sigma^2(s)}{2} C^2(s,T)\, ds$$

Because $A(T,T)$ equals to zero

$$A(t,T) = \int_t^T a(s)C(s,T) - \frac{\sigma^2(s)}{2} C^2(s,T)\, ds \quad (6.5.11)$$

6.8 Feynman-Kac equation

Consider the PDE

$$\left(\frac{\partial u}{\partial t} + \mu \frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2}{\partial x^2} - V(x,t) \right) u(x,t) + f(x,t) = 0$$

The solution can be written as a conditional expectation

$$u(x,t) = E^{x,t}\left[\int_t^T e^{-\int_t^r V(X_s,s)ds} f(X_r,r)\,dr + e^{-\int_t^T V(X_r,r)dr} \psi(X_T) \right]$$

where $X_t = \mu(X_t,t)\,dt + \sigma(X_t,t)\,dW(t)$

$\psi(x) = u(x,T)$

Method I:

transition density $p(t, T, x, y)$

proof: $p(y, T \mid x, t) = E^{x,t}[\delta(X_T - y)]$

$$E^{x,t}[\delta(X_T - y)] = E \int_{-\infty}^{+\infty} \delta(X_T - y)\, p(X_T, T \mid x, t)\, dX_T$$

$$= p(y, T \mid x, t)$$

Then, $V(x,t) = f(x,t) = 0$ substitution into equation given:

$$\left(\frac{\partial}{\partial t} + \mu(x,t)\frac{\partial}{\partial x} + \frac{\sigma^2(x,t)}{2}\frac{\partial^2}{\partial x^2}\right) p(y, T \mid x, t) = 0$$

Method II:

$$g(t,x) = E^{x,t}[h(X_T)] = \int_0^{+\infty} h(X_T)\, p(X_T, T \mid x, t)\, dX_T$$

From Feynman-Kac equation

$$g_t(t,x) + \mu(x,t) g_x(t,x) + \frac{\sigma^2(x,t)}{2} g_{xx}(t,x) = 0 \quad (6.9.45)$$

$$g_t(t,x) = \int_0^{+\infty} h(X_T) \frac{\partial}{\partial t} p(X_T, T \mid x, t)\, dX_T$$
$$g_x(t,x) = \int_0^{+\infty} h(X_T) \frac{\partial}{\partial x} p(X_T, T \mid x, t)\, dX_T$$
$$g_{xx}(t,x) = \int_0^{+\infty} h(X_T) \frac{\partial^2}{\partial x^2} p(X_T, T \mid x, t)\, dX_T$$

$$\int_0^{+\infty} h(X_T)\left[\frac{\partial}{\partial t} + \mu(x,t)\frac{\partial}{\partial x} + \frac{\sigma^2(x,t)}{2}\frac{\partial^2}{\partial x^2}\right] p(X_T, T \mid x, t)\, dX_T = 0$$

The equation can hold regardless of the choice of the function $h(X_T)$ is for $p(X_T, T \mid x, t)$ to satisfy the Kolmogorov backward equation.

11.1 $M(t) = N(t) - \lambda t$

$$E[N(t)-N(s)] \overset{\text{memorylessness}}{=} E[N(t-s)] = \sum_{k=0}^{\infty} P\{N(t-s)=k\} = \lambda(t-s)$$

$$M^2(t) = N^2(t) + \lambda^2 t^2 - 2N(t)\lambda t$$

$$E[M^2(t) \mid \mathcal{F}_s] = \cancel{E\,(N^2(t)+N^2(s))}\; E[M^2(t) - M^2(s) + M^2(s) \mid \mathcal{F}_s]$$

