Jeremy Schumacher

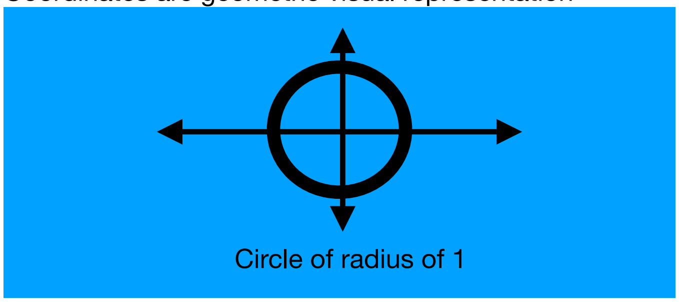
Chapter 13

3D Space

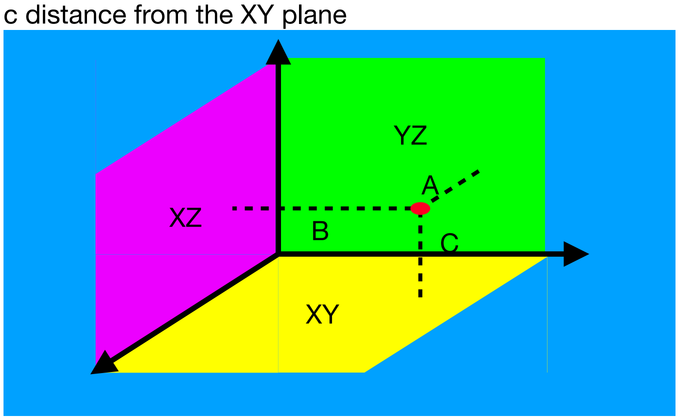
Equations are algebraic

$$x^2 + y^2 = 1$$

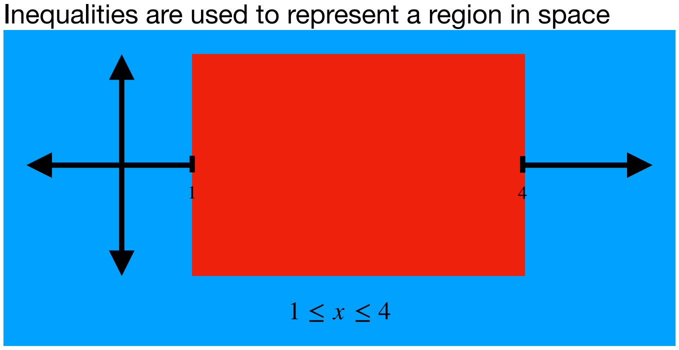
Coordinates are geometric visual representation



In 3D space a coordinate (a,b,c) is a point a distance way from the ZY plane b distance from the XZ point and c distance from the XY plane



Inequalities are used to represent a region in space



Circle $x^2 + y^2 = 1$

Sphere $x^2 + y^2 + z^2 = 1$

Ball $x^2 + y^2 + z^2 \leq 1$

Upper half of sphere $x^2 + y^2 + z^2 \leq 1, \; z \geq 0$

Disk $x^2 + y^2 + \leq 1, \; 0 \leq z \leq 1$

Vectors

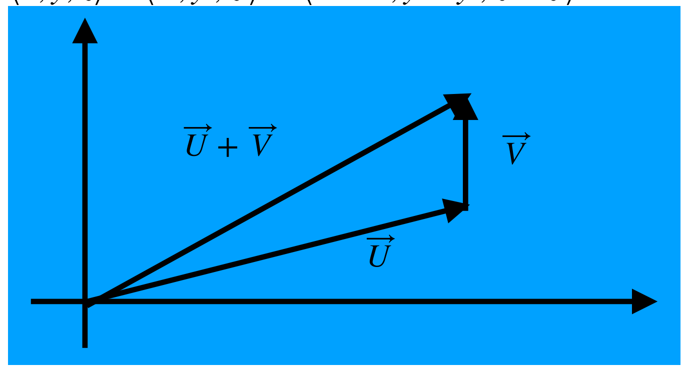
Vectors- 2 ordered points $\langle a, b, c \rangle$

Have 2 properties a direction and magnitude

$\overrightarrow{PQ}$ is a vector from point P to point Q

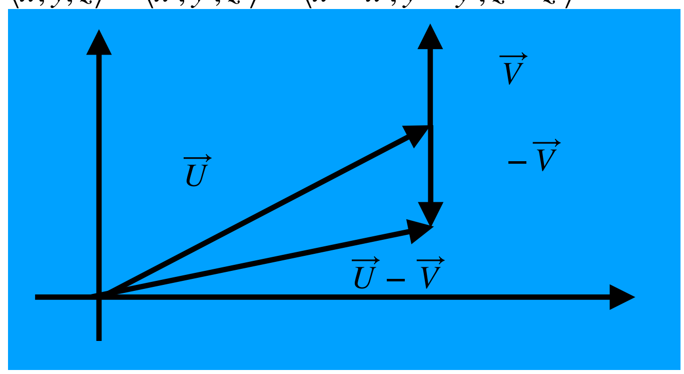
Adding Vectors

$\langle x, y, z \rangle + \langle x', y', z' \rangle = \langle x + x', y + y', z + z' \rangle$

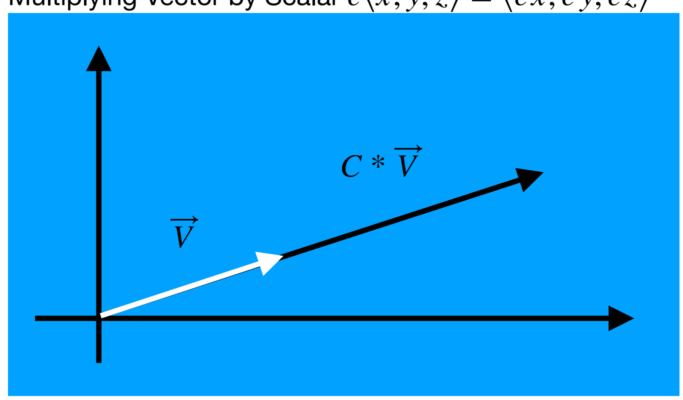


Subtracting Vectors

$\langle x, y, z \rangle - \langle x', y', z' \rangle = \langle x - x', y - y', z - z' \rangle$



