Title: Calculus Made Easy
Being a very-simplest introduction to those beautiful
methods which are generally called by the terrifying names
of the Differentia

Author: Silvanus Thompson

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CÁLCULO FÁCIL

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TORONTO

CÁLCULO FÁCIL:

SIENDO UNA INTRODUCCIÓN MUY SIMPLE A ESOS HERMOSOS MÉTODOS DE CÁLCULO QUE GENERALMENTE SE LLAMAN CON LOS NOMBRES TERRORÍFICOS DEL

CÁLCULO DIFERENCIAL

Y EL

CÁLCULO INTEGRAL.

BY

F. R. S.

SECOND EDITION, ENLARGED

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Lo que un tonto puede hacer, otro también puede hacerlo.

(*Ancient Simian Proverb.*)

PREFACIO A LA SEGUNDA EDICIÓN.

El sorprendente éxito de esta obra ha llevado al autor a agregar un número considerable de ejemplos resueltos y ejercicios. También se ha aprovechado la oportunidad para ampliar ciertas partes donde la experiencia mostró que explicaciones adicionales serían útiles.

El autor reconoce con gratitud las muchas valiosas sugerencias y cartas recibidas de profesores, estudiantes y—críticos.

Octubre, 1914.

CONTENIDO.

PRÓLOGO.

Teniendo en cuenta cuántos tontos saben calcular, es sorprendente que se considere una tarea difícil o tediosa para cualquier otro tonto aprender a dominar los mismos trucos.

Algunos trucos de cálculo son bastante fáciles. Otros son enormemente difíciles. Los tontos que escriben los libros de texto de matemáticas avanzadas, y en su mayoría son tontos inteligentes, rara vez se molestan en mostrarte lo fáciles que son los cálculos fáciles. Por el contrario, parecen desear impresionarte con su tremenda inteligencia abordándolos de la manera más difícil.

Siendo yo mismo un tipo notablemente estúpido, he tenido que desaprender las dificultades, y ahora ruego presentar a mis compañeros tontos las partes que no son difíciles. Domínalas a fondo y el resto vendrá por sí solo. Lo que un tonto puede hacer, otro también.

CAPÍTULO I.

PARA LIBRARTE DE LOS TERRORES PRELIMINARES.

El terror preliminar, que impide a la mayoría de los chicos de quinto curso incluso intentar aprender a calcular, puede abolirse de una vez por todas simplemente explicando cuál es el significado, en términos de sentido común, de los dos símbolos principales que se utilizan en el cálculo.

Estos temibles símbolos son:

(1) d que simplemente significa "un poco de."

Así dx significa un poco de x; o du significa un poco de u. Los matemáticos ordinarios piensan que es más educado decir "un elemento de," en lugar de "un poco de." Como gustes. Pero encontrarás que estos pequeños pedazos (o elementos) pueden considerarse como indefinidamente pequeños.

(2) $\int$ que es simplemente una S larga, y puede llamarse (si gustas) "la suma de."

Así $\int dx$ significa la suma de todos los pequeños pedazos de x; o $\int dt$ significa la suma de todos los pequeños pedazos de t. Los matemáticos ordinarios llaman a este símbolo "la integral de." Ahora cualquier tonto puede ver que si x se considera como compuesta de muchos pequeños pedazos, cada uno de los cuales se llama dx, si los sumas todos juntos obtienes la suma de todas las dx's, (que es lo mismo que todo el x). La palabra "integral" simplemente significa "el todo." Si piensas en la duración del tiempo durante una hora, puedes (si gustas) pensar en ella como cortada en 3600 pequeños pedazos llamados segundos. El total de los 3600 pequeños pedazos sumados juntos hacen una hora.

Cuando veas una expresión que comience con este símbolo terrorífico, de ahora en adelante sabrás que está puesto allí simplemente para darte instrucciones de que ahora debes realizar la operación (si puedes) de sumar todos los pequeños pedazos que se indican por los símbolos que siguen.

Eso es todo.

CAPÍTULO II.

SOBRE DIFERENTES GRADOS DE PEQUEÑEZ.

Encontraremos que en nuestros procesos de cálculo debemos tratar con pequeñas cantidades de varios grados de pequeñez.

También tendremos que aprender bajo qué circunstancias podemos considerar que las pequeñas cantidades son tan diminutas que podemos omitirlas de la consideración. Todo depende de la pequeñez relativa.

Antes de fijar cualquier regla, pensemos en algunos casos familiares. Hay 60 minutos en la hora, 24 horas en el día, 7 días en la semana. Por lo tanto hay 1440 minutos en el día y 10080 minutos en la semana.

Obviamente 1 minuto es una cantidad muy pequeña de tiempo comparada con una semana entera. De hecho, nuestros antepasados la consideraban pequeña comparada con una hora, y la llamaban "un minùte," significando una fracción diminuta—a saber, una sesentava parte—de una hora. Cuando llegaron a requerir subdivisiones aún más pequeñas del tiempo, dividieron cada minuto en 60 partes aún más pequeñas, que, en los días de la Reina Isabel, llamaron "segundos minùtes" (*i.e.* pequeñas cantidades del segundo orden de pequeñez). Hoy en día llamamos a estas pequeñas cantidades del segundo orden de pequeñez "segundos." Pero poca gente sabe *por qué* se llaman así.

Ahora, si un minuto es tan pequeño comparado con un día entero,

¡cuánto más pequeño en comparación es un segundo!

De nuevo, piensa en un cuarto de penique comparado con una libra esterlina: apenas vale más de $\frac{1}{1000}$ parte. Un cuarto de penique más o menos es de muy poca importancia comparado con una libra esterlina: ciertamente puede considerarse como una cantidad *pequeña*. Pero compara un cuarto de penique con £1000: relativamente a esta suma mayor, el cuarto de penique no tiene más importancia que $\frac{1}{1000}$ de un cuarto de penique tendría para una libra esterlina. Incluso una libra esterlina de oro es relativamente una cantidad despreciable en la riqueza de un millonario.

Ahora, si fijamos cualquier fracción numérica como constituyendo la proporción que para cualquier propósito llamamos relativamente pequeña, podemos fácilmente establecer otras fracciones de un grado superior de pequeñez. Así, si, para el propósito del tiempo, $\frac{1}{60}$ se llama una fracción *pequeña*, entonces $\frac{1}{60}$ de $\frac{1}{60}$ (siendo una fracción *pequeña* de una fracción *pequeña*) puede considerarse como una *cantidad pequeña del segundo orden* de pequeñez.*

O, si para cualquier propósito fuéramos a tomar 1 por ciento (*i.e.* $\frac{1}{100}$) como una fracción *pequeña*, entonces 1 por ciento de 1 por ciento (*i.e.* $\frac{1}{10{,}000}$) sería una fracción pequeña del segundo orden de pequeñez; y $\frac{1}{1{,}000{,}000}$ sería una fracción pequeña del tercer orden de pequeñez, siendo 1 por ciento de 1 por ciento de 1 por ciento.

Por último, supongamos que para algún propósito muy preciso de-

*Los matemáticos hablan sobre el segundo orden de "magnitud" (*i.e.* grandeza) cuando realmente quieren decir segundo orden de *pequeñez*. Esto es muy confuso para los principiantes.

beríamos considerar $\frac{1}{1{,}000{,}000}$ como "pequeño." Así, si un cronómetro de primera clase no debe perder o ganar más de medio minuto en un año, debe mantener el tiempo con una precisión de 1 parte en $1,051,200$. Ahora, si para tal propósito, consideramos $\frac{1}{1{,}000{,}000}$ (o una millonésima) como una cantidad pequeña, entonces $\frac{1}{1{,}000{,}000}$ de $\frac{1}{1{,}000{,}000}$, es decir $\frac{1}{1{,}000{,}000{,}000{,}000}$ (o una billonésima) será una cantidad pequeña del segundo orden de pequeñez, y puede despreciarse por completo, en comparación.

Entonces vemos que cuanto más pequeña es una cantidad pequeña en sí misma, más despreciable se vuelve la cantidad pequeña correspondiente del segundo orden. Por lo tanto sabemos que *en todos los casos estamos justificados en despreciar las cantidades pequeñas del segundo—o tercer* (o superior)*—órdenes*, si solo tomamos la cantidad pequeña del primer orden suficientemente pequeña en sí misma.

Pero, debe recordarse, que las cantidades pequeñas si ocurren en nuestras expresiones como factores multiplicados por algún otro factor, pueden volverse importantes si el otro factor es en sí mismo grande. Incluso un cuarto de penique se vuelve importante si solo se multiplica por unos pocos cientos.

Ahora en el cálculo escribimos dx para un pequeño pedazo de x. Estas cosas como dx, y du, y dy, se llaman "diferenciales," la diferencial de x, o de u, o de y, según sea el caso. [Las *lees* como *de-equis*, o *de-u*, o *de-i griega*.] Si dx es un pequeño pedazo de x, y relativamente pequeño en sí mismo, no se sigue que cantidades como $x \cdot dx$, o $x^2\,dx$, o $a^x\,dx$ sean despreciables. Pero $dx \times dx$ sería despreciable, siendo una cantidad pequeña del segundo orden.

Un ejemplo muy simple servirá como ilustración.

Pensemos en x como una cantidad que puede crecer por una pequeña cantidad para convertirse en $x+dx$, donde dx es el pequeño incremento añadido por el crecimiento. El cuadrado de esto es $x^2 + 2x \cdot dx + (dx)^2$. El segundo término no es despreciable porque es una cantidad de primer orden; mientras que el tercer término es del segundo orden de pequeñez, siendo un pedazo de un pedazo de x^2. Así, si tomáramos dx signifique numéricamente, digamos, $\frac{1}{60}$ de x, entonces el segundo término sería $\frac{2}{60}$ de x^2, mientras que el tercer término sería $\frac{1}{3600}$ de x^2. Este último término es claramente menos importante que el segundo. Pero si vamos más lejos y tomamos dx para significar solo $\frac{1}{1000}$ de x, entonces el segundo término será $\frac{2}{1000}$ de x^2, mientras que el tercer término será solo $\frac{1}{1{,}000{,}000}$ de x^2.

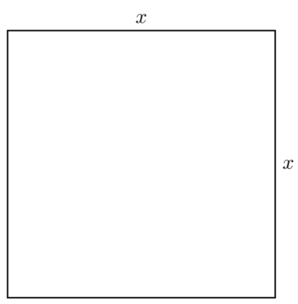


Fig. 1.

Geométricamente esto puede representarse de la siguiente manera: Dibuje un cuadrado (Fig. 1) cuyo lado tomaremos para representar x. Ahora supongamos que el cuadrado crece agregando un poco dx a su tamaño en cada dirección. El cuadrado ampliado está compuesto del cuadrado original x^2, los dos rectángulos en la parte superior y a la derecha, cada uno de los cuales tiene un área $x \cdot dx$ (o juntos $2x \cdot dx$), y el pequeño cuadrado en la esquina superior derecha que es $(dx)^2$. En Fig. 2 hemos tomado dx como una fracción bastante grande de

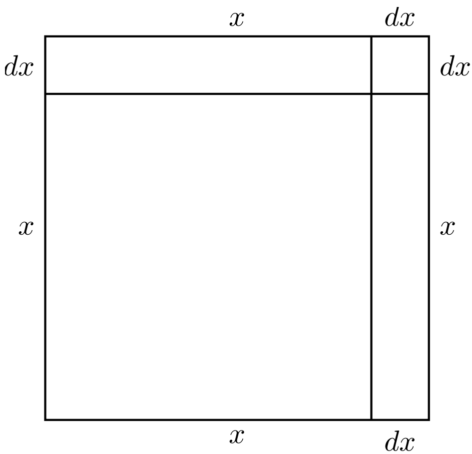


Fig. 2.

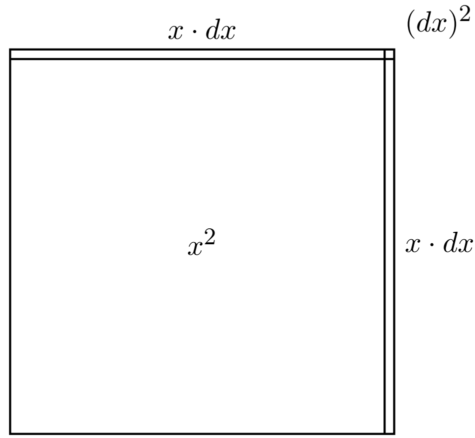


Fig. 3.

x—aproximadamente $\frac{1}{5}$. Pero supongamos que la hubiéramos tomado solo como $\frac{1}{100}$—aproximadamente el grosor de una línea entintada dibujada con una pluma fina. Entonces el pequeño cuadrado de la esquina tendría un área de solo $\frac{1}{10{,}000}$ de x^2, y sería prácticamente invisible. Claramente $(dx)^2$ es despreciable si solo consideramos que el incremento dx sea en sí mismo suficientemente pequeño.

Consideremos un símil.

Supongamos que un millonario fuera a decir a su secretario: la próxima semana te daré una pequeña fracción de cualquier dinero que me llegue. Supongamos que el secretario fuera a decir a su muchacho: te daré una pequeña fracción de lo que recibo. Supongamos que la fracción en cada caso sea $\frac{1}{100}$ parte. Ahora, si el Sr. Millonario recibiera durante la próxima semana £1000, el secretario recibiría £10 y el muchacho 2 chelines. Diez libras sería una cantidad pequeña comparada con £1000; pero dos chelines es una cantidad muy pequeña de hecho, de un orden muy secundario. Pero ¿cuál sería la desproporción si la fracción, en lugar de ser $\frac{1}{100}$, hubiera sido establecida en $\frac{1}{1000}$ parte? Entonces, mientras el Sr. Millonario obtuvo sus £1000, el Sr. Secretario obtendría solo £1, ¡y el muchacho menos de un cuarto de penique!

El ingenioso Deán Swift* una vez escribió:

> "Así, los Nat'ralistas observan, una Pulga
> "Tiene Pulgas menores que la devoran.
> "Y estas tienen Pulgas menores que las muerdan,
> "Y así prosiguen *ad infinitum.*"

Un buey podría preocuparse por una pulga de tamaño ordinario—una pequeña criatura del primer orden de pequeñez. Pero probablemente no se molestaría por la pulga de una pulga; siendo del segundo orden de pequeñez, sería despreciable. Incluso una gruesa de pulgas de pulgas no sería de mucha importancia para el buey.

* *On Poetry: a Rhapsody* (p. 20), impreso en 1733—usualmente mal citado.

CAPÍTULO III.

SOBRE CRECIMIENTOS RELATIVOS.

A través de todo el cálculo estamos tratando con cantidades que están creciendo, y con tasas de crecimiento. Clasificamos todas las cantidades en dos clases: *constantes* y *variables*. Aquellas que consideramos de valor fijo, y llamamos *constantes*, generalmente las denotamos algebraicamente por letras del principio del alfabeto, tales como a, b, o c; mientras que aquellas que consideramos como capaces de crecer, o (como dicen los matemáticos) de "variar," las denotamos por letras del final del alfabeto, tales como x, y, z, u, v, w, o a veces t.

Además, usualmente estamos tratando con más de una variable a la vez, y pensando en la manera en que una variable depende de la otra: por ejemplo, pensamos en la manera en que la altura alcanzada por un proyectil depende del tiempo de alcanzar esa altura. O se nos pide considerar un rectángulo de área dada, e investigar cómo cualquier aumento en la longitud de él obligará a una disminución correspondiente en el ancho de él. O pensamos en la manera en que cualquier variación en la inclinación de una escalera causará que la altura que alcanza, varíe.

Supongamos que tenemos dos variables tales que dependen una de la otra. Una alteración en una traerá una alteración en la otra, *debido* *a* esta dependencia. Llamemos a una de las variables x, y a la otra que depende de ella y.

Supongamos que hacemos que x varíe, es decir, la alteramos o imaginamos que se altera, agregándole un poco que llamamos dx. Así estamos causando que x se convierta en $x+dx$. Entonces, porque x ha sido alterada, y también habrá cambiado, y se habrá convertido en $y+dy$. Aquí el poco dy puede ser en algunos casos positivo, en otros negativo; y no será (excepto por un milagro) del mismo tamaño que dx.

Tomemos dos ejemplos.

(1) Sean x e y respectivamente la base y la altura de un triángulo rectángulo (Fig. 4), del cual la pendiente del otro lado está fija en 30°.

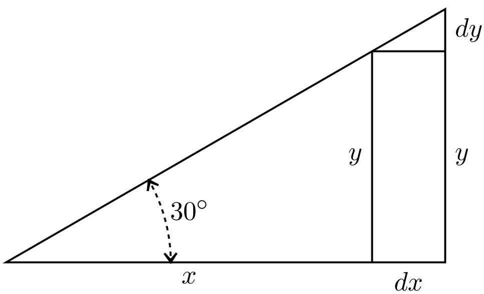


Fig. 4.

Si suponemos que este triángulo se expande y aún mantiene sus ángulos iguales que al principio, entonces, cuando la base crece para convertirse en $x+dx$, la altura se convierte en $y+dy$. Aquí, aumentar x resulta en un aumento de y. El pequeño triángulo, cuya altura es dy, y cuya base es dx, es similar al triángulo original; y es obvio que el valor de la razón $\dfrac{dy}{dx}$ es el mismo que el de la razón $\dfrac{y}{x}$. Como el ángulo es 30° se verá que aquí

$$\frac{dy}{dx} = \frac{1}{1.73}.$$

(2) Sea x representar, en Fig. 5, la distancia horizontal, desde una pared, del extremo inferior de una escalera, AB, de longitud fija; y sea

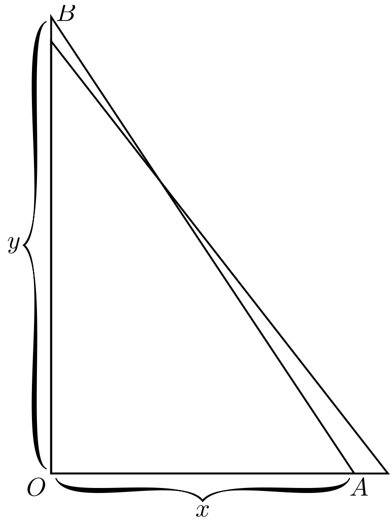


Fig. 5.

y la altura que alcanza en la pared. Ahora y claramente depende de x. Es fácil ver que, si tiramos del extremo inferior A un poco más lejos de la pared, el extremo superior B bajará un poco más. Expresemos esto en lenguaje científico. Si aumentamos x a $x + dx$, entonces y se convertirá en $y - dy$; es decir, cuando x recibe un incremento positivo el incremento que resulta en y es negativo.

Sí, ¿pero cuánto? Supongamos que la escalera era tan larga que cuando el extremo inferior A estaba a 19 pulgadas de la pared, el extremo superior B alcanzaba justamente 15 pies del suelo. Ahora, si fueras a tirar del extremo inferior 1 pulgada más, ¿cuánto bajaría el extremo superior? Pongámoslo todo en pulgadas: $x = 19$ pulgadas, $y = 180$ pulgadas. Ahora el incremento de x que llamamos dx, es 1 pulgada: o $x + dx = 20$ pulgadas.

¿Cuánto disminuirá y? La nueva altura será $y - dy$. Si calculamos la altura por Euclides I. 47, entonces podremos encontrar cuánto será dy. La longitud de la escalera es

$$\sqrt{(180)^2 + (19)^2} = 181 \text{ pulgadas.}$$

Claramente entonces, la nueva altura, que es $y - dy$, será tal que

$$(y - dy)^2 = (181)^2 - (20)^2 = 32761 - 400 = 32361,$$
$$y - dy = \sqrt{32361} = 179.89 \text{ pulgadas.}$$

Ahora y es 180, así que dy es $180 - 179.89 = 0.11$ pulgada.

Así vemos que hacer dx un aumento de 1 pulgada ha resultado en hacer dy una disminución de 0.11 pulgada.

Y la razón de dy a dx puede expresarse así:

$$\frac{dy}{dx} = -\frac{0.11}{1}.$$

También es fácil ver que (excepto en una posición particular) dy será de un tamaño diferente al de dx.

Ahora, a través de todo el cálculo diferencial estamos buscando, buscando, buscando una cosa curiosa, una mera razón, a saber, la proporción que dy tiene con dx cuando ambas son indefinidamente pequeñas.

Debe notarse aquí que solo podemos encontrar esta razón $\frac{dy}{dx}$ cuando y y x están relacionadas entre sí de alguna manera, de modo que siempre que x varíe y también varíe. Por ejemplo, en el primer ejemplo recién tomado, si la base x del triángulo se hace más larga, la altura y del triángulo también se hace mayor, y en el segundo ejemplo, si la distancia x del pie de la escalera desde la pared se hace aumentar, la altura y alcanzada por la escalera disminuye de una manera correspondiente, lentamente al principio, pero más y más rápidamente a medida que x se hace mayor. En estos casos la relación entre x y y es perfectamente definida, puede expresarse matemáticamente, siendo $\frac{y}{x} = \tan 30^\circ$ y $x^2 + y^2 = l^2$ (donde l es la longitud de la escalera) respectivamente, y $\frac{dy}{dx}$ tiene el significado que encontramos en cada caso.

Si, mientras x es, como antes, la distancia del pie de la escalera desde la pared, y es, en lugar de la altura alcanzada, la longitud horizontal de la pared, o el número de ladrillos en ella, o el número de años desde que fue construida, cualquier cambio en x naturalmente no causaría ningún cambio en y; en este caso $\frac{dy}{dx}$ no tiene ningún significado, y no es posible encontrar una expresión para ello. Cuando usamos diferenciales dx, dy, dz, etc., se implica la existencia de algún tipo de relación entre x, y, z, etc., y esta relación se llama una "función" en x, y, z, etc.; las dos expresiones dadas arriba, por ejemplo, a saber $\frac{y}{x} = \tan 30^\circ$ y $x^2+y^2 = l^2$, son funciones de x y y. Tales expresiones contienen implícitamente (es decir, contienen sin mostrarlo distintamente) los medios para expresar ya sea x en términos de y o y en términos de x, y por esta razón se llaman *funciones implícitas* en x y y; pueden ponerse respectivamente en las formas

$$y = x \tan 30^\circ \quad \text{o} \quad x = \frac{y}{\tan 30^\circ}$$

y

$$y = \sqrt{l^2 - x^2} \quad \text{o} \quad x = \sqrt{l^2 - y^2}.$$

Estas últimas expresiones establecen explícitamente (es decir, distintamente) el valor de x en términos de y, o de y en términos de x, y por esta razón se llaman *funciones explícitas* de x o y. Por ejemplo $x^2+3=2y-7$ es una función implícita en x y y; puede escribirse como $y=\dfrac{x^2+10}{2}$ (función explícita de x) o $x=\sqrt{2y-10}$ (función explícita de y). Vemos que una función explícita en x, y, z, etc., es simplemente algo cuyo valor cambia cuando x, y, z, etc., están cambiando, ya sea uno a la vez o varios juntos. Debido a esto, el valor de la función explícita se llama la *variable dependiente*, ya que depende de las valor de las otras cantidades variables en la función; estas otras variables se llaman las *variables independientes* porque su valor no está determinado por el valor asumido por la función. Por ejemplo, si $u=x^2\sin\theta$, x y θ son las variables independientes, y u es la variable dependiente.

A veces la relación exacta entre varias cantidades x, y, z no se conoce o no es conveniente establecerla; solo se sabe, o es conveniente establecer, que hay algún tipo de relación entre estas variables, de modo que uno no puede alterar ni x ni y ni z individualmente sin afectar las otras cantidades; la existencia de una función en x, y, z se indica entonces por la notación $F(x,y,z)$ (función implícita) o por $x=F(y,z)$, $y=F(x,z)$ o $z=F(x,y)$ (función explícita). A veces se usa la letra f o ϕ en lugar de F, de modo que $y=F(x)$, $y=f(x)$ y $y=\phi(x)$ todas significan lo mismo, a saber, que el valor de y depende del valor de x de alguna manera que no se especifica.

Llamamos a la razón $\dfrac{dy}{dx}$ "el *coeficiente diferencial* de y con respecto a x." Es un nombre científico solemne para esta cosa muy simple. Pero no vamos a asustarnos por nombres solemnes, cuando las cosas mismas son tan fáciles. En lugar de asustarnos simplemente pronunciaremos una breve maldición sobre la estupidez de dar nombres largos y trabalenguas; y, habiendo aliviado nuestras mentes, seguiremos con la cosa simple misma, a saber la razón $\frac{dy}{dx}$.

En el álgebra ordinaria que aprendiste en la escuela, siempre estabas buscando alguna cantidad desconocida que llamabas x o y; o a veces había dos cantidades desconocidas que buscar simultáneamente. Ahora tienes que aprender a ir cazando de una manera nueva; siendo el zorro ahora ni x ni y. En lugar de esto tienes que cazar este curioso cachorro llamado $\frac{dy}{dx}$. El proceso de encontrar el valor de $\frac{dy}{dx}$ se llama "diferenciar." Pero, recuerda, lo que se quiere es el valor de esta razón cuando tanto dy como dx son ellas mismas indefinidamente pequeñas. El verdadero valor del coeficiente diferencial es aquel al que se aproxima en el caso límite cuando cada uno de ellos se considera como infinitesimalmente diminuto.

Aprendamos ahora cómo ir en busca de $\frac{dy}{dx}$.

NOTA AL CAPÍTULO III.

Cómo leer Diferenciales.

Nunca está bien caer en el error escolar de pensar que dx significa d por x, porque d no es un factor—significa "un elemento de" o "un poco de" lo que sigue. Uno lee dx así: "de-equis."

En caso de que el lector no tenga a nadie que lo guíe en tales asuntos, aquí puede decirse simplemente que uno lee coeficientes diferenciales de la siguiente manera. El coeficiente diferencial

$\dfrac{dy}{dx}$ se lee "*de-i-griega entre de-equis,*" o "*de-i-griega sobre de-equis.*"

Así también $\dfrac{du}{dt}$ se lee "*de-u entre de-te.*"

Los coeficientes diferenciales de segundo orden se encontrarán más adelante. Son como esto:

$\dfrac{d^2y}{dx^2}$; que se lee "*de-dos-i-griega sobre de-equis-al-cuadrado,*"

y significa que la operación de diferenciar y con respecto a x ha sido (o tiene que ser) realizada dos veces.

Otra manera de indicar que una función ha sido diferenciada es poniendo un acento al símbolo de la función. Así si $y = F(x)$, lo que significa que y es alguna función no especificada de x (véase p. 13), podemos escribir $F'(x)$ en lugar de $\dfrac{d\big(F(x)\big)}{dx}$. Similarmente, $F''(x)$ significará que la función original $F(x)$ ha sido diferenciada dos veces con respecto a x.

CAPÍTULO IV.

CASOS MÁS SIMPLES.

Ahora veamos cómo, partiendo de primeros principios, podemos diferenciar alguna expresión algebraica simple.

Case 1.

Comencemos con la expresión simple $y = x^2$. Ahora recuerda que la noción fundamental sobre el cálculo es la idea de *crecimiento*. Los matemáticos lo llaman *variación*. Ahora, como y y x^2 son iguales entre sí, está claro que si x crece, x^2 también crecerá. Y si x^2 crece, entonces y también crecerá. Lo que tenemos que averiguar es la proporción entre el crecimiento de y y el crecimiento de x. En otras palabras, nuestra tarea es encontrar la razón entre dy y dx, o, en resumen, encontrar el valor de $\dfrac{dy}{dx}$.

Que x, entonces, crezca un poquito más y se convierta en $x + dx$; similarmente, y crecerá un poco más y se convertirá en $y+dy$. Entonces, claramente, aún será cierto que la y aumentada será igual al cuadrado de la x aumentada. Escribiendo esto, tenemos:

$$y + dy = (x + dx)^2.$$

Haciendo el cuadrado obtenemos:

$$y + dy = x^2 + 2x \cdot dx + (dx)^2.$$

¿Qué significa $(dx)^2$? Recuerda que dx significaba un pedazo—un pequeño pedazo—de x. Entonces $(dx)^2$ significará un pequeño pedazo de un pequeño pedazo de x; es decir, como se explicó arriba (p. 4), es una cantidad pequeña del segundo orden de pequeñez. Por lo tanto puede descartarse como bastante despreciable en comparación con los otros términos. Dejándolo fuera, entonces tenemos:

$$y + dy = x^2 + 2x \cdot dx.$$

Ahora $y = x^2$; entonces restemos esto de la ecuación y nos queda

$$dy = 2x \cdot dx.$$

Dividiendo entre dx, obtenemos

$$\frac{dy}{dx} = 2x.$$

Ahora *esto** es lo que nos propusimos encontrar. La razón del crecimiento de y al crecimiento de x es, en el caso que tenemos delante, $2x$.

* *N.B.*—Esta razón $\frac{dy}{dx}$ es el resultado de diferenciar y con respecto a x. Diferenciar significa encontrar el coeficiente diferencial. Supongamos que tuviéramos alguna otra función de x, como, por ejemplo, $u = 7x^2 + 3$. Entonces si nos dijeran que diferenciáramos esto con respecto a x, tendríamos que encontrar $\frac{du}{dx}$, o, lo que es lo mismo, $\frac{d(7x^2+3)}{dx}$. Por otro lado, podemos tener un caso en el que el tiempo

Ejemplo numérico.

Supongamos $x = 100$ y $\therefore\ y = 10,000$. Entonces dejemos que x crezca hasta que se convierta en 101 (es decir, sea $dx = 1$). Entonces la y ampliada será $101 \times 101 = 10,201$. Pero si acordamos que podemos ignorar pequeñas cantidades del segundo orden, 1 puede rechazarse comparado con 10,000; así que podemos redondear la y ampliada a 10,200. y ha crecido de 10,000 a 10,200; la parte agregada es dy, que por lo tanto es 200.

$\dfrac{dy}{dx} = \dfrac{200}{1} = 200$. Según el trabajo algebraico del párrafo anterior, encontramos $\dfrac{dy}{dx} = 2x$. Y así es; porque $x = 100$ y $2x = 200$.

Pero, dirás, descuidamos una unidad completa.

Bien, prueba otra vez, haciendo dx un poco aún más pequeño.

Prueba $dx = \frac{1}{10}$. Entonces $x + dx = 100.1$, y

$$(x + dx)^2 = 100.1 \times 100.1 = 10,020.01.$$

Ahora la última cifra 1 es solo una millonésima parte del 10,000, y es completamente despreciable; así que podemos tomar 10,020 sin el pequeño decimal al final. Y esto hace $dy = 20$; y $\dfrac{dy}{dx} = \dfrac{20}{0.1} = 200$, que es aún lo mismo que $2x$.

Case 2.

Trata de diferenciar $y = x^3$ de la misma manera.

Dejamos que y crezca a $y + dy$, mientras x crece a $x + dx$.

fuera la variable independiente (véase p. 14), tal como este: $y = b + \frac{1}{2}at^2$. Entonces, si nos dijeran que lo diferenciáramos, eso significa que debemos encontrar su coeficiente diferencial con respecto a t. De modo que entonces nuestro trabajo sería tratar de encontrar $\dfrac{dy}{dt}$, es decir, encontrar $\dfrac{d(b + \frac{1}{2}at^2)}{dt}$.

Entonces tenemos

$$y + dy = (x + dx)^3.$$

Haciendo el cubo obtenemos

$$y + dy = x^3 + 3x^2 \cdot dx + 3x(dx)^2 + (dx)^3.$$

Ahora sabemos que podemos despreciar pequeñas cantidades del segundo y tercer orden; ya que, cuando dy y dx se hacen ambas indefinidamente pequeñas, $(dx)^2$ y $(dx)^3$ se volverán indefinidamente más pequeñas por comparación. Así, considerándolas como despreciables, nos queda:

$$y + dy = x^3 + 3x^2 \cdot dx.$$

Pero $y = x^3$; y, restando esto, tenemos:

$$dy = 3x^2 \cdot dx,$$

and
$$\frac{dy}{dx} = 3x^2.$$

Case 3.

Trata de diferenciar $y = x^4$. Comenzando como antes dejando que tanto y como x crezcan un poco, tenemos:

$$y + dy = (x + dx)^4.$$

Desarrollando la elevación a la cuarta potencia, obtenemos

$$y + dy = x^4 + 4x^3\,dx + 6x^2(dx)^2 + 4x(dx)^3 + (dx)^4.$$

Entonces eliminando los términos que contienen todas las potencias superiores de dx, por ser despreciables en comparación, tenemos

$$y + dy = x^4 + 4x^3\, dx.$$

Restando la $y = x^4$ original, nos queda

$$dy = 4x^3\, dx,$$

and $$\frac{dy}{dx} = 4x^3.$$

Ahora todos estos casos son bastante fáciles. Recopilemos los resultados para ver si podemos inferir alguna regla general. Pongámoslos en dos columnas, los valores de y en una y los valores correspondientes encontrados para $\dfrac{dy}{dx}$ en la otra: así

y	$\dfrac{dy}{dx}$
x^2	$2x$
x^3	$3x^2$
x^4	$4x^3$

Simplemente observa estos resultados: la operación de diferenciar parece haber tenido el efecto de disminuir la potencia de x en 1 (por ejemplo en el último caso reduciendo x^4 a x^3), y al mismo tiempo multiplicar por un número (el mismo número de hecho que originalmente aparecía como la potencia). Ahora, cuando hayas visto esto una vez, podrías fácilmente conjeturar cómo los otros resultarán. Esperarías que diferenciando x^5 se obtuviera $5x^4$, o diferenciando x^6 se obtuviera $6x^5$. Si dudas, prueba uno de estos, y ve si la conjetura resulta correcta.

Prueba $y = x^5$.

$$\text{Entonces}\quad y + dy = (x + dx)^5$$
$$= x^5 + 5x^4\,dx + 10x^3(dx)^2 + 10x^2(dx)^3$$
$$+ 5x(dx)^4 + (dx)^5.$$

Despreciando todos los términos que contienen pequeñas cantidades de los órdenes superiores, nos queda

$$y + dy = x^5 + 5x^4\,dx,$$

y restando $\quad y = x^5$ nos deja

$$dy = 5x^4\,dx,$$

de donde $\quad \dfrac{dy}{dx} = 5x^4,$ exactamente como supusimos.

Siguiendo lógicamente nuestra observación, deberíamos concluir que si queremos tratar con cualquier potencia superior,—llamémosla n—podríamos abordarla de la misma manera.

Sea $\quad y = x^n,$

entonces, deberíamos esperar encontrar que

$$\frac{dy}{dx} = nx^{(n-1)}.$$

Por ejemplo, sea $n = 8$, entonces $y = x^8$; y diferenciándolo daría $\dfrac{dy}{dx} = 8x^7$.

Y, de hecho, la regla de que diferenciar x^n da como resultado nx^{n-1} es verdadera para todos los casos donde n es un número entero y positivo. [Expandir $(x + dx)^n$ por el teorema binomial mostrará esto inmediatamente.] Pero la cuestión de si es verdadero para casos donde n tiene valores negativos o fraccionarios requiere ulterior consideración.

Caso de una potencia negativa.

Sea $y = x^{-2}$. Entonces procede como antes:

$$\begin{aligned} y + dy &= (x + dx)^{-2} \\ &= x^{-2}\left(1 + \frac{dx}{x}\right)^{-2}. \end{aligned}$$

Expandiendo esto por el teorema binomial (véase p. 140), obtenemos

$$\begin{aligned} &= x^{-2}\left[1 - \frac{2\,dx}{x} + \frac{2(2+1)}{1 \times 2}\left(\frac{dx}{x}\right)^2 - \text{etc.}\right] \\ &= x^{-2} - 2x^{-3} \cdot dx + 3x^{-4}(dx)^2 - 4x^{-5}(dx)^3 + \text{etc.} \end{aligned}$$

Así, despreciando las pequeñas cantidades de órdenes superiores de pequeñez, tenemos:

$$y + dy = x^{-2} - 2x^{-3} \cdot dx.$$

Restando la $y = x^{-2}$ original, encontramos

$$\begin{aligned} dy &= -2x^{-3}dx, \\ \frac{dy}{dx} &= -2x^{-3}. \end{aligned}$$

Y esto está aún de acuerdo con la regla inferida arriba.

Caso de una potencia fraccionaria.

Sea $y = x^{\frac{1}{2}}$. Entonces, como antes,

$$y + dy = (x + dx)^{\frac{1}{2}} = x^{\frac{1}{2}}\left(1 + \frac{dx}{x}\right)^{\frac{1}{2}}$$

$$= \sqrt{x} + \frac{1}{2}\frac{dx}{\sqrt{x}} - \frac{1}{8}\frac{(dx)^2}{x\sqrt{x}} + \text{términos con potencias superiores de } dx.$$

Restando la $y = x^{\frac{1}{2}}$ original, y despreciando potencias superiores tenemos:

$$dy = \frac{1}{2}\frac{dx}{\sqrt{x}} = \frac{1}{2}x^{-\frac{1}{2}} \cdot dx,$$

y $\dfrac{dy}{dx} = \dfrac{1}{2}x^{-\frac{1}{2}}$. De acuerdo con la regla general.

Resumen. Veamos hasta dónde hemos llegado. Hemos llegado a la siguiente regla: Para diferenciar x^n, multiplica por la potencia y reduce la potencia en uno, dándonos así nx^{n-1} como resultado.

Exercises I. (Véase p. 255 para las Respuestas.)

Diferencia lo siguiente:

(1) $y = x^{13}$

(2) $y = x^{-\frac{3}{2}}$

(3) $y = x^{2a}$

(4) $u = t^{2.4}$

(5) $z = \sqrt[3]{u}$

(6) $y = \sqrt[3]{x^{-5}}$

(7) $u = \sqrt[5]{\dfrac{1}{x^8}}$

(8) $y = 2x^a$

(9) $y = \sqrt[q]{x^3}$

(10) $y = \sqrt[n]{\dfrac{1}{x^m}}$

Ahora has aprendido cómo diferenciar potencias de x. ¡Qué fácil es!

CAPÍTULO V.

SIGUIENTE ETAPA. QUÉ HACER CON LAS CONSTANTES.

En nuestras ecuaciones hemos considerado que x crece, y como resultado de hacer que x crezca, y también cambió su valor y creció. Usualmente pensamos en x como una cantidad que podemos variar; y, considerando la variación de x como una especie de *causa*, consideramos la variación resultante de y como un *efecto*. En otras palabras, consideramos que el valor de y depende del de x. Tanto x como y son variables, pero x es aquella sobre la que operamos, y y es la "variable dependiente." En todo el capítulo precedente hemos estado tratando de encontrar reglas para la proporción que la variación dependiente en y guarda con la variación independientemente hecha en x.

Nuestro siguiente paso es averiguar qué efecto sobre el proceso de diferenciar es causado por la presencia de *constantes*, es decir, de números que no cambian cuando x o y cambian sus valores.

Constantes Sumadas.

Comencemos con algún caso simple de una constante sumada, así:

Sea $$y = x^3 + 5.$$

Justo como antes, supongamos que x crece a $x+dx$ y y crece a $y+dy$.

$$\begin{aligned}\text{Entonces:}\quad y+dy &= (x+dx)^3+5\\ &= x^3+3x^2\,dx+3x(dx)^2+(dx)^3+5.\end{aligned}$$

Despreciando las pequeñas cantidades de órdenes superiores, esto se convierte en

$$y+dy = x^3+3x^2\cdot dx+5.$$

Resta la $y=x^3+5$ original, y nos queda:

$$\begin{aligned}dy &= 3x^2\,dx.\\ \frac{dy}{dx} &= 3x^2.\end{aligned}$$

Así que el 5 ha desaparecido completamente. No agregó nada al crecimiento de x, y no entra en el coeficiente diferencial. Si hubiéramos puesto 7, o 700, o cualquier otro número, en lugar de 5, habría desaparecido. Así que si tomamos la letra a, o b, o c para representar cualquier constante, simplemente desaparecerá cuando diferenciemos.

Si la constante adicional hubiera sido de valor negativo, tal como -5 o $-b$, igualmente habría desaparecido.

Constantes Multiplicadas.

Toma como un experimento simple este caso:

Sea $y=7x^2$.

Entonces al proceder como antes obtenemos:

$$\begin{aligned} y + dy &= 7(x + dx)^2 \\ &= 7\{x^2 + 2x \cdot dx + (dx)^2\} \\ &= 7x^2 + 14x \cdot dx + 7(dx)^2. \end{aligned}$$

Entonces, restando la $y = 7x^2$ original, y despreciando el último término, tenemos

$$\begin{aligned} dy &= 14x \cdot dx. \\ \frac{dy}{dx} &= 14x. \end{aligned}$$

Ilustremos este ejemplo desarrollando las gráficas de las ecuaciones $y = 7x^2$ y $\dfrac{dy}{dx} = 14x$, asignando a x un conjunto de valores sucesivos, 0, 1, 2, 3, etc., y encontrando los valores correspondientes de y y de $\dfrac{dy}{dx}$.

Estos valores los tabulamos como sigue:

x	0	1	2	3	4	5	-1	-2	-3
y	0	7	28	63	112	175	7	28	63
$\dfrac{dy}{dx}$	0	14	28	42	56	70	-14	-28	-42

Ahora grafica estos valores en alguna escala conveniente, y obtenemos las dos curvas, Figs. 6 y 6a.

Compara cuidadosamente las dos figuras, y verifica por inspección que la altura de la ordenada de la curva derivada, Fig. 6a, es proporcional a la *pendiente* de la curva original,* Fig. 6, en el valor correspondiente de x. A la izquierda del origen, donde la curva original tiene

*Véase p. 79 acerca de las *pendientes* de las curvas.

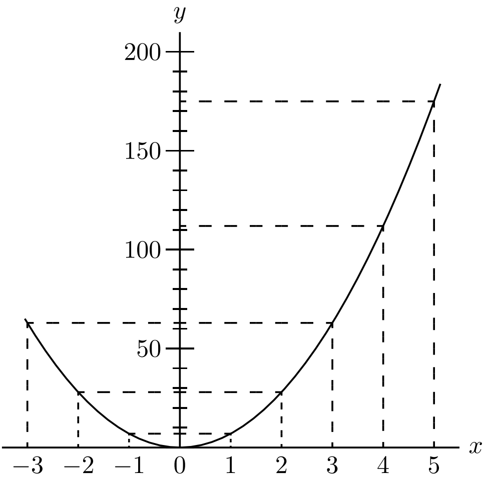


FIG. 6.—Graph of $y = 7x^2$.

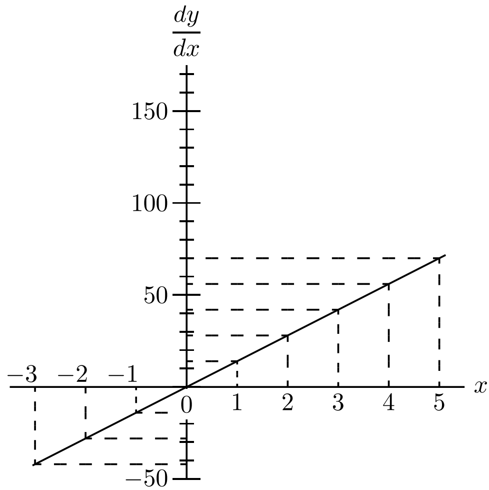


FIG. 6*a*.—Graph of $\dfrac{dy}{dx} = 14x$.

pendiente negativa (es decir, hacia abajo de izquierda a derecha) las ordenadas correspondientes de la curva derivada son negativas.

Ahora si miramos hacia atrás en p. 18, veremos que simplemente diferenciar x^2 nos da $2x$. Así que el coeficiente diferencial de $7x^2$ es justamente 7 veces tan grande como el de x^2. Si hubiéramos tomado $8x^2$, el coeficiente diferencial habría resultado ocho veces tan grande como el de x^2. Si ponemos $y = ax^2$, obtendremos

$$\frac{dy}{dx} = a \times 2x.$$

Si hubiéramos comenzado con $y = ax^n$, deberíamos haber tenido $\dfrac{dy}{dx} = a \times nx^{n-1}$. Así que cualquier mera multiplicación por una constante reaparece como una mera multiplicación cuando la cosa se diferencia. Y, lo que es verdadero acerca de la multiplicación es igualmente verdadero acerca de la *división*: porque si, en el ejemplo anterior, hubiéramos tomado como constante $\frac{1}{7}$ en lugar de 7, deberíamos haber tenido la misma $\frac{1}{7}$ salir en el resultado después de la diferenciación.

Algunos Ejemplos Adicionales.

Los siguientes ejemplos adicionales, completamente desarrollados, te permitirán dominar completamente el proceso de diferenciación tal como se aplica a las expresiones algebraicas ordinarias, y te permitirán resolver por ti mismo los ejemplos dados al final de este capítulo.

(1) Diferencia $y = \dfrac{x^5}{7} - \dfrac{3}{5}$.

$\dfrac{3}{5}$ es una constante sumada y desaparece (véase p. 26).

Podemos entonces escribir inmediatamente

$$\frac{dy}{dx} = \frac{1}{7} \times 5 \times x^{5-1},$$

or

$$\frac{dy}{dx} = \frac{5}{7}x^4.$$

(2) Diferencia $y = a\sqrt{x} - \dfrac{1}{2}\sqrt{a}$.

El término $\dfrac{1}{2}\sqrt{a}$ desaparece, siendo una constante sumada; y como $a\sqrt{x}$, en la forma de índice, se escribe $ax^{\frac{1}{2}}$, tenemos

$$\frac{dy}{dx} = a \times \frac{1}{2} \times x^{\frac{1}{2}-1} = \frac{a}{2} \times x^{-\frac{1}{2}},$$

or

$$\frac{dy}{dx} = \frac{a}{2\sqrt{x}}.$$

(3) Si $ay + bx = by - ax + (x+y)\sqrt{a^2 - b^2}$,

encuentra el coeficiente diferencial de y con respecto a x.

Como regla, una expresión de este tipo necesitará un poco más de conocimiento del que hemos adquirido hasta ahora; sin embargo, siempre vale la pena intentar si la expresión puede ponerse en una forma más simple.

Primero debemos intentar llevarla a la forma $y =$ alguna expresión que involucre solo x.

La expresión puede escribirse

$$(a-b)y+(a+b)x=(x+y)\sqrt{a^2-b^2}.$$

Elevando al cuadrado, obtenemos

$$(a-b)^2y^2+(a+b)^2x^2+2(a+b)(a-b)xy=(x^2+y^2+2xy)(a^2-b^2),$$

que se simplifica a

$$(a-b)^2y^2+(a+b)^2x^2=x^2(a^2-b^2)+y^2(a^2-b^2);$$

o
$$[(a-b)^2-(a^2-b^2)]y^2=[(a^2-b^2)-(a+b)^2]x^2,$$

es decir
$$2b(b-a)y^2=-2b(b+a)x^2;$$

por lo tanto
$$y=\sqrt{\frac{a+b}{a-b}}x \quad \text{y} \quad \frac{dy}{dx}=\sqrt{\frac{a+b}{a-b}}.$$

(4) El volumen de un cilindro de radio r y altura h está dado por la fórmula $V=\pi r^2 h$. Encuentra la tasa de variación del volumen con el radio cuando $r=5.5$ pulg. y $h=20$ pulg. Si $r=h$, encuentra las dimensiones del cilindro de modo que un cambio de 1 pulg. en el radio cause un cambio de 400 pulg. cúb. en el volumen.

