Chapter 1- First Order Differential Equations

Basic Models

Differential Equation is an equation relating an unknown function to one or more of its derivatives

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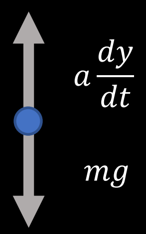
$$\frac{dy}{dt} = 2y + 3t$$

$$y^\prime = y\left(4 - y\right)$$

$$y^{\prime\prime} + 3y^\prime + y = 0$$

Mathematical Model- differential equation that describes physical model

- Population: $\frac{dp}{dt} = \mathrm{rp} = \mathrm{rate} \ast \mathrm{population}$
- Interest: $\frac{dm}{dt} = \mathrm{rm} = \mathrm{rate} \ast \mathrm{money}$
- Newton's law of cooling and heating: $\frac{dT}{dt} = \mathrm{k}\left(T - A\right)$
  - $\mathrm{k} > 0$ heating
  - $\mathrm{k} < 0$ cooling
- Newton's second law: $\mathrm{m}\frac{dv}{dt} = \mathrm{cv} - \mathrm{mg}$



$$m\frac{d^2y}{dt^2} = mg - a\frac{dy}{dt}$$

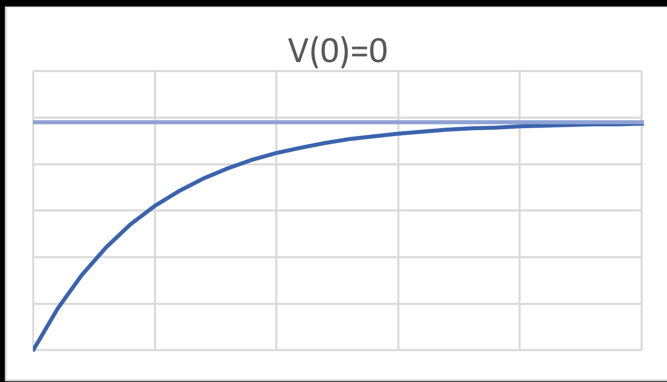
$$m\frac{dv}{dt} = mg - av$$

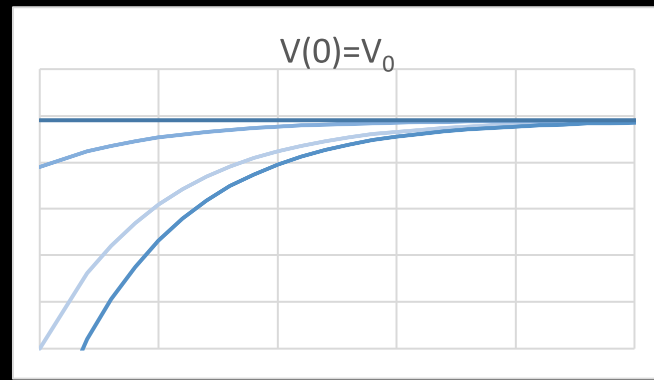
$$\frac{mdv}{v - \frac{mg}{a}} = -adt$$

$$m\,ln\left|v - \frac{mg}{a}\right| = -at + C$$

$$v - \frac{mg}{a} = Ce^{-\frac{at}{m}}$$

$$v = Ce^{-\frac{at}{m}} + \frac{mg}{a}$$





If $v_0 > v_{terminal}$ then drag > weight so object will lose speed until drag = weight (only approaches value)

If $v_0 < v_{terminal}$ then drag < weight so object will gain speed until drag = weight (only approaches value)

Initial Value Problem- combination of a differential equation and an initial condition to product a particular solution

- $y = y'$ and $y(0) = 2$ which gives the particular solution $y = 2e^x$

Existence and Uniqueness Theorem- for a given Initial value Problem ($y' = f(x, y)$ and $y(x_0) = y_0$) if $f(x, y)$ and $\frac{df(x, y)}{dt}$ are continuous functions in a rectangular region containing point (x_0, y_0) then the Initial Value Problem has a unique solution

Classifications of Differential Equations

Ordinary Differential Equations- equations with only ordinary derivatives

Partial Differential Equations- equations with at least one partial derivative ($\frac{du}{dt} = k\frac{\partial^2 u}{\partial x^2}$ heat equation)

Order of differential Equation- order of highest derivative in equation

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$y'' + y' + y = 0$ *second order*

$\frac{du}{dt} = k\frac{\partial^2 u}{\partial x^2}$ *first order*

Linear Differential Equation- (easy to solve) all its derivatives of derivatives of dependent variable are not multiplied, not raised to any power, not part of any transcendental function ($e^x, \sin(x), \cos(x), \log(x)$ etc)

Nonlinear differential equation- (hard to solve) derivatives are either multiplied, raised to power or in a transcendental function

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$$y^{\prime\prime} + y^\prime + y = 0 \text{ linear}$$

$$y^{\prime\prime} + {y^\prime}^2 + y = 0 \text{ nonlinear}$$

$$y^{\prime\prime} + y^\prime \ast y = 0 \text{ nonlinear}$$

$$y^{\prime\prime} + y^\prime + y = \sin{\left(t\right)} \text{ linear}$$

Solution- any function that satisfies differential equation

- General solution includes all possible solution and usually an arbitrary constant or arbitrary function
- Particular solution- solution that does not include any arbitrary constants or functions and satisfies the initial condition along with the differential equation

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$$\text{Differential equation:} \frac{dy}{dt} = y$$

$$\text{solutioon: } y = Ce^t$$

Slope/Direction Fields

For a given Differential Equation it is a way to visualize solution qualitatively as a plot of small/short line segments drawn at various points in XY-plane showing the slowing of the solution curves are those points in a slope/direction field of the Differential Equations

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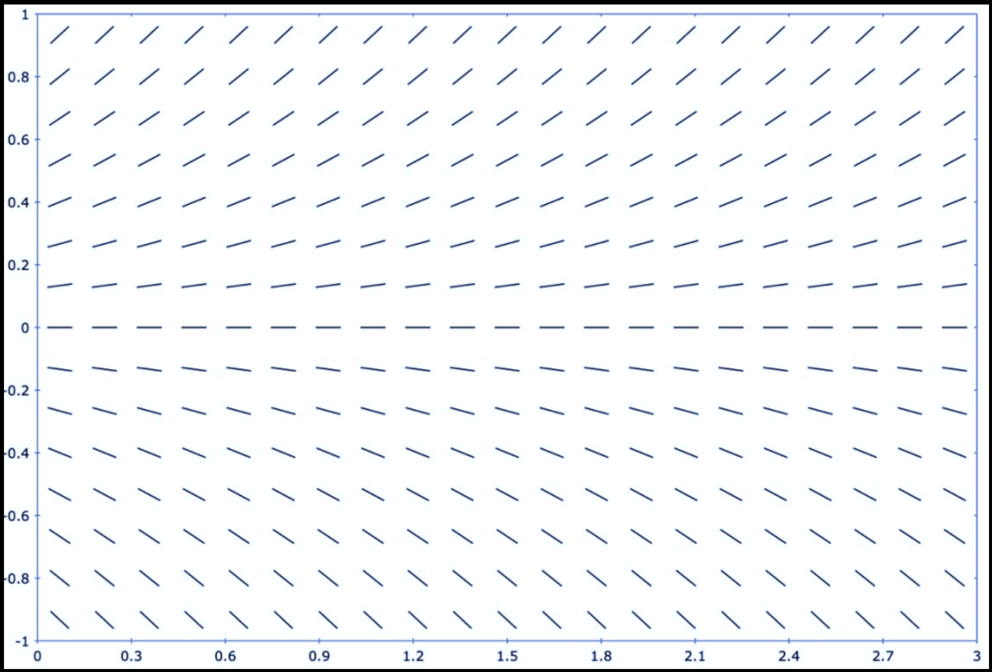
$$\frac{dy}{dt} = y \text{ so the slope of a solution} = y$$

$$\text{When y=0 then } \frac{dy}{dt} = 0 \text{ and when y=1 then } \frac{dy}{dt} = 1$$



Equilibrium occurs when $y^\prime = 0$

Solving $y^\prime = a + by$

$$m\frac{d^2y}{dt^2} = mg - a\frac{dy}{dt}$$

$$m\frac{dv}{dt} = mg - av$$

$$\frac{mdv}{v - \frac{mg}{a}} = -adt$$

$$m\ln{\left|v - \frac{mg}{a}\right|} = -at + C$$

$$v - \frac{mg}{a} = {Ce}^{-\frac{at}{m}}$$

$$v = {Ce}^{-\frac{at}{m}} + \frac{mg}{a}$$

Linear Equations and Linear factors

Solving equations in the form $y^\prime = p\left(t\right)y = g\left(t\right)$ which is a first order linear Differential Equation

$$y^\prime + \frac{2}{t}y = \frac{4}{t} \text{ multipy both sides by } t^2$$

$$t^2y^\prime + 2ty = 4t$$

$$\frac{d}{dt}\left(uv\right) = u^\prime v + uv^\prime$$

$$u = t^2 \text{ and } v = y$$

$$\frac{d}{dt}\left(t^2y\right) = 4t$$

$$t^2y =\int4tdt = 2t^2 + C$$

$$y = 2 +\frac{C}{t^2}$$

t^2 is the integrating factor (different for each equation)

Integrating factor $\mu\left(t\right)$- function that entire linear first order differential equation is multiplied to solve for y

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$$y^\prime + p\left(t\right)y = g\left(t\right)$$

$$\mu\left(t\right)y^\prime +\mu\left(t\right)p\left(t\right)y =\mu\left(t\right)g\left(t\right)$$

$$\mu\left(t\right)y^\prime +\mu\left(t\right)p\left(t\right)y =\frac{d}{dt}\left(\mu\left(t\right)y\right) =\ \mu\left(t\right)y^\prime +\ \mu^\prime\left(t\right)y$$

$$\mu\left(t\right)y^\prime +\mu\left(t\right)p\left(t\right)y =\ \mu\left(t\right)y^\prime +\ \mu^\prime\left(t\right)y$$

$$\mu\left(t\right)p\left(t\right) =\mu^\prime\left(t\right)$$

$$p\left(t\right) =\mu^\prime\left(t\right)\frac{1}{\mu\left(t\right)}$$

$$\ln{\left|\mu\left(t\right)\right|} =\int p\left(t\right)dt$$

$$\mu\left(t\right) = e^{\int p\left(t\right)dt}$$

There $\int p\left(t\right)dt$ ends up with a C value but its arbitrary choose whatever is easiest ($C = 0$)

After finding the integrating factor multiply the entire equation by $\mu\left(t\right)$ then solve for y

