1.2 (i) Since $A=\{\omega=\omega_1\omega_2\omega_3\cdots ; \omega_1=\omega_2, \omega_3=\omega_4\cdots\}$

Construct a mapping $\Phi : A\to A'$ $\quad A'=\{\omega'=\omega_1'\omega_2'\cdots\}$

As A' equals to Ω_∞ which is uncountable

Then A is uncountable

(ii) $\mathbb{P}(A)=\int_A d\mathbb{P}$

Method I : Proof $P_n=\cancel{\frac{\#(A_n)}{\#(\Omega_n)}}\to 0$ (finite set Ω_n)

Let $A_1=\{\omega_1\omega_2 \quad \omega_1=\omega_2\}$ $\quad \Omega_1=\{\omega_1\omega_2\}$

$A_2=\{\omega_1\omega_2\omega_3\omega_4 ,\ \omega_1=\omega_2\ \omega_3=\omega_4)$ $\quad \Omega_2=\{\omega_1\omega_2\omega_3\omega_4)$

$\vdots \qquad \vdots$

$P_1 = \cancel{\frac{\#A_1}{\#\Omega_1}} = p^2+(1-p)^2$

$P_2 = [p^2+(1-p)^2]^2$

$\vdots$

$P_n = [p^2+(1-p)^2]^n$

$\cancel{\lim_{n\to\infty}\mathbb{P}_n = \lim_{n\to\infty}\int_{A_n} d\mathbb{P}_n = \lim_{n\to\infty}\int_{\Omega_n}\chi_{A_n}d\mathbb{P}_n}$

$\lim_{n\to\infty}\mathbb{P}_n = \lim_{n\to\infty}\int_{A_n} d\underset{\Delta}{\mathbb{P}_n}$

As we cannot handle $\underset{\Delta}{\mathbb{P}_n}$ (it change as Ω_n)

We cannot infer $\mathbb{P}(A)=\lim_{n\to\infty}\mathbb{P}_n$

Abortion ! ><

Method II. Fix Ω (uncountable) and construct a sequence of $\{A_n\}$

Let $A_1=\{\omega_1\omega_2\cdots : \omega_1=\omega_2\}$

$A_2=\{\omega_1\omega_2\omega_3\cdots : \omega_1=\omega_2 ,\ \omega_3=\omega_4\}$

$\vdots$

$A_n=\{\omega_1\omega_2\cdots : \omega_1=\omega_2, \omega_3=\omega_4,\cdots, \omega_{2n-1}=\omega_{2n}\}$

$\mathbb{P}(A_n)=\int_{A_n} d\mathbb{P} = [(1-p)^2+p^2]^n = \int_\Omega \chi_{A_n} d\mathbb{P}$

Add limitation Symbol : $\lim_{n\to\infty}\mathbb{P}(A_n) = \lim_{n\to\infty}[(1-p)^2+p^2]^n \overset{?}{=} 0$ (As $0<p<1$) (1)

Moreover, $\lim_{n\to\infty}\int_\Omega \chi_{A_n} d\mathbb{P} = \int_\Omega \lim \chi_{A_n} d\mathbb{P}$ (As Lebesgue Dominated Convergence Theorem and $|\chi_{A_n}|\le 1$)

So $\lim_{n\to\infty} \mathbb{P}(A_n) = \int_\Omega \lim_{n\to\infty} \chi_{A_n} \, d\mathbb{P}$

$$= \int_\Omega \chi_A \, d\mathbb{P} = \int_A d\mathbb{P} = \mathbb{P}(A) \quad (2)$$

$\chi_{A_n} \to \chi_A$ (convergence in every point) for any $\omega \in \Omega$

that means $\forall \varepsilon > 0, \exists$ we can always

find $n > N$ $\quad |\chi_{A_n}(\omega) - \chi_A(\omega)| < \varepsilon$

or $\chi_{A_n}(\omega) = \chi_A(\omega)$

Compare (1) and (2)

