Linear Notes

Linear Systems, Matrix Equations, Vector Equations and Solutions

Linear equation- equation in the form $a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_px_p = b$

- Where a_1, a_2, a_3...a_p are all constants
- Where x_1, x_2, x_3...x_p are all variables

Linear system- set of one or more linear equations that have the same variable set (can have any number of equations an any number of variables)

Linear System is equivalent to the matrix equation $Ax = b$

- $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$
- $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$
- $b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$

Linear system is equivalent to the vector equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$

Linear system, matrix equation and vector equation are all equivalent (this means they all have the same solution)

Solution of a system- tuple of n values that satisfies each equation in the system

- Tuple- ordered list

Solution set- set of all solutions

- If set is empty, then the system has no solutions and is inconsistent
- If set has one or more solutions, then the system in constant
- If two systems have the same number of systems, they are the same

Solving matrix equation $Ax = b$

1. Rows reduce the augmented matrix $[A\ b]$ into Row Echelon Form (REF)

a. If there is one row of the REF that has the form $[0 \quad 0 \quad \dots \quad 0 \quad b]$ where $b \neq 0$ then $Ax = b$ is inconsistent and does not have a solution

b. Otherwise, equation $Ax = b$ is consistent and the equation has a solution (can be unique or have infinitely many solutions

2. Rows reduce the REF form of augmented matrix into reduced Row echelon form (RREF)

    a. If there are no free variables, then there is a unique solution

    b. If there is at least one free variable infinitely many solutions

If A is an $m \times n$ matrix and equation $Ax = b$is consistent then:

- The solution is in $\mathbb{R}^n$
- If the number of pivot positions is smaller than n the equation has infinitely many solutions
- The number of pivot positions is not greater than m
- If $b = 0$ the homogeneous equation $Ax = 0$ is always consistent

Linear Combination, Linear Dependent, Independent Set, Span

Assume that v_1, v_2, …, v_p are p vectors in $\mathbb{R}^n$ and $S = \{v_1, v_2, \dots, v_p\}$ is an indexed set

- If a vector in $\mathbb{R}^n$ can be represented as $w = c_1v_1 + c_2v_2 + \dots + c_pv_p$ where c_1, c_2, …, c_p are scalars, then w is a linear combination of p vectors v_1, v_2, …, v_p with the weights c_1, c_2, …, c_p
- Otherwise, if there does not exist c_1, c_2, …, c_p such that w has the representation, then w is not a linear combination of v_1, v_2, …, v_p
- The set S is linearly dependent provided $0 = c_1v_1 + c_2v_2 + \dots + c_pv_p$ if and only if $c_1 = c_2 = c_p = 0$
- The set S is linearly dependent provided there exist p constants c_1, c_2, …, c_p <u>not all zeros</u> such that $0 = c_1v_1 + c_2v_2 + \dots + c_pv_p$
- $span\{v_1, \dots, v_p\}$ is the set of all linear combinations of v_1, v_2, …, v_p

Denote by $A = [v_1, v_2, \dots v_p]$ the $n \times p$ matrix and w is a vector in $\mathbb{R}^n$

- W is a linear combination of v_1, v_2, …, v_p if an only if equation $Ax = w$ is consistent. Otherwise, w is not a linear combination of v_1, v_2, …, v_p
- W is in $span\{v_1, \dots, v_p\}$ if and only if $Ax = w$ is consistent

- The set $S = \{v_1, v_2, \dots, v_p\}$ is linearly independent if equation $Ax = 0$ only has trivial solution
- The set $S = \{v_1, v_2, \dots, v_p\}$ is linearly independent if matrix A has p pivot positions
- The set $S = \{v_1, v_2, \dots, v_p\}$ is linearly dependent if equation $Ax = 0$ only has infinitely many solution
- The set $S = \{v_1, v_2, \dots, v_p\}$ is linearly dependent if matrix A has less than p pivot positions
- If $p > n$ any set of p vectors in $\mathbb{R}^n$ is linearly dependent
- If A is an $m \times n$ matrix with $m > n$ then the set of all columns of A is linearly independent
- Any vector set containing zero vector is linearly dependent
- If there exists at least one vector in a set $S = \{v_1, v_2, \dots, v_p\}$ is a linear combination of the others, then the set S is linearly dependent

If A is an $m \times n$ matrix, then the following statements are logically equivalent