Multiplying Vector by Scalar $c\langle x, y, z \rangle = \langle cx, cy, cz \rangle$



Dot Product

- $\langle x, y, z \rangle \cdot \langle x', y', z' \rangle = x * x' + y * y' + z * z'$
- $\vec{U} \cdot \vec{V} = |\vec{U}||\vec{V}| cos\theta$ (theta is the angle between vectors)
- If dot product is zero then the factors are orthogonal
- $\vec{U} \cdot \vec{V} = \vec{V} \cdot \vec{U}$
- $(c * \vec{U}) \cdot \vec{V} = c(\vec{U} \cdot \vec{V})$
- $\vec{U} \cdot \vec{U} = |\vec{U}|^2$
- $(\vec{U} + \vec{V}) \cdot \vec{W} = \vec{U} \cdot \vec{W} + \vec{V} \cdot \vec{W}$
-

Projection of vector A onto vector B

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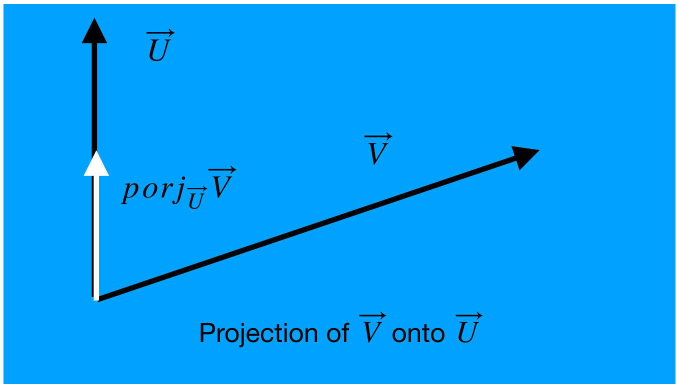


$$\frac{\vec{A} * \vec{B}}{|\vec{B}|} = \vec{A}cos\theta$$

Cross Product- gives the vector perpendicular to the plane containing the other two vectors

- $\langle x,y,z\rangle \times \langle x',y',z'\rangle = \begin{bmatrix} i & j & k \\ x & y & z \\ x' & y' & z' \end{bmatrix} = \langle y*z' - y'*z, -(x*z' - x'*z), x*y' - x'*y\rangle$
- $|\vec{U} \times \vec{V}| = |\vec{U}||\vec{V}|sin\theta$
- $\vec{U} \times (\vec{V} \times \vec{W}) = (\vec{U} \cdot \vec{W}) \cdot \vec{V} - (\vec{U} \cdot \vec{V}) \cdot \vec{W}$
- $(c\vec{U}) \times \vec{V} = c(\vec{U} \times \vec{V})$
- $(\vec{V} + \vec{U}) \times \vec{W} = \vec{U} \times \vec{W} + \vec{V} \times \vec{W}$
- $\vec{U} \cdot (\vec{V} \times \vec{W}) = \vec{W} \cdot (\vec{V} \times \vec{U})$
- $\vec{V} \times \vec{U} = -\vec{V} \times \vec{U}$

Vector Properties

$\vec{U} + \vec{V} = \vec{V} + \vec{U}$

$\vec{U} + 0 = 0 + \vec{U}$

$\vec{U} - \vec{U} = 0$

$1 * \vec{U} = \vec{U}$

$c * (\vec{U} + \vec{V}) = c * \vec{U} + c * \vec{V}$

$(c + d) * \vec{U} = c * \vec{U} + d * \vec{U}$

Magnitude (or norm)-

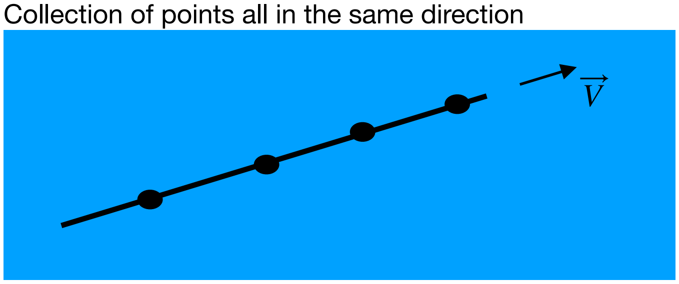
$$|\vec{U}| = |\langle x,y,z\rangle| = \sqrt{x^2 + y^2 + z^2}$$

Unit Vector $\vec{T} = \dfrac{1}{|\vec{U}|}\vec{U}$

Vector IJK Form $\langle x,y,z\rangle = xi + yj + zk$

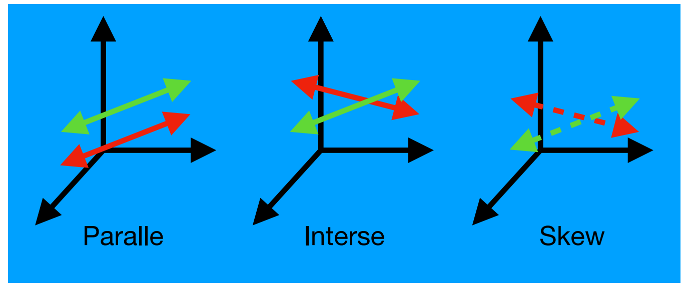
Lines

Collection of points all in the same direction



In 2D lines either intersect or are parallel

In 3D lines either intersect are linear or pass are skew (pass by each other)



Lines are only parallel in 3D when there direction vectors are the same

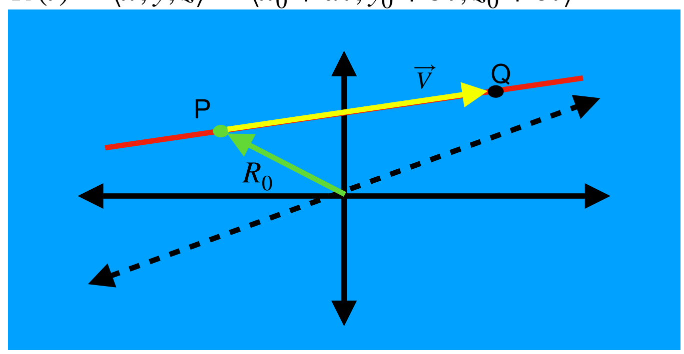
R_0- position vector one point on line

Move along $\vec{V}$ to find next point

$\vec{R}(t) = R_0 + t\vec{V}$

$\vec{R}(t) = \langle x,y,z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a,b,c\rangle$