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$$\mu\left(t\right)y^\prime +\mu\left(t\right)p\left(t\right)y =\mu\left(t\right)g\left(t\right)$$

$$\frac{d}{dt}\left(\mu\left(t\right)y\right) =\mu\left(t\right)g\left(t\right)$$

$$\mu\left(t\right)y =\int\mu\left(t\right)g\left(t\right)dt + C$$

$$y =\frac{\int\mu\left(t\right)g\left(t\right)dt + C}{\mu\left(t\right)}$$

C depends on the initial condition

If we don’t have an initial condition (C=?) this is the general solution

Separable Equation

Variables are separated multiplication and division not by addition or subtraction thus all the y variables can be on one side of the equation and all the x variables can be on the other side of the equation

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$$y^\prime =\frac{dy}{dx} =\frac{x^2}{y\left(1 + x^2\right)}$$ *(y is the dependent variable and x is the independent variable)*

$$ydy =\frac{x^2}{\left(1 + x^2\right)}dx \rightarrow \int y\, d y\ =\int{\frac{x^2}{\left(1 + x^2\right)}dx}$$

$$\frac{1}{2}y^2 =\frac{1}{3}\ln{\left|1 + x^3\right|} + C$$

$$y =\left(\frac{2}{3}\ln{\left|1 + x^3\right|} + C\right)^\frac{1}{2}$$

Separable Differential Equations can be written as $p\left(y\right)\frac{dy}{dx} = q\left(x\right)\rightarrow p\left(y\right)dy = q\left(x\right)dx$ **and therefore the general solution to the separable equation is** $\int{p\left(y\right)dy =\int q\left(x\right)dx + C}$ **where C is an arbitrary constant**

Intervals where solutions are valid- where both y and y' are both continuous

Substitution

For nonlinear differential equations like $\frac{dy}{dx} = G\left(ax + by + c\right)$ we can use substitution to solve

$u = ax + by + c$ then $G\left(ax + by + c\right) = G\left(u\right)$ and $\frac{du}{dx} = a + b\frac{dy}{dx}$ so $\frac{1}{b}\left(\frac{du}{dx} - a\right) = G\left(u\right)$

This can be rewritten as $\frac{du}{dx} = bG\left(u\right) + a$ which is a separable function of u

$$\int\frac{du}{bG\left(u\right) + a} =\int dx + C$$

Bernoulli's Equation

$$\frac{dy}{dx} + p\left(x\right)y = q\left(x\right)y^n$$

$$\frac{dy}{dx}y^{-n} + p\left(x\right)y^{1-n\ } = q\left(x\right)$$

$$V = y^{1-n} \text{ and } \frac{dV}{dx} =\left(1-n\right)y^{-n}\frac{dy}{dx}$$

$$\left(\frac{1}{1-n}\right)\frac{dV}{dx} + p\left(x\right)V = q\left(x\right)$$

$$\frac{d}{dt}\left(\frac{1}{1-n}V\right) = q\left(x\right)$$

$$\frac{1}{1-n}V = \int q\left(x\right)dx + C$$

$$V = \left(1-n\right)\int q\left(x\right)dx + C$$

$$y = x\left(1-n\right)\int q\left(x\right)dx + xC$$

Homogeneous Equations

An equation that can be expressed by the faction $\frac{y}{x}$

$$\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2} = 1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2$$

Homogeneous equations can be turned into separable equations by make a change where $y = x\ast v\left(x\right)$ because $v\left(x\right) = \frac{y}{x}$ and therefore $y^\prime = \frac{d}{dx}\left(x\ast v\left(x\right)\right) = v\left(x\right) + x\ast v'(x)$

- Solve equation for $v\left(x\right)$
- Solve for y

$$\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2} = 1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2$$

$$v\left(x\right) + x\ast v'(x) = 1 + v(x) + v^2(x)$$

$$x\ast\frac{dv}{dx} = 1 + v^2\left(x\right)$$

$$\frac{dv}{1+v^2} = \frac{1}{x}dx$$

$${tan}^{-1}{v} = \ln{\left|x\right|} + C = tan(ln|x| + C)$$

$$y = x\tan{\left(\ln{\left|x\right|} + C\right)}$$

Homogeneous equations are functions of $\frac{y}{x}$ so $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ so the slope of the function depends on $\frac{y}{x}$ so (2,2) and (4,4) have the same slope

- Any line though the origin has the same $\frac{dy}{dx}$ value

Exact Differential Equation

Differential equation in the form $M\left(x,y\right)dx + N\left(x,y\right)dy = 0$ (or $M\left(x,y\right) + N\left(x,y\right)\frac{dy}{dx} = 0$) where there exists a function $\phi\left(x,y\right)$ such that $\frac{d\phi}{dx} = M\left(x,y\right)$ and $\frac{d\phi}{dy} = N\left(x,y\right)$

If a differential equation is exact, then $\frac{d\phi}{dx}dx + \frac{d\phi}{dy}dy = 0$ and thus $\phi^\prime = \frac{d\phi}{dx}dx + \frac{d\phi}{dy}dy$

$\phi\left(x,y\right)$ is a potential function of the differential equation

For $M\left(x,y\right)dx + N\left(x,y\right)dy = 0$ to be exact $\frac{dM\left(x,y\right)}{dy} = \frac{dN\left(x,y\right)}{dx}$

If $M\left(x,y\right)dx + N\left(x,y\right)dy = 0$ is exact, then $\phi = \int{M\left(x,y\right)dx + g\left(y\right)} = \int{N\left(x,y\right)dy + f\left(x\right)}$

So, the solution of the differential equation is $\phi = C$

Reducible 2nd Order Differential Equations

General form of second order derivatives $F\left(y^{\prime\prime}, y^\prime, y, x\right) = 0$

Special case- y is missing

- $F\left(y^{\prime\prime}, y^\prime, x\right) = 0$
- Substitute $v = \frac{dy}{dx}$ and $v^\prime = \frac{d^2y}{dx^2}$
- Solve for v
- Plug $\frac{dy}{dx}$ back into the equation and solve for y

Special case – x is missing

- $F\left(y^{\prime\prime}, y^\prime, y\right) = 0$
- Substitute $v = y^\prime$ and $\frac{dv}{dy}v = y^{\prime\prime}$
- Solve for v
- Plug $\frac{dy}{dx}$ back into the equation and solve for y

Chapter 2- Numerical and Mathematical Models

Exponential Population model

$\frac{dp}{dt} = kp$ where k is a constant

This is a first order linear and a separable equation

$$\int\frac{dp}{p} = \int k\, d\, t + C$$

$$\ln{\left|p\right|} = kt + C$$

$$p = Ce^{kt}$$

Logistic Population Model

$\frac{dp}{dt} = kp\left(m - p\right)$ where k and m are constants

$p\left(t\right) = \frac{m}{1 - \frac{c-m}{c}e^{-kmt}}$ m is called the carrying capacity and it the value the population approaches

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$$\frac{dp}{dt} = 0.0004p\left(150 - p\right) \text{ and } p\left(0\right) = 200$$

$$\int\frac{dp}{p\left(150 - p\right)} = \int{-0.0004dt} + C$$

$$\frac{1}{p\left(150 - p\right)} = \frac{A}{p} + \frac{B}{150 - p}$$

$$1 = A\left(150 - p\right) + Bp$$

Set P to various values and solving the system of equations (in this case 0 and 150) to find A and B

$$\int{-\frac{1}{150p}dp} + \int{\frac{1}{150\left(p - 150\right)}dp} = \int{-0.0004dt} + C$$

$$\frac{1}{150}\ln{\left|\frac{p - 150}{p}\right|} = -0.0004t + C$$

$$p = \frac{150}{1 - \frac{1}{4}e^{-0.06t}}$$

Autonomous Differential equation

A differential equation that is only a function of the dependent variable and thus has the form $\frac{dy}{dx} = f\left(y\right)$

Equilibrium Solutions

a constant y value solution which causes $\frac{dy}{dx} = 0$

these are also called the critical points of a differential equation

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$$\frac{dy}{dx} = x\left(x - 1\right)y\left(y - x\right)\left(y - 5\right)\left(y - 10\right)^2$$

The equilibrium solutions are $y = 0,\ \ y = 5,$ *and* $y = 10$

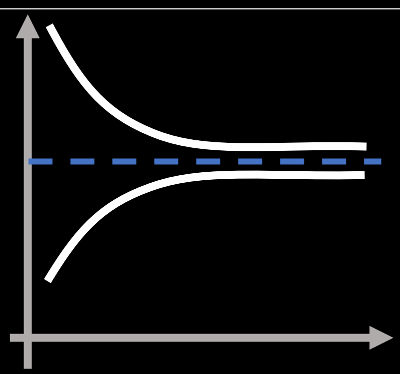
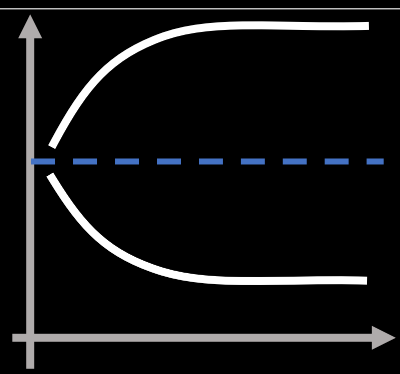
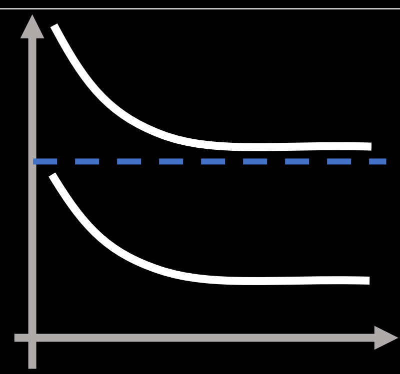
$y = x$ *is not a equilibrium solution because y is not constant*

Stability

A **stable critical point** of a differential equation is the equilibrium solution of the differential equation with the property that solution curves lying on both sides tend to approach it

An **unstable critical point** of a differential equation is the equilibrium solution of the differential equation with the property that solution curves lying on both sizes tend to depart from it

A **semistable critical point** of a Differential equation is the equilibrium solution of the differential equation with the property that solutions curve lying on one side tends to approach it but the other side tends to depart from it

Stable	Unstable	Semistable
		

Acceleration Velocity models

No Air Resistance

$$F_{net} = ma$$

$$\text{If } F_{net} = F_g = -mg \text{ so } a = -g$$

$$v(t) = velocity = at + v_o = -gt + v_0$$

$$y(t) = position = \frac{1}{2}at^2 + v_ot + y_0 = -\frac{1}{2}gt^t + v_0t + y_0$$