- For each b in $\mathbb{R}^m$ equation $Ax = b$ has at least one solution
- Each b in $\mathbb{R}^m$ is a linear combination of the columns in AA
- The columns of A span $\mathbb{R}^m$ that is $\mathbb{R}^m = span\{columns\ of\ A\}$
- A has n pivot positions

Linear transformation, one-to-one mapping, onto mapping, composition mapping, invertible mapping

A transformation (or mapping) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ s a rule that assigns each vector x in $\mathbb{R}^n$ to a vector $T(x)$ in $\mathbb{R}^m$

A linear transformation T is a transformation which satisfies two following properties

- $T(U + V) = T(U) + T(V)$
- $T(cU) = cT(U)$

A one-to-one mapping is a mapping (or transformation) satisfying:

- Each b in $\mathbb{R}^m$ there exists at most one x in $\mathbb{R}^n$ such that $T(x) = b$

An onto mapping $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a mapping satisfying:

- Each b in $\mathbb{R}^m$ there exists at least one x in $\mathbb{R}^n$ such that $T(x) = b$

If $T: \mathbb{R}^n \rightarrow \mathbb{R}^p$ is a mapping that maps each x in $\mathbb{R}^n$ to $T(x) = y$ in $\mathbb{R}^p$ and $S: \mathbb{R}^p \rightarrow \mathbb{R}^m$ is a mapping that maps each y in $\mathbb{R}^p$ to $S(y) = z$ in $\mathbb{R}^m$ then the composition mapping of T and S demoted by S.T maps X in $\mathbb{R}^n$ to Z in $\mathbb{R}^m$

- $S.T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ and is defined as $S.T(X) = S(T(X))$

If A is an $m \times n$ matrix, then the transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ defined by $T(x) = Ax$ is linear

If the transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear then $T(0) = 0$

For each linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ there is a unique standard matr5ix A such that: $T(x) = Ax$

Let e_j be the j-th column of the identity matrix in $\mathbb{R}^n$. If the $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation, then the standard matrix A is given by $A = [T(e_1) \quad T(e_2) \quad \dots \quad T(e_n)]$

If a transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^p$is linear and A is the standard matrix for T and a transformation $S: \mathbb{R}^p \rightarrow \mathbb{R}^m$ is linear and B is the standard matrix for S. Then the standard matrix for $S.T$ is BA

If a transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^p$is linear and a is the standard matrix for T, then

- T is invertible if and only if A is invertible
- The standard matrix for T^{-1} (the inverse of T) is A^{-1}

Characterizations of one-to-one mapping- Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation A be the standard matrix for T

1. T is one-to-one if and only if equation $Ax = 0$ has only the trivial solution
2. T is one-to-one if and only if the columns of A for a linearly independent set
3. T is one-to-one if and only if A has n pivot positions (every column of A is a pivot column
4. If T is one-to-one, then $m \geq n$

Characterizations of onto mapping- Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation and A be the standard matrix for T

- T is onto if and only if for each b in $\mathbb{R}^m$ the equation $Ax = b$ as at least one solution

- T is onto if and only if the columns of a span $\mathbb{R}^m$

- T is onto if and only if A has pivot positions in every row
- If T is onto then $m \leq n$

Matrix Multiplications, Inverse matrices

Let A be an $m \times p$ matrix and let B be and $p \times n$ matrix then the multiplication AB is an $m \times n$ matrix determined by $[Ab_1 \quad Ab_2 \quad \dots \quad Ab_n]$

Let A ben an $n \times n$ matrix A is said to be invertible if and only if there exists $n \times n$ matrix C such that $AC = CA = I$

- This matrix C is said to be the inverse of A denoted by A^{-1}
- $AA^{-1} = A^{-1}A = I$

$AB \neq BA$ furthermore AB being defined does not guarantee that BA is defined

If A is an $m \times n$ matrix then A^T is an $n \times m$ matrix then $A^T A$ is a square $m \times m$ matrix and AA^T is a square $n \times n$ matrix

If $AB = 0$ this does **not** imply that $A = 0$ and does not imply that $B = 0$

Let a be an $n \times n$ matrix

- Row reduce the augmented matrix $[A\, I]$
- If A is row equivalent to I then $[A \quad I]$ is row equivalent to $[I\, A^{-1}]$
- Otherwise A is not invertible
- If $\det(A) \neq 0$ then:

$$A^{(-1)} = \frac{1}{\det(A)} \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \dots & \dots & \dots & \dots \\ C_{1m} & C_{2m} & \dots & C_{nm} \end{bmatrix}$$

