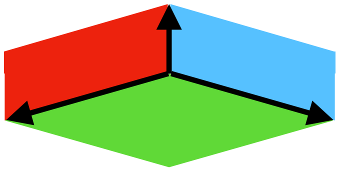
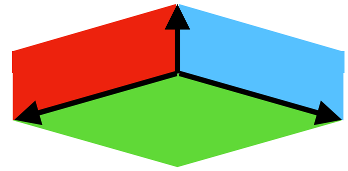
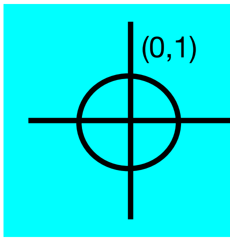


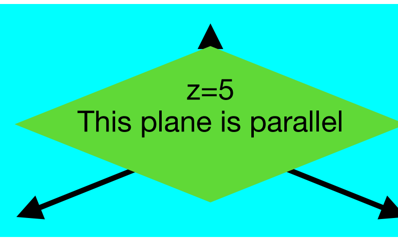
3D Surfaces

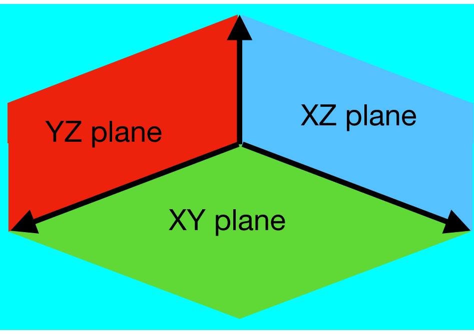


Coordinates are needed to write equations

Coordinates are geometric visual representations
(ie circle with a radius of 1)





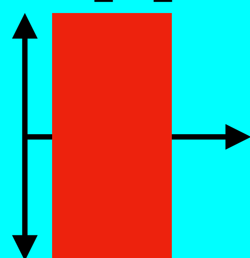


Equations are algebraic
(ie $x^2 + y^2 = 1$)

Equation for sphere of radius r centered at (a,b,c)
$x^2 + y^2 + z^2 = 1$

Use inequalities to represent region

$1 \le x \le 4$



Ball is $x^2 + y^2 + z^2 \le 1$

Disk is $x^2 + y^2 \le 1$

Circle is $x^2 + y^2 = 1$

Upper half of sphere
$x^2 + y^2 + z^2 \le 4, z \ge 0$

3 numbers (a,b,c) mean there is a point with coordinates a,b,c

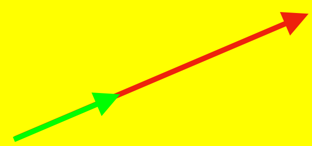
Vectors have 2 properties a direction and a magnitude

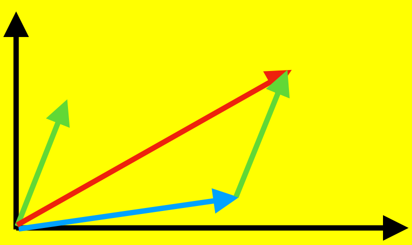
Vectors- 2 ordered points ⟨a,b,c⟩

$\overrightarrow{PQ}$ is a vector from point P to point Q

Adding Vectors
$\langle x, y, z\rangle + \langle x', y', z'\rangle = \langle x + x', y + y', z + z'\rangle$

Scalars
$c\langle x, y, z\rangle = \langle cx, cy, cz\rangle$





$\vec{V} + \vec{U} = \vec{U} + \vec{V}$

$0 + \vec{U} = \vec{U} + 0$

$\vec{V} - \vec{V} = 0$

$1 * \vec{V} = \vec{V}$

$c(\vec{V} + \vec{U}) = c\vec{U} + c\vec{V}$

$(c + d)\vec{U} = c\vec{U} + d\vec{U}$

$c|\vec{U}| = |c\vec{U}|$

IJK form
$\langle x, y, z\rangle = xi = yj + zk$

Unit Vectors- magnitude is 1
$$\vec{U_v} = \frac{1}{|v|}\vec{V}$$

Cross product (on 3D)
$\vec{U} \times \vec{V} = |\vec{U}||\vec{V}| sin\theta$

$$\langle x, y, z\rangle \times \langle x', y', z'\rangle = \begin{pmatrix} i & j & k \\ x & y & z \\ x' & y' & z' \end{pmatrix}$$

