About homework

1. Robust the VB code in Excel

Sheet 2, 3 1) Optimize the original BinomialTree code

Sheet 2.3 2) Add Trinomial Tree both Stock pricing and Option pricing

Sheet 3 3) Add BlackSchole Option pricing

Sheet 1 4) Add Greeks calculator

Sheet 3 5) Add Residual tree of [Binomial Option prices − Black-Scholes option prices]

Sheet 3 6) Add Mean Square Error calculator of [Binomial Option prices − Black-Scholes option price]

2 Results (Analysis)

1)
- Price Call << Price Put When Strike > $S(0)$
- Price Call > Price Put When Strike = $S(0)$
- Price Call > Price Put When Strike < $S(0)$

2) As n increased three prices convergent

3) $C-P = S_0 - e^{-RT}K$ put-call parity

4) Price Binomial < Price Trinomial < Price BSM

There are underestimations in Binomial and Trinomial Model

Analysis

1) Why the velocity of V_c convergenting to 0 so quiker than V_p when Strike > S_0 ?

Considering BSM model, when $K > S_0$, $d_2 < d_1 < 0$

as $V_p = Ke^{-r(T-t)} N(-d_2) - SN(-d_1)$

$V_c = -Ke^{-r(T-t)} N(d_2) + SN(d_1)$

Whence $N(d_{1,2}) \approx 0$ $\quad N(-d_{1,2}) \approx 1$

So $V_c \approx 0$

It's a very rough provement, just point out qualitative analysis.

2) Rigor Provement, see below (6 pages)

3) Easy to prove when it is Binomial ~~or~~ Trinomial condition

Skip that conditions, let's see BSM model :

$V_c - V_p = -Ke^{-r(T-t)} [N(-d_2) + N(d_2)] + S\cancel{N}(N(d_1) + N(-d_1))$

As $N(x) = 1 - N(-x)$, subtitud to equation before,

$V_c - V_p = -Ke^{-r(T-t)} + S\cancel{N} = S - e^{-rt}K$

4) ~~BSM is the~~ Just the fact Binomial tree and Trinomial tre are approximate models of BSM. See below.

(discrete)

$\boxed{\text{Lemma \& Preparation}}$

Symmetric Random Walk $\quad M_k = \sum_{j=1}^{k} X_j$

$$Var[M_\ell - M_k] = \ell - k$$

Martingale $\quad \mathbb{E}[M_\ell \mid \mathcal{F}_k] = M_k$

$$[M, M]_k = k$$

Scaled Symmetric Random Walk $\quad W^{(n)}(t) = \frac{1}{\sqrt{n}} M_{nt}$

$\downarrow$

Brownian Motion

$$Var[W^{(n)}(t) - W^{(n)}(s)] = t - s$$

$$\mathbb{E}[W^{(n)}(t) \mid \mathcal{F}_s] = W^{(n)}(s)$$

$$[W^{(n)}, W^{(n)}](t) = t$$

$n \to \infty$, the distribution of $W^{(n)} \sim N(0, t)$

Binomial Tree

$\downarrow$

Geometric Brownian Motion

1) $u_n = 1 + \frac{\sigma}{\sqrt{n}} \quad d_n = 1 - \frac{\sigma}{\sqrt{n}} \quad r = 0$ risk-neutral

$\Rightarrow \quad S(t) = S(0) \exp\{\sigma W - \frac{1}{2}\sigma^2 t\}$

2) $u_n = e^{\sigma/\sqrt{n}} \quad d_n = e^{-\sigma/\sqrt{n}} \quad r_n = \frac{r}{n}$ risk-neutral probabilities

$$\tilde{p}_n = \frac{(\frac{r}{n}+1) - e^{-\sigma/\sqrt{n}}}{e^{\sigma/\sqrt{n}} - e^{-\sigma/\sqrt{n}}} \qquad \tilde{q}_n = \frac{e^{\sigma/\sqrt{n}} - (1+\frac{r}{n})}{e^{\sigma/\sqrt{n}} - e^{-\sigma/\sqrt{n}}}$$

$$M_{nt}^{(n)} = \sum_{k=1}^{nt} X_{k}^{(n)} \qquad H_{nt} = \frac{1}{2}(nt + M_{nt}) \quad T_{nt} = \frac{1}{2}(nt - M_{nt})$$

$$S_n(t) = S(0)\, u_n^{H_{nt}}\, d_n^{T_{nt}}$$

$$= S(0) \exp\left\{\frac{\sigma}{\sqrt{n}} M_{nt}^{(n)}\right\}$$

By Motion-Generating Function $\varphi_n(u) = \mathbb{E}\left(e^{u \frac{\sigma}{\sqrt{n}} M_{nt}^{(n)}}\right)$

$$\lim_{n\to\infty} \varphi_n(u) = \exp\{u\mu + \tfrac{1}{2}u^2\sigma^2\}$$

$\mu \to (r - \frac{1}{2}\sigma^2)t \qquad \sigma^2 \to \sigma^2 t$

$$S(t) = S(0) \exp\{\sigma W + (r - \tfrac{1}{2}\sigma^2)t\}$$

Brownian Motion

$$[W,W](t) = t$$
$$\mathbb{E}[W(t)|\mathcal{F}_s] = W(s)$$
$$Var(W(t) - W(s)) = t-s$$