$C_{ij} = (-1)^{ij} \det\left(A_{ij}\right)$ and A_{ij} is the $(n-1) \times (n-1)$ matrix obtained by removing i-th row and j-th column from the matrix A

Invertible matrix Theorem- let A be a square $n \times n$ matrix. Then the following statements are equivalent:

- A is an invertible matrix
- A is row equivalent to the $n \times n$ identity matrix
- A has n pivot positions
- The equation $Ax = 0$ has only the trivial solution

- The columns of A form a linearly independent set
- The linear transformation $x \to Ax$ is one-to-one
- The equation $Ax = b$ has at least one solution for each b in $\mathbb{R}^n$
- The columns of A span $\mathbb{R}^n$
- The linear transformation $x \to Ax$ maps $\mathbb{R}^n$ to $\mathbb{R}^n$
- There is an $n \times n$ mtrix C such that $CA = I$
- There is an $n \times n$ matrix D such that $AD = I$
- A^T is an invertible matrix

Subspaces, Column Space, Null Space, Basis, Dimension, and Rank

A subspace of $\mathbb{R}^n$ is aa set H in $\mathbb{R}^n$ that has the three following properties

- The zero vector is in H
- For each U and V in H the sum u and v $(u + v)$ is in H
- For each u in H and each scalar c the vector cu is in H

For $p \geq 1$, $S = \{v_1, \ldots, v_p\}$ is a set of p vectors in $\mathbb{R}^n$ then $H = span\{v_1, \ldots, v_p\}$ is a subspace of $\mathbb{R}^n$

A basis for a subspace H of $\mathbb{R}^n$ is a linearly independent set in H that spans H

Let A be an $m \times n$ matrix. The column space of A is the set of all linear combination of the columns of A denoted by ColA

- $ColA = span\{all\ columns\ in\ A\}$
- ColA is a subspace of $\mathbb{R}^m$
- A basis of ColA is the set of all pivot columns

Let A be an $m \times n$ matrix. The null space of A is the set of all solutions of the equation $Ax = 0$ denoted by NulA

- NulA is a subspace of $\mathbb{R}^n$
- To find a basis for NulA solve the equation $Ax = 0$ then write the general solution into the parametric vector form

The dimension of a nonzero subspace H is the number of vectors in the basis for H

The dimension of the zero subspace $\{0\}$ is 0

The rank of a matrix A is the dimension of $ColA$ that is $RankA = dimColA$

- $RankA =$ $pivot\ columns$
- To find Rank A row reduce A into REF

Let A be an $m \times n$ matrix

- $RankA + dimNulA = n$
- If mapping $x \to Ax$ is one-to-one, then $\dim NulA = 0$
- If mapping $x \to Ax$ is onto then RankA = m

Determinates

The determinant of a matrix A is denoted by det A or $[A]$

For a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $[A] = ad - bc$

For an $n \times n$ matrix $A = [a_{ij}]$ where $n \geq 3$

- Denote by A_{ij} matrix obtained by removing the i-th row and j-th column from A
- Denoted by C_{ij} the (i,j) cofactor given by $C_{ij} = (-1)^{i+j} \det(A_{ij})$

$$detA = \sum_{j=1}^{n} a_{1j} c_{1j}$$

The determinant can be computed by a cofactor expansion across any row or down any column

Properties- Let A be an $n \times n$ matrix. If a matrix B is product from A by:

- Adding a multiple of one row (or column) to another row (or column) then $detA = detB$
- Interchanging two rows (or two columns) then $detB = -detA$
- Multiplying one row (or one column) by a constant k then $detB = kdetA$
- $detA^T = detA$
- $\det(AB) = detA * detB$
- $\det(A^{-1}) = \frac{1}{detA}$
- $\det(A + B) \neq detA + detB$

If A is a 2×2 matrix, then the area of the parallelogram determined by the columns of A is $|detA|$ (absolute value of $detA$)

If A is a 3×3 matrix, then the volume of the parallelepiped determined by the columns of A is $|detA|$ (absolute value of $detA$)

Let D be a finite area region $\mathbb{R}^2$ and $T: \mathbb{R}^2 \to \mathbb{R}^2$, $x \to Ax$ be a linear transformation then $\{Area\ of\ T(D)\} = |detA| * \{area\ of\ D\}$

Let D be a finite volume in $\mathbb{R}^3$ and $T: \mathbb{R}^3 \to \mathbb{R}^3$, $x \to Ax$ be a linear transformation then $\{Volume\ of\ T(D)\} = |detA| * \{volume\ of\ D\}$