Air Resistance

$$F_{net} = ma$$

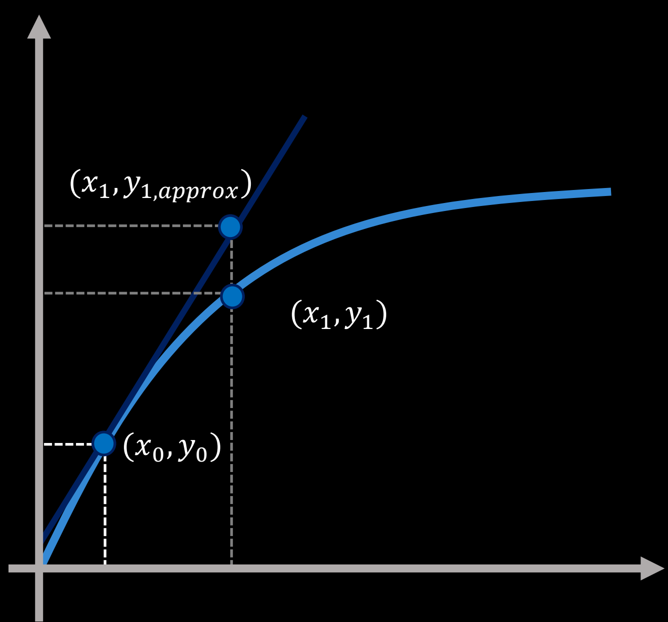
$$F_g = -mg \text{ and } F_{air} - kv$$

$$F_{net} = F_g + F_{air} = ma = -mg - kv \text{ so } a = \frac{dv}{dt} = -g - \frac{k}{m}v$$

$\frac{dv}{dt} = -g - \frac{k}{m}v$ is a first order linear equation which ends up being $v = -\frac{mg}{k} + \left(v_0 + \frac{mg}{k}\right)e^{-\frac{k}{m}t}$

$\lim_{t\rightarrow\infty} v(t) = \frac{mg}{k}$ $= dy/dt == -mg/k + (v_0 + mg/k)\ e^{(-k/m\ t)}$ this if a function of t so to find y just integrate

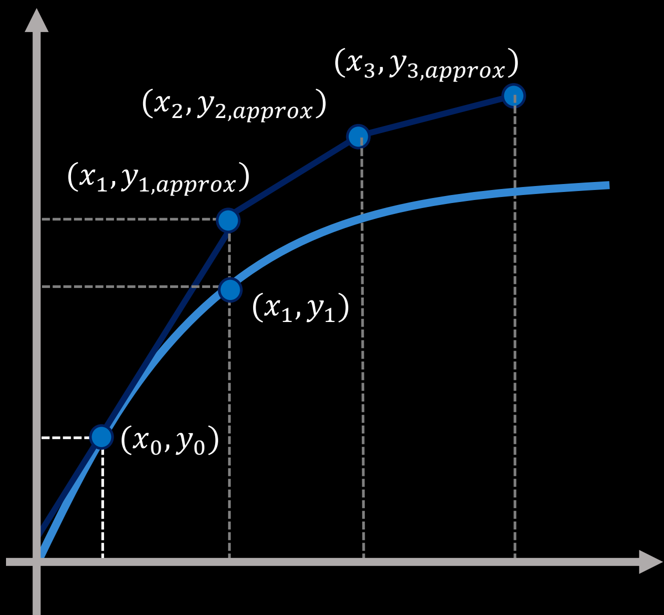
Simple Approximation using tangent line



$$y_{1,\ approx} = y_0 + \left(x_1 - x_0\right)f\left(x_0, y_0\right)$$

Euler's Method

First order method that uses $\frac{dy}{dx} = f(x, y)$ and an additional value $y(x_0) = y_0$ to predict the value $y(x_1)$



Find $y_{1,\ approx}$ through $y_{1,\ approx} = y_0 + \left(x_1 - x_0\right) f\left(x_0, y_0\right)$

Find $y_{2,\ approx}$ through $y_{2,\ approx} = y_{1,approx} + \left(x_2 - x_1\right) f\left(x_1, y_{1,approx}\right)$

Therefore $y_{n,\ approx} = y_{n-1,approx} + \left(x_n - x_{n-1}\right) f\left(x_{n-1}, y_{n-1,approx}\right)$

Predictor Corrector Method

Improvement on Euler's method by adjusting the solve of the tangent line

Start by using $f\left(x_0, y_0\right),\ x_0\ \text{and}\ y_0$ to calculate the value t_1

Calculate $f\left(x_1, t_1\right)$

Find the average of $m = \frac{f\left(x_0, y_0\right) + f\left(x_1, t_1\right)}{2}$

Calculate $y_{1,approx} = y_0 + m\left(x_1 - x_0\right)$

Use $f\left(x_1, y_1\right),\ x_1\ \text{and}\ y_1$ to calculate the value t_2

Calculate $f\left(x_2, t_2\right)$

Find the average of $m = \frac{f\left(x_1, y_1\right) + f\left(x_2, t_2\right)}{2}$

Calculate $y_{2,approx} = y_1 + m\left(x_2 - x_1\right)$

Repeat using general form $y_{n,approx} = y_{n-1} + \frac{f\left(x_{n-1}, y_{n-1}\right) + f\left(x_n, t_n\right)}{2\left(x_n - x_{n-1}\right)}$

Chapter 3: Linear Derivative of Higher Order

2nd Order Linear Differential Equations

General form of 1st order equation $f\left(x,y,y^\prime\right)=0$

General form of 2nd order equation $f\left(x,y,y^\prime,y^{\prime\prime}\right)=0$

General form of 1st order linear equation $y^\prime+p\left(x\right)y=q\left(x\right)$

General form of 2nd order linear differential equation $A\left(x\right)y^{\prime\prime}+B\left(x\right)y^\prime+C\left(x\right)y=F\left(x\right)$ where $A\left(x\right)\neq0$ which are also in the form $y^{\prime\prime}+p\left(x\right)y^\prime+q\left(x\right)y=g\left(x\right)$

If $F\left(x\right)=0$ or $g\left(x\right)=0$ then the second order linear differential equation is homogeneous otherwise it is nonhomogeneous

- First order homogeneous is $y^\prime=f\left(\frac{y}{x}\right)$

Existence and Uniqueness Theorem for Second Order- consider Initial value problem $y^{\prime\prime}+p\left(x\right)y^\prime+q\left(x\right)y=f\left(x\right)$ **where** $y\left(x_0\right)=y_0$ **and** $y^\prime\left(x_0\right)=y_1$. **If** $p\left(x\right),\ q\left(x\right),\ \text{and}\ g\left(x\right)$ **are all continuous function over an interval containing,** x_0 **then the IVP solution is unique**

Principle Superposition Theorem- if y_1 **and** y_2 **are two solutions to** $y^{\prime\prime}+p\left(x\right)y^\prime+q\left(x\right)y=0$ **then the general solution y is equal to the linear combination of** y_1 **and** y_2

- The linear combination of y_1 and y_2 is $c_1y_1+c_2y_2$ where c_1 and c_2 are constants

Linearly Independence- $f\left(x\right)$ and $g\left(x\right)$ are linearly independent if neither is scalar multiple of the other $f\left(x\right)\neq kg\left(x\right)$

Wronskian of two functions $f\left(x\right)$ and $g\left(x\right)$ is defined by $W=w\left(f\left(x\right),g\left(x\right)\right)=\left|\begin{matrix}f\left(x\right)&g\left(x\right)\\f^\prime\left(x\right)&g^\prime\left(x\right)\\\end{matrix}\right|=f\left(x\right)g^\prime\left(x\right)-g\left(x\right)f^\prime\left(x\right)$

- If $W=0$ then $f\left(x\right)$ and $g\left(x\right)$ are linearly dependent

If y_1 and y_2 are linearly independent, then the general solution of $y^{\prime\prime}+p\left(x\right)y^\prime+q\left(x\right)y=0$ is $y=c_1y_1+c_2y_2$

The linearly independent set of solutions to the differential equation is known as the functional solution set this is also called the basis which is expressed as $\left\{y_1,y_2\right\}$

2nd Order Homogeneous Linear Differential Equation with Constant Coefficients

$ay^{\prime\prime}+by^\prime+cy=0$ where $a,\ b,\ and\ c$ are constants where $a\neq0$

The solution to this equation is $y=e^{rx}$ and thus $y^\prime=re^{rx}$ and $y^{\prime\prime}=r^2e^{rx}$

$$ar^2e^{rx}+bre^{rx}+ce^{rx}=0$$

$$e^{rx}\left(ar^2+br+c\right)=0$$

$\left(ar^2+br+c\right)=0$ is the characteristic equation of $ay^{\prime\prime}+by^\prime+cy=0$

$$\Delta=b^2-4ac$$

If r has 2 distinct roots then $\Delta>0$ and $y\left(x\right)=c_1e^{rx}+c_2e^{rx}$

If r has 1 repeated real root then $\Delta=0$ and $y\left(x\right)=c_1e^{rx}+c_2xe^{rx}$

If r has two complex roots then $\Delta<0$ and $y\left(x\right)=c_1e^{rx}+c_2e^{rx}$

General Solutions of Linear Differential Equations

An nth order linear differential equation has the form $y^{\left(n\right)}+p_1\left(x\right)y^{\left(n-1\right)}+\ldots+p_{\left(n-1\right)}y^\prime+p_ny=f\left(x\right)$

- If $f\left(x\right) = 0$ then the differential equation is homogeneous
- If $f\left(x\right) \neq 0$ then the differential equation is non homogeneous

Existence and Uniqueness Theorem- Consider $y^{\left(n\right)} + p_1\left(x\right)y^{\left(n-1\right)} + \ldots + p_{\left(n-1\right)}y^\prime + p_ny = f\left(x\right)$**,** $y\left(x_0\right) = y_0$**,** $y^\prime\left(x_0\right) = y_1$**,…,**$y^{\left(n-1\right)}\left(x_0\right) = y_{n-1}$ **If the functions** $p_1\left(x\right)\ldots p_{n-1}\left(x\right)$ **and** $f\left(x\right)$ **are continuous functions on an interval containing** x_0 **then the Initial value Problem has a unique solution**

Principle Superposition Theorem for Homogeneous Differential Equations- if $y_1\ldots y_n$ **are n solutions to** $y^{\left(n\right)} + p_1\left(x\right)y^{\left(n-1\right)} + \ldots + p_{\left(n-1\right)}y^\prime + p_ny = 0$ **then** $y = c_1y_1 + c_2y_2 + \ldots + c_ny_n$ **is also a solution if and only if** $y_1\ldots y_n$ **linearly independent**

Linear Independence- n functions $f_1, f_2\ldots f_n$ **are said to be linearly independent if** $c_1f_1 + \ldots + c_nf_n = 0$ **only when** $c_1 = \ldots = c_n = 0$