Geometric Brownian Motion

$$S(t) = S(0)\exp\{\sigma W - (\alpha - \tfrac{\sigma^2}{2})t\}$$
$$Z(t) = \exp\{\sigma W - \tfrac{1}{2}\sigma^2 t\}$$
$$\mathbb{E}(Z(t)|\mathcal{F}_s) = Z(s)$$

Itô process

$$X = \int \sigma dW + \int (\alpha - \tfrac{1}{2}\sigma^2)\, dt$$

Percentage drift

Percentage volatility

Itô - integral

$S(t)$ = geometric Brownian $= e^{X} S(0) \Rightarrow \int dS(t) = \int \alpha S(t)\, dt + \int \sigma S(t)\, dW$

volatility

instantaneous rate of return

$\Downarrow$ Solution

$$S(t) = S(0)\exp\{\sigma W + (\alpha - \tfrac{1}{2}\sigma^2)t\}$$

$\mathbb{E}(S(t)) = e^{\alpha t} S(0)$ Mean rate of return is α

Analysis : Itô integral $I = \int_0^T \Delta(t) dW(t)$ $\Delta(t)$ is an adapted stochastic process

① continuity ② Adaptivity ③ Linearity

④ Martingale ⑤ Itô Isometry ⑥ Quadratic variation

$\mathbb{E}[I_t|\mathcal{F}_s] = I_s$ $\mathbb{E}I^2 = \mathbb{E}\int_0^t \Delta^2 du$ $[I,I](t) = \int_0^t \Delta^2 du$

Why the price calculated by Binomial-tree and Black-Scholes Model are (L_2) convergent? (Their mean square error convergent to 0)

To figure out why, I make an anatomy in this question using Stochastic-Calculus knowledge.

1. Our Assumption = Risk-neutral measure

In a complete market, a derivative's price is the discounted expected value of the future payoff under the unique risk-neutral measure.

2. Binomial tree model =

$$u = \exp\left\{\frac{\sigma\sqrt{t}}{\sqrt{n}}\right\} \qquad d = \exp\left\{-\frac{\sigma\sqrt{t}}{\sqrt{n}}\right\} \qquad (2.1)$$

Using risk-neutral assumption = $P_u = \dfrac{e^{\alpha(t/n)} - d}{u - d}$ $\quad P_d = \dfrac{u - e^{\alpha(t/n)}}{u-d}$ (2.2)

Stock Price $\quad S(t) = S(0)\cdot u^{H^{(n)}(t)} d^{T^{(n)}(t)}$ (2.3) H = number of "up" $\quad T$ = number of "down"

(Call) Option Price $\quad f^{(n)}_k(t_k) = [P_u f^{(n)}_u(t_{k+1}) + P_d f^{(n)}_d(t_{k+1})]$ $\quad f(t) = [S(t) - K]^+$ (2.4)

3. Black-Scholes-Merton Model =

1) Portfolio Value = Capital Gain + Interest Earning on cash position

a) Capital Gain $= \int \Delta\, dS$ in which Δ is the shares of Stock

Stock Price $= S(T) = \int_0^T \alpha S(t)\,dt + \int_0^T \sigma S(t)\,dW$ or: $dS = \alpha S\,dt + \sigma S\,dW$

(3.1)

b) Interest Earning $= \int r(X - \Delta S)\,dt$ $\qquad$ (3.2)

Combine a) b): $\quad dX(t) = rX(t)\,dt + (\alpha - r)\Delta S(t)\,dt + \sigma \Delta S(t)\,dW$

(3.3)

Call
2) Option Value $c = (S(T) - K)^+$

Assumption : Option Value is only the function depends on time and the value of the stock price at that time

Right ? → (It can be automatically inherit from risk-neutral assumption)

$c = c(t, S(t))$

3) Equating the Evolution (the risk-neutral assumption)

$$e^{-rt} X(t) \equiv e^{-rt} c(t, S(t)) \qquad (3.4)$$

$$d\left(e^{-rt} X(t)\right) = d\left[e^{-rt} c(t,S)\right] \quad (\text{BSM equation}) \qquad (3.5)$$

