ECEN315 Assignment 3. Xiaobin(Jerry) Zhuang
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Root Locus.

1. ① $G(s) = \dfrac{1}{s^3+2s^2+5s} = \dfrac{1}{s(s^2+2s+5)}$

$\because$ Quadratic formula (s^2+2s+5)

$$\text{root} = \frac{-2\pm\sqrt{4-4\times5}}{2} = \frac{-2\pm4i}{2} = -1\pm2i$$

$$\therefore G(s) = \frac{1}{s(s+(1-2i))(s+(1+2i))}$$

Poles: 0, $-1+2i$, $-1-2i$

$$\sigma_a = \frac{\sum_{i=1}^{P} P_i - \sum_{j=1}^{Z} Z_j}{P-Z} = \frac{0-1+2i-1-2i}{3} = -\frac{2}{3}$$

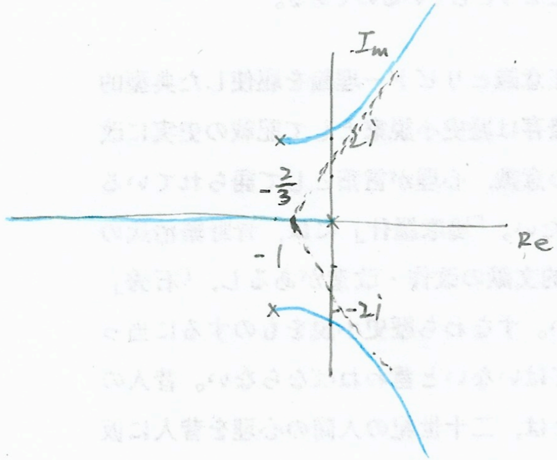
$$\theta_a = \frac{(2k+1)180}{P-Z} = \frac{(2k+1)180}{3}$$

when $k=0$, $\theta_a = 60°$

$k=1$, $\theta_a = 180°$

$k=-1$, $\theta_a = -60°$

sketch:



② $G(s) = \dfrac{s^2+4s+8}{s^2+5s+4}$

$\because$ Quadratic formula of s^2+4s+8

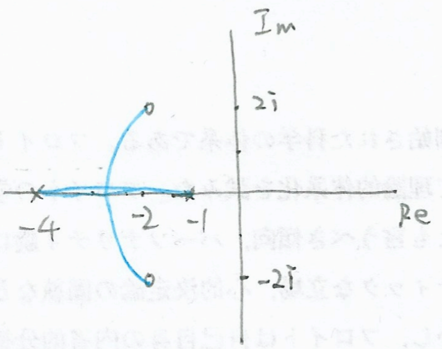
$$\text{root} = \frac{-4\pm\sqrt{16-32}}{2} = \frac{-4\pm4i}{2} = -2\pm2i$$

$$\therefore G(s) = \frac{(s+(2+2i))(s+(2-2i))}{(s+1)(s+4)}$$

poles: -1, -4. zeros: $-2+2i$, $-2-2i$

$\therefore p - z = 2 - 2 = 0$

$\therefore$ Sketch:



③ $$G(s) = \frac{4}{(s+2)(s+5)(s+6)(s+9)}$$

Poles: $-2, -5, -6, -9$

$$\sigma_a = \frac{\sum_{i=1}^{p} p_i - \sum_{i=1}^{z} z_i}{p - z} = \frac{-2-5-6-9}{4} = -\frac{11}{2}$$

$$\theta_a = \frac{(2k+1)180}{p-z} = \frac{(2k+1)180}{4}$$

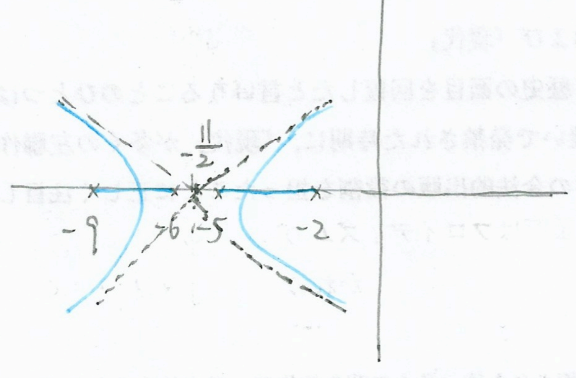
when $k=0$, $\theta_a = 45^\circ$

$k=1$, $\theta_a = 135^\circ$

$k=-1$, $\theta_a = -45^\circ$

$k=2$, $\theta_a = 225^\circ$

Sketch:



2. a. $G(s) = \frac{(s-2)(s-6)}{(s+1)(s+4)}$

Poles: $-1, -4$; zeros: $2, 6$

$\because s$ is real

$\therefore G(\sigma) = \frac{(\sigma-2)(\sigma-6)}{(\sigma+1)(\sigma+4)}$

Algebraic method:

$\because \sum_{i=1}^{z} \frac{1}{\sigma_b - z_i} = \sum_{j=1}^{p} \frac{1}{\sigma_b - p_j}$

$\therefore \frac{1}{\sigma_b - 2} + \frac{1}{\sigma_b - 6} = \frac{1}{\sigma_b + 1} + \frac{1}{\sigma_b + 4}$

$\Rightarrow \frac{(\sigma-6)+(\sigma-2)}{(\sigma-2)(\sigma-6)} = \frac{(\sigma+4)+(\sigma+1)}{(\sigma+1)(\sigma+4)}$

$((\sigma-6)+(\sigma-2))(\sigma^2+5\sigma+4) = ((\sigma+4)+(\sigma+1))(\sigma^2-8\sigma+12)$

$(2\sigma-8)(\sigma^2+5\sigma+4) = (2\sigma+5)(\sigma^2-8\sigma+12)$

$2\sigma^3+10\sigma^2+8\sigma-8\sigma^2-40\sigma-32 = 2\sigma^3-16\sigma^2+24\sigma+5\sigma^2-40\sigma+60$

$2\sigma^2-32\sigma-32 = -11\sigma^2-16\sigma+60$

$\Rightarrow 13\sigma^2-16\sigma-92=0$

$\because$ Quadratic formular

$Root = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{+16 \pm \sqrt{16^2+4\times13\times92}}{2\times13} = \frac{8 \pm 6\sqrt{35}}{13}$

$\therefore$ Break away point $= -2.12$ Break in point $= 3.35$

Derivative method:

$\because 1+KG(s)=0$ $\therefore K = -\frac{1}{G(s)}$

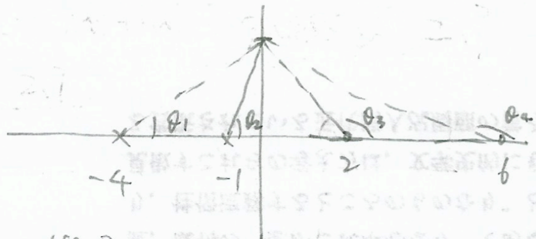
$\because s$ is real. $\therefore K = -\frac{1}{G(\sigma)}$

$\therefore K = -\frac{(\sigma+1)(\sigma+4)}{(\sigma-2)(\sigma-6)}$

Take derivative of K, according to WolframAlpha's calculation, we have $\frac{dk}{d\sigma} = \frac{13\sigma^2-16\sigma-92}{(\sigma-6)^2(\sigma-2)^2}$, set numerator to 0, we can get same equation with Algebraic method $13\sigma^2-16\sigma-92=0$

Also use quadratic formular, then we can get same answer.

Finding Imaginary axis crossing point:



$\therefore \angle KG(s) = -180°$

Take an initial guess 2.7:

$\theta_1 = \arctan\left(\frac{2.7}{-4}\right) = -34.02°$

$\theta_2 = \arctan\left(\frac{2.7}{-1}\right) = -69.68°$

$\theta_3 = \arctan\left(\frac{2.7}{2}\right) = 53.47$

$\theta_4 = \arctan\left(\frac{2.7}{6}\right) = 24.22$

$\therefore \theta_3 + \theta_4 - \theta_1 - \theta_2 = 24.23 + 53.47 + 69.68 + 34.02 = 181.4$, which is a little high. Try a little bit lower: 2.65 guess

$\theta_1 = \arctan\left(\frac{2.65}{-4}\right) = -33.52$

$\theta_2 = \arctan\left(\frac{2.65}{-1}\right) = -69.33$

$\theta_4 = \arctan\left(\frac{2.65}{2}\right) = 52.96$

$\theta_4 = \arctan\left(\frac{2.65}{6}\right) = 23.83$

$\therefore \theta_3 + \theta_4 - \theta_1 - \theta_2 = 23.83 + 52.96 + 69.33 + 33.52 = 179.64$

