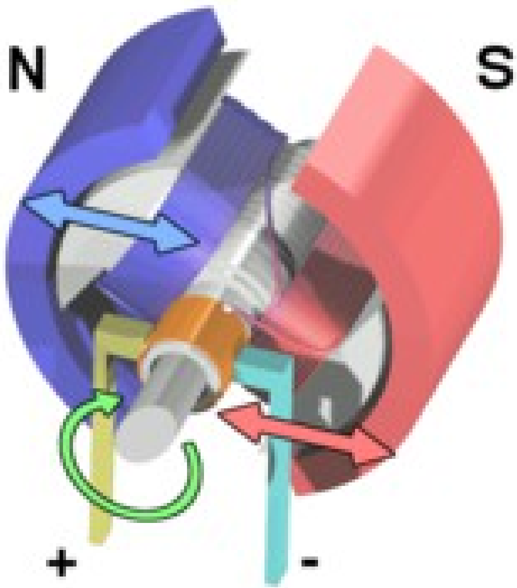
Understanding small, permanent magnet brushed DC motors.



Gideon Gouws
March 2008

Last Updated
21st April 2015

Variables

B = Fixed magnetic field (Tesla)

$i_a(t)$ = armature current, function of time (A)

$i_a(\infty)$ = steady state armature current (A)

J_L = Inertia of the load

D_L = Viscous damping coefficient of the load

J_a = Inertia of the armature and rotor

J_L = Inertia of the load.

J_M = Total inertia presented to the motor

D_a = Viscous damping coefficient of the armature and rotor

D_L = Viscous damping coefficient of the load.

D_M = Viscous damping coefficient of the armature plus load.

K_b = back emf constant

K_m = Motor constant

K_t = torque constant (N.m/A)

l = length of the conductor (m)

L_a = armature inductance (Henries)

N_1 = Gear on motor side

N_2 = Gear on load side

P_E = Electrical power (Watt)

P_M = Mechanical power (Watt)

R_a = Armature resistance (ohm)

Rm = Speed regulation constant

R_s = Series resistance (ohm)

T(t) = Torque (newton.meter)

T_{stall} = Stall torque of motor (newton.meter)

$v_a(t)$ = applied armature voltage (volts)

$v_b(t)$ = induced back emf (volts)

$\theta_m(t)$ = angular displacement of the motor (radians)

$\omega_m(t)$ = angular velocity of motor (radians/second)

ω_0 = no-load speed of motor (radians/second)

Abbreviations

emf: electro motive force

LT: Laplace transform

TF: Transfer function

1. What does the title mean ?

In the "who's who in the electric motor zoo", the group of motors that can be classified as small, permanent magnet brushed DC motors is just one of many groupings. The document title thus implies that we are looking at only this one group, which is:

Small - of limited power output, typically also small in size.
Permanent magnet – The magnetic field is permanent, typically formed by permanent magnets.
Brushed – the commutator arrangement uses brushes for electrical contact and reversal of the armature current.
DC - use direct current for operation.
Understanding - left to the reader .....

This type of motor is generally inexpensive, easy to use and is typically used in mechatronic application.

2. Principle of Operation and Construction.

A DC motor is a good example of an electromechanical device, consisting of both an electrical and a mechanical subsystem. It functions as a bi-directional transducer, converting electrical energy into mechanical energy (motor application), or can also convert mechanical energy into electrical (generator application).

Motors exploit the phenomenon described by Maxwell's equations; a current flowing within a conductor will establish a magnetic field around it. If this current-carrying conductor is now placed in a permanent magnetic field, the interaction between the two magnetic field will produce a force on the conductor. In the motor, the production of this force is used to create a torque that turns the rotor.

The construction of this type of motor can be divided into the following parts:

- The stator, or stationary part that consists of the permanent magnets attached to the motor housing.
- The rotor (shaft) that also carries the armature or coil windings.

- The commutator and brushes that provide electrical contact to coil while the motor is rotating and provides a method of switching the direction of current in the armature.
- Bearings that the shaft runs on.