$\vec{R}(t) = \langle x,y,z\rangle = \langle x_0 + at, y_0 + bt, z_0 + ct\rangle$



Planes

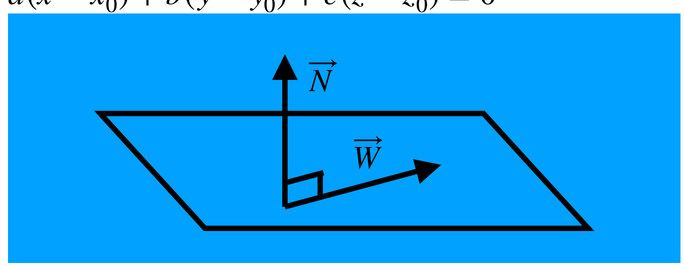
$\vec{N}$ is normal vector to plane

$\vec{W} = \langle x - x_0, y - y_0, z - z_0\rangle$

$0 = \langle a,b,c\rangle\langle x - x_0, y - y_0, z - z_0\rangle$

$0 = \langle a,b,c\rangle\langle x - x_0, y - y_0, z - z_0\rangle$

$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$



$ax + by + cz = d$

Find the line where 2 planes intersect

- Use the cross product of the two normal vectors to find $\vec{V}$
- Find R_0 by setting one variable to zero and solving the system of equations

Angle between two planes- dot product between the two normal vectors

Surfaces

Free Variable- in 3D sometimes a variable is missing so its not tied to any other variable

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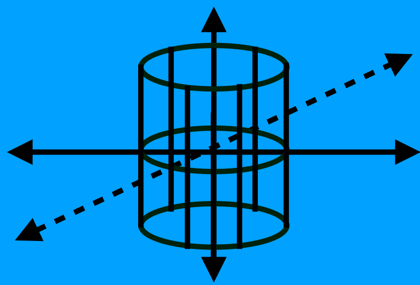
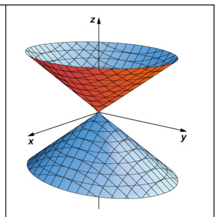
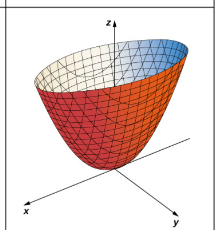
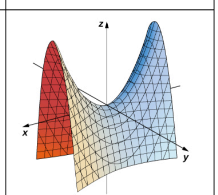
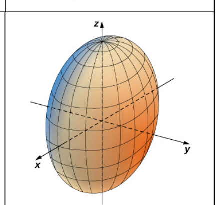
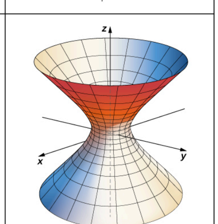
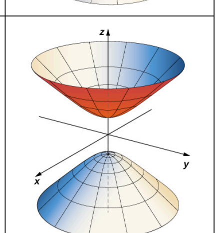
In equation $x^2 + y^2 = 1$ is a free variable



Traces- cross section view of 3D shape to help people identify what the shape is
When trying to draw shape find X Y and Z intercepts by setting other variables to zero
Draw traces of XY YZ and XZ plane
If shape is still unclear do additional traces at x,y,z=k

Characteristics of Common Quadric Surfaces

Elliptic Cone $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$ *Traces* In plane $z = p$: an ellipse In plane $y = q$: a hyperbola In plane $x = r$: a hyperbola In the xz – plane: a pair of lines that intersect at the origin In the yz – plane: a pair of lines that intersect at the origin The axis of the surface corresponds to the variable with a negative coefficient. The traces in the coordinate planes parallel to the axis are intersecting lines.	
Elliptic Paraboloid $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ *Traces* In plane $z = p$: an ellipse In plane $y = q$: a parabola In plane $x = r$: a parabola The axis of the surface corresponds to the linear variable.	
Hyperbolic Paraboloid $z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ *Traces* In plane $z = p$: a hyperbola In plane $y = q$: a parabola In plane $x = r$: a parabola The axis of the surface corresponds to the linear variable.	
Ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ *Traces* In plane $z = p$: an ellipse In plane $y = q$: an ellipse In plane $x = r$: an ellipse If $a = b = c$, then this surface is a sphere.	
Hyperboloid of One Sheet $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ *Traces* In plane $z = p$: an ellipse In plane $y = q$: a hyperbola In plane $x = r$: a hyperbola In the equation for this surface, two of the variables have positive coefficients and one has a negative coefficient. The axis of the surface corresponds to the variable with the negative coefficient.	
Hyperboloid of Two Sheets $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ *Traces* In plane $z = p$: an ellipse or the empty set (no trace) In plane $y = q$: a hyperbola In plane $x = r$: a hyperbola In the equation for this surface, two of the variables have negative coefficients and one has a positive coefficient. The axis of the surface corresponds to the variable with a positive coefficient. The surface does not intersect the coordinate plane perpendicular to the axis.	