Space, Subspace, Linearly Independent/dependent, Basis, Dimension

A vector space V is non empty set of objects called vectors on which two operations called addition and multiplication by scalars are defined subject to the following 10 axioms:

1. $\vec{u} + \vec{v}$ is in the set
2. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
3. $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$
4. $\vec{0}$ is defined such that $\vec{u} + \vec{0} = \vec{u}$
5. $-\vec{u}$ is defined such that $\vec{u} + -\vec{u} = \vec{0}$
6. $c\vec{u}$ is in the set
7. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$
8. $(c + d)\,\vec{u} = c\vec{u} + d\vec{u}$
9. $(cd)\,\vec{u} = c(d\vec{u})$
10. $1\,\vec{u} = \vec{u}$

A subspace H of a vector space V is a subset H of V that satisfies three properties:

a. $\vec{0}$ is defined (4)
b. $\vec{u} + \vec{v}$ is in the set (1)
c. $c\vec{u}$ is in the set (6)

- If there is at least one of these three properties not satisfied, then H is NOT a subspace of V
- If $v_1, \dots, v_p$ are p vectors in V then $H = span\{v_1, \dots, v_p\}$ is a subspace of B
  - $span\{v\_1, \dots, v\_p\} = \{c\_1\, v\_1 + \cdots + c\_p\, v\_p \mid c\_1, \dots, c\_p\ \ are\ all\ scalars\}$
- Any vector space. V is a subspace of itself

An indexed set of vectors $\{v_1, \dots, v_p\}$ is only linearly independent if and only if the vector equation $c_1 v_1 + \cdots + c_p v_p = 0$ has only the trivial solution ($c_1 = \cdots = c_p = 0$)

The set $\{v_1, \ldots, v_p\}$ is linearly dependent if and only if equation had nontrivial solution

- Any set of vectors containing the zero vector is linearly dependent
- An indexed set $\{v_1, \ldots, v_p\}$, $v \neq 0$ is linearly dependent if and only if some v_j $(j > 1)$ is a linear combination of the preceding vectors
- An indexed set $\{v_1, \ldots, v_p\}$ is linearly independent if NO vector is a linear combination of preceding vectors

If H is a subspace of V. A set of vectors B is a basis for H if two properties are satisfied

a. B is a linearly independent set

b. H is spanned by B $H = Span\{B\}$

If a vector space B is span by a finite set, then B is said to be finite dimensional. The dimension of V denoted by $dimV$ is the number of vectors in a basis for V

- Dimension of the zero-vector space $\{0\}$ is zero
- If V is not spanned by a finite set, then V is said to be infinite-dimensional

If V is a n-dimensional space $(dimV = n)$ then any linearly independent set of n vectors in V is a basis for V

Let $H = span\{v_1, \ldots, v_p\}$ be a subspace of V if the set; $B = \{v_1, \ldots, v_p\}$ is linearly independent, the B is a basis for H and $dimH = p$

- Otherwise B is not a basis for H

Let $B = \{v_1, \ldots, v_p\}$ and $H = spanB$. If one of vectors in B say v_k is a linear combination of the remaining vectors in H then $H = span\, B_1$ where $B_1 = B \backslash \{v_k\}$

Let $B = \{v_1, \ldots, v_p\}$ and $H = spanB$. If B s a linear dependent set, then to find a bas for H remove all vectors in B that are linear combination of the remaining vectors in B, from B until only the vectors that remain are for an independent set

Null Spaces, Column Spaces, Row Spaces

Let A be an $m \times n$ matrix

- $NulA = \{x \text{ in } \mathbb{R}^n : Ax = 0\}$- the set of all solutions of the equation $Ax = 0$
  - $NulA$ is a subspace of $\mathbb{R}^n$

- $ColA = \{b \text{ in } \mathbb{R}^m : b = Ax \text{ for some } x \text{ in } \mathbb{R}^n\} = span\{columns \text{ of } A\}$
    - $ColA$ is a subspace of $\mathbb{R}^m$
- $RowA = span\{rows \text{ of } A\} = ColA^T$
    - $RowA$ is a subspace of $\mathbb{R}^n$

$dimColA + dimNulA = numer \text{ of columns of } A$ or $rankA + nullityA = n$ (A s an $m \times n$ matrix)