La tasa de variación de V con respecto a r es

$$\frac{dV}{dr}=2\pi rh.$$

Si $r = 5.5$ pulg. y $h = 20$ pulg. esto se convierte en 690.8. Esto significa que un cambio de radio de 1 pulgada causará un cambio de volumen de 690.8 pulg. cúb. Esto puede ser fácilmente verificado, pues los volúmenes con $r = 5$ y $r = 6$ son 1570 pulg. cúb. y 2260.8 pulg. cúb. respectivamente, y $2260.8 - 1570 = 690.8$.

También, si

$$r = h, \quad \frac{dV}{dr} = 2\pi r^2 = 400 \quad \text{y} \quad r = h = \sqrt{\frac{400}{2\pi}} = 7.98 \text{ pulg.}$$

(5) La lectura θ de un pirómetro de radiación de Féry está relacionada con la temperatura Centígrada t del cuerpo observado por la relación

$$\frac{\theta}{\theta_1} = \left(\frac{t}{t_1}\right)^4,$$

donde θ_1 es la lectura correspondiente a una temperatura conocida t_1 del cuerpo observado.

Compara la sensibilidad del pirómetro a temperaturas 800° C., 1000° C., 1200° C., dado que leyó 25 cuando la temperatura era 1000° C.

La sensibilidad es la tasa de variación de la lectura con la temperatura, es decir $\frac{d\theta}{dt}$. La fórmula puede escribirse

$$\theta = \frac{\theta_1}{t_1^4} t^4 = \frac{25t^4}{1000^4},$$

y tenemos

$$\frac{d\theta}{dt} = \frac{100t^3}{1000^4} = \frac{t^3}{10,000,000,000}.$$

Cuando $t = 800$, 1000 y 1200, obtenemos $\frac{d\theta}{dt} = 0.0512$, 0.1 y 0.1728 respectivamente.

La sensibilidad se duplica aproximadamente de 800° a 1000°, y se vuelve tres cuartos tan grande hasta los 1200°.

Exercises II. (Ver p. 255 para las Respuestas.)

Diferencia las siguientes:

(1) $y = ax^3 + 6$.

(2) $y = 13x^{\frac{3}{2}} - c$.

(3) $y = 12x^{\frac{1}{2}} + c^{\frac{1}{2}}$.

(4) $y = c^{\frac{1}{2}} x^{\frac{1}{2}}$.

(5) $u = \dfrac{az^n - 1}{c}$.

(6) $y = 1.18t^2 + 22.4$.

Inventa algunos otros ejemplos por ti mismo y prueba tu habilidad para diferenciarlos.

(7) Si l_t y l_0 son las longitudes de una barra de hierro a las temperaturas t° C. y 0° C. respectivamente, entonces $l_t = l_0(1 + 0.000012t)$. Encuentra el cambio de longitud de la barra por grado centígrado.

(8) Se ha encontrado que si c es la potencia en candelas de una lámpara eléctrica incandescente, y V es el voltaje, $c = aV^b$, donde a y b son constantes.

Encuentra la tasa de cambio de la potencia en candelas con el voltaje, y calcula el cambio de potencia en candelas por voltio a 80, 100 y 120 voltios en el caso de una lámpara para la cual $a = 0.5 \times 10^{-10}$ y $b = 6$.

(9) La frecuencia n de vibración de una cuerda de diámetro D, longitud L y gravedad específica σ, tensada con una fuerza T, está dada

por

$$n = \frac{1}{DL}\sqrt{\frac{gT}{\pi\sigma}}.$$

Encuentra la tasa de cambio de la frecuencia cuando D, L, σ y T son variados individualmente.

(10) La mayor presión externa P que un tubo puede soportar sin colapsar está dada por

$$P = \left(\frac{2E}{1-\sigma^2}\right)\frac{t^3}{D^3},$$

donde E y σ son constantes, t es el espesor del tubo y D es su diámetro. (Esta fórmula asume que $4t$ es pequeño comparado con D.)

Compara la tasa a la cual P varía para un pequeño cambio de espesor y para un pequeño cambio de diámetro ocurriendo por separado.

(11) Encuentra, desde primeros principios, la tasa a la cual las siguientes varían con respecto a un cambio en el radio:

(a) la circunferencia de un círculo de radio r;

(b) el área de un círculo de radio r;

(c) el área lateral de un cono de dimensión inclinada l;

(d) el volumen de un cono de radio r y altura h;

(e) el área de una esfera de radio r;

(f) el volumen de una esfera de radio r.

(12) La longitud L de una barra de hierro a la temperatura T siendo dada por $L = l_t\big[1 + 0.000012(T-t)\big]$, donde l_t es la longitud a la temperatura t, encuentra la tasa de variación del diámetro D de una

llanta de hierro adecuada para ser encogida sobre una rueda, cuando la temperatura T varía.

CAPÍTULO VI.

SUMAS, DIFERENCIAS, PRODUCTOS Y COCIENTES.

Hemos aprendido cómo diferenciar funciones algebraicas simples tales como $x^2 + c$ o ax^4, y ahora tenemos que considerar cómo abordar la *suma* de dos o más funciones.

Por ejemplo, sea

$$y = (x^2 + c) + (ax^4 + b);$$

¿cuál será su $\dfrac{dy}{dx}$? ¿Cómo procederemos con este nuevo trabajo?

La respuesta a esta pregunta es bastante simple: simplemente diferéncialas, una tras otra, así:

$$\frac{dy}{dx} = 2x + 4ax^3. \quad (\textit{Resp.})$$

Si tienes alguna duda de si esto es correcto, prueba un caso más general, trabajándolo desde primeros principios. Y esta es la manera.

Sea $y = u + v$, donde u es cualquier función de x, y v cualquier otra función de x. Entonces, permitiendo que x aumente a $x + dx$, y aumentará a $y + dy$; y u aumentará a $u + du$; y v a $v + dv$.

Y tendremos:

$$y + dy = u + du + v + dv.$$

Restando la $y = u + v$ original, obtenemos

$$dy = du + dv,$$

y dividiendo por dx, obtenemos:

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}.$$

Esto justifica el procedimiento. Diferencias cada función por separado y sumas los resultados. Así que si ahora tomamos el ejemplo del párrafo anterior, y ponemos los valores de las dos funciones, tendremos, usando la notación mostrada (p. 16),

$$\begin{aligned} \frac{dy}{dx} &= \frac{d(x^2+c)}{dx} + \frac{d(ax^4+b)}{dx} \\ &= 2x \qquad\quad + 4ax^3, \end{aligned}$$

exactamente como antes.

Si hubiera tres funciones de x, que podemos llamar u, v y w, de modo que

$$y = u + v + w;$$

entonces
$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx}.$$

En cuanto a la *sustracción*, se sigue inmediatamente; pues si la función v hubiera tenido ella misma un signo negativo, su coeficiente diferencial también sería negativo; así que al diferenciar

$$y = u - v,$$

deberíamos obtener
$$\frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}.$$

Pero cuando llegamos a tratar con *Productos*, la cosa no es tan simple.

Supón que nos pidieran diferenciar la expresión

$$y = (x^2 + c) \times (ax^4 + b),$$

¿qué debemos hacer? El resultado ciertamente *no* será $2x \times 4ax^3$; pues es fácil ver que ni $c \times ax^4$, ni $x^2 \times b$, habrían sido tomados en ese producto.

Ahora hay dos maneras en las que podemos proceder.

Primera manera. Hace la multiplicación primero, y, habiendo trabajado esto, luego diferencia.

Por consiguiente, multiplicamos juntos $x^2 + c$ y $ax^4 + b$.

Esto da $ax^6 + acx^4 + bx^2 + bc$.

Ahora diferencia, y obtenemos:

$$\frac{dy}{dx} = 6ax^5 + 4acx^3 + 2bx.$$

Segunda manera. Regresa a primeros principios, y considera la ecuación

$$y = u \times v;$$

donde u es una función de x, y v es cualquier otra función de x. Entonces, si x crece a ser $x + dx$; y y a $y + dy$; y u se convierte en $u + du$, y v se convierte en $v + dv$, tendremos:

$$\begin{aligned} y + dy &= (u + du) \times (v + dv) \\ &= u \cdot v + u \cdot dv + v \cdot du + du \cdot dv. \end{aligned}$$

Ahora $du \cdot dv$ es una cantidad pequeña del segundo orden de pequeñez, y por lo tanto en el límite puede ser descartada, dejando

$$y + dy = u \cdot v + u \cdot dv + v \cdot du.$$

Entonces, restando la $y = u \cdot v$ original, nos queda

$$dy = u \cdot dv + v \cdot du;$$

y, dividiendo por dx, obtenemos el resultado:

$$\frac{dy}{dx} = u\,\frac{dv}{dx} + v\,\frac{du}{dx}.$$

Esto muestra que nuestras instrucciones serán las siguientes: *Para diferenciar el producto de dos funciones, multiplica cada función por el coeficiente diferencial de la otra, y suma los dos productos así obtenidos.*

Debes notar que este proceso equivale a lo siguiente: Trata u como constante mientras diferencias v; luego trata v como constante mientras diferencias u; y todo el coeficiente diferencial $\dfrac{dy}{dx}$ será la suma de estos dos tratamientos.

Ahora, habiendo encontrado esta regla, aplícala al ejemplo concreto que fue considerado arriba.

Queremos diferenciar el producto

$$(x^2 + c) \times (ax^4 + b).$$

Llama $(x^2 + c) = u$; y $(ax^4 + b) = v$.

Entonces, por la regla general recién establecida, podemos escribir:

$$\begin{aligned}
\frac{dy}{dx} &= (x^2+c)\,\frac{d(ax^4+b)}{dx} && + (ax^4+b)\,\frac{d(x^2+c)}{dx} \\
&= (x^2+c)\,4ax^3 && + (ax^4+b)\,2x \\
&= 4ax^5 + 4acx^3 && + 2ax^5 + 2bx, \\
\frac{dy}{dx} &= 6ax^5 + 4acx^3 && + 2bx,
\end{aligned}$$

exactamente como antes.

Finalmente, tenemos que diferenciar *cocientes*.

Piensa en este ejemplo, $y = \dfrac{bx^5+c}{x^2+a}$. En tal caso no sirve de nada tratar de hacer la división de antemano, porque x^2+a no dividirá a bx^5+c, ni tampoco tienen algún factor común. Así que no queda otra opción más que regresar a primeros principios, y encontrar una regla.

Así que pondremos $\qquad y = \dfrac{u}{v};$

donde u y v son dos funciones diferentes de la variable independiente x. Entonces, cuando x se convierte en $x+dx$, y se convertirá en $y+dy$; y u se convertirá en $u+du$; y v se convertirá en $v+dv$. Así entonces

$$y + dy = \frac{u+du}{v+dv}.$$

Ahora realiza la división algebraica, así:

$$
\begin{array}{r|l|l}
v+dv & u+du & \dfrac{u}{v}+\dfrac{du}{v}-\dfrac{u\cdot dv}{v^2} \\ \cline{1-1}\cline{3-3}
 & \underline{u+\dfrac{u\cdot dv}{v}} & \\
 & \quad du-\dfrac{u\cdot dv}{v} & \\
 & \quad \underline{du+\dfrac{du\cdot dv}{v}} & \\
 & \qquad -\dfrac{u\cdot dv}{v}-\dfrac{du\cdot dv}{v} & \\
 & \qquad \underline{-\dfrac{u\cdot dv}{v}-\dfrac{u\cdot dv\cdot dv}{v^2}} & \\
 & \qquad\qquad -\dfrac{du\cdot dv}{v}+\dfrac{u\cdot dv\cdot dv}{v^2}. &
\end{array}
$$

Como ambos residuos son cantidades pequeñas del segundo orden, pueden ser despreciados, y la división puede detenerse aquí, ya que cualquier residuo adicional sería de magnitudes aún menores.

Así que hemos obtenido:

$$y+dy=\frac{u}{v}+\frac{du}{v}-\frac{u\cdot dv}{v^2};$$

que puede escribirse

$$=\frac{u}{v}+\frac{v\cdot du-u\cdot dv}{v^2}.$$

Ahora resta la $y = \dfrac{u}{v}$ original, y nos queda:

$$dy = \frac{v \cdot du - u \cdot dv}{v^2};$$

de donde
$$\frac{dy}{dx} = \frac{v\,\dfrac{du}{dx} - u\,\dfrac{dv}{dx}}{v^2}.$$

Esto nos da nuestras instrucciones sobre *cómo diferenciar un cociente* de dos funciones. *Multiplica la función divisor por el coeficiente diferencial de la función dividendo; luego multiplica la función dividendo por el coeficiente diferencial de la función divisor; y resta. Finalmente divide por el cuadrado de la función divisor.*

Regresando a nuestro ejemplo $y = \dfrac{bx^5 + c}{x^2 + a}$,

escribe
$$bx^5 + c = u;$$
y
$$x^2 + a = v.$$

Entonces

$$\begin{aligned}
\frac{dy}{dx} &= \frac{(x^2 + a)\,\dfrac{d(bx^5 + c)}{dx} - (bx^5 + c)\,\dfrac{d(x^2 + a)}{dx}}{(x^2 + a)^2} \\
&= \frac{(x^2 + a)(5bx^4) - (bx^5 + c)(2x)}{(x^2 + a)^2}, \\
\frac{dy}{dx} &= \frac{3bx^6 + 5abx^4 - 2cx}{(x^2 + a)^2}. \quad (\textit{Respuesta.})
\end{aligned}$$

El desarrollo de cocientes es a menudo tedioso, pero no hay nada difícil al respecto.

Algunos ejemplos adicionales completamente desarrollados se dan a continuación.

(1) Diferencia $y = \frac{a}{b^2}x^3 - \frac{a^2}{b}x + \frac{a^2}{b^2}$.

Siendo una constante, $\frac{a^2}{b^2}$ se desvanece, y tenemos

$$\frac{dy}{dx} = \frac{a}{b^2} \times 3 \times x^{3-1} - \frac{a^2}{b} \times 1 \times x^{1-1}.$$

Pero $x^{1-1} = x^0 = 1$; así que obtenemos:

$$\frac{dy}{dx} = \frac{3a}{b^2}x^2 - \frac{a^2}{b}.$$

(2) Diferencia $y = 2a\sqrt{bx^3} - \frac{3b\sqrt[3]{a}}{x} - 2\sqrt{ab}$.

Poniendo x en la forma de índice, obtenemos

$$y = 2a\sqrt{b}x^{\frac{3}{2}} - 3b\sqrt[3]{a}x^{-1} - 2\sqrt{ab}.$$

Ahora

$$\frac{dy}{dx} = 2a\sqrt{b} \times \tfrac{3}{2} \times x^{\frac{3}{2}-1} - 3b\sqrt[3]{a} \times (-1) \times x^{-1-1};$$

o,

$$\frac{dy}{dx} = 3a\sqrt{bx} + \frac{3b\sqrt[3]{a}}{x^2}.$$

(3) Diferencia $z = 1.8\sqrt[3]{\frac{1}{\theta^2}} - \frac{4.4}{\sqrt[5]{\theta}} - 27°$.

Esto puede escribirse: $z = 1.8\,\theta^{-\frac{2}{3}} - 4.4\,\theta^{-\frac{1}{5}} - 27°$.

El 27° se desvanece, y tenemos

$$\frac{dz}{d\theta} = 1.8 \times -\tfrac{2}{3} \times \theta^{-\frac{2}{3}-1} - 4.4 \times \left(-\tfrac{1}{5}\right)\theta^{-\frac{1}{5}-1};$$

o,

$$\frac{dz}{d\theta} = -1.2\,\theta^{-\frac{5}{3}} + 0.88\,\theta^{-\frac{6}{5}};$$

o,

$$\frac{dz}{d\theta} = \frac{0.88}{\sqrt[5]{\theta^6}} - \frac{1.2}{\sqrt[3]{\theta^5}}.$$

(4) Diferencia $v = (3t^2 - 1.2t + 1)^3$.

Una manera directa de hacer esto se explicará más adelante (ver p. 69); pero aún así podemos manejarlo ahora sin ninguna dificultad.

Desarrollando el cubo, obtenemos

$$v = 27t^6 - 32.4t^5 + 39.96t^4 - 23.328t^3 + 13.32t^2 - 3.6t + 1;$$

por tanto

$$\frac{dv}{dt} = 162t^5 - 162t^4 + 159.84t^3 - 69.984t^2 + 26.64t - 3.6.$$

(5) Diferencia $y = (2x - 3)(x + 1)^2$.

$$\begin{aligned}\frac{dy}{dx} &= (2x-3)\,\frac{d\big[(x+1)(x+1)\big]}{dx} + (x+1)^2\,\frac{d(2x-3)}{dx}\\ &= (2x-3)\left[(x+1)\,\frac{d(x+1)}{dx} + (x+1)\,\frac{d(x+1)}{dx}\right]\\ &\qquad\qquad + (x+1)^2\,\frac{d(2x-3)}{dx}\\ &= 2(x+1)\big[(2x-3)+(x+1)\big] = 2(x+1)(3x-2);\end{aligned}$$

o, más simplemente, multiplica y luego diferencia.

(6) Diferencia $y = 0.5x^3(x - 3)$.

$$\begin{aligned}\frac{dy}{dx} &= 0.5\left[x^3\frac{d(x-3)}{dx} + (x-3)\frac{d(x^3)}{dx}\right]\\ &= 0.5\left[x^3 + (x-3)\times 3x^2\right] = 2x^3 - 4.5x^2.\end{aligned}$$

Las mismas observaciones que para el ejemplo anterior.

(7) Diferencia $w = \left(\theta + \dfrac{1}{\theta}\right)\left(\sqrt{\theta} + \dfrac{1}{\sqrt{\theta}}\right)$.

Esto puede escribirse

$$w = (\theta + \theta^{-1})(\theta^{\frac{1}{2}} + \theta^{-\frac{1}{2}}).$$

$$\begin{aligned}
\frac{dw}{d\theta} &= (\theta + \theta^{-1})\frac{d(\theta^{\frac{1}{2}} + \theta^{-\frac{1}{2}})}{d\theta} + (\theta^{\frac{1}{2}} + \theta^{-\frac{1}{2}})\frac{d(\theta + \theta^{-1})}{d\theta} \\
&= (\theta + \theta^{-1})(\tfrac{1}{2}\theta^{-\frac{1}{2}} - \tfrac{1}{2}\theta^{-\frac{3}{2}}) + (\theta^{\frac{1}{2}} + \theta^{-\frac{1}{2}})(1 - \theta^{-2}) \\
&= \tfrac{1}{2}(\theta^{\frac{1}{2}} + \theta^{-\frac{3}{2}} - \theta^{-\frac{1}{2}} - \theta^{-\frac{5}{2}}) + (\theta^{\frac{1}{2}} + \theta^{-\frac{1}{2}} - \theta^{-\frac{3}{2}} - \theta^{-\frac{5}{2}}) \\
&= \tfrac{3}{2}\left(\sqrt{\theta} - \frac{1}{\sqrt{\theta^5}}\right) + \tfrac{1}{2}\left(\frac{1}{\sqrt{\theta}} - \frac{1}{\sqrt{\theta^3}}\right).
\end{aligned}$$

Esto, nuevamente, podría obtenerse más simplemente multiplicando primero los dos factores, y diferenciando después. Esto no es, sin embargo, siempre posible; ve, por ejemplo, p. 173, ejemplo 8, en el cual la regla para diferenciar un producto *debe* ser usada.

(8) Diferencia $y = \dfrac{a}{1 + a\sqrt{x} + a^2x}$.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{(1 + ax^{\frac{1}{2}} + a^2x) \times 0 - a\dfrac{d(1 + ax^{\frac{1}{2}} + a^2x)}{dx}}{(1 + a\sqrt{x} + a^2x)^2} \\
&= -\frac{a(\frac{1}{2}ax^{-\frac{1}{2}} + a^2)}{(1 + ax^{\frac{1}{2}} + a^2x)^2}.
\end{aligned}$$

(9) Diferencia $y = \dfrac{x^2}{x^2 + 1}$.

$$\frac{dy}{dx} = \frac{(x^2 + 1)\,2x - x^2 \times 2x}{(x^2 + 1)^2} = \frac{2x}{(x^2 + 1)^2}.$$

(10) Diferencia $y = \dfrac{a + \sqrt{x}}{a - \sqrt{x}}$.

En la forma indexada, $y = \dfrac{a + x^{\frac{1}{2}}}{a - x^{\frac{1}{2}}}$.

$$\frac{dy}{dx} = \frac{(a - x^{\frac{1}{2}})(\frac{1}{2}x^{-\frac{1}{2}}) - (a + x^{\frac{1}{2}})(-\frac{1}{2}x^{-\frac{1}{2}})}{(a - x^{\frac{1}{2}})^2} = \frac{a - x^{\frac{1}{2}} + a + x^{\frac{1}{2}}}{2(a - x^{\frac{1}{2}})^2\, x^{\frac{1}{2}}};$$

por tanto $$\frac{dy}{dx} = \frac{a}{(a - \sqrt{x})^2\,\sqrt{x}}.$$

(11) Diferencia $$\theta = \frac{1 - a\sqrt[3]{t^2}}{1 + a\sqrt[2]{t^3}}.$$

Ahora $$\theta = \frac{1 - at^{\frac{2}{3}}}{1 + at^{\frac{3}{2}}}.$$

$$\begin{aligned}\frac{d\theta}{dt} &= \frac{(1 + at^{\frac{3}{2}})(-\frac{2}{3}at^{-\frac{1}{3}}) - (1 - at^{\frac{2}{3}}) \times \frac{3}{2}at^{\frac{1}{2}}}{(1 + at^{\frac{3}{2}})^2} \\ &= \frac{5a^2\sqrt[6]{t^7} - \dfrac{4a}{\sqrt[3]{t}} - 9a\sqrt[2]{t}}{6(1 + a\sqrt[2]{t^3})^2}.\end{aligned}$$

(12) Un depósito de sección transversal cuadrada tiene lados inclinados en un ángulo de 45° con la vertical. El lado del fondo es 200 pies. Encuentra una expresión para la cantidad que entra o sale cuando la profundidad del agua varía en 1 pie; de ahí encuentra, en galones, la cantidad retirada por hora cuando la profundidad se reduce de 14 a 10 pies en 24 horas.

El volumen de un tronco de pirámide de altura H, y de bases A y a, es $V = \dfrac{H}{3}(A + a + \sqrt{Aa})$. Es fácil ver que, siendo la pendiente 45°, si la profundidad es h, la longitud del lado de la superficie cuadrada del agua es $200 + 2h$ pies, de modo que el volumen de agua es

$$\frac{h}{3}[200^2 + (200 + 2h)^2 + 200(200 + 2h)] = 40,000h + 400h^2 + \frac{4h^3}{3}.$$

$\dfrac{dV}{dh} = 40,000 + 800h + 4h^2 =$ pies cúbicos por pie de variación de profundidad. El nivel medio de 14 a 10 pies es 12 pies, cuando $h = 12$, $\dfrac{dV}{dh} = 50,176$ pies cúbicos.

Galones por hora correspondientes a un cambio de profundidad de 4 pie en 24 horas $= \dfrac{4 \times 50,176 \times 6.25}{24} = 52,267$ galones.

(13) La presión absoluta, en atmósferas, P, del vapor saturado a la temperatura $t°$ C. está dada por Dulong como siendo $P = \left(\dfrac{40+t}{140}\right)^5$ mientras t esté por encima de 80°. Encuentra la tasa de variación de la presión con la temperatura a 100° C.

Expande el numerador por el teorema binomial (ver p. 140).

$$P = \frac{1}{140^5}(40^5 + 5 \times 40^4 t + 10 \times 40^3 t^2 + 10 \times 40^2 t^3 + 5 \times 40 t^4 + t^5);$$

por tanto $\dfrac{dP}{dt} = \dfrac{1}{537,824 \times 10^5}$

$$(5 \times 40^4 + 20 \times 40^3 t + 30 \times 40^2 t^2 + 20 \times 40 t^3 + 5t^4),$$

cuando $t = 100$ esto se convierte en 0.036 atmósfera por grado centígrado de cambio de temperatura.

Exercises III. (Ver las Respuestas en p. 256.)

(1) Diferencia

(a) $u = 1 + x + \dfrac{x^2}{1 \times 2} + \dfrac{x^3}{1 \times 2 \times 3} + \cdots.$

(b) $y = ax^2 + bx + c.$ (c) $y = (x + a)^2.$

(d) $y = (x + a)^3.$

(2) Si $w = at - \frac{1}{2}bt^2$, encuentra $\dfrac{dw}{dt}$.

(3) Encuentra el coeficiente diferencial de

$$y = (x + \sqrt{-1}) \times (x - \sqrt{-1}).$$

(4) Diferencia

$$y = (197x - 34x^2) \times (7 + 22x - 83x^3).$$

(5) Si $x = (y + 3) \times (y + 5)$, encuentra $\dfrac{dx}{dy}$.

(6) Diferencia $y = 1.3709x \times (112.6 + 45.202x^2)$.

Encuentra los coeficientes diferenciales de

(7) $y = \dfrac{2x + 3}{3x + 2}$. (8) $y = \dfrac{1 + x + 2x^2 + 3x^3}{1 + x + 2x^2}$.

(9) $y = \dfrac{ax + b}{cx + d}$. (10) $y = \dfrac{x^n + a}{x^{-n} + b}$.

(11) La temperatura t del filamento de una lámpara eléctrica incandescente está conectada con la corriente que pasa a través de la lámpara por la relación

$$C = a + bt + ct^2.$$

Encuentra una expresión que dé la variación de la corriente correspondiente a una variación de temperatura.

(12) Las siguientes fórmulas han sido propuestas para expresar la relación entre la resistencia eléctrica R de un alambre a la temperatura $t°$ C., y la resistencia R_0 de ese mismo alambre a 0° centígrado, siendo a, b, c constantes.

$$R = R_0(1 + at + bt^2).$$
$$R = R_0(1 + at + b\sqrt{t}).$$
$$R = R_0(1 + at + bt^2)^{-1}.$$

Encuentra la tasa de variación de la resistencia con respecto a la temperatura como está dada por cada una de estas fórmulas.

(13) La fuerza electromotriz E de un cierto tipo de celda estándar se ha encontrado que varía con la temperatura t de acuerdo con la relación

$$E = 1.4340\big[1 - 0.000814(t - 15) + 0.000007(t - 15)^2\big] \text{ voltios.}$$

Encuentra el cambio de fuerza electromotriz por grado, a 15°, 20° y 25°.

(14) La fuerza electromotriz necesaria para mantener un arco eléctrico de longitud l con una corriente de intensidad i ha sido encontrada por la Sra. Ayrton como siendo

$$E = a + bl + \frac{c + kl}{i},$$

donde a, b, c, k son constantes.

Encuentra una expresión para la variación de la fuerza electromotriz (a) con respecto a la longitud del arco; (b) con respecto a la intensidad de la corriente.

CAPÍTULO VII.

DIFERENCIACIÓN SUCESIVA.

PROBEMOS el efecto de repetir varias veces la operación de diferenciar una función (ver p. 13). Comencemos con un caso concreto.

Sea $y = x^5$.

$$\begin{array}{lll}
\text{Primera diferenciación,} & 5x^4. & \\
\text{Segunda diferenciación,} & 5 \times 4x^3 & = 20x^3. \\
\text{Tercera diferenciación,} & 5 \times 4 \times 3x^2 & = 60x^2. \\
\text{Cuarta diferenciación,} & 5 \times 4 \times 3 \times 2x & = 120x. \\
\text{Quinta diferenciación,} & 5 \times 4 \times 3 \times 2 \times 1 & = 120. \\
\text{Sexta diferenciación,} & & = 0.
\end{array}$$

Hay una cierta notación, con la cual ya estamos familiarizados (ver p. 14), usada por algunos escritores, que es muy conveniente. Esta es emplear el símbolo general $f(x)$ para cualquier función de x. Aquí el símbolo $f(\ )$ se lee como “función de,” sin decir qué función particular se refiere. Así la declaración $y = f(x)$ simplemente nos dice que y es una función de x, puede ser x^2 o ax^n, o $\cos x$ o cualquier otra función complicada de x.

El símbolo correspondiente para el coeficiente diferencial es $f'(x)$, que es más simple de escribir que $\dfrac{dy}{dx}$. Esta se llama la "función derivada" de x.

Supón que diferenciamos otra vez, obtendremos la "segunda función derivada" o segundo coeficiente diferencial, que se denota por $f''(x)$; y así sucesivamente.

Ahora generalicemos.

Sea $y = f(x) = x^n$.

Primera diferenciación, $\quad f'(x) = nx^{n-1}$.

Segunda diferenciación, $\quad f''(x) = n(n-1)x^{n-2}$.

Tercera diferenciación, $\quad f'''(x) = n(n-1)(n-2)x^{n-3}$.

Cuarta diferenciación, $\quad f''''(x) = n(n-1)(n-2)(n-3)x^{n-4}$.

etc., etc.

Pero esta no es la única manera de indicar diferenciaciones sucesivas. Pues,

si la función original es $\quad y = f(x)$;

diferenciando una vez da $\quad \dfrac{dy}{dx} = f'(x)$;

diferenciando dos veces da $\quad \dfrac{d\left(\dfrac{dy}{dx}\right)}{dx} = f''(x)$;

y esto se escribe más convenientemente como $\dfrac{d^2y}{(dx)^2}$, o más usualmente $\dfrac{d^2y}{dx^2}$. Similarmente, podemos escribir como el resultado de diferenciar tres veces, $\dfrac{d^3y}{dx^3} = f'''(x)$.

Examples.

Ahora probemos $y = f(x) = 7x^4 + 3.5x^3 - \frac{1}{2}x^2 + x - 2$.

$$\begin{aligned}
\frac{dy}{dx} &= f'(x) = 28x^3 + 10.5x^2 - x + 1, \\
\frac{d^2y}{dx^2} &= f''(x) = 84x^2 + 21x - 1, \\
\frac{d^3y}{dx^3} &= f'''(x) = 168x + 21, \\
\frac{d^4y}{dx^4} &= f''''(x) = 168, \\
\frac{d^5y}{dx^5} &= f'''''(x) = 0.
\end{aligned}$$

De manera similar si $y = \phi(x) = 3x(x^2 - 4)$,

$$\begin{aligned}
\phi'(x) &= \frac{dy}{dx} = 3\big[x \times 2x + (x^2 - 4) \times 1\big] = 3(3x^2 - 4), \\
\phi''(x) &= \frac{d^2y}{dx^2} = 3 \times 6x = 18x, \\
\phi'''(x) &= \frac{d^3y}{dx^3} = 18, \\
\phi''''(x) &= \frac{d^4y}{dx^4} = 0.
\end{aligned}$$

Exercises IV. (Ver page 256 para las Respuestas.)

Encontrar $\dfrac{dy}{dx}$ y $\dfrac{d^2y}{dx^2}$ para las siguientes expresiones:

(1) $y = 17x + 12x^2$.

(2) $y = \dfrac{x^2 + a}{x + a}$.

(3) $y = 1 + \dfrac{x}{1} + \dfrac{x^2}{1 \times 2} + \dfrac{x^3}{1 \times 2 \times 3} + \dfrac{x^4}{1 \times 2 \times 3 \times 4}$.

(4) Encontrar las funciones derivadas 2da y 3ra en los Ejercicios III. (p. 47), No. 1 al No. 7, y en los Ejemplos dados (p. 43), No. 1 al No. 7.

CAPÍTULO VIII.

CUANDO EL TIEMPO VARÍA.

ALGUNOS de los problemas más importantes del cálculo son aquellos donde el tiempo es la variable independiente, y tenemos que pensar sobre los valores de alguna otra cantidad que varía cuando el tiempo varía. Algunas cosas crecen más grandes conforme pasa el tiempo; algunas otras cosas se vuelven más pequeñas. La distancia que un tren ha recorrido desde su lugar de partida continúa aumentando conforme el tiempo avanza. Los árboles crecen más altos conforme pasan los años. ¿Cuál está creciendo a mayor velocidad; una planta de 12 pulgadas de altura que en un mes se vuelve de 14 pulgadas de altura, o un árbol de 12 pies de altura que en un año se vuelve de 14 pies de altura?

En este capítulo vamos a hacer mucho uso de la palabra *velocidad*. Nada que ver con impuesto sobre los pobres, o tarifa del agua (excepto que aún aquí la palabra sugiere una proporción—una razón—tantos peniques por libra). Nada que ver incluso con tasa de natalidad o tasa de mortalidad, aunque estas palabras sugieren tantos nacimientos o muertes por mil de la población. Cuando un automóvil pasa velozmente junto a nosotros, decimos: ¡Qué velocidad tan terrible! Cuando un derrochador está despilfarrando su dinero, comentamos que ese joven está viviendo a una velocidad prodigiosa. ¿Qué queremos decir con *ve-*

locidad? En ambos casos estamos haciendo una comparación mental de algo que está ocurriendo, y la duración de tiempo que toma para que ocurra. Si el automóvil vuela junto a nosotros yendo a 10 yardas por segundo, un simple cálculo mental nos mostrará que esto es equivalente—mientras dure—a una velocidad de 600 yardas por minuto, o más de 20 millas por hora.

Ahora ¿en qué sentido es cierto que una velocidad de 10 yardas por segundo es la misma que 600 yardas por minuto? Diez yardas no es lo mismo que 600 yardas, ni un segundo es lo mismo que un minuto. Lo que queremos decir al afirmar que la *velocidad* es la misma, es esto: que la proporción que existe entre la distancia recorrida y el tiempo tomado para recorrerla, es la misma en ambos casos.

Tomemos otro ejemplo. Un hombre puede tener solo unas cuantas libras en su posesión, y aún así ser capaz de gastar dinero a la velocidad de millones al año—siempre que continúe gastando dinero a esa velocidad por solo unos pocos minutos. Supongamos que entregas un chelín sobre el mostrador para pagar algunas mercancías; y supongamos que la operación dura exactamente un segundo. Entonces, durante esa breve operación, estás separándote de tu dinero a la velocidad de 1 chelín por segundo, que es la misma velocidad que £3 por minuto, o £180 por hora, o £4320 por día, o £$1,576,800$ por año! Si tienes £10 en tu bolsillo, puedes continuar gastando dinero a la velocidad de un millón al año por exactamente $5\frac{1}{4}$ minutos.

Se dice que Sandy no había estado en Londres más de cinco minutos cuando "bang went saxpence." Si fuera a gastar dinero a esa velocidad todo el día, digamos por 12 horas, estaría gastando 6 chelines por hora, o £3. 12*s*. por día, o £21. 12*s*. a la semana, sin contar el sabbath.

Ahora trata de poner algunas de estas ideas en notación diferencial.

Sea y en este caso representar dinero, y sea t representar tiempo.

Si estás gastando dinero, y la cantidad que gastas en un tiempo corto dt se llama dy, la *velocidad* de gastarlo será $\dfrac{dy}{dt}$, o mejor dicho, debería escribirse con un signo menos, como $-\dfrac{dy}{dt}$, porque dy es un *decremento*, no un incremento. Pero el dinero no es un buen ejemplo para el cálculo, porque generalmente viene y se va a saltos, no por un flujo continuo—puedes ganar £200 al año, pero no sigue corriendo todo el día en una corriente delgada; viene solo semanalmente, o mensualmente, o trimestralmente, en bultos: y tu gasto también sale en pagos súbitos.

Una ilustración más apropiada de la idea de velocidad es proporcionada por la velocidad de un cuerpo en movimiento. Desde Londres (estación Euston) hasta Liverpool son 200 millas. Si un tren sale de Londres a las 7 en punto, y llega a Liverpool a las 11 en punto, sabes que, ya que ha viajado 200 millas en 4 horas, su velocidad promedio debe haber sido 50 millas por hora; porque $\frac{200}{4} = \frac{50}{1}$. Aquí realmente estás haciendo una comparación mental entre la distancia recorrida y el tiempo tomado para recorrer esa distancia. Estás dividiendo una por la otra. Si y es toda la distancia, y t todo el tiempo, claramente la velocidad promedio es $\dfrac{y}{t}$. Ahora la velocidad no fue realmente constante todo el camino: al arrancar, y durante la desaceleración al final del viaje, la velocidad fue menor. Probablemente en alguna parte, cuando bajaba una colina, la velocidad fue más de 60 millas por hora. Si, durante cualquier elemento particular de tiempo dt, el elemento correspondiente de distancia recorrida fue dy, entonces en esa parte del viaje la velocidad fue $\frac{dy}{dt}$. La *velocidad* a la cual una cantidad (en la presente instancia, *distancia*) está cambiando en relación a la otra cantidad (en este caso, *tiempo*) se expresa apropiadamente, entonces, al establecer el coeficiente diferencial de una con respecto a la otra. Una *velocidad*, expresada científicamente, es la velocidad a la cual una distancia muy pequeña en cualquier dirección dada está siendo recorrida; y por lo tanto puede escribirse

$$v = \frac{dy}{dt}.$$

Pero si la velocidad v no es uniforme, entonces debe estar ya sea aumentando o disminuyendo. La velocidad a la cual una velocidad está aumentando se llama la *aceleración.* Si un cuerpo en movimiento está, en cualquier instante particular, ganando una velocidad adicional dv en un elemento de tiempo dt, entonces la aceleración a en ese instante puede escribirse

$$a = \frac{dv}{dt};$$

pero dv es en sí misma $d\left(\frac{dy}{dt}\right)$. Por lo tanto podemos poner

$$a = \frac{d\left(\frac{dy}{dt}\right)}{dt};$$

y esto usualmente se escribe $a = \frac{d^2y}{dt^2}$;
o la aceleración es el segundo coeficiente diferencial de la distancia, con respecto al tiempo. La aceleración se expresa como un cambio de velocidad en unidad de tiempo, por ejemplo, como tantos pies por segundo por segundo; la notación usada siendo pies $\div$ segundo^2.

Cuando un tren de ferrocarril acaba de empezar a moverse, su velocidad v es pequeña; pero está ganando velocidad rápidamente—está siendo acelerado, o acelerado, por el esfuerzo de la máquina. Así que su $\dfrac{d^2y}{dt^2}$ es grande. Cuando ha alcanzado su velocidad máxima ya no está siendo acelerado, de modo que entonces $\dfrac{d^2y}{dt^2}$ ha caído a cero. Pero cuando se acerca a su lugar de parada su velocidad comienza a disminuir; puede, de hecho, disminuir muy rápidamente si se aplican los frenos, y durante este período de *desaceleración* o aflojamiento del paso, el valor de $\dfrac{dv}{dt}$, es decir, de $\dfrac{d^2y}{dt^2}$ será negativo.

Para acelerar una masa m se requiere la aplicación continua de fuerza. La fuerza necesaria para acelerar una masa es proporcional a la masa, y también es proporcional a la aceleración que se está impartiendo. Por lo tanto podemos escribir para la fuerza f, la expresión

$$f = ma;$$

o

$$f = m\frac{dv}{dt};$$

o

$$f = m\frac{d^2y}{dt^2}.$$

El producto de una masa por la velocidad a la cual está yendo se llama su *momentum*, y en símbolos es mv. Si diferenciamos el momentum con respecto al tiempo obtendremos $\dfrac{d(mv)}{dt}$ para la velocidad de cambio del momentum. Pero, ya que m es una cantidad constante, esto puede escribirse $m\dfrac{dv}{dt}$, que vemos arriba es lo mismo que f. Es decir, la fuerza puede expresarse ya sea como masa por aceleración, o como velocidad de cambio de momentum.

Nuevamente, si una fuerza se emplea para mover algo (contra una contra-fuerza igual y opuesta), hace *trabajo*; y la cantidad de trabajo hecho se mide por el producto de la fuerza por la distancia (en su propia dirección) a través de la cual su punto de aplicación se mueve hacia adelante. Así que si una fuerza f se mueve hacia adelante a través de una longitud y, el trabajo hecho (que podemos llamar w) será

$$w = f \times y;$$

donde tomamos f como una fuerza constante. Si la fuerza varía en diferentes partes del rango y, entonces debemos encontrar una expresión para su valor de punto a punto. Si f es la fuerza a lo largo del pequeño elemento de longitud dy, la cantidad de trabajo hecho será $f \times dy$. Pero como dy es solo un elemento de longitud, solo un elemento de trabajo será hecho. Si escribimos w para trabajo, entonces un elemento de trabajo será dw; y tenemos

$$dw = f \times dy;$$

que puede escribirse

$$dw = ma \cdot dy;$$

o

$$dw = m\frac{d^2y}{dt^2} \cdot dy;$$

o

$$dw = m\frac{dv}{dt} \cdot dy.$$

Además, podemos trasponer la expresión y escribir

$$\frac{dw}{dy} = f.$$

Esto nos da aún una tercera definición de *fuerza*; que si se está usando para producir un desplazamiento en cualquier dirección, la fuerza (en esa dirección) es igual a la velocidad a la cual el trabajo está siendo hecho por unidad de longitud en esa dirección. En esta última oración la palabra *velocidad* claramente no se usa en su sentido temporal, sino en su significado como razón o proporción.

Sir Isaac Newton, quien fue (junto con Leibnitz) un inventor de los métodos del cálculo, consideraba todas las cantidades que estaban variando como *fluyendo*; y la razón que hoy en día llamamos el coeficiente diferencial él la consideraba como la velocidad de flujo, o la *fluxión* de la cantidad en cuestión. No usó la notación de dy y dx, y dt (esto se debió a Leibnitz), sino que tenía en su lugar una notación propia. Si y era una cantidad que variaba, o "fluía," entonces su símbolo para su velocidad de variación (o "fluxión") era $\dot{y}$. Si x era la variable, entonces su fluxión se llamaba $\dot{x}$. El punto sobre la letra indicaba que había sido diferenciada. Pero esta notación no nos dice cuál es la variable independiente con respecto a la cual la diferenciación ha sido efectuada. Cuando vemos $\dfrac{dy}{dt}$ sabemos que y debe ser diferenciada con respecto a t. Si vemos $\dfrac{dy}{dx}$ sabemos que y debe ser diferenciada con respecto a x. Pero si vemos meramente $\dot{y}$, no podemos decir sin mirar el contexto si esto debe significar $\dfrac{dy}{dx}$ o $\dfrac{dy}{dt}$ o $\dfrac{dy}{dz}$, o cuál es la otra variable. Así, por lo tanto, esta notación fluxional es menos informativa que la notación diferencial, y en consecuencia ha caído en gran medida en desuso. Pero su simplicidad le da una ventaja si solo acordamos usarla para aquellos casos exclusivamente donde el *tiempo* es la variable independiente. En ese caso $\dot{y}$ significará $\dfrac{dy}{dt}$ y $\dot{u}$ significará $\dfrac{du}{dt}$; y $\ddot{x}$ significará $\dfrac{d^2x}{dt^2}$.

Adoptando esta notación fluxional podemos escribir las ecuaciones mecánicas consideradas en los párrafos anteriores, como sigue:

distancia	x,
velocidad	$v = \dot{x}$,
aceleración	$a = \dot{v} = \ddot{x}$,
fuerza	$f = m\dot{v} = m\ddot{x}$,
trabajo	$w = x \times m\ddot{x}$.

Examples.

(1) Un cuerpo se mueve de tal manera que la distancia x (en pies), que viaja desde cierto punto O, está dada por la relación $x = 0.2t^2 + 10.4$, donde t es el tiempo en segundos transcurrido desde cierto instante. Encontrar la velocidad y aceleración 5 segundos después de que el cuerpo comenzó a moverse, y también encontrar los valores correspondientes cuando la distancia recorrida es 100 pies. Encontrar también la velocidad promedio durante los primeros 10 segundos de su movimiento. (Suponer que las distancias y el movimiento hacia la derecha son positivos.)

Ahora $$x = 0.2t^2 + 10.4$$

$$v = \dot{x} = \frac{dx}{dt} = 0.4t; \quad \text{y} \quad a = \ddot{x} = \frac{d^2x}{dt^2} = 0.4 = \text{constante}.$$

Cuando $t = 0$, $x = 10.4$ y $v = 0$. El cuerpo comenzó desde un punto 10.4 pies a la derecha del punto O; y el tiempo se contó desde el instante en que el cuerpo comenzó.

Cuando $t = 5$, $v = 0.4 \times 5 = 2$ pies/seg.; $a = 0.4$ pies/seg^2.

Cuando $x = 100$, $100 = 0.2t^2 + 10.4$, o $t^2 = 448$, y $t = 21.17$ seg.; $v = 0.4 \times 21.17 = 8.468$ pies/seg.

Cuando $t = 10$,

$$\text{distancia recorrida} = 0.2 \times 10^2 + 10.4 - 10.4 = 20 \text{ pies.}$$

$$\text{Velocidad promedio} = \tfrac{20}{10} = 2 \text{ pies/seg.}$$

(Es la misma velocidad que la velocidad en el medio del intervalo, $t = 5$; porque, siendo la aceleración constante, la velocidad ha variado uniformemente desde cero cuando $t = 0$ hasta 4 pies/seg. cuando $t = 10$.)

(2) En el problema anterior supongamos

$$x = 0.2t^2 + 3t + 10.4.$$

$$v = \dot{x} = \frac{dx}{dt} = 0.4t + 3; \quad a = \ddot{x} = \frac{d^2x}{dt^2} = 0.4 = \text{constante.}$$

Cuando $t = 0$, $x = 10.4$ y $v = 3$ pies/seg, el tiempo se cuenta desde el instante en que el cuerpo pasó por un punto 10.4 pies del punto O, siendo su velocidad entonces ya 3 pies/seg. Para encontrar el tiempo transcurrido desde que comenzó a moverse, sea $v = 0$; entonces $0.4t + 3 = 0$, $t = -\frac{3}{.4} = -7.5$ seg. El cuerpo comenzó a moverse 7.5 seg. antes de que el tiempo comenzara a ser observado; 5 segundos después de esto da $t = -2.5$ y $v = 0.4 \times -2.5 + 3 = 2$ pies/seg.

Cuando $x = 100$ pies,

$$100 = 0.2t^2 + 3t + 10.4; \quad \text{o } t^2 + 15t - 448 = 0;$$

por lo tanto $t = 14.95$ seg., $v = 0.4 \times 14.95 + 3 = 8.98$ pies/seg.

Para encontrar la distancia recorrida durante los 10 primeros segundos del movimiento uno debe saber qué tan lejos estaba el cuerpo del punto O cuando comenzó.

Cuando $t = -7.5$,

$$x = 0.2 \times (-7.5)^2 - 3 \times 7.5 + 10.4 = -0.85 \text{ pies},$$

es decir 0.85 pies a la izquierda del punto O.