Wronskian of n functions $f_1,\ldots f_n$ **is** $W = w\left(f_1\ldots f_n\right) = \left|\left[\begin{matrix}f_1&\cdots&f_n\\\vdots&\ddots&\vdots\\f_1^{n-1}&\cdots&f_n^{n-1}\\\end{matrix}\right]\right|$

- If $W = 0$ then $f_1\ldots f_n$ are linearly dependent
- If $W \neq 0$ then $f_1\ldots f_n$ are linearly independent

Method of Reduction of Order for Solving

$y^{\prime\prime} + p\left(x\right)y^\prime + q\left(x\right)y = 0$ where one solution y_1 is either given or can be determined this is the method for finding the 2nd solution y_2 such that $\left\{y_1, y_2\right\}$ is a fundamental set of solutions

Let $y = vy_1$ then $y^\prime = v^\prime y_1 + vy_1^\prime$ and $y^{\prime\prime} = v^{\prime\prime}y_1 + 2v^\prime y_1^\prime + vy_1^{\prime\prime}$

$$y^{\prime\prime} + p\left(x\right)y^\prime + q\left(x\right)y = 0$$

$$v^{\prime\prime}y_1 + 2v^\prime y_1^\prime + vy_1^{\prime\prime} + p\left(x\right)\left(v^\prime y_1 + vy_1^\prime\right) + q\left(x\right)\left(vy_1\right) = 0$$

$$v^{\prime\prime}y_1 + v^\prime\left(2y_1^\prime + p\left(x\right)y_1\right) + v\left(y_1^{\prime\prime} + p\left(x\right)y_1^\prime + q\left(x\right)y_1\right) = 0$$

$$v^{\prime\prime}y_1 + v^\prime\left(2y_1^\prime + p\left(x\right)y_1\right) = 0$$

$$v' = w \text{ and } v'' = w'$$

$$\frac{dw}{dx}y_1 + w\left(2y_1' + p\left(x\right)y_1\right) = 0$$

Solve for w $\frac{dw}{dx} = -\frac{w\left(2y_1' + p\left(x\right)y_1\right)}{y_1} \rightarrow \frac{dw}{w} = \frac{\left(2y_1' + p\left(x\right)y_1\right)}{y_1}dx$

Solve for v

Solve for y_2 $vy_1 = c_1y_1 + c_2y_2$

2nd Order Euler Equation

$$ax^2y'' + bxy' + cy = 0$$

Solution has the form $y = x^r$ so $y' = rx^{r-1}$ and $y'' = r\left(r-1\right)x^{r-2}$

Plug y, y’ and y’’ into the Differential Equation

$$ax^2r\left(r-1\right)x^{r-2} + bxrx^{r-1} + cx^r = 0$$

$x^r\left(ar\left(r-1\right) + br + c\right) = 0 \ ar^2 + \left(b-a\right)r + c = 0$ is the characteristic equation

If the equation has two distinct roots (r_1 and r_2) then $y = c_1x^{r_1} + c_2x^{r_2}$

Homogeneous Differential Equations with Constant Coefficients

$a_ny^{\left(n\right)} + a_{n-1}y^{\left(n-1\right)} + \ldots + a_1y' + a_0y = 0$ where all a values are constants

This creates the characteristic equation $p\left(r\right) = a_nr^n + a_{n-1}r^{n-1} + \ldots + a_1r + a_0 = 0$

Including multiplicities $p\left(r\right)$ has n roots which can. Be used too find $y_1 \ldots y_n$ solutions

General Solution $y = c_1y_1 + \ldots + c_ny_n$

Distinct real roots

If $r_1 \ldots r_n$ are n distinct roots, then the general solution is

$$y = c_1e^{r_1x} + \ldots + c_ne^{r_nx}$$

Repeated real roots

If r is a repeated root of $p\left(r\right)$ of multiplicity m then the m solutions of the Differential euqation corresponding to r are

$$y_1 = e^{rx}\ y_2 = xe^{rx}\ \ldots\ \ y = x^{m-1}e^{rx}$$

Defining D as Differential Operator

- $Dy = y^\prime$ and $D^2y = y^{\prime\prime}$ … $D^ny = y^n$
- Let $L = a_nD^n + a_{n-1}D^{n-1} +\ldots + a_1D + a_0$ then $Ly =\left(a_nD^n + a_{n-1}D^{n-1} +\ldots + a_1D + a_0\right)$
- Differential equation $Ly = 0$ has the characteristic equation $p\left(r\right) = a_nr^n + a_{n-1}r^{n-1} +\ldots + a_1r + a_0 = 0$ where we may replace D in L with r to get p(r)

Complex Numbers

$z = x + iy$

X- real part

y- imaginary part

$$z = x + iy = r\left(cos\theta + isin\theta\right) = |z| = \sqrt{(x^2 + y^2)}$$

$$x = rcos\theta$$

$$x = rsin\theta$$

$tan\theta =\frac{y}{x}$*lers Formula*

$$e^{i\theta} = cos\theta + isin\theta^{(-i\theta)} = cos\theta - isin\theta$$

$$e^{iB\theta} = cosB\theta + isinB\theta$$

$$e^{-iB\theta} = cosB\theta - isinB\theta$$

Complex Roots

If characteristic equation of a Differential Equation has 2 complex conjugate roots $r =\alpha\pm Bi$ then the two solutions corresponding to these two roots are $y_1 = e^{r_1x} = e^{\left(\alpha + \beta i\right)x} = e^{\alpha x}\left(cos\beta x + isin\beta x\right)$ d $y_2 = e^{r_2x} = e^{\left(\alpha -\beta i\right)x} = e^{\alpha x}\left(cos\beta x - isin\beta x\right)$ en $\left\{y_1, y_2\right\}$ are the fundamental set of solutions

$$y = c_1y_1 + c_2y_2$$
$$=\left(c_1 + c_2\right)e^{\alpha x}\cos{\beta x} +\left(c_1 - c_2\right)ie^{\alpha x}\sin{\beta x}$$

= d_1e^{\alpha x}\cos{\beta x} + d_2e^{\alpha x}\sin{\beta x}

Complex roots Theorem- if r =\alpha\pm bi are complex conjugate roots of characteristic equations then the solutions corresponding to these roots are e^{\alpha x}cos\beta xd e^{\alpha x}sin\beta xpeated Complex roots

If r =\alpha\pmBi is a repeatd complex rooot with multiplic m then the 2m solutionos of Linear Independence to. The 2m roots are

e^{\alpha x}cos\beta xe^\alpha x\ sin\beta x

e^{\alpha x}xcos\beta x *e^{\alpha x}xsin\beta x*

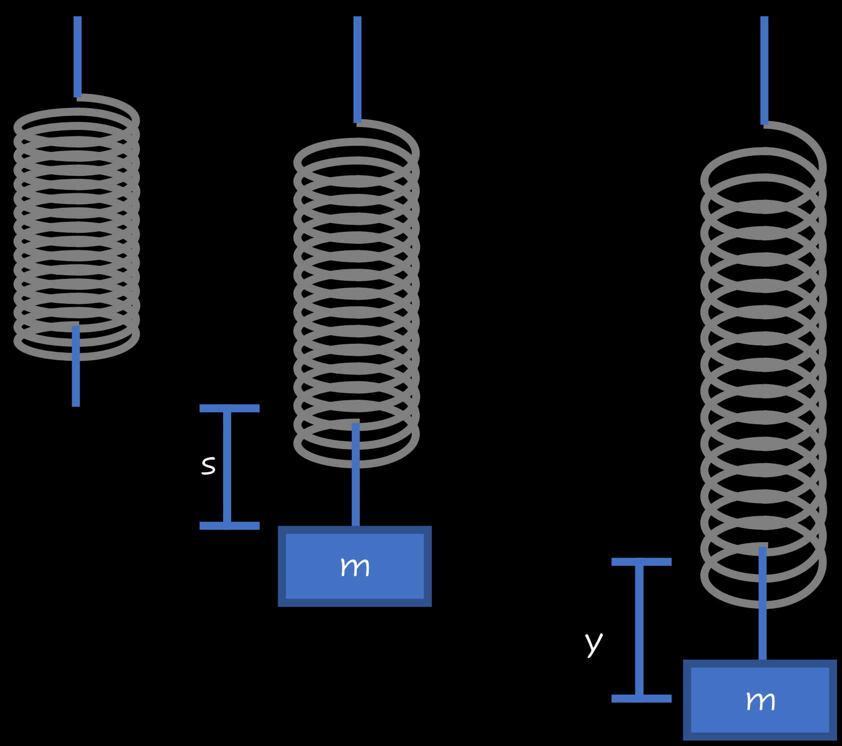
... ...

e^{\alpha x}x^{m − 1\ }cos\beta x^\alpha x\ x^(m − 1)\ sin\beta x

Mechanical Vibrations

Applications of 2nd order homogeneous linear Differential Equations with constant coefficients

mass and string system (y\left(t\right) or x\left(t\right) which are the position functions)



- F_g- gravitational force
- F_s- spring force
- F_r- resistance force
- $F(t)$- any other external force
- m- mass
- c- resistance constant
- k- spring constant
- y_o- initial position
- v_0- initial velocity
- s- the elongation of the spring after attaching an object with mass m

$$F_s = F_g \rightarrow\ -kS = mg$$

$$F_{net} = F_g + F\left(t\right) + F_s + F_r \rightarrow my^{\prime\prime} + cy^\prime + ky = F\left(t\right)$$

$$F_r = cv$$

$$F_s = kS$$

Free undamped Motion ($F\left(t\right) = 0\ \text{and}\ c = 0$)

$$mx^{\prime\prime} + kx = 0,\ x\left(0\right) = x_0, x^\prime\left(0\right) = v_0$$

Find the characteristic equation $mr^2 + k = 0$ so $r = \pm\sqrt{\frac{k}{m}}i$

Let $w_0 = \sqrt{\frac{k}{m}}$ so $x\left(t\right) = c_1\cos{\left(w_0t\right)} + c_2\sin{\left(w_0t\right)}$

$$x\left(t\right) = \sqrt{c_1^2 + c_2^2}\left(\frac{c_1}{\sqrt{c_1^2 + c_2^2}}\cos{\left(w_0t\right)} + \frac{c_2}{\sqrt{c_1^2 + c_2^2}}\sin{\left(w_0t\right)}\right)$$

$$\cos{\alpha} = \frac{c_1}{\sqrt{c_1^2 + c_2^2}}, \sin{\alpha} = \frac{c_2}{\sqrt{c_1^2 + c_2^2}}, \tan{\alpha} = \frac{c_2}{c_1}$$

$$x\left(t\right) = c_1cos{\left(w_0t\right)} + c_2sin{\left(w_0t\right)} = Acos\left(w_0t - \alpha\right)$$

$$A = \sqrt{c_2^2 + c_1^2}$$

$$w_0 = \sqrt{\frac{k}{m}}$$

$$\alpha = \begin{cases} c_1 > 0\ \ c_2 > 0\ \ then\, tan^{-1}\frac{c_2}{c_1} \\ c_1 < 0\ then\ \pi + tan^{-1}\frac{c_2}{c_1} \\ c_2 < 0\ then\ 2\pi + tan^{-1}\frac{c_2}{c_1} \end{cases} \_0$$

$$F = \frac{1}{T} = \frac{w_0}{2\pi}$$

$$\sigma - time\ lag = \frac{\alpha}{w_0}$$

Damped Free motion ($c\neq0,\ F\left(t\right) = 0$)

$$mx^{\prime\prime} + cx^\prime + kx = 0,\ x\left(0\right) = x_0, x^\prime\left(0\right) = v_0$$