$$= E[(M(t)+M(s))(M(t)-M(s)) + M^2(s) \mid \mathcal{F}_s]$$

$$= M^2(s) + E[(M(t)+M(s))(M(t)-M(s)) \mid \mathcal{F}_s]$$

$$\cancel{E[M(t)-M(s) \mid \mathcal{F}_s] = E[M(t-s)] = 0}$$

$$= M^2(s) + \cancel{E[M(t-s)]\,E[M(t)+M(s) \mid \mathcal{F}_s]}$$

$$= M^2(s) + \cancel{E[M(t-s)]\,(M(s) + E[M(t) \mid \mathcal{F}_s])}$$

Martingale property $= M^2(s) + \cancel{E[M(t-s)]\,2M(s)}$

$$= M^2(s) \cancel{+ 2\lambda(t-s)M(s)}\; E[M^2(t) - M^2(s) \mid \mathcal{F}_s]$$

$$M^2(t) - M^2(s) = (N(t)+N(s)+\lambda(t+s))(N(t)-N(s)+\lambda(t-s))$$

$$= (N(t)-N(s)+\lambda(t+s)+2N(s))(N(t)-N(s)+\lambda(t-s))$$

$$E[M^2(t)-M^2(s) \mid \mathcal{F}_s] = E[(N(t)-N(s)+\lambda(t+s))(N(t)-N(s)+\lambda(t-s)) + 2N(s)(N(t)-N(s)+\lambda(t-s)) \mid \mathcal{F}_s]$$

$$= E[(N(t)-N(s)+\lambda(t+s))(N(t)-N(s)+\lambda(t-s))]$$

$$+ 2N(s)\,E[N(t)-N(s)+\lambda(t-s)]$$

$$= E\left[(N(t)-N(s))^2 + 2\lambda t (N(t)-N(s)) + \lambda(t^2-s^2) \right] + 4\lambda s(\lambda(t-s))$$

$$= Var(N(t)-N(s)) + E[N(t)-N(s)]^2 + 2\lambda t \cdot \lambda(t-s) + \lambda(t^2-s^2) + 4\lambda^2 s(t-s)$$

$$= \lambda(t-s) + \lambda^2(t-s)^2 + 2\lambda^2 t(t-s) + \lambda(t^2-s^2) + 4\lambda^2 s(t-s)$$

$$= \lambda^2\left[(t-s)^2 + 2t^2 - 2st + 4st - 4s^2\right] + \lambda(t-s+t^2-s^2)$$

$$= \lambda^2[3t^2 - 3s^2 \cancel{-4ts+2}] + \lambda(t-s)(1+t+s)$$

$$= 3(t-s)(t+s)\lambda^2 + (t-s)(1+t+s)\lambda$$

$$= (t-s)\left[3(t+s)\lambda^2 + (1+t+s)\lambda\right] > 0$$

So, $E[M^2(t) \mid \mathcal{F}_s] > M^2(s)$ its a submartingale

Method II : Conditional Jensen's inequality

if $\varphi(x)$ is a convex function then

$$E[\varphi(X) \mid \mathcal{G}] \geq \varphi(E[X \mid \mathcal{G}])$$

suppose $\varphi(x) = x^2$, $t \geq s$

$$E[X_t^2 \mid \mathcal{F}_s] \geq E[X_t \mid \mathcal{F}_s]^2 = X_s^2$$

Conclusion : Any martingale X , X^{2n} is a submartingale

(iii) $E[M^2(t) - \lambda t \mid \mathcal{F}_s]$

$$E[M^2(t) \mid \mathcal{F}_s] = E[M^2(t) - M^2(s) + M^2(s) \mid \mathcal{F}_s]$$

$$= M_s^2 + E[M_t^2 - M_s^2 \mid \mathcal{F}_s]$$

$$E[M_t^2 - M_s^2 \mid \mathcal{F}_s] = E[(M_t - M_s)^2 + 2M_tM_s - 2M_s^2 \mid \mathcal{F}_s]$$

$$= E[M_{t-s}^2] + E[2M_s(M_t - M_s) \mid \mathcal{F}_s]$$

$$= Var[M_{t-s}] + E[M_{t-s}]^2 + 2M_s E[M_t - M_s \mid \mathcal{F}_s]$$

$$= \lambda(t-s) + 0 + 2M_s E[M_{t-s}] = \lambda(t-s) > 0$$

so $M^2(t)$ a submartingale

Moreover $E[M^2(t) - \lambda t \mid \mathcal{F}_s] = \cancel{E}\, M^2(s) - \lambda s$