$\mathbb{P}(A) = \lim \mathbb{P}(A_n) = 0$

1.6 (i) $\mathbb{E} e^{uX} = \int_{\mathbb{R}} e^{ux} \, d\mathbb{P}^X(x) = \int_{\mathbb{R}} e^{ux} f(x) \, dx$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2} + ux} \, dx$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} \exp\left(- \frac{x^2 + \mu^2 - 2\mu x - 2\sigma^2 u x}{2\sigma^2} \right) dx$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} \exp\left(- \frac{x^2 - 2(\mu + u\sigma^2)x + (\mu + u\sigma^2)^2 - (\mu + u\sigma^2)^2 + \mu^2}{2\sigma^2} \right) dx$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} \exp\left(- \frac{[x - (\mu + u\sigma^2)]^2}{2\sigma^2} \right) \exp\left(+ \frac{(u\sigma^2)^2 + 2\mu u \sigma^2}{2\sigma^2} \right) dx$$

$$= 1 \cdot \exp\left(\frac{u^2\sigma^2}{2} + \mu u \right)$$

(ii) Proof $\mathbb{E}\varphi(X) \geq \varphi(\mathbb{E}X)$:

$\mathbb{E}\{\varphi X\} = \mathbb{E}e^{uX} = e^{u\mu + \frac{1}{2}u^2\sigma^2}$ as we proofed in (i)

$\varphi(\mathbb{E}X) = \varphi\left(\int_R \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \right)$

$= \varphi(\mu) = e^{u\mu}$

As $\mathbb{E}\{\varphi X\} = \varphi(\mathbb{E}X) \cdot \exp\left(\frac{1}{2}u^2\sigma^2\right)$

$\mathbb{E}\{\varphi X\} > 0$, $\varphi(\mathbb{E}X) > 0$ $\exp\left(\frac{1}{2}u^2\sigma^2\right) \geq 1$

$\Rightarrow$ $\mathbb{E}\{\varphi X\} \geq \varphi(\mathbb{E}X)$ (they are equal only when $u=0$)

1.8 (i) $\lim_{n\to\infty} \mathbb{E}Y_n = \lim_{n\to\infty} \mathbb{E}\left[\frac{e^{tX} - e^{s_n X}}{t - s_n}\right]$ ~~Y_n~~ ...... (1)

Beacause of Mean Value Theorem ~~lim~~ $\frac{e^{tX} - e^{s_n X}}{t - s_n} = Xe^{\theta_n X}$ ... (2)

Combine (1) and (2)

$\lim_{n\to\infty} \mathbb{E}\, Y_n = \lim_{n\to\infty} \mathbb{E}(Xe^{\theta_n X}) = \lim_{n\to\infty} \int_\Omega x e^{\theta_n x} \, d\mathbb{P}(x)$

Since Lebesgue Dominated Convergence required $\exists F > 0$ ~~st~~ $|f_n| = |xe^{\theta_n x}| \leq F$ ①

& F is absolutely integrable ②

Proof ①② : As $\theta_n \in [\inf\{s_n\}, \sup\{s_n\}]$

and $X \geq 0$

Then $|f_n| \leq x e^{\sup\{s_n\} x}$

Define $F = x e^{\sup\{s_n\} x}$

Lets Proof F is absolutely integrable :

As $\mathbb{E}[Xe^{tX}] < \infty$ for $t \in \mathbb{R}$

~~We can~~

Whence $\mathbb{E}[Xe^{\sup\{s_n\}X}] < \infty$ Proof done

Therefore, using Lebesgue Dominated Convergence.

$$\lim_{n\to\infty} \int_\Omega x e^{\theta_n x} \, d\mathbb{P}(x) = \int_\Omega \lim_{n\to\infty} x e^{\theta_n x} \, d\mathbb{P}(x)$$

$$= \int_\Omega x e^{tx} \, d\mathbb{P}$$

$$= \mathbb{E}[Xe^{tX}]$$

Since we choose s_n randomly. $\lim_{s\to t} \mathbb{E}\left[\frac{e^{tX} - e^{sX}}{t-s}\right] = \lim_{s_n\to t} \mathbb{E}\left[\frac{e^{tX} - e^{s_n X}}{t-s_n}\right]$

$$= E[Xe^{tX}]$$

Hence, $\varphi'(t) = E[Xe^{tX}]$ Proof Done!