Characteristic equation $mr^2 + c_r + k = 0$ so $r = \frac{-c\pm\sqrt{c^2 - 4mk}}{2m}$

$$\mathrm{\Delta} = c_2 - 4mk \begin{cases} > 0 \\ = 0 \\ < 0 \end{cases}$$

$\Delta > 0$ overdamped

$r_1 \neq r_2$ *distint real values* $r_1 < 0, r_2 < 0$

$$x = c_1e^{r_1t} + c_2e^{r_2t}$$

$\Delta = 0$ critical damped

$r_1 = r_2$ *repeated real values* $r_1 = r_2 = -\frac{c}{2m} < 0$

$$x = c_1e^{-\frac{c}{2m}t} + c_2{xe}^{-\frac{c}{2m}t}$$

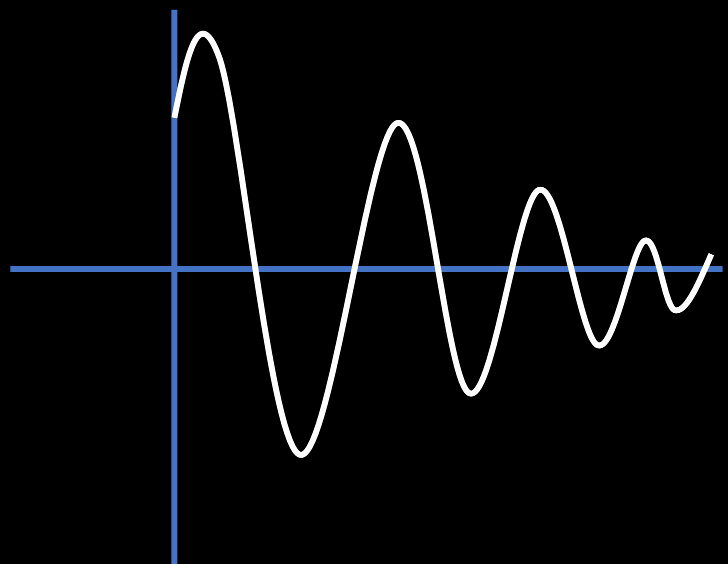
Goes to zero as t goes to infinity

$\Delta < 0$ underdamped

$r_1 = -\frac{c}{2m}\pm\frac{\sqrt{4mk - c^2}}{2m}i = -\frac{c}{2m}\pm\beta i$ *repeated real values* $r_1 = r_2 = -\frac{c}{2m} < 0$

$$x = c_1e^{-\frac{c}{2m}t}\cos{\beta t} + c_2e^{-\frac{c}{2m}t}\sin{\beta t}$$

$x = e^{-\frac{c}{2m}t}\left(c_1\cos\beta t + c_2\sin\beta t\right)$ *to zero as t goes to infinity*



Time varying amplitude $e^{-\frac{c}{2m}t}A$

Pseudo frequency β

Pseudo period $T = \frac{2\pi}{\beta}$

Time lag $\sigma = \frac{\alpha}{\beta}$

Nonhomogeneous Differential Equations

The general solution of the nonhomogeneous Differential Equation $a_n y^{\left(n\right)} + a_{n-1} y^{\left(n-1\right)} + \ldots + a_1 y^\prime + a_0 y = f\left(x\right)$ is of the form $y = y_c + y_p$ where y_c is the general solution of $a_n y^{\left(n\right)} + a_{n-1} y^{\left(n-1\right)} + \ldots + a_1 y^\prime + a_0 y = 0$ and y_p is a particular solution the homogeneous Differential Equation

Undetermined Coefficients

works better if $f\left(x\right)$ is of the following types

- Polynomial of x
- Exponential function of x
- $\sin\beta x$, $\cos\beta x$ or $c_1 \cos\beta x + c_2 \sin\beta x$f product of the above 3 types

Form of f(x)	Form of y_p	Coefficients to be determined
$a_n x^n + a_{n-1\ } x^{n-1} + \ldots + a_1 x + a_0$	$A_n x^n + A_{n-1\ } x^{n-1} + \ldots + A_1 x + A_0$	$A_n, A_{n-1\ }, \ldots, A_0$
Ce^{ax}	Ae^{ax}	A
$\sin\beta x$, $\cos\beta x$ or $c_1 \cos\beta x + c_2 \sin\beta x$	$A\cos\beta x + B\sin\beta x$	$A,\ B$

If the assumed form of y_p duplicates a function in, y_c then we need to multiply the function in y_p by multiplying by x. If it is not sufficient to remove all the duplications, then we need to keep multiplying by x until there are not duplications

Find y_c

Use f(x) to determine a particular solution y_p, modify it by multiplying x (or x^2, x^3, ...) if needed

Plug y_p into the nonhomogeneous Differential Equations to determine the coefficients in y_p

General solution to the nonhomogeneous Differential Equation is $y = y_c + y_p$

If the given problem is an initial value problem then you need to use the initial conditions to find these arbitrary constants in y_c

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Variation of parameter

For other nonhomogeneous Differential Equations we may use this method

Can be used to solve any nonhomogeneous Differential Equation

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Consider the nonhomogeneous Differential Equation $y^{\prime\prime} + p\left(x\right)y^\prime + q\left(x\right)y = f\left(x\right)$

Suppose $y_c = c_1y_1 + c_2y_2$ is known

Let $W = w\left(y_1, y_2\right) = \left|\begin{matrix}y_1&y_2\\y_1^\prime&y_2^\prime\\ \end{matrix}\right|$

Then the particular solution to the Differential Equation is $y_p = u_1y_1 + u_2y_2$

$u_1^\prime = -\frac{y_2f\left(t\right)}{w}\rightarrow u_1 = \int{-\frac{y_2f}{w}dx}$ *(no arbitrary constant needed)*

$u_2^\prime = -\frac{y_1f\left(t\right)}{w}\rightarrow u_1 = \int{\frac{y_1f}{w}dx}$ *(no arbitrary coonstant needed)*

$$y = y_c + y_p = c_1y_1 + c_2y_2 + u_1y_1 + u_2y_2 = \left(c_1 + u_1\right)y_1 + \left(c_2 + u_2\right)y_2$$

Forced Oscillation and Resonance

Occurs when $mx^{\prime\prime} + cx^\prime + kx = F\left(t\right),\ x\left(0\right) = x_0, x^\prime\left(0\right) = x_1$ where $F\left(t\right)\neq0$

Undamped ($c = 0$) Forced ($F\left(t\right)\neq0$) Oscillation

---

$$mx^{\prime\prime} + kx = F\left(t\right)\neq0$$

Homogeneous $mx^{\prime\prime} + kx = 0\rightarrow mr^2 + k = 0\rightarrow r = \pm\sqrt{\frac{k}{m}}i$

$w_0 = \sqrt{\frac{k}{m}}$ *- - natural frequency*

$$x_c = c_1\cos{\left(w_0t\right)} + c_2\sin{\left(w_0t\right)}$$

L;et $F\left(t\right) = \sin{\left(wt\right)}$ or $F\left(t\right) = \cos{\left(wt\right)}$ where w is the frequency of the external force

$$x_p = \begin{cases} A\cos(wt) + B\sin(wt) & w \neq w_0 \\ At\cos(wt) + Bt\sin(wt) & w = w_0 \end{cases}$$

If $c = 0$ (undamped) Resonance happens when $w = w_0 = \sqrt{\frac{k}{m}}$

Resonance- a phenomenon that the amplitude of the oscillation becomes unbounded as time goes on

Damped ($c \neq 0$) forced ($F\left(t\right) \neq 0$) Oscillations

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$$mx^{\prime\prime} + cx^\prime + kx = F\left(t\right) \neq 0$$

Overdamped ($\mathrm{\Delta} = c^2 - 4mk > 0$) *so* $r_1 \neq r_2 < 0$ *real distinct values*

$x_c = c_1e^{r_1t} + c_2e^{r_2t}$ *which goes to 0 as t goes to infinity*

Critical Damping ($\mathrm{\Delta} = c^2 - 4mk = 0$) *so* $r_1 = r_2 = -\frac{c}{2m} < 0$ *real distinct values*

$x_c = e^{-\frac{c}{2m}t}\left(c_1 + c_2t\right)$ *which goes to 0 as t goes to infinity*

Under Damping ($\mathrm{\Delta} = c^2 - 4mk < 0$) *so* $r_1 = r_2 = -\frac{c}{2m} \pm \beta i$

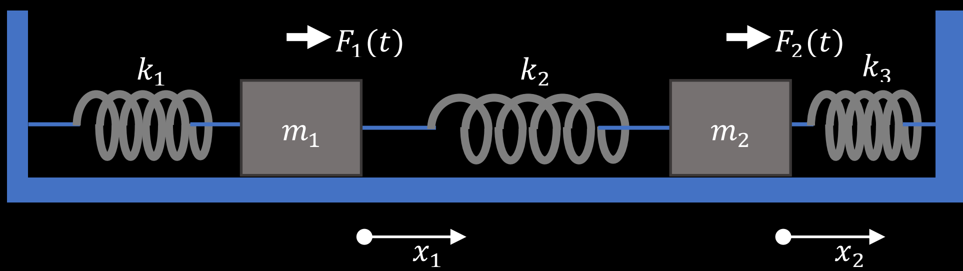
$x_c = e^{-\frac{c}{2m}t}\left(c_1\cos{\left(\beta t\right)} + c_2\sin{\left(\beta t\right)}\right)$ *which goes to 0 as t goes to infinity*

If $F\left(t\right) = F\cos\left(wt\right)$ where F is a constant, then $mx_p^{\prime\prime} + cx_p^\prime + kx_p = \left(kA + CBw - mAw^2\right)\cos{\left(wt\right)} + \left(kB - CAw - mBw^2\right)\sin{\left(wt\right)} = F\cos\left(wt\right)$