$M^2(t) - \lambda t$ is a martingale

11.2 $P\{N(s+t)=k \mid N(s)=k\} = \cancel{\dfrac{P\{N(s+t)=k \cap N(s)=k\}}{P\{N(s)=k\}}}$

$= P\{N(s+t)-N(s)=0 \mid N(s)=k\}$

Memorylessness $= P\{N(s+t)-N(s)=0\} = P\{N(t)=0\} = e^{-\lambda t}$

$= 1 - \lambda t + O(t^2)$

$P\{N(s+t)=k+1 \mid N(s)=k\} = P\{N(s+t)-N(s)=1 \mid N(s)=k\}$

Memorylessness $= P\{N(t)=1\} = \dfrac{(\lambda t)^k}{k!} e^{-\lambda t}\Big|_{k=1} = \lambda t e^{-\lambda t}$

$= \lambda t + O(t^2)$

$P\{N(s+t) \geq k+2 \mid N(s)=k\} = P\{N(s+t)-N(s)=2 \mid N(s)=k\}$

$= P\{N(t)=2\} = \frac{1}{2}(\lambda t)^2 e^{-\lambda t} = O(t^2)$

Additional Problem

Suppose the portforlio only invest in stock market and money market satisfies the form :

$$dX(t) = \Delta(t)\, dS(t) + (X(t) - \Delta(t) S(t)) R(t)\, dt$$

Assume the stock is a geometric Brownian motion under risk-neutral probability

$$dS(t) = R(t) S(t)\, dt + \sigma(t) S(t)\, d\tilde{W}(t)$$

in which $d\tilde{W}(t) = \Theta(t)\, dt + dW(t) = \frac{\alpha(t) - R(t)}{\sigma(t)} dt + dW(t)$

Our goal is to gain $S^2(T)$ in maturity T almost surely. Let the power derivative denoted as $V(t, S(t))$ which is a Marcov process of $(t, S(t))$. Then,

$d(D(t) X(t)) = d(D(t) V(t, S(t)))$ and $D(t)$ is the discounted factor.

$X(T) = V(T, S(T)) = S^2(T)$ (terminal condition)

Derivation :

$$d(D(t) X(t)) = \Delta(t)\, d(D(t) S(t)) = \Delta(t) D(t) [-R(t) S(t)\, dt + R(t) S(t)\, dt + \sigma(t) S(t)\, d\tilde{W}(t)] = \Delta(t) D(t) S(t) \sigma(t)\, d\tilde{W}(t)$$

$$d(D(t) V(t, S(t))) = -R(t) D(t) V(t, S(t))\, dt + D(t)\, dV(t, S(t))$$

$$= D(t) \left[\frac{\partial}{\partial t} + S(t) R(t) \frac{\partial}{\partial x} + \frac{S^2(t) \sigma^2(t)}{2} \frac{\partial^2}{\partial x^2} - R(t) \right] V(t, S(t))\, dt + D(t) S(t) \sigma(t) \frac{\partial}{\partial x} V(t, S(t))\, d\tilde{W}(t)$$

Equating the evolutions

$$d(D(t)X(t)) = d(D(t)V(t, S(t)))$$

We have two equation as follows:

1) $\left(\frac{\partial}{\partial t} + \overset{R(t)S(t)}{\frac{\partial}{\partial x}} + \frac{\sigma^2(t)S^2(t)}{2}\frac{\partial^2}{\partial x^2} - R(t)\right)V(t, S(t)) \equiv 0$