Ahora, cuando $t = 2.5$,

$$x = 0.2 \times 2.5^2 + 3 \times 2.5 + 10.4 = 19.15.$$

Así, en 10 segundos, la distancia recorrida fue $19.15+0.85 = 20$ pies, y

$$\text{la velocidad promedio } = \tfrac{20}{10} = 2 \text{ pies/seg.}$$

(3) Considerar un problema similar cuando la distancia está dada por $x = 0.2t^2 - 3t + 10.4$. Entonces $v = 0.4t - 3$, $a = 0.4 =$ constante. Cuando $t = 0$, $x = 10.4$ como antes, y $v = -3$; de modo que el cuerpo se estaba moviendo en la dirección opuesta a su movimiento en los casos previos. Como la aceleración es positiva, sin embargo, vemos que esta velocidad disminuirá conforme pase el tiempo, hasta que se vuelva cero, cuando $v = 0$ o $0.4t - 3 = 0$; o $t = 7.5$ seg. Después de esto, la velocidad se vuelve positiva; y 5 segundos después de que el cuerpo comenzó, $t = 12.5$, y

$$v = 0.4 \times 12.5 - 3 = 2 \text{ pies/seg.}$$

Cuando $x = 100$,

$$100 = 0.2t^2 - 3t + 10.4, \quad \text{o } t^2 - 15t - 448 = 0,$$

y $$t = 29.95; \; v = 0.4 \times 29.95 - 3 = 8.98 \text{ pies/seg.}$$

Cuando v es cero, $x = 0.2 \times 7.5^2 - 3 \times 7.5 + 10.4 = -0.85$, informándonos que el cuerpo se mueve de vuelta a 0.85 pies más allá del punto O antes de detenerse. Diez segundos después

$$t = 17.5 \text{ y } x = 0.2 \times 17.5^2 - 3 \times 17.5 + 10.4 = 19.15.$$

La distancia recorrida $= .85 + 19.15 = 20.0$, y la velocidad promedio es nuevamente 2 pies/seg.

(4) Considerar aún otro problema del mismo tipo con $x = 0.2t^3 - 3t^2 + 10.4$; $v = 0.6t^2 - 6t$; $a = 1.2t - 6$. La aceleración ya no es constante.

Cuando $t = 0$, $x = 10.4$, $v = 0$, $a = -6$. El cuerpo está en reposo, pero listo para moverse con una aceleración negativa, es decir para ganar una velocidad hacia el punto O.

(5) Si tenemos $x = 0.2t^3 - 3t + 10.4$, entonces $v = 0.6t^2 - 3$, y $a = 1.2t$.

Cuando $t = 0$, $x = 10.4$; $v = -3$; $a = 0$.

El cuerpo se está moviendo hacia el punto O con una velocidad de 3 pies/seg., y justo en ese instante la velocidad es uniforme.

Vemos que las condiciones del movimiento siempre pueden determinarse inmediatamente de la ecuación tiempo-distancia y sus primera y segunda funciones derivadas. En los últimos dos casos la velocidad media durante los primeros 10 segundos y la velocidad 5 segundos después del inicio ya no serán iguales, porque la velocidad no está aumentando uniformemente, ya que la aceleración no es más constante.

(6) El ángulo θ (en radianes) girado por una rueda está dado por $\theta = 3 + 2t - 0.1t^3$, donde t es el tiempo en segundos desde cierto instante; encontrar la velocidad angular ω y la aceleración angular α, (a) después

de 1 segundo; (b) después de que ha realizado una revolución. ¿En qué momento está en reposo, y cuántas revoluciones ha realizado hasta ese instante?

Escribiendo para la aceleración

$$\omega = \dot{\theta} = \frac{d\theta}{dt} = 2 - 0.3t^2, \quad \alpha = \ddot{\theta} = \frac{d^2\theta}{dt^2} = -0.6t.$$

Cuando $t = 0$, $\theta = 3$; $\omega = 2$ rad./seg.; $\alpha = 0$.

Cuando $t = 1$,

$$\omega = 2 - 0.3 = 1.7 \text{ rad./seg.}; \quad \alpha = -0.6 \text{ rad./seg}^2.$$

Esto es una retardación; la rueda se está desacelerando.

Después de 1 revolución

$$\theta = 2\pi = 6.28; \quad 6.28 = 3 + 2t - 0.1t^3.$$

Graficando la gráfica, $\theta = 3 + 2t - 0.1t^3$, podemos obtener el valor o valores de t para los cuales $\theta = 6.28$; estos son 2.11 y 3.03 (hay un tercer valor negativo).

Cuando $t = 2.11$,

$$\theta = 6.28; \quad \omega = 2 - 1.34 = 0.66 \text{ rad./seg.};$$
$$\alpha = -1.27 \text{ rad./seg}^2.$$

Cuando $t = 3.03$,

$$\theta = 6.28; \quad \omega = 2 - 2.754 = -0.754 \text{ rad./seg.};$$
$$\alpha = -1.82 \text{ rad./seg}^2.$$

La velocidad está invertida. La rueda evidentemente está en reposo entre estos dos instantes; está en reposo cuando $\omega = 0$, es decir cuando $0 = 2 - 0.3t^3$, o cuando $t = 2.58$ seg., ha realizado

$$\frac{\theta}{2\pi} = \frac{3 + 2 \times 2.58 - 0.1 \times 2.58^3}{6.28} = 1.025 \text{ revoluciones.}$$

Exercises V. (Ver page 258 para las Respuestas.)

(1) Si $y = a + bt^2 + ct^4$; encontrar $\dfrac{dy}{dt}$ y $\dfrac{d^2y}{dt^2}$.

$$\textit{Resp.}\ \frac{dy}{dt} = 2bt + 4ct^3; \quad \frac{d^2y}{dt^2} = 2b + 12ct^2.$$

(2) Un cuerpo cayendo libremente en el espacio describe en t segundos un espacio s, en pies, expresado por la ecuación $s = 16t^2$. Dibujar una curva mostrando la relación entre s y t. También determinar la velocidad del cuerpo en los siguientes tiempos desde que se deja caer: $t = 2$ segundos; $t = 4.6$ segundos; $t = 0.01$ segundo.

(3) Si $x = at - \frac{1}{2}gt^2$; encontrar $\dot{x}$ y $\ddot{x}$.

(4) Si un cuerpo se mueve de acuerdo a la ley

$$s = 12 - 4.5t + 6.2t^2,$$

encontrar su velocidad cuando $t = 4$ segundos; s estando en pies.

(5) Encontrar la aceleración del cuerpo mencionado en el ejemplo precedente. ¿Es la aceleración la misma para todos los valores de t?

(6) El ángulo θ (en radianes) girado por una rueda giratoria está conectado con el tiempo t (en segundos) que ha transcurrido desde el inicio; por la ley

$$\theta = 2.1 - 3.2t + 4.8t^2.$$

Encontrar la velocidad angular (en radianes por segundo) de esa rueda cuando han transcurrido $1\frac{1}{2}$ segundos. Encontrar también su aceleración angular.

(7) Un deslizador se mueve de tal manera que, durante la primera parte de su movimiento, su distancia s en pulgadas desde su punto de partida está dada por la expresión

$$s = 6.8t^3 - 10.8t; \quad t \text{ estando en segundos.}$$

Encontrar la expresión para la velocidad y la aceleración en cualquier momento; y de ahí encontrar la velocidad y la aceleración después de 3 segundos.

(8) El movimiento de un globo que se eleva es tal que su altura h, en millas, está dada en cualquier instante por la expresión $h = 0.5 + \frac{1}{10}\sqrt[3]{t - 125}$; t estando en segundos.

Encontrar una expresión para la velocidad y la aceleración en cualquier momento. Dibujar curvas para mostrar la variación de altura, velocidad y aceleración durante los primeros diez minutos del ascenso.

(9) Una piedra se lanza hacia abajo al agua y su profundidad p en metros en cualquier instante t segundos después de alcanzar la superficie

del agua está dada por la expresión

$$p = \frac{4}{4+t^2} + 0.8t - 1.$$

Encontrar una expresión para la velocidad y la aceleración en cualquier momento. Encontrar la velocidad y aceleración después de 10 segundos.

(10) Un cuerpo se mueve de tal manera que los espacios descritos en el tiempo t desde el inicio están dados por $s = t^n$, donde n es una constante. Encontrar el valor de n cuando la velocidad se duplica del 5to al 10mo segundo; encontrarlo también cuando la velocidad es numéricamente igual a la aceleración al final del 10mo segundo.

CAPÍTULO IX.

INTRODUCIENDO UN TRUCO ÚTIL.

A veces uno se encuentra en apuros al encontrar que la expresión a ser diferenciada es demasiado complicada para abordarla directamente.

Así, la ecuación

$$y = (x^2 + a^2)^{\frac{3}{2}}$$

es incómoda para un principiante.

Ahora el truco para vencer la dificultad es este: Escribir algún símbolo, como u, para la expresión $x^2 + a^2$; entonces la ecuación se vuelve

$$y = u^{\frac{3}{2}},$$

la cual puedes manejar fácilmente; porque

$$\frac{dy}{du} = \frac{3}{2} u^{\frac{1}{2}}.$$

Entonces aborda la expresión

$$u = x^2 + a^2,$$

y diferénciarla con respecto a x,

$$\frac{du}{dx} = 2x.$$

Entonces todo lo que queda es navegar en aguas tranquilas;

porque $$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx};$$

es decir, $$\begin{aligned} \frac{dy}{dx} &= \frac{3}{2} u^{\frac{1}{2}} \times 2x \\ &= \tfrac{3}{2}(x^2 + a^2)^{\frac{1}{2}} \times 2x \\ &= 3x(x^2 + a^2)^{\frac{1}{2}}; \end{aligned}$$

y así el truco está hecho.

Con el tiempo, cuando hayas aprendido cómo lidiar con senos, y cosenos, y exponenciales, encontrarás este truco de utilidad creciente.

Examples.

Practiquemos este truco en unos cuantos ejemplos.

(1) Diferenciar $y = \sqrt{a + x}$.

Sea $a + x = u$.

$$\frac{du}{dx} = 1; \quad y = u^{\frac{1}{2}}; \quad \frac{dy}{du} = \tfrac{1}{2} u^{-\frac{1}{2}} = \tfrac{1}{2}(a + x)^{-\frac{1}{2}}.$$
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{2\sqrt{a + x}}.$$

(2) Diferenciar $y = \dfrac{1}{\sqrt{a + x^2}}$.

Sea $a + x^2 = u$.

$$\frac{du}{dx} = 2x; \quad y = u^{-\frac{1}{2}}; \quad \frac{dy}{du} = -\tfrac{1}{2} u^{-\frac{3}{2}}.$$
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -\frac{x}{\sqrt{(a + x^2)^3}}.$$

(3) Diferenciar $y = \left(m - nx^{\frac{2}{3}} + \dfrac{p}{x^{\frac{4}{3}}}\right)^a$.

Sea $m - nx^{\frac{2}{3}} + px^{-\frac{4}{3}} = u$.

$$\frac{du}{dx} = -\tfrac{2}{3}nx^{-\frac{1}{3}} - \tfrac{4}{3}px^{-\frac{7}{3}};$$

$$y = u^a; \quad \frac{dy}{du} = au^{a-1}.$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -a\left(m - nx^{\frac{2}{3}} + \frac{p}{x^{\frac{4}{3}}}\right)^{a-1} (\tfrac{2}{3}nx^{-\frac{1}{3}} + \tfrac{4}{3}px^{-\frac{7}{3}}).$$

(4) Diferenciar $y = \dfrac{1}{\sqrt{x^3 - a^2}}$.

Sea $u = x^3 - a^2$.

$$\frac{du}{dx} = 3x^2; \quad y = u^{-\frac{1}{2}}; \quad \frac{dy}{du} = -\frac{1}{2}(x^3 - a^2)^{-\frac{3}{2}}.$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -\frac{3x^2}{2\sqrt{(x^3 - a^2)^3}}.$$

(5) Diferenciar $y = \sqrt{\dfrac{1 - x}{1 + x}}$.

Escribir esto como $y = \dfrac{(1 - x)^{\frac{1}{2}}}{(1 + x)^{\frac{1}{2}}}$.

$$\frac{dy}{dx} = \frac{(1 + x)^{\frac{1}{2}} \dfrac{d(1 - x)^{\frac{1}{2}}}{dx} - (1 - x)^{\frac{1}{2}} \dfrac{d(1 + x)^{\frac{1}{2}}}{dx}}{1 + x}.$$

(También podemos escribir $y = (1-x)^{\frac{1}{2}}(1+x)^{-\frac{1}{2}}$ y diferenciar como un producto.)

Procediendo como en el ejemplo (1) arriba, obtenemos

$$\frac{d(1 - x)^{\frac{1}{2}}}{dx} = -\frac{1}{2\sqrt{1 - x}}; \quad \text{y} \quad \frac{d(1 + x)^{\frac{1}{2}}}{dx} = \frac{1}{2\sqrt{1 + x}}.$$

Por lo tanto

$$\frac{dy}{dx} = -\frac{(1+x)^{\frac{1}{2}}}{2(1+x)\sqrt{1-x}} - \frac{(1-x)^{\frac{1}{2}}}{2(1+x)\sqrt{1+x}}$$
$$= -\frac{1}{2\sqrt{1+x}\sqrt{1-x}} - \frac{\sqrt{1-x}}{2\sqrt{(1+x)^3}};$$

o
$$\frac{dy}{dx} = -\frac{1}{(1+x)\sqrt{1-x^2}}.$$

(6) Diferenciar $y = \sqrt{\dfrac{x^3}{1+x^2}}$.

Podemos escribir esto

$$y = x^{\frac{3}{2}}(1+x^2)^{-\frac{1}{2}};$$

$$\frac{dy}{dx} = \tfrac{3}{2}x^{\frac{1}{2}}(1+x^2)^{-\frac{1}{2}} + x^{\frac{3}{2}} \times \frac{d\left[(1+x^2)^{-\frac{1}{2}}\right]}{dx}.$$

Diferenciando $(1+x^2)^{-\frac{1}{2}}$, como se muestra en el ejemplo (2) arriba, obtenemos

$$\frac{d\left[(1+x^2)^{-\frac{1}{2}}\right]}{dx} = -\frac{x}{\sqrt{(1+x^2)^3}};$$

de modo que

$$\frac{dy}{dx} = \frac{3\sqrt{x}}{2\sqrt{1+x^2}} - \frac{\sqrt{x^5}}{\sqrt{(1+x^2)^3}} = \frac{\sqrt{x}(3+x^2)}{2\sqrt{(1+x^2)^3}}.$$

(7) Diferenciar $y = (x + \sqrt{x^2+x+a})^3$.

Sea $x + \sqrt{x^2+x+a} = u$.

$$\frac{du}{dx} = 1 + \frac{d\left[(x^2+x+a)^{\frac{1}{2}}\right]}{dx}.$$

$$y = u^3; \quad \text{y} \quad \frac{dy}{du} = 3u^2 = 3\left(x + \sqrt{x^2+x+a}\right)^2.$$

Ahora sea $(x^2+x+a)^{\frac{1}{2}} = v$ y $(x^2+x+a) = w$.

$$\frac{dw}{dx} = 2x+1; \quad v = w^{\frac{1}{2}}; \quad \frac{dv}{dw} = \tfrac{1}{2}w^{-\frac{1}{2}}.$$

$$\frac{dv}{dx} = \frac{dv}{dw} \times \frac{dw}{dx} = \tfrac{1}{2}(x^2+x+a)^{-\frac{1}{2}}(2x+1).$$

Por lo tanto $$\frac{du}{dx} = 1 + \frac{2x+1}{2\sqrt{x^2+x+a}},$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= 3\left(x+\sqrt{x^2+x+a}\right)^2\left(1+\frac{2x+1}{2\sqrt{x^2+x+a}}\right).$$

(8) Diferencia $y = \sqrt{\dfrac{a^2+x^2}{a^2-x^2}}\sqrt[3]{\dfrac{a^2-x^2}{a^2+x^2}}$.

Obtenemos

$$y = \frac{(a^2+x^2)^{\frac{1}{2}}(a^2-x^2)^{\frac{1}{3}}}{(a^2-x^2)^{\frac{1}{2}}(a^2+x^2)^{\frac{1}{3}}} = (a^2+x^2)^{\frac{1}{6}}(a^2-x^2)^{-\frac{1}{6}}.$$

$$\frac{dy}{dx} = (a^2+x^2)^{\frac{1}{6}}\frac{d\left[(a^2-x^2)^{-\frac{1}{6}}\right]}{dx} + \frac{d\left[(a^2+x^2)^{\frac{1}{6}}\right]}{(a^2-x^2)^{\frac{1}{6}}\,dx}.$$

Sea $u = (a^2-x^2)^{-\frac{1}{6}}$ y $v = (a^2-x^2)$.

$$u = v^{-\frac{1}{6}}; \quad \frac{du}{dv} = -\frac{1}{6}v^{-\frac{7}{6}}; \quad \frac{dv}{dx} = -2x.$$

$$\frac{du}{dx} = \frac{du}{dv} \times \frac{dv}{dx} = \frac{1}{3}x(a^2-x^2)^{-\frac{7}{6}}.$$

Sea $w = (a^2+x^2)^{\frac{1}{6}}$ y $z = (a^2+x^2)$.

$$w = z^{\frac{1}{6}}; \quad \frac{dw}{dz} = \frac{1}{6}z^{-\frac{5}{6}}; \quad \frac{dz}{dx} = 2x.$$

$$\frac{dw}{dx} = \frac{dw}{dz} \times \frac{dz}{dx} = \frac{1}{3}x(a^2+x^2)^{-\frac{5}{6}}.$$

Hence

$$\frac{dy}{dx} = (a^2 + x^2)^{\frac{1}{6}} \frac{x}{3(a^2 - x^2)^{\frac{7}{6}}} + \frac{x}{3(a^2 - x^2)^{\frac{1}{6}}(a^2 + x^2)^{\frac{5}{6}}};$$

o $$\frac{dy}{dx} = \frac{x}{3}\left[\sqrt[6]{\frac{a^2 + x^2}{(a^2 - x^2)^7}} + \frac{1}{\sqrt[6]{(a^2 - x^2)(a^2 + x^2)^5]}}\right].$$

(9) Diferencia y^n con respecto a y^5.

$$\frac{d(y^n)}{d(y^5)} = \frac{ny^{n-1}}{5y^{5-1}} = \frac{n}{5}y^{n-5}.$$

(10) Encuentra el primer y segundo coeficientes diferenciales de $y = \dfrac{x}{b}\sqrt{(a-x)x}$.

$$\frac{dy}{dx} = \frac{x}{b}\,\frac{d\left\{\left[(a-x)x\right]^{\frac{1}{2}}\right\}}{dx} + \frac{\sqrt{(a-x)x}}{b}.$$

Sea $\left[(a-x)x\right]^{\frac{1}{2}} = u$ y sea $(a-x)x = w$; entonces $u = w^{\frac{1}{2}}$.

$$\frac{du}{dw} = \frac{1}{2}w^{-\frac{1}{2}} = \frac{1}{2w^{\frac{1}{2}}} = \frac{1}{2\sqrt{(a-x)x}}.$$

$$\frac{dw}{dx} = a - 2x.$$

$$\frac{du}{dw} \times \frac{dw}{dx} = \frac{du}{dx} = \frac{a-2x}{2\sqrt{(a-x)x}}.$$

Por tanto

$$\frac{dy}{dx} = \frac{x(a-2x)}{2b\sqrt{(a-x)x}} + \frac{\sqrt{(a-x)x}}{b} = \frac{x(3a-4x)}{2b\sqrt{(a-x)x}}.$$

Ahora

$$\frac{d^2y}{dx^2} = \frac{2b\sqrt{(a-x)x}\,(3a-8x) - \dfrac{(3ax-4x^2)b(a-2x)}{\sqrt{(a-x)x}}}{4b^2(a-x)x}$$
$$= \frac{3a^2-12ax+8x^2}{4b(a-x)\sqrt{(a-x)x}}.$$

(Necesitaremos estos dos últimos coeficientes diferenciales más adelante. Ver Ej. X. No. 11.)

Exercises VI. (Ver page 258 para las Respuestas.)

Diferencia lo siguiente:

(1) $y = \sqrt{x^2+1}$.

(2) $y = \sqrt{x^2+a^2}$.

(3) $y = \dfrac{1}{\sqrt{a+x}}$.

(4) $y = \dfrac{a}{\sqrt{a-x^2}}$.

(5) $y = \dfrac{\sqrt{x^2-a^2}}{x^2}$.

(6) $y = \dfrac{\sqrt[3]{x^4+a}}{\sqrt[2]{x^3+a}}$.

(7) $y = \dfrac{a^2+x^2}{(a+x)^2}$.

(8) Diferencia y^5 con respecto a y^2.

(9) Diferencia $y = \dfrac{\sqrt{1-\theta^2}}{1-\theta}$.

El proceso puede extenderse a tres o más coeficientes diferenciales, de modo que $\dfrac{dy}{dx} = \dfrac{dy}{dz} \times \dfrac{dz}{dv} \times \dfrac{dv}{dx}$.

Examples.

(1) Si $z = 3x^4$; $\quad v = \dfrac{7}{z^2}$; $\quad y = \sqrt{1+v}$, encuentra $\dfrac{dv}{dx}$.

Tenemos

$$\frac{dy}{dv} = \frac{1}{2\sqrt{1+v}}; \quad \frac{dv}{dz} = -\frac{14}{z^3}; \quad \frac{dz}{dx} = 12x^3.$$

$$\frac{dy}{dx} = -\frac{168x^3}{(2\sqrt{1+v})z^3} = -\frac{28}{3x^5\sqrt{9x^8+7}}.$$

(2) Si $t = \dfrac{1}{5\sqrt{\theta}}$; $\quad x = t^3 + \dfrac{t}{2}$; $\quad v = \dfrac{7x^2}{\sqrt[3]{x-1}}$, encuentra $\dfrac{dv}{d\theta}$.

$$\frac{dv}{dx} = \frac{7x(5x-6)}{3\sqrt[3]{(x-1)^4}}; \quad \frac{dx}{dt} = 3t^2 + \tfrac{1}{2}; \quad \frac{dt}{d\theta} = -\frac{1}{10\sqrt{\theta^3}}.$$

Por tanto
$$\frac{dv}{d\theta} = -\frac{7x(5x-6)(3t^2+\frac{1}{2})}{30\sqrt[3]{(x-1)^4}\sqrt{\theta^3}},$$

una expresión en la que x debe ser reemplazada por su valor, y t por su valor en términos de θ.

(3) Si $\theta = \dfrac{3a^2x}{\sqrt{x^3}}$; $\quad \omega = \dfrac{\sqrt{1-\theta^2}}{1+\theta}$; $\quad$ y $\phi = \sqrt{3} - \dfrac{1}{\omega\sqrt{2}}$, encuentra $\dfrac{d\phi}{dx}$.

Obtenemos

$$\theta = 3a^2x^{-\frac{1}{2}}; \quad \omega = \sqrt{\frac{1-\theta}{1+\theta}}; \quad \text{y} \quad \phi = \sqrt{3} - \frac{1}{\sqrt{2}}\omega^{-1}.$$

$$\frac{d\theta}{dx} = -\frac{3a^2}{2\sqrt{x^3}}; \quad \frac{d\omega}{d\theta} = -\frac{1}{(1+\theta)\sqrt{1-\theta^2}}$$

(ver ejemplo 5, p. 71); y

$$\frac{d\phi}{d\omega} = \frac{1}{\sqrt{2}\omega^2}.$$

De modo que $\dfrac{d\theta}{dx} = \dfrac{1}{\sqrt{2}\times\omega^2} \times \dfrac{1}{(1+\theta)\sqrt{1-\theta^2}} \times \dfrac{3a^2}{2\sqrt{x^3}}$.

Reemplaza ahora primero ω, luego θ por su valor.

Exercises VII. Ahora puedes intentar exitosamente lo siguiente. (Ver page 259 para las Respuestas.)

(1) Si $u = \frac{1}{2}x^3$; $v = 3(u + u^2)$; y $w = \dfrac{1}{v^2}$, encuentra $\dfrac{dw}{dx}$.

(2) Si $y = 3x^2 + \sqrt{2}$; $z = \sqrt{1 + y}$; y $v = \dfrac{1}{\sqrt{3} + 4z}$, encuentra $\dfrac{dv}{dx}$.

(3) Si $y = \dfrac{x^3}{\sqrt{3}}$; $z = (1 + y)^2$; y $u = \dfrac{1}{\sqrt{1 + z}}$, encuentra $\dfrac{du}{dx}$.

CAPÍTULO X.

SIGNIFICADO GEOMÉTRICO DE LA DIFERENCIACIÓN.

Es útil considerar qué significado geométrico se puede dar al coeficiente diferencial.

En primer lugar, cualquier función de x, tal, por ejemplo, como x^2, o $\sqrt{x}$, o $ax + b$, puede graficarse como a curve; and nowadays every schoolboy is familiar with the process of curve-plotting.

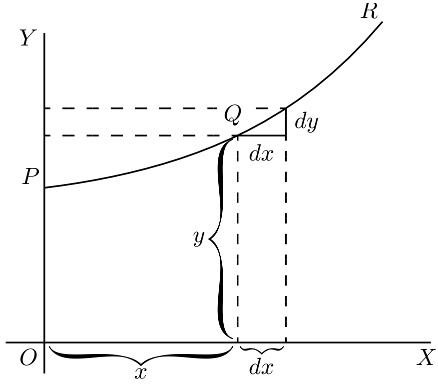


Fig. 7.

Let PQR, in Fig. 7, be a portion of a curve plotted with respect to the axes of coordinates OX and OY. Consider any point Q on this curve, where the abscissa of the point is x and its ordinate is y. Now observe how y changes when x is varied. If x is made to increase by a small increment dx, to the right, it will be observed that y also (in *this* particular curve) increases by a small increment dy (because this particular curve happens to be an *ascending* curve). Then the ratio of dy to dx is a measure of the degree to which the curve is sloping up between the two points Q and T. As a matter of fact, it can be seen on the figure that the curve between Q and T has many different slopes, so that we cannot very well speak of the slope of the curve between Q and T. If, however, Q and T are so near each other that the small portion QT of the curve is practically straight, then it is true to say that the ratio $\dfrac{dy}{dx}$ is the slope of the curve along QT. The straight line QT produced on either side touches the curve along the portion QT only, and if this portion is indefinitely small, the straight line will touch the curve at practically one point only, and be therefore a *tangent* to the curve.

This tangent to the curve has evidently the same slope as QT, so that $\dfrac{dy}{dx}$ is the slope of the tangent to the curve at the point Q for which the value of $\dfrac{dy}{dx}$ is found.

We have seen that the short expression "the slope of a curve" has no precise meaning, because a curve has so many slopes—in fact, every small portion of a curve has a different slope. "The slope of a curve *at a point*" is, however, a perfectly defined thing; it is the slope of a very small portion of the curve situated just at that point; and we have seen that this is the same as "the slope of the tangent to the curve at that point."

Observe that dx is a short step to the right, and dy the corresponding short step upwards. These steps must be considered as short as possible—in fact indefinitely short,—though in diagrams we have to represent them by bits that are not infinitesimally small, otherwise they could not be seen.

We shall hereafter make considerable use of this circumstance that $\dfrac{dy}{dx}$ *represents the slope of the curve at any point.*

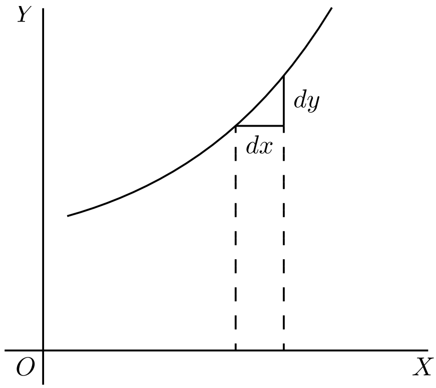


Fig. 8.

If a curve is sloping up at 45° at a particular point, as in Fig. 8, dy and dx will be equal, and the value of $\dfrac{dy}{dx} = 1$.

If the curve slopes up steeper than 45° (Fig. 9), $\dfrac{dy}{dx}$ will be greater than 1.

If the curve slopes up very gently, as in Fig. 10, $\dfrac{dy}{dx}$ will be a fraction smaller than 1.

For a horizontal line, or a horizontal place in a curve, $dy = 0$, and therefore $\dfrac{dy}{dx} = 0$.

If a curve slopes *downward*, as in Fig. 11, dy will be a step down, and must therefore be reckoned of negative value; hence $\dfrac{dy}{dx}$ will have negative sign also.

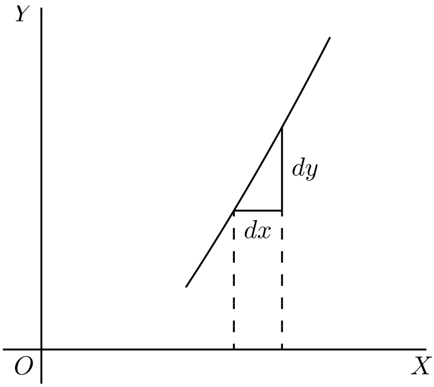


Fig. 9.

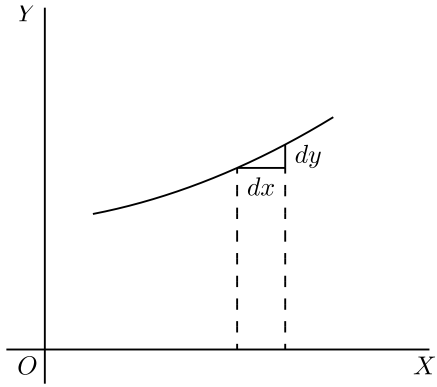


Fig. 10.

If the "curve" happens to be a straight line, like that in Fig. 12, the value of $\dfrac{dy}{dx}$ will be the same at all points along it. In other words its *slope* is constant.

If a curve is one that turns more upwards as it goes along to the right, the values of $\dfrac{dy}{dx}$ will become greater and greater with the increasing steepness, as in Fig. 13.

If a curve is one that gets flatter and flatter as it goes along, the values of $\dfrac{dy}{dx}$ will become smaller and smaller as the flatter part is reached, as in Fig. 14.

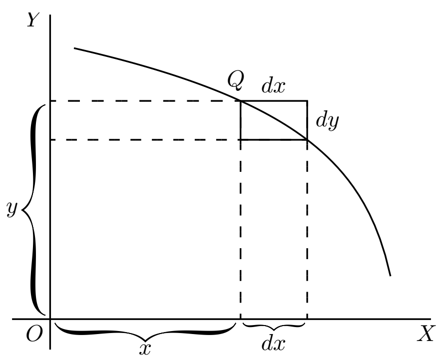


Fig. 11.

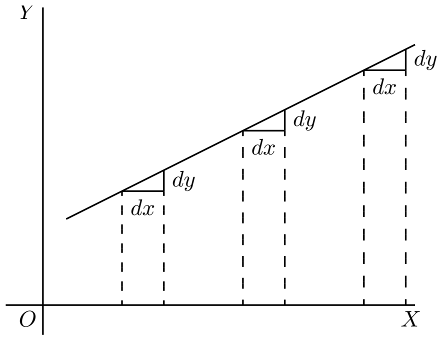


Fig. 12.

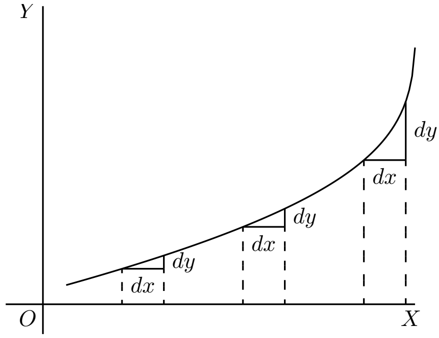


Fig. 13.

If a curve first descends, and then goes up again, as in Fig. 15, presenting a concavity upwards, then clearly $\dfrac{dy}{dx}$ will first be negative, with diminishing values as the curve flattens, then will be zero at the point where the bottom of the trough of the curve is reached; and from this point onward $\dfrac{dy}{dx}$ will have positive values that go on increasing. In such a case y is said to pass by a *minimum*. The minimum value of y is not necessarily the smallest value of y, it is that value of y corresponding to the bottom of the trough; for instance, in Fig. 28 (p. 102), the value of y corresponding to the bottom of the trough is 1, while y takes elsewhere values which are smaller than this. The characteristic of a

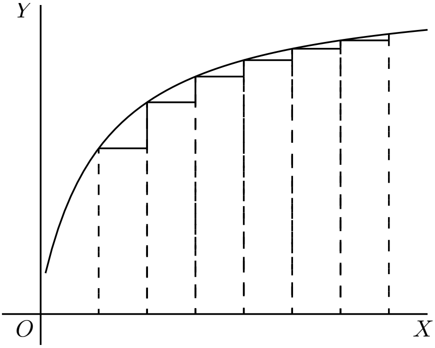


Fig. 14.

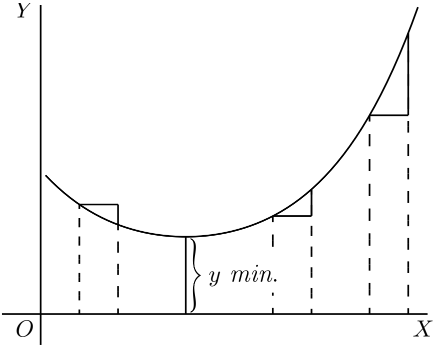


Fig. 15.

minimum is that y must increase *on either side* of it.

N.B.—For the particular value of x that makes y *a minimum*, the value of $\dfrac{dy}{dx} = 0$.

If a curve first ascends and then descends, the values of $\dfrac{dy}{dx}$ will be positive at first; then zero, as the summit is reached; then negative, as the curve slopes downwards, as in Fig. 16. In this case y is said to pass by a *maximum*, but the maximum value of y is not necessarily the greatest value of y. In Fig. 28, the maximum of y is $2\frac{1}{3}$, but this is by no means the greatest value y can have at some other point of the curve.

N.B.—For the particular value of x that makes y *a maximum*, the value of $\dfrac{dy}{dx} = 0$.

If a curve has the peculiar form of Fig. 17, the values of $\dfrac{dy}{dx}$ will always be positive; but there will be one particular place where the slope is least steep, where the value of $\dfrac{dy}{dx}$ will be a minimum; that is, less than it is at any other part of the curve.

If a curve has the form of Fig. 18, the value of $\dfrac{dy}{dx}$ will be negative

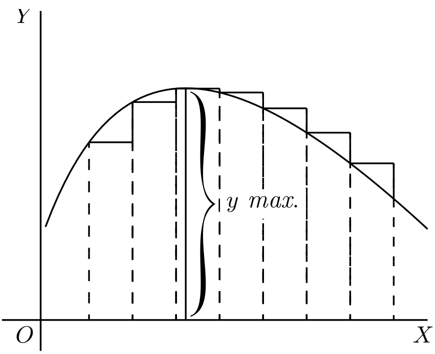


Fig. 16.

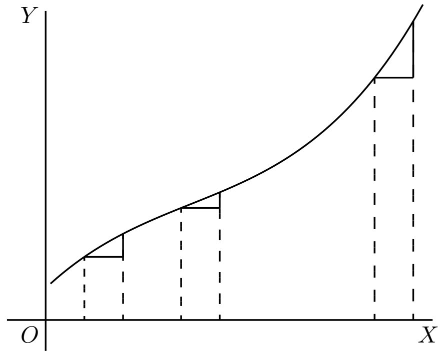


Fig. 17.

in the upper part, and positive in the lower part; while at the nose of the curve where it becomes actually perpendicular, the value of $\frac{dy}{dx}$ will be infinitely great.

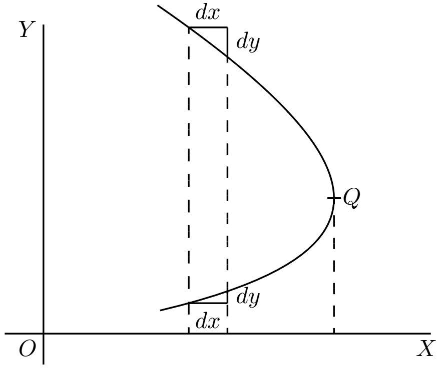


Fig. 18.

Now that we understand that $\frac{dy}{dx}$ measures the steepness of a curve at any point, let us turn to some of the equations which we have already learned how to differentiate.

(1) As the simplest case take this:

$$y = x + b.$$

It is plotted out in Fig. 19, using equal scales for x and y. If we put $x = 0$, then the corresponding ordinate will be $y = b$; that is to say, the "curve" crosses the y-axis at the height b. From here it ascends at 45°;

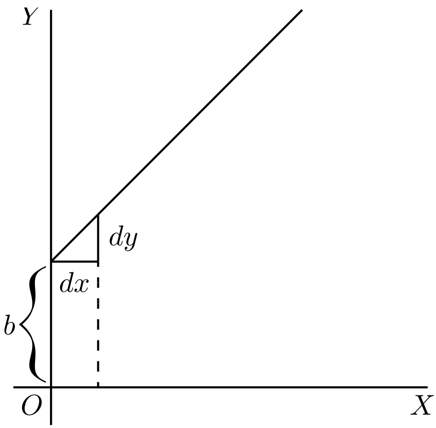


Fig. 19.

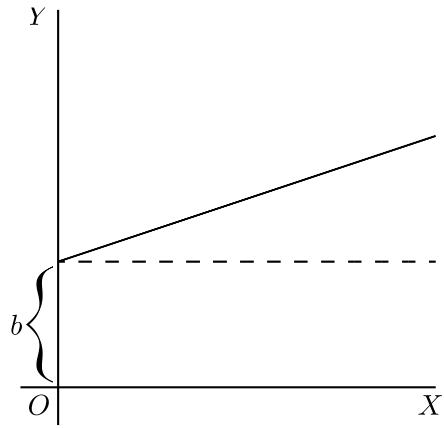


Fig. 20.

for whatever values we give to x to the right, we have an equal y to ascend. The line has a gradient of 1 in 1.

Now differentiate $y = x + b$, by the rules we have already learned (pp. 21 and 26 *ante*), and we get $\dfrac{dy}{dx} = 1$.

The slope of the line is such that for every little step dx to the right, we go an equal little step dy upward. And this slope is constant—always the same slope.

(2) Take another case:

$$y = ax + b.$$

We know that this curve, like the preceding one, will start from a height b on the y-axis. But before we draw the curve, let us find its slope by differentiating; which gives $\dfrac{dy}{dx} = a$. The slope will be constant, at an angle, the tangent of which is here called a. Let us assign to a some numerical value—say $\frac{1}{3}$. Then we must give it such a slope that it ascends 1 in 3; or dx will be 3 times as great as dy; as magnified in Fig. 21. So, draw the line in Fig. 20 at this slope.

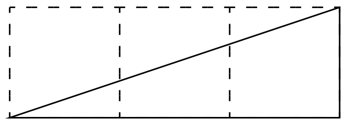


FIG. 21.

(3) Now for a slightly harder case.

Let $$y = ax^2 + b.$$

Again the curve will start on the y-axis at a height b above the origin.

Now differentiate. [If you have forgotten, turn back to p. 26; or, rather, *don't* turn back, but think out the differentiation.]

$$\frac{dy}{dx} = 2ax.$$

This shows that the steepness will not be constant: it increases as x increases. At the starting point P, where $x = 0$, the curve (Fig. 22) has no steepness—that is, it is level. On the left of the origin, where x has negative values, $\dfrac{dy}{dx}$ will also have negative values, or will descend from left to right, as in the Figure.

Let us illustrate this by working out a particular instance. Taking the equation

$$y = \tfrac{1}{4}x^2 + 3,$$

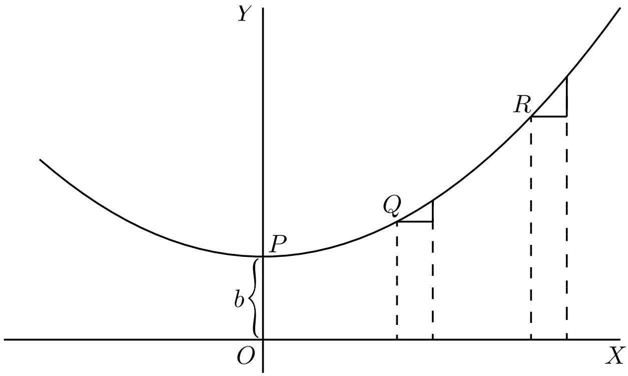


FIG. 22.

and differentiating it, we get

$$\frac{dy}{dx} = \tfrac{1}{2}x.$$

Now assign a few successive values, say from 0 to 5, to x; and calculate the corresponding values of y by the first equation; and of $\frac{dy}{dx}$ from the second equation. Tabulating results, we have:

x	0	1	2	3	4	5
y	3	$3\frac{1}{4}$	4	$5\frac{1}{4}$	7	$9\frac{1}{4}$
$\frac{dy}{dx}$	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$

Then plot them out in two curves, Figs. 23 and 24, in Fig. 23 plotting the values of y against those of x and in Fig. 24 those of $\frac{dy}{dx}$ against those of x. For any assigned value of x, the *height* of the ordinate in the second curve is proportional to the *slope* of the first curve.

If a curve comes to a sudden cusp, as in Fig. 25, the slope at that point suddenly changes from a slope upward to a slope downward. In

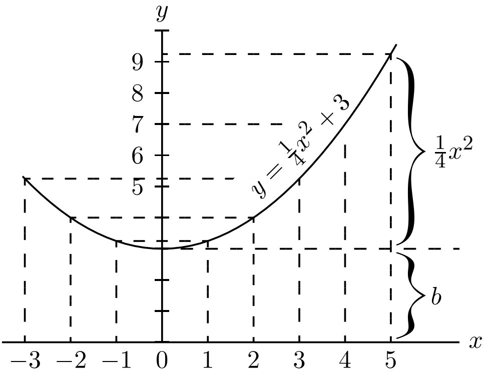


Fig. 23.

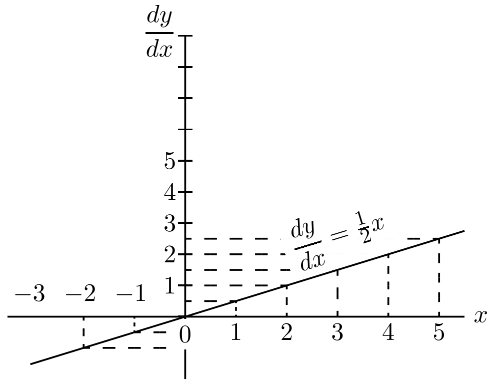


Fig. 24.

that case $\dfrac{dy}{dx}$ will clearly undergo an abrupt change from a positive to a negative value.

The following examples show further applications of the principles just explained.

(4) Find the slope of the tangent to the curve

$$y = \frac{1}{2x} + 3,$$

at the point where $x = -1$. Find the angle which this tangent makes

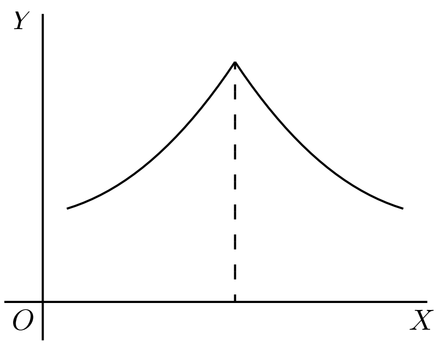


Fig. 25.

with the curve $y = 2x^2 + 2$.

The slope of the tangent is the slope of the curve at the point where they touch one another (see p. 79); that is, it is the $\frac{dy}{dx}$ of the curve for that point. Here $\frac{dy}{dx} = -\frac{1}{2x^2}$ and for $x = -1$, $\frac{dy}{dx} = -\frac{1}{2}$, which is the slope of the tangent and of the curve at that point. The tangent, being a straight line, has for equation $y = ax + b$, and its slope is $\frac{dy}{dx} = a$, hence $a = -\frac{1}{2}$. Also if $x = -1$, $y = \frac{1}{2(-1)} + 3 = 2\frac{1}{2}$; and as the tangent passes by this point, the coordinates of the point must satisfy the equation of the tangent, namely

$$y = -\frac{1}{2}x + b,$$

so that $2\frac{1}{2} = -\frac{1}{2} \times (-1) + b$ and $b = 2$; the equation of the tangent is therefore $y = -\frac{1}{2}x + 2$.

Now, when two curves meet, the intersection being a point common to both curves, its coordinates must satisfy the equation of each one of the two curves; that is, it must be a solution of the system of simultaneous equations formed by coupling together the equations of the curves. Here the curves meet one another at points given by the solution of

$$\begin{cases} y = 2x^2 + 2, \\ y = -\frac{1}{2}x + 2 \quad \text{or} \quad 2x^2 + 2 = -\frac{1}{2}x + 2; \end{cases}$$

that is,
$$x(2x + \tfrac{1}{2}) = 0.$$

This equation has for its solutions $x = 0$ and $x = -\frac{1}{4}$. The slope of the curve $y = 2x^2 + 2$ at any point is

$$\frac{dy}{dx} = 4x.$$

For the point where $x = 0$, this slope is zero; the curve is horizontal. For the point where

$$x = -\frac{1}{4}, \quad \frac{dy}{dx} = -1;$$

hence the curve at that point slopes downwards to the right at such an angle θ with the horizontal that $\tan\theta = 1$; that is, at 45° to the horizontal.

The slope of the straight line is $-\frac{1}{2}$; that is, it slopes downwards to the right and makes with the horizontal an angle ϕ such that $\tan\phi = \frac{1}{2}$; that is, an angle of $26^\circ\ 34'$. It follows that at the first point the curve cuts the straight line at an angle of $26^\circ\ 34'$, while at the second it cuts it at an angle of $45^\circ - 26^\circ\ 34' = 18^\circ\ 26'$.

(5) A straight line is to be drawn, through a point whose coordinates are $x = 2$, $y = -1$, as tangent to the curve $y = x^2 - 5x + 6$. Find the coordinates of the point of contact.

The slope of the tangent must be the same as the $\frac{dy}{dx}$ of the curve; that is, $2x - 5$.

The equation of the straight line is $y = ax+b$, and as it is satisfied for the values $x = 2$, $y = -1$, then $-1 = a\times 2+b$; also, its $\frac{dy}{dx} = a = 2x-5$.

The x and the y of the point of contact must also satisfy both the equation of the tangent and the equation of the curve.

We have then

$$\begin{cases} y = x^2 - 5x + 6, & \text{(i)} \\ y = ax + b, & \text{(ii)} \\ -1 = 2a + b, & \text{(iii)} \\ a = 2x - 5, & \text{(iv)} \end{cases}$$

four equations in a, b, x, y.

Equations (i) and (ii) give $x^2 - 5x + 6 = ax + b$.

Replacing a and b by their value in this, we get

$$x^2 - 5x + 6 = (2x - 5)x - 1 - 2(2x - 5),$$

which simplifies to $x^2 - 4x + 3 = 0$, the solutions of which are: $x = 3$ and $x = 1$. Replacing in (i), we get $y = 0$ and $y = 2$ respectively; the two points of contact are then $x = 1$, $y = 2$, and $x = 3$, $y = 0$.