To find the basis for $NulA$ solve the homogeneous equation $Ax = 0$

To find the basis for $ColA$ find all the pivot columns of A

To fine the basis for $H = span\{v_1, \ldots, v_p\}$ where $v_1, \ldots, v_p$ are in $\mathbb{R}^n$ find pivot columns of matrix$B = [v_1 \ldots v_p]$

Coordinate Systems, Linear Transformation and matrix of a Linear Transformation

Coordinate Systems- Let V be a n-dimensional vector space and $B = \{v_1, \ldots, v_n\}$ be a basis for V. For each vector w in V we represent w as a linear combination of $v_1, \ldots, v_n$ as:

- $w = c_1v_1 + \cdots + c_nv_n$ where $c_1, \ldots, c_n$ in $\mathbb{R}$
- Denoted by $[w]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ a vector formed by weights $c_1, \ldots, c_n$. Then $[w]_B$ is said to be the B-coordinate of w or coordinate of w relative to B ($[w]_B$ is in $\mathbb{R}^n$)

Linear Transformation- Let V, W be two vector spaces and A be a linear transformation $T: V \to W$ is a rule that assigns to each vector v in W to a vector $T(v)$ in W such that:

- $T(u + v) = T(u) + T(v)$
- $T(cu) = cT(u)$
- $Kernel\, T = \{v \text{ in } V : T(v) = 0\}$ (Subspace of V)
- $Range\, T = \{T(v) \text{ for all } v \text{ in } V\}$ (subspace of W)

Matrix of a Linear Transformation- Let V be an n-dimensional vector space with a basis $B = \{v_1, \ldots, v_n\}$

- If $T: V \to V$ is a linear transformation then the matrix of T relative to bis B, denoted by $[T]_B$ is an $n \times n$ matrix who's the jth-column is the B-cordinate of the image of the jth-vector in the basis B under T. that is

$$[T]_B = [[T(v_1)]_B \quad \ldots \quad [T(v_1)]_B]$$

- In $\mathbb{R}^n$ if $T: \mathbb{R}^n \to \mathbb{R}^n, x \to Ax$ is a linear transformation
- If $B = \{b_1, \dots, b_n\}$ any basis for $\mathbb{R}^n$ then the matrix C for the transformation T relative to B satisfied $A = PCP^{-1}$, where P is the $n \times n$ invertible matrix whose columned formed by B
- Suppose $A = PDP^{-1}$ where D is the diagonal $n \times n$ matrix/ If B is the basis for $\mathbb{R}^n$ formed the columns of P, then D is the matrix relative to B for the transportation $\mathrm{x} \to \mathrm{Ax}$

Eigenvalues, Eigenvectors

Eigenvectors- vector that when operated on returns a scalar of itself

- Same direction different magnitude

Eigenvalues- the scaled factors after transformation

If $\vec{x}$ is an eigenvector and λ is the corresponding eigenvalue then $A\vec{x} = \lambda\vec{x}$

To find eigenvectors

Eigenvalues are the roots of the characteristic equation ($\det(A - \lambda I) = 0$). If A is an $n \times n$ matrix, then there are exactly n eigenvalues including multiplicity and complex

Eigenvectors corresponding to eigenvalues are the nonzero solution to $(A - I\lambda)\vec{v} = \vec{0}$

If $v_1, \dots, v_p$ are eigenvectors corresponding to distinct eigenvalues $\lambda_1, \dots, \lambda_p$ of an $n \times n$ matrix A then the set $\{v_1, \dots, v_p\}$ are linearly independent

Similarity, Diagonalization and Applications

If A and B are two $n \times n$ matrices, then A is similar to B if there is an invertible matrix P such that $A = PBP^{-1}$

If $n \times n$ matrices A and B are similar, then:

- $detA = detB$
- A and B have the same characteristic polynomials and therefore the same eigenvalues (with the same multiplicities)

Matrix A is said to be diagonizable if and only if A has n linearly independent eigenvectors

If an $n \times n$ matrix A has n distinct real eigenvalues, then A is diagonizable

If an $n \times n$ matrix A has n real eigenvalues (including multiplicity) where $\lambda_1, \dots, \lambda_p$ (p<n) are distinct, then A is diagonizable if and only if the dimension of eigenspace corresponding to λ_i equals the multiplicity of λ_i or, sum of dimension of all eigenspace corresponding to $\lambda_1, \dots, \lambda_p$ equals to n

$$nullity(A - \lambda_1 I) + \cdots + nullity(A - \lambda_p I) = n$$

If an $n \times n$ matrix A has less than. N linearly independent eigenvectors then A is NOT digonizable