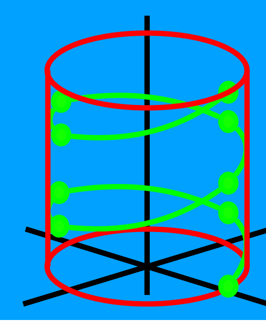
Chapter 14

Vector Valued Functions

An objects position in 3D space is defined by
$\vec{r} = \langle x, y, z \rangle$
Full trajectory of an object is $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$
All 3 variables x, y, and z change with respect to a single variable t
Circle with radius r is $\vec{r}(t) = \langle rcos(t), rsin(t) \rangle$

$\vec{r}(t) = \langle cos(t), sin(t), t \rangle$

T	X	Y	Z
0	1	0	0
$\pi/2$	0	1	$\pi/2$
π	-1	0	π
$3\pi/2$	0	-1	$3\pi/2$
2π	1	0	2π



Domain of a vector valued function must work for x(t), y(t) and z(t)

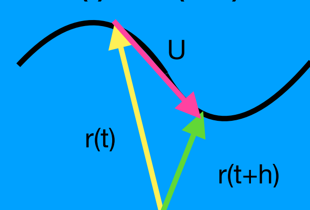
Derivative

For f(t) $f'(t) = \lim_{h \to 0} \frac{f(h+t) - f(t)}{h}$

For r(t) $\vec{r}'(t) = \lim_{h \to 0} \frac{\vec{r}(h+t) - \vec{r}(t)}{h}$ So

$\vec{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$

r(t)+u=r(t+h)



if $\vec{r}(t)$ is a position vector than $\vec{r}'(t)$ is the velocity vector
if $\vec{r}(t)$ is a position vector then $|\vec{r}'(t)|$ is speed (scalar)
if $\vec{r}(t)$ is a position vector then $\vec{r}''(t)$ is acceleration vector

Quotient Rule

$f(x)\frac{u}{v}$ then $f'(x) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$f'(x) = \frac{LoDeHi - HiDeLo}{(Lo)^2}$

Chain Rule

$h(x) = f(g(x))$ then $h'(x) = f'(g(x)) * g'(x)$

Power Rule

$f(x) = x^n$ then $f'(x) = n x^{n-1}$

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Product Rule

$f(x) = uv$ then $f'(x) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

Derivative Rules

$f(x) = k\sqrt{x}$ then $f'(x) = \dfrac{k}{2\sqrt{x}}$

$f(x) = \dfrac{k}{x}$ then $f'(x) = -\dfrac{k}{x^2}$

$f(x) = kx$ then $f'(x) = k$

$f(x) = k$ then $f(x) = 0$

$f(x) = x$ then $f'(x) = 1$

Special Derivatives

$f(x)a^x \ then \ f'(x) = a^x ln(a)$

$f(x) = ln(x) \ then \ f'(x) = \dfrac{1}{x}$

$f(x) = e^x \ then \ f'(x) = e^x$

Trig Functions	
function	**Derivative**
$sin(x)$	$cos(x)$
$cos(x)$	$-sin(x)$
$tan(x)$	$sec^2(x)$
$sec(x)$	$sec(x)tan(x)$
$csc(x)$	$-csc(x)cot(x)$
$cot(x)$	$csc^2(x)$

Integrals

Indefinite- $\int \vec{r}(t)dt = \overrightarrow{R} + \overrightarrow{c}$

Definite $\int_a^b \vec{r}(t)dt = \overrightarrow{R}(b) - \overrightarrow{R}(a)$

Integral is an accumulation of all the tangent vectors over interval [a,b]

if $\vec{r}(t)$ is position vector than $\vec{r}'(t)$ is velocity vector

if $\vec{r}(t)$ is position vector then $|\vec{r}'(t)|$ is speed (scalar)

if $\vec{r}(t)$ is position vector then $\vec{r}''(t)$ is acceleration vector

Integral Rules

$\dfrac{dy}{dt} = ky$

$\int a^x dx = \dfrac{a^x}{lna} + C$

$\int e^x dx = e^x + C$

$\int \dfrac{dx}{x} = ln|x| + c$

$\int x^n dx = \dfrac{x^{n+1}}{n+1} + C$

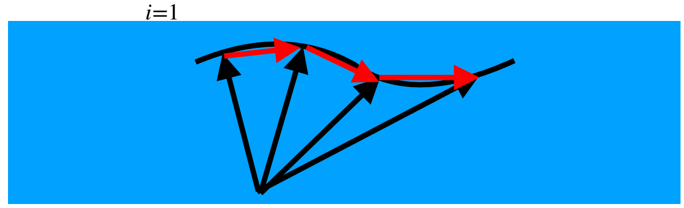
$\int k dx = kx + C$

$\int u(dv) = uv - \int v(du)$

$\int ln(x)dx = xln(x) - x + C$

Length

Estimated- $\sum_{i=1}^{n} |\vec{r}'(t)|\Delta t$



Actual- $\lim_{n\to\infty} \sum_{i=1}^{n} |\vec{r}'(t)|\Delta t = \int_a^b |\vec{r}'(t)|dt$

$s(t) = \int_a^t |\vec{r}'(t)|dt$ relates length and time

$r(s)$ tells us where we are after a given length but to do this we need s(t)

Unit tangent vector $T(t) = \dfrac{r'(t)}{r(t)}$

T does not change based on magnitude so a change in T means direction changes

- So $|\dfrac{dT}{ds}|$ is how much the curve turns
  - If $|\dfrac{dT}{ds}|$ its big then the curve turns a lot on very sharply
  - If $|\dfrac{dT}{ds}|$ its small then the curve turns very gradually
  - If $|\dfrac{dT}{ds}| = 0$ then the function is a straight line
- $\dfrac{ds}{dt} = |r'(t)|$ so $\dfrac{dT}{ds} = \dfrac{\frac{dT}{ds}}{\frac{ds}{dt}} = |\dfrac{T'(t)}{r'(t)}| = \kappa$

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Chapter 15

Domain and Range

Domain- set of all possible input values

- For two variable functions the domain is a set of ordered pairs (a region non the XY plane)

Range- set of all possible output values

- For two variable variable functions range is still one value

Level Curve- for a function $z = f(x, y)$ we set z equal to a constant reviewing a function where $z_0 = f(x, y)$ (basically a trace)

If $\lim_{x \to a} f(x) = L$ then the closer x gets to a the closer f(x) gets to L

Limits and Continuity

Conditions of Continuity

- $f(c)$ exists
- $\lim_{x \to c} f(x)$ exists
- $f(c) = \lim_{x \to c} f(x)$ exists

If $\lim_{x \to a} f(x) = f(a)$ then $f(x)$ is continuous at $x = a$

Polynomials, sin, cosine and exponential functions are continuous functions

Rational and log functions are continues where they are defined

$$\lim_{(x,y) \to a,b} f(x, y) = L$$

Approaching a two function limit is more complicated because there are infinite paths (along x=a y=b along y=b/ax, parabola, exponential etc)