(ii) In this case, $|Xe^{tX}| = X^+ e^{tX^+} + X^- e^{-tX^-}$

$$Xe^{tX} = X^+ e^{tX^+} - X^- e^{-tX^-}$$

Why need use this notation?

$$|Xe^{tX}| = |X| e^{tX} \le |X| e^{t|X|}$$

Since we can also build up $F := |X| e^{\sup\{s_n\} |X|}$

Then, similar to (i) $\varphi'(t) = E[Xe^{tX}]$

1.10 (i) Proof $\tilde{\mathbb{P}}(\Omega) = 1$ ①

$m \neq n$ $A_m \cap A_n = \phi$, $\tilde{\mathbb{P}}(\bigcup_i A_i) = \sum \tilde{\mathbb{P}}(A_i)$ ②

Firstly $\tilde{\mathbb{P}}(\Omega) = \int_\Omega Z \, d\mathbb{P} = \int_{[0,1]} Z \, d\mathbb{P} = \int_{[\frac{1}{2},1]} 2 \, d\mathbb{P} + \int_{[0,\frac{1}{2})} 0 \, d\mathbb{P}$

Since $\mathbb{P}$ is uniform measure $= 2 \cdot |(1-\frac{1}{2})| = 1$

Secondly Let $\forall \{A_i\} \in$ Borel sets in Ω $m \neq n$ $A_m \cap A_n = \phi$ $\tilde{\mathbb{P}}(\bigcup_i^\infty A_i) = \int_{\bigcup_i^\infty A_i} Z \, d\mathbb{P}$

$= \int_\Omega Z(\sum \chi_{A_i}) \, d\mathbb{P}$

As $Z\chi_{A_i}$ are nonnegative measurable function, we can put sigma symbol outside the integral symbol.

$= \sum_{i=1}^\infty \int_\Omega Z\chi_{A_i} \, d\mathbb{P} = \sum_{i=1}^\infty \tilde{\mathbb{P}}(A_i)$

(ii). $\tilde{\mathbb{P}}(A) = \int_A Z \, d\mathbb{P} = \int_\Omega Z\chi_A \, d\mathbb{P}$

$= \int_{[0,\frac{1}{2})} 0 \, \chi_{A \cap [0,\frac{1}{2})} \, d\mathbb{P} + \int_{[\frac{1}{2},1]} 2 \, \chi_{A \cap [\frac{1}{2},1]} \, d\mathbb{P}$

if $\mathbb{P}(A) = 0$, ~~then~~ As $d\mathbb{P} \geq 0 \Rightarrow d\mathbb{P} = 0$ in A

so $\tilde{\mathbb{P}}(A) = 0 + \int_{A \cap [\frac{1}{2},1]} 2 \, d\mathbb{P} = 0 + 0 = 0$

(iii). Let us make a Partition $A_1 = A \cap [0, \frac{1}{2})$ $A_2 = A \cap [\frac{1}{2}, 1]$

As we infer in (ii) $\tilde{P} = \int_{A_1} 0 \cdot d\mathbb{P} + \int_{A_2} 2 \, d\mathbb{P}$

so. if $A_1 \neq \phi$ and $A_2 = \phi$ we have :

$\tilde{P}(A) = 0$ and $P(A) > 0$

In other word, if $A \subset [0, \frac{1}{2})$ we have $P(A) > 0$ and $\tilde{P}(A) = 0$

So $\tilde{\mathbb{P}}$ and $\mathbb{P}$ are not ~~equalva~~ equivalent.