Magnitude (or norm) = $|\vec{U}|$

θ = angle between vectors

Dot product
$$\vec{U} \cdot \vec{V} = |\vec{U}||\vec{V}| cos\theta$$
$$cos\theta = \frac{\vec{U} \cdot \vec{V}}{|\vec{U}||\vec{V}|}$$
$$\vec{U} \cdot \vec{V} = xi * x'i + yj + y'j + zk * z'k$$

Cross product gives the vector perpendicular to the plane contains the other two vectors

If dot product equals 0 then the vectors are orthogonal

$\vec{U} \times (\vec{V} \times \vec{W}) = (\vec{U} \cdot \vec{W}) \cdot \vec{V} - (\vec{U} \cdot \vec{V}) \cdot \vec{W}$

$(c\vec{U}) \times \vec{V} = c(\vec{U} \times \vec{V})$

$(c\vec{U}) \times \vec{V} = c(\vec{U} \times \vec{V})$

$(\vec{V} + \vec{U}) \times \vec{W} = \vec{U} \times \vec{W} + \vec{V} \times \vec{W}$

$\vec{U} \cdot (\vec{V} \times \vec{W}) = \vec{W} \cdot (\vec{V} \times \vec{U})$

$\vec{V} \times \vec{U} = - \vec{V} \times \vec{U}$

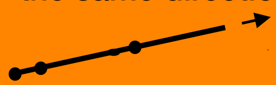
$\vec{U} \cdot \vec{V} = \vec{V} \cdot \vec{U}$

$(c * \vec{U}) \cdot \vec{V} = c(\vec{U} \cdot \vec{V})$

$\vec{U} \cdot \vec{U} = |\vec{U}|^2$

$(\vec{U} + \vec{V}) \cdot \vec{W} = \vec{U} \cdot \vec{W} + \vec{V} \cdot \vec{W}$

Lines- collection of points that all lie along the same direction

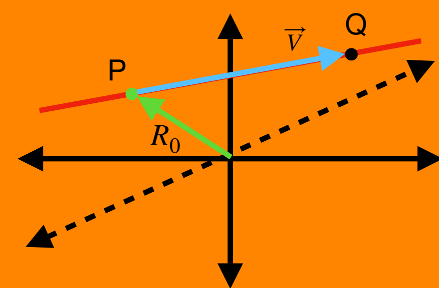


In 2D lines either intersect or are parallel

Lines are only parallel in 3D when there direction vectors are the same

In 3D lines either intersect are parallel or pas by each other

Finding the equation of a Line



R_0- position vector one point on line

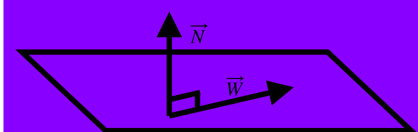
Move along $\vec{V}$ to find next point

$\vec{R}(t) = R_0 + t\vec{V}$

$\vec{R}(t) = \langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$

$\vec{R}(t) = \langle x, y, z \rangle = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$

Equation of Plane



$\vec{N}$ is normal vector to plane

$\vec{W} = \langle x - x_0, y - y_0, z - z_0 \rangle$

$0 = \langle a, b, c \rangle \langle x - x_0, y - y_0, z - z_0 \rangle$

$0 = \langle a, b, c \rangle \langle x - x_0, y - y_0, z - z_0 \rangle$

$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

$ax + by + cz = d$

Find the line where 2 planes intersect

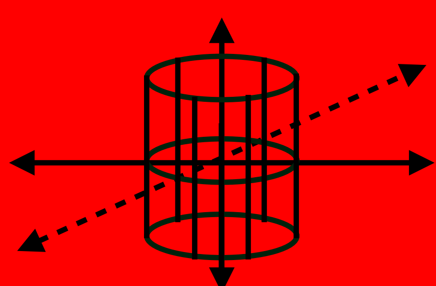
Use the cross product of the two normal vectors to find $\vec{V}$

Find R_0 by setting one variable to zero and solving the system of equations

Angle between two planes- dot product between the two normal vectors

Free variable- in 3D sometimes a variable is missing not tied to any other variable

In equation $x^2 + y^2 = 1$ is a free variable



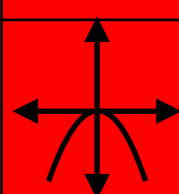
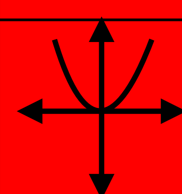
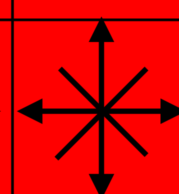
Traces- cross section view of 3D shape to help people identify what the shape is