If the limit of two paths are different then limit does not exist

Check the type of function it is and if its continuous at f(a,b) than $\lim_{(x,y) \to a,b} f(x, y) = f(a, b)$

Partial Derivatives

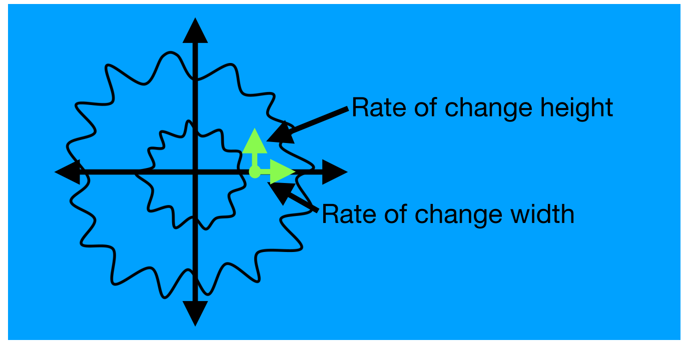
For $z = f(x, y)$is affected by x and y so we cannot take the derivative with both changing

We can hold one variable constant and allow the other to change to get the partial derivative

Partial Derivative- the rate of change of z with respect to the variable that is changing

$$F_x = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$F_y = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$



F_x and F_y tell us the effect of varying on variables at a time

Clairaut's Theorem- $F_{xy} = F_{yx}$ whenever $f(x, y)$ is defined as F_x and F_y are continuous

Chain Rule- $y = f(u) \quad u = g(t)$ and $f(g(t))$ then $y' = f'(g(t)) * g'(t)$

To find out how fast z is changing use the chain rule $z = f(x, y)$ and $y = g(t)$ and $x = h(t)$ then

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Implicit Differentiation- If we have F(x,y) we can differentiate implicitly to find $\frac{dy}{dx} = -\frac{F_x}{F_y}$

Directional Derivative

Let $\vec{U} = \langle a, b \rangle$ be a unit vector (giving direction). Than the rate of change in that direction is

$$D\vec{u}f(x, y) = \lim_{h \to 0} \frac{f(x + ah, y + bh) - f(x, y)}{h}$$

Max directional derivative occurs when $\theta = 0$ and the value of $D\vec{u}f(x, y) = \vec{\nabla} f$

- $Gradient = \nabla = \langle F_x, F_y \rangle$
- Gradient is perpendicular to the level curve

Min directional derivative occurs when $\theta = 180$ and the value of $D\vec{u}f(x, y) = -\vec{\nabla} f$

$D\vec{u}f(x, y) = 0 \; if \; \vec{\nabla} f \perp \vec{U} \; \theta = \frac{\pi}{2} \; or \; \frac{3\pi}{2}$ is the only directions you go to see no change

$D\vec{u}f(x, y) = \langle F_x, F_y \rangle \cdot \langle a, b \rangle$

If the level curve is $\vec{r}(t) = \langle x(t), y(t) \rangle$ then (x_0, y_0)

$$\langle F_x, F_y \rangle \cdot \frac{\langle \frac{dx}{dt}, \frac{dy}{dt} \rangle}{|\langle \frac{dx}{dt}, \frac{dy}{dt} \rangle|} = 0 \text{ and therefore } \frac{dy}{dx} = -\frac{F_x}{F_y}$$

Tangent Plane

z=f(x,y) is a surface so instead of finding a tangent line at point (x_0, y_0, z_0) we actually have infinitely meany tangent lines which form tangent plane

When very near to (x_0, y_0, z_0) tangent plane roughly equals true surface

Infinitely many curves form curve $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ and their tangent vectors are

are $\vec{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$ and the normal vector is perpendicular to all the tangent vectors
The normal vector of the tangent plane is the gradient
The smaller dx and dy are the closer dz is
$\% x = \frac{dx}{x}$
Equation for tangent plane

- $z - z_0 = F_x(x - x_0) + F_y(y - y_0)$
- $z - z_0 = F_x(\Delta X) + F_y(\Delta Y)$

Max and Mins

Critical points occur when F_x and F_y are equal to zero

Second derivative test $D = F_{xx} * F_{yy} - F_{xy}^2$

- If D>0 and Fx<0 its a maximum
- If D>0 and Fx>0 its a minimum
- If D<0 its a saddle point
- If D=0 test is inconclusive and you have to check points around it

To find absolute max and main find all critical points in regions find all end critical points and find all corner points
Plug them all into the equation and see which yields the largest and smallest value

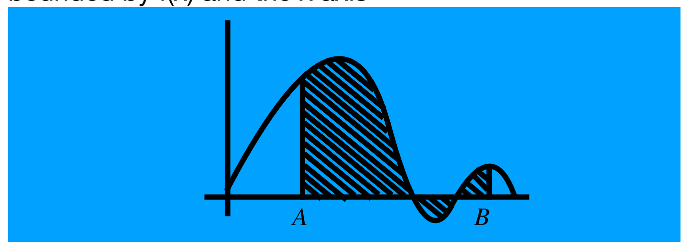
Lagrange Multipliers

used when finding the max and min of constrained optimization problems
f(x,y) is the objective
g(x,y)=0 is the constraint
At critical point the level curve and the constraint curve are tangent to each other
At the point where f and g are tangent the $\vec{\nabla} f \,||\, \vec{\nabla} g$
To find max and mins of f(x,y) subject to g(x,y) we solve for location where $\vec{\nabla} f = \lambda \vec{\nabla} g$

Chapter 16

Integration

If $f(x)\ \ a \le x \le b$ then $\int_a^b f(x)dx$ gives us the area bounded by f(x) and the x axis



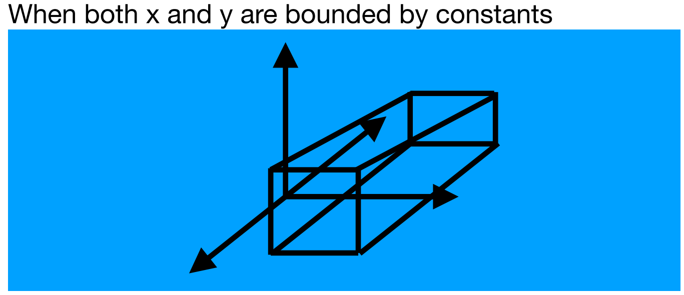
$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta X$$