1.13

(i) Since X is a standart normal random variable on $(\Omega, \tilde{\mathcal{F}}, \mathbb{P})$

$$\mathbb{P}\{X \in B\} = \int_B f(x)\, dx = \int_B \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

As $B = [x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2}]$, it is "small"

So $\mathbb{P}\{X \in B\} \approx \frac{1}{\sqrt{2\pi}} e^{-\frac{X^2}{2}} |B|$, and fix in $\bar{\omega}$.

Whence $\mathbb{P}\{X \in B\} \approx \frac{1}{\sqrt{2\pi}} e^{-\frac{X^2}{2}} \cdot \varepsilon$

$$\frac{1}{\varepsilon} \mathbb{P}\{X \in B\} \approx \frac{1}{\sqrt{2\pi}} e^{-\frac{X^2}{2}} \quad \omega \text{ fix in } \bar{\omega}$$

(ii) Exactly same as (i) Just change X to Y

(iii) Since $Y = X + \theta$

$\{X \in B(x, \varepsilon)\} = \{Y - \theta \in B(x, \varepsilon)\} = \{Y - \theta \in B(y - \theta, \varepsilon)\}$

$= \{Y \in B(y, \varepsilon)\}$

(iv) $$\frac{\tilde{\mathbb{P}}(A)}{\mathbb{P}(A)} \approx \frac{\tilde{\mathbb{P}}\{X\in B(x,\varepsilon)\}}{\tilde{\mathbb{P}}\{Y\in B(y,\varepsilon)\}} \approx \exp\left\{\frac{X^2(\bar\omega)-Y^2(\bar\omega)}{2}\right\}$$

since $Y = X+\theta$ $\quad Y^2 - X^2 = \theta^2+2\theta X$

so, $$\frac{\tilde{\mathbb{P}}(A)}{\mathbb{P}(A)} \approx \exp\left\{-\left(\theta^2+2\theta X(\bar\omega)\right)\right\}$$

22 (i) Since $\sigma(X)$ is of form $\{X \in B\}$

atoms definition?

Whence atoms of $\sigma(X)$ are $A = \{\omega \in \Omega : X(\omega) = 1\}$,

$B = \{\omega \in \Omega : X(\omega) = 0\}$

Is it proper for me to describe like that?

Whats the relation between partition and atoms

So, we make a partition in Ω

$A = \{HT, TH\}$, $B = \{HH, TT\}$

As ϕ. Ω_2 are both in $\sigma(X)$, $A^c = B$, $B^c = A$, $A \cup B = \Omega$

$\sigma(X) = \{\phi, \{HT, TH\}, \{HH, TT\}, \Omega_2\}$

(ii) Similar to (i), we have atoms $C = \{HH, HT\}$ $D = \{TH, TT\}$

$\sigma(S_1) = \{\phi, \{HH, HT\}, \{TH, TT\}, \Omega_2\}$

(iii) Proof. $\tilde{\mathbb{P}}(A \cap C) = \tilde{\mathbb{P}}(\{HT\}) = \frac{1}{4}$

$\tilde{\mathbb{P}}(A \cap D) = \tilde{\mathbb{P}}(\{TH\}) = \frac{1}{4}$

$\tilde{\mathbb{P}}(B \cap C) = \tilde{\mathbb{P}}(\{HH\}) = \frac{1}{4}$

$\tilde{\mathbb{P}}(B \cap D) = \tilde{\mathbb{P}}(\{TT\}) = \frac{1}{4}$

Moreover, $\tilde{\mathbb{P}}(A) = \tilde{\mathbb{P}}(B) = \tilde{\mathbb{P}}(C) = \tilde{\mathbb{P}}(D) = \frac{1}{2}$

Hence, For any set from $\sigma(X)$ Set1 and any set from $\sigma(S_1)$ Set2

We have $\tilde{\mathbb{P}}(\text{Set1} \cap \text{Set 2}) = \tilde{\mathbb{P}}(\text{Set1})\, \tilde{\mathbb{P}}(\text{Set 2})$

So, $\sigma(X)$ and $\sigma(S_1)$ are independent under the probability measure $\tilde{\mathbb{P}}$