- $kA + cBw - mAw^2 = F$
- $kB - CAw - mBw^2 = 0$
- $x = x_c + x_p = x_c + A\cos\left(wt\right) + B\sin\left(wt\right)$
- $x_c \rightarrow 0$ as $t \rightarrow \infty$
- x_c- transient solution
- $x_p = A\cos\left(wt\right) + B\sin\left(wt\right) = R\cos\left(wt - \alpha\right)$- steady periodic solution (steady state solution)
    - $R = \sqrt{A^2 + B^2}$
    - R (amplitude) gets large if $w = \sqrt{\frac{k}{m} - \frac{c^2}{2m^2}}$ this is called practice resonance

Chapter 4: Introduction to Systems of Differential Equations

1st Order systems and Applications



$$F\_\{s1\} = -kx\_1$$

$$F\_\{s2\} = k\_2\backslash left(x\_1 - x\_2\backslash right)$$

$$F\_\{s3\} = \{-k\}\_3\backslash left(x\_2\backslash right)$$

$$\begin{aligned} m\_1a\_1 &= F\_\{s1\} + F\_\{s2\} + F\_1\backslash left(t\backslash right) \\ &= -kx\_1 + k\_2\backslash left(x\_1 - x\_2\backslash right) + F\_1\backslash left(t\backslash right) \end{aligned}$$

$$\begin{aligned} m\_2a\_2 &= F\_\{s2\} + F\_\{s3\} + F\_2\backslash left(t\backslash right) \\ &= \{-k\}\_2\backslash left(x\_1 - x\_2\backslash right) + k\_3\backslash left(x\_2\backslash right) + F\_2\backslash left(t\backslash right) \end{aligned}$$

$$\begin{cases} m_1\dfrac{d^2x_1}{dt^2} = -kx_1 + k_2(x_1 - x_2) + F_1(t) \\ m_2\dfrac{d^2x_2}{dt^2} = -k_2(x_1 - x_2) - k_3x_2 + F_2(t) \end{cases}$$

The general form of a system of n 1^{st} order differential equations is

$$\begin{cases} x_1' = f_{1(x_1,x_2,\ldots,x_n)} \\ x_2' = f_{2(x_1,x_2,\ldots,x_n)} \\ \qquad\vdots \\ x_n' = f_{n(x_1,x_2,\ldots,x_n)} \end{cases}$$

- Highest derivative in each equation is first order
- $x\_i\hat{}\backslash prime$ only appears in the ith equation on the left side

In general, a single higher order nth order differential equation can be written as a system on n first order differential equations by introducing n new variables $(\mathrm{x\_1}, \mathrm{x\_2}, \backslash ldots, \mathrm{x\_n})$ and a system of n first order differential equations can be converted into a single higher order nth order derivatives by getting rid of n-1 variables

- The first n-1 differential equations follow the pattern $x_i^\prime = x_{i+1}$
- The last equation depends on the given Differential Equation

Simple 2-dimensional systems of 1st order can be solved by elimination

Method of Elimination

$$\begin{cases} x' = 4x - 3y \\ y' = 6x - 7y \end{cases}$$

$$y = \frac{x^\prime - 4x}{-3}$$

$$y^\prime = 6x - 7y \rightarrow \ \left(\frac{x^\prime - 4x}{-3}\right)^\prime = 6x - 7\left(\frac{x^\prime - 4x}{-3}\right)$$

$$\ \frac{x^{\prime\prime} - 4x^\prime}{-3} = 6x - 7\left(\frac{x^\prime - 4x}{-3}\right)$$

$$x^{\prime\prime} + 3x^\prime - 10x = 0$$

$$r^2 + 3r - 10 = 0 \rightarrow R = -5,2$$

$$\boldsymbol{x} = \boldsymbol{c}_\mathbf{1}\boldsymbol{e}^{-\mathbf{5}\boldsymbol{t}} + \boldsymbol{c}_\mathbf{2}\boldsymbol{e}^{\mathbf{2}\boldsymbol{t}}$$

$$\boldsymbol{y} = \mathbf{3}\boldsymbol{c}_\mathbf{1}\boldsymbol{e}^{-\mathbf{5}\boldsymbol{t}} + \frac{\mathbf{2}}{\mathbf{3}}\boldsymbol{c}_\mathbf{2}\boldsymbol{e}^{\mathbf{2}\boldsymbol{t}}$$

Polynomial Differential Operator

$\mathrm{L} = a_nD^n + a_{n-1}D^{n-1} + \ldots + a_1\mathrm{D} + a_0$ then any system of 2 linear Differential Equations with constant coefficients can be written as

$$\begin{cases} L_1x + L_2y = f_1(t) \\ L_3x + L_4y = f_2(t) \end{cases}$$

Cramer's rule

$$\left|\begin{matrix}L_1&L_2\\L_3&L_4\\\end{matrix}\right|\mathrm{x} = \left|\begin{matrix}f_1\left(t\right)&L_2\\f_2\left(t\right)&L_4\\\end{matrix}\right|$$

$$\left|\begin{matrix}L_1&L_2\\L_3&L_4\\\end{matrix}\right|\mathrm{y} = \left|\begin{matrix}L_1&f_1\left(t\right)\\L_3&f_2\left(t\right)\\\end{matrix}\right|$$

- If system is homogeneous then

$$\left|\begin{matrix}L_1&L_2\\L_3&L_4\\\end{matrix}\right|x = 0$$

$$\left|\begin{matrix}L_1&L_2\\L_3&L_4\\\end{matrix}\right|y = 0$$

Chapter 5: Linear Systems of Differential Equations

Matrices and Linear Systems

Matrix- a rectangular array of numbers with m rows and n columns

$$A=\left[a_{ij}\right]_{m\times n}=\left[\begin{matrix}a_{11}&\cdots&a_{1n}\\\vdots&\ddots&\vdots\\a_{m1}&\cdots&a_{mn}\\\end{matrix}\right]_{m\times n}$$

$$A^T=\left[a_{ji}\right]_{n\times m}$$

$$\overline{A}=\left[\overline{a_{ji}}\right]_{n\times m}$$

$$I_n=\left[\begin{matrix}1&\cdots&0\\\vdots&\ddots&\vdots\\0&\cdots&1\\\end{matrix}\right]_{n\times n}$$

$$0_n=\left[\begin{matrix}0&\cdots&0\\\vdots&\ddots&\vdots\\0&\cdots&0\\\end{matrix}\right]_{n\times n}$$

Matrix Algebra

$$A\pm B=\left[a_{ij}+b_{ij}\right]$$

$$kA=\left[a_{ij}\right]$$

AB is defined if the number of rows in B is equal to the number of columns in A (A is size mXn and B is size nXp)

$$AB\neq BA$$

$$A^\prime\left(t\right)=\frac{dA}{dt}=\left[a_{ij}^\prime\left(t\right)\right]$$

Determinant of $\boldsymbol{n}\times\boldsymbol{n}$ matrix A (detA or $\left|\boldsymbol{A}\right|$)

N=2 $A=\left[\begin{matrix}a&b\\c&d\\\end{matrix}\right]\rightarrow \mathrm{det}A=ad-bc$

N=3 $A=\left[\begin{matrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\\\end{matrix}\right]\rightarrow \mathrm{det}A=a_{11}\left|\begin{matrix}a_{22}&a_{23}\\a_{32}&a_{33}\\\end{matrix}\right|-a_{12}\left|\begin{matrix}a_{21}&a_{23}\\a_{31}&a_{33}\\\end{matrix}\right|+a_{13}\left|\begin{matrix}a_{21}&a_{22}\\a_{31}&a_{32}\\\end{matrix}\right|$

Inverse matrix (\mathbit{A}^{−\mathbf{1}})

Exists if A is an nXn matrix

A is said to be nonsingular/invertible if there exists another nXn matrix b such that AB = BA = I_n

Then B is called the inverse of A and is denoted as B = A^{−1} or B^{−1} = A

If A =\left[\begin{matrix}a&b\\c&d\\\end{matrix}\right] and \det{A} = ad − bc\neq0 Then A^{−1} =\frac{\left[\begin{matrix}d& − b\\−c&a\\\end{matrix}\right]}{ad − bc}

Systems Expressed as Matrixes

$\begin{cases} x_1' = f_{1(x_1,x_2,\ldots,x_n)} \\ x_2' = f_{2(x_1,x_2,\ldots,x_n)} \\ \quad\vdots \\ x_n' = f_{n(x_1,x_2,\ldots,x_n)} \end{cases}$ can also be expressed as {\vec{x}}^\prime =\left[\begin{matrix}p_{11\left(t\right)}&\cdots&p_{1n\left(t\right)}\\\vdots&\ddots&\vdots\\p_{n1\left(t\right)}&\cdots&p_{nn\left(t\right)}\\\end{matrix}\right]\vec{x} +\vec{f\left(t\right)} where {\vec{x}}^\prime =\left[\begin{matrix}x_1^\prime\\\vdots\\x_n^\prime\\\end{matrix}\right], \vec{x} =\left[\begin{matrix}x_1\\\vdots\\x_n\\\end{matrix}\right] *and* f\left(t\right) = \left[\begin{matrix}f_{1\left(t\right)}\\\vdots\\f_{n\left(t\right)}\\\end{matrix}\right]

If \vec{f\left(t\right)}\neq0 *the system is nonhomogeneous*

If \vec{f\left(t\right)} = 0 *the system is homogeneous*

Superposition theorem- If {\vec{\mathbit{x}}}_\mathbf{1}\left(\mathbit{t}\right),\ \ {\vec{\mathbit{x}}}_\mathbf{2}\left(\mathbit{t}\right),\ \ \ldots,\ \ {\vec{\mathbit{x}}}_\mathbit{n}\left(\mathbit{t}\right) **are. N solutions to** {\vec{\mathbit{x}}}^\prime = **\mathbit{p}**\left(\mathbit{t}\right)\vec{\mathbit{x}}**. Then** \mathbit{c}_\mathbf{1}{\vec{\mathbit{x}}}_\mathbf{1}\left(\mathbit{t}\right) +\mathbit{c}_\mathbf{2}{\vec{\mathbit{x}}}_\mathbf{2}\left(\mathbit{t}\right) +\ldots +\mathbit{c}_\mathbit{n}{\vec{\mathbit{x}}}_\mathbit{n}\left(\mathbit{t}\right) **is also a solution**

Linearly Independent vector functions- the n vector functions {\vec{\mathbit{x}}}_\mathbf{1}\left(\mathbit{t}\right),\ {\vec{\mathbit{x}}}_\mathbf{2}\left(\mathbit{t}\right),\ldots,\ {\vec{\mathbit{x}}}_\mathbit{n}\left(\mathbit{t}\right) **if** \mathbit{c}_\mathbf{1}{\vec{\mathbit{x}}}_\mathbf{1}\left(\mathbit{t}\right) +\mathbit{c}_\mathbf{2}{\vec{\mathbit{x}}}_\mathbf{2}\left(\mathbit{t}\right) +\ldots +\mathbit{c}_\mathbit{n}{\vec{\mathbit{x}}}_\mathbit{n}\left(\mathbit{t}\right) =**\mathbf{0}** **only when** \mathbit{c}_\mathbf{1} =\mathbit{c}_\mathbf{2} =\ldots =\mathbit{c}_\mathbit{n} =**\mathbf{0}**