2) $\Delta(t) = \frac{\partial}{\partial x}V(t, S(t))$

The SDE is satisfied in all pathes, we have the PDE:

$$\left(\frac{\partial}{\partial t} + R(t)x\frac{\partial}{\partial x} + \frac{\sigma^2(t)x^2}{2}\frac{\partial^2}{\partial x^2} - R(t)\right)V(t, x) = 0 \quad t\in[0,T)$$

with the terminal condition: $V(T, x) = x^2$

From Feynman-Kac theorem, the PDE can be solved by a conditional expectation be of the form:

$$V(t,x) = \tilde{E}^{x,t}\left[\exp\left(-\int_t^T R(s)\,ds\right)V(T, S_T)\right]$$
$$= \tilde{E}^{x,t}\left[\exp\left(-\int_t^T R(s)\,ds\right)S_T^2\right]$$

is a Brownian Motion

$\tilde{E}$ means it's the expectation under risk-neutral such that $dS(t) = \dots d\tilde{W}(t)$

SDE for $S(t)$: $dS(t) = S(t)R(t)\,dt + S(t)\sigma(t)\,d\tilde{W}(t)$

Using Doléans-Dade exponential

$$S(t) = S(0)\,\mathcal{E}(X)_t = S(0)\exp\left(\int_0^t R(u)\,du + \int_0^t \sigma(u)\,d\tilde{W}(u) - \int_0^t \frac{1}{2}\sigma^2(u)\,du\right)$$

$$S(t) = S(0)\exp\left\{ \int_0^t \left(R(u) - \frac{\sigma^2(u)}{2}\right) du + \int_0^t \sigma(u)\, d\tilde{W}(u) \right\}$$

In this case : $R(u) \equiv r \quad \sigma(u) \equiv \sigma$

$$S(t) = S(0)\exp\left\{ \left(r - \frac{\sigma^2}{2}\right)t + \sigma\left(W(t) - W(0)\right) \right\}$$

$$S(T) = S(t)\exp\left\{ \left(r - \frac{\sigma^2}{2}\right)\tau + \sigma\left(W(T) - W(t)\right) \right\} ; \ \tau = T - t$$

$$V(t,x) = \tilde{E}^{x,t}\left[e^{-r\tau} S^2(t) \exp^2\left\{ \left(r - \frac{\sigma^2}{2}\right)\tau + \sigma\left(W(T) - W(t)\right) \right\} \right]$$

$$= \tilde{E}\left[e^{-r\tau} x^2 \exp^2\left\{ \left(r - \frac{\sigma^2}{2}\right)\tau + \sigma\left(W(T) - W(t)\right) \right\} \Big| \mathcal{F}_t \right]$$

let $\frac{W(t) - W(T)}{\cancel{\sigma}\sqrt{\tau}} = Y$, then $Y \sim N(0,1)$

$$V(t,x) = e^{-r\tau} x^2 e^{2\left(r - \frac{\sigma^2}{2}\right)\tau} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left[\overset{-2\sigma\sqrt{\tau}Y}{\cancel{2\sigma(W(T) - W(t))}} \right] e^{-\frac{Y^2}{2}} dY$$

$$= x^2 e^{(r - \sigma^2)\tau} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{(Y + 2\sigma\sqrt{\tau})^2}{2}} e^{2\sigma^2\tau} dY$$

$$= x^2 e^{(r + \sigma^2)\tau} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{y^2}{2}} dy$$

$$= x^2 e^{(r + \sigma^2)\tau}$$

The derivative can be priced as $S^2(t)\, e^{(r + \sigma^2)(T - t)}$

$V(0, S(0)) = S^2(0)\, e^{(r + \sigma^2)T}$

The shares = $\Delta(t) = \frac{\partial}{\partial x} V(t, S(t))$

$\Delta(t) = 2x\, e^{(r + \sigma^2)\tau} \Big|_{x = S(t)} = 2S(t)\, e^{(r + \sigma^2)(T - t)}$