3. Derivation of the transfer function.

The current $i_a(t)$ flowing in the armature will produce an interaction with the magnetic field, so that the armature will experience a force f(t),

$$f(t) = B.l.i_a(t)$$

This force will cause the armature to move (rotate), which will lead to a back emf being induced due to the magnetic field:

$$v_b(t) = K_b \frac{d\theta(t)}{dt} = K_b \omega(t) \quad \rightarrow \quad (1)$$

The voltage-current relationship in the armature is then given by:

$$v_a(t) = i_a(t)R + L_a \frac{di(t)}{dt} + v_b(t) \quad \rightarrow \quad (2)$$

This can be written in LT form:

$$V_a(s) = RI(s) + L_a sI(s) + V_b(s) \quad \rightarrow \quad (3)$$

The torque the motor develops will be proportional to the armature current, so that:

$$T_m(s) = K_t I_a(s) \quad \rightarrow \quad (4)$$

or $$I_a(s) = \frac{T_m(s)}{K_t} \quad \rightarrow \quad (5)$$

where K_t is the torque constant.

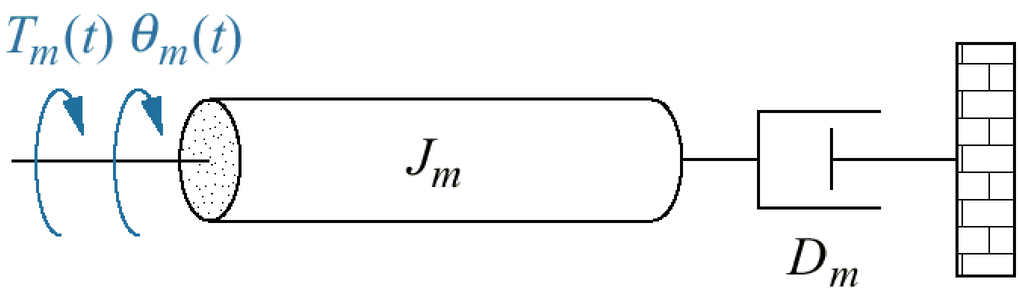
Substitute (2) and (5) into (4):

$$\frac{T_m(s)(R_a + L_a s)}{K_t} + K_b s\theta_m(s) = V_a(s) \quad \rightarrow \quad (6a)$$

or writing the same equation in terms the angular velocity ω(t):

$$\frac{T_m(s)(R_a + L_a s)}{K_t} + K_b \omega_m(s) = V_a(s) \qquad \rightarrow \qquad (6b)$$

Consider the mechanical subsystem of the motor, where the torque produced by the armature current now turns the rotor-armature combination. This combination has an inertia J_m and a linear damping (viscous) D_m. Other possible damping effects are neglected.



$$T_m(t) = J_m \frac{d^2\theta(t)}{dt^2} + D_m \frac{d\theta(t)}{dt}$$

or in LT format:

$$T(s)=J_m s^2\theta(s) + D_m s\ \theta(s) = (J_m s^2 + D_m s)\ \theta(s) \qquad \rightarrow \qquad (7a)$$

Similarly, in terms of the angular velocity $\Omega(s)$:

$$T(s)=J_m s\ \Omega(s) + D_m\ \Omega(s) = (J_m s + D_m)\ \Omega(s) \qquad \rightarrow \qquad (7b)$$

Substituting (7) into (6) to produce:

$$\frac{(J_m s^2 + D_m s)(R_a + L_a s)\theta_m(s)}{K_t} + K_b s\theta_m(s) = V_a(s) \qquad \rightarrow (8)$$

The transfer functions can then be derived in terms of angular displacement or angular velocity as:

$$\frac{\theta_m(s)}{V_a(s)} = \frac{K_t/(J_m L_a)}{s^3 + \left(\frac{J_m R_a + D_m L_a}{J_m L_a}\right)s^2 + \left(\frac{R_a D_m + K_t K_b}{J_m L_a}\right)s} \qquad \rightarrow (9a)$$

or

$$\frac{\Omega_m(s)}{V_a(s)} = \frac{K_t/(J_m L_a)}{s^2 + \left(\frac{J_m R_a + D_m L_a}{J_m L_a}\right)s + \left(\frac{R_a D_m + K_t K_b}{J_m L_a}\right)} \qquad \rightarrow (9b)$$