Note.—In all exercises dealing with curves, students will find it extremely instructive to verify the deductions obtained by actually plotting the curves.

Exercises VIII. (See page 259 for Answers.)

(1) Plot the curve $y = \frac{3}{4}x^2 - 5$, using a scale of millimetres. Measure at points corresponding to different values of x, the angle of its slope.

Find, by differentiating the equation, the expression for slope; and see, from a Table of Natural Tangents, whether this agrees with the measured angle.

(2) Find what will be the slope of the curve

$$y = 0.12x^3 - 2,$$

at the particular point that has as abscissa $x = 2$.

(3) If $y = (x - a)(x - b)$, show that at the particular point of the curve where $\dfrac{dy}{dx} = 0$, x will have the value $\frac{1}{2}(a + b)$.

(4) Find the $\dfrac{dy}{dx}$ of the equation $y = x^3 + 3x$; and calculate the numerical values of $\dfrac{dy}{dx}$ for the points corresponding to $x = 0$, $x = \frac{1}{2}$, $x = 1$, $x = 2$.

(5) In the curve to which the equation is $x^2 + y^2 = 4$, find the values of x at those points where the slope $= 1$.

(6) Find the slope, at any point, of the curve whose equation is $\dfrac{x^2}{3^2} + \dfrac{y^2}{2^2} = 1$; and give the numerical value of the slope at the place where $x = 0$, and at that where $x = 1$.

(7) The equation of a tangent to the curve $y = 5 - 2x + 0.5x^3$, being of the form $y = mx + n$, where m and n are constants, find the value of m and n if the point where the tangent touches the curve has $x = 2$ for abscissa.

(8) At what angle do the two curves

$$y = 3.5x^2 + 2 \quad \text{and} \quad y = x^2 - 5x + 9.5$$

cut one another?

(9) Tangents to the curve $y = \pm\sqrt{25 - x^2}$ are drawn at points for which $x = 3$ and $x = 4$. Find the coordinates of the point of intersection of the tangents and their mutual inclination.

(10) A straight line $y = 2x - b$ touches a curve $y = 3x^2 + 2$ at one point. What are the coordinates of the point of contact, and what is the value of b?

CAPÍTULO XI.

MÁXIMOS Y MÍNIMOS.

One of the principal uses of the process of differentiating is to find out under what conditions the value of the thing differentiated becomes a maximum, or a minimum. This is often exceedingly important in engineering questions, where it is most desirable to know what conditions will make the cost of working a minimum, or will make the efficiency a maximum.

Now, to begin with a concrete case, let us take the equation

$$y = x^2 - 4x + 7.$$

By assigning a number of successive values to x, and finding the corresponding values of y, we can readily see that the equation represents a curve with a minimum.

x	0	1	2	3	4	5
y	7	4	3	4	7	12

These values are plotted in Fig. 26, which shows that y has apparently a minimum value of 3, when x is made equal to 2. But are you sure that the minimum occurs at 2, and not at $2\frac{1}{4}$ or at $1\frac{3}{4}$?

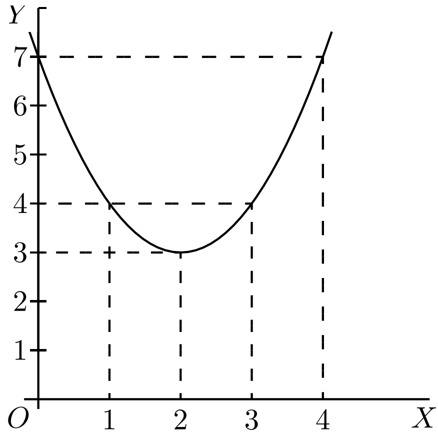


Fig. 26.

Of course it would be possible with any algebraic expression to work out a lot of values, and in this way arrive gradually at the particular value that may be a maximum or a minimum.

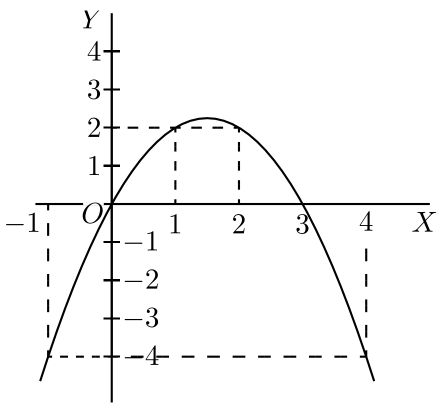


Fig. 27.

Here is another example:

Let $$y = 3x - x^2.$$

Calculate a few values thus:

x	-1	0	1	2	3	4	5
y	-4	0	2	2	0	-4	-10

Plot these values as in Fig. 27.

It will be evident that there will be a maximum somewhere between $x = 1$ and $x = 2$; and the thing *looks* as if the maximum value of y ought to be about $2\frac{1}{4}$. Try some intermediate values. If $x = 1\frac{1}{4}$, $y = 2.187$; if $x = 1\frac{1}{2}$, $y = 2.25$; if $x = 1.6$, $y = 2.24$. How can we be sure that 2.25 is the real maximum, or that it occurs exactly when $x = 1\frac{1}{2}$?

Now it may sound like juggling to be assured that there is a way by which one can arrive straight at a maximum (or minimum) value without making a lot of preliminary trials or guesses. And that way depends on differentiating. Look back to an earlier page (81) for the remarks about Figs. 14 and 15, and you will see that whenever a curve gets either to its maximum or to its minimum height, at that point its $\dfrac{dy}{dx} = 0$. Now this gives us the clue to the dodge that is wanted. When there is put before you an equation, and you want to find that value of x that will make its y a minimum (or a maximum), *first differentiate it*, and having done so, write its $\dfrac{dy}{dx}$ as *equal to zero*, and then solve for x. Put this particular value of x into the original equation, and you will then get the required value of y. This process is commonly called "equating to zero."

To see how simply it works, take the example with which this chapter opens, namely

$$y = x^2 - 4x + 7.$$

Differentiating, we get:

$$\frac{dy}{dx} = 2x - 4.$$

Now equate this to zero, thus:

$$2x - 4 = 0.$$

Solving this equation for x, we get:

$$2x = 4,$$
$$x = 2.$$

Now, we know that the maximum (or minimum) will occur exactly when $x = 2$.

Putting the value $x = 2$ into the original equation, we get

$$\begin{aligned} y &= 2^2 - (4 \times 2) + 7 \\ &= 4 - 8 + 7 \\ &= 3. \end{aligned}$$

Now look back at Fig. 26, and you will see that the minimum occurs when $x = 2$, and that this minimum of $y = 3$.

Try the second example (Fig. 24), which is

$$y = 3x - x^2.$$

Differentiating, $$\frac{dy}{dx} = 3 - 2x.$$

Equating to zero,

$$3 - 2x = 0,$$

whence $$x = 1\tfrac{1}{2};$$

and putting this value of x into the original equation, we find:

$$y = 4\tfrac{1}{2} - (1\tfrac{1}{2} \times 1\tfrac{1}{2}),$$
$$y = 2\tfrac{1}{4}.$$

This gives us exactly the information as to which the method of trying a lot of values left us uncertain.

Now, before we go on to any further cases, we have two remarks to make. When you are told to equate $\dfrac{dy}{dx}$ to zero, you feel at first (that is if you have any wits of your own) a kind of resentment, because you know that $\dfrac{dy}{dx}$ has all sorts of different values at different parts of the curve, according to whether it is sloping up or down. So, when you are suddenly told to write

$$\frac{dy}{dx} = 0,$$

you resent it, and feel inclined to say that it can't be true. Now you will have to understand the essential difference between "an equation," and "an equation of condition." Ordinarily you are dealing with equations that are true in themselves, but, on occasions, of which the present are examples, you have to write down equations that are not necessarily true, but are only true if certain conditions are to be fulfilled; and you write them down in order, by solving them, to find the conditions which make them true. Now we want to find the particular value that x has when the curve is neither sloping up nor sloping down, that is, at the particular place where $\dfrac{dy}{dx} = 0$. So, writing $\dfrac{dy}{dx} = 0$ does *not* mean that it always is $= 0$; but you write it down *as a condition* in order to see how much x will come out if $\dfrac{dy}{dx}$ is to be zero.

The second remark is one which (if you have any wits of your own) you will probably have already made: namely, that this much-belauded process of equating to zero entirely fails to tell you whether the x that you thereby find is going to give you a *maximum* value of y or a *minimum* value of y. Quite so. It does not of itself discriminate; it finds for you the right value of x but leaves you to find out for yourselves whether the corresponding y is a maximum or a minimum. Of course, if you have plotted the curve, you know already which it will be.

For instance, take the equation:

$$y = 4x + \frac{1}{x}.$$

Without stopping to think what curve it corresponds to, differentiate it, and equate to zero:

$$\frac{dy}{dx} = 4 - x^{-2} = 4 - \frac{1}{x^2} = 0;$$

whence $x = \frac{1}{2}$;

and, inserting this value,

$$y = 4$$

will be either a maximum or else a minimum. But which? You will hereafter be told a way, depending upon a second differentiation, (see Chap. XII., p. 112). But at present it is enough if you will simply try any other value of x differing a little from the one found, and see whether with this altered value the corresponding value of y is less or greater than that already found.

Try another simple problem in maxima and minima. Suppose you were asked to divide any number into two parts, such that the product was a maximum? How would you set about it if you did not know the trick of equating to zero? I suppose you could worry it out by the rule of try, try, try again. Let 60 be the number. You can try cutting it into two parts, and multiplying them together. Thus, 50 times 10 is 500; 52 times 8 is 416; 40 times 20 is 800; 45 times 15 is 675; 30 times 30 is 900. This looks like a maximum: try varying it. 31 times 29 is 899, which is not so good; and 32 times 28 is 896, which is worse. So it seems that the biggest product will be got by dividing into two equal halves.

Now see what the calculus tells you. Let the number to be cut into two parts be called n. Then if x is one part, the other will be $n - x$, and the product will be $x(n - x)$ or $nx - x^2$. So we write $y = nx - x^2$. Now differentiate and equate to zero;

$$\frac{dy}{dx} = n - 2x = 0$$

Solving for x, we get

$$\frac{n}{2} = x.$$

So now we *know* that whatever number n may be, we must divide it into two equal parts if the product of the parts is to be a maximum; and the value of that maximum product will always be $= \frac{1}{4}n^2$.

This is a very useful rule, and applies to any number of factors, so that if $m + n + p =$ a constant number, $m \times n \times p$ is a maximum when $m = n = p$.

Test Case.

Let us at once apply our knowledge to a case that we can test.

Let $$y = x^2 - x;$$

and let us find whether this function has a maximum or minimum; and if so, test whether it is a maximum or a minimum.

Differentiating, we get

$$\frac{dy}{dx} = 2x - 1.$$

Equating to zero, we get

$$2x - 1 = 0,$$

whence $$2x = 1,$$

or $$x = \tfrac{1}{2}.$$

That is to say, when x is made $= \frac{1}{2}$, the corresponding value of y will be either a maximum or a minimum. Accordingly, putting $x = \frac{1}{2}$ in the original equation, we get

$$y = (\tfrac{1}{2})^2 - \tfrac{1}{2},$$

or $$y = -\tfrac{1}{4}.$$

Is this a maximum or a minimum? To test it, try putting x a little bigger than $\frac{1}{2}$,—say make $x = 0.6$. Then

$$y = (0.6)^2 - 0.6 = 0.36 - 0.6 = -0.24,$$

which is higher up than -0.25; showing that $y = -0.25$ is a *minimum.*

Plot the curve for yourself, and verify the calculation.

Further Examples.

A most interesting example is afforded by a curve that has both a maximum and a minimum. Its equation is:

$$y = \tfrac{1}{3}x^3 - 2x^2 + 3x + 1.$$

Now $$\frac{dy}{dx} = x^2 - 4x + 3.$$

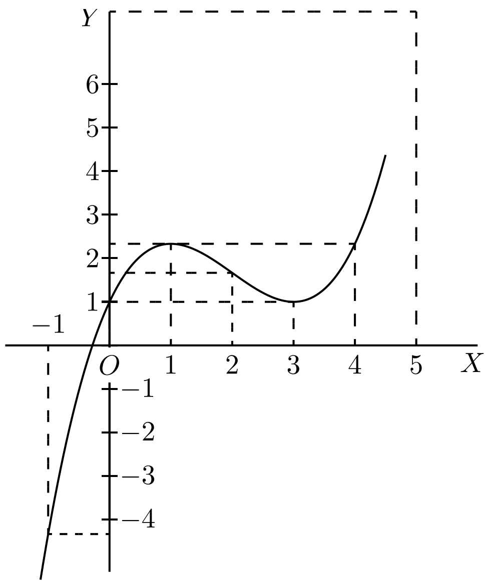


FIG. 28.

Equating to zero, we get the quadratic,

$$x^2 - 4x + 3 = 0;$$

and solving the quadratic gives us *two* roots, viz.

$$\begin{cases} x = 3 \\ x = 1. \end{cases}$$

Now, when $x = 3$, $y = 1$; and when $x = 1$, $y = 2\frac{1}{3}$. The first of these is a minimum, the second a maximum.

The curve itself may be plotted (as in Fig. 28) from the values calculated, as below, from the original equation.

x	-1	0	1	2	3	4	5	6
y	$-4\frac{1}{3}$	1	$2\frac{1}{3}$	$1\frac{2}{3}$	1	$2\frac{1}{3}$	$7\frac{2}{3}$	19

A further exercise in maxima and minima is afforded by the following example:

The equation to a circle of radius r, having its centre C at the point whose coordinates are $x = a$, $y = b$, as depicted in Fig. 29, is:

$$(y - b)^2 + (x - a)^2 = r^2.$$

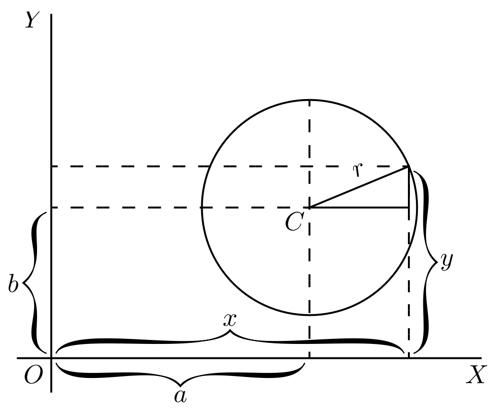


FIG. 29.

This may be transformed into

$$y = \sqrt{r^2 - (x - a)^2} + b.$$

Now we know beforehand, by mere inspection of the figure, that when $x = a$, y will be either at its maximum value, $b + r$, or else at its minimum value, $b - r$. But let us not take advantage of this knowledge; let us set about finding what value of x will make y a maximum or a minimum, by the process of differentiating and equating to zero.

$$\frac{dy}{dx} = \frac{1}{2} \frac{1}{\sqrt{r^2 - (x - a)^2}} \times (2a - 2x),$$

which reduces to

$$\frac{dy}{dx} = \frac{a - x}{\sqrt{r^2 - (x - a)^2}}.$$

Then the condition for y being maximum or minimum is:

$$\frac{a - x}{\sqrt{r^2 - (x - a)^2}} = 0.$$

Since no value whatever of x will make the denominator infinite, the only condition to give zero is

$$x = a.$$

Inserting this value in the original equation for the circle, we find

$$y = \sqrt{r^2} + b;$$

and as the root of r^2 is either $+r$ or $-r$, we have two resulting values of y,

$$\begin{cases} y = b + r \\ y = b - r. \end{cases}$$

The first of these is the maximum, at the top; the second the minimum, at the bottom.

If the curve is such that there is no place that is a maximum or minimum, the process of equating to zero will yield an impossible result. For instance:

Let $$y = ax^3 + bx + c.$$

Then $$\frac{dy}{dx} = 3ax^2 + b.$$

Equating this to zero, we get $3ax^2 + b = 0$,

$$x^2 = \frac{-b}{3a}, \quad \text{and} \quad x = \sqrt{\frac{-b}{3a}}, \text{ which is impossible.}$$

Therefore y has no maximum nor minimum.

A few more worked examples will enable you to thoroughly master this most interesting and useful application of the calculus.

(1) What are the sides of the rectangle of maximum area inscribed in a circle of radius R?

If one side be called x,

$$\text{the other side} = \sqrt{(\text{diagonal})^2 - x^2};$$

and as the diagonal of the rectangle is necessarily a diameter, the other side $= \sqrt{4R^2 - x^2}$.

Then, area of rectangle $S = x\sqrt{4R^2 - x^2}$,

$$\frac{dS}{dx} = x \times \frac{d\left(\sqrt{4R^2 - x^2}\,\right)}{dx} + \sqrt{4R^2 - x^2} \times \frac{d(x)}{dx}.$$

If you have forgotten how to differentiate $\sqrt{4R^2 - x^2}$, here is a hint: write $4R^2 - x^2 = w$ and $y = \sqrt{w}$, and seek $\frac{dy}{dw}$ and $\frac{dw}{dx}$; fight it out, and only if you can't get on refer to page 69.

You will get

$$\frac{dS}{dx} = x \times -\frac{x}{\sqrt{4R^2 - x^2}} + \sqrt{4R^2 - x^2} = \frac{4R^2 - 2x^2}{\sqrt{4R^2 - x^2}}.$$

For maximum or minimum we must have

$$\frac{4R^2 - 2x^2}{\sqrt{4R^2 - x^2}} = 0;$$

that is, $4R^2 - 2x^2 = 0$ and $x = R\sqrt{2}$.

The other side $= \sqrt{4R^2 - 2R^2} = R\sqrt{2}$; the two sides are equal; the figure is a square the side of which is equal to the diagonal of the square constructed on the radius. In this case it is, of course, a maximum with which we are dealing.

(2) What is the radius of the opening of a conical vessel the sloping side of which has a length l when the capacity of the vessel is greatest?

If R be the radius and H the corresponding height, $H = \sqrt{l^2 - R^2}$.

$$\text{Volume } V = \pi R^2 \times \frac{H}{3} = \pi R^2 \times \frac{\sqrt{l^2 - R^2}}{3}.$$

Proceeding as in the previous problem, we get

$$\begin{aligned}\frac{dV}{dR} &= \pi R^2 \times -\frac{R}{3\sqrt{l^2 - R^2}} + \frac{2\pi R}{3}\sqrt{l^2 - R^2} \\ &= \frac{2\pi R(l^2 - R^2) - \pi R^3}{3\sqrt{l^2 - R^2}} = 0\end{aligned}$$

for maximum or minimum.

Or, $2\pi R(l^2 - R^2) - \pi R^2 = 0$, and $R = l\sqrt{\frac{2}{3}}$, for a maximum, obviously.

(3) Find the maxima and minima of the function

$$y = \frac{x}{4 - x} + \frac{4 - x}{x}.$$

We get

$$\frac{dy}{dx} = \frac{(4-x)-(-x)}{(4-x)^2} + \frac{-x-(4-x)}{x^2} = 0$$

for maximum or minimum; or

$$\frac{4}{(4-x)^2} - \frac{4}{x^2} = 0 \quad \text{and} \quad x = 2.$$

There is only one value, hence only one maximum or minimum.

$$\begin{aligned} \text{For} \quad x &= 2, & y &= 2, \\ \text{for} \quad x &= 1.5, & y &= 2.27, \\ \text{for} \quad x &= 2.5, & y &= 2.27; \end{aligned}$$

it is therefore a minimum. (It is instructive to plot the graph of the function.)

(4) Find the maxima and minima of the function $y = \sqrt{1+x} + \sqrt{1-x}$. (It will be found instructive to plot the graph.)

Differentiating gives at once (see example No. 1, p. 70)

$$\frac{dy}{dx} = \frac{1}{2\sqrt{1+x}} - \frac{1}{2\sqrt{1-x}} = 0$$

for maximum or minimum.

Hence $\sqrt{1+x} = \sqrt{1-x}$ and $x = 0$, the only solution

For $x = 0$, $\quad y = 2$.

For $x = \pm 0.5$, $y = 1.932$, so this is a maximum.

(5) Find the maxima and minima of the function

$$y = \frac{x^2-5}{2x-4}.$$

We have

$$\frac{dy}{dx} = \frac{(2x-4) \times 2x - (x^2-5)2}{(2x-4)^2} = 0$$

for maximum or minimum; or

$$\frac{2x^2 - 8x + 10}{(2x-4)^2} = 0;$$

or $x^2 - 4x + 5 = 0$; which has for solutions

$$x = \tfrac{5}{2} \pm \sqrt{-1}.$$

These being imaginary, there is no real value of x for which $\dfrac{dy}{dx} = 0$; hence there is neither maximum nor minimum.

(6) Find the maxima and minima of the function

$$(y - x^2)^2 = x^5.$$

This may be written $y = x^2 \pm x^{\frac{5}{2}}$.

$$\frac{dy}{dx} = 2x \pm \tfrac{5}{2} x^{\frac{3}{2}} = 0 \quad \text{for maximum or minimum;}$$

that is, $x(2 \pm \frac{5}{2} x^{\frac{1}{2}}) = 0$, which is satisfied for $x = 0$, and for $2 \pm \frac{5}{2} x^{\frac{1}{2}} = 0$, that is for $x = \frac{16}{25}$. So there are two solutions.

Taking first $x = 0$. If $x = -0.5$, $y = 0.25 \pm \sqrt[2]{-(.5)^5}$, and if $x = +0.5$, $y = 0.25 \pm \sqrt[2]{(.5)^5}$. On one side y is imaginary; that is, there is no value of y that can be represented by a graph; the latter is therefore entirely on the right side of the axis of y (see Fig. 30).

On plotting the graph it will be found that the curve goes to the origin, as if there were a minimum there; but instead of continuing beyond, as it should do for a minimum, it retraces its steps (forming what is called a "cusp"). There is no minimum, therefore, although the condition for a minimum is satisfied, namely $\frac{dy}{dx} = 0$. It is necessary therefore always to check by taking one value on either side.

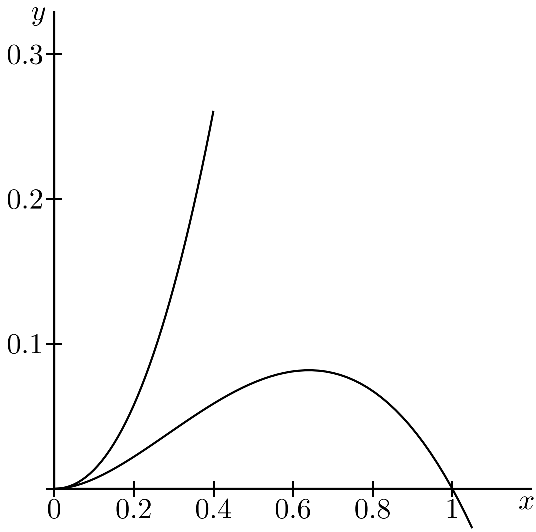


FIG. 30.

Now, if we take $x = \frac{16}{25} = 0.64$. If $x = 0.64$, $y = 0.7373$ and $y = 0.0819$; if $x = 0.6$, y becomes 0.6389 and 0.0811; and if $x = 0.7$, y becomes 0.8996 and 0.0804.

This shows that there are two branches of the curve; the upper one does not pass through a maximum, but the lower one does.

(7) A cylinder whose height is twice the radius of the base is increasing in volume, so that all its parts keep always in the same proportion to each other; that is, at any instant, the cylinder is *similar* to the original cylinder. When the radius of the base is r feet, the surface area is increasing at the rate of 20 square inches per second; at what

rate is its volume then increasing?

$$
\begin{aligned}
\text{Area} = S = 2(\pi r^2) + 2\pi r \times 2r = 6\pi r^2.\\
\text{Volume} = V = \pi r^2 \times 2r = 2\pi r^3.\\
\frac{dS}{dr} = 12\pi r, \quad \frac{dV}{dr} = 6\pi r^2,\\
dS = 12\pi r\, dr = 20, \quad dr = \frac{20}{12\pi r},\\
dV = 6\pi r^2\, dr = 6\pi r^2 \times \frac{20}{12\pi r} = 10r.
\end{aligned}
$$

The volume changes at the rate of $10r$ cubic inches.

Make other examples for yourself. There are few subjects which offer such a wealth for interesting examples.

Exercises IX. (See page 260 for Answers.)

(1) What values of x will make y a maximum and a minimum, if $y = \dfrac{x^2}{x+1}$?

(2) What value of x will make y a maximum in the equation $y = \dfrac{x}{a^2 + x^2}$?

(3) A line of length p is to be cut up into 4 parts and put together as a rectangle. Show that the area of the rectangle will be a maximum if each of its sides is equal to $\frac{1}{4}p$.

(4) A piece of string 30 inches long has its two ends joined together and is stretched by 3 pegs so as to form a triangle. What is the largest triangular area that can be enclosed by the string?

(5) Plot the curve corresponding to the equation

$$y = \frac{10}{x} + \frac{10}{8-x};$$

also find $\dfrac{dy}{dx}$, and deduce the value of x that will make y a minimum; and find that minimum value of y.

(6) If $y = x^5 - 5x$, find what values of x will make y a maximum or a minimum.

(7) What is the smallest square that can be inscribed in a given square?

(8) Inscribe in a given cone, the height of which is equal to the radius of the base, a cylinder (a) whose volume is a maximum; (b) whose lateral area is a maximum; (c) whose total area is a maximum.

(9) Inscribe in a sphere, a cylinder (a) whose volume is a maximum; (b) whose lateral area is a maximum; (c) whose total area is a maximum.

(10) A spherical balloon is increasing in volume. If, when its radius is r feet, its volume is increasing at the rate of 4 cubic feet per second, at what rate is its surface then increasing?

(11) Inscribe in a given sphere a cone whose volume is a maximum.

(12) The current C given by a battery of N similar voltaic cells is $C = \dfrac{n \times E}{R + \dfrac{rn^2}{N}}$, where E, R, r, are constants and n is the number of cells coupled in series. Find the proportion of n to N for which the current is greatest.

CAPÍTULO XII.

CURVATURA DE CURVAS.

Returning to the process of successive differentiation, it may be asked: Why does anybody want to differentiate twice over? We know that when the variable quantities are space and time, by differentiating twice over we get the acceleration of a moving body, and that in the geometrical interpretation, as applied to curves, $\frac{dy}{dx}$ means the *slope* of the curve. But what can $\frac{d^2y}{dx^2}$ mean in this case? Clearly it means the rate (per unit of length x) at which the slope is changing—in brief, it is *a measure of the curvature of the slope.*

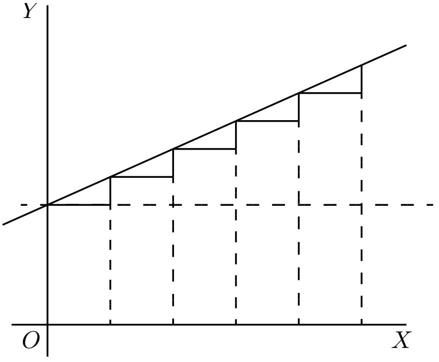


Fig. 31.

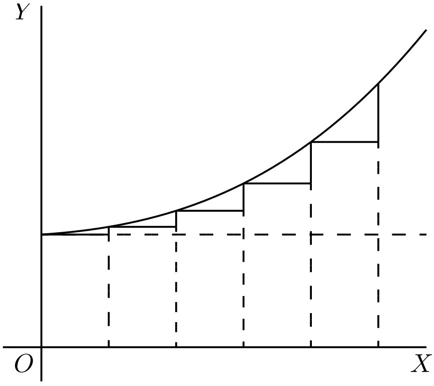


Fig. 32.

Suppose a slope constant, as in Fig. 31.

Here, $\dfrac{dy}{dx}$ is of constant value.

Suppose, however, a case in which, like Fig. 32, the slope itself is getting greater upwards, then $\dfrac{d\left(\dfrac{dy}{dx}\right)}{dx}$, that is, $\dfrac{d^2y}{dx^2}$, will be *positive*.

If the slope is becoming less as you go to the right (as in Fig. 14, p. 83), or as in Fig. 33, then, even though the curve may be going upward, since the change is such as to diminish its slope, its $\dfrac{d^2y}{dx^2}$ will be *negative*.

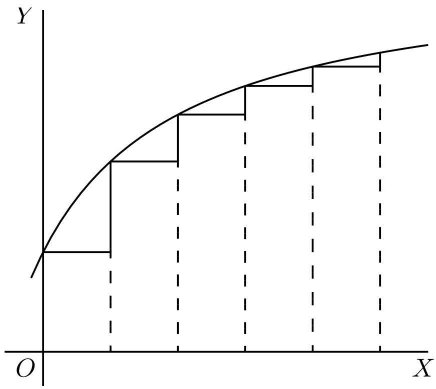


FIG. 33.

It is now time to initiate you into another secret—how to tell whether the result that you get by "equating to zero" is a maximum or a minimum. The trick is this: After you have differentiated (so as to get the expression which you equate to zero), you then differentiate a second time, and look whether the result of the second differentiation is *positive* or *negative*. If $\dfrac{d^2y}{dx^2}$ comes out *positive*, then you know that the value of y which you got was a *minimum*; but if $\dfrac{d^2y}{dx^2}$ comes out *negative*, then the value of y which you got must be a *maximum*. That's the rule.

The reason of it ought to be quite evident. Think of any curve that has a minimum point in it (like Fig. 15, p. 83), or like Fig. 34, where the point of minimum y is marked M, and the curve is *concave* upwards. To the left of M the slope is downward, that is, negative, and is getting less negative. To the right of M the slope has become upward, and is getting more and more upward. Clearly the change of slope as the

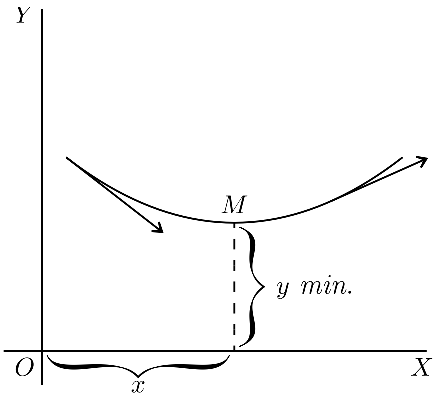


Fig. 34.

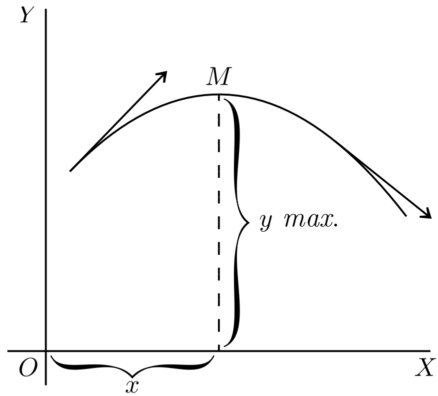


Fig. 35.

curve passes through M is such that $\dfrac{d^2y}{dx^2}$ is *positive*, for its operation, as x increases toward the right, is to convert a downward slope into an upward one.

Similarly, consider any curve that has a maximum point in it (like Fig. 16, p. 84), or like Fig. 35, where the curve is *convex*, and the maximum point is marked M. In this case, as the curve passes through M from left to right, its upward slope is converted into a downward or negative slope, so that in this case the "slope of the slope" $\dfrac{d^2y}{dx^2}$ is *negative*.

Go back now to the examples of the last chapter and verify in this way the conclusions arrived at as to whether in any particular case there is a maximum or a minimum. You will find below a few worked out examples.

---

(1) Find the maximum or minimum of

$$(a)\quad y = 4x^2 - 9x - 6; \qquad (b)\quad y = 6 + 9x - 4x^2;$$

and ascertain if it be a maximum or a minimum in each case.

$$(a)\quad \frac{dy}{dx} = 8x - 9 = 0; \quad x = 1\tfrac{1}{8}, \quad \text{and } y = -11.065.$$
$$\frac{d^2y}{dx^2} = 8; \quad \text{it is } +\text{; hence it is a minimum.}$$
$$(b)\quad \frac{dy}{dx} = 9 - 8x = 0; \quad x = 1\tfrac{1}{8}; \quad \text{and } y = +11.065.$$
$$\frac{d^2y}{dx^2} = -8; \quad \text{it is } -\text{; hence it is a maximum.}$$

(2) Find the maxima and minima of the function $y = x^3 - 3x + 16$.

$$\frac{dy}{dx} = 3x^2 - 3 = 0; \quad x^2 = 1; \quad \text{and } x = \pm 1.$$
$$\frac{d^2y}{dx^2} = 6x; \quad \text{for } x = 1\text{; it is } +;$$

hence $x = 1$ corresponds to a minimum $y = 14$. For $x = -1$ it is $-$; hence $x = -1$ corresponds to a maximum $y = +18$.

(3) Find the maxima and minima of $y = \dfrac{x-1}{x^2+2}$.

$$\frac{dy}{dx} = \frac{(x^2+2)\times 1 - (x-1)\times 2x}{(x^2+2)^2} = \frac{2x - x^2 + 2}{(x^2+2)^2} = 0;$$

or $x^2 - 2x - 2 = 0$, whose solutions are $x = +2.73$ and $x = -0.73$.

$$\begin{aligned}\frac{d^2y}{dx^2} &= -\frac{(x^2+2)^2 \times (2x-2) - (x^2-2x-2)(4x^3+8x)}{(x^2+2)^4}\\ &= -\frac{2x^5 - 6x^4 - 8x^3 - 8x^2 - 24x + 8}{(x^2+2)^4}.\end{aligned}$$

The denominator is always positive, so it is sufficient to ascertain the sign of the numerator.

If we put $x = 2.73$, the numerator is negative; the maximum, $y = 0.183$.

If we put $x = -0.73$, the numerator is positive; the minimum, $y = -0.683$.

(4) The expense C of handling the products of a certain factory varies with the weekly output P according to the relation $C = aP + \dfrac{b}{c+P} + d$, where a, b, c, d are positive constants. For what output will the expense be least?

$$\frac{dC}{dP} = a - \frac{b}{(c+P)^2} = 0 \quad \text{for maximum or minimum;}$$

hence $a = \dfrac{b}{(c+P)^2}$ and $P = \pm\sqrt{\dfrac{b}{a}} - c$.

As the output cannot be negative, $P = +\sqrt{\dfrac{b}{a}} - c$.

Now $$\frac{d^2C}{dP^2} = +\frac{b(2c+2P)}{(c+P)^4},$$

which is positive for all the values of P; hence $P = +\sqrt{\dfrac{b}{a}} - c$ corresponds to a minimum.

(5) The total cost per hour C of lighting a building with N lamps of a certain kind is

$$C = N\left(\frac{C_l}{t} + \frac{EPC_e}{1000}\right),$$

where E is the commercial efficiency (watts per candle),

P is the candle power of each lamp,

t is the average life of each lamp in hours,

C_l = cost of renewal in pence per hour of use,

C_e = cost of energy per 1000 watts per hour.

Moreover, the relation connecting the average life of a lamp with the commercial efficiency at which it is run is approximately $t = mE^n$, where m and n are constants depending on the kind of lamp.

Find the commercial efficiency for which the total cost of lighting will be least.

We have
$$C = N\left(\frac{C_l}{m}E^{-n} + \frac{PC_e}{1000}E\right),$$
$$\frac{dC}{dE} = \frac{PC_e}{1000} - \frac{nC_l}{m}E^{-(n+1)} = 0$$

for maximum or minimum.

$$E^{n+1} = \frac{1000 \times nC_l}{mPC_e} \quad \text{and} \quad E = \sqrt[n+1]{\frac{1000 \times nC_l}{mPC_e}}.$$

This is clearly for minimum, since

$$\frac{d^2C}{dE^2} = (n+1)\frac{nC_l}{m}E^{-(n+2)},$$

which is positive for a positive value of E.

For a particular type of 16 candle-power lamps, $C_l = 17$ pence, $C_e = 5$ pence; and it was found that $m = 10$ and $n = 3.6$.

$$E = \sqrt[4.6]{\frac{1000 \times 3.6 \times 17}{10 \times 16 \times 5}} = 2.6 \text{ watts per candle-power.}$$

Exercises X. (You are advised to plot the graph of any numerical example.) (See p. 261 for the Answers.)

(1) Find the maxima and minima of

$$y = x^3 + x^2 - 10x + 8.$$

(2) Given $y = \frac{b}{a}x - cx^2$, find expressions for $\frac{dy}{dx}$, and for $\frac{d^2y}{dx^2}$, also find the value of x which makes y a maximum or a minimum, and show whether it is maximum or minimum.

(3) Find how many maxima and how many minima there are in the curve, the equation to which is

$$y = 1 - \frac{x^2}{2} + \frac{x^4}{24};$$

and how many in that of which the equation is

$$y = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}.$$

(4) Find the maxima and minima of

$$y = 2x + 1 + \frac{5}{x^2}.$$

(5) Find the maxima and minima of

$$y = \frac{3}{x^2 + x + 1}.$$

(6) Find the maxima and minima of

$$y = \frac{5x}{2 + x^2}.$$

(7) Find the maxima and minima of

$$y = \frac{3x}{x^2 - 3} + \frac{x}{2} + 5.$$

(8) Divide a number N into two parts in such a way that three times the square of one part plus twice the square of the other part shall be a minimum.

(9) The efficiency u of an electric generator at different values of output x is expressed by the general equation:

$$u = \frac{x}{a + bx + cx^2};$$

where a is a constant depending chiefly on the energy losses in the iron and c a constant depending chiefly on the resistance of the copper parts. Find an expression for that value of the output at which the efficiency will be a maximum.

(10) Suppose it to be known that consumption of coal by a certain steamer may be represented by the formula $y = 0.3 + 0.001v^3$; where y is the number of tons of coal burned per hour and v is the speed expressed in nautical miles per hour. The cost of wages, interest on capital, and depreciation of that ship are together equal, per hour, to the cost of

1 ton of coal. What speed will make the total cost of a voyage of 1000 nautical miles a minimum? And, if coal costs 10 shillings per ton, what will that minimum cost of the voyage amount to?

(11) Find the maxima and minima of

$$y = \pm \frac{x}{6}\sqrt{x(10-x)}.$$

(12) Find the maxima and minima of

$$y = 4x^3 - x^2 - 2x + 1.$$

CAPÍTULO XIII.

OTROS TRUCOS ÚTILES.

Partial Fractions.

We have seen that when we differentiate a fraction we have to perform a rather complicated operation; and, if the fraction is not itself a simple one, the result is bound to be a complicated expression. If we could split the fraction into two or more simpler fractions such that their sum is equivalent to the original fraction, we could then proceed by differentiating each of these simpler expressions. And the result of differentiating would be the sum of two (or more) differentials, each one of which is relatively simple; while the final expression, though of course it will be the same as that which could be obtained without resorting to this dodge, is thus obtained with much less effort and appears in a simplified form.

Let us see how to reach this result. Try first the job of adding two fractions together to form a resultant fraction. Take, for example, the two fractions $\dfrac{1}{x+1}$ and $\dfrac{2}{x-1}$. Every schoolboy can add these together and find their sum to be $\dfrac{3x+1}{x^2-1}$. And in the same way he can add together three or more fractions. Now this process can certainly be reversed: that is to say, that if this last expression were given, it is certain that it can somehow be split back again into its original components or partial fractions. Only we do not know in every case that may be presented to us *how* we can so split it. In order to find this out we shall consider a simple case at first. But it is important to bear in mind that all which follows applies only to what are called "proper" algebraic fractions, meaning fractions like the above, which have the numerator of *a lesser degree* than the denominator; that is, those in which the highest index of x is less in the numerator than in the denominator. If we have to deal with such an expression as $\dfrac{x^2+2}{x^2-1}$, we can simplify it by division, since it is equivalent to $1+\dfrac{3}{x^2-1}$; and $\dfrac{3}{x^2-1}$ is a proper algebraic fraction to which the operation of splitting into partial fractions can be applied, as explained hereafter.

Case I. If we perform many additions of two or more fractions the denominators of which contain only terms in x, and no terms in x^2, x^3, or any other powers of x, we *always* find that *the denominator of the final resulting fraction is the product of the denominators* of the fractions which were added to form the result. It follows that by factorizing the denominator of this final fraction, we can find every one of the denominators of the partial fractions of which we are in search.

Suppose we wish to go back from $\dfrac{3x+1}{x^2-1}$ to the components which we know are $\dfrac{1}{x+1}$ and $\dfrac{2}{x-1}$. If we did not know what those components were we can still prepare the way by writing:

$$\frac{3x+1}{x^2-1} = \frac{3x+1}{(x+1)(x-1)} = \frac{\quad}{x+1} + \frac{\quad}{x-1},$$

leaving blank the places for the numerators until we know what to put there. We always may assume the sign between the partial fractions to be *plus*, since, if it be *minus*, we shall simply find the corresponding numerator to be negative. Now, since the partial fractions are *proper* fractions, the numerators are mere numbers without x at all, and we can call them A, B, C ... as we please. So, in this case, we have:

$$\frac{3x+1}{x^2-1} = \frac{A}{x+1} + \frac{B}{x-1}.$$

If now we perform the addition of these two partial fractions, we get $\dfrac{A(x-1)+B(x+1)}{(x+1)(x-1)}$; and this must be equal to $\dfrac{3x+1}{(x+1)(x-1)}$. And, as the denominators in these two expressions are the same, the numerators must be equal, giving us:

$$3x+1 = A(x-1) + B(x+1).$$

Now, this is an equation with two unknown quantities, and it would seem that we need another equation before we can solve them and find A and B. But there is another way out of this difficulty. The equation must be true for all values of x; therefore it must be true for such values of x as will cause $x-1$ and $x+1$ to become zero, that is for $x=1$ and for $x=-1$ respectively. If we make $x=1$, we get $4 = (A\times 0)+(B\times 2)$, so that $B=2$; and if we make $x=-1$, we get $-2 = (A\times -2)+(B\times 0)$, so that $A=1$. Replacing the A and B of the partial fractions by these new values, we find them to become $\dfrac{1}{x+1}$ and $\dfrac{2}{x-1}$; and the thing is done.

As a farther example, let us take the fraction $\dfrac{4x^2+2x-14}{x^3+3x^2-x-3}$. The denominator becomes zero when x is given the value 1; hence $x-1$ is a factor of it, and obviously then the other factor will be x^2+4x+3;

and this can again be decomposed into $(x+1)(x+3)$. So we may write the fraction thus:

$$\frac{4x^2+2x-14}{x^3+3x^2-x-3} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{x+3},$$

making three partial factors.

Proceeding as before, we find

$$4x^2+2x-14 = A(x-1)(x+3) + B(x+1)(x+3) + C(x+1)(x-1).$$

Now, if we make $x=1$, we get:

$$-8 = (A \times 0) + B(2 \times 4) + (C \times 0); \quad \text{that is, } B = -1.$$

If $x=-1$, we get:

$$-12 = A(-2 \times 2) + (B \times 0) + (C \times 0); \quad \text{whence } A = 3.$$

If $x=-3$, we get:

$$16 = (A \times 0) + (B \times 0) + C(-2 \times -4); \quad \text{whence } C = 2.$$

So then the partial fractions are:

$$\frac{3}{x+1} - \frac{1}{x-1} + \frac{2}{x+3},$$

which is far easier to differentiate with respect to x than the complicated expression from which it is derived.

Case II. If some of the factors of the denominator contain terms in x^2, and are not conveniently put into factors, then the corresponding numerator may contain a term in x, as well as a simple number; and hence it becomes necessary to represent this unknown numerator not by the symbol A but by $Ax+B$; the rest of the calculation being made as before.

Try, for instance: $\quad \dfrac{-x^2-3}{(x^2+1)(x+1)}.$

$$\frac{-x^2-3}{(x^2+1)(x+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1};$$

$$-x^2-3 = (Ax+B)(x+1) + C(x^2+1).$$

Putting $x=-1$, we get $-4 = C \times 2$; and $C=-2$;

hence $\quad -x^2-3 = (Ax+B)(x+1) - 2x^2 - 2;$

and $\quad x^2-1 = Ax(x+1) + B(x+1).$

Putting $x=0$, we get $-1=B$;
hence

$$x^2-1 = Ax(x+1) - x - 1; \quad \text{or } x^2+x = Ax(x+1);$$

and $\quad x+1 = A(x+1),$

so that $A=1$, and the partial fractions are:

$$\frac{x-1}{x^2+1} - \frac{2}{x+1}.$$

Take as another example the fraction

$$\frac{x^3-2}{(x^2+1)(x^2+2)}.$$

We get

$$\frac{x^3-2}{(x^2+1)(x^2+2)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2}$$
$$= \frac{(Ax+B)(x^2+2)+(Cx+D)(x^2+1)}{(x^2+1)(x^2+2)}.$$

In this case the determination of A, B, C, D is not so easy. It will be simpler to proceed as follows: Since the given fraction and the fraction found by adding the partial fractions are equal, and have *identical* denominators, the numerators must also be identically the same. In such a case, and for such algebraical expressions as those with which we are dealing here, *the coefficients of the same powers of x are equal and of same sign.*

Hence, since

$$x^3-2 = (Ax+B)(x^2+2)+(Cx+D)(x^2+1)$$
$$= (A+C)x^3+(B+D)x^2+(2A+C)x+2B+D,$$

we have $1 = A+C$; $0 = B+D$ (the coefficient of x^2 in the left expression being zero); $0 = 2A+C$; and $-2 = 2B+D$. Here are four equations, from which we readily obtain $A=-1$; $B=-2$; $C=2$; $D=0$; so that the partial fractions are $\dfrac{2(x+1)}{x^2+2} - \dfrac{x+2}{x^2+1}$. This method can always be used; but the method shown first will be found the quickest in the case of factors in x only.

Case III. When, among the factors of the denominator there are some which are raised to some power, one must allow for the possible existence of partial fractions having for denominator the several powers of that factor up to the highest. For instance, in splitting the fraction $\dfrac{3x^2-2x+1}{(x+1)^2(x-2)}$ we must allow for the possible existence of a denominator $x+1$ as well as $(x+1)^2$ and $(x-2)$.

It maybe thought, however, that, since the numerator of the fraction the denominator of which is $(x+1)^2$ may contain terms in x, we must allow for this in writing $Ax+B$ for its numerator, so that

$$\frac{3x^2-2x+1}{(x+1)^2(x-2)} = \frac{Ax+B}{(x+1)^2} + \frac{C}{x+1} + \frac{D}{x-2}.$$

If, however, we try to find A, B, C and D in this case, we fail, because we get four unknowns; and we have only three relations connecting them, yet

$$\frac{3x^2-2x+1}{(x+1)^2(x-2)} = \frac{x-1}{(x+1)^2} + \frac{1}{x+1} + \frac{1}{x-2}.$$

But if we write

$$\frac{3x^2-2x+1}{(x+1)^2(x-2)} = \frac{A}{(x+1)^2} + \frac{B}{x+1} + \frac{C}{x-2},$$

we get

$$3x^2-2x+1 = A(x-2) + B(x+1)(x-2) + C(x+1)^2,$$

which gives $C=1$ for $x=2$. Replacing C by its value, transposing, gathering like terms and dividing by $x-2$, we get $-2x = A+B(x+1)$, which gives $A=-2$ for $x=-1$. Replacing A by its value, we get

$$2x = -2 + B(x+1).$$

Hence $B=2$; so that the partial fractions are:

$$\frac{2}{x+1} - \frac{2}{(x+1)^2} + \frac{1}{x-2},$$

instead of $\frac{1}{x+1}+\frac{x-1}{(x+1)^2}+\frac{1}{x-2}$ stated above as being the fractions from which $\frac{3x^2-2x+1}{(x+1)^2(x-2)}$ was obtained. The mystery is cleared if we observe that $\frac{x-1}{(x+1)^2}$ can itself be split into the two fractions $\frac{1}{x+1}-\frac{2}{(x+1)^2}$, so that the three fractions given are really equivalent to

$$\frac{1}{x+1}+\frac{1}{x+1}-\frac{2}{(x+1)^2}+\frac{1}{x-2}=\frac{2}{x+1}-\frac{2}{(x+1)^2}+\frac{1}{x-2},$$

which are the partial fractions obtained.