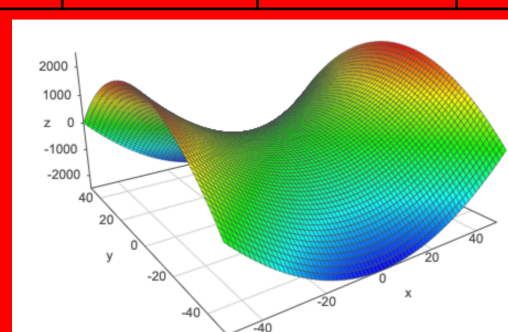
When trying to draw shape find X Y and Z intercepts by setting other variables to zero

Draw traces of XY YZ and XZ plane

If shape is still unclear do additional traces at x,y,z=k

$x^2 - y^2 = z$

	X	Y	Z
Intercept	0	0	0
Trace equation (var=0)	$z = -y^2$	$z = x^2$	$x = \pm y$
Trace picture			



3D space | Vectors | Dot product | Cross product | Lines | Planes | Surfaces

Vector Valued Functions

An objects position in 3D space is defined by $\vec{r} = \langle x, y, z \rangle$

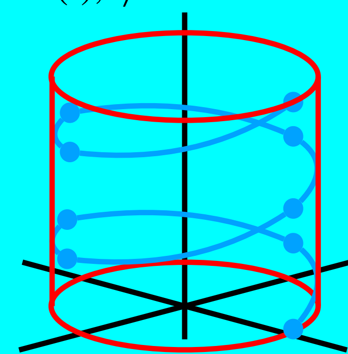
Full trajectory of an object is $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

All 3 variables x, y, and z change with respect to a single variable t

Circle with radius r is $\vec{r}(t) = \langle rcos(t), rsin(t) \rangle$

$\vec{r}(t) = \langle cos(t), sin(t), t \rangle$

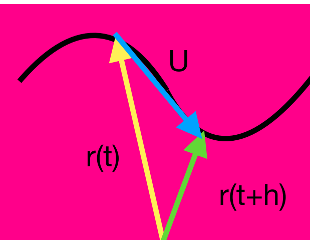
T	X	Y	Z
0	1	0	0
$\pi/2$	0	1	$\pi/2$
π	-1	0	π
$3\pi/2$	0	-1	$3\pi/2$
2π	1	0	2π



Circle with radius r is to see if surfaces interest put one equation into another

Domain of a vector valued function must work for x(t), y(t) and z(t)

For f(t) $f'(t) = \lim_{h \to 0} \frac{f(h+t) - f(t)}{h}$



r(t)+u=r(t+h)

For r(t) $\vec{r}'(t) = \lim_{h \to 0} \frac{\vec{r}(h+t) - \vec{r}(t)}{h}$ So $\vec{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$

Integral $\int \vec{r}(t)dt = \vec{R} + \vec{c}$

$\int_a^b \vec{r}(t)dt = \vec{R}(b) - \vec{R}(a)$

Integral is an accumulation of all the tangent vectors over inverval [a,b]

if $\vec{r}(t)$ is a position vector than $\vec{r}'(t)$ is the velocity vector

if $|\vec{r}'(t)|$ is speed (scalar)

$\vec{r}''(t)$ is acceleration vector

$f(x) = \frac{k}{x}$ then $f'(x) = -\frac{k}{x^2}$

$f(x) = kx$ then $f'(x) = k$

$f(x) = k$ then $f'(x) = 0$

$f(x) = x$ then $f'(x) = 1$

$f(x) = k\sqrt{x}$ then $f'(x) = \frac{k}{2\sqrt{x}}$

Quotient Rule

$f(x)\frac{u}{v}$ then $f'(x) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$f'(x) = \frac{LoDeHi - HiDeLo}{(Lo)^2}$

Chain Rule

$h(x) = f(g(x))$ then $h'(x) = f'(g(x)) * g'(x)$

Power Rule
$f(x) = x^n$ then $f'(x) = nx^{n-1}$

Product Rule
$f(x) = uv$ then $f'(x) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