Diagonalization of A: $A = PDP^{-1}$ where $P = [v_1 \quad \cdots \quad v_n]$ and $D = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$ where $v_1, \dots, v_n$ are n linearly independent eigenvectors corresponding to eigenvalues $\lambda_1, \dots, \lambda_p$

If $A = PDP^{-1}$ $then$ $A^k = PD^k P^{-1}$

A complex number $z = a \pm bi$ is equivalent to a point (a, b) in $\mathbb{R}^2$

- Polar coordinate $z = r(cos\phi + isin\phi) = re^{i\phi} = rcos\phi$ and $b = sin\phi$

Complex vector- if $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbb{C}^n$ where n entries $v_1, \dots, v_n$ are complex numbers where $v = Re(v) + i * Im(v) = \begin{bmatrix} Re(v_1) \\ \vdots \\ Re(v_n) \end{bmatrix} + i \begin{bmatrix} Im(v_1) \\ \vdots \\ Im(v_n) \end{bmatrix}$

Application to Differential Equations, Initial Value problem Gallery of Solution Curves

Let A be a 2×2 matrix. Supposed that vector x_0 in $\mathbb{R}^2$ is given. The initial value problem is defined as

> $x'(t) = Ax(t)$ and $x(0) = x_0$ where $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$. Solving this equation means fined vector x(t) that satisfies both equations

1. Find the general solution of the equation $x'(t) = Ax(t)$ which depends on two parameters c_1, c_2

2. Apply the initial condition to find the exact values for c_1, c_2

Case 1: matrix A has two distinct eigenvalues $\lambda_{<\lambda_2}$ with corresponding eigenvectors v_1, v_2

- The general solution of equation $x'(t) = Ax(t)$ is $x(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$
- Apply the initial condition $x(0) = x_0$ gives: $x_0 = c_1 v_1 + c_2 v_2$ or $\left(\begin{bmatrix} v_1 & v_2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = x_0\right)$
- Gallery of Solutions
  - If $\lambda_1 < 0 < \lambda_2$: The origin is a saddle point
  - If $0 < \lambda_1 < \lambda_2$: The origin is a repelled (or source)
  - If $\lambda_1 < \lambda_2 < 0$: The origin is an attractor (or sink)

Case 2: matrix A has two complex eigenvalues $\lambda = a + bi$ and $\bar{z} = a - bi$. Let $v = u + iw$ be the eigenvector corresponding the eigenvector $\lambda = a + bi$

- The REAL solution of $x'(t) = Ax(t)$ is $x(t) = c_1 e^{at}(ucosbt - wsinbt) + c_2 e^{at}(ucosbt + wsinbt)$
- Apply the initial condition $x(0) = x_0$ gives: $x_0 = c_1 u + c_2 w$ or $\left(\begin{bmatrix} u & w \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = x_0\right)$
- Gallery of Solutions
  - If $a \neq 0$: The origin is a spiral point
  - If $a = 0$: the origin is a center

Inner product In $\mathbb{R}^n$, orthogonality, orthogonal sets, orthogonal projects, the best approximations, the Gram-Schmidt process

Inner product in $\mathbb{R}^n$: Let u and v be two column vectors in $\mathbb{R}^n$, the inner product (or dot product) of u and v, denoted by $u \cdot v = u^T \cdot v = u_1 v_1 + \cdots + u_n v_n$

The dot product satisfies 4 axions of an inner product in general:

- $u \cdot v = v \cdot u$
- $(u + v) \cdot w = u \cdot w + v \cdot w$
- $(cu) \cdot v = c(u \cdot v) = u \cdot (cv)$
- $u \cdot u \geq 0$ and $u \cdot u = 0$ if and only if $u = 0$

Length (norm) of a vector is the nonnegative scalar $\|v\| = \sqrt{v \cdot v}$

- $\|cv\| = |c|\|v\|$

Distance between u and v in $\mathbb{R}^n$ the distance between u and v $dist(u, v)$ is the
Two vectors u and v in $\mathbb{R}^n$ are orthogonal if $u \cdot v = 0$

- We can also check this by $\|u + v\|^2 = \|u\|^2 + \|v\|^2$ (Pythagorean theorem)

Orthogonal Complements ($W^\perp$)- the set of all vectors that ae orthogonal to W