Wronskian of n vector functions ($\ {\vec{\boldsymbol{x}}}_\mathbf{1}\left(\boldsymbol{t}\right),\ \ {\vec{\boldsymbol{x}}}_\mathbf{2}\left(\boldsymbol{t}\right),\ \ \ldots,\ \ {\vec{\boldsymbol{x}}}_\boldsymbol{n}\left(\boldsymbol{t}\right)$**) is defined by** $\boldsymbol{W}=\boldsymbol{w}\left(\boldsymbol{t}\right)=\left[\begin{matrix}{\vec{\boldsymbol{x}}}_\mathbf{1}\left(\boldsymbol{t}\right)&\cdots&{\vec{\boldsymbol{x}}}_\boldsymbol{n}\left(\boldsymbol{t}\right)\\\end{matrix}\right]$

If $w\neq0$ then ${\vec{x}}_1\left(t\right),\ \ {\vec{x}}_2\left(t\right),\ \ \ldots,\ \ {\vec{x}}_n\left(t\right)$ are linearly independent and the general solution to ${\vec{\boldsymbol{x}}}^\prime=\boldsymbol{p}\left(\boldsymbol{t}\right)\vec{\boldsymbol{x}}$ **is** $\boldsymbol{x}=\boldsymbol{c}_\mathbf{1}{\vec{\boldsymbol{x}}}_\mathbf{1}\left(\boldsymbol{t}\right)+\boldsymbol{c}_\mathbf{2}{\vec{\boldsymbol{x}}}_\mathbf{2}\left(\boldsymbol{t}\right)+\ldots+\boldsymbol{c}_\boldsymbol{n}{\vec{\boldsymbol{x}}}_\boldsymbol{n}\left(\boldsymbol{t}\right)=\left[\begin{matrix}{\vec{\boldsymbol{x}}}_\mathbf{1}\left(\boldsymbol{t}\right)&\cdots&{\vec{\boldsymbol{x}}}_\boldsymbol{n}\left(\boldsymbol{t}\right)\\\end{matrix}\right]\left[\begin{matrix}c_1\\\vdots\\c_n\\\end{matrix}\right]=\mathrm{x}\left(t\right)\vec{c}$ and $\mathrm{x}\left(t\right)=\left[\begin{matrix}{\vec{\boldsymbol{x}}}_\mathbf{1}\left(\boldsymbol{t}\right)&\cdots&{\vec{\boldsymbol{x}}}_\boldsymbol{n}\left(\boldsymbol{t}\right)\\\end{matrix}\right]=\boldsymbol{w}$ **which is called the fundamental matrix**

If $w=0$ then ${\vec{x}}_1\left(t\right),\ \ {\vec{x}}_2\left(t\right),\ \ \ldots,\ \ {\vec{x}}_n\left(t\right)$ are linearly dependents

Eigenvalue method for homogeneous systems

Let A be an $n\times n$ matrix then an eigenvector of matrix A is a nonzero vector $\vec{v}$ such that $A\vec{v}=\lambda\vec{v}$ for some scalar λ

The scalar λ is called an eigenvalue of matrix A

$\lambda,\vec{v}$ are an eigenvalue-eigenvector pair of matrix A

To find the eigenvectors of matrix A solve the equation $\det{\left(A-\lambda I\right)}=0$ which is called the characteristic equation and the roots of $\det{\left(A-\lambda I\right)}=0$ are the eigenvalues of A

Steps for finding eigenvalues and eigenvectors of A

1. Solve $\det{\left(A-\lambda I\right)}=0$ for λ
2. For each eigenvalue λ and solve $\left(A-\lambda I\right)\vec{v}=\vec{0}$

The general solution to the homogeneous equation $x^\prime=Ax$ is $\vec{x}=c_1{\vec{v}}_1e^{\lambda_1t}+\ldots+c_n{\vec{v}}_ne^{\lambda_nt}$

If A has n distinct eigenvalues, $\lambda_1\ldots\lambda_n$ then the corresponding eigenvectors $v_1\ldots v_n$ are linearly independent and the n solutions $x_1 = v_1e^{\lambda_1t}\ldots x_n = v_ne^{\lambda_nt}$

Complex Eigenvalues

Let A be a square matrix with real entries

1. If $\lambda_1 = \mathrm{p} - \mathrm{qi}$ is an eigenvalue of A then $\bar{\lambda_2} = \mathrm{p} - \mathrm{qi}$ is also an eigenvector of A
2. If $\vec{v_1}$ is an eigenvector corresponding to λ_1 then $\vec{v_2} = \bar{\vec{v_1}}$ which corresponds to $\lambda_2 = \bar{\lambda_1}$
3. If $x_1\left(t\right) = v_1e^{\lambda_1t}$ is a solution to $x^\prime = \mathrm{Ax}$ then $x_2\left(t\right) = v_2e^{\lambda_2t}$ is another solution to to $x^\prime = \mathrm{Ax}$

Let A be a 2×2 matrix with complex conjugate eigenvalues and $x_1\left(t\right) = v_1e^{\lambda_1t} = \mathrm{A}\left(t\right) + \mathrm{iB}\left(t\right)$ then $x_2\left(t\right) = \mathrm{A}\left(t\right) - \mathrm{iB}\left(t\right)$ and the general solution to $x^\prime = \mathrm{Ax}$ is $\mathrm{x}\left(t\right) = C_1\mathrm{A}\left(t\right) + C_2\mathrm{B}\left(t\right)$

Gallery of Solution Curves of Linear Systems

Consider the homogeneous system $x^\prime = \mathrm{Ax}$

- If A is a 1×1 matrix that is $\mathrm{A} = \mathrm{a}$ then $x^\prime = \mathrm{ax}$ so if $\mathrm{a}\neq0$ then $x^\prime = \mathrm{Ax}$ is an autonomus differential equation and $\mathrm{x} = 0$ is an equilibrium solution so we can use the phase line to check the stability of $\mathrm{x} = 0$
    - If $\mathrm{a} > 0$ then $\mathrm{x} = 0$ is unstable
    - If $\mathrm{a} < 0$ is stable
- If A is a 2×2 matrix and $\left|A\right|\neq0$ then the homogeneous system $x^\prime = \mathrm{Ax}$ has $\vec{x} = \vec{0}$ as the equilibrium solution. Let $\vec{x} = \left[\begin{matrix}x_1\\x_2\\\end{matrix}\right]$ we will use the phase place ($x_1 - x_2$ place) to determine the stability of the equilibrium solution (origin on phase plane)

Repeated real eigenvalues

- if A has 2 linearly independent eigenvalues $\vec{v_1}$ and $\vec{v_2}$ then the general solution too $x^\prime = \mathrm{Ax}$ where $\mathrm{x}\left(t\right) = \left(c_1\vec{v_1} + c_2\vec{v_2}\right)e^{\lambda t}$
- If A has only one eigenvector, ${\vec{v}}_1$ then let the generalized eigenvector $\vec{v_2} = \left[\begin{matrix}1\\0\\\end{matrix}\right]$ and solve

from ${\vec{v}}_1 =\left(A -\lambda I\right){\vec{v}}_2$ then the general solution is $\vec{x\left(t\right)} =\left(c_1v_1 + c_2v_2 + c_2tv_1\right)e^{\lambda t}$

	λ_1 and λ_2	$\vec{x} =\vec{0}$ (type of stability
λ_1 and λ_2 are distinct real values	$\lambda_1 < 0$ and $\lambda_2 > 0$	Saddle point (unstable)
	$\lambda_1 < 0$ and $\lambda_2 < 0$	Improper nodal sink (stable)
	$\lambda_1 > 0$ and $\lambda_2 > 0$	Improper nodal source (unstable)
$\lambda_1 =\lambda_2$ repeated real value with 2 independent eigenvectors	$\lambda_1 =\lambda_2 < 0$	Proper nodal sink (stable)
	$\lambda_1 =\lambda_2 > 0$	Proper nodal source (unstable)
$\lambda_1 =\lambda_2$ repeated real value with 1 independent eigenvectors	$\lambda_1 < 0$ and $\lambda_2 > 0$	Improper nodal sink (stable)
	$\lambda_1 < 0$ and $\lambda_2 > 0$	Improper nodal source (unstable)
Complex conjugate eigenvalues $\lambda = p\pm qi$	$p = 0$	Center (stable but not asymptotically stable)
	$p < 0$	Spiral source (unstable)
	$p > 0$	Spiral sink (stable)

$\vec{x} =\vec{0}$ is stable (asymptotically stable) if A has two negative real eigenvalues or 2 complex conjugate eigenvalues with negative real parts

Multiple Eigenvalue solution

Let A be an $n\times n$ matrix then the total number of eigenvalues of A (counting multiplicities) is n, but the total number of linearly independent eigenvectors may be equal to or less than n

Defective matrix- an $n\times n$ matrix that has less than n linearly independent eigenvalues

nondeffective matrix- an $n\times n$ matrix with n linearly independent eigenvalues *if A is a nondeffective matrix then $x^\prime = Ax$ has the general solution*

$$\vec{x\left(t\right)} = c_1v_1e^{\lambda_1t} + \ldots + c_nv_ne^{\lambda_nt}$$

Generalize eigenvector- let a be a 2×2 matrix with repeated eigenvalues $\lambda_1 = \lambda_2 = \lambda$ if A is defective and ${\vec{v}}_1$ be the eigenvector corresponding to λ_1 then $\left(A - I\lambda\right){\vec{v}}_1 = 0$ and the generalized vector $\vec{v_2}$ satisfies the equation $\left(A - \lambda_1I\right){\vec{v}}_2 = {\vec{v}}_1$

Let A be a 2×2 defective matrix with eigenvalue λ_1 and eigenvector ${\vec{v}}_1$ and let ${\vec{v}}_2$ be the generalized eigenvector of A then $\vec{x}\left(t\right) = e^{\lambda t}\left(c_1v_1 + c_2v_2 + c_2tv_1\right)$

Matrix Exponents and linear Systems

Consider $x^\prime = \mathrm{Ax}$ let $\vec{x_1\left(t\right)}\ldots\ \vec{x_n\left(t\right)}$ by n linearly independent solutions then the general solution is $\phi\left(t\right)\vec{c} = \left[\begin{matrix}x_1&\ldots&x_n\\\end{matrix}\right]\left[\begin{matrix}c_1\\\vdots\\c_n\\\end{matrix}\right]$ where $\phi\left(t\right)$ is the fundamental matrix

$$\mathrm{x}\left(t\right) = \phi\left(t\right)\ \phi^{-1}\left(t\right)\vec{x_0}$$