Volume

$$\lim_{n,m\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{n} f(x_i, y_j)\Delta A$$

$$\int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy$$

When both x and y are bounded by constants



Switching the order of integration can make it easier
Integrate variable with constants bounds last

Average Value of a Function

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)dx$$

$$f_{avg} = \frac{1}{b-a}\frac{1}{d-c}\int_a^b \int_c^d f(x,y)dydx$$

Integral General region

When regions are non rectangular swapping order is not so easily
We can still swap order but we need to reformulate R
Type I- x bounded by constants y bounded by vars
Type II- y bounded by constants x bounded by vars
In Cartesian form point is expressed by (x,y) distance from y axis and distance from x axis

Integral Polar Region

In polar form point is expressed by (r, θ) where r is displacement from origin and theta is the angle of the point to the origin from x axis
$x = rcos(\theta)$

$y = rsin(\theta)$

$\iint_R f(r,\theta)dA = \iint_R f(r,\theta)rdrd\theta$

Triple Integrals

$\iiint_D dV$ accumulates Volume over volume D

Start with a projection on a plane (generally XY)

Integrate var bounded by constants last and the variable bounded by the other variables first

Cylindrical Integral

Cylindrical $\iiint f(z,r,\theta) * rdzdrd\theta$

$dV = rdzdrd\theta$

Spherical Integral

$\iiint f(\rho,\phi,\theta) * \rho^2 sin\phi d\rho d\phi d\theta$

$x = \rho sin\phi cos\theta$

$y = \rho sin\phi sin\theta$

$z = \rho cos\phi$

$dV = \rho^2 sin\phi$

$\rho^2 = z^2 + x^2 + y^2$

$0 \leq \rho \leq \infty$

$0 \leq \phi \leq \pi$

$0 \leq \theta \leq 2\pi$

Center of Mass

$\bar{x} = \frac{1}{m}\iint xpdA$

$\bar{y} = \frac{1}{m}\iint ypdA$

$m = \iint pdA$

$\bar{x} = \frac{1}{m}\iiint xpdV$

$\bar{y} = \frac{1}{m}\iiint ypdV$

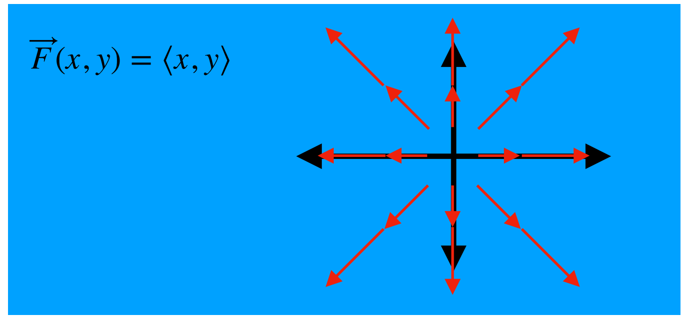
$\bar{z} = \frac{1}{m}\iiint zpdV$

Chapter 17

Vector Fields

A function that assigns a factor to each point in space

- Acceleration- vector function



Scalar fields- assigns scalar each point (has no direction)

- Temp- scalar function

Vector field has a domain

If $\vec{F}$ is gradient of a scalar function than the scalar field is the potential function of the vector field $\vec{\nabla} U = \vec{F}$

The gradient is perpendicular to the level curve

When the level curves are closer the vector field has larger magnitude (SHARPER CHANGES)

Not every vector field has a potential function because not every vector file is gradient of a function

If $\vec{F}$ is a force and has a potential function U than the force is said to be **conservative**

If a function is conservative than the work done by force is independent of path (like gravity which is path independent)

Line Integrals

Length of $\vec{r}'(t)$ from $a \leq t \leq b$ is

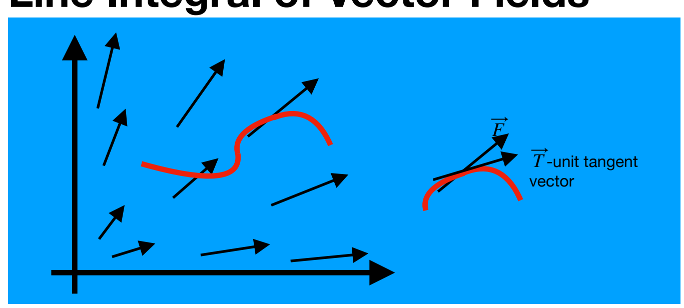
$\int_a^b |\vec{r}'(t)| dt = \int_c ds$

$ds = |\vec{r}'(t)| dt$ and c is the re-parameterized curve length. Is a special case of line integral $\int_c f(x,y)ds$ where $f(x,y) = 1$

If $f(x,y)$ its density than $\int_c f(x,y)ds$ gives the mass

The parameterization of C does not change the line integral

Line Integral of Vector Fields



Jeremy Schumacher

We want to accumulate some component of $\vec{F}$ along path most typically vector field along unite tangent of path

So line integral $\int_c \vec{F} \cdot \vec{T} ds$ parameterized C as $\vec{r}(t)$

over $a \leq t \leq b$ and $\vec{T} = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}$ and $ds = |\vec{r}'(t)|dt$

$$\int_c \vec{F} \cdot \vec{T} ds = \int_c \vec{F} \cdot \vec{r}'(t) dt = \int_c \vec{F} \cdot d\vec{r} = \int_c (fx' + gy')dt$$

The parameterization of C does not affect the answer (do what is easiest

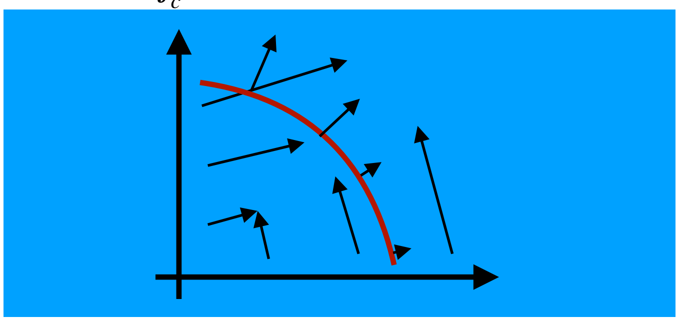
If C is a closed loop (start and end are the same) then the line integral. Is called the circulation of $\vec{F}$ onto C (but we do the same thing)