We see that it is sufficient to allow for one numerical term in each numerator, and that we always get the ultimate partial fractions.

When there is a power of a factor of x^2 in the denominator, however, the corresponding numerators must be of the form $Ax+B$; for example,

$$\frac{3x-1}{(2x^2-1)^2(x+1)}=\frac{Ax+B}{(2x^2-1)^2}+\frac{Cx+D}{2x^2-1}+\frac{E}{x+1},$$

which gives

$$3x-1=(Ax+B)(x+1)+(Cx+D)(x+1)(2x^2-1)+E(2x^2-1)^2.$$

For $x=-1$, this gives $E=-4$. Replacing, transposing, collecting like terms, and dividing by $x+1$, we get

$$16x^3-16x^2+3=2Cx^3+2Dx^2+x(A-C)+(B-D).$$

Hence $2C=16$ and $C=8$; $2D=-16$ and $D=-8$; $A-C=0$ or $A-8=0$ and $A=8$, and finally, $B-D=3$ or $B=-5$. So that we obtain as the partial fractions:

$$\frac{(8x-5)}{(2x^2-1)^2}+\frac{8(x-1)}{2x^2-1}-\frac{4}{x+1}.$$

It is useful to check the results obtained. The simplest way is to replace x by a single value, say $+1$, both in the given expression and in the partial fractions obtained.

Whenever the denominator contains but a power of a single factor, a very quick method is as follows:

Taking, for example, $\dfrac{4x+1}{(x+1)^3}$, let $x+1=z$; then $x=z-1$.

Replacing, we get

$$\frac{4(z-1)+1}{z^3}=\frac{4z-3}{z^3}=\frac{4}{z^2}-\frac{3}{z^3}.$$

The partial fractions are, therefore,

$$\frac{4}{(x+1)^2}-\frac{3}{(x+1)^3}.$$

Application to differentiation. Let it be required to differentiate $y=\dfrac{5-4x}{6x^2+7x-3}$; we have

$$\begin{aligned}\frac{dy}{dx}&=-\frac{(6x^2+7x-3)\times 4+(5-4x)(12x+7)}{(6x^2+7x-3)^2}\\&=\frac{24x^2-60x-23}{(6x^2+7x-3)^2}.\end{aligned}$$

If we split the given expression into

$$\frac{1}{3x-1}-\frac{2}{2x+3},$$

we get, however,

$$\frac{dy}{dx}=-\frac{3}{(3x-1)^2}+\frac{4}{(2x+3)^2},$$

which is really the same result as above split into partial fractions. But the splitting, if done after differentiating, is more complicated, as will easily be seen. When we shall deal with the *integration* of such expressions, we shall find the splitting into partial fractions a precious auxiliary (see p. 231).

Exercises XI. (See page 262 for Answers.)

Split into fractions:

(1) $\dfrac{3x+5}{(x-3)(x+4)}$. (2) $\dfrac{3x-4}{(x-1)(x-2)}$.

(3) $\dfrac{3x+5}{x^2+x-12}$. (4) $\dfrac{x+1}{x^2-7x+12}$.

(5) $\dfrac{x-8}{(2x+3)(3x-2)}$. (6) $\dfrac{x^2-13x+26}{(x-2)(x-3)(x-4)}$.

(7) $\dfrac{x^2-3x+1}{(x-1)(x+2)(x-3)}$.

(8) $\dfrac{5x^2+7x+1}{(2x+1)(3x-2)(3x+1)}$.

(9) $\dfrac{x^2}{x^3-1}$. (10) $\dfrac{x^4+1}{x^3+1}$.

(11) $\dfrac{5x^2+6x+4}{(x+1)(x^2+x+1)}$. (12) $\dfrac{x}{(x-1)(x-2)^2}$.

(13) $\dfrac{x}{(x^2-1)(x+1)}$. (14) $\dfrac{x+3}{(x+2)^2(x-1)}$.

(15) $\dfrac{3x^2+2x+1}{(x+2)(x^2+x+1)^2}$. (16) $\dfrac{5x^2+8x-12}{(x+4)^3}$.

(17) $\dfrac{7x^2+9x-1}{(3x-2)^4}$. (18) $\dfrac{x^2}{(x^3-8)(x-2)}$.

Differential of an Inverse Function.

Consider the function (see p. 13) $y = 3x$; it can be expressed in the form $x = \dfrac{y}{3}$; this latter form is called the *inverse function* to the one originally given.

If $y = 3x$, $\dfrac{dy}{dx} = 3$; if $x = \dfrac{y}{3}$, $\dfrac{dx}{dy} = \dfrac{1}{3}$, and we see that

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \quad \text{or} \quad \frac{dy}{dx} \times \frac{dx}{dy} = 1.$$

Consider $y = 4x^2$, $\dfrac{dy}{dx} = 8x$; the inverse function is

$$x = \frac{y^{\frac{1}{2}}}{2}, \quad \text{and} \quad \frac{dx}{dy} = \frac{1}{4\sqrt{y}} = \frac{1}{4 \times 2x} = \frac{1}{8x}.$$

Here again $$\frac{dy}{dx} \times \frac{dx}{dy} = 1.$$

It can be shown that for all functions which can be put into the inverse form, one can always write

$$\frac{dy}{dx} \times \frac{dx}{dy} = 1 \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}.$$

It follows that, being given a function, if it be easier to differentiate the inverse function, this may be done, and the reciprocal of the differential coefficient of the inverse function gives the differential coefficient of the given function itself.

As an example, suppose that we wish to differentiate $y = \sqrt[2]{\frac{3}{x} - 1}$. We have seen one way of doing this, by writing $u = \frac{3}{x} - 1$, and finding $\frac{dy}{du}$ and $\frac{du}{dx}$. This gives

$$\frac{dy}{dx} = -\frac{3}{2x^2\sqrt{\frac{3}{x} - 1}}.$$

If we had forgotten how to proceed by this method, or wished to check our result by some other way of obtaining the differential coefficient, or for any other reason we could not use the ordinary method, we can proceed as follows: The inverse function is $x = \frac{3}{1+y^2}$.

$$\frac{dx}{dy} = -\frac{3 \times 2y}{(1+y^2)^2} = -\frac{6y}{(1+y^2)^2};$$

hence

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = -\frac{(1+y^2)^2}{6y} = -\frac{\left(1 + \frac{3}{x} - 1\right)^2}{6 \times \sqrt[2]{\frac{3}{x} - 1}} = -\frac{3}{2x^2\sqrt{\frac{3}{x} - 1}}.$$

Let us take as an other example $y = \frac{1}{\sqrt[3]{\theta + 5}}$.

The inverse function is $\theta = \frac{1}{y^3} - 5$ or $\theta = y^{-3} - 5$, and

$$\frac{d\theta}{dy} = -3y^{-4} = -3\sqrt[3]{(\theta + 5)^4}.$$

It follows that $\dfrac{dy}{dx} = -\dfrac{1}{3\sqrt{(\theta+5)^4}}$, as might have been found otherwise.

We shall find this dodge most useful later on; meanwhile you are advised to become familiar with it by verifying by its means the results obtained in Exercises I. (p. 24), Nos. 5, 6, 7; Examples (p. 70), Nos. 1, 2, 4; and Exercises VI. (p. 75), Nos. 1, 2, 3 and 4.

You will surely realize from this chapter and the preceding, that in many respects the calculus is an *art* rather than a *science*: an art only to be acquired, as all other arts are, by practice. Hence you should work many examples, and set yourself other examples, to see if you can work them out, until the various artifices become familiar by use.

CAPÍTULO XIV.

SOBRE EL VERDADERO INTERÉS COMPUESTO Y LA LEY DEL CRECIMIENTO ORGÁNICO.

Let there be a quantity growing in such a way that the increment of its growth, during a given time, shall always be proportional to its own magnitude. This resembles the process of reckoning interest on money at some fixed rate; for the bigger the capital, the bigger the amount of interest on it in a given time.

Now we must distinguish clearly between two cases, in our calculation, according as the calculation is made by what the arithmetic books call "simple interest," or by what they call "compound interest." For in the former case the capital remains fixed, while in the latter the interest is added to the capital, which therefore increases by successive additions.

(1) *At simple interest.* Consider a concrete case. Let the capital at start be £100, and let the rate of interest be 10 per cent. per annum. Then the increment to the owner of the capital will be £10 every year. Let him go on drawing his interest every year, and hoard it by putting it by in a stocking, or locking it up in his safe. Then, if he goes on for 10 years, by the end of that time he will have received 10 increments

of £10 each, or £100, making, with the original £100, a total of £200 in all. His property will have doubled itself in 10 years. If the rate of interest had been 5 per cent., he would have had to hoard for 20 years to double his property. If it had been only 2 per cent., he would have had to hoard for 50 years. It is easy to see that if the value of the yearly interest is $\frac{1}{n}$ of the capital, he must go on hoarding for n years in order to double his property.

Or, if y be the original capital, and the yearly interest is $\frac{y}{n}$, then, at the end of n years, his property will be

$$y + n\frac{y}{n} = 2y.$$

(2) *At compound interest.* As before, let the owner begin with a capital of £100, earning interest at the rate of 10 per cent. per annum; but, instead of hoarding the interest, let it be added to the capital each year, so that the capital grows year by year. Then, at the end of one year, the capital will have grown to £110; and in the second year (still at 10%) this will earn £11 interest. He will start the third year with £121, and the interest on that will be £12. 2*s.*; so that he starts the fourth year with £133. 2*s.*, and so on. It is easy to work it out, and find that at the end of the ten years the total capital will have grown to £259. 7*s.* 6*d.* In fact, we see that at the end of each year, each pound will have earned $\frac{1}{10}$ of a pound, and therefore, if this is always added on, each year multiplies the capital by $\frac{11}{10}$; and if continued for ten years (which will multiply by this factor ten times over) will multiply the original capital by 2.59374. Let us put this into symbols. Put y_0 for the original capital; $\frac{1}{n}$ for the fraction added on at

each of the n operations; and y_n for the value of the capital at the end of the n^{th} operation. Then

$$y_n = y_0 \left(1 + \frac{1}{n}\right)^n .$$

But this mode of reckoning compound interest once a year, is really not quite fair; for even during the first year the £100 ought to have been growing. At the end of half a year it ought to have been at least £105, and it certainly would have been fairer had the interest for the second half of the year been calculated on £105. This would be equivalent to calling it 5% per half-year; with 20 operations, therefore, at each of which the capital is multiplied by $\frac{21}{20}$. If reckoned this way, by the end of ten years the capital would have grown to £265. 6*s.* 7*d.*; for

$$(1 + \tfrac{1}{20})^{20} = 2.653.$$

But, even so, the process is still not quite fair; for, by the end of the first month, there will be some interest earned; and a half-yearly reckoning assumes that the capital remains stationary for six months at a time. Suppose we divided the year into 10 parts, and reckon a one-per-cent. interest for each tenth of the year. We now have 100 operations lasting over the ten years; or

$$y_n = \pounds 100 \left(1 + \tfrac{1}{100}\right)^{100};$$

which works out to £270. 9*s.* $7\frac{1}{2}d.$

Even this is not final. Let the ten years be divided into 1000 periods, each of $\frac{1}{100}$ of a year; the interest being $\frac{1}{10}$ per cent. for each such period; then

$$y_n = \pounds 100 \left(1 + \tfrac{1}{1000}\right)^{1000};$$

which works out to £271. 13*s.* 10*d.*

Go even more minutely, and divide the ten years into $10,000$ parts, each $\frac{1}{1000}$ of a year, with interest at $\frac{1}{100}$ of 1 per cent. Then

$$y_n = \pounds 100 \left(1 + \tfrac{1}{10{,}000}\right)^{10{,}000};$$

which amounts to £271. 16*s.* $3\frac{1}{2}$*d.*

Finally, it will be seen that what we are trying to find is in reality the ultimate value of the expression $\left(1 + \dfrac{1}{n}\right)^n$, which, as we see, is greater than 2; and which, as we take n larger and larger, grows closer and closer to a particular limiting value. However big you make n, the value of this expression grows nearer and nearer to the figure

$$2.71828\ldots$$

a number *never to be forgotten.*

Let us take geometrical illustrations of these things. In Fig. 36, OP stands for the original value. OT is the whole time during which the value is growing. It is divided into 10 periods, in each of which there is an equal step up. Here $\dfrac{dy}{dx}$ is a constant; and if each step up is $\frac{1}{10}$ of the original OP, then, by 10 such steps, the height is doubled. If we had taken 20 steps, each of half the height shown, at the end the height would still be just doubled. Or n such steps, each of $\dfrac{1}{n}$ of the original height OP, would suffice to double the height. This is the case of simple interest. Here is 1 growing till it becomes 2.

In Fig. 37, we have the corresponding illustration of the geometrical progression. Each of the successive ordinates is to be $1 + \dfrac{1}{n}$, that is, $\dfrac{n+1}{n}$ times as high as its predecessor. The steps up are not equal,

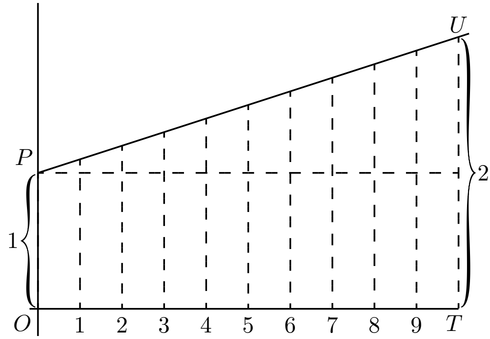


Fig. 36.

because each step up is now $\dfrac{1}{n}$ of the ordinate *at that part* of the curve. If we had literally 10 steps, with $\left(1+\frac{1}{10}\right)$ for the multiplying factor, the final total would be $(1+\frac{1}{10})^{10}$ or 2.594 times the original 1. But if only we take n sufficiently large (and the corresponding $\dfrac{1}{n}$ sufficiently small), then the final value $\left(1+\dfrac{1}{n}\right)^n$ to which unity will grow will be 2.71828.

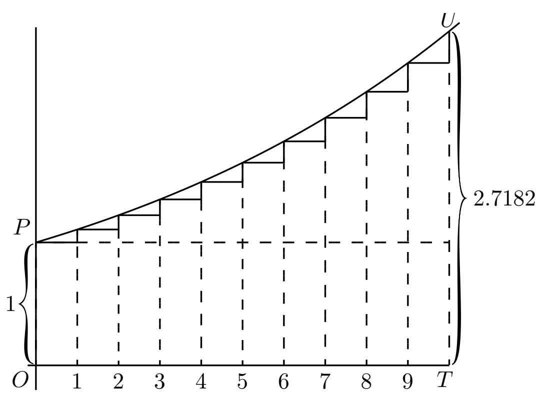


Fig. 37.

Epsilon. To this mysterious number 2.7182818 etc., the mathematicians have assigned as a symbol the Greek letter ϵ (pronounced *ep-*

silon). All schoolboys know that the Greek letter π (called *pi*) stands for 3.141592 etc.; but how many of them know that *epsilon* means 2.71828? Yet it is an even more important number than π!

What, then, is *epsilon*?

Suppose we were to let 1 grow at simple interest till it became 2; then, if at the same nominal rate of interest, and for the same time, we were to let 1 grow at true compound interest, instead of simple, it would grow to the value *epsilon*.

This process of growing proportionately, at every instant, to the magnitude at that instant, some people call *a logarithmic rate* of growing. Unit logarithmic rate of growth is that rate which in unit time will cause 1 to grow to 2.718281. It might also be called the *organic rate* of growing: because it is characteristic of organic growth (in certain circumstances) that the increment of the organism in a given time is proportional to the magnitude of the organism itself.

If we take 100 per cent. as the unit of rate, and any fixed period as the unit of time, then the result of letting 1 grow *arithmetically* at unit rate, for unit time, will be 2, while the result of letting 1 grow *logarithmically* at unit rate, for the same time, will be 2.71828 . . . .

A little more about Epsilon. We have seen that we require to know what value is reached by the expression $\left(1+\dfrac{1}{n}\right)^n$, when n becomes indefinitely great. Arithmetically, here are tabulated a lot of values (which anybody can calculate out by the help of an ordinary table of logarithms) got by assuming $n = 2$; $n = 5$; $n = 10$; and so on, up to

$n = 10,000$.

$$
\begin{aligned}
(1 + \tfrac{1}{2})^2 &= 2.25.\\
(1 + \tfrac{1}{5})^5 &= 2.488.\\
(1 + \tfrac{1}{10})^{10} &= 2.594.\\
(1 + \tfrac{1}{20})^{20} &= 2.653.\\
(1 + \tfrac{1}{100})^{100} &= 2.705.\\
(1 + \tfrac{1}{1000})^{1000} &= 2.7169.\\
(1 + \tfrac{1}{10,000})^{10,000} &= 2.7181.
\end{aligned}
$$

It is, however, worth while to find another way of calculating this immensely important figure.

Accordingly, we will avail ourselves of the binomial theorem, and expand the expression $\left(1 + \dfrac{1}{n}\right)^n$ in that well-known way.

The binomial theorem gives the rule that

$$
\begin{aligned}
(a+b)^n = a^n + n\frac{a^{n-1}b}{1!} &+ n(n-1)\frac{a^{n-2}b^2}{2!}\\
&+ n(n-1)(n-2)\frac{a^{n-3}b^3}{3!} + \text{etc.}
\end{aligned}
$$

Putting $a = 1$ and $b = \dfrac{1}{n}$, we get

$$
\begin{aligned}
\left(1 + \frac{1}{n}\right)^n = 1 + 1 + \frac{1}{2!}\left(\frac{n-1}{n}\right) &+ \frac{1}{3!}\frac{(n-1)(n-2)}{n^2}\\
&+ \frac{1}{4!}\frac{(n-1)(n-2)(n-3)}{n^3} + \text{etc.}
\end{aligned}
$$

Now, if we suppose n to become indefinitely great, say a billion, or a billion billions, then $n-1$, $n-2$, and $n-3$, etc., will all be sensibly equal to n; and then the series becomes

$$\epsilon = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \text{etc.}\dots$$

By taking this rapidly convergent series to as many terms as we please, we can work out the sum to any desired point of accuracy. Here is the working for ten terms:

	1.000000
dividing by 1	1.000000
dividing by 2	0.500000
dividing by 3	0.166667
dividing by 4	0.041667
dividing by 5	0.008333
dividing by 6	0.001389
dividing by 7	0.000198
dividing by 8	0.000025
dividing by 9	0.000002
Total	2.718281

ϵ is incommensurable with 1, and resembles π in being an interminable non-recurrent decimal.

The Exponential Series. We shall have need of yet another series.

Let us, again making use of the binomial theorem, expand the expression $\left(1+\frac{1}{n}\right)^{nx}$, which is the same as ϵ^x when we make n indefinitely great.

$$\begin{aligned}
\epsilon^x &= 1^{nx} + nx\frac{1^{nx-1}\left(\dfrac{1}{n}\right)}{1!} + nx(nx-1)\frac{1^{nx-2}\left(\dfrac{1}{n}\right)^2}{2!} \\
&\qquad + nx(nx-1)(nx-2)\frac{1^{nx-3}\left(\dfrac{1}{n}\right)^3}{3!} + \text{etc.} \\
&= 1 + x + \frac{1}{2!}\cdot\frac{n^2x^2 - nx}{n^2} + \frac{1}{3!}\cdot\frac{n^3x^3 - 3n^2x^2 + 2nx}{n^3} + \text{etc.} \\
&= 1 + x + \frac{x^2 - \dfrac{x}{n}}{2!} + \frac{x^3 - \dfrac{3x^2}{n} + \dfrac{2x}{n^2}}{3!} + \text{etc.}
\end{aligned}$$

But, when n is made indefinitely great, this simplifies down to the following:

$$\epsilon^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \text{etc.}\dots$$

This series is called *the exponential series.*

The great reason why ϵ is regarded of importance is that ϵ^x possesses a property, not possessed by any other function of x, that *when you differentiate it its value remains unchanged*; or, in other words, its differential coefficient is the same as itself. This can be instantly seen by differentiating it with respect to x, thus:

$$\begin{aligned}
\frac{d(\epsilon^x)}{dx} &= 0 + 1 + \frac{2x}{1\cdot 2} + \frac{3x^2}{1\cdot 2\cdot 3} + \frac{4x^3}{1\cdot 2\cdot 3\cdot 4} \\
&\qquad + \frac{5x^4}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \text{etc.}
\end{aligned}$$

or
$$= 1 + x + \frac{x^2}{1\cdot 2} + \frac{x^3}{1\cdot 2\cdot 3} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.},$$

which is exactly the same as the original series.

Now we might have gone to work the other way, and said: Go to; let us find a function of x, such that its differential coefficient is the same as itself. Or, is there any expression, involving only powers of x, which is unchanged by differentiation? Accordingly; let us *assume* as a general expression that

$$y = A + Bx + Cx^2 + Dx^3 + Ex^4 + \text{etc.},$$

(in which the coefficients A, B, C, etc. will have to be determined), and differentiate it.

$$\frac{dy}{dx} = B + 2Cx + 3Dx^2 + 4Ex^3 + \text{etc.}$$

Now, if this new expression is really to be the same as that from which it was derived, it is clear that A *must* $= B$; that $C = \dfrac{B}{2} = \dfrac{A}{1 \cdot 2}$; that $D = \dfrac{C}{3} = \dfrac{A}{1 \cdot 2 \cdot 3}$; that $E = \dfrac{D}{4} = \dfrac{A}{1 \cdot 2 \cdot 3 \cdot 4}$, etc.

The law of change is therefore that

$$y = A\left(1 + \frac{x}{1} + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \text{etc.}\right).$$

If, now, we take $A = 1$ for the sake of further simplicity, we have

$$y = 1 + \frac{x}{1} + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \text{etc.}$$

Differentiating it any number of times will give always the same series over again.

If, now, we take the particular case of $A = 1$, and evaluate the series, we shall get simply

when $x = 1$,	$y = 2.718281$ etc.;	that is, $y = \epsilon$;
when $x = 2$,	$y = (2.718281 \text{ etc.})^2$;	that is, $y = \epsilon^2$;
when $x = 3$,	$y = (2.718281 \text{ etc.})^3$;	that is, $y = \epsilon^3$;

and therefore

$$\text{when } x = x, \quad y = (2.718281 \text{ etc.})^x; \quad \text{that is, } y = \epsilon^x,$$

thus finally demonstrating that

$$\epsilon^x = 1 + \frac{x}{1} + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \text{etc.}$$

[NOTE.—*How to read exponentials.* For the benefit of those who have no tutor at hand it may be of use to state that ϵ^x is read as "*epsilon to the eksth power*;" or some people read it "*exponential eks.*" So ϵ^{pt} is read "*epsilon to the pee-teeth-power*" or "*exponential pee tee.*" Take some similar expressions:—Thus, ϵ^{-2} is read "*epsilon to the minus two power*" or "*exponential minus two.*" ϵ^{-ax} is read "*epsilon to the minus ay-eksth*" or "*exponential minus ay-eks.*"]

Of course it follows that ϵ^y remains unchanged if differentiated with respect to y. Also ϵ^{ax}, which is equal to $(\epsilon^a)^x$, will, when differentiated with respect to x, be $a\epsilon^{ax}$, because a is a constant.

Natural or Naperian Logarithms.

Another reason why ϵ is important is because it was made by Napier, the inventor of logarithms, the basis of his system. If y is the value of

ϵ^x, then x is the *logarithm*, to the base ϵ, of y. Or, if

$$y = \epsilon^x,$$

then

$$x = \log_\epsilon y.$$

The two curves plotted in Figs. 38 and 39 represent these equations.

The points calculated are:

For Fig. 38

x	0	0.5	1	1.5	2
y	1	1.65	2.71	4.50	7.39

For Fig. 39

y	1	2	3	4	8
x	0	0.69	1.10	1.39	2.08

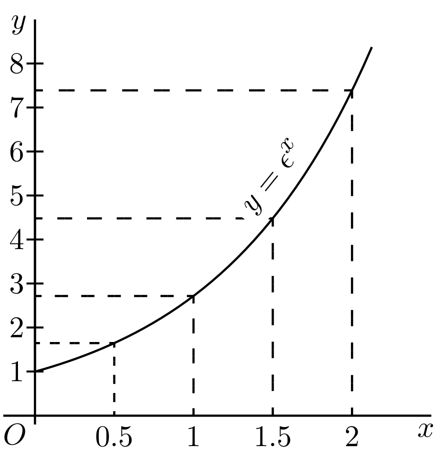


Fig. 39.

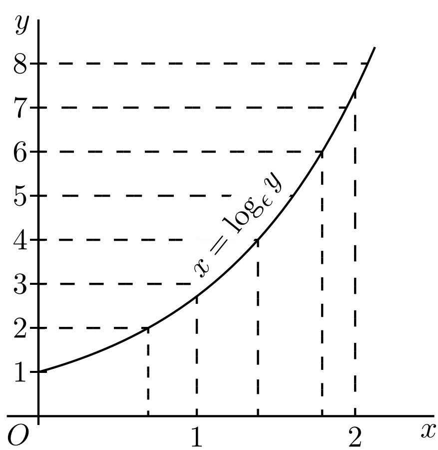


Fig. 38.

It will be seen that, though the calculations yield different points for plotting, yet the result is identical. The two equations really mean the same thing.

As many persons who use ordinary logarithms, which are calculated to base 10 instead of base ϵ, are unfamiliar with the "natural" logarithms, it may be worth while to say a word about them. The ordinary rule that adding logarithms gives the logarithm of the product still holds good; or

$$\log_\epsilon a + \log_\epsilon b = \log_\epsilon ab.$$

Also the rule of powers holds good;

$$n \times \log_\epsilon a = \log_\epsilon a^n.$$

But as 10 is no longer the basis, one cannot multiply by 100 or 1000 by merely adding 2 or 3 to the index. One can change the natural logarithm to the ordinary logarithm simply by multiplying it by 0.4343; or

$$\log_{10} x = 0.4343 \times \log_\epsilon x,$$

and conversely, $$\log_\epsilon x = 2.3026 \times \log_{10} x.$$

Exponential and Logarithmic Equations.

Now let us try our hands at differentiating certain expressions that contain logarithms or exponentials.

Take the equation:

$$y = \log_\epsilon x.$$

First transform this into

$$\epsilon^y = x,$$

whence, since the differential of ϵ^y with regard to y is the original function unchanged (see p. 142),

$$\frac{dx}{dy} = \epsilon^y,$$

A Useful Table of "Naperian Logarithms"

(Also called Natural Logarithms or Hyperbolic Logarithms)

Number	$\log_\epsilon$	Number	$\log_\epsilon$
1	0.0000	6	1.7918
1.1	0.0953	7	1.9459
1.2	0.1823	8	2.0794
1.5	0.4055	9	2.1972
1.7	0.5306	10	2.3026
2.0	0.6931	20	2.9957
2.2	0.7885	50	3.9120
2.5	0.9163	100	4.6052
2.7	0.9933	200	5.2983
2.8	1.0296	500	6.2146
3.0	1.0986	1,000	6.9078
3.5	1.2528	2,000	7.6009
4.0	1.3863	5,000	8.5172
4.5	1.5041	10,000	9.2103
5.0	1.6094	20,000	9.9035

and, reverting from the inverse to the original function,

$$\frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}} = \frac{1}{\epsilon^y} = \frac{1}{x}.$$

Now this is a very curious result. It may be written

$$\frac{d(\log_\epsilon x)}{dx} = x^{-1}.$$

Note that x^{-1} is a result that we could never have got by the rule for differentiating powers. That rule (page 24) is to multiply by the power, and reduce the power by 1. Thus, differentiating x^3 gave us $3x^2$; and differentiating x^2 gave $2x^1$. But differentiating x^0 does not give us x^{-1} or $0 \times x^{-1}$, because x^0 is itself $= 1$, and is a constant. We shall have to come back to this curious fact that differentiating $\log_\epsilon x$ gives us $\dfrac{1}{x}$ when we reach the chapter on integrating.

Now, try to differentiate

$$y = \log_\epsilon (x + a),$$

that is $$\epsilon^y = x + a;$$

we have $\dfrac{d(x+a)}{dy} = \epsilon^y$, since the differential of ϵ^y remains ϵ^y.

This gives $$\frac{dx}{dy} = \epsilon^y = x + a;$$

hence, reverting to the original function (see p. 131), we get

$$\frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}} = \frac{1}{x+a}.$$

Next try $$y = \log_{10} x.$$

First change to natural logarithms by multiplying by the modulus 0.4343. This gives us

$$y = 0.4343 \log_\epsilon x;$$

whence $$\frac{dy}{dx} = \frac{0.4343}{x}.$$

The next thing is not quite so simple. Try this:

$$y = a^x.$$

Taking the logarithm of both sides, we get

$$\log_\epsilon y = x \log_\epsilon a,$$

or $$x = \frac{\log_\epsilon y}{\log_\epsilon a} = \frac{1}{\log_\epsilon a} \times \log_\epsilon y.$$

Since $\dfrac{1}{\log_\epsilon a}$ is a constant, we get

$$\frac{dx}{dy} = \frac{1}{\log_\epsilon a} \times \frac{1}{y} = \frac{1}{a^x \times \log_\epsilon a};$$

hence, reverting to the original function.

$$\frac{dy}{dx} = \frac{1}{\;\dfrac{dx}{dy}\;} = a^x \times \log_\epsilon a.$$

We see that, since

$$\frac{dx}{dy} \times \frac{dy}{dx} = 1 \quad \text{and} \quad \frac{dx}{dy} = \frac{1}{y} \times \frac{1}{\log_\epsilon a}, \quad \frac{1}{y} \times \frac{dy}{dx} = \log_\epsilon a.$$

We shall find that whenever we have an expression such as $\log_\epsilon y =$ a function of x, we always have $\dfrac{1}{y}\,\dfrac{dy}{dx}$ = the differential coefficient of the function of x, so that we could have written at once, from $\log_\epsilon y = x \log_\epsilon a$,

$$\frac{1}{y}\,\frac{dy}{dx} = \log_\epsilon a \quad \text{and} \quad \frac{dy}{dx} = a^x \log_\epsilon a.$$

Let us now attempt further examples.

Examples.

(1) $y = \epsilon^{-ax}$. Let $-ax = z$; then $y = \epsilon^z$.

$$\frac{dy}{dx} = \epsilon^z; \quad \frac{dz}{dx} = -a; \quad \text{hence} \quad \frac{dy}{dx} = -a\epsilon^{-ax}.$$

Or thus:

$$\log_\epsilon y = -ax; \quad \frac{1}{y}\,\frac{dy}{dx} = -a; \quad \frac{dy}{dx} = -ay = -a\epsilon^{-ax}.$$

(2) $y = \epsilon^{\frac{x^2}{3}}$. Let $\dfrac{x^2}{3} = z$; then $y = \epsilon^z$.

$$\frac{dy}{dz} = \epsilon^z; \quad \frac{dz}{dx} = \frac{2x}{3}; \quad \frac{dy}{dx} = \frac{2x}{3}\,\epsilon^{\frac{x^2}{3}}.$$

Or thus:

$$\log_\epsilon y = \frac{x^2}{3}; \quad \frac{1}{y}\,\frac{dy}{dx} = \frac{2x}{3}; \quad \frac{dy}{dx} = \frac{2x}{3}\,\epsilon^{\frac{x^2}{3}}.$$

(3) $y = \epsilon^{\frac{2x}{x+1}}$.

$$\log_\epsilon y = \frac{2x}{x+1}, \quad \frac{1}{y}\,\frac{dy}{dx} = \frac{2(x+1)-2x}{(x+1)^2};$$

hence
$$\frac{dy}{dx} = \frac{2}{(x+1)^2}\epsilon^{\frac{2x}{x+1}}.$$

Check by writing $\dfrac{2x}{x+1} = z$.

(4) $y = \epsilon^{\sqrt{x^2+a}}$. $\log_\epsilon y = (x^2+a)^{\frac{1}{2}}$.

$$\frac{1}{y}\,\frac{dy}{dx} = \frac{x}{(x^2+a)^{\frac{1}{2}}} \quad \text{and} \quad \frac{dy}{dx} = \frac{x \times \epsilon^{\sqrt{x^2+a}}}{(x^2+a)^{\frac{1}{2}}}.$$

For if $(x^2+a)^{\frac{1}{2}} = u$ and $x^2+a = v$, $u = v^{\frac{1}{2}}$,

$$\frac{du}{dv} = \frac{1}{2v^{\frac{1}{2}}}; \quad \frac{dv}{dx} = 2x; \quad \frac{du}{dx} = \frac{x}{(x^2+a)^{\frac{1}{2}}}.$$

Check by writing $\sqrt{x^2+a} = z$.

(5) $y = \log(a+x^3)$. Let $(a+x^3) = z$; then $y = \log_\epsilon z$.

$$\frac{dy}{dz} = \frac{1}{z}; \quad \frac{dz}{dx} = 3x^2; \quad \text{hence} \quad \frac{dy}{dx} = \frac{3x^2}{a+x^3}.$$

(6) $y = \log_\epsilon\{3x^2 + \sqrt{a+x^2}\}$. Let $3x^2 + \sqrt{a+x^2} = z$; then $y = \log_\epsilon z$.

$$\frac{dy}{dz} = \frac{1}{z}; \quad \frac{dz}{dx} = 6x + \frac{x}{\sqrt{x^2+a}};$$

$$\frac{dy}{dx} = \frac{6x + \dfrac{x}{\sqrt{x^2+a}}}{3x^2 + \sqrt{a+x^2}} = \frac{x(1+6\sqrt{x^2+a})}{(3x^2+\sqrt{x^2+a})\sqrt{x^2+a}}.$$

(7) $y = (x+3)^2\sqrt{x-2}$.

$$\begin{aligned}
\log_\epsilon y &= 2\log_\epsilon(x+3) + \tfrac{1}{2}\log_\epsilon(x-2).\\
\frac{1}{y}\,\frac{dy}{dx} &= \frac{2}{(x+3)} + \frac{1}{2(x-2)};\\
\frac{dy}{dx} &= (x+3)^2\sqrt{x-2}\left\{\frac{2}{x+3} + \frac{1}{2(x-2)}\right\}.
\end{aligned}$$

(8) $y = (x^2+3)^3(x^3-2)^{\frac{2}{3}}$.

$$\begin{aligned}
&\log_\epsilon y = 3\log_\epsilon(x^2+3) + \tfrac{2}{3}\log_\epsilon(x^3-2);\\
&\frac{1}{y}\,\frac{dy}{dx} = 3\frac{2x}{(x^2+3)} + \frac{2}{3}\,\frac{3x^2}{x^3-2} = \frac{6x}{x^2+3} + \frac{2x^2}{x^3-2}.
\end{aligned}$$

For if $y = \log_\epsilon(x^2+3)$, let $x^2+3 = z$ and $u = \log_\epsilon z$.

$$\frac{du}{dz} = \frac{1}{z}; \quad \frac{dz}{dx} = 2x; \quad \frac{du}{dx} = \frac{2x}{x^2+3}.$$

Similarly, if $v = \log_\epsilon(x^3-2)$, $\dfrac{dv}{dx} = \dfrac{3x^2}{x^3-2}$ and

$$\frac{dy}{dx} = (x^2+3)^3(x^3-2)^{\frac{2}{3}}\left\{\frac{6x}{x^2+3} + \frac{2x^2}{x^3-2}\right\}.$$

(9) $y = \dfrac{\sqrt[2]{x^2+a}}{\sqrt[3]{x^3-a}}$.

$$\begin{aligned}
\log_\epsilon y &= \frac{1}{2}\log_\epsilon(x^2+a) - \frac{1}{3}\log_\epsilon(x^3-a).\\
\frac{1}{y}\,\frac{dy}{dx} &= \frac{1}{2}\,\frac{2x}{x^2+a} - \frac{1}{3}\,\frac{3x^2}{x^3-a} = \frac{x}{x^2+a} - \frac{x^2}{x^3-a}
\end{aligned}$$

and
$$\frac{dy}{dx} = \frac{\sqrt[2]{x^2+a}}{\sqrt[3]{x^3-a}}\left\{\frac{x}{x^2+a} - \frac{x^2}{x^3-a}\right\}.$$

(10) $y = \dfrac{1}{\log_\epsilon x}$

$$\frac{dy}{dx} = \frac{\log_\epsilon x \times 0 - 1 \times \dfrac{1}{x}}{\log_\epsilon^2 x} = -\frac{1}{x \log_\epsilon^2 x}.$$

(11) $y = \sqrt[3]{\log_\epsilon x} = (\log_\epsilon x)^{\frac{1}{3}}$. Let $z = \log_\epsilon x$; $y = z^{\frac{1}{3}}$.

$$\frac{dy}{dz} = \frac{1}{3} z^{-\frac{2}{3}}; \quad \frac{dz}{dx} = \frac{1}{x}; \quad \frac{dy}{dx} = \frac{1}{3x\sqrt[3]{\log_\epsilon^2 x}}.$$

(12) $y = \left(\dfrac{1}{a^x}\right)^{ax}$.

$$\log_\epsilon y = ax(\log_\epsilon 1 - \log_\epsilon a^x) = -ax \log_\epsilon a^x.$$

$$\frac{1}{y}\frac{dy}{dx} = -ax \times a^x \log_\epsilon a - a \log_\epsilon a^x.$$

and
$$\frac{dy}{dx} = -\left(\frac{1}{a^x}\right)^{ax} (x \times a^{x+1} \log_\epsilon a + a \log_\epsilon a^x).$$

Try now the following exercises.

Exercises XII. (See page 263 for Answers.)

(1) Differentiate $y = b(\epsilon^{ax} - \epsilon^{-ax})$.

(2) Find the differential coefficient with respect to t of the expression $u = at^2 + 2 \log_\epsilon t$.

(3) If $y = n^t$, find $\dfrac{d(\log_\epsilon y)}{dt}$.

(4) Show that if $y = \dfrac{1}{b} \cdot \dfrac{a^{bx}}{\log_\epsilon a}$, $\quad \dfrac{dy}{dx} = a^{bx}$.

(5) If $w = pv^n$, find $\dfrac{dw}{dv}$.

Differentiate

(6) $y = \log_\epsilon x^n$. (7) $y = 3\epsilon^{-\frac{x}{x-1}}$.

(8) $y = (3x^2+1)\epsilon^{-5x}$. (9) $y = \log_\epsilon(x^a + a)$.

(10) $y = (3x^2-1)(\sqrt{x}+1)$.

(11) $y = \dfrac{\log_\epsilon(x+3)}{x+3}$. (12) $y = a^x \times x^a$.

(13) It was shown by Lord Kelvin that the speed of signalling through a submarine cable depends on the value of the ratio of the external diameter of the core to the diameter of the enclosed copper wire. If this ratio is called y, then the number of signals s that can be sent per minute can be expressed by the formula

$$s = ay^2 \log_\epsilon \frac{1}{y};$$

where a is a constant depending on the length and the quality of the materials. Show that if these are given, s will be a maximum if $y = 1 \div \sqrt{\epsilon}$.

(14) Find the maximum or minimum of

$$y = x^3 - \log_\epsilon x.$$

(15) Differentiate $y = \log_\epsilon(ax\epsilon^x)$.

(16) Differentiate $y = (\log_\epsilon ax)^3$.

The Logarithmic Curve.

Let us return to the curve which has its successive ordinates in geometrical progression, such as that represented by the equation $y = bp^x$.

We can see, by putting $x = 0$, that b is the initial height of y.

Then when

$$x = 1, \quad y = bp; \qquad x = 2, \quad y = bp^2; \qquad x = 3, \quad y = bp^3, \quad \text{etc.}$$

Also, we see that p is the numerical value of the ratio between the height of any ordinate and that of the next preceding it. In Fig. 40, we have taken p as $\frac{6}{5}$; each ordinate being $\frac{6}{5}$ as high as the preceding one.

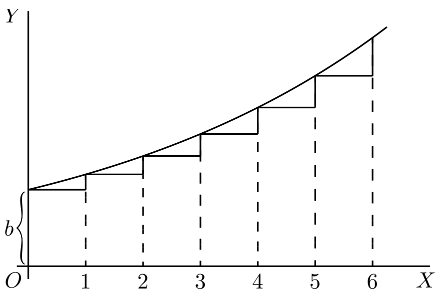


FIG. 40.

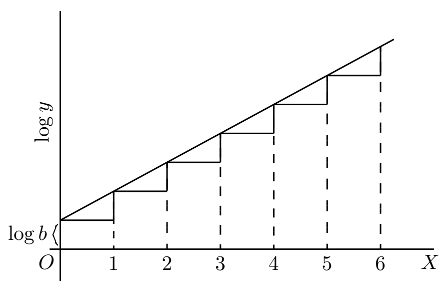


FIG. 41.

If two successive ordinates are related together thus in a constant ratio, their logarithms will have a constant difference; so that, if we should plot out a new curve, Fig. 41, with values of $\log_\epsilon y$ as ordinates, it would be a straight line sloping up by equal steps. In fact, it follows from the equation, that

$$\log_\epsilon y = \log_\epsilon b + x \cdot \log_\epsilon p,$$

whence $$\log_\epsilon y - \log_\epsilon b = x \cdot \log_\epsilon p.$$

Now, since $\log_\epsilon p$ is a mere number, and may be written as $\log_\epsilon p = a$, it follows that

$$\log_\epsilon \frac{y}{b} = ax,$$

and the equation takes the new form

$$y = b\epsilon^{ax}.$$

The Die-away Curve.

If we were to take p as a proper fraction (less than unity), the curve would obviously tend to sink downwards, as in Fig. 42, where each successive ordinate is $\frac{3}{4}$ of the height of the preceding one.

The equation is still

$$y = bp^x;$$

but since p is less than one, $\log_\epsilon p$ will be a negative quantity, and may be written $-a$; so that $p = \epsilon^{-a}$, and now our equation for the curve takes the form

$$y = b\epsilon^{-ax}.$$

The importance of this expression is that, in the case where the independent variable is *time*, the equation represents the course of a

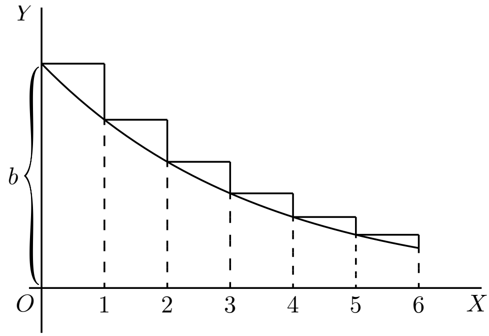


FIG. 42.

great many physical processes in which something is *gradually dying away.* Thus, the cooling of a hot body is represented (in Newton's celebrated "law of cooling") by the equation

$$\theta_t = \theta_0 \epsilon^{-at};$$

where θ_0 is the original excess of temperature of a hot body over that of its surroundings, θ_t the excess of temperature at the end of time t, and a is a constant—namely, the constant of decrement, depending on the amount of surface exposed by the body, and on its coefficients of conductivity and emissivity, etc.

A similar formula,

$$Q_t = Q_0 \epsilon^{-at},$$

is used to express the charge of an electrified body, originally having a charge Q_0, which is leaking away with a constant of decrement a; which constant depends in this case on the capacity of the body and on the resistance of the leakage-path.

Oscillations given to a flexible spring die out after a time; and the dying-out of the amplitude of the motion may be expressed in a similar way.

In fact ϵ^{-at} serves as a *die-away factor* for all those phenomena in which the rate of decrease is proportional to the magnitude of that which is decreasing; or where, in our usual symbols, $\dfrac{dy}{dt}$ is proportional at every moment to the value that y has at that moment. For we have only to inspect the curve, Fig. 42 above, to see that, at every part of it, the slope $\dfrac{dy}{dx}$ is proportional to the height y; the curve becoming flatter as y grows smaller. In symbols, thus

$$y = b\epsilon^{-ax}$$

or $$\log_\epsilon y = \log_\epsilon b - ax \log_\epsilon \epsilon = \log_\epsilon b - ax,$$

and, differentiating, $$\frac{1}{y}\frac{dy}{dx} = -a;$$

hence $$\frac{dy}{dx} = b\epsilon^{-ax} \times (-a) = -ay;$$

or, in words, the slope of the curve is downward, and proportional to y and to the constant a.

We should have got the same result if we had taken the equation in the form

$$y = bp^x;$$

for then $$\frac{dy}{dx} = bp^x \times \log_\epsilon p.$$

But $$\log_\epsilon p = -a;$$

giving us $$\frac{dy}{dx} = y \times (-a) = -ay,$$

as before.

The Time-constant. In the expression for the "die-away factor" ϵ^{-at}, the quantity a is the reciprocal of another quantity known as "*the time-constant*," which we may denote by the symbol T. Then the die-away factor will be written $\epsilon^{-\frac{t}{T}}$; and it will be seen, by making $t = T$ that the meaning of T $\left(\text{or of } \dfrac{1}{a}\right)$ is that this is the length of time which it takes for the original quantity (called θ_0 or Q_0 in the preceding instances) to die away $\dfrac{1}{\epsilon}$th part—that is to 0.3678—of its original value.

The values of ϵ^{x} and ϵ^{-x} are continually required in different branches of physics, and as they are given in very few sets of mathematical tables, some of the values are tabulated on p. 160 for convenience.

As an example of the use of this table, suppose there is a hot body cooling, and that at the beginning of the experiment (*i.e.* when $t = 0$) it is 72° hotter than the surrounding objects, and if the time-constant of its cooling is 20 minutes (that is, if it takes 20 minutes for its excess of temperature to fall to $\dfrac{1}{\epsilon}$ part of 72°), then we can calculate to what it will have fallen in any given time t. For instance, let t be 60 minutes. Then $\dfrac{t}{T} = 60 \div 20 = 3$, and we shall have to find the value of ϵ^{-3}, and then multiply the original 72° by this. The table shows that ϵ^{-3} is 0.0498. So that at the end of 60 minutes the excess of temperature will have fallen to $72° \times 0.0498 = 3.586°$.

x	ϵ^x	ϵ^{-x}	$1-\epsilon^{-x}$
0.00	1.0000	1.0000	0.0000
0.10	1.1052	0.9048	0.0952
0.20	1.2214	0.8187	0.1813
0.50	1.6487	0.6065	0.3935
0.75	2.1170	0.4724	0.5276
0.90	2.4596	0.4066	0.5934
1.00	2.7183	0.3679	0.6321
1.10	3.0042	0.3329	0.6671
1.20	3.3201	0.3012	0.6988
1.25	3.4903	0.2865	0.7135
1.50	4.4817	0.2231	0.7769
1.75	5.755	0.1738	0.8262
2.00	7.389	0.1353	0.8647
2.50	12.182	0.0821	0.9179
3.00	20.086	0.0498	0.9502
3.50	33.115	0.0302	0.9698
4.00	54.598	0.0183	0.9817
4.50	90.017	0.0111	0.9889
5.00	148.41	0.0067	0.9933
5.50	244.69	0.0041	0.9959
6.00	403.43	0.00248	0.99752
7.50	1808.04	0.00055	0.99947
10.00	22026.5	0.000045	0.999955

Further Examples.