- A vector x is in $W^\perp$ if and only if x is orthogonal to every vector in a set that spans W
- $W^\perp$ is a subspace of $\mathbb{R}^n$
- Let A be an $m \times n$ matric. The orthogonal complement of the row space of A is the null space of A and the orthogonal complement of the column space of A is the null space of A^T
    - $(RowA)^T = NulA$
    - $(ColA)^T = NulA^T$

Orthogonal Set- a set $\{v_1, \dots, v_p\}$ in $\mathbb{R}^n$ is said to be an orthogonal set if $v_i \cdot v_j = 0$ whenever $i \neq j$

Denote $S = \{v_1, \dots, v_p\}$ a set in $\mathbb{R}^n$ and $W = span\{S\}$ if the set is orthogonal then:

- S is linearly independent and thus S is an orthogonal basis for W
- For each vector w in W

$$w = \frac{w \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{w \cdot v_2}{v_2 \cdot v_2} v_2 + \dots + \frac{w \cdot v_n}{v_n \cdot v_n} v_n$$

Orthonormal set- a set $\{u_1, \dots, u_p\}$ is $\mathbb{R}^n$ is said to be orthonormal set if it is an orthogonal set of all unit vectors

- By normalizing all vectors in an orthogonal set one obtains the orthonormal set
- Let $S = \{u_1, \dots, u_p\}$ be an orthonormal set in $\mathbb{R}^n$ and let $U = [u_1 \quad \cdots \quad u_p]$
    - $U^T U = I$
    - $\|Ux\| = \|x\|$ for each x in $\mathbb{R}^n$
    - $(Ux) \cdot (Uy) = x \cdot y$ for each x and y in $\mathbb{R}^n$
    - $(Ux) \cdot (Uy) = 0$ if and only if $x \cdot y = 0$

Orthogonal projections- let W be a p-dimensional subspace of $\mathbb{R}^n$. If $\{v_1, \dots, v_p\}$ is an orthogonal bases of W, then the orthogonal projection of vector y in $\mathbb{R}^n$ onto the subspace Y is denoted by $\hat{y} = proj_W y = \frac{y \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{y \cdot v_2}{v_2 \cdot v_2} v_2 + \dots + \frac{y \cdot v_p}{v_p \cdot v_p} v_p$

- Orthogonal Decomposition Theorem- let W be a subspace of $\mathbb{R}^n$ Then each y in $\mathbb{R}^n$ can be decomposed uniquely in the form $y = \hat{y} + z$ where $\hat{y}$ s the orthogonal projection of y onto W and $z = y - \hat{y}$ belongs to $W^{\perp}$ (z is orthogonal to $\hat{y}$
- Best Approximation- let W be a subspace of $\mathbb{R}^n$ and let y be any vector in $\mathbb{R}^n$. Then the orthogonal projection of y onto W (denoted by $\hat{y}$) is the closest point in W to y

The Gram-Schmidt Process- Let W be a nonzero subspace of $\mathbb{R}^n$ and a basis $\{x_1, \dots, x_p\}$:

$$v_1 = x_1$$

$$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 * v_1} v_1$$

$$v_3 = x_3 - \frac{x_3 \cdot v_1}{v_1 * v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 * v_2} v_2$$

$$\dots$$

$$v_p = x_p - \frac{x_p \cdot v_1}{v_1 * v_1} v_1 - \frac{x_p \cdot v_2}{v_2 * v_2} v_2 - \dots - \frac{x_p \cdot v_{p-1}}{v_{p-1} * v_{p-1}} v_{p-1}$$

Then $\{v_1, \dots, v_p\}$ is an orthogonal basis for W. Furthermore $span\{v_1, \dots, v_k\} = span\{x_1, \dots, x_k\}$ for $1 \le k \le p$

Least-Square Problems

If A is an $m \times n$ matrix and b is in $\mathbb{R}^m$, a least-squares solution of $Ax - b$ is an $\hat{x}$ in $\mathbb{R}^n$ such that $\|b - A\hat{x}\| \le \|b - Ax\|$ for all x in $\mathbb{R}^n$

The set of least-squares solutions of $Ax = b$ coincides with the nonempty set of solutions of the normal equation $A^T Ax = A^T b$

Let A be an $m \times n$ matrix. The following statements are logically equivalent

- Equation $Ax = b$ has a unique squares solution
- The columns of A are linearly independent
- The matrix $A^T A$ is invertible

When these statements are true the least squares solution $\hat{\hat{x}} = (A^T A)^{-1} A^T b$