$f(x)a^x\ then\ f'(x) = a^x ln(a)$

$f(x) = ln(x)\ then\ f'(x) = \dfrac{1}{x}$

$f(x) = e^x\ then\ f'(x) = e^x$

$\dfrac{dy}{dt} = ky$

$\int a^x dx = \dfrac{a^x}{lna} + C$

$\int e^x dx = e^x + C$

$\int \dfrac{dx}{x} = ln|x| + c$

$\int x^n dx = \dfrac{x^{n+1}}{n+1} + C$

$\int k dx = kx + C$

$\int u(dv) = uv - \int v(du)$

$\int ln(x)dx = xln(x) - x + C$

Trig Functions

function	**Derivative**
$sin(x)$	$cos(x)$
$cos(x)$	$-sin(x)$
$tan(x)$	$sec^2(x)$
$sec(x)$	$sec(x)tan(x)$
$csc(x)$	$-csc(x)cot(x)$
$cot(x)$	$csc^2(x)$

Estimated Length of curve
$\sum_{i=1}^{n} |\vec{r}'(t)|\Delta t$

Actual Length of Curve
$\lim_{n\to\infty} \sum_{i=1}^{n} |\vec{r}'(t)|\Delta t = \int_a^b |\vec{r}'(t)|dt$

$s(t) = \int_a^t |\vec{r}'(t)|dt$ Shows how length and time are related

$r(s)$ tells us where we are after a given length but to do this we need s(t)

Unit tangent vector $T(t) = \dfrac{r'(t)}{r(t)}$ T dies bot change based on magnitude so a change in T means direction changes

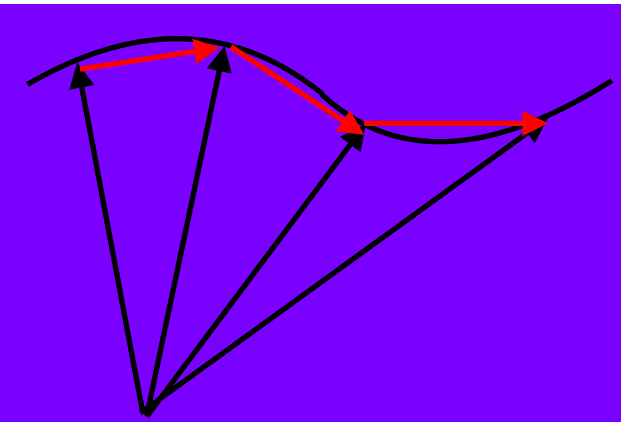
So $|\dfrac{dT}{ds}|$ is how much the curve turns

If $|\dfrac{dT}{ds}|$ its big then the curve turns a lot on very sharply

If $|\dfrac{dT}{ds}|$ its small then the curve turns very gradually

If $|\dfrac{dT}{ds}| = 0$ then the function is a straight line

$\dfrac{ds}{dt} = |r'(t)|$ so $\dfrac{dT}{ds} = \dfrac{\frac{dT}{ds}}{\frac{ds}{dt}} = |\dfrac{T'(t)}{r'(t)}| = \kappa$



Vector functions	Derivatives	Integrals	Derivative rules	Special derivatives

unit tangent vector	Length	Domain of function	Motion in space

Functions of Several Variables

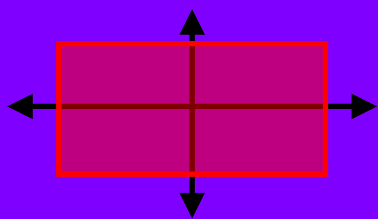
Domain- set of all possible input values

Range- set of all possible output values

For two variable functions domain is set of ordered pairs (a region on the XY plane)

For two variable functions range is still a one value

Domain of 2 variables



If doesn't include endpoints we use dotted lines

We know z=f(x,y) is a surface so if we set z equal to a constant then the graph $z_0 = f(x, y)$ is called a level curve (basically a trace)

If $\lim\limits_{x \to a} f(x) = L$ then the closer a gets to x the closer f(x) gets to L

Conditions of continuity

1. f(c) exists
2. $\lim\limits_{x \to c} f(x)$ exists
   - (All paths lead to the same limit)
3. $\lim\limits_{x \to c} f(x)$ =f(c)

If $\lim\limits_{x \to a} f(x) = f(a)$ then f(x) is continuous at x=a

Polynomials, sine, cosine and exponential functions are continuous functions

Rational and log functions are continues where they are defined

$$\lim_{(x,y) \to a,b} f(x, y) = L$$

Approaching a two function limit is more complicated because their are infinite paths (along x=a, y=b along b/ax parabola exponential etc