(1) The strength of an electric current in a conductor at a time t secs. after the application of the electromotive force producing it is given by the expression $C = \frac{E}{R}\left\{1 - \epsilon^{-\frac{Rt}{L}}\right\}$.

The time constant is $\frac{L}{R}$.

If $E = 10$, $R = 1$, $L = 0.01$; then when t is very large the term $\epsilon^{-\frac{Rt}{L}}$ becomes 1, and $C = \frac{E}{R} = 10$; also

$$\frac{L}{R} = T = 0.01.$$

Its value at any time may be written:

$$C = 10 - 10\epsilon^{-\frac{t}{0.01}},$$

the time-constant being 0.01. This means that it takes 0.01 sec. for the variable term to fall by $\frac{1}{\epsilon} = 0.3678$ of its initial value $10\epsilon^{-\frac{0}{0.01}} = 10$.

To find the value of the current when $t = 0.001$ sec., say, $\frac{t}{T} = 0.1$, $\epsilon^{-0.1} = 0.9048$ (from table).

It follows that, after 0.001 sec., the variable term is $0.9048 \times 10 = 9.048$, and the actual current is $10 - 9.048 = 0.952$.

Similarly, at the end of 0.1 sec.,

$$\frac{t}{T} = 10; \quad \epsilon^{-10} = 0.000045;$$

the variable term is $10 \times 0.000045 = 0.00045$, the current being 9.9995.

(2) The intensity I of a beam of light which has passed through a thickness l cm. of some transparent medium is $I = I_0\epsilon^{-Kl}$, where I_0 is the initial intensity of the beam and K is a "constant of absorption."

This constant is usually found by experiments. If it be found, for instance, that a beam of light has its intensity diminished by 18% in passing through 10 cms. of a certain transparent medium, this means that $82 = 100 \times \epsilon^{-K\times 10}$ or $\epsilon^{-10K} = 0.82$, and from the table one sees that $10K = 0.20$ very nearly; hence $K = 0.02$.

To find the thickness that will reduce the intensity to half its value, one must find the value of l which satisfies the equality $50 = 100\times\epsilon^{-0.02l}$, or $0.5 = \epsilon^{-0.02l}$. It is found by putting this equation in its logarithmic form, namely,

$$\log 0.5 = -0.02 \times l \times \log\epsilon,$$

which gives

$$l = \frac{-0.3010}{-0.02 \times 0.4343} = 34.7 \text{ centimetres nearly.}$$

(3) The quantity Q of a radio-active substance which has not yet undergone transformation is known to be related to the initial quantity Q_0 of the substance by the relation $Q = Q_0\epsilon^{-\lambda t}$, where λ is a constant and t the time in seconds elapsed since the transformation began.

For "Radium A," if time is expressed in seconds, experiment shows that $\lambda = 3.85 \times 10^{-3}$. Find the time required for transforming half the substance. (This time is called the "mean life" of the substance.)

We have $0.5 = \epsilon^{-0.00385t}$.

$$\log 0.5 = -0.00385t \times \log\epsilon;$$

and $$t = 3 \text{ minutes very nearly.}$$

Exercises XIII. (See page 263 for Answers.)

(1) Draw the curve $y = b\epsilon^{-\frac{t}{T}}$; where $b = 12$, $T = 8$, and t is given various values from 0 to 20.

(2) If a hot body cools so that in 24 minutes its excess of temperature has fallen to half the initial amount, deduce the time-constant, and find how long it will be in cooling down to 1 per cent. of the original excess.

(3) Plot the curve $y = 100(1 - \epsilon^{-2t})$.

(4) The following equations give very similar curves:

$$\text{(i) } y = \frac{ax}{x+b};$$
$$\text{(ii) } y = a(1 - \epsilon^{-\frac{x}{b}});$$
$$\text{(iii) } y = \frac{a}{90^\circ} \operatorname{arc}\tan\left(\frac{x}{b}\right).$$

Draw all three curves, taking $a = 100$ millimetres; $b = 30$ millimetres.

(5) Find the differential coefficient of y with respect to x, if

$$(a)\ y = x^x; \quad (b)\ y = (\epsilon^x)^x; \quad (c)\ y = \epsilon^{x^x}.$$

(6) For "Thorium A," the value of λ is 5; find the "mean life," that is, the time taken by the transformation of a quantity Q of "Thorium A" equal to half the initial quantity Q_0 in the expression

$$Q = Q_0\epsilon^{-\lambda t};$$

t being in seconds.

(7) A condenser of capacity $K = 4 \times 10^{-6}$, charged to a potential $V_0 = 20$, is discharging through a resistance of $10,000$ ohms. Find the potential V after (a) 0.1 second; (b) 0.01 second; assuming that the fall of potential follows the rule $V = V_0 \epsilon^{-\frac{t}{KR}}$.

(8) The charge Q of an electrified insulated metal sphere is reduced from 20 to 16 units in 10 minutes. Find the coefficient μ of leakage, if $Q = Q_0 \times \epsilon^{-\mu t}$; Q_0 being the initial charge and t being in seconds. Hence find the time taken by half the charge to leak away.

(9) The damping on a telephone line can be ascertained from the relation $i = i_0 \epsilon^{-\beta l}$, where i is the strength, after t seconds, of a telephonic current of initial strength i_0; l is the length of the line in kilometres, and β is a constant. For the Franco-English submarine cable laid in 1910, $\beta = 0.0114$. Find the damping at the end of the cable (40 kilometres), and the length along which i is still 8% of the original current (limiting value of very good audition).

(10) The pressure p of the atmosphere at an altitude h kilometres is given by $p = p_0 \epsilon^{-kh}$; p_0 being the pressure at sea-level (760 millimetres).

The pressures at 10, 20 and 50 kilometres being 199.2, 42.2, 0.32 respectively, find k in each case. Using the mean value of k, find the percentage error in each case.

(11) Find the minimum or maximum of $y = x^x$.

(12) Find the minimum or maximum of $y = x^{\frac{1}{x}}$.

(13) Find the minimum or maximum of $y = xa^{\frac{1}{x}}$.

CAPÍTULO XV.

CÓMO TRATAR CON SENOS Y COSENOS.

Greek letters being usual to denote angles, we will take as the usual letter for any variable angle the letter θ ("theta").

Let us consider the function

$$y = \sin\theta.$$

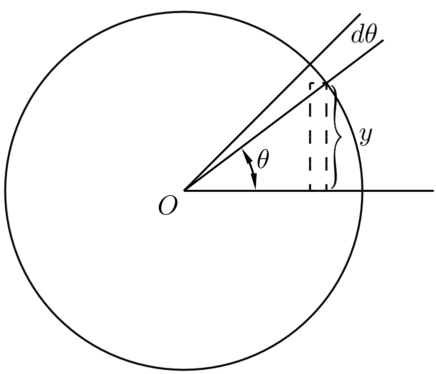


Fig. 43.

What we have to investigate is the value of $\dfrac{d(\sin\theta)}{d\theta}$; or, in other words, if the angle θ varies, we have to find the relation between the increment of the sine and the increment of the angle, both increments being indefinitely small in themselves. Examine Fig. 43, wherein, if the radius of the circle is unity, the height of y is the sine, and θ is the angle. Now, if θ is supposed to increase by the addition to it of the small angle $d\theta$—an element of angle—the height of y, the sine, will be increased by a small element dy. The new height $y+dy$ will be the sine of the new angle $\theta + d\theta$, or, stating it as an equation,

$$y + dy = \sin(\theta + d\theta);$$

and subtracting from this the first equation gives

$$dy = \sin(\theta + d\theta) - \sin\theta.$$

The quantity on the right-hand side is the difference between two sines, and books on trigonometry tell us how to work this out. For they tell us that if M and N are two different angles,

$$\sin M - \sin N = 2\cos\frac{M+N}{2} \cdot \sin\frac{M-N}{2}.$$

If, then, we put $M = \theta + d\theta$ for one angle, and $N = \theta$ for the other, we may write

$$dy = 2\cos\frac{\theta + d\theta + \theta}{2} \cdot \sin\frac{\theta + d\theta - \theta}{2},$$

or, $$dy = 2\cos(\theta + \tfrac{1}{2}d\theta) \cdot \sin\tfrac{1}{2}d\theta.$$

But if we regard $d\theta$ as indefinitely small, then in the limit we may neglect $\frac{1}{2}d\theta$ by comparison with θ, and may also take $\sin\frac{1}{2}d\theta$ as being the same as $\frac{1}{2}d\theta$. The equation then becomes:

$$dy = 2\cos\theta \times \tfrac{1}{2}d\theta;$$

$$dy = \cos\theta \cdot d\theta,$$

and, finally, $$\frac{dy}{d\theta} = \cos\theta.$$

The accompanying curves, Figs. 44 and 45, show, plotted to scale, the values of $y = \sin\theta$, and $\dfrac{dy}{d\theta} = \cos\theta$, for the corresponding values of θ.

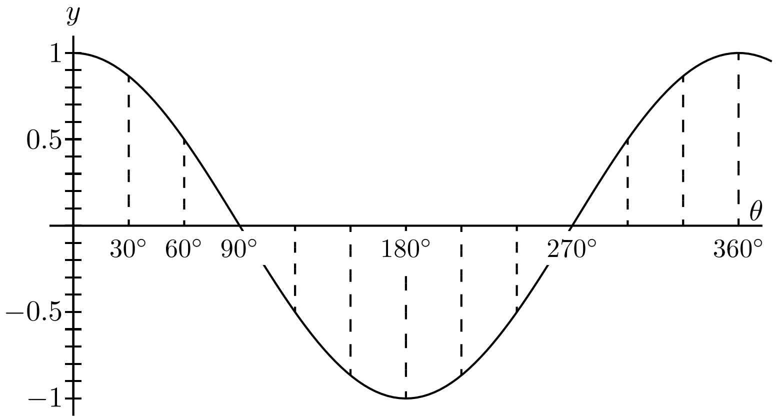


Fig. 44.

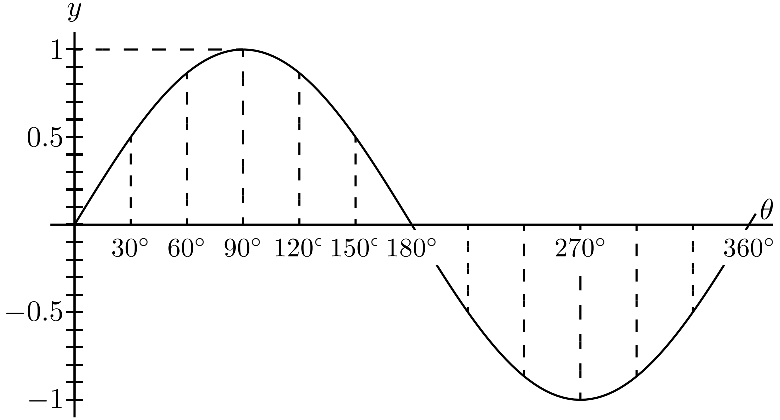


Fig. 45.

Take next the cosine.

Let $y = \cos\theta$.

Now $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$.

Therefore

$$\begin{aligned} dy = d\left(\sin\left(\frac{\pi}{2} - \theta\right)\right) &= \cos\left(\frac{\pi}{2} - \theta\right) \times d(-\theta), \\ &= \cos\left(\frac{\pi}{2} - \theta\right) \times (-d\theta), \end{aligned}$$

$$\frac{dy}{d\theta} = -\cos\left(\frac{\pi}{2} - \theta\right).$$

And it follows that

$$\frac{dy}{d\theta} = -\sin\theta.$$

Lastly, take the tangent.

Let $$y = \tan\theta,$$

$$dy = \tan(\theta + d\theta) - \tan\theta.$$

Expanding, as shown in books on trigonometry,

$$\tan(\theta + d\theta) = \frac{\tan\theta + \tan d\theta}{1 - \tan\theta \cdot \tan d\theta};$$

whence
$$\begin{aligned} dy &= \frac{\tan\theta + \tan d\theta}{1 - \tan\theta \cdot \tan d\theta} - \tan\theta \\ &= \frac{(1 + \tan^2\theta)\tan d\theta}{1 - \tan\theta \cdot \tan d\theta}. \end{aligned}$$

Now remember that if $d\theta$ is indefinitely diminished, the value of $\tan d\theta$ becomes identical with $d\theta$, and $\tan\theta \cdot d\theta$ is negligibly small compared with 1, so that the expression reduces to

$$dy = \frac{(1+\tan^2\theta)\, d\theta}{1},$$

so that
$$\frac{dy}{d\theta} = 1 + \tan^2\theta,$$

or
$$\frac{dy}{d\theta} = \sec^2\theta.$$

Collecting these results, we have:

y	$\dfrac{dy}{d\theta}$
$\sin\theta$	$\cos\theta$
$\cos\theta$	$-\sin\theta$
$\tan\theta$	$\sec^2\theta$

Sometimes, in mechanical and physical questions, as, for example, in simple harmonic motion and in wave-motions, we have to deal with angles that increase in proportion to the time. Thus, if T be the time of one complete *period*, or movement round the circle, then, since the angle all round the circle is 2π radians, or 360°, the amount of angle moved through in time t, will be

$$\theta = 2\pi\frac{t}{T}, \quad \text{in radians,}$$

or
$$\theta = 360\frac{t}{T}, \quad \text{in degrees.}$$

If the *frequency*, or number of periods per second, be denoted by n, then $n = \frac{1}{T}$, and we may then write:

$$\theta = 2\pi nt.$$

Then we shall have

$$y = \sin 2\pi nt.$$

If, now, we wish to know how the sine varies with respect to time, we must differentiate with respect, not to θ, but to t. For this we must resort to the artifice explained in Capítulo IX., p. 69, and put

$$\frac{dy}{dt} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dt}.$$

Now $\frac{d\theta}{dt}$ will obviously be $2\pi n$; so that

$$\begin{aligned}\frac{dy}{dt} &= \cos\theta \times 2\pi n \\ &= 2\pi n \cdot \cos 2\pi nt.\end{aligned}$$

Similarly, it follows that

$$\frac{d(\cos 2\pi nt)}{dt} = -2\pi n \cdot \sin 2\pi nt.$$

Second Differential Coefficient of Sine or Cosine.

We have seen that when $\sin\theta$ is differentiated with respect to θ it becomes $\cos\theta$; and that when $\cos\theta$ is differentiated with respect to θ it becomes $-\sin\theta$; or, in symbols,

$$\frac{d^2(\sin\theta)}{d\theta^2} = -\sin\theta.$$

So we have this curious result that we have found a function such that if we differentiate it twice over, we get the same thing from which we started, but with the sign changed from + to −.

The same thing is true for the cosine; for differentiating $\cos\theta$ gives us $-\sin\theta$, and differentiating $-\sin\theta$ gives us $-\cos\theta$; or thus:

$$\frac{d^2(\cos\theta)}{d\theta^2} = -\cos\theta.$$

Sines and cosines are the only functions of which the second differential coefficient is equal (and of opposite sign to) *the original function.*

Examples.

With what we have so far learned we can now differentiate expressions of a more complex nature.

(1) $y = \operatorname{arc}\sin x$.

If y is the arc whose sine is x, then $x = \sin y$.

$$\frac{dx}{dy} = \cos y.$$

Passing now from the inverse function to the original one, we get

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\cos y}.$$

Now $$\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2};$$

hence $$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}},$$

a rather unexpected result.

(2) $y = \cos^3\theta$.

This is the same thing as $y = (\cos\theta)^3$.

Let $\cos\theta = v$; then $y = v^3$; $\dfrac{dy}{dv} = 3v^2$.

$$\begin{aligned}\frac{dv}{d\theta} &= -\sin\theta.\\ \frac{dy}{d\theta} &= \frac{dy}{dv} \times \frac{dv}{d\theta} = -3\cos^2\theta\sin\theta.\end{aligned}$$

(3) $y = \sin(x+a)$.

Let $x+a = v$; then $y = \sin v$.

$$\frac{dy}{dv} = \cos v; \qquad \frac{dv}{dx} = 1 \quad \text{and} \quad \frac{dy}{dx} = \cos(x+a).$$

(4) $y = \log_\epsilon \sin\theta$.

Let $\sin\theta = v$; $y = \log_\epsilon v$.

$$\begin{aligned}\frac{dy}{dv} &= \frac{1}{v}; \quad \frac{dv}{d\theta} = \cos\theta;\\ \frac{dy}{d\theta} &= \frac{1}{\sin\theta} \times \cos\theta = \cot\theta.\end{aligned}$$

(5) $y = \cot\theta = \dfrac{\cos\theta}{\sin\theta}$.

$$\begin{aligned}\frac{dy}{d\theta} &= \frac{-\sin^2\theta - \cos^2\theta}{\sin^2\theta}\\ &= -(1+\cot^2\theta) = -\operatorname{cosec}^2\theta.\end{aligned}$$

(6) $y = \tan 3\theta$.

Let $3\theta = v$; $y = \tan v$; $\dfrac{dy}{dv} = \sec^2 v$.

$$\frac{dv}{d\theta} = 3; \quad \frac{dy}{d\theta} = 3\sec^2 3\theta.$$

(7) $y = \sqrt{1 + 3\tan^2\theta}$; $y = (1 + 3\tan^2\theta)^{\frac{1}{2}}$.

Let $3\tan^2\theta = v$.

$$y = (1 + v)^{\frac{1}{2}}; \quad \frac{dy}{dv} = \frac{1}{2\sqrt{1 + v}} \text{ (see p. 70)};$$

$$\frac{dv}{d\theta} = 6\tan\theta\sec^2\theta$$

(for, if $\tan\theta = u$,

$$v = 3u^2; \quad \frac{dv}{du} = 6u; \quad \frac{du}{d\theta} = \sec^2\theta;$$

hence $$\frac{dv}{d\theta} = 6(\tan\theta\sec^2\theta)$$

hence $$\frac{dy}{d\theta} = \frac{6\tan\theta\sec^2\theta}{2\sqrt{1 + 3\tan^2\theta}}.$$

(8) $y = \sin x\cos x$.

$$\begin{aligned} \frac{dy}{dx} &= \sin x(-\sin x) + \cos x \times \cos x \\ &= \cos^2 x - \sin^2 x. \end{aligned}$$

Exercises XIV. (See page 264 for Answers.)

(1) Differentiate the following:

$$\text{(i)} \quad y = A\sin\left(\theta - \frac{\pi}{2}\right).$$

$$\text{(ii)} \quad y = \sin^2\theta; \quad \text{and } y = \sin 2\theta.$$

$$\text{(iii)} \quad y = \sin^3\theta; \quad \text{and } y = \sin 3\theta.$$

(2) Find the value of θ for which $\sin\theta \times \cos\theta$ is a maximum.

(3) Differentiate $y = \dfrac{1}{2\pi} \cos 2\pi nt$.

(4) If $y = \sin a^x$, find $\dfrac{dy}{dx}$.

(5) Differentiate $y = \log_\epsilon \cos x$.

(6) Differentiate $y = 18.2 \sin(x + 26^\circ)$.

(7) Plot the curve $y = 100 \sin(\theta - 15^\circ)$; and show that the slope of the curve at $\theta = 75^\circ$ is half the maximum slope.

(8) If $y = \sin\theta \cdot \sin 2\theta$, find $\dfrac{dy}{d\theta}$.

(9) If $y = a \cdot \tan^m(\theta^n)$, find the differential coefficient of y with respect to θ.

(10) Differentiate $y = \epsilon^x \sin^2 x$.

(11) Differentiate the three equations of Exercises XIII. (p. 163), No. 4, and compare their differential coefficients, as to whether they are equal, or nearly equal, for very small values of x, or for very large values of x, or for values of x in the neighbourhood of $x = 30$.

(12) Differentiate the following:

(i) $y = \sec x$.

(ii) $y = \operatorname{arc} \cos x$.

(iii) $y = \operatorname{arc} \tan x$.

(iv) $y = \operatorname{arc} \sec x$.

(v) $y = \tan x \times \sqrt{3 \sec x}$.

(13) Differentiate $y = \sin(2\theta + 3)^{2.3}$.

(14) Differentiate $y = \theta^3 + 3\sin(\theta + 3) - 3^{\sin\theta} - 3^\theta$.

(15) Find the maximum or minimum of $y = \theta \cos\theta$.

CAPÍTULO XVI.

DIFERENCIACIÓN PARCIAL.

WE sometimes come across quantities that are functions of more than one independent variable. Thus, we may find a case where y depends on two other variable quantities, one of which we will call u and the other v. In symbols

$$y = f(u, v).$$

Take the simplest concrete case.

Let $$y = u \times v.$$

What are we to do? If we were to treat v as a constant, and differentiate with respect to u, we should get

$$dy_v = v\,du;$$

or if we treat u as a constant, and differentiate with respect to v, we should have:

$$dy_u = u\,dv.$$

The little letters here put as subscripts are to show which quantity has been taken as constant in the operation.

Another way of indicating that the differentiation has been performed only *partially*, that is, has been performed only with respect to *one* of the independent variables, is to write the differential coefficients with Greek deltas, like ∂, instead of little d. In this way

$$\frac{\partial y}{\partial u} = v,$$
$$\frac{\partial y}{\partial v} = u.$$

If we put in these values for v and u respectively, we shall have

$$\left.\begin{aligned} dy_v &= \frac{\partial y}{\partial u}\,du, \\ dy_u &= \frac{\partial y}{\partial v}\,dv, \end{aligned}\right\} \quad \text{which are \textit{partial differentials.}}$$

But, if you think of it, you will observe that the total variation of y depends on *both* these things at the same time. That is to say, if both are varying, the real dy ought to be written

$$dy = \frac{\partial y}{\partial u}\,du + \frac{\partial y}{\partial v}\,dv;$$

and this is called a *total differential.* In some books it is written $dy = \left(\dfrac{dy}{du}\right)\,du + \left(\dfrac{dy}{dv}\right)\,dv$.

Example (1). Find the partial differential coefficients of the expression $w = 2ax^2 + 3bxy + 4cy^3$. The answers are:

$$\left.\begin{aligned} \frac{\partial w}{\partial x} &= 4ax + 3by. \\ \frac{\partial w}{\partial y} &= 3bx + 12cy^2. \end{aligned}\right\}$$

The first is obtained by supposing y constant, the second is obtained by supposing x constant; then

$$dw = (4ax + 3by)\,dx + (3bx + 12cy^2)\,dy.$$

Example (2). Let $z = x^y$. Then, treating first y and then x as constant, we get in the usual way

$$\left.\begin{aligned}\frac{\partial z}{\partial x} &= yx^{y-1},\\ \frac{\partial z}{\partial y} &= x^y \times \log_\epsilon x,\end{aligned}\right\}$$

so that $dz = yx^{y-1}\,dx + x^y \log_\epsilon x\,dy$.

Example (3). A cone having height h and radius of base r has volume $V = \frac{1}{3}\pi r^2 h$. If its height remains constant, while r changes, the ratio of change of volume, with respect to radius, is different from ratio of change of volume with respect to height which would occur if the height were varied and the radius kept constant, for

$$\left.\begin{aligned}\frac{\partial V}{\partial r} &= \frac{2\pi}{3} rh,\\ \frac{\partial V}{\partial h} &= \frac{\pi}{3} r^2.\end{aligned}\right\}$$

The variation when both the radius and the height change is given by $dV = \dfrac{2\pi}{3} rh\,dV + \dfrac{\pi}{3} r^2\,dh$.

Example (4). In the following example F and f denote two arbitrary functions of any form whatsoever. For example, they may be sine-functions, or exponentials, or mere algebraic functions of the two independent variables, t and x. This being understood, let us take the expression

$$y = F(x+at) + f(x-at),$$

or,
$$y = F(w) + f(v);$$

where
$$w = x + at, \quad \text{and} \quad v = x - at.$$

Then
$$\begin{aligned}\frac{\partial y}{\partial x} &= \frac{\partial F(w)}{\partial w} \cdot \frac{\partial w}{\partial x} + \frac{\partial f(v)}{\partial v} \cdot \frac{\partial v}{\partial x}\\ &= F'(w) \cdot 1 + f'(v) \cdot 1\end{aligned}$$

(where the figure 1 is simply the coefficient of x in w and v);

and
$$\frac{\partial^2 y}{\partial x^2} = F''(w) + f''(v).$$

Also
$$\begin{aligned}\frac{\partial y}{\partial t} &= \frac{\partial F(w)}{\partial w} \cdot \frac{\partial w}{\partial t} + \frac{\partial f(v)}{\partial v} \cdot \frac{\partial v}{\partial t}\\ &= F'(w) \cdot a - f'(v)a;\end{aligned}$$

and
$$\frac{\partial^2 y}{\partial t^2} = F''(w)a^2 + f''(v)a^2;$$

whence
$$\frac{\partial^2 y}{\partial t^2} = a^2\,\frac{\partial^2 y}{\partial x^2}.$$

This differential equation is of immense importance in mathematical physics.

Maxima and Minima of Functions of two Independent Variables.

Example (5). Let us take up again Exercise IX., p. 110, No. 4.

Let x and y be the length of two of the portions of the string. The third is $30 - (x + y)$, and the area of the triangle is $A =$

$\sqrt{s(s-x)(s-y)(s-30+x+y)}$, where s is the half perimeter, 15, so that $A = \sqrt{15P}$, where

$$\begin{aligned} P &= (15-x)(15-y)(x+y-15) \\ &= xy^2 + x^2y - 15x^2 - 15y^2 - 45xy + 450x + 450y - 3375. \end{aligned}$$

Clearly A is maximum when P is maximum.

$$dP = \frac{\partial P}{\partial x}\,dx + \frac{\partial P}{\partial y}\,dy.$$

For a maximum (clearly it will not be a minimum in this case), one must have simultaneously

$$\frac{\partial P}{\partial x} = 0 \quad \text{and} \quad \frac{\partial P}{\partial y} = 0;$$

that is,
$$\left.\begin{aligned} 2xy - 30x + y^2 - 45y + 450 = 0, \\ 2xy - 30y + x^2 - 45x + 450 = 0. \end{aligned}\right\}$$

An immediate solution is $x = y$.

If we now introduce this condition in the value of P, we find

$$P = (15-x)^2(2x-15) = 2x^3 - 75x^2 + 900x - 3375.$$

For maximum or minimum, $\dfrac{dP}{dx} = 6x^2 - 150x + 900 = 0$, which gives $x = 15$ or $x = 10$.

Clearly $x = 15$ gives minimum area; $x = 10$ gives the maximum, for $\dfrac{d^2P}{dx^2} = 12x - 150$, which is $+30$ for $x = 15$ and -30 for $x = 10$.

Example (6). Find the dimensions of an ordinary railway coal truck with rectangular ends, so that, for a given volume V the area of sides and floor together is as small as possible.

The truck is a rectangular box open at the top. Let x be the length and y be the width; then the depth is $\dfrac{V}{xy}$. The surface area is $S = xy + \dfrac{2V}{x} + \dfrac{2V}{y}$.

$$dS = \frac{\partial S}{\partial x}\,dx + \frac{\partial S}{\partial y}\,dy = \left(y - \frac{2V}{x^2}\right)dx + \left(x - \frac{2V}{y^2}\right)dy.$$

For minimum (clearly it won't be a maximum here),

$$y - \frac{2V}{x^2} = 0, \quad x - \frac{2V}{y^2} = 0.$$

Here also, an immediate solution is $x = y$, so that $S = x^2 + \dfrac{4V}{x}$, $\dfrac{dS}{dx} = 2x - \dfrac{4V}{x^2} = 0$ for minimum, and

$$x = \sqrt[3]{2V}.$$

Exercises XV. (See page 266 for Answers.)

(1) Differentiate the expression $\dfrac{x^3}{3} - 2x^3y - 2y^2x + \dfrac{y}{3}$ with respect to x alone, and with respect to y alone.

(2) Find the partial differential coefficients with respect to x, y and z, of the expression

$$x^2yz + xy^2z + xyz^2 + x^2y^2z^2.$$

(3) Let $r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$.

Find the value of $\dfrac{\partial r}{\partial x} + \dfrac{\partial r}{\partial y} + \dfrac{\partial r}{\partial z}$. Also find the value of $\dfrac{\partial^2 r}{\partial x^2} + \dfrac{\partial^2 r}{\partial y^2} + \dfrac{\partial^2 r}{\partial z^2}$.

(4) Find the total differential of $y = u^v$.

(5) Find the total differential of $y = u^3 \sin v$; of $y = (\sin x)^u$; and of $y = \dfrac{\log_\epsilon u}{v}$.

(6) Verify that the sum of three quantities x, y, z, whose product is a constant k, is maximum when these three quantities are equal.

(7) Find the maximum or minimum of the function

$$u = x + 2xy + y.$$

(8) The post-office regulations state that no parcel is to be of such a size that its length plus its girth exceeds 6 feet. What is the greatest volume that can be sent by post (a) in the case of a package of rectangular cross section; (b) in the case of a package of circular cross section.

(9) Divide π into 3 parts such that the continued product of their sines may be a maximum or minimum.

(10) Find the maximum or minimum of $u = \dfrac{\epsilon^{x+y}}{xy}$.

(11) Find maximum and minimum of

$$u = y + 2x - 2\log_\epsilon y - \log_\epsilon x.$$

(12) A telpherage bucket of given capacity has the shape of a horizontal isosceles triangular prism with the apex underneath, and the

opposite face open. Find its dimensions in order that the least amount of iron sheet may be used in its construction.

CAPÍTULO XVII.

INTEGRACIÓN.

El gran secreto ya ha sido revelado que este símbolo misterioso $\displaystyle\int$, que después de todo es solo una S larga, simplemente significa "la suma de," o "la suma de todas las cantidades tales como." Por lo tanto se asemeja a ese otro símbolo $\sum$ (la *Sigma* griega), que también es un signo de suma. Hay esta diferencia, sin embargo, en la práctica de los hombres matemáticos en cuanto al uso de estos signos, que mientras $\sum$ generalmente se usa para indicar la suma de un número de cantidades finitas, el signo integral $\displaystyle\int$ generalmente se usa para indicar la suma de un vasto número de pequeñas cantidades de magnitud indefinidamente diminuta, meros elementos de hecho, que van a formar el total requerido. Así $\displaystyle\int dy = y$, y $\displaystyle\int dx = x$.

Cualquiera puede entender cómo el total de cualquier cosa puede concebirse como formado por muchos pequeños pedazos; y cuanto más pequeños los pedazos, más de ellos habrá. Así, una línea de una pulgada de largo puede concebirse como formada por 10 piezas, cada una de $\frac{1}{10}$ de pulgada de largo; o de 100 partes, cada parte siendo $\frac{1}{100}$ de pulgada de largo; o de $1,000,000$ partes, cada una de las cuales es $\frac{1}{1{,}000{,}000}$ de pulgada de largo; o, llevando el pensamiento a los límites de la concebibilidad, puede considerarse como formada por un número infinito de elementos cada uno de los cuales es infinitesimalmente pequeño.

Sí, dirás, pero ¿cuál es el uso de pensar of anything that way? Why not think of it straight off, as a whole? The simple reason is that there are a vast number of cases in which one cannot calculate the bigness of the thing as a whole without reckoning up the sum of a lot of small parts. The process of "*integrating*" is to enable us to calculate totals that otherwise we should be unable to estimate directly.

Let us first take one or two simple cases to familiarize ourselves with this notion of summing up a lot of separate parts.

Consider the series:

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} + \tfrac{1}{64} + \text{etc.}$$

Here each member of the series is formed by taking it half the value of the preceding. What is the value of the total if we could go on to an infinite number of terms? Every schoolboy knows that the answer is 2. Think of it, if you like, as a line. Begin with one inch; add a half

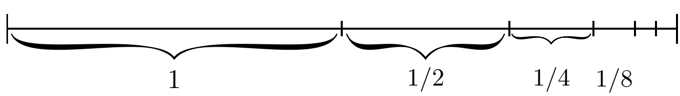


Fig. 46.

inch, add a quarter; add an eighth; and so on. If at any point of the operation we stop, there will still be a piece wanting to make up the whole 2 inches; and the piece wanting will always be the same size as the last piece added. Thus, if after having put together 1, $\frac{1}{2}$, and $\frac{1}{4}$, we stop, there will be $\frac{1}{4}$ wanting. If we go on till we have added $\frac{1}{64}$, there will still be $\frac{1}{64}$ wanting. The remainder needed will always be equal to the last term added. By an infinite number of operations only should we reach the actual 2 inches. Practically we should reach it when we got to pieces so small that they could not be drawn—that would be after about 10 terms, for the eleventh term is $\frac{1}{1024}$. If we want to go so far that not even a Whitworth's measuring machine would detect it, we should merely have to go to about 20 terms. A microscope would not show even the 18^{th} term! So the infinite number of operations is no such dreadful thing after all. The *integral* is simply the whole lot. But, as we shall see, there are cases in which the integral calculus enables us to get at the *exact* total that there would be as the result of an infinite number of operations. In such cases the integral calculus gives us a *rapid* and easy way of getting at a result that would otherwise require an interminable lot of elaborate working out. So we had best lose no time in learning *how to integrate*.

Slopes of Curves, and the Curves themselves.

Let us make a little preliminary enquiry about the slopes of curves. For we have seen that differentiating a curve means finding an expression for its slope (or for its slopes at different points). Can we perform the reverse process of reconstructing the whole curve if the slope (or slopes) are prescribed for us?

Go back to case (2) on p. 85. Here we have the simplest of curves, a sloping line with the equation

$$y = ax + b.$$

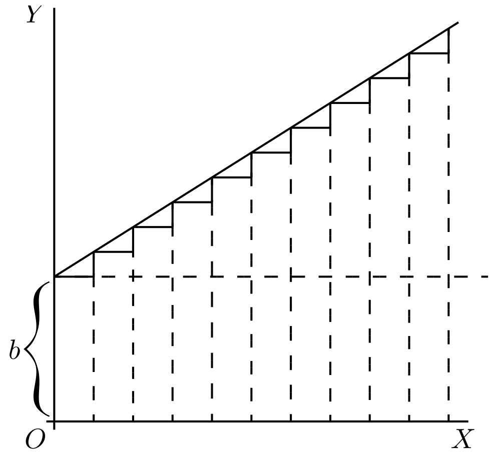
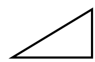
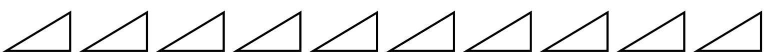


Fig. 47.

We know that here b represents the initial height of y when $x = 0$, and that a, which is the same as $\dfrac{dy}{dx}$, is the "slope" of the line. The line has a constant slope. All along it the elementary triangles 

have the same proportion between height and base. Suppose we were to take the dx's, and dy's of finite magnitude, so that 10 dx's made up one inch, then there would be ten little triangles like



Now, suppose that we were ordered to reconstruct the "curve," starting merely from the information that $\dfrac{dy}{dx} = a$. What could we do? Still taking the little d's as of finite size, we could draw 10 of them, all with the same slope, and then put them together, end to end, like this: And, as the slope is the same for all, they would join to make, as in Fig. 48, a sloping line sloping with the correct slope $\dfrac{dy}{dx} = a$. And whether we take the dy's and dx's as finite or infinitely small, as they

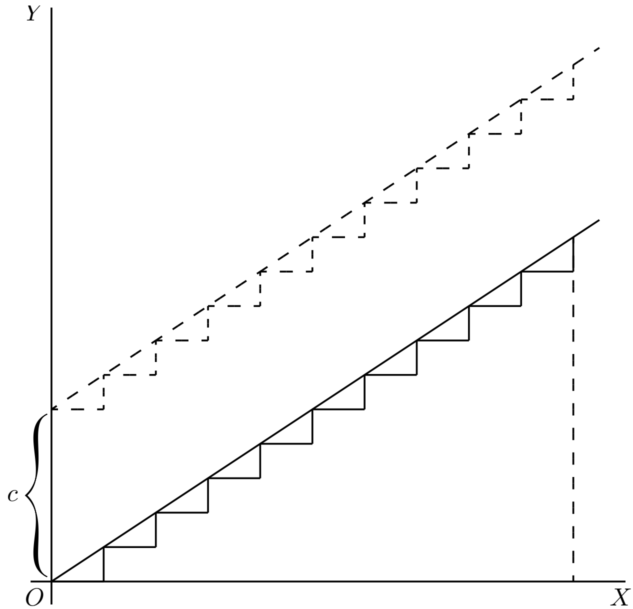


FIG. 48.

are all alike, clearly $\dfrac{y}{x} = a$, if we reckon y as the total of all the dy's, and x as the total of all the dx's. But whereabouts are we to put this sloping line? Are we to start at the origin O, or higher up? As the only information we have is as to the slope, we are without any instructions as to the particular height above O; in fact the initial height is undetermined. The slope will be the same, whatever the initial height. Let us therefore make a shot at what may be wanted, and start the sloping line at a height C above O. That is, we have the equation

$$y = ax + C.$$

It becomes evident now that in this case the added constant means the particular value that y has when $x = 0$.

Now let us take a harder case, that of a line, the slope of which is not constant, but turns up more and more. Let us assume that the upward slope gets greater and greater in proportion as x grows. In symbols this is:

$$\frac{dy}{dx} = ax.$$

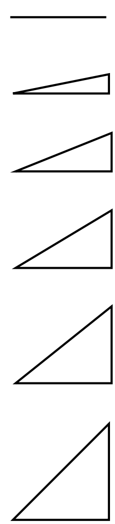
Or, to give a concrete case, take $a = \frac{1}{5}$, so that

$$\frac{dy}{dx} = \tfrac{1}{5}x.$$

Then we had best begin by calculating a few of the values of the slope at different values of x, and also draw little diagrams of them.

When

$$\begin{aligned}
x &= 0, & \frac{dy}{dx} &= 0, \\
x &= 1, & \frac{dy}{dx} &= 0.2, \\
x &= 2, & \frac{dy}{dx} &= 0.4, \\
x &= 3, & \frac{dy}{dx} &= 0.6, \\
x &= 4, & \frac{dy}{dx} &= 0.8, \\
x &= 5, & \frac{dy}{dx} &= 1.0.
\end{aligned}$$



Now try to put the pieces together, setting each so that the middle of its base is the proper distance to the right, and so that they fit together at the corners; thus (Fig. 49). The result is, of course, not a smooth curve: but it is an approximation to one. If we had taken

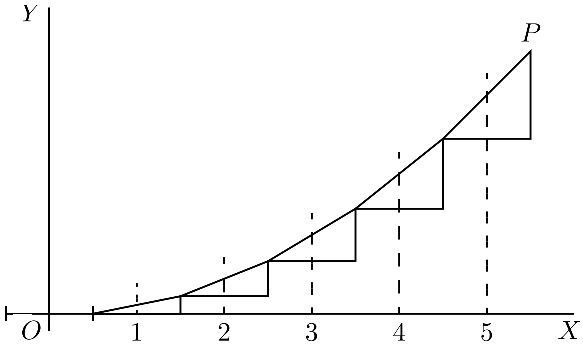


Fig. 49.

bits half as long, and twice as numerous, like Fig. 50, we should have a better approximation. But for a perfect curve we ought to take each dx and its corresponding dy infinitesimally small, and infinitely numerous.

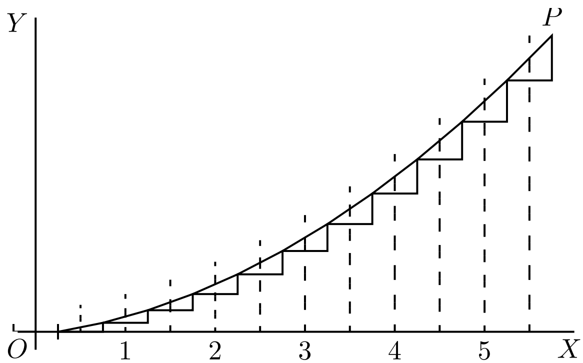


Fig. 50.

Then, how much ought the value of any y to be? Clearly, at any point P of the curve, the value of y will be the sum of all the little dy's from 0 up to that level, that is to say, $\int dy = y$. And as each dy is equal to $\frac{1}{5}x \cdot dx$, it follows that the whole y will be equal to the sum of all such bits as $\frac{1}{5}x \cdot dx$, or, as we should write it, $\int \frac{1}{5}x \cdot dx$.

Now if x had been constant, $\int \frac{1}{5}x \cdot dx$ would have been the same as $\frac{1}{5}x \int dx$, or $\frac{1}{5}x^2$. But x began by being 0, and increases to the particular value of x at the point P, so that its average value from 0 to that point is $\frac{1}{2}x$. Hence $\int \frac{1}{5}x\,dx = \frac{1}{10}x^2$; or $y = \frac{1}{10}x^2$.

But, as in the previous case, this requires the addition of an undetermined constant C, because we have not been told at what height above the origin the curve will begin, when $x = 0$. So we write, as the equation of the curve drawn in Fig. 51,

$$y = \tfrac{1}{10}x^2 + C.$$

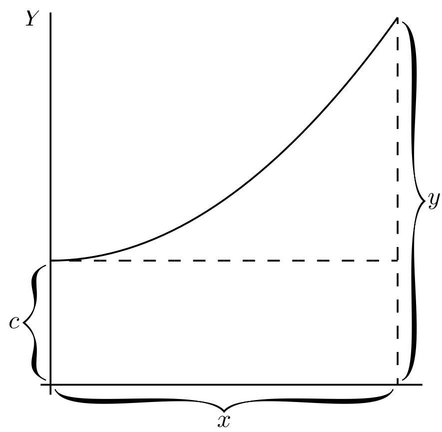


FIG. 51.

Exercises XVI. (See page 267 for Answers.)

(1) Find the ultimate sum of $\frac{2}{3} + \frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \frac{1}{24} +$ etc.

(2) Show that the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7}$ etc., is convergent, and find its sum to 8 terms.

(3) If $\log_\epsilon(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \text{etc.}$, find $\log_\epsilon 1.3$.

(4) Following a reasoning similar to that explained in this chapter, find y,

$$(a) \text{ if } \frac{dy}{dx} = \tfrac{1}{4}x; \quad (b) \text{ if } \frac{dy}{dx} = \cos x.$$

(5) If $\dfrac{dy}{dx} = 2x + 3$, find y.

CAPÍTULO XVIII.

INTEGRANDO COMO EL REVERSO DE DIFERENCIAR.

Differentiating is the process by which when y is given us (as a function of x), we can find $\dfrac{dy}{dx}$.

Like every other mathematical operation, the process of differentiation may be reversed; thus, if differentiating $y = x^4$ gives us $\dfrac{dy}{dx} = 4x^3$; if one begins with $\dfrac{dy}{dx} = 4x^3$ one would say that reversing the process would yield $y = x^4$. But here comes in a curious point. We should get $\dfrac{dy}{dx} = 4x^3$ if we had begun with *any* of the following: x^4, or $x^4 + a$, or $x^4 + c$, or x^4 with *any* added constant. So it is clear that in working backwards from $\dfrac{dy}{dx}$ to y, one must make provision for the possibility of there being an added constant, the value of which will be undetermined until ascertained in some other way. So, if differentiating x^n yields nx^{n-1}, going backwards from $\dfrac{dy}{dx} = nx^{n-1}$ will give us $y = x^n + C$; where C stands for the yet undetermined possible constant.

Clearly, in dealing with powers of x, the rule for working backwards will be: Increase the power by 1, then divide by that increased power, and add the undetermined constant.

So, in the case where

$$\frac{dy}{dx} = x^n,$$

working backwards, we get

$$y = \frac{1}{n+1} x^{n+1} + C.$$

If differentiating the equation $y = ax^n$ gives us

$$\frac{dy}{dx} = anx^{n-1},$$

it is a matter of common sense that beginning with

$$\frac{dy}{dx} = anx^{n-1},$$

and reversing the process, will give us

$$y = ax^n.$$

So, when we are dealing with a multiplying constant, we must simply put the constant as a multiplier of the result of the integration.

Thus, if $\dfrac{dy}{dx} = 4x^2$, the reverse process gives us $y = \frac{4}{3}x^3$.

But this is incomplete. For we must remember that if we had started with

$$y = ax^n + C,$$

where C is any constant quantity whatever, we should equally have found

$$\frac{dy}{dx} = anx^{n-1}.$$

So, therefore, when we reverse the process we must always remember to add on this undetermined constant, even if we do not yet know what its value will be.

This process, the reverse of differentiating, is called *integrating*; for it consists in finding the value of the whole quantity y when you are given only an expression for dy or for $\dfrac{dy}{dx}$. Hitherto we have as much as possible kept dy and dx together as a differential coefficient: henceforth we shall more often have to separate them.

If we begin with a simple case,

$$\frac{dy}{dx} = x^2.$$

We may write this, if we like, as

$$dy = x^2\,dx.$$

Now this is a "differential equation" which informs us that an element of y is equal to the corresponding element of x multiplied by x^2. Now, what we want is the integral; therefore, write down with the proper symbol the instructions to integrate both sides, thus:

$$\int dy = \int x^2\,dx.$$

[Note as to reading integrals: the above would be read thus:

"*Integral dee-wy* equals *integral eks-squared dee-eks.*"]

We haven't yet integrated: we have only written down instructions to integrate—if we can. Let us try. Plenty of other fools can do it—why not we also? The left-hand side is simplicity itself. The sum of all the bits of y is the same thing as y itself. So we may at once put:

$$y = \int x^2\,dx.$$

But when we come to the right-hand side of the equation we must remember that what we have got to sum up together is not all the dx's, but all such terms as $x^2\,dx$; and this will *not* be the same as $x^2 \int dx$, because x^2 is not a constant. For some of the dx's will be multiplied by big values of x^2, and some will be multiplied by small values of x^2, according to what x happens to be. So we must bethink ourselves as to what we know about this process of integration being the reverse of differentiation. Now, our rule for this reversed process—see p. 192 *ante*—when dealing with x^n is "increase the power by one, and divide by the same number as this increased power." That is to say, $x^2\,dx$ will be changed* to $\frac{1}{3}x^3$. Put this into the equation; but don't forget to add the "constant of integration" C at the end. So we get:

$$y = \tfrac{1}{3}x^3 + C.$$

You have actually performed the integration. How easy!

Let us try another simple case.