(iv) Let we calculate $\mathbb{P}(A \cap C)$ ~~and~~

$$\mathbb{P}(A \cap C) = \mathbb{P}(\{HT\}) = \frac{2}{9}$$

$$\mathbb{P}(A) = \mathbb{P}(\{HT, TH\}) = \mathbb{P}(HT) + \mathbb{P}(TH) = \frac{4}{9}$$

$$\mathbb{P}(C) = \mathbb{P}(\{HH\ HT\}) = \mathbb{P}(HT) + \mathbb{P}(HH) = \frac{6}{9}$$

As $\mathbb{P}(A) \cdot \mathbb{P}(C) = \frac{24}{81} = \frac{8}{27} \neq \mathbb{P}(A \cap C)$

$\sigma(X)$ and $\sigma(S_1)$ are not independent under the probability measure $\tilde{\mathbb{P}}$

(V) $\mathbb{P}\{S_1 = 8\} = \mathbb{P}\{HH, HT\} = \mathbb{P}\{HH\} + \mathbb{P}\{HT\} = \frac{6}{9} = \frac{2}{3}$

$\mathbb{P}\{S_1 = 2\} = \mathbb{P}\{TH, TT\} = \mathbb{P}\{TH\} + \mathbb{P}\{TT\} = \frac{3}{9} = \frac{1}{3}$

Yes, beacause X and S_2 are not independent under $\mathbb{P}$

2.4 (i) $\mathbb{E}\{e^{uX+vY}\} = \mathbb{E}\{e^{uX+ZvX}\} = \mathbb{E}\{e^{uX+ZvX} \mid Z=1\} \mathbb{P}(Z=1)$

$+\mathbb{E}$

$+\mathbb{E}\{e^{uX+ZvX} \mid Z=-1\} \mathbb{P}(Z=-1) = \mathbb{E}\{e^{(u+v)X}\}\frac{1}{2} + \mathbb{E}\{e^{(u-v)X}\}\frac{1}{2}$

As Normal Distribution $= e^{(u+v)\mu + \frac{1}{2}\sigma^2(u+v)^2}\frac{1}{2} + e^{(u-v)\mu + \frac{1}{2}\sigma^2(u-v)^2}\frac{1}{2}$

$\mu = 0$, $\sigma = 1$

$$= \frac{1}{2}e^{\frac{1}{2}(u+v)^2} + \frac{1}{2}e^{\frac{1}{2}(u-v)^2}$$

$$= \frac{1}{2}e^{\left(\frac{u^2+v^2}{2} + uv\right)} + \frac{1}{2}e^{\left(\frac{u^2+v^2}{2} - uv\right)}$$

$$= \frac{1}{2}e^{\frac{u^2+v^2}{2}} \cdot (e^{uv} + e^{-uv})$$

$$\mathbb{E}(e^{uX})\cancel{\mathbb{E}(e^{vY})} = \frac{1}{2} \cdot e^{\left(\frac{u^2}{2}\sigma^2 + u\mu\right)} \cdot \cancel{\frac{1}{2}e^{\frac{u^2\sigma^2}{2}}} = \frac{1}{2}e^{\frac{u^2}{2}}$$

$\mathbb{E}$

~~(iii)~~ $\mathbb{E}\{e^{uY}\} = \mathbb{E}\{e^{uZX}\} = \mathbb{E}\{e^{uZX} \mid Z=1\} \mathbb{P}(Z=1)$
$+ \mathbb{E}\{e^{uZX} \mid Z=-1\} \mathbb{P}(Z=-1) = \mathbb{E}\{e^{vX}\}\frac{1}{2} + \mathbb{E}\{e^{-vX}\}\frac{1}{2}$

$$= \frac{1}{2} e^{v\mu + \frac{1}{2}\sigma^2 v^2} + \frac{1}{2} e^{-v\mu + \frac{1}{2}\sigma^2 v^2} = e^{\frac{1}{2}\sigma^2 v^2} = e^{\frac{1}{2}v^2}$$