With boundary condition, we get the solution :

$$c(t,x) = x N(d_+) - K e^{-r(T-t)} N(d_-) \qquad (3.6)$$

When $t=T$, $c(T,x) = (x - K)^+$ (3.7)

Stock Price Analysis : the formula above from 1, 2, 3 are used in our price modeling

Let us figure out what would happen when we increase the "steps n"

1) Stock Price ~~S~~ From (2.3) combined with (2.1)

$$S(t) = S(0) \cdot u^{H^{(n)}(t)} d^{T^{(n)}(t)}$$

$$= S(0) \exp\left\{\frac{\sigma\sqrt{t}}{\sqrt{n}} \cdot H^{(n)}(t)\right\} \exp\left\{\frac{-\sigma\sqrt{t}}{\sqrt{n}} \cdot T^{(n)}(t)\right\} \qquad (4.1)$$

Moreover the Symetric Random Walk $M_k = \sum_{j=1}^{k} X_j \quad X_j = \begin{cases} 1 \\ -1 \end{cases}$

We have $n = H + T$

$M = H - T$ which is a random walk, a stochastic process

So

$$\begin{cases} H^{(n)}(t) = \frac{1}{2}\left(M^{(n)}_{(t)} + n\right) \\ T^{(n)}(t) = \frac{-1}{2}\left(M^{(n)}_{(t)} - n\right) \end{cases} \qquad (4.2)$$

Substitute (4.2) into (4.1), we got

$$S^{(n)}(t) = S^{(n)}(0)\exp\left\{\frac{\sigma\sqrt{t}}{\sqrt{n}} M^{(n)}(t)\right\} \qquad (4.3)$$

As Pu and Pd are not equal to $\frac{1}{2}$, we cannot make the equation that $M^{(n)}(t)/\sqrt{n} = W^{(n)}(t)$ which is a random Walk.

So, ~~Besides the effort to~~ Rather than making make similarity, I calculate the Moment-generating function of $S^{(n)}(t)$

$$\varphi^{(n)}(u) = \mathbb{E}[e^{uS}] = \mathbb{E}\left[\exp\left\{u \cdot S^{(n)}(0)\exp\left\{\frac{\sigma\sqrt{t}\,M^{(n)}(t)}{\sqrt{n}}\right\}\right\}\right] \qquad (4.4)$$

$$\lim_{n\to\infty}\varphi^{(n)}(u) = \exp\left\{u\left[(r-\tfrac{1}{2}\sigma^2)t\right] + \tfrac{1}{2}u^2(\sigma^2 t)\right\} \qquad (4.5)$$

So as $n\to\infty$, $S^{(n)}(t)$ is log Normal $\sim N((r-\frac{1}{2}\sigma^2)t, \sigma\sqrt{t})$

Moreover $S(t) = \lim_{n\to\infty} S^{(n)}(t) = S(0)\exp\{\sigma W + (r-\frac{1}{2}\sigma^2)t\}$ (4.6)

Using Ito-Doeblin Formula:

$$dS(t) = S(t)\sigma dW + rS(t)dt \qquad (4.7)$$

Comparing (4.7) with (3.1) what we used in Black Scholes Model

The Binomial tree we used is the Special Case that $\alpha = r$

As we discussed above, in BSM model Option price ~~has no~~ is irrelevant (uncorrelated) with α. It can also be concluded from Risk-Neutral Assumption.

Moreover. focuse on Option Pricing

5. Option Price Analysis

Option Pricing

1) In Binomial tree, we used the risk-neutral Assumption that

$$f(t_k) = e^{-r\Delta t}\left[P_u f_u(t_{k+1}) + P_d f_d(t_{k+1})\right] \cdots (2.4).$$

2) Boundary Condition $f(T) = (S(T) - K)^+$

1) In Black-Scholes-Merton, we also used the risk-neutral Assumption

~~(3.5)~~

$$d\left(e^{-rt} X\right) = d\left(e^{-rt} c(t, S(t))\right) \cdots (3.5)$$

2) Boundary Condition $c(T, (S(T))) = (S(T) - K)^+$

6. Conclusion.

1) The stock price in both models ~~are~~ ~~distr~~ convergent to or submit to geometric Brownian Motion

$$S(t) = S(0)\exp\left\{\sigma W + \left(r - \frac{\sigma^2}{2}\right)t\right\}$$

$$dS(t) = \alpha S(t)\, dt + \sigma S(t)\, dW$$

2) The option price in both models obey to risk-neutral theory and ~~same~~ has the same boundary condition.

Hence, it's not that surprised that ~~why~~ price estimated by Binomial tree and BSM moder are so equal as n increased.