Let
$$\frac{dy}{dx} = ax^{12},$$

where a is any constant multiplier. Well, we found when differentiating (see p. 29) that any constant factor in the value of y reappeared

*You may ask, what has become of the little dx at the end? Well, remember that it was really part of the differential coefficient, and when changed over to the right-hand side, as in the $x^2\,dx$, serves as a reminder that x is the independent variable with respect to which the operation is to be effected; and, as the result of the product being totalled up, the power of x has increased by *one*. You will soon become familiar with all this.

unchanged in the value of $\frac{dy}{dx}$. In the reversed process of integrating, it will therefore also reappear in the value of y. So we may go to work as before, thus

$$\begin{aligned} dy &= ax^{12} \cdot dx, \\ \int dy &= \int ax^{12} \cdot dx, \\ \int dy &= a \int x^{12}\, dx, \\ y &= a \times \tfrac{1}{13} x^{13} + C. \end{aligned}$$

So that is done. How easy!

We begin to realize now that integrating is a process of *finding our way back*, as compared with differentiating. If ever, during differentiating, we have found any particular expression—in this example ax^{12}—we can find our way back to the y from which it was derived. The contrast between the two processes may be illustrated by the following remark due to a well-known teacher. If a stranger were set down in Trafalgar Square, and told to find his way to Euston Station, he might find the task hopeless. But if he had previously been personally conducted from Euston Station to Trafalgar Square, it would be comparatively easy to him to find his way back to Euston Station.

Integration of the Sum or Difference of two Functions.

Let $$\frac{dy}{dx} = x^2 + x^3,$$

then $$dy = x^2\, dx + x^3\, dx.$$

There is no reason why we should not integrate each term separately: for, as may be seen on p. 36, we found that when we differentiated the sum of two separate functions, the differential coefficient was simply the sum of the two separate differentiations. So, when we work backwards, integrating, the integration will be simply the sum of the two separate integrations.

Our instructions will then be:

$$\begin{aligned} \int dy &= \int (x^2 + x^3)\, dx \\ &= \int x^2\, dx + \int x^3\, dx \\ y &= \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + C. \end{aligned}$$

If either of the terms had been a negative quantity, the corresponding term in the integral would have also been negative. So that differences are as readily dealt with as sums.

How to deal with Constant Terms.

Suppose there is in the expression to be integrated a constant term—such as this:

$$\frac{dy}{dx} = x^n + b.$$

This is laughably easy. For you have only to remember that when you differentiated the expression $y = ax$, the result was $\dfrac{dy}{dx} = a$. Hence, when you work the other way and integrate, the constant reappears multiplied by x. So we get

$$dy = x^n\,dx + b \cdot dx,$$
$$\int dy = \int x^n\,dx + \int b\,dx,$$
$$y = \frac{1}{n+1}x^{n+1} + bx + C.$$

Here are a lot of examples on which to try your newly acquired powers.

Examples.

(1) Given $\dfrac{dy}{dx} = 24x^{11}$. Find y. *Ans.* $y = 2x^{12} + C$.

(2) Find $\displaystyle\int (a+b)(x+1)\,dx$. It is $(a+b)\displaystyle\int (x+1)\,dx$

or $(a+b)\left[\displaystyle\int x\,dx + \int dx\right]$ or $(a+b)\left(\dfrac{x^2}{2} + x\right) + C$.

(3) Given $\dfrac{du}{dt} = gt^{\frac{1}{2}}$. Find u. *Ans.* $u = \frac{2}{3}gt^{\frac{3}{2}} + C$.

(4) $\dfrac{dy}{dx} = x^3 - x^2 + x$. Find y.

$$dy = (x^3 - x^2 + x)\,dx \quad \text{or}$$
$$dy = x^3\,dx - x^2\,dx + x\,dx; \quad y = \int x^3\,dx - \int x^2\,dx + \int x\,dx;$$

and
$$y = \tfrac{1}{4}x^4 - \tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 + C.$$

(5) Integrate $9.75x^{2.25}\,dx$. *Ans.* $y = 3x^{3.25} + C$.

All these are easy enough. Let us try another case.

Let $$\frac{dy}{dx} = ax^{-1}.$$

Proceeding as before, we will write

$$dy = ax^{-1} \cdot dx, \qquad \int dy = a \int x^{-1}\, dx.$$

Well, but what is the integral of $x^{-1}\, dx$?

If you look back amongst the results of differentiating x^2 and x^3 and x^n, etc., you will find we never got x^{-1} from any one of them as the value of $\frac{dy}{dx}$. We got $3x^2$ from x^3; we got $2x$ from x^2; we got 1 from x^1 (that is, from x itself); but we did not get x^{-1} from x^0, for two very good reasons. *First*, x^0 is simply $= 1$, and is a constant, and could not have a differential coefficient. *Secondly*, even if it could be differentiated, its differential coefficient (got by slavishly following the usual rule) would be $0 \times x^{-1}$, and that multiplication by zero gives it zero value! Therefore when we now come to try to integrate $x^{-1}\, dx$, we see that it does not come in anywhere in the powers of x that are given by the rule:

$$\int x^n\, dx = \frac{1}{n+1} x^{n+1}.$$

It is an exceptional case.

Well; but try again. Look through all the various differentials obtained from various functions of x, and try to find amongst them x^{-1}. A sufficient search will show that we actually did get $\frac{dy}{dx} = x^{-1}$ as the result of differentiating the function $y = \log_\epsilon x$ (see p. 148).

Then, of course, since we know that differentiating $\log_\epsilon x$ gives us x^{-1}, we know that, by reversing the process, integrating $dy = x^{-1}\,dx$ will give us $y = \log_\epsilon x$. But we must not forget the constant factor a that was given, nor must we omit to add the undetermined constant of integration. This then gives us as the solution to the present problem,

$$y = a\log_\epsilon x + C.$$

N.B.—Here note this very remarkable fact, that we could not have integrated in the above case if we had not happened to know the corresponding differentiation. If no one had found out that differentiating $\log_\epsilon x$ gave x^{-1}, we should have been utterly stuck by the problem how to integrate $x^{-1}\,dx$. Indeed it should be frankly admitted that this is one of the curious features of the integral calculus:—that you can't integrate anything before the reverse process of differentiating something else has yielded that expression which you want to integrate. No one, even to-day, is able to find the general integral of the expression,

$$\frac{dy}{dx} = a^{-x^2},$$

because a^{-x^2} has never yet been found to result from differentiating anything else.

Another simple case.

Find $\int (x+1)(x+2)\,dx$.

On looking at the function to be integrated, you remark that it is the product of two different functions of x. You could, you think, integrate $(x+1)\,dx$ by itself, or $(x+2)\,dx$ by itself. Of course you could. But what to do with a product? None of the differentiations you have learned have yielded you for the differential coefficient a product like this. Failing such, the simplest thing is to multiply up the two functions, and then integrate. This gives us

$$\int (x^2 + 3x + 2)\, dx.$$

And this is the same as

$$\int x^2\, dx + \int 3x\, dx + \int 2\, dx.$$

And performing the integrations, we get

$$\tfrac{1}{3}x^3 + \tfrac{3}{2}x^2 + 2x + C.$$

Some other Integrals.

Now that we know that integration is the reverse of differentiation, we may at once look up the differential coefficients we already know, and see from what functions they were derived. This gives us the following integrals ready made:

x^{-1}	(p. 148);	$\int x^{-1}\, dx$	$= \log_\epsilon x + C.$
$\dfrac{1}{x+a}$	(p. 148);	$\int \dfrac{1}{x+a}\, dx$	$= \log_\epsilon (x+a) + C.$
ϵ^x	(p. 142);	$\int \epsilon^x\, dx$	$= \epsilon^x + C.$
ϵ^{-x}		$\int \epsilon^{-x}\, dx$	$= -\epsilon^{-x} + C$

(for if $y = -\dfrac{1}{\epsilon^x}$, $\quad \dfrac{dy}{dx} = -\dfrac{\epsilon^x \times 0 - 1 \times \epsilon^x}{\epsilon^{2x}} = \epsilon^{-x}$).

$$\sin x \quad (\text{p. } 168); \qquad \int \sin x\,dx = -\cos x + C.$$

$$\cos x \quad (\text{p. } 166); \qquad \int \cos x\,dx = \sin x + C.$$

Also we may deduce the following:

$$\log_\epsilon x; \qquad \int \log_\epsilon x\,dx = x(\log_\epsilon x - 1) + C$$

(for if $y = x\log_\epsilon x - x$, $\quad \dfrac{dy}{dx} = \dfrac{x}{x} + \log_\epsilon x - 1 = \log_\epsilon x$).

$$\log_{10} x; \qquad \int \log_{10} x\,dx = 0.4343x(\log_\epsilon x - 1) + C.$$

$$a^x \quad (\text{p. } 149); \qquad \int a^x\,dx = \frac{a^x}{\log_\epsilon a} + C.$$

$$\cos ax; \qquad \int \cos ax\,dx = \frac{1}{a}\sin ax + C$$

(for if $y = \sin ax$, $\dfrac{dy}{dx} = a\cos ax$; hence to get $\cos ax$ one must differentiate $y = \dfrac{1}{a}\sin ax$).

$$\sin ax; \qquad \int \sin ax\,dx = -\frac{1}{a}\cos ax + C.$$

Try also $\cos^2\theta$; a little dodge will simplify matters:

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1;$$

hence $$\cos^2\theta = \tfrac{1}{2}(\cos 2\theta + 1),$$

and
$$\begin{aligned}\int \cos^2\theta\,d\theta &= \tfrac{1}{2}\int(\cos 2\theta + 1)\,d\theta \\ &= \tfrac{1}{2}\int \cos 2\theta\,d\theta + \tfrac{1}{2}\int d\theta. \\ &= \frac{\sin 2\theta}{4} + \frac{\theta}{2} + C.\end{aligned}$$
(See also p. 228).

See also the Tabla de Formas Estándar on pp. 252–254. You should make such a table for yourself, putting in it only the general functions which you have successfully differentiated and integrated. See to it that it grows steadily!

On Double and Triple Integrals.

In many cases it is necessary to integrate some expression for two or more variables contained in it; and in that case the sign of integration appears more than once. Thus,

$$\iint f(x, y,)\,dx\,dy$$

means that some function of the variables x and y has to be integrated for each. It does not matter in which order they are done. Thus, take the function $x^2 + y^2$. Integrating it with respect to x gives us:

$$\int (x^2 + y^2)\,dx = \tfrac{1}{3}x^3 + xy^2.$$

Now, integrate this with respect to y:

$$\int (\tfrac{1}{3}x^3 + xy^2)\,dy = \tfrac{1}{3}x^3y + \tfrac{1}{3}xy^3,$$

to which of course a constant is to be added. If we had reversed the order of the operations, the result would have been the same.

In dealing with areas of surfaces and of solids, we have often to integrate both for length and breadth, and thus have integrals of the form

$$\iint u \cdot dx\, dy,$$

where u is some property that depends, at each point, on x and on y. This would then be called a *surface-integral*. It indicates that the value of all such elements as $u \cdot dx \cdot dy$ (that is to say, of the value of u over a little rectangle dx long and dy broad) has to be summed up over the whole length and whole breadth.

Similarly in the case of solids, where we deal with three dimensions. Consider any element of volume, the small cube whose dimensions are $dx\ dy\ dz$. If the figure of the solid be expressed by the function $f(x, y, z)$, then the whole solid will have the *volume-integral*,

$$\text{volume} = \iiint f(x, y, z) \cdot dx \cdot dy \cdot dz.$$

Naturally, such integrations have to be taken between appropriate limits* in each dimension; and the integration cannot be performed unless one knows in what way the boundaries of the surface depend on x, y, and z. If the limits for x are from x_1 to x_2, those for y from y_1 to y_2, and those for z from z_1 to z_2, then clearly we have

$$\text{volume} = \int_{z1}^{z2} \int_{y1}^{y2} \int_{x1}^{x2} f(x, y, z) \cdot dx \cdot dy \cdot dz.$$

*See p. 209 for integration between limits.

There are of course plenty of complicated and difficult cases; but, in general, it is quite easy to see the significance of the symbols where they are intended to indicate that a certain integration has to be performed over a given surface, or throughout a given solid space.

Exercises XVII. (See p. 267 for the Answers.)

(1) Find $\int y\,dx$ when $y^2 = 4ax$.

(2) Find $\int \dfrac{3}{x^4}\,dx$.

(3) Find $\int \dfrac{1}{a}x^3\,dx$.

(4) Find $\int (x^2+a)\,dx$.

(5) Integrate $5x^{-\frac{7}{2}}$.

(6) Find $\int (4x^3+3x^2+2x+1)\,dx$.

(7) If $\dfrac{dy}{dx} = \dfrac{ax}{2} + \dfrac{bx^2}{3} + \dfrac{cx^3}{4}$; find y.

(8) Find $\int \left(\dfrac{x^2+a}{x+a}\right) dx$.

(9) Find $\int (x+3)^3\,dx$.

(10) Find $\int (x+2)(x-a)\,dx$.

(11) Find $\int (\sqrt{x} + \sqrt[3]{x})3a^2\,dx$.

(12) Find $\int (\sin\theta - \tfrac{1}{2})\,\dfrac{d\theta}{3}$.

(13) Find $\int \cos^2 a\theta\,d\theta$.

(14) Find $\int \sin^2\theta\,d\theta$.

(15) Find $\int \sin^2 a\theta \, d\theta$.

(16) Find $\int \epsilon^{3x} \, dx$.

(17) Find $\int \frac{dx}{1+x}$.

(18) Find $\int \frac{dx}{1-x}$.

CAPÍTULO XIX.

SOBRE ENCONTRAR ÁREAS POR INTEGRACIÓN.

One use of the integral calculus is to enable us to ascertain the values of areas bounded by curves.

Let us try to get at the subject bit by bit.

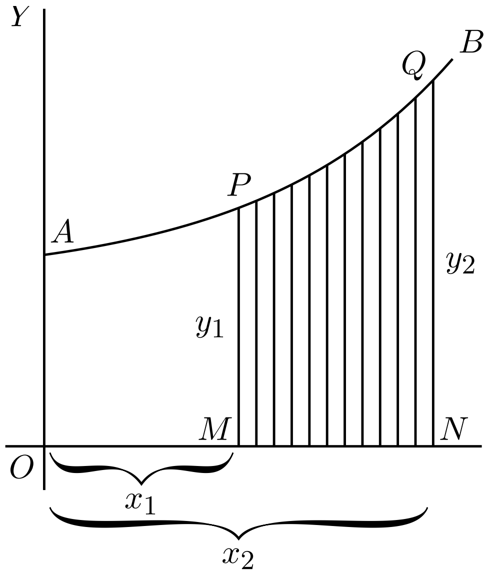
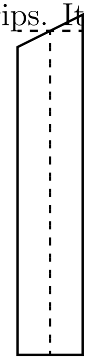


Fig. 52.

Let AB (Fig. 52) be a curve, the equation to which is known. That is, y in this curve is some known function of x. Think of a piece of the curve from the point P to the point Q.

Let a perpendicular PM be dropped from P, and another QN from the point Q. Then call $OM = x_1$ and $ON = x_2$, and the ordinates $PM = y_1$ and $QN = y_2$. We have thus marked out the area $PQNM$ that lies beneath the piece PQ. The problem is, *how can we calculate the value of this area*?

The secret of solving this problem is to conceive the area as being divided up into a lot of narrow strips, each of them being of the width dx. The smaller we take dx, the more of them there will be between x_1 and x_2. Now, the whole area is clearly equal to the sum of the areas of all such strips. Our business will then be to discover an expression for the area of any one narrow strip, and to integrate it so as to add together all the strips. Now think of any one of the strips. It will be like this: being bounded between two vertical sides, with a flat bottom dx, and with a slightly curved sloping top. Suppose we take its *average* height as being y; then, as its width is dx, its area will be $y\,dx$. And seeing that we may take the width as narrow as we please, if we only take it narrow enough its average height will be the same as the height at the middle of it. Now let us call the unknown value of the whole area S, meaning surface. The area of one strip will be simply a bit of the whole area, and may therefore be called dS. So we may write



$$\text{area of 1 strip} = dS = y \cdot dx.$$

If then we add up all the strips, we get

$$\text{total area } S = \int dS = \int y\,dx.$$

So then our finding S depends on whether we can integrate $y \cdot dx$ for the particular case, when we know what the value of y is as a function of x.

For instance, if you were told that for the particular curve in question $y = b + ax^2$, no doubt you could put that value into the expression and say: then I must find $\int (b + ax^2)\, dx$.

That is all very well; but a little thought will show you that something more must be done. Because the area we are trying to find is not the area under the whole length of the curve, but only the area limited on the left by PM, and on the right by QN, it follows that we must do something to define our area between those '*limits.*'

This introduces us to a new notion, namely that of *integrating between limits.* We suppose x to vary, and for the present purpose we do not require any value of x below x_1 (that is OM), nor any value of x above x_2 (that is ON). When an integral is to be thus defined between two limits, we call the lower of the two values *the inferior limit*, and the upper value *the superior limit.* Any integral so limited we designate as a *definite integral*, by way of distinguishing it from a *general integral* to which no limits are assigned.

In the symbols which give instructions to integrate, the limits are marked by putting them at the top and bottom respectively of the sign of integration. Thus the instruction

$$\int_{x=x_1}^{x=x_2} y \cdot dx$$

will be read: find the integral of $y \cdot dx$ between the inferior limit x_1 and the superior limit x_2.

Sometimes the thing is written more simply

$$\int_{x_1}^{x_2} y \cdot dx.$$

Well, but *how* do you find an integral between limits, when you have got these instructions?

Look again at Fig. 52 (p. 207). Suppose we could find the area under the larger piece of curve from A to Q, that is from $x = 0$ to $x = x_2$, naming the area $AQNO$. Then, suppose we could find the area under the smaller piece from A to P, that is from $x = 0$ to $x = x_1$, namely the area $APMO$. If then we were to subtract the smaller area from the larger, we should have left as a remainder the area $PQNM$, which is what we want. Here we have the clue as to what to do; the definite integral between the two limits is *the difference* between the integral worked out for the superior limit and the integral worked out for the lower limit.

Let us then go ahead. First, find the general integral thus:

$$\int y\, dx,$$

and, as $y = b + ax^2$ is the equation to the curve (Fig. 52),

$$\int (b + ax^2)\, dx$$

is the general integral which we must find.

Doing the integration in question by the rule (p. 196), we get

$$bx + \frac{a}{3}x^3 + C;$$

and this will be the whole area from 0 up to any value of x that we may assign.

Therefore, the larger area up to the superior limit x_2 will be

$$bx_2 + \frac{a}{3}x_2^3 + C;$$

and the smaller area up to the inferior limit x_1 will be

$$bx_1 + \frac{a}{3}x_1^3 + C.$$

Now, subtract the smaller from the larger, and we get for the area S the value,

$$\text{area } S = b(x_2 - x_1) + \frac{a}{3}(x_2^3 - x_1^3).$$

This is the answer we wanted. Let us give some numerical values. Suppose $b = 10$, $a = 0.06$, and $x_2 = 8$ and $x_1 = 6$. Then the area S is equal to

$$\begin{aligned}
&10(8-6) + \frac{0.06}{3}(8^3 - 6^3) \\
&= 20 + 0.02(512 - 216) \\
&= 20 + 0.02 \times 296 \\
&= 20 + 5.92 \\
&= 25.92.
\end{aligned}$$

Let us here put down a symbolic way of stating what we have ascertained about limits:

$$\int_{x=x_1}^{x=x_2} y\,dx = y_2 - y_1,$$

where y_2 is the integrated value of $y\,dx$ corresponding to x_2, and y_1 that corresponding to x_1.

All integration between limits requires the difference between two values to be thus found. Also note that, in making the subtraction the added constant C has disappeared.

Examples.

(1) To familiarize ourselves with the process, let us take a case of which we know the answer beforehand. Let us find the area of the triangle (Fig. 53), which has base $x = 12$ and height $y = 4$. We know beforehand, from obvious mensuration, that the answer will come 24.

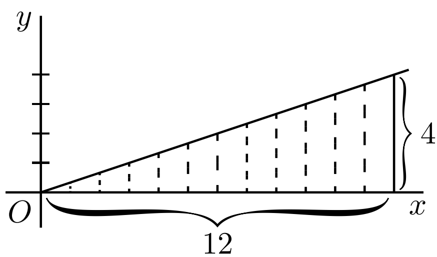


FIG. 53.

Now, here we have as the "curve" a sloping line for which the equation is

$$y = \frac{x}{3}.$$

The area in question will be

$$\int_{x=0}^{x=12} y \cdot dx = \int_{x=0}^{x=12} \frac{x}{3} \cdot dx.$$

Integrating $\frac{x}{3}\,dx$ (p. 195), and putting down the value of the general integral in square brackets with the limits marked above and below, we get

$$
\begin{aligned}
\text{area} &= \left[\frac{1}{3}\cdot\frac{1}{2}x^2\right]_{x=0}^{x=12} + C \\
&= \left[\frac{x^2}{6}\right]_{x=0}^{x=12} + C \\
&= \left[\frac{12^2}{6}\right] - \left[\frac{0^2}{6}\right] \\
&= \frac{144}{6} = 24. \quad \textit{Ans.}
\end{aligned}
$$

Let us satisfy ourselves about this rather surprising dodge of calculation, by testing it on a simple example. Get some squared paper, preferably some that is ruled in little squares of one-eighth inch or one-

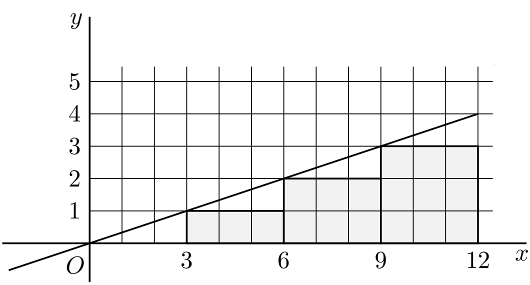


FIG. 54.

tenth inch each way. On this squared paper plot out the graph of the equation,

$$y = \frac{x}{3}.$$

The values to be plotted will be:

x	0	3	6	9	12
y	0	1	2	3	4

The plot is given in Fig. 54.

Now reckon out the area beneath the curve *by counting the little squares* below the line, from $x = 0$ as far as $x = 12$ on the right. There are 18 whole squares and four triangles, each of which has an area equal to $1\frac{1}{2}$ squares; or, in total, 24 squares. Hence 24 is the numerical value of the integral of $\frac{x}{3}\,dx$ between the lower limit of $x = 0$ and the higher limit of $x = 12$.

As a further exercise, show that the value of the same integral between the limits of $x = 3$ and $x = 15$ is 36.

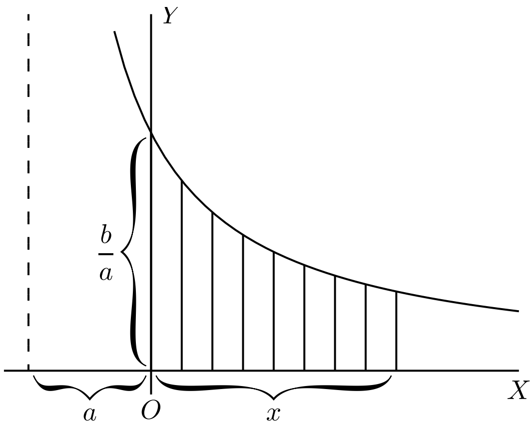


FIG. 55.

(2) Find the area, between limits $x = x_1$ and $x = 0$, of the curve $y = \dfrac{b}{x+a}$.

$$\begin{aligned}\text{Area} = \int_{x=0}^{x=x_1} y \cdot dx &= \int_{x=0}^{x=x_1} \frac{b}{x+a}\,dx \\ &= b\Big[\log_\epsilon(x+a)\Big]_0^{x_1} + C \\ &= b\Big[\log_\epsilon(x_1+a) - \log_\epsilon(0+a)\Big]\end{aligned}$$

$$= b \log_\epsilon \frac{x_1 + a}{a}. \quad \textit{Ans.}$$

N.B.—Notice that in dealing with definite integrals the constant C always disappears by subtraction.

Let it be noted that this process of subtracting one part from a larger to find the difference is really a common practice. How do you find the area of a plane ring (Fig. 56), the outer radius of which is r_2

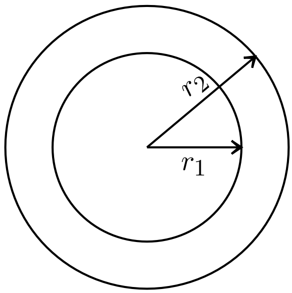


FIG. 56.

and the inner radius is r_1? You know from mensuration that the area of the outer circle is πr_2^2; then you find the area of the inner circle, πr_1^2; then you subtract the latter from the former, and find area of ring $= \pi(r_2^2 - r_1^2)$; which may be written

$$\pi(r_2 + r_1)(r_2 - r_1)$$

= mean circumference of ring × width of ring.

(3) Here's another case—that of the *die-away curve* (p. 156). Find the area between $x = 0$ and $x = a$, of the curve (Fig. 57) whose equation is

$$y = b\epsilon^{-x}.$$

$$\text{Area} = b \int_{x=0}^{x=a} \epsilon^{-x} \cdot dx.$$

The integration (p. 201) gives

$$\begin{aligned} &= b\left[-\epsilon^{-x}\right]_0^a \\ &= b\big[-\epsilon^{-a} - (-\epsilon^{-0})\big] \\ &= b(1 - \epsilon^{-a}). \end{aligned}$$

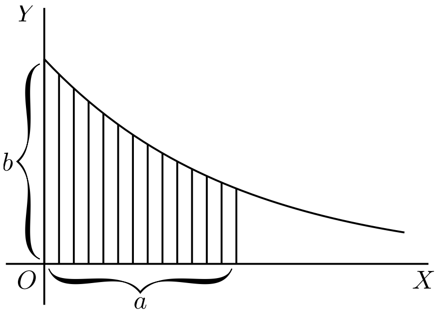


FIG. 57.

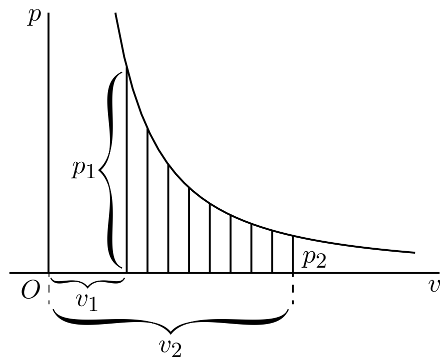


FIG. 58.

(4) Another example is afforded by the adiabatic curve of a perfect gas, the equation to which is $pv^n = c$, where p stands for pressure, v for volume, and n is of the value 1.42 (Fig. 58).

Find the area under the curve (which is proportional to the work done in suddenly compressing the gas) from volume v_2 to volume v_1.

Here we have

$$\begin{aligned} \text{area} &= \int_{v=v_1}^{v=v_2} cv^{-n} \cdot dv \\ &= c\left[\frac{1}{1-n} v^{1-n}\right]_{v_1}^{v_2} \\ &= c\frac{1}{1-n}(v_2^{1-n} - v_1^{1-n}) \\ &= \frac{-c}{0.42}\left(\frac{1}{v_2^{0.42}} - \frac{1}{v_1^{0.42}}\right). \end{aligned}$$

An Exercise.

Prove the ordinary mensuration formula, that the area A of a circle whose radius is R, is equal to πR^2.

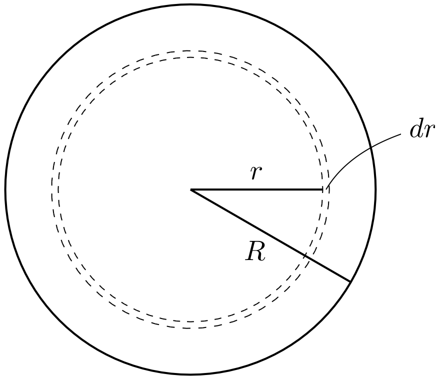


Fig. 59.

Consider an elementary zone or annulus of the surface (Fig. 59), of breadth dr, situated at a distance r from the centre. We may consider the entire surface as consisting of such narrow zones, and the whole area A will simply be the integral of all such elementary zones from centre to margin, that is, integrated from $r = 0$ to $r = R$.

We have therefore to find an expression for the elementary area dA of the narrow zone. Think of it as a strip of breadth dr, and of a length that is the periphery of the circle of radius r, that is, a length of $2\pi r$. Then we have, as the area of the narrow zone,

$$dA = 2\pi r\, dr.$$

Hence the area of the whole circle will be:

$$A = \int dA = \int_{r=0}^{r=R} 2\pi r \cdot dr = 2\pi \int_{r=0}^{r=R} r \cdot dr.$$

Now, the general integral of $r \cdot dr$ is $\frac{1}{2}r^2$. Therefore,

$$A = 2\pi \left[\tfrac{1}{2}r^2\right]_{r=0}^{r=R};$$

or
$$A = 2\pi \left[\tfrac{1}{2}R^2 - \tfrac{1}{2}(0)^2\right];$$

whence
$$A = \pi R^2.$$

Another Exercise.

Let us find the mean ordinate of the positive part of the curve $y = x - x^2$, which is shown in Fig. 60. To find the mean ordinate, we

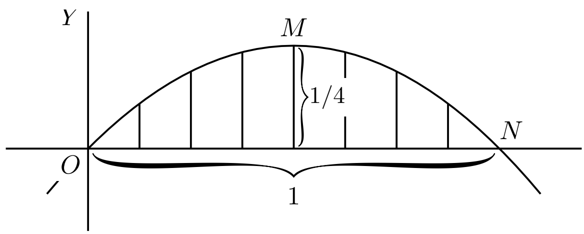


FIG. 60.

shall have to find the area of the piece OMN, and then divide it by the length of the base ON. But before we can find the area we must ascertain the length of the base, so as to know up to what limit we are to integrate. At N the ordinate y has zero value; therefore, we must look at the equation and see what value of x will make $y = 0$. Now, clearly, if x is 0, y will also be 0, the curve passing through the origin O; but also, if $x = 1$, $y = 0$; so that $x = 1$ gives us the position of the point N.

Then the area wanted is

$$\begin{aligned}
&= \int_{x=0}^{x=1} (x - x^2)\, dx \\
&= \left[\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right]_0^1 \\
&= \left[\tfrac{1}{2} - \tfrac{1}{3}\right] - [0 - 0] \\
&= \tfrac{1}{6}.
\end{aligned}$$

But the base length is 1.

Therefore, the average ordinate of the curve $= \frac{1}{6}$.

[*N.B.*—It will be a pretty and simple exercise in maxima and minima to find by differentiation what is the height of the maximum ordinate. It *must* be greater than the average.]

The mean ordinate of any curve, over a range from $x = 0$ to $x = x_1$, is given by the expression,

$$\text{mean } y = \frac{1}{x_1} \int_{x=0}^{x=x_1} y \cdot dx.$$

One can also find in the same way the surface area of a solid of revolution.

Example.

The curve $y = x^2 - 5$ is revolving about the axis of x. Find the area of the surface generated by the curve between $x = 0$ and $x = 6$.

A point on the curve, the ordinate of which is y, describes a circumference of length $2\pi y$, and a narrow belt of the surface, of width dx, corresponding to this point, has for area $2\pi y\, dx$. The total area is

$$2\pi \int_{x=0}^{x=6} y\, dx = 2\pi \int_{x=0}^{x=6} (x^2 - 5)\, dx = 2\pi \left[\frac{x^3}{3} - 5x\right]_0^6$$

$$= 6.28 \times 42 = 263.76.$$

Areas in Polar Coordinates.

When the equation of the boundary of an area is given as a function of the distance r of a point of it from a fixed point O (see Fig. 61) called the *pole*, and of the angle which r makes with the positive horizontal

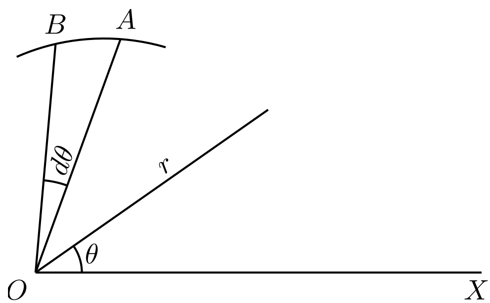


Fig. 61.

direction OX, the process just explained can be applied just as easily, with a small modification. Instead of a strip of area, we consider a small triangle OAB, the angle at O being $d\theta$, and we find the sum of all the little triangles making up the required area.

The area of such a small triangle is approximately $\dfrac{AB}{2} \times r$ or $\dfrac{r\,d\theta}{2} \times r$; hence the portion of the area included between the curve and two positions of r corresponding to the angles θ_1 and θ_2 is given by

$$\tfrac{1}{2} \int_{\theta=\theta_1}^{\theta=\theta_2} r^2\, d\theta.$$

Examples.

(1) Find the area of the sector of 1 radian in a circumference of radius a inches.

The polar equation of the circumference is evidently $r = a$. The area is

$$\tfrac{1}{2}\int_{\theta=\theta_1}^{\theta=\theta_2} a^2\,d\theta = \frac{a^2}{2}\int_{\theta=0}^{\theta=1} d\theta = \frac{a^2}{2}.$$

(2) Find the area of the first quadrant of the curve (known as "Pascal's Snail"), the polar equation of which is $r = a(1+\cos\theta)$.

$$\begin{aligned}
\text{Area} &= \tfrac{1}{2}\int_{\theta=0}^{\theta=\frac{\pi}{2}} a^2(1+\cos\theta)^2\,d\theta\\
&= \frac{a^2}{2}\int_{\theta=0}^{\theta=\frac{\pi}{2}} (1+2\cos\theta+\cos^2\theta)\,d\theta\\
&= \frac{a^2}{2}\left[\theta+2\sin\theta+\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right]_0^{\frac{\pi}{2}}\\
&= \frac{a^2(3\pi+8)}{8}.
\end{aligned}$$

Volumes by Integration.

What we have done with the area of a little strip of a surface, we can, of course, just as easily do with the volume of a little strip of a solid. We can add up all the little strips that make up the total solid, and find its volume, just as we have added up all the small little bits that made up an area to find the final area of the figure operated upon.

Examples.

(1) Find the volume of a sphere of radius r.

A thin spherical shell has for volume $4\pi x^2\,dx$ (see Fig. 59, p. 217); summing up all the concentric shells which make up the sphere, we have

$$\text{volume sphere} = \int_{x=0}^{x=r} 4\pi x^2\,dx = 4\pi\left[\frac{x^3}{3}\right]_0^r = \tfrac{4}{3}\pi r^3.$$

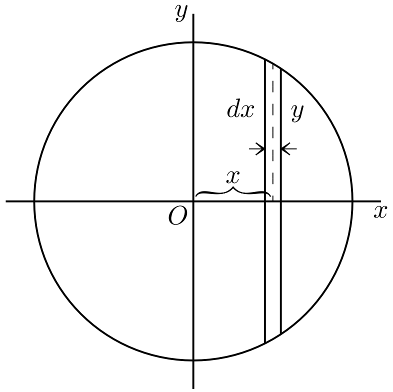


FIG. 62.

We can also proceed as follows: a slice of the sphere, of thickness dx, has for volume $\pi y^2\,dx$ (see Fig. 62). Also x and y are related by the expression

$$y^2 = r^2 - x^2.$$

$$\begin{aligned}
\text{Hence}\quad \text{volume sphere} &= 2\int_{x=0}^{x=r} \pi(r^2 - x^2)\,dx \\
&= 2\pi\left[\int_{x=0}^{x=r} r^2\,dx - \int_{x=0}^{x=r} x^2\,dx\right] \\
&= 2\pi\left[r^2 x - \frac{x^3}{3}\right]_0^r = \frac{4\pi}{3}r^3.
\end{aligned}$$

(2) Find the volume of the solid generated by the revolution of the curve $y^2 = 6x$ about the axis of x, between $x = 0$ and $x = 4$.

The volume of a strip of the solid is $\pi y^2\, dx$.

Hence $$\text{volume} = \int_{x=0}^{x=4} \pi y^2\, dx = 6\pi \int_{x=0}^{x=4} x\, dx$$
$$= 6\pi \left[\frac{x^2}{2}\right]_0^4 = 48\pi = 150.8.$$

On Quadratic Means.

In certain branches of physics, particularly in the study of alternating electric currents, it is necessary to be able to calculate the *quadratic mean* of a variable quantity. By "quadratic mean" is denoted the square root of the mean of the squares of all the values between the limits considered. Other names for the quadratic mean of any quantity are its "virtual" value, or its "R.M.S." (meaning root-mean-square) value. The French term is *valeur efficace.* If y is the function under consideration, and the quadratic mean is to be taken between the limits of $x = 0$ and $x = l$; then the quadratic mean is expressed as

$$\sqrt[2]{\frac{1}{l}\int_0^l y^2\, dx}.$$

Examples.

(1) To find the quadratic mean of the function $y = ax$ (Fig. 63).

Here the integral is $\int_0^l a^2x^2\, dx$, which is $\frac{1}{3}a^2l^3$.

Dividing by l and taking the square root, we have

$$\text{quadratic mean} = \frac{1}{\sqrt{3}}\, al.$$

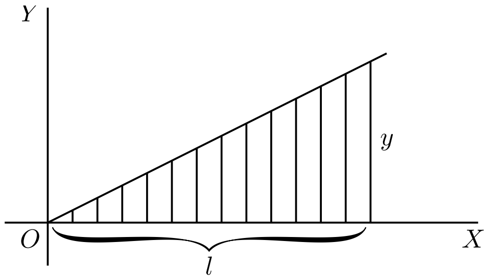


Fig. 63.

Here the arithmetical mean is $\frac{1}{2}al$; and the ratio of quadratic to arithmetical mean (this ratio is called the *form-factor*) is $\dfrac{2}{\sqrt{3}} = 1.155$.

(2) To find the quadratic mean of the function $y = x^a$.

The integral is $\displaystyle\int_{x=0}^{x=l} x^{2a}\, dx$, that is $\dfrac{l^{2a+1}}{2a+1}$.

Hence
$$\text{quadratic mean} = \sqrt[2]{\frac{l^{2a}}{2a+1}}.$$

(3) To find the quadratic mean of the function $y = a^{\frac{x}{2}}$.

The integral is $\displaystyle\int_{x=0}^{x=l} (a^{\frac{x}{2}})^2\, dx$, that is $\displaystyle\int_{x=0}^{x=l} a^x\, dx$,

or
$$\left[\frac{a^x}{\log_\epsilon a}\right]_{x=0}^{x=l},$$

which is $\dfrac{a^l - 1}{\log_\epsilon a}$.

Hence the quadratic mean is $\sqrt[2]{\dfrac{a^l - 1}{l \log_\epsilon a}}$.

Exercises XVIII. (See p. 268 for Answers.)

(1) Find the area of the curve $y = x^2 + x - 5$ between $x = 0$ and $x = 6$, and the mean ordinates between these limits.

(2) Find the area of the parabola $y = 2a\sqrt{x}$ between $x = 0$ and $x = a$. Show that it is two-thirds of the rectangle of the limiting ordinate and of its abscissa.

(3) Find the area of the positive portion of a sine curve and the mean ordinate.

(4) Find the area of the positive portion of the curve $y = \sin^2 x$, and find the mean ordinate.

(5) Find the area included between the two branches of the curve $y = x^2 \pm x^{\frac{5}{2}}$ from $x = 0$ to $x = 1$, also the area of the positive portion of the lower branch of the curve (see Fig. 30, p. 109).

(6) Find the volume of a cone of radius of base r, and of height h.

(7) Find the area of the curve $y = x^3 - \log_\epsilon x$ between $x = 0$ and $x = 1$.

(8) Find the volume generated by the curve $y = \sqrt{1 + x^2}$, as it revolves about the axis of x, between $x = 0$ and $x = 4$.

(9) Find the volume generated by a sine curve revolving about the axis of x. Find also the area of its surface.

(10) Find the area of the portion of the curve $xy = a$ included between $x = 1$ and $x = a$. Find the mean ordinate between these limits.

(11) Show that the quadratic mean of the function $y = \sin x$, between the limits of 0 and π radians, is $\dfrac{\sqrt{2}}{2}$. Find also the arithmetical mean of the same function between the same limits; and show that the form-factor is $= 1.11$.

(12) Find the arithmetical and quadratic means of the function $x^2 + 3x + 2$, from $x = 0$ to $x = 3$.

(13) Find the quadratic mean and the arithmetical mean of the function $y = A_1 \sin x + A_1 \sin 3x$.

(14) A certain curve has the equation $y = 3.42\epsilon^{0.21x}$. Find the area included between the curve and the axis of x, from the ordinate at $x = 2$ to the ordinate at $x = 8$. Find also the height of the mean ordinate of the curve between these points.

(15) Show that the radius of a circle, the area of which is twice the area of a polar diagram, is equal to the quadratic mean of all the values of r for that polar diagram.

(16) Find the volume generated by the curve $y = \pm\dfrac{x}{6}\sqrt{x(10 - x)}$ rotating about the axis of x.

CAPÍTULO XX.

TRUCOS, TRAMPAS Y TRIUNFOS.

Dodges. A great part of the labour of integrating things consists in licking them into some shape that can be integrated. The books—and by this is meant the serious books—on the Integral Calculus are full of plans and methods and dodges and artifices for this kind of work. The following are a few of them.

Integration by Parts. This name is given to a dodge, the formula for which is

$$\int u\,dx = ux - \int x\,du + C.$$

It is useful in some cases that you can't tackle directly, for it shows that if in any case $\int x\,du$ can be found, then $\int u\,dx$ can also be found. The formula can be deduced as follows. From p. 39, we have,

$$d(ux) = u\,dx + x\,du,$$

which may be written

$$u(dx) = d(ux) - x\,du,$$

which by direct integration gives the above expression.

Examples.

(1) Find $\int w \cdot \sin w \, dw$.

Write $u = w$, and for $\sin w \cdot dw$ write dx. We shall then have $du = dw$, while $\int \sin w \cdot dw = -\cos w = x$.

Putting these into the formula, we get

$$\begin{aligned}\int w \cdot \sin w \, dw &= w(-\cos w) - \int -\cos w \, dw \\ &= -w \cos w + \sin w + C.\end{aligned}$$

(2) Find $\int x\epsilon^x \, dx$.

Write $\qquad u = x, \qquad \epsilon^x \, dx = dv;$

then $\qquad du = dx, \qquad v = \epsilon^x,$

and
$$\begin{aligned}\int x\epsilon^x \, dx &= x\epsilon^x - \int \epsilon^x \, dx \quad \text{(by the formula)} \\ &= x\epsilon^x - \epsilon^x = \epsilon^x(x-1) + C.\end{aligned}$$

(3) Try $\int \cos^2 \theta \, d\theta$.

$$u = \cos\theta, \qquad \cos\theta \, d\theta = dv.$$

Hence $\qquad du = -\sin\theta \, d\theta, \qquad v = \sin\theta,$

$$\begin{aligned}\int \cos^2 \theta \, d\theta &= \cos\theta \sin\theta + \int \sin^2 \theta \, d\theta \\ &= \frac{2\cos\theta\sin\theta}{2} + \int (1 - \cos^2\theta) \, d\theta \\ &= \frac{\sin 2\theta}{2} + \int d\theta - \int \cos^2\theta \, d\theta.\end{aligned}$$

Hence $$2\int \cos^2\theta\,d\theta = \frac{\sin 2\theta}{2} + \theta$$

and $$\int \cos^2\theta\,d\theta = \frac{\sin 2\theta}{4} + \frac{\theta}{2} + C.$$

(4) Find $\int x^2 \sin x\,dx$.

Write $x^2 = u, \qquad \sin x\,dx = dv;$

then $du = 2x\,dx, \qquad v = -\cos x,$

$$\int x^2 \sin x\,dx = -x^2\cos x + 2\int x\cos x\,dx.$$

Now find $\int x\cos x\,dx$, integrating by parts (as in Example 1 above):

$$\int x\cos x\,dx = x\sin x + \cos x + C.$$

Hence

$$\begin{aligned}\int x^2\sin x\,dx &= -x^2\cos x + 2x\sin x + 2\cos x + C' \\ &= 2\left[x\sin x + \cos x\left(1 - \frac{x^2}{2}\right)\right] + C'.\end{aligned}$$

(5) Find $\int \sqrt{1-x^2}\,dx$.

Write $u = \sqrt{1-x^2}, \quad dx = dv;$

then $du = -\dfrac{x\,dx}{\sqrt{1-x^2}}$ (see Chap. IX., p. 69)

and $x = v$; so that

$$\int \sqrt{1-x^2}\, dx = x\sqrt{1-x^2} + \int \frac{x^2\, dx}{\sqrt{1-x^2}}.$$

Here we may use a little dodge, for we can write

$$\int \sqrt{1-x^2}\, dx = \int \frac{(1-x^2)\, dx}{\sqrt{1-x^2}} = \int \frac{dx}{\sqrt{1-x^2}} - \int \frac{x^2\, dx}{\sqrt{1-x^2}}.$$

Adding these two last equations, we get rid of $\int \frac{x^2\, dx}{\sqrt{1-x^2}}$, and we have

$$2\int \sqrt{1-x^2}\, dx = x\sqrt{1-x^2} + \int \frac{dx}{\sqrt{1-x^2}}.$$

Do you remember meeting $\frac{dx}{\sqrt{1-x^2}}$? it is got by differentiating $y = \arcsin x$ (see p. 171); hence its integral is $\arcsin x$, and so

$$\int \sqrt{1-x^2}\, dx = \frac{x\sqrt{1-x^2}}{2} + \tfrac{1}{2}\arcsin x + C.$$

You can try now some exercises by yourself; you will find some at the end of this chapter.

Substitution. This is the same dodge as explained in Chap. IX., p. 69. Let us illustrate its application to integration by a few examples.

(1) $\int \sqrt{3+x}\, dx.$

Let $$3 + x = u, \quad dx = du;$$

replace $$\int u^{\frac{1}{2}}\, du = \tfrac{2}{3}u^{\frac{3}{2}} = \tfrac{2}{3}(3+x)^{\frac{3}{2}}.$$

(2) $\displaystyle\int \frac{dx}{\epsilon^x + \epsilon^{-x}}$.

Let $$\epsilon^x = u, \quad \frac{du}{dx} = \epsilon^x, \quad \text{and} \quad dx = \frac{du}{\epsilon^x};$$

so that $$\int \frac{dx}{\epsilon^x + \epsilon^{-x}} = \int \frac{du}{\epsilon^x(\epsilon^x + \epsilon^{-x})} = \int \frac{du}{u\left(u + \dfrac{1}{u}\right)} = \int \frac{du}{u^2 + 1}.$$

$\dfrac{du}{1+u^2}$ is the result of differentiating $\arctan x$.

Hence the integral is $\arctan \epsilon^x$.

(3) $$\int \frac{dx}{x^2 + 2x + 3} = \int \frac{dx}{x^2 + 2x + 1 + 2} = \int \frac{dx}{(x+1)^2 + (\sqrt{2})^2}.$$

Let $$x + 1 = u, \quad dx = du;$$

then the integral becomes $\displaystyle\int \frac{du}{u^2 + (\sqrt{2})^2}$; but $\dfrac{du}{u^2 + a^2}$ is the result of differentiating $u = \dfrac{1}{a} \arctan \dfrac{u}{a}$.