$\mathbb{E}\{e^{uX+vY}\} \neq \mathbb{E}\{e^{uX}\}\mathbb{E}\{e^{vY}\}$ So, X, Y are not independent.

$$\begin{aligned} Cov(X,Y) &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \\ &= \mathbb{E}[ZX^2] - \mathbb{E}[X]\mathbb{E}[ZX] \\ &= \mathbb{E}[ZX^2 \mid Z=1]\mathbb{P}(Z=1) + \mathbb{E}[ZX^2 \mid Z=-1]\mathbb{P}(Z=-1) - \cdots \\ &= 0 - \mathbb{E}[X]\mathbb{E}[ZX] = 0 - 0 = 0 \end{aligned}$$

Hence, X, Y are uncorrelated

(ii) See (i)

(iii) See (i)

2.6 (i) $\sigma(X)$ is of form $\{X \in B\}$ or $\{\omega \in \Omega;\ X(\omega) \in B\}$

So, the sets in $\sigma(X)$ are:

atoms = $\{a, b\}$, $\{c, d\}$, and ϕ, Ω

Always atoms and ϕ, Ω?

~~(ii).~~ $\mathbb{E}[Y \mid X] = \cancel{\mathbb{E}[Y \mid X=1]\,\mathbb{P}(X=1) + \mathbb{E}[Y \mid X=-1]\,\mathbb{P}(X=-1)}$

$= \cancel{\mathbb{E}[Y \mid \{a,b\}]\tfrac{1}{2} + \mathbb{E}[Y \mid \{c,d\}]\tfrac{1}{2}}$

$= 0$

$E[Y|\sigma(X)]$ What exactly mean ?

(ii). $E[Y|X] = E[Y|\sigma(X)]$ $\quad P(a|\{a,b\})$ $\quad P(b|\{a,b\})$

$$E[Y|X=1] = E[Y|\{a,b\}] = P\{a\}\,Y(a) + P\{b\}\,Y(b)$$
$$= \frac{1}{3}\cdot 1 + \frac{2}{3}\cdot(-1) = -\frac{1}{3}$$

$$E[Y|X=-1] = E[Y|\{c,d\}] = P(c|\{c,d\})\,Y(c) + P(d|\{c,d\})\,Y(d)$$
$$= \frac{1}{2}\cdot 1 + \frac{1}{2}(-1) = 0$$

Moreover $E[Y|X=1] = \sum_{y\in\{1,-1\}} y\, P(Y=y|X=1)$

$$= \sum_{y\in\{1,-1\}} y\,\frac{P(Y=y, X=1)}{P(X=1)}$$

$$= 1\cdot\frac{\frac{1}{6}}{\frac{1}{6}+\frac{1}{3}} + (-1)\,\frac{\frac{1}{3}}{\frac{1}{6}+\frac{1}{3}}$$

$$= \frac{\frac{1}{6}}{\frac{3}{6}} + \frac{-\frac{2}{6}}{\frac{3}{6}} = \frac{1}{3} - \frac{2}{3} = -\frac{1}{3}$$

$\cancel{E[(E[Y|X] - X)Y_g]}$

$$E[E(Y|X)] = E[E(Y|X=1)P(X=1) + E(Y|X=-1)P(X=-1)]$$
$$= \left(-\frac{1}{3}\right)\cdot\left(\frac{1}{6}+\frac{1}{3}\right) + 0 = -\frac{1}{6}$$

$$EY = \sum_{\omega\in\Omega} Y(\omega)P(\omega) = 1\cdot\frac{1}{6} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} = -\frac{1}{6}$$

So, $E[E(Y|X)] = EY$ Partial averaging property

(iii) $\mathbb{E}[Z|X] = \mathbb{E}[(X+Y)|X] = X + \mathbb{E}[Y|X]$

Proof $\mathbb{E}[\mathbb{E}[Z|X]\chi_A] = \mathbb{E}[Z\chi_A]$ $\quad$ χ_A is a indicator function