If the limit of two different paths don't mach than the limit does not exist (but we can't check all)

Check the type of function it is and if its continuous at f(a,b) than

$$\lim_{(x,y) \to a,b} f(x, y) = f(a, b)$$

For z=f(x,y) z is affected by x and y so we can not take the derivative with both changing
We can hold one variable constant and allow the other to change to get the partial derivative

Partial derivative- the rate of change of z with respect to the variable that is changing

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

$$\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

F_x and F_y tell us the effect of varying on variable at a time

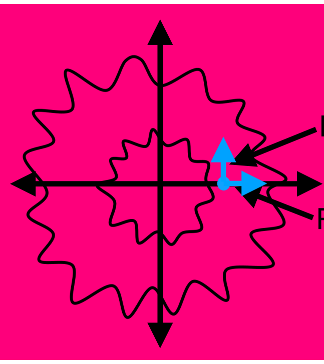
Clairaut's Theorem- $F_{xy} = F_{yx}$ wherever f(x,y) is defined and F_x and F_y are continuous

$y = f(u) \quad u = g(t)$ and $f(g(t))$ then $y' = f'(g(t)) * g'(t)$

To find out how fast z is changing use the chain rule $z = f(x, y)$ and $y = g(t)$ and $x = h(t)$

Then $\dfrac{dz}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$

Implicit Differentiation
If we have F(x,y) we can differentiate implicitly to find $\dfrac{dy}{dx} = -\dfrac{F_x}{F_y}$



Directional Derivative

Let $\overrightarrow{U} = \langle a, b \rangle$ be a unit vector (giving direction). Than the rate of change in that direction is

$$D\overrightarrow{u}f(x,y) = \lim_{h \to 0} \frac{f(x+ah, y+bh) - f(x,y)}{h}$$

$Gradient = \nabla = \langle F_x, F_y \rangle$

$D\overrightarrow{u}f(x,y) = \langle F_x, F_y \rangle * \langle a, b \rangle$

Max directional derivative occurs when $\theta = 0$ and the value of $D\overrightarrow{u}f(x,y) = \overrightarrow{\nabla}f$

Along the level curve the height does not change so must move along level curve (tangent to level curve)

Min directional derivative occurs when $\theta = 180$ and the value of $D\overrightarrow{u}f(x,y) = -\overrightarrow{\nabla}f$

Gradient is perpendicular to the level curve

$D\overrightarrow{u}f(x,y) = 0\ if\ \overrightarrow{\nabla}f \perp \overrightarrow{U}\ \theta = \frac{\pi}{2}\ or\ \frac{3\pi}{2}$ is the only directions you go to see no change

If the level curve is $\vec{r}(t) = \langle x(t), y(t) \rangle$ then

$$(x_0, y_0)\ \langle F_x, F_y \rangle \cdot \frac{\langle \frac{dx}{dt}, \frac{dy}{dt} \rangle}{|\langle \frac{dx}{dt}, \frac{dy}{dt} \rangle|} = 0$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

z=f(x,y) is a surface so instead of finding a tangent line at point (x_0, y_0, z_0) we actually have infinitely meany tangent lines which form tangent plane

When very near to (x_0, y_0, z_0) tangent plane roughly equals true surface

Infinitely many curves form curve$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ and their tangent vectors are are $\vec{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$ and the normal vector is perpendicular to all the tangent vectors

The normal vector of the tangent plane is the gradient

The smaller dx and dy are the closer dz is

$\%x = \frac{dx}{x}$

Equation for tangent plane

$$z - z_0 = F_x(x - x_0) + F_y(y - y_0)$$

$$z - z_0 = F_x(\Delta X) + F_y(\Delta Y)$$

Critical points occur when F_x and F_y are both equal to zero

If D>0 and Fx<0 its a maximum

Second derivative test $D = F_{xx} * F_{yy} - F_{xy}^2$

If D>0 and Fx>0 its a minimum

If D=0 test is inconclusive and you have to check points around it

If D<0 its a saddle point

To find absolute max and main find all critical points in regions find all end critical points and find all corner points
Plug them all into the equation and see which yields the largest and smallest value

Domain and Range | Directional Derivatives | Limits | Continuity | Chain Rule

Partial Derivatives | Tangent Plane and Partial Derivative | Max and Mins