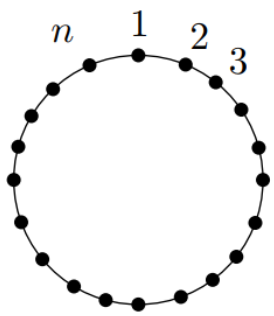
Homework 6:
Spectral graph theory

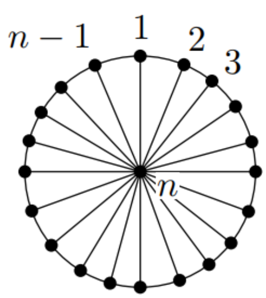
Problem 1: Graph Laplacians

(a) [4 points]

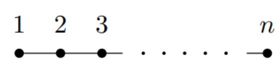
Consider the graphs given below. For this part, take $n = 7$. For each graph, write down the Laplacian matrix $L = D - A$ where D is the diagonal degree matrix and A is the adjacency matrix. (Actually write out or print out the matrices.)



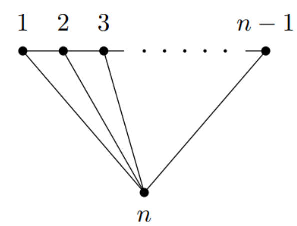
A cycle graph



A spoke and wheel graph



A line graph



Line graph + a point

(b) [4 points]

For each of the graphs in part (a), compute the eigenvectors and eigenvalues of the Laplacian matrix L and the adjacency matrix A when there are $n = 150$ vertices. Describe the smallest and the second-smallest eigenvalues (for both L and A) and plot the corresponding eigenvectors. Describe the largest and second-largest eigenvalues (for both L and A), and plot the corresponding eigenvectors.

Include four plots for each graph and clearly label the plots. When plotting an eigenvector $\mathbf{v}$, the x-axis ranges from 1 to n and the ith vertex is plotted at location $(i, \mathbf{v}(i))$.

(c) [3 points]

You should've seen that the largest eigenvalue of the Laplacian for some of these graphs is much larger than for the others. Explain why this is the case.

(d) [8 points]

For every eigenvector you computed in part (b), explain why the structure of the eigenvector makes sense.

(f) [4 points]

As a function of n, find the expression for the Frobenius norm of the difference between the Laplacian of the cycle graph and the Laplacian of each of the other three graphs. Recall that for any matrix $A \in \mathbb{R}^{n\times n}$, the Frobenius norm of A is given by

$$\|A\|_F = \left(\sum_{i,j=1}^{n} A_{ij}^2\right)^{1/2}.$$

Explain why this is related to any similarities and/or differences between the structures of the eigenvectors and eigenvalues you found earlier.

(g) [6 points]

Pick 600 random points in the unit square by independently choosing their x and y coordinates from the interval $[0, 2]$. Form a graph by adding an edge between every pair of points whose Euclidean distance is at most $1/2$. Compute the eigenvectors of the Laplacian of this graph and plot the embedding of the graph onto the 2nd and 3rd smallest eigenvectors. This time, do *not* overlay the edges of the graph, just plot the vertices. For all points in the original graph with x and y coordinates both less than 1, plot their images in a different color. What do you observe? Why does this make sense?

(h) [5 points]

Consider the 100×100 grid graph whose 10,000 vertices are $\{(i,j) : 1 \le i \le 100, 1 \le j \le 100\}$ and which has an edge $\{(i,j),(i',j')\}$ whenever $|i-i'| \le 1, |j-j'| \le 1$. Draw the spectral embedding (with edges present) as in part (g) using the second and third eigenvectors. Now remove 100 random vertices from this graph and again draw the spectral embedding. What do you observe?

Problem 2: Finding Friends

In this part, you will analyze part of the (anonymized) Facebook friend network. The file `friends.csv` contains a graph with 1495 vertices and 61796 rows each containing a pair (i,j) and representing the fact that person i is friends with person j.

(a) [2 points]

Compute the smallest 12 eigenvalues and corresponding eigenvectors of the Laplacian of the friendship graph. Print a list of the 12 smallest eigenvalues.

(b) [7 points]

How many connected components does this graph have? Justify your answer using the eigenvalues you computed in part (a). (Keep in mind that your numerical linear algebra package might have some rounding errors and a value less than 10^{-12} should probably be regarded as 0.) Using the eigenvectors, how can you tell which nodes are in which components?

(c) [7 points]

The *conductance* of a set of nodes in a graph is a measure of how tightly knit that set is, with lower conductance indicating a more insular set.

$$\text{cond}(S) = \frac{\sum_{u \in S} \sum_{v \in V \setminus S} A_{uv}}{\min(A(S), A(V \setminus S))} \,,$$

where $A(S)$ is the sum of the degrees of vertices inside S.

Using ideas from spectral graph theory, find at least 3 sets S_1, S_2, S_3 of people in the friendship graph such that each set has at least 200 people and conductance at most 0.1. Explain how you found this set.

(d) [8 points]

Now select a random set of 200 nodes and compute the conductance of that set. Do the sets you found in part (c) seem tight-knit compared to a random set of people?