$A \in \sigma(X) = \{\phi, \{a,b\}, \{c,d\}, \Omega\}$

left-hand side $= \mathbb{E}[(X + \mathbb{E}[Y|X])\chi_A]$

As Partial-averaging property $= \mathbb{E}[(X+Y)\chi_A]$

right-hand side $= \cancel{\mathbb{E}[(X+Y)|X)\chi_A]} = \cancel{X + \mathbb{E}(Y|X)}$

$= \cancel{\mathbb{E}[\chi_A X|X] Y \otimes X + \mathbb{E}[Y|X\chi_A]}$

$= \mathbb{E}[(X+Y)\chi_A]$

(iv) $\mathbb{E}[Z|X] - \mathbb{E}[Y|X] = \mathbb{E}[(X+Y)|X] - \mathbb{E}[Y|X]$

(i) Linearity of conditional expectation $= \mathbb{E}[X|X] + \mathbb{E}[Y|X] - \mathbb{E}[Y|X]$

(ii) Taking out what is known $= X + 0$

X is $\sigma(X)$-measurable

$= X$

2.8 Let Z is a $\sigma(X)$-measurable random variable

Proof $\mathbb{E}[Y_2 Z] - \mathbb{E}[Y_2]\mathbb{E}[Z] = 0$

$$\mathbb{E}[Y_2 Z] = \mathbb{E}[(Y - \mathbb{E}(Y|X))Z]$$

linearity $= \mathbb{E}(YZ) - \mathbb{E}[E(Y|X)\,Z]$

Reverse of Putting out what is known $= \mathbb{E}(YZ) - \mathbb{E}[E(YZ|X)]$

(?)

Partial averaging $= \mathbb{E}(YZ) - \mathbb{E}(YZ)$

Properly ?

$= 0$

$$\mathbb{E}[Y_2]\mathbb{E}[Z] = \mathbb{E}[Y - E(Y|X)]\,\mathbb{E}(Z)$$

Partial averaging $= 0 \cdot \mathbb{E}(Z)$

$= 0$

Then $\mathbb{E}[Y_2 Z] - \mathbb{E}[Y_2]\mathbb{E}[Z] = 0 - 0 = 0$

2.10

$$\int_A g(X)\,d\mathbb{P} = \int_{\cancel{A}} g(X)\,\chi_A\,d\mathbb{P}$$

$$= \int_{-\infty}^{+\infty} g(x)\,\chi_A\,f_X(x)\,dx$$

$$= \iint_{\mathbb{R}^2} \frac{y f_{X,Y}(x,y)}{f_X(x)}\,f_X(x)\,\chi_A\,dx\,dy$$

$$= \iint y\,f_{X,Y}(x,y)\,\chi_A\,dx\,dy$$

$$= \int_A Y\,d\mathbb{P}$$

Bonus :

A. $M_1 = X_1$

$E[M_1] = E[X_1] = \sum_{\omega \in \{H,T\}} X(\omega)P(\omega) = 2 \cdot p + (-1) \cdot q = 2p - q$

As for $E[M_1] = E[X_1] = 2p - q = M_0 = 0$ ①

$p + q = 1$ ②

From ①② Then $p = \frac{1}{3}$ $q = \frac{2}{3}$

B. $\cancel{E[M_2 | M_1]} =$ $\mathbb{E}[M_{n+1} - M_n | M_n] = \mathbb{E}[X_{n+1} | M_n]$

Independent $= \mathbb{E}[X_{n+1}]$

$= \sum_{\omega} P_\omega X_\omega = 0$

Moreover, ~~So~~ $\mathbb{E}[M_{n+1} - M_n | M_n] = \mathbb{E}[M_{n+1} | M_n] - M_n$

So $\mathbb{E}[M_{n+1} | M_n] = M_n$

It is a Martingale

Also $\mathbb{E}[M_{n+2} - M_n | M_n] = \cancel{M}_n$

So $\mathbb{E}[M_{302} | M_{300} = 60] = M_{300} = 60$