If it is assumed that the armature inductance J_a is very small compared to the armature resistance R_a, it can be shown (from eqn (8)) that the transfer function reduce to:

$$\frac{\theta_m(s)}{V_a(s)} = \frac{K_t/(J_m R_a)}{s\left[s + \frac{1}{J_m}\left(D_m + \frac{K_t K_b}{R_a}\right)\right]} \qquad \rightarrow (10a)$$

or

$$\frac{\Omega_m(s)}{V_a(s)} = \frac{K_t/(J_m R_a)}{\left[s + \frac{1}{J_m}\left(D_m + \frac{K_t K_b}{R_a}\right)\right]} \qquad \rightarrow (10b)$$

4. Relationship between K_t and K_b.

The back emf constant of a motor, K_b, (also called the speed constant or the voltage constant) was defined as:

$K_b = \frac{v_b}{\omega}$ with units V.s.rad^{-1}.

Similarly, the torque constant of the motor has been defined as relating the motor torque to the armature current, with:

$K_t = \frac{T_m(t)}{i_a(t)}$ and units N.mA^{-1}.

The value of both these constants are determined by the geometrical and physical properties of the motor, with factors such as the physical dimensions, number of turns on the coil winding and the magnetic flux density all contributing.

The relationship between these constants can be best seen by considering the motor in a generator application. Also neglect non-ideal mechanical and electrical losses associated with the motor/generator operation. If this approximation is made, the mechanical power on the input of the generator should equal the electrical power at the generator output. The mechanical power is given by:

P_m = (Torque x angular displacement)/time = $T_m\omega_m$.

The electrical power generated is defined as:

P_e = output voltage x output current = $v_b i_a$.

So that:

$$T_m\omega_m = v_b i_a \qquad \text{and} \qquad \frac{T_m}{i_a} = \frac{v_b}{\omega_m}$$

This implies that $K_b = K_t$ when both are measured in consistent units. This is true because the same factors (geometrical and physical) that govern the value of one constant also determine the value of the other. These constants describe the fundamental coupling between mechanical and electrical power and should be no different for the direction of energy conversion.

We can then simply define a new constant, K_m, the motor constant as:

$K_m = K_b = K_t$

and the transfer functions previous derived can be modified to reflect this definition.

5. The torque-speed curve.

The mechanical constants of the motor can be obtained from a dynamometer test – measuring the rotational speed and torque of the motor at constant applied voltage while changing the load.

Substituting (2) and (5) into (4) yields:

$$R_a \frac{T_m(s)}{K_t} + L_a s \frac{T_m(s)}{K_t} + K_b \omega_m(s) = V_a(s)$$

With the approximation $L_a \approx 0$, this will reduce to:

$$\frac{R_a T_m(s)}{K_t} + K_b \Omega(s) = V_a(s)$$

If we transform to the time-domain and rearrange and considering steady state conditions:

$$T_m(\infty) = -\frac{K_b K_t}{R_a}\omega(\infty) + \frac{K_t v_a}{R_a} \qquad \rightarrow (12)$$

This will be in the form of a straight line y = mx + c when we plot T_m vs ω. In practise, we will produce a family of lines relating T to ω for different values of the applied voltage.

Two points are of significance on this graph:

(1) The stall torque (T_{stall}) presents the point where the load becomes so great that the motor stalls ($\omega_m = 0$). This presents the maximum torque that the motor can deliver as well as the maximum current that the motor draws.

(2) The no-load speed of the motor is the speed with no applied load (maximum speed possible for this applied voltage). The only torque is that needed to turn the armature and rotor and the current drawn is a minimum (maximum back emf). It can be assumed that the torque of the motor at this point is ~ 0.

The y-intercept of the graph from (12) occurs when the motor stalls, so that:

$$T_{stall} = \frac{K_t v_a}{R_a}$$

If we can then measure the stall torque, we can calculate K_t.