If A is a $p \times n$ matrix, then

- In order to find least-squares solution of equation $Ax = b$ we solve $A^T ax = A^T b$
- If columns of A form an orthogonal set, then the least squares solution is given by $\hat{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_2 \end{bmatrix}$ where $x_j = \frac{b \cdot a_j}{a_j \cdot a_j}$ and a_j is the jth column of A

Inner product spaces

An inner product on a vector space V is a function that to each pair ofr vectors u and v, associated a real number $\langle u, v \rangle$ and satisfies the following axioms for all u, v, w in V and scalar c

- $\langle u, v \rangle = \langle v, u \rangle$
- $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
- $\langle cu, v \rangle = c\langle u, v \rangle$
- $\langle u, u \rangle \geq 0$ and $\langle u, v \rangle = 0$ if and only if $u = 0$

A vector space with an inner product is called an inner product space

The orthogonal projection, the best approximation, and the Gram Schmidt process are similar to $\mathbb{R}^n$ (replace the dot product by an inner product $\langle u, v \rangle$ and replace $\mathbb{R}^n$ by a vector space V). That is, let V be an n-dimensional vector space with an inner product $\langle u, v \rangle$. Let W be a subspace of V and $\{v_1, \dots, v_p\}$ be an orthogonal basis for W. Then:

- For each vector u in W, the orthogonal projection of u onto W is $\hat{u} = \frac{\langle u, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 + \dots + \frac{\langle u, v_p \rangle}{\langle v_p, v_p \rangle} v_p$
- The best approximation to u by a vector in W is $\hat{u}$ (the projection of u onto W)
- Let W be aa subspace of an inner product space V with an inner product $\langle u, v \rangle$ and $\{x_1, \dots, x_p\}$ be a basis for W. The Gram-Schmidt process to construct an orthogonal basis $\{v_1, \dots, v_p\}$ for W is:

$$v_1 = x_1$$

$$v_2 = x_2 - \frac{\langle u, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$$

$$\dots$$

$$v_p = x_p - \frac{\langle u, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle u, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 - \dots - \frac{\langle u, v_{p-1} \rangle}{\langle v_{p-1}, v_{p-1} \rangle} v_{p-1}$$

The Cauchy-Schwartz Inequality- $\|\langle u, v\rangle\| < \|u\| * \|v\|$ for all u, v in V

The Triangle Inequality- for all u and v $\|u+v\| \leq \|u\| + \|v\|$

- $\|u+v\|^2 \leq \|u\|^2 + \|v\|^2$ is NOT TRUE but $\|u+v\|^2 \leq 2(\|u\|^2 + \|v\|^2)$

Diagonalization of Symmetric Matrices

A symmetric matrix is a square matrix A such that $A^T = A$ $(a_{ij} = a_{ji})$

Diagonalize

- $\det(A - \lambda I) = 0$ to find λ (eigenvalues)
- $(A - \lambda I)\vec{v} = 0$ to find $\vec{v}$ (eigenvector)
- Normalize eigenvectors

The eigenvectors form different eigenvalues are orthogonal (always true for symmetric matrix)

Orthogonal matrix- columns of matrix are orthonormal

- Its transpose is its inverse $P^T = P^{-1}$
- $PDP^T = PDP^{-1}$

Orthogonally diagonizable- when a matrix A can be diagonalized, so P is orthogonal

Matrix a is orthogonally diagonizable if and only if A is symmetric

- Dimension of eigenspace=multiplicity of eigenvalues
- Make sure all eigenvectors are orthogonal (gram-Schmidt process)

If A is symmetric $A^T = A$ then $A = PDP^{-1} = PDP^T$ where P is the orthonormal eigenvectors of A and D is the diagonal matrix when eigenvalues on the diagonal

Properties

- If A is symmetric then $(A\vec{x}) \cdot \vec{y} = \vec{x} \cdot (A\vec{y})$
- If A is symmetric then A^k is symmetric

Spectral decomposition- set of eigenvectors is a spectrum of A

$$A = PDP^T = [u_1 \quad \dots \quad u_n] \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} u_1^T \\ \vdots \\ u_n^T \end{bmatrix} = \lambda_1 u_1 u_1^T + \cdots + \lambda_n u_n u_n^T$$

$$Ax = \lambda_1 u_1 u_1^T \vec{x} + \cdots + \lambda_n u_n u_n^T \vec{x}$$

$u_1 u_1^T c$ is projectioin of $\vec{x}$ onto subspace spanned by u_1