- We may need to do multiple parameterizations if it was a sharp corner

If $\vec{F}$ is conservative than the circulation is 0

If $\vec{r}'(t) = \langle x, y \rangle$and $\vec{F}'(t) = \langle f, g \rangle$ then $\int_c fdx + gdy$

If we replace $\vec{T}$ (unit tangent vector) with $\vec{N}$ (normal vector) then $\int_c \vec{F} \cdot \vec{N} ds$ becomes the flux integral



If C is a closed loop choose N that points outward

If C is not a closed loop choose $\vec{N}$ to the right of $\vec{T}$

Conservative Vector Fields and Fundamental Theorem

If $\vec{F}$ is conservative than $\vec{F} = \vec{\nabla} U$

Given U finding f is easy

Remember if we have function $G(x, y)$ then $G_{xy} = G_{yx}$

So if $\vec{F} = \langle f, g \rangle$ is conservative than $\vec{f}_y = \vec{g}_x$

To find U if $\vec{F} = \langle f, g \rangle$ is conservative

$\vec{F} = \vec{\nabla} U = \langle \dfrac{\partial U}{\partial x}, \dfrac{\partial U}{\partial y} \rangle$ so $\int f + C = U$ but C is a function of the other variable(s) and therefore

$$\frac{\partial \int f + C}{\partial y} = g$$

In 3D for $\vec{F} = \langle f, g, h \rangle$ to be conservative

- $f_z = h_x$
- $f_y = g_x$
- $g_z = h_y$

Fundamental Theorem of Line Integrals

If $\vec{F}$ is conservative then

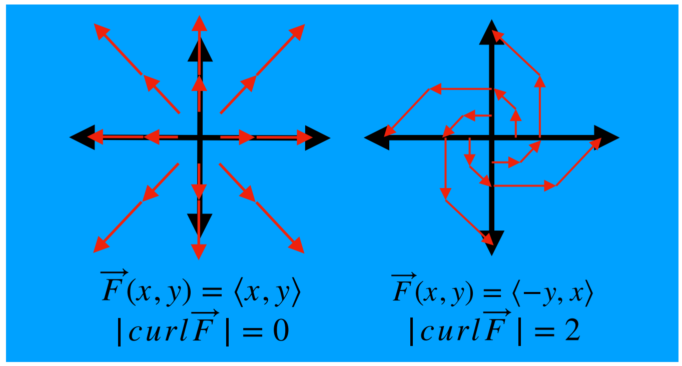
$$\int_c \vec{F} \cdot \vec{T} ds = \int_c \vec{F} \cdot \vec{r}'(t) dt = \int_c \vec{F} \cdot d\vec{r} = U(B) - U(A)$$

Green's Theorem

Let $\vec{F} = \langle f, g \rangle$ be a 2D vector field

Then the quantity $(\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y})k$ or $\langle 0,0, \dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y} \rangle$ is called the curl of $\vec{F}$ $(curl \vec{F})$

$|curl \vec{F}| = g_x - f_y$ is the mess sure of rotation in a vector field

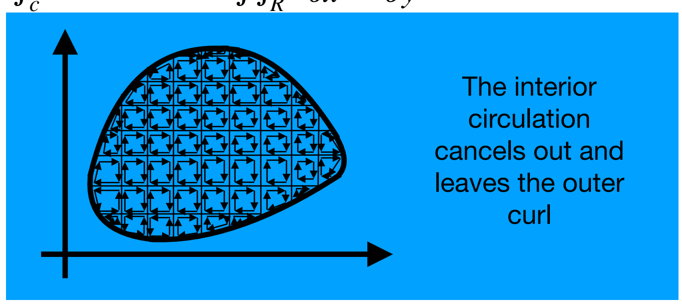


Recall if $g_x - f_y = 0$ than $\vec{F} = \langle f, g \rangle$ is conservative

Therefore if $|curl \vec{F}| = 0$ than $\vec{F}$ is conservative and is irrotational

Greens Theorem- if $\vec{F} = \langle f, g \rangle$and c is a simple closed path traveled once in a counter clockwise direction than

$$\oint_c fdx + gdy = \iint_R (\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y})dA$$



Greens Theorem allows us to calculate the area of the region bounded by C

$\oint_c fdx + gdy = \iint_R (\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y})dA$ if $\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y} = 1$ then we get $A = \dfrac{1}{2} \oint_c xdx - ydy$

If we replace $\vec{T}$ (unit tangent vector) with $\vec{N}$ (normal vector) we get $\oint_c fdy - gdx = \iint_R (\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y})dA$

allowing us to calculate the flux integral as a double integral

Jeremy Schumacher

$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$ is the divergence of $\vec{F} = \langle f, g \rangle$ which measures the change of a small volume as it flows in the vector field

Del Operator

$\vec{\nabla} F = \langle F_x, F_y, F_z \rangle$ and F is a scaler but the result is a vector therefore $\vec{\nabla}$ must be a vector like operator

$\vec{\nabla}$ is the 'del operator' and is defined as $\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$

$\vec{\nabla}$ does not mean python until it is applied to something

Curl

Scalar value

$|curl\vec{F}| = \vec{\nabla} \times \vec{F}$

We know if $\vec{F}$ is conservative than $\vec{F} = \nabla U$ so than

$\vec{\nabla} \times \vec{\nabla} U = \langle 0,0,0 \rangle$

Divergence

Scalar value

$|div\vec{F}| = \vec{\nabla} \cdot \vec{F} = (\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y})$

$\vec{F} = \langle f, g \rangle$

$\frac{\partial f}{\partial x}$- rate of change of f as x increases (does x get bigger as we move to the right)

$\frac{\partial g}{\partial y}$- rate of change of g as y increase (does y get bigger as we move up)