Fundamental matrix solution- let $\phi\left(t\right)$ be a fundamental matrix for the system $x^\prime = \mathrm{Ax}$ then the unique solution to the initial value problem $x^\prime = \mathrm{Ax}$ where $\mathrm{x}\left(0\right) = x_0$

Matrix Exponential

$$e^t = 1 + \mathrm{t} + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots + \frac{t^n}{n!} + \ldots$$

$$e^{At} = \mathrm{I} + \mathrm{At} + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \ldots + A^n\frac{t^n}{n!} + \ldots$$

So if $A^n = 0$ (0 matrix) for some positive integer n then A is said to be nipotent

If B is a diagonal matrix $\mathrm{B} = \left[\begin{matrix}b_1&\cdots&0\\\vdots&\ddots&\vdots\\0&\cdots&b_n\\\end{matrix}\right]$ then $e^{Bt} = \left[\begin{matrix}e^{b_1t}&\cdots&0\\\vdots&\ddots&\vdots\\0&\cdots&e^{b_nt}\\\end{matrix}\right]$

If A=B+C where b is a multiple of the identity matrix and C is a nilpotent matrix then $e^{At} = e^{Bt}\ast e^{Ct}$

$$e^{At} = \phi\left(t\right)\phi^{-1}\left(0\right)$$

Matrix exponential solution- the solution to the initial value problem $x^\prime = \mathrm{Ax}$ where $\mathrm{x}\left(0\right) = x_0$ can be given by $\mathrm{x}\left(t\right) = e^{At}x_0$

Nonhomogeneous

The general solution to $x^\prime = Ax + f\left(t\right)$ where A is a constant matrix is $x = x_c + x_p$ where x_c is the general solution to $x^\prime = Ax$ and x_p is the particular solution to f(t)

Method of Undefined Coefficients

> Consider $x^\prime = Ax + f\left(t\right)$ and $f\left(t\right)$ is a vector function with 3 types of functions (polynomial, exponential, sin/cos) then we may choose a similar form of f(t) to write x_p in terms of different types of vector functions

Variation of Parameters

> Consider $x^\prime = Ax + f\left(t\right)$ let $x_{c\left(t\right)} = c_1x_1 + c_2x_2$ be the general solution to $x^\prime = Ax$ and let $\phi\left(t\right) = \left[\begin{matrix}x_1&\ldots&x_n\\\end{matrix}\right]$ then $x_c = \phi\left(t\right)c$ and $x_c = \phi\left(t\right)u$ where $u^\prime = \phi^{-1}\left(t\right)f\left(t\right)$ so $u = \int\phi^{-1}\left(t\right)\vec{f\left(t\right)}dt$ so $x = x_c + x_p = \phi\left(t\right)c + \phi\left(t\right)u$

Steps for solving variation of parameters

1. Solve $x^\prime = Ax$ to find $\phi\left(t\right) = \left[\begin{matrix}x_1&x_2\\\end{matrix}\right]$
2. Find $\phi^{-1}\left(t\right)$
3. Find $u^\prime = \phi^{-1}\left(t\right)f\left(t\right)$
4. Find $u = \int\phi^{-1}\left(t\right)f\left(t\right)dt$
5. $x_p = \phi\left(t\right)u$
6. General solution is $x = x_c + x_p = \phi\left(t\right)c + \phi\left(t\right)u$

$$x\left(t\right) = e^{At}x_0 + e^{At\int e^{-As}f\left(s\right)ds}$$

Chapter 7: Laplace Transformations

Laplace Transformations and Inverse transforms

Laplace Transform- given a function f(t) defined for all $t\geq0$ the Laplace transforms of f(t) is the function F(s) defined as $f\left(t\right){\buildrel L \over \rightarrow}F\left(s\right)$

$F\left(s\right)=L\left\{f\left(t\right)\right\}=\int_{0}^{\infty}{e^{-st}f\left(t\right)dt}$ for all values oof s such that the improper integral converges

Linearly of Laplace Transformations- let $L\left\{f_1\left(t\right)\right\}=F_1\left(s\right)$ and $L\left\{f_2\left(t\right)\right\}=F_2\left(t\right)$ then $L\left\{c_1f_1\left(t\right)+c_2f_2\left(t\right)\right\}=c_1F_1\left(s\right)+c_2F_2\left(s\right)$

Inverse Laplace transformation- $L^{-1}\left\{F\left(s\right)\right\}=f\left(t\right)$ and $L^{-1}\left\{c_1F_1\left(s\right)+c_2F_2\left(s\right)\right\}=c_1f_1\left(t\right)+c_2f_2\left(t\right)$

f(t)	F(s)
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$
t^p where $(p>-1)$	$\frac{\Gamma\left(p+1\right)}{s^{p+1\ }}$
$\sin{\left(at\right)}$	$\frac{a}{s^2+a^2}$
$\cos{\left(at\right)}$	$\frac{s}{s^2+a^2}$
$\sinh{\left(at\right)}$	$\frac{a}{s^2-a^2}$

$\cosh{\left(at\right)}$	$\frac{s}{s^2-a^2}$
$e^{at}\sin{\left(bt\right)}$	$\frac{b}{\left(s-a\right)^2+b^2}$
$e^{at}\cos{\left(bt\right)}$	$\frac{s-a}{\left(s-a\right)^2+b^2}$
$e^{at}t^n$	$\frac{n!}{\left(s-a\right)^{n+1}}$
$u_c\left(t\right)$	$\frac{e^{-cs}}{s}$
$u_c\left(t\right)f\left(t-c\right)$	$e^{-cs}F\left(s\right)$
$f\left(ct\right)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$ when c>0
$\int_{0}^{t}f\left(t-r\right)g\left(r\right)dr$	$F\left(s\right)G\left(s\right)$
$\delta\left(t-c\right)$	e^{-cs}
$f^{\left(n\right)}\left(t\right)$	$sF\left(s\right)-s^{n-1\ }f\left(0\right)-\ldots-sf^{\left(n-2\right)}\left(0\right)-f^{\left(n-1\right)}\left(0\right)$
$\left(-t\right)^nF\left(s\right)$	$F^{\left(n\right)}s$

Translation and Partial fractions

$$\left\{\begin{matrix}ax^{\prime\prime}+bx^\prime+cx=f\left(t\right)\\x\left(0\right)=x_0\ and\ x\left(0\right)=x_1\\\end{matrix}\right.\rightarrow x\left(s\right)=\frac{\left(\left(as+b\right)x_0+ax_1+F\left(s\right)\right)}{as^2+bs+c}\rightarrow x\left(t\right)=L^{-1}\left\{x\left(s\right)\right\}$$

For $x\left(s\right)=\frac{D\left(s\right)}{as^2+bs+c}$ where $\Delta=b^2-4ac$

- If $\Delta>0$ we can use partial fraction decomposition to solve
- If $\Delta<0$ we can complete the square

Rules for Partial Fraction decomposition (for $\frac{p\left(s\right)}{q\left(s\right)}$)

- If $\left(s-a\right)^n$ is a factor of Q(s) then the partial fraction decomposition corresponding to $\left(s-a\right)^n$ is $\frac{A_1}{s-a}+\frac{A_2}{\left(s-a\right)^2}+\ldots+\frac{A_n}{\left(s-a\right)^n}$
- If $\left[\left(s-a\right)^2+b^2\right]^n$ is a factor of Q(s) then the partial fraction decomposition corresponding to $\left[\left(s-a\right)^2+b^2\right]^n$ is $\frac{A_1s+B_1}{\left(s-a\right)^2+b^2}+\frac{A_2s+B_2}{\left[\left(s-a\right)^2+b^2\right]^2}+\ldots+\frac{A_ns+B_n}{\left[\left(s-a\right)^2+b^2\right]^n}$

Translation on the S-axis- if $\mathrm{L}\left\{f\left(t\right)\right\}=\mathrm{F}\left(s\right)$ exists for $\mathrm{s}>\mathrm{c}$ then $\mathrm{L}\left\{e^{at}f\left(t\right)\right\}$ exists for $\mathrm{s}>\mathrm{a}+\mathrm{c}$ and $\mathrm{L}\left\{e^{at}f\left(t\right)\right\}$ for $\mathrm{s}>\mathrm{a}+\mathrm{c}\ $ and $\mathrm{L}\left\{e^{at}f\left(t\right)\right\}=\mathrm{F}\left(s-a\right)$

Derivatives, Integral and Products of Transform

Convolution property $\mathrm{L}\left\{f\left(t\right)\ast g\left(t\right)\right\}=\mathrm{F}\left(s\right)\mathrm{G}\left(s\right)$

Transform of derivatives $\mathrm{L}\left\{f^\prime\left(t\right)\right\}=\mathrm{sF}\left(s\right)-\mathrm{f}\left(0\right)$

Transform of integrals $\mathrm{L}\left\{\int_{0}^{t}f\left(r\right)dr\right\}=\frac{F\left(s\right)}{s}$

Differentiation of Transforms $L^{-1}\left\{F^{\left(n\right)}\left(s\right)\right\}=\left(-1\right)^nt^n\mathrm{f}\left(t\right)$

Integration of Transforms $L^{-1}\left\{\int_{s}^{\infty}f\left(\sigma\right)d\sigma\right\}=\frac{f\left(t\right)}{t}$

Periodic and Peicewise continuous Functions

$u_a\left(t\right)$ is a piecewise continuous function $\left\{\begin{matrix}0 & t<a\\ 1 & t\geq a\end{matrix}\right.$

Translation on the t-axis- $\mathrm{L}\left\{u\left(t-a\right)f\left(t-a\right)\right\}=e^{-as}\mathrm{F}\left(s\right)$

Impulses and delta functions

The total impulse of the force function f(t) over an interval [a,b] is defined by $\mathrm{P}=\int_{a}^{b}f\left(t\right)dt$ and is a measure of the strength of the force

Dirac Delta function (\delta_a\left(t\right) =\delta\left(t − a\right))- \left\{\ begin{matrix}\ \ \ \ \ \infty\ \ \ \ \ \ \ \ \ \ \ \ \ t = a\\\ \ \ \ \ \ 0\ \ \ \ \ \ \ \ \ \ \ \ \ \ t\neq a\\\ end{matrix}\right. function that is zer everywhere but 1 point and \int_{0}^{\infty}{\delta_a\left(t\ right)}dt = 1\

L\left\{\delta_a\left(t\right)\right\} = e^{−as}

\delta(t − a) = u^\prime\ (t − a)