Try a little bit higher: 2.66 guess

$\theta_1 = \arctan\left(\frac{2.66}{-4}\right) = -33.62$

$\theta_2 = \arctan\left(\frac{2.66}{-1}\right) = -69.40$

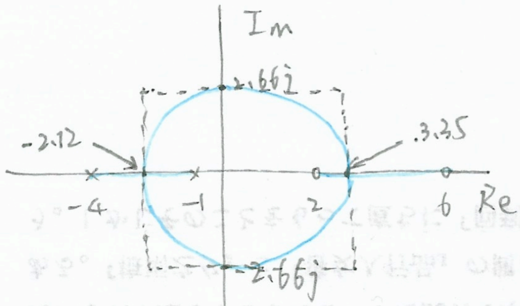
$\theta_3 = \arctan\left(\frac{2.66}{2}\right) = 53.06$

$\theta_4 = \arctan\left(\frac{2.66}{6}\right) = 23.91$

$\therefore \theta_3 + \theta_4 - \theta_1 - \theta_2 = 23.91 + 53.06 + 33.62 + 69.40 = 179.99$, which is close enough to 180.

$\therefore$ Imaginary axis crossing point $= \pm 2.66j$.

sketch:



b. $G(s) = \dfrac{s^2-2s+5}{s^2+3s+2}$

Quadratic formula for s^2-2s+5:

$\dfrac{-b \pm \sqrt{b^2-4ac}}{2a} = \dfrac{+2 \pm \sqrt{4-4\times 5}}{2} = \dfrac{2 \pm 4i}{2} = 1 \pm 2i$

$\therefore G(s) = \dfrac{(s-(1+2i))(s-(1-2i))}{(s+1)(s+2)}$

poles: $-1, -2$, zeros: $1+2i$, $1-2i$

① Algebraic:

$\because$ s is real $\therefore G(s) = G(\sigma) = \dfrac{(s-(1+2i))(s-(1-2i))}{(s+1)(s+2)}$

$\sum_{i=1}^{z} \dfrac{1}{\sigma - z_i} = \sum_{j=1}^{p} \dfrac{1}{\sigma - p_j}$

$\therefore \dfrac{1}{\sigma-(1+2i)} + \dfrac{1}{\sigma-(1-2i)} = \dfrac{1}{\sigma+1} + \dfrac{1}{\sigma+2}$

$\dfrac{\sigma-(1+2i)+\sigma-(1-2i)}{(\sigma-(1+2i))(\sigma-(1-2i))} = \dfrac{(\sigma+2)+(\sigma+1)}{(\sigma+1)(\sigma+2)}$

$(\sigma-(1+2i)+\sigma-(1-2i))(\sigma^2+3\sigma+2) = ((\sigma+2)+(\sigma+1))(\sigma^2-2\sigma+5)$

$(2\sigma-2)(\sigma^2+3\sigma+2) = (2\sigma+3)(\sigma^2-2\sigma+5)$

$2\sigma^3+6\sigma^2+4\sigma-2\sigma^2-6\sigma-4 = 2\sigma^3-4\sigma^2+10\sigma+3\sigma^2-6\sigma+15$

$4\sigma^2-2\sigma-4 = -\sigma^2+4\sigma+15$

$5\sigma^2-6\sigma-19 = 0$

Use Quadratic formula:

$\dfrac{-b \pm \sqrt{b^2-4ac}}{2a} = \dfrac{+6 \pm \sqrt{36+4\times 5\times 19}}{2\times 5} = \dfrac{3 \pm 2\sqrt{26}}{5}$

$\therefore$ Breakaway point $= -1.44$ Breakin point $= 2.64$

② Derivative method:

$\because 1 + kG(s) = 0 \quad \therefore k = -\frac{1}{G(s)}$

$\because s$ is real, $\therefore k = -\frac{1}{G(\sigma)}$

$\therefore k = -\frac{(s+1)(s+2)}{(s-(1+2i))(s-(1-2i))}$

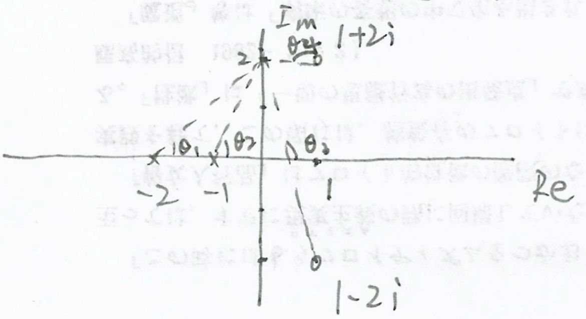
Take the derivative of k, we got:

$\frac{dk}{d\sigma} = \frac{5s^2-6s-19}{(s^2-2s+5)^2}$, set $5s^2-6s-19=0$

$\frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{+6 \pm \sqrt{36+4\times5\times19}}{2\times5} = -1.44,\ 2.64$

$\therefore$ Breakaway point $= -1.44$. Breakin $= 2.64$.