Surface Integrals

If $f(x, y)$ is 1 line integral gives length

If $f(x, y)$ is density line integral gives mass

Surface integral is the accumulation of $f(x, y, z)$ on surface

$$\iint_S f(x, y, z)dS$$

We have to parameterize S as $\vec{r}(u, v)$

Parameterization- mathematically locating each point on surface

- If we know x and y we can find z let u be x & v be y

$dS = |\vec{r}_u \times \vec{r}_v| du dv$

$$\iint_S f(u, v)|\vec{r}_u \times \vec{r}_v| dS$$

When $z = f(x, y)$the surface integral

$$\iint_S f(x, y)\sqrt{1 + f_x^2 + f_y^2} dx dy$$

If we parameterize in one coordinate system and then switch to another then we need to manually adjust dA otherwise we don't

We can calculate the average value of $f(x, y, z)$ on S

$$f_{avg} = \frac{\iint_S f(x, y, z)dS}{\iint_S dS}$$

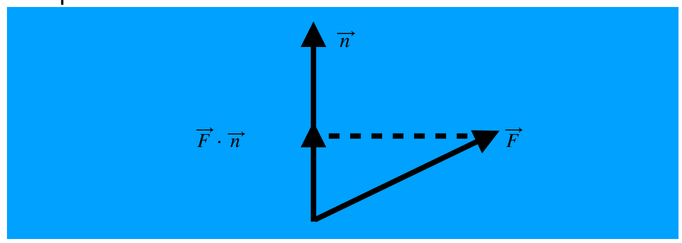
Surface Integrals in Vector Fields

In scalar field order of $|\vec{r}_u \times \vec{r}_v|$ is irrelevant but in vector field the direction is important

Conventionally we choose upward or outward normal vector

The surface integral in vector field is

$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} dS.$ This is also called the flux integral (for surfaces) which accumulates the components of $\vec{F}$ in the same direction of $\vec{n}$



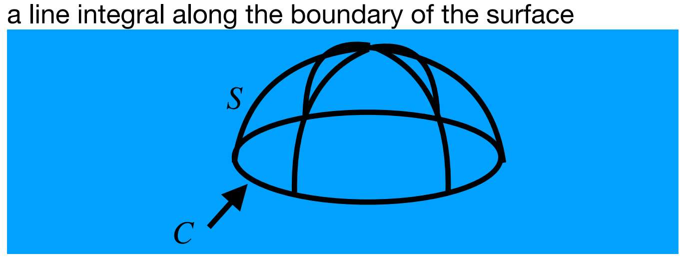
$$\iint_S \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) dA$$

If $z = f(x, y)$ and $\vec{F} = \langle f, g, h \rangle$

$$\iint_R (-f * z_x - g * z_y + h) dA$$

Stokes Theorem

Relates the surface integral of the curl of a vector file to a line integral along the boundary of the surface



The boundary curve is the open part of the surface

Both the surface and the boundary curve are oriented their orientations obey the right hand rule

Turn is in the direction of the normal vector then figures curl in the direction of the boundary curve C

$$\iint_S curl\vec{F} \cdot d\vec{S} = \oint \vec{F} \cdot d\vec{r}$$

S- surface

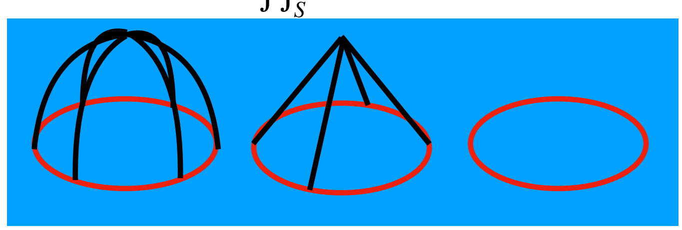
C- boundary curve

This is Green's Theorem but on steroids

Greens theorem- the surface must be flat

Stoke's- surface can be flat or curved

This means that two surfaces with the same boundary C must have the same $\iint_S curl\vec{F} \cdot d\vec{S}$



Jeremy Schumacher

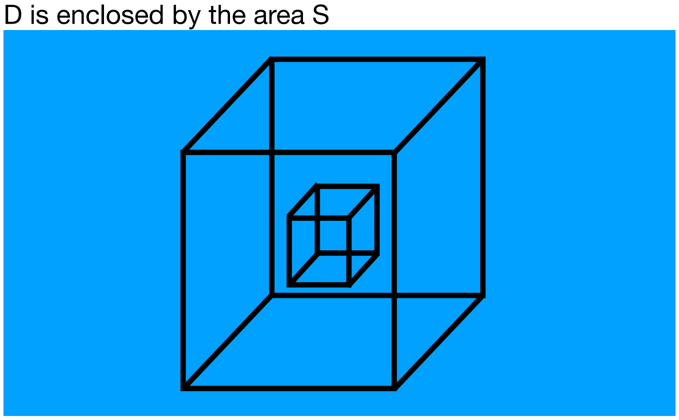
The $curl\overrightarrow{F} = \langle f, g, h \rangle$ tell us atet to get the maximum paddle wheel spin we want to agin its axis so that it matches the curl

Divergence Theorem

If the divergence is positive then the volume increase as it moves through the vector field
Box expands
If divergence is negative then the volume decreases as it moves through the vector field
Box shrinks
If the box is not flexible but the surface is porous (things flow through it) then if the divergences is positive each face will have a greater magnitude leaving than entering
So divergences is change in volume/ flux through surface

$$\iint_S \overrightarrow{F} \cdot d\overrightarrow{S} = \iint_S \overrightarrow{F} \cdot \overrightarrow{n} dS = \iiint_D div\overrightarrow{F} dV$$

D is enclosed by the area S



The normal vector of the outer box points outward
The normal vector of the inner box points inward (is negative)

$$\iint_{S_1} \overrightarrow{F} \cdot \overrightarrow{n} dS - \iint_{S_2} \overrightarrow{F} \cdot \overrightarrow{n} dS = \iiint_D div\overrightarrow{F} dV$$

D is the space between surfaces

$$\iiint_D div\overrightarrow{F} dV = \iiint_{D_2} div\overrightarrow{F} dV - \iiint_{D_1} div\overrightarrow{F} dV$$

If $div\overrightarrow{F}$ is not defined at a point put a little. Bible around the origin with a radius r that $\lim\limits_{\epsilon \to 0}$