The slope of this line will be:

$$\frac{\Delta T}{\Delta \omega} = \frac{K_b K_t}{R_a} \ ,$$

The inverse of the slope is sometimes called the speed regulation constant, defined as:

$$R_m = \frac{R_a}{K_t K_B}$$

The stall torque can then also be written as:

$$T_{stall} = \frac{\omega_{no-load}}{R_m} \qquad \text{(check out)}$$

The minimum torque point (T ~ 0) on the x-axis will occur at:

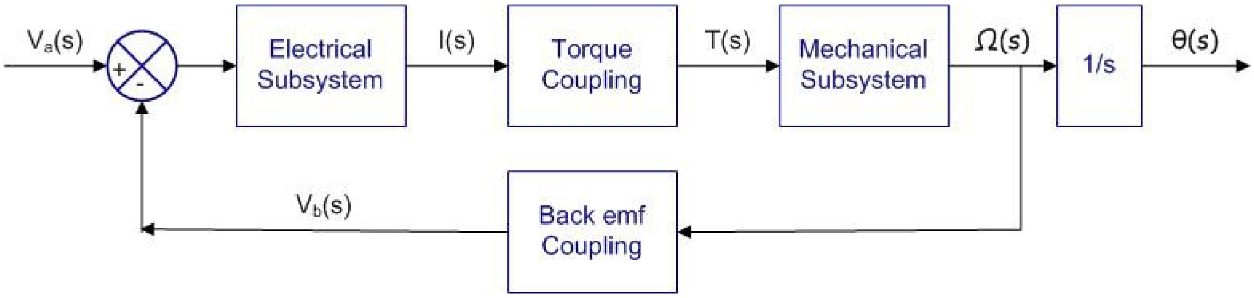
$$\omega_{no-load} = \frac{v_a}{K_b}$$

As a minimum, only the two conditions of T_{stall} and $\omega_{no-load}$ need to be measured to generate this line.

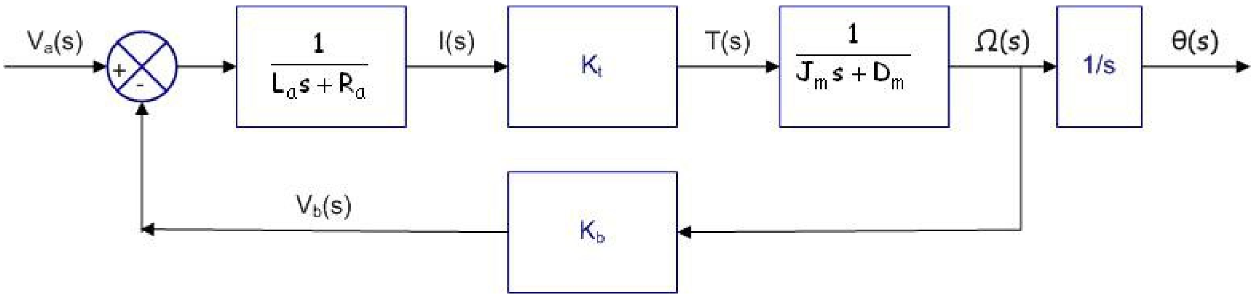
In practice, it is important to keep in mind that a stall occurs whenever the motor is attempting to move against a force that is bigger than the torque it can generate. However, a stall condition exists every time the motor starts from a resting position or any time the motor reverses direction. This must be taken into account when designing motor driver circuits.

6. Block diagram representation of the DC motor.

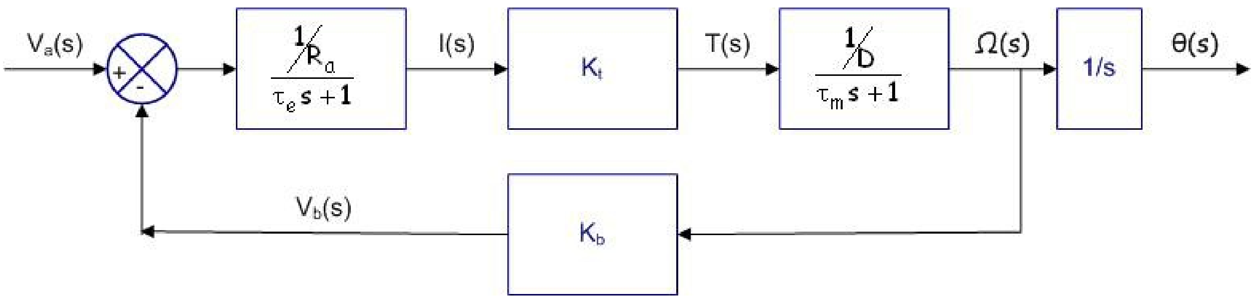
The motor can be represented in block diagram form as shown below:



If the transfer function of each individual block is determined as was described in section 2, it will produce:



Both the electrical and mechanical subsystems are thus first order and can be presented in terms of their time constants as"



where $\tau_m = \frac{J_m}{D_m}$ and $\tau_e = \frac{L_a}{R_a}$

If we simplify and assume that $\tau_e \ll \tau_m$, the transfer function can be reduced to:

$$\frac{\Omega(s)}{V_a(s)} = \frac{K_s}{\tau_s + 1}$$

where K_s is the simplified gain

$$K_s = \frac{K_t}{R_a D_m + K_t K_b}$$

and τ_s is the simplified time constant

$$\tau_s = \frac{J_a R_a}{R_a D_m + K_t K_b}$$

7. Transfer function estimation by explicit measurement.

In order to obtain the motor transfer function, we must be able to measure the following parameters:
R_a = Armature resistance
L_a = Armature inductance
K_t = Torque constant
K_b = Back emf constant
J_m = Inertia of the motor
D_m = Damping of the motor.

7.1 Measuring R_a.

The armature resistance can simply be measured by measuring the resistance across the motor terminals. However, measure for several positions of rotation and average in order to take into account the effect a varying contact of the brushes on the commutator.

The resistance can also be obtained by clamping the motor (preventing rotation) and measuring the steady state current to a specific applied voltage. This ensures that no back emf is induced. Again, measure at several positions of rotation.

Typical values of R_a should be in the 0.5 – 3 Ω range.

7.2 Measuring L_a.

The inductance of the armature can be measured from a measure of the time constant of the RL circuit that forms the electrical subsystem. This again entails clamping the motor and then subjecting it to steps in the input voltage. An external resistor, R_s, should be connected in series with the armature so that the voltage drop across R_s can be measured in order to calculate the current.

The time constant of this step response is then given by:

$$\tau_e = \frac{L_a}{(R_a + R_s)}$$

and the value of L_a can be calculated if R_a is known.

Typical values for L_a should be in the 10^{-3} H range.

7.3 Measurement of K_b (and K_t).

The current-voltage relationship in the armature at steady state condition is given by:

$$v_a(\infty) = i_a(\infty)R + K_b\omega(\infty)$$

If the current and the rotational speed can then be measured, K_b can be calculated as:

$$K_b = \frac{v_a - i_a R}{\omega}$$

The angular velocity can be measured with a tachometer or by an optical sensor.

As K_t should equal K_b, we can assume at this stage that we have $K_b = K_t = K_m$.

7.4 Measuring D_m.
From the current – torque relationship

$$T_m = K_t i_a(t)$$

and the equation for the mechanical subsystem

$$T_m = J\frac{d\omega(t)}{dt} + D_m\omega(t)$$

we can write an expression for the steady state conditions ($d\omega/dt = 0$) as:

$K_t i_a = D_m \omega$

By measuring the steady state rotational velocity and the current and from knowledge of K_t we can then calculate the damping coefficient of the motor.

7.5 Measuring J_m.

The motor inertia can be obtained from a measure of the mechanical time constant of the system. This can be done by switching off the motor by producing an open circuit in the electrical input. From the decay in angular velocity with time, the mechanical time constant can be measured. This is given by:

$$\tau_m = \frac{J_m}{D_m}$$

and J_m can be calculated if D_m is known.

7.6 Some further points.

In section 6 the simplified (ignoring inductance) gain of the motor was shown to be:

$$K_s = \frac{K_t}{R_a D_m + K_t K_b} = \frac{\omega(\infty)}{v_a(\infty)}$$

If we then test the motor response to an input voltage step and measure the steady state rotational velocity, we can calculate this factor K_s. Note that the value of K_s is independent of the inertia of the motor and is only determined by the damping of the motor (and the motor constants as well as the armature resistance). If we know the other factors, we can then calculate Dm from the steady state response.

It can be shown that this equation for the damping reduces to the same as that used in Section 7.4

$$\tau_s = \frac{J_m R_a}{R_a D_m + K_t K_b}$$

*****