Find Imaginary axis crossing point:



First guess: 2.

$\theta_1 = \arctan(\frac{2}{-2}) = -45$

$\theta_2 = \arctan(\frac{2}{-1}) = -63.45$

$\theta_3 = \arctan(\frac{2+2}{-1}) = -75.96$

$\theta_4 = \arctan(\frac{2-2}{-1}) = 0$

$\therefore 63.43 + 45 + 75.96 = 184.39$, which is a little bit high.

Try a little bit lower guess. 1.8.

$\theta_1 = \arctan(\frac{1.8}{-2}) = -41.99$

$\theta_2 = \arctan(\frac{1.8}{-1}) = -60.95$

$\theta_3 = \arctan(\frac{1.8+2}{-1}) = -75.26$

$\theta_4 = \arctan(\frac{1.8-2}{-1}) = 11.31$

$\therefore \theta_1 + \theta_2 + \theta_3 + \theta_4$

$= 41.99 + 60.95 + 75.26 - 11.31$

$= 166.89$

which is far more lower than 180.

∴ Try a little higher guess. 1.95.

$\theta_1 = \arctan\left(\frac{1.95}{-2}\right) = -44.27$

$\theta_2 = \arctan\left(\frac{1.95}{-1}\right) = -62.85$

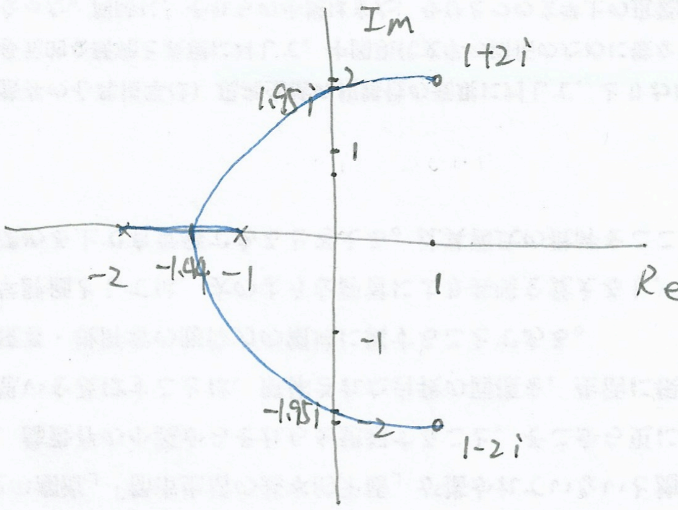
$\theta_3 = \arctan\left(\frac{1.95+2}{-1}\right) = -75.79$

$\theta_4 = \arctan\left(\frac{1.95-2}{-1}\right) = 2.86$

∴ $\theta_1 + \theta_2 + \theta_3 + \theta_4 = 44.27 + 62.85 + 75.79 - 2.86 = 180.23$

which is close enough.

∴ Imaginary axis crossing point $= \pm 1.95i$



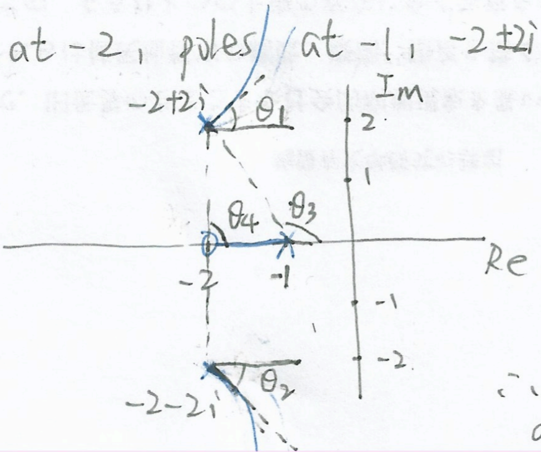
3.

$$G(s) = \frac{(s+2)}{(s^2+4s+8)(s+1)}$$

using quadratic formula for s^2+4s+8 to find root:

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-4 \pm \sqrt{16-4\times 8}}{2} = \frac{-4 \pm 4i}{2} = -2 \pm 2i$$

∴ zero at -2, poles at -1, $-2 \pm 2i$



$-180° = -\theta_1 - \theta_2 - \theta_3 + \theta_4$

$= -\theta_1 - 90° - (180° - \arctan\frac{2}{1}) + 90°$

∴ $\theta_1 = 180° - 116.57°$

$= 63.43°$

∴ The angle of departure from the pole at $-2+2i$ is $63.43°$