Hence one has finally $\dfrac{1}{\sqrt{2}} \arctan \dfrac{x+1}{\sqrt{2}}$ for the value of the given integral.

Formulæ of Reduction are special forms applicable chiefly to binomial and trigonometrical expressions that have to be integrated, and have to be reduced into some form of which the integral is known.

Rationalization, and *Factorization of Denominator* are dodges applicable in special cases, but they do not admit of any short or general explanation. Much practice is needed to become familiar with these preparatory processes.

The following example shows how the process of splitting into partial fractions, which we learned in Chap. XIII., p. 121, can be made use of in integration.

Take again $\displaystyle\int \frac{dx}{x^2+2x+3}$; if we split $\dfrac{1}{x^2+2x+3}$ into partial fractions, this becomes (see p. 233):

$$\frac{1}{2\sqrt{-2}}\left[\int \frac{dx}{x+1-\sqrt{-2}} - \int \frac{dx}{x+1+\sqrt{-2}}\right]$$
$$= \frac{1}{2\sqrt{-2}} \log_\epsilon \frac{x+1-\sqrt{-2}}{x+1+\sqrt{-2}}.$$

Notice that the same integral can be expressed sometimes in more than one way (which are equivalent to one another).

Pitfalls. A beginner is liable to overlook certain points that a practised hand would avoid; such as the use of factors that are equivalent to either zero or infinity, and the occurrence of indeterminate quantities such as $\frac{0}{0}$. There is no golden rule that will meet every possible case. Nothing but practice and intelligent care will avail. An example of a pitfall which had to be circumvented arose in Chap. XVIII., p. 192, when we came to the problem of integrating $x^{-1}\,dx$.

Triumphs. By triumphs must be understood the successes with which the calculus has been applied to the solution of problems otherwise intractable. Often in the consideration of physical relations one is able to build up an expression for the law governing the interaction of the parts or of the forces that govern them, such expression being naturally in the form of a *differential equation*, that is an equation containing differential coefficients with or without other algebraic quantities. And when such a differential equation has been found, one can get no further until it has been integrated. Generally it is much easier to state the appropriate differential equation than to solve it:—the real trouble begins then only when one wants to integrate, unless indeed the equation is seen to possess some standard form of which the integral is known, and then the triumph is easy. The equation which results from integrating a differential equation is called* its "solution"; and it is quite astonishing how in many cases the solution looks as if it had no relation to the differential equation of which it is the integrated form. The solution often seems as different from the original expression as a butterfly does from the caterpillar that it was. Who would have supposed that such an innocent thing as

$$\frac{dy}{dx} = \frac{1}{a^2 - x^2}$$

could blossom out into

$$y = \frac{1}{2a} \log_\epsilon \frac{a+x}{a-x} + C?$$

yet the latter is the *solution* of the former.

As a last example, let us work out the above together.

By partial fractions,

$$\begin{aligned} \frac{1}{a^2 - x^2} &= \frac{1}{2a(a+x)} + \frac{1}{2a(a-x)}, \\ dy &= \frac{dx}{2a(a+x)} + \frac{dx}{2a(a-x)}, \\ y &= \frac{1}{2a}\left(\int \frac{dx}{a+x} + \int \frac{dx}{a-x}\right) \end{aligned}$$

*This means that the actual result of solving it is called its "solution." But many mathematicians would say, with Professor Forsyth, "every differential equation *is considered as solved* when the value of the dependent variable is expressed as a function of the independent variable by means either of known functions, or of integrals, whether the integrations in the latter can or cannot be expressed in terms of functions already known."

$$= \frac{1}{2a}\left(\log_\epsilon(a+x) - \log_\epsilon(a-x)\right)$$
$$= \frac{1}{2a}\log_\epsilon \frac{a+x}{a-x} + C.$$

Not a very difficult metamorphosis!

There are whole treatises, such as Boole's *Differential Equations*, devoted to the subject of thus finding the "solutions" for different original forms.

Exercises XIX. (See p. 269 for Answers.)

(1) Find $\int \sqrt{a^2 - x^2}\, dx$.

(2) Find $\int x \log_\epsilon x\, dx$.

(3) Find $\int x^a \log_\epsilon x\, dx$.

(4) Find $\int \epsilon^x \cos \epsilon^x\, dx$.

(5) Find $\int \frac{1}{x}\cos(\log_\epsilon x)\, dx$.

(6) Find $\int x^2 \epsilon^x\, dx$.

(7) Find $\int \frac{(\log_\epsilon x)^a}{x}\, dx$.

(8) Find $\int \frac{dx}{x \log_\epsilon x}$.

(9) Find $\int \frac{5x+1}{x^2+x-2}\, dx$.

(10) Find $\int \frac{(x^2-3)\, dx}{x^3 - 7x + 6}$.

(11) Find $\int \frac{b\, dx}{x^2 - a^2}$.

(12) Find $\int \frac{4x\, dx}{x^4 - 1}$.

(13) Find $\int \frac{dx}{1 - x^4}$.

(14) Find $\int \frac{dx}{x\sqrt{a - bx^2}}$.

CAPÍTULO XXI.

ENCONTRANDO ALGUNAS SOLUCIONES.

In this chapter we go to work finding solutions to some important differential equations, using for this purpose the processes shown in the preceding chapters.

The beginner, who now knows how easy most of those processes are in themselves, will here begin to realize that integration is *an art*. As in all arts, so in this, facility can be acquired only by diligent and regular practice. He who would attain that facility must work out examples, and more examples, and yet more examples, such as are found abundantly in all the regular treatises on the Calculus. Our purpose here must be to afford the briefest introduction to serious work.

Example 1. Find the solution of the differential equation

$$ay + b\frac{dy}{dx} = 0.$$

Transposing we have

$$b\frac{dy}{dx} = -ay.$$

Now the mere inspection of this relation tells us that we have got to do with a case in which $\dfrac{dy}{dx}$ is proportional to y. If we think of the curve which will represent y as a function of x, it will be such that its slope at any point will be proportional to the ordinate at that point, and will be a negative slope if y is positive. So obviously the curve will be a die-away curve (p. 156), and the solution will contain ϵ^{-x} as a factor. But, without presuming on this bit of sagacity, let us go to work.

As both y and dy occur in the equation and on opposite sides, we can do nothing until we get both y and dy to one side, and dx to the other. To do this, we must split our usually inseparable companions dy and dx from one another.

$$\frac{dy}{y} = -\frac{a}{b}\,dx.$$

Having done the deed, we now can see that both sides have got into a shape that is integrable, because we recognize $\dfrac{dy}{y}$, or $\dfrac{1}{y}\,dy$, as a differential that we have met with (p. 146) when differentiating logarithms. So we may at once write down the instructions to integrate,

$$\int \frac{dy}{y} = \int -\frac{a}{b}\,dx;$$

and doing the two integrations, we have:

$$\log_\epsilon y = -\frac{a}{b}x + \log_\epsilon C,$$

where $\log_\epsilon C$ is the yet undetermined constant* of integration. Then,

*We may write down any form of constant as the "constant of integration," and the form $\log_\epsilon C$ is adopted here by preference, because the other terms in this line of equation are, or are treated as logarithms; and it saves complications afterward if the added constant be *of the same kind*.

delogarizing, we get:

$$y = C\epsilon^{-\frac{a}{b}x},$$

which is *the solution* required. Now, this solution looks quite unlike the original differential equation from which it was constructed: yet to an expert mathematician they both convey the same information as to the way in which y depends on x.

Now, as to the C, its meaning depends on the initial value of y. For if we put $x = 0$ in order to see what value y then has, we find that this makes $y = C\epsilon^{-0}$; and as $\epsilon^{-0} = 1$ we see that C is nothing else than the particular value* of y at starting. This we may call y_0, and so write the solution as

$$y = y_0\epsilon^{-\frac{a}{b}x}.$$

Example 2.

Let us take as an example to solve

$$ay + b\frac{dy}{dx} = g,$$

where g is a constant. Again, inspecting the equation will suggest, (1) that somehow or other ϵ^x will come into the solution, and (2) that if at any part of the curve y becomes either a maximum or a minimum, so that $\dfrac{dy}{dx} = 0$, then y will have the value $= \dfrac{g}{a}$. But let us go to work as before, separating the differentials and trying to transform the thing

*Compare what was said about the "constant of integration," with reference to Fig. 48 on p. 187, and Fig. 51 on p. 190.

into some integrable shape.

$$\begin{aligned} b\frac{dy}{dx} &= g - ay; \\ \frac{dy}{dx} &= \frac{a}{b}\left(\frac{g}{a} - y\right); \\ \frac{dy}{y - \dfrac{g}{a}} &= -\frac{a}{b}\,dx. \end{aligned}$$

Now we have done our best to get nothing but y and dy on one side, and nothing but dx on the other. But is the result on the left side integrable?

It is of the same form as the result on p. 148; so, writing the instructions to integrate, we have:

$$\int \frac{dy}{y - \dfrac{g}{a}} = -\int \frac{a}{b}\,dx;$$

and, doing the integration, and adding the appropriate constant,

$$\log_\epsilon\left(y - \frac{g}{a}\right) = -\frac{a}{b}x + \log_\epsilon C;$$

whence $$y - \frac{g}{a} = C\epsilon^{-\frac{a}{b}x};$$

and finally, $$y = \frac{g}{a} + C\epsilon^{-\frac{a}{b}x},$$

which is *the solution*.

If the condition is laid down that $y = 0$ when $x = 0$ we can find C; for then the exponential becomes $= 1$; and we have

$$0 = \frac{g}{a} + C,$$

or $$C = -\frac{g}{a}.$$

Putting in this value, the solution becomes

$$y = \frac{g}{a}(1 - \epsilon^{-\frac{a}{b}x}).$$

But further, if x grows indefinitely, y will grow to a maximum; for when $x = \infty$, the exponential $= 0$, giving $y_{\text{max.}} = \dfrac{g}{a}$. Substituting this, we get finally

$$y = y_{\text{max.}}(1 - \epsilon^{-\frac{a}{b}x}).$$

This result is also of importance in physical science.

Example 3.

Let $$ay + b\frac{dy}{dt} = g \cdot \sin 2\pi nt.$$

We shall find this much less tractable than the preceding. First divide through by b.

$$\frac{dy}{dt} + \frac{a}{b}y = \frac{g}{b}\sin 2\pi nt.$$

Now, as it stands, the left side is not integrable. But it can be made so by the artifice—and this is where skill and practice suggest a plan—of multiplying all the terms by $\epsilon^{\frac{a}{b}t}$, giving us:

$$\frac{dy}{dt}\epsilon^{\frac{a}{b}t} + \frac{a}{b}y\epsilon^{\frac{a}{b}t} = \frac{g}{b}\epsilon^{\frac{a}{b}t} \cdot \sin 2\pi nt,$$

which is the same as

$$\frac{dy}{dt}\epsilon^{\frac{a}{b}t} + y\frac{d(\epsilon^{\frac{a}{b}t})}{dt} = \frac{g}{b}\epsilon^{\frac{a}{b}t} \cdot \sin 2\pi nt;$$

and this being a perfect differential may be integrated thus:—since, if $u = y\epsilon^{\frac{a}{b}t}$, $\dfrac{du}{dt} = \dfrac{dy}{dt}\epsilon^{\frac{a}{b}t} + y\dfrac{d(\epsilon^{\frac{a}{b}t})}{dt}$,

$$y\epsilon^{\frac{a}{b}t} = \frac{g}{b}\int \epsilon^{\frac{a}{b}t} \cdot \sin 2\pi nt \cdot dt + C,$$

or
$$y = \frac{g}{b}\epsilon^{-\frac{a}{b}t}\int \epsilon^{\frac{a}{b}t} \cdot \sin 2\pi nt \cdot dt + C\epsilon^{-\frac{a}{b}t}. \qquad \text{[A]}$$

The last term is obviously a term which will die out as t increases, and may be omitted. The trouble now comes in to find the integral that appears as a factor. To tackle this we resort to the device (see p. 227) of integration by parts, the general formula for which is $\int udv = uv - \int vdu$. For this purpose write

$$\begin{cases} u = \epsilon^{\frac{a}{b}t}; \\ dv = \sin 2\pi nt \cdot dt. \end{cases}$$

We shall then have

$$\begin{cases} du = \epsilon^{\frac{a}{b}t} \times \dfrac{a}{b}\, dt; \\ v = -\dfrac{1}{2\pi n}\cos 2\pi nt. \end{cases}$$

Inserting these, the integral in question becomes:

$$\begin{aligned}
&\int \epsilon^{\frac{a}{b}t} \cdot \sin 2\pi nt \cdot dt \\
&\qquad = -\frac{1}{2\pi n} \cdot \epsilon^{\frac{a}{b}t} \cdot \cos 2\pi nt - \int -\frac{1}{2\pi n}\cos 2\pi nt \cdot \epsilon^{\frac{a}{b}t} \cdot \frac{a}{b}\, dt \\
&\qquad = -\frac{1}{2\pi n}\epsilon^{\frac{a}{b}t}\cos 2\pi nt + \frac{a}{2\pi nb}\int \epsilon^{\frac{a}{b}t} \cdot \cos 2\pi nt \cdot dt. \qquad \text{[B]}
\end{aligned}$$

The last integral is still irreducible. To evade the difficulty, repeat the integration by parts of the left side, but treating it in the reverse way by writing:

$$\begin{cases} u = \sin 2\pi nt; \\ dv = \epsilon^{\frac{a}{b}t} \cdot dt; \end{cases}$$

whence
$$\begin{cases} du = 2\pi n \cdot \cos 2\pi nt \cdot dt; \\ v = \dfrac{b}{a}\epsilon^{\frac{a}{b}t} \end{cases}$$

Inserting these, we get

$$\int \epsilon^{\frac{a}{b}t} \cdot \sin 2\pi nt \cdot dt$$
$$= \frac{b}{a} \cdot \epsilon^{\frac{a}{b}t} \cdot \sin 2\pi nt - \frac{2\pi nb}{a} \int \epsilon^{\frac{a}{b}t} \cdot \cos 2\pi nt \cdot dt. \qquad [\text{C}]$$

Noting that the final intractable integral in [C] is the same as that in [B], we may eliminate it, by multiplying [B] by $\frac{2\pi nb}{a}$, and multiplying [C] by $\frac{a}{2\pi nb}$, and adding them.

The result, when cleared down, is:

$$\int \epsilon^{\frac{a}{b}t} \cdot \sin 2\pi nt \cdot dt = \epsilon^{\frac{a}{b}t} \left\{ \frac{ab \cdot \sin 2\pi nt - 2\pi nb^2 \cdot \cos 2\pi nt}{a^2 + 4\pi^2 n^2 b^2} \right\} \qquad [\text{D}]$$

Inserting this value in [A], we get

$$y = g \left\{ \frac{a \cdot \sin 2\pi nt - 2\pi nb \cdot \cos 2\pi nt}{a^2 + 4\pi^2 n^2 b^2} \right\}.$$

To simplify still further, let us imagine an angle ϕ such that $\tan \phi =$

$\dfrac{2\pi nb}{a}$.

Then $$\sin\phi = \frac{2\pi nb}{\sqrt{a^2+4\pi^2n^2b^2}},$$
and $$\cos\phi = \frac{a}{\sqrt{a^2+4\pi^2n^2b^2}}.$$

Substituting these, we get:

$$y = g\frac{\cos\phi\cdot\sin 2\pi nt - \sin\phi\cdot\cos 2\pi nt}{\sqrt{a^2+4\pi^2n^2b^2}},$$

which may be written

$$y = g\frac{\sin(2\pi nt-\phi)}{\sqrt{a^2+4\pi^2n^2b^2}},$$

which is *the solution* desired.

This is indeed none other than the equation of an alternating electric current, where g represents the amplitude of the electromotive force, n the frequency, a the resistance, b the coefficient of self-induction of the circuit, and ϕ is an angle of lag.

Example 4.

Suppose that $$M\,dx + N\,dy = 0.$$

We could integrate this expression directly, if M were a function of x only, and N a function of y only; but, if both M and N are functions that depend on both x and y, how are we to integrate it? Is it itself an exact differential? That is: have M and N each been formed by partial differentiation from some common function U, or not? If they have, then

$$\begin{cases} \dfrac{\partial U}{\partial x} = M, \\ \dfrac{\partial U}{\partial y} = N. \end{cases}$$

And if such a common function exists, then

$$\frac{\partial U}{\partial x}\,dx + \frac{\partial U}{\partial y}\,dy$$

is an exact differential (compare p. 175).

Now the test of the matter is this. If the expression is an exact differential, it must be true that

$$\frac{dM}{dy} = \frac{dN}{dx};$$

for then
$$\frac{d(dU)}{dx\,dy} = \frac{d(dU)}{dy\,dx},$$

which is necessarily true.

Take as an illustration the equation

$$(1 + 3xy)\,dx + x^2\,dy = 0.$$

Is this an exact differential or not? Apply the test.

$$\begin{cases} \dfrac{d(1 + 3xy)}{dy} = 3x, \\ \dfrac{d(x^2)}{dx} \qquad = 2x, \end{cases}$$

which do not agree. Therefore, it is not an exact differential, and the two functions $1 + 3xy$ and x^2 have not come from a common original function.

It is possible in such cases to discover, however, *an integrating factor*, that is to say, a factor such that if both are multiplied by this factor, the expression will become an exact differential. There is no one rule for discovering such an integrating factor; but experience will usually suggest one. In the present instance $2x$ will act as such. Multiplying by $2x$, we get

$$(2x + 6x^2y)\,dx + 2x^3\,dy = 0.$$

Now apply the test to this.

$$\begin{cases} \dfrac{d(2x + 6x^2y)}{dy} = 6x^2, \\ \qquad \dfrac{d(2x^3)}{dx} \qquad = 6x^2, \end{cases}$$

which agrees. Hence this is an exact differential, and may be integrated. Now, if $w = 2x^3y$,

$$dw = 6x^2y\,dx + 2x^3\,dy.$$

Hence $$\int 6x^2y\,dx + \int 2x^3\,dy = w = 2x^3y;$$

so that we get $$U = x^2 + 2x^3y + C.$$

Example 5. Let $\dfrac{d^2y}{dt^2} + n^2y = 0$.

In this case we have a differential equation of the second degree, in which y appears in the form of a second differential coefficient, as well as in person.

Transposing, we have $\dfrac{d^2y}{dt^2} = -n^2y$.

It appears from this that we have to do with a function such that its second differential coefficient is proportional to itself, but with reversed sign. In Chapter XV. we found that there was such a function—namely, the *sine* (or the *cosine* also) which possessed this property. So, without further ado, we may infer that the solution will be of the form $y = A\sin(pt+q)$. However, let us go to work.

Multiply both sides of the original equation by $2\dfrac{dy}{dt}$ and integrate, giving us $2\dfrac{d^2y}{dt^2}\,\dfrac{dy}{dt} + 2x^2y\dfrac{dy}{dt} = 0$, and, as

$$2\frac{d^2y}{dt^2}\,\frac{dy}{dt} = \frac{d\left(\dfrac{dy}{dt}\right)^2}{dt}, \qquad \left(\frac{dy}{dt}\right)^2 + n^2(y^2 - C^2) = 0,$$

C being a constant. Then, taking the square roots,

$$\frac{dy}{dt} = -n\sqrt{y^2 - C^2} \quad \text{and} \quad \frac{dy}{\sqrt{C^2 - y^2}} = n \cdot dt.$$

But it can be shown that (see p. 171)

$$\frac{1}{\sqrt{C^2 - y^2}} = \frac{d(\arcsin\dfrac{y}{C})}{dy};$$

whence, passing from angles to sines,

$$\arcsin\frac{y}{C} = nt + C_1 \quad \text{and} \quad y = C\sin(nt + C_1),$$

where C_1 is a constant angle that comes in by integration.

Or, preferably, this may be written

$$y = A\sin nt + B\cos nt, \text{ which is the solution.}$$

Example 6. $$\frac{d^2y}{dt^2} - n^2y = 0.$$

Here we have obviously to deal with a function y which is such that its second differential coefficient is proportional to itself. The only function we know that has this property is the exponential function (see p. 142), and we may be certain therefore that the solution of the equation will be of that form.

Proceeding as before, by multiplying through by $2\dfrac{dy}{dx}$, and integrating, we get $2\dfrac{d^2y}{dx^2}\,\dfrac{dy}{dx} - 2x^2y\dfrac{dy}{dx} = 0$,

and, as $$2\frac{d^2y}{dx^2}\,\frac{dy}{dx} = \frac{d\left(\dfrac{dy}{dx}\right)^2}{dx}, \quad \left(\frac{dy}{dx}\right)^2 - n^2(y^2+c^2) = 0,$$

$$\frac{dy}{dx} - n\sqrt{y^2+c^2} = 0,$$

where c is a constant, and $\dfrac{dy}{\sqrt{y^2+c^2}} = n\,dx$.

Now, if $w = \log_\epsilon(y + \sqrt{y^2+c^2}) = \log_\epsilon u$,

$$\frac{dw}{du} = \frac{1}{u}, \quad \frac{du}{dy} = 1 + \frac{y}{\sqrt{y^2+c^2}} = \frac{y+\sqrt{y^2+c^2}}{\sqrt{y^2+c^2}}$$

and $$\frac{dw}{dy} = \frac{1}{\sqrt{y^2+c^2}}.$$

Hence, integrating, this gives us

$$\log_\epsilon(y+\sqrt{y^2+c^2}) = nx + \log_\epsilon C,$$

$$y + \sqrt{y^2+c^2} = C\epsilon^{nx}. \tag{1}$$

Now $$(y+\sqrt{y^2+c^2}) \times (-y + \sqrt{y^2+c^2}) = c^2;$$

whence $$-y + \sqrt{y^2+c^2} = \frac{c^2}{C}\epsilon^{-nx}. \tag{2}$$

Subtracting (2) from (1) and dividing by 2, we then have

$$y = \frac{1}{2}C\epsilon^{nx} - \frac{1}{2}\frac{c^2}{C}\epsilon^{-nx},$$

which is more conveniently written

$$y = A\epsilon^{nx} + B\epsilon^{-nx}.$$

Or, the solution, which at first sight does not look as if it had anything to do with the original equation, shows that y consists of two terms, one of which grows logarithmically as x increases, and of a second term which dies away as x increases.

Example 7.

Let $$b\frac{d^2y}{dt^2} + a\frac{dy}{dt} + gy = 0.$$

Examination of this expression will show that, if $b = 0$, it has the form of Example 1, the solution of which was a negative exponential. On the other hand, if $a = 0$, its form becomes the same as that of Example 6, the solution of which is the sum of a positive and a negative exponential. It is therefore not very surprising to find that the solution of the present example is

$$y = (\epsilon^{-mt})(A\epsilon^{nt} + B\epsilon^{-nt}),$$

where $$m = \frac{a}{2b} \quad \text{and} \quad n = \sqrt{\frac{a^2}{4b^2} - \frac{g}{b}}.$$

The steps by which this solution is reached are not given here; they may be found in advanced treatises.

Example 8.

$$\frac{d^2y}{dt^2} = a^2\frac{d^2y}{dx^2}.$$

It was seen (p. 177) that this equation was derived from the original

$$y = F(x + at) + f(x - at),$$

where F and f were any arbitrary functions of t.

Another way of dealing with it is to transform it by a change of variables into

$$\frac{d^2y}{du \cdot dv} = 0,$$

where $u = x + at$, and $v = x - at$, leading to the same general solution. If we consider a case in which F vanishes, then we have simply

$$y = f(x - at);$$

and this merely states that, at the time $t = 0$, y is a particular function of x, and may be looked upon as denoting that the curve of the relation of y to x has a particular shape. Then any change in the value of t is equivalent simply to an alteration in the origin from which x is reckoned. That is to say, it indicates that, the form of the function being conserved, it is propagated along the x direction with a uniform velocity a; so that whatever the value of the ordinate y at any particular time t_0 at any particular point x_0, the same value of y will appear at the subsequent time t_1 at a point further along, the abscissa of which is $x_0 + a(t_1 - t_0)$. In this case the simplified equation represents the propagation of a wave (of any form) at a uniform speed along the x direction.

If the differential equation had been written

$$m\frac{d^2y}{dt^2} = k\,\frac{d^2y}{dx^2},$$

the solution would have been the same, but the velocity of propagation would have had the value

$$a = \sqrt{\frac{k}{m}}.$$

You have now been personally conducted over the frontiers into the enchanted land. And in order that you may have a handy reference to the principal results, the author, in bidding you farewell, begs to present you with a passport in the shape of a convenient collection of standard forms (see pp. 252–254). In the middle column are set down a number of the functions which most commonly occur. The results of differentiating them are set down on the left; the results of integrating them are set down on the right. May you find them useful!

EPÍLOGO Y APÓLOGO.

PUEDE asumirse con confianza que cuando este tratado "Cálculo Fácil" caiga en las manos de los matemáticos profesionales, ellos (si no son demasiado perezosos) se levantarán como un solo hombre, y lo condenarán como siendo un libro completamente malo. De eso no puede haber, desde su punto de vista, ninguna posible manera de duda en absoluto. Comete varios errores muy graves y deplorables.

Primero, muestra cuán ridículamente fáciles son realmente la mayoría de las operaciones del cálculo.

Segundo, revela tantos secretos comerciales. Al mostrarte que *lo que un tonto puede hacer, otros tontos también pueden hacer*, te permite ver que estos presuntuosos matemáticos, que se enorgullecen de haber dominado un tema tan terriblemente difícil como el cálculo, no tienen una razón tan grande para estar engreídos. Les gusta que pienses cuán terriblemente difícil es, y no quieren que esa superstición sea rudamente disipada.

Tercero, entre las cosas terribles que dirán sobre "Tan Fácil" está esto: que hay un completo fracaso de parte del autor para demostrar con rigurosa y satisfactoria completitud la validez de varios métodos que él ha presentado de manera simple, ¡y incluso se ha *atrevido a usar* en la resolución de problemas! Pero why should he not? You don't forbid the use of a watch to every person who does not know how to make one? You don't object to the musician playing on a violin that he has not himself constructed. You don't teach the rules of syntax to children until they have already become fluent in the *use* of speech. It would be equally absurd to require general rigid demonstrations to be expounded to beginners in the calculus.

One other thing will the professed mathematicians say about this thoroughly bad and vicious book: that the reason why it is *so easy* is because the author has left out all the things that are really difficult. And the ghastly fact about this accusation is that—*it is true!* That is, indeed, why the book has been written—written for the legion of innocents who have hitherto been deterred from acquiring the elements of the calculus by the stupid way in which its teaching is almost always presented. Any subject can be made repulsive by presenting it bristling with difficulties. The aim of this book is to enable beginners to learn its language, to acquire familiarity with its endearing simplicities, and to grasp its powerful methods of solving problems, without being compelled to toil through the intricate out-of-the-way (and mostly irrelevant) mathematical gymnastics so dear to the unpractical mathematician.

There are amongst young engineers a number on whose ears the adage that *what one fool can do, another can*, may fall with a familiar sound. They are earnestly requested not to give the author away, nor to tell the mathematicians what a fool he really is.

TABLA DE FORMAS ESTÁNDAR.

$\dfrac{dy}{dx}$	$\longleftarrow \quad y \quad \longrightarrow$	$\displaystyle\int y\,dx$
	Algebraic.	
1	x	$\frac{1}{2}x^2 + C$
0	a	$ax + C$
1	$x \pm a$	$\frac{1}{2}x^2 \pm ax + C$
a	ax	$\frac{1}{2}ax^2 + C$
$2x$	x^2	$\frac{1}{3}x^3 + C$
nx^{n-1}	x^n	$\dfrac{1}{n+1}x^{n+1} + C$
$-x^{-2}$	x^{-1}	$\log_\epsilon x + C$
$\dfrac{du}{dx} \pm \dfrac{dv}{dx} \pm \dfrac{dw}{dx}$	$u \pm v \pm w$	$\int u\,dx \pm \int v\,dx \pm \int w\,dx$
$u\,\dfrac{dv}{dx} + v\,\dfrac{du}{dx}$	uv	No general form known
$\dfrac{v\,\dfrac{du}{dx} - u\,\dfrac{dv}{dx}}{v^2}$	$\dfrac{u}{v}$	No general form known
$\dfrac{du}{dx}$	u	$ux - \int x\,du + C$

$\frac{dy}{dx}$	$\longleftarrow \quad y \quad \longrightarrow$	$\int y\,dx$
Exponential and Logarithmic.		
ϵ^x	ϵ^x	$\epsilon^x + C$
x^{-1}	$\log_\epsilon x$	$x(\log_\epsilon x - 1) + C$
$0.4343 \times x^{-1}$	$\log_{10} x$	$0.4343x(\log_\epsilon x - 1) + C$
$a^x \log_\epsilon a$	a^x	$\frac{a^x}{\log_\epsilon a} + C$
Trigonometrical.		
$\cos x$	$\sin x$	$-\cos x + C$
$-\sin x$	$\cos x$	$\sin x + C$
$\sec^2 x$	$\tan x$	$-\log_\epsilon \cos x + C$
Circular (Inverse).		
$\frac{1}{\sqrt{(1-x^2)}}$	$\operatorname{arc}\sin x$	$x \cdot \operatorname{arc}\sin x + \sqrt{1-x^2} + C$
$-\frac{1}{\sqrt{(1-x^2)}}$	$\operatorname{arc}\cos x$	$x \cdot \operatorname{arc}\cos x - \sqrt{1-x^2} + C$
$\frac{1}{1+x^2}$	$\operatorname{arc}\tan x$	$x \cdot \operatorname{arc}\tan x - \frac{1}{2}\log_\epsilon(1+x^2) + C$
Hyperbolic.		
$\cosh x$	$\sinh x$	$\cosh x + C$
$\sinh x$	$\cosh x$	$\sinh x + C$
$\operatorname{sech}^2 x$	$\tanh x$	$\log_\epsilon \cosh x + C$

$\dfrac{dy}{dx}$	$\longleftarrow \quad y \quad \longrightarrow$	$\int y\,dx$
	Miscellaneous.	
$-\dfrac{1}{(x+a)^2}$	$\dfrac{1}{x+a}$	$\log_\epsilon(x+a)+C$
$-\dfrac{x}{(a^2+x^2)^{\frac{3}{2}}}$	$\dfrac{1}{\sqrt{a^2+x^2}}$	$\log_\epsilon(x+\sqrt{a^2+x^2})+C$
$\mp\dfrac{b}{(a\pm bx)^2}$	$\dfrac{1}{a\pm bx}$	$\pm\dfrac{1}{b}\log_\epsilon(a\pm bx)+C$
$-\dfrac{3a^2x}{(a^2+x^2)^{\frac{5}{2}}}$	$\dfrac{a^2}{(a^2+x^2)^{\frac{3}{2}}}$	$\dfrac{x}{\sqrt{a^2+x^2}}+C$
$a\cdot\cos ax$	$\sin ax$	$-\dfrac{1}{a}\cos ax+C$
$-a\cdot\sin ax$	$\cos ax$	$\dfrac{1}{a}\sin ax+C$
$a\cdot\sec^2 ax$	$\tan ax$	$-\dfrac{1}{a}\log_\epsilon\cos ax+C$
$\sin 2x$	$\sin^2 x$	$\dfrac{x}{2}-\dfrac{\sin 2x}{4}+C$
$-\sin 2x$	$\cos^2 x$	$\dfrac{x}{2}+\dfrac{\sin 2x}{4}+C$
$n\cdot\sin^{n-1}x\cdot\cos x$	$\sin^n x$	$-\dfrac{\cos x}{n}\sin^{n-1}x+\dfrac{n-1}{n}\int\sin^{n-2}x\,dx+C$
$-\dfrac{\cos x}{\sin^2 x}$	$\dfrac{1}{\sin x}$	$\log_\epsilon\tan\dfrac{x}{2}+C$
$-\dfrac{\sin 2x}{\sin^4 x}$	$\dfrac{1}{\sin^2 x}$	$-\operatorname{cotan}x+C$
$\dfrac{\sin^2 x-\cos^2 x}{\sin^2 x\cdot\cos^2 x}$	$\dfrac{1}{\sin x\cdot\cos x}$	$\log_\epsilon\tan x+C$
$n\cdot\sin mx\cdot\cos nx+$ $m\cdot\sin nx\cdot\cos mx$	$\sin mx\cdot\sin nx$	$\frac{1}{2}\cos(m-n)x-\frac{1}{2}\cos(m+n)x+C$
$2a\cdot\sin 2ax$	$\sin^2 ax$	$\dfrac{x}{2}-\dfrac{\sin 2ax}{4a}+C$
$-2a\cdot\sin 2ax$	$\cos^2 ax$	$\dfrac{x}{2}+\dfrac{\sin 2ax}{4a}+C$

RESPUESTAS.

Exercises I. (p. 24.)

(1) $\dfrac{dy}{dx} = 13x^{12}$. (2) $\dfrac{dy}{dx} = -\dfrac{3}{2}x^{-\frac{5}{2}}$. (3) $\dfrac{dy}{dx} = 2ax^{(2a-1)}$.

(4) $\dfrac{du}{dt} = 2.4t^{1.4}$. (5) $\dfrac{dz}{du} = \dfrac{1}{3}u^{-\frac{2}{3}}$. (6) $\dfrac{dy}{dx} = -\dfrac{5}{3}x^{-\frac{8}{3}}$.

(7) $\dfrac{du}{dx} = -\dfrac{8}{5}x^{-\frac{13}{5}}$. (8) $\dfrac{dy}{dx} = 2ax^{a-1}$.

(9) $\dfrac{dy}{dx} = \dfrac{3}{q}x^{\frac{3-q}{q}}$. (10) $\dfrac{dy}{dx} = -\dfrac{m}{n}x^{-\frac{m+n}{n}}$.

Exercises II. (p. 33.)

(1) $\dfrac{dy}{dx} = 3ax^2$. (2) $\dfrac{dy}{dx} = 13 \times \frac{3}{2}x^{\frac{1}{2}}$. (3) $\dfrac{dy}{dx} = 6x^{-\frac{1}{2}}$.

(4) $\dfrac{dy}{dx} = \dfrac{1}{2}c^{\frac{1}{2}}x^{-\frac{1}{2}}$. (5) $\dfrac{du}{dz} = \dfrac{an}{c}z^{n-1}$. (6) $\dfrac{dy}{dt} = 2.36t$.

(7) $\dfrac{dl_t}{dt} = 0.000012 \times l_0$.

(8) $\dfrac{dC}{dV} = abV^{b-1}$, 0.98, 3.00 and 7.47 candle power per volt respectively.

(9) $\dfrac{dn}{dD} = -\dfrac{1}{LD^2}\sqrt{\dfrac{gT}{\pi\sigma}}$, $\quad \dfrac{dn}{dL} = -\dfrac{1}{DL^2}\sqrt{\dfrac{gT}{\pi\sigma}}$,

$\dfrac{dn}{d\sigma} = -\dfrac{1}{2DL}\sqrt{\dfrac{gT}{\pi\sigma^3}}$, $\quad \dfrac{dn}{dT} = \dfrac{1}{2DL}\sqrt{\dfrac{g}{\pi\sigma T}}$.

(10) $$\frac{\text{Rate of change of } P \text{ when } t \text{ varies}}{\text{Rate of change of } P \text{ when } D \text{ varies}} = -\frac{D}{t}.$$

(11) 2π, $2\pi r$, πl, $\frac{2}{3}\pi rh$, $8\pi r$, $4\pi r^2$.

(12) $$\frac{dD}{dT} = \frac{0.000012 l_t}{\pi}.$$

Exercises III. (p. 47.)

(1) (*a*) $1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots$ (*b*) $2ax + b$. (*c*) $2x + 2a$.

(*d*) $3x^2 + 6ax + 3a^2$.

(2) $$\frac{dw}{dt} = a - bt.$$

(3) $$\frac{dy}{dx} = 2x.$$

(4) $14110x^4 - 65404x^3 - 2244x^2 + 8192x + 1379$.

(5) $$\frac{dx}{dy} = 2y + 8.$$

(6) $185.9022654x^2 + 154.36334$.

(7) $$\frac{-5}{(3x+2)^2}.$$

(8) $$\frac{6x^4 + 6x^3 + 9x^2}{(1 + x + 2x^2)^2}.$$

(9) $$\frac{ad - bc}{(cx+d)^2}.$$

(10) $$\frac{anx^{-n-1} + bnx^{n-1} + 2nx^{-1}}{(x^{-n} + b)^2}.$$

(11) $b + 2ct$.

(12) $$R_0(a+2bt), \quad R_0\left(a + \frac{b}{2\sqrt{t}}\right), \quad -\frac{R_0(a+2bt)}{(1+at+bt^2)^2} \quad \text{or} \quad \frac{R^2(a+2bt)}{R_0}.$$

(13) $1.4340(0.000014t - 0.001024)$, -0.00117, -0.00107, -0.00097.

(14) $$\frac{dE}{dl} = b + \frac{k}{i}, \quad \frac{dE}{di} = -\frac{c + kl}{i^2}.$$

Exercises IV. (p. 52.)

(1) $17+24x;\quad 24.$

(2) $\dfrac{x^2+2ax-a}{(x+a)^2};\quad \dfrac{2a(a+1)}{(x+a)^3}.$

(3) $1+x+\dfrac{x^2}{1\times 2}+\dfrac{x^3}{1\times 2\times 3};\quad 1+x+\dfrac{x^2}{1\times 2}.$

(4) (*Exercises III.*):

(1) (a) $\dfrac{d^2y}{dx^2}=\dfrac{d^3y}{dx^3}=1+x+\tfrac{1}{2}x^2+\tfrac{1}{6}x^3+\dots.$

(b) $2a$, 0. (c) 2, 0. (d) $6x+6a$, 6.

(2) $-b$, 0. (3) 2, 0.

(4) $56440x^3-196212x^2-4488x+8192.$

$$169320x^2-392424x-4488.$$

(5) 2, 0. (6) $371.80453x$, 371.80453.

(7) $\dfrac{30}{(3x+2)^3},\quad -\dfrac{270}{(3x+2)^4}.$

(*Examples*, p. 43):

(1) $\dfrac{6a}{b^2}x,\quad \dfrac{6a}{b^2}.$ (2) $\dfrac{3a\sqrt{b}}{2\sqrt{x}}-\dfrac{6b\sqrt[3]{a}}{x^3},\quad \dfrac{18b\sqrt[3]{a}}{x^4}-\dfrac{3a\sqrt{b}}{4\sqrt{x^3}}.$

(3) $\dfrac{2}{\sqrt[3]{\theta^8}}-\dfrac{1.056}{\sqrt[5]{\theta^{11}}},\quad \dfrac{2.3232}{\sqrt[5]{\theta^{16}}}-\dfrac{16}{3\sqrt[3]{\theta^{11}}}.$

(4) $810t^4-648t^3+479.52t^2-139.968t+26.64.$

$$3240t^3-1944t^2+959.04t-139.968.$$

(5) $12x+2$, 12. (6) $6x^2-9x,\quad 12x-9.$

(7) $\dfrac{3}{4}\left(\dfrac{1}{\sqrt{\theta}}+\dfrac{1}{\sqrt{\theta^5}}\right)+\dfrac{1}{4}\left(\dfrac{15}{\sqrt{\theta^7}}-\dfrac{1}{\sqrt{\theta^3}}\right).$

$\dfrac{3}{8}\left(\dfrac{1}{\sqrt{\theta^5}}-\dfrac{1}{\sqrt{\theta^3}}\right)-\dfrac{15}{8}\left(\dfrac{7}{\sqrt{\theta^9}}+\dfrac{1}{\sqrt{\theta^7}}\right).$

Exercises V. (p. 66.)

(2) 64; 147.2; and 0.32 feet per second.

(3) $x = a - gt$; $\ddot{x} = -g$.

(4) 45.1 feet per second.

(5) 12.4 feet per second per second. Yes.

(6) Angular velocity $= 11.2$ radians per second; angular acceleration $= 9.6$ radians per second per second.

(7) $v = 20.4t^2 - 10.8$. $a = 40.8t$. 172.8 in./sec., 122.4 in./sec^2.

(8) $v = \dfrac{1}{30\sqrt[3]{(t-125)^2}}$, $a = -\dfrac{1}{45\sqrt[3]{(t-125)^5}}$.

(9) $v = 0.8 - \dfrac{8t}{(4+t^2)^2}$, $a = \dfrac{24t^2-32}{(4+t^2)^3}$, 0.7926 and 0.00211.

(10) $n = 2$, $n = 11$.

Exercises VI. (p. 75.)

(1) $\dfrac{x}{\sqrt{x^2+1}}$.

(2) $\dfrac{x}{\sqrt{x^2+a^2}}$.

(3) $-\dfrac{1}{2\sqrt{(a+x)^3}}$.

(4) $\dfrac{ax}{\sqrt{(a-x^2)^3}}$.

(5) $\dfrac{2a^2-x^2}{x^3\sqrt{x^2-a^2}}$.

(6) $\dfrac{\frac{3}{2}x^2\left[\frac{8}{9}x\left(x^3+a\right)-\left(x^4+a\right)\right]}{\left(x^4+a\right)^{\frac{2}{3}}\left(x^3+a\right)^{\frac{3}{2}}}$ (7) $\dfrac{2a\left(x-a\right)}{(x+a)^3}$.

(8) $\frac{5}{2}y^3$. (9) $\dfrac{1}{(1-\theta)\sqrt{1-\theta^2}}$.

Exercises VII. (p. 77.)

(1) $\dfrac{dw}{dx}=\dfrac{3x^2\left(3+3x^3\right)}{27\left(\frac{1}{2}x^3+\frac{1}{4}x^6\right)^3}$.

(2) $\dfrac{dv}{dx}=-\dfrac{12x}{\sqrt{1+\sqrt{2}+3x^2}\left(\sqrt{3}+4\sqrt{1+\sqrt{2}+3x^2}\right)^2}$.

(3) $\dfrac{du}{dx}=-\dfrac{x^2\left(\sqrt{3}+x^3\right)}{\sqrt{\left[1+\left(1+\dfrac{x^3}{\sqrt{3}}\right)^2\right]^3}}$

Exercises VIII. (p. 91.)

(2) 1.44.

(4) $\dfrac{dy}{dx}=3x^2+3$; and the numerical values are: 3, $3\frac{3}{4}$, 6, and 15.

(5) $\pm\sqrt{2}$.

(6) $\dfrac{dy}{dx}=-\dfrac{4}{9}\dfrac{x}{y}$. Slope is zero where $x=0$; and is $\mp\dfrac{1}{3\sqrt{2}}$ where $x=1$.

(7) $m = 4$, $n = -3$.

(8) Intersections at $x = 1$, $x = -3$. Angles $153° \, 26'$, $2° \, 28'$.

(9) Intersection at $x = 3.57$, $y = 3.50$. Angle $16° \, 16'$.

(10) $x = \frac{1}{3}$, $y = 2\frac{1}{3}$, $b = -\frac{5}{3}$.

Exercises IX. (p. 110.)

(1) Min.: $x = 0$, $y = 0$; max.: $x = -2$, $y = -4$.

(2) $x = a$.

(4) $25\sqrt{3}$ square inches.

(5) $\dfrac{dy}{dx} = -\dfrac{10}{x^2} + \dfrac{10}{(8-x)^2}$; $x = 4$; $y = 5$.

(6) Max. for $x = -1$; min. for $x = 1$.

(7) Join the middle points of the four sides.

(8) $r = \frac{2}{3}R$, $r = \dfrac{R}{2}$, no max.

(9) $r = R\sqrt{\dfrac{2}{3}}$, $r = \dfrac{R}{\sqrt{2}}$, $r = 0.8506R$.

(10) At the rate of $\dfrac{8}{r}$ square feet per second.

(11) $r = \dfrac{R\sqrt{8}}{3}$.

(12) $n = \sqrt{\dfrac{NR}{r}}$.

Exercises X. (p. 118.)

(1) Max.: $x = -2.19$, $y = 24.19$; min.:, $x = 1.52$, $y = -1.38$.

(2) $\dfrac{dy}{dx} = \dfrac{b}{a} - 2cx$; $\dfrac{d^2y}{dx^2} = -2c$; $x = \dfrac{b}{2ac}$ (*a maximum*).

(3) (a) One maximum and two minima.

(b) One maximum. ($x = 0$; other points unreal.)

(4) Min.: $x = 1.71$, $y = 6.14$.

(5) Max: $x = -.5$, $y = 4$.

(6) Max.: $x = 1.414$, $y = 1.7675$.

Min.: $x = -1.414$, $y = 1.7675$.

(7) Max.: $x = -3.565$, $y = 2.12$.

Min.: $x = +3.565$, $y = 7.88$.

(8) $0.4N$, $0.6N$.

(9) $x = \sqrt{\dfrac{a}{c}}$.

(10) Speed 8.66 nautical miles per hour. Time taken 115.47 hours. Minimum cost £112. 12*s.*

(11) Max. and min. for $x = 7.5$, $y = \pm 5.414$. (See example no. 10, p. 74.)

(12) Min.: $x = \frac{1}{2}$, $y = 0.25$; max.: $x = -\frac{1}{3}$, $y = 1.408$.

Exercises XI. (p. 130.)

(1) $\dfrac{2}{x-3}+\dfrac{1}{x+4}$. (2) $\dfrac{1}{x-1}+\dfrac{2}{x-2}$. (3) $\dfrac{2}{x-3}+\dfrac{1}{x+4}$.

(4) $\dfrac{5}{x-4}-\dfrac{4}{x-3}$. (5) $\dfrac{19}{13(2x+3)}-\dfrac{22}{13(3x-2)}$.

(6) $\dfrac{2}{x-2}+\dfrac{4}{x-3}-\dfrac{5}{x-4}$.

(7) $\dfrac{1}{6(x-1)}+\dfrac{11}{15(x+2)}+\dfrac{1}{10(x-3)}$.

(8) $\dfrac{7}{9(3x+1)}+\dfrac{71}{63(3x-2)}-\dfrac{5}{7(2x+1)}$.

(9) $\dfrac{1}{3(x-1)}+\dfrac{2x+1}{3(x^2+x+1)}$. (10) $x+\dfrac{2}{3(x+1)}+\dfrac{1-2x}{3(x^2-x+1)}$.

(11) $\dfrac{3}{(x+1)}+\dfrac{2x+1}{x^2+x+1}$. (12) $\dfrac{1}{x-1}-\dfrac{1}{x-2}+\dfrac{2}{(x-2)^2}$.

(13) $\dfrac{1}{4(x-1)}-\dfrac{1}{4(x+1)}+\dfrac{1}{2(x+1)^2}$.

(14) $\dfrac{4}{9(x-1)}-\dfrac{4}{9(x+2)}-\dfrac{1}{3(x+2)^2}$.

(15) $\dfrac{1}{x+2}-\dfrac{x-1}{x^2+x+1}-\dfrac{1}{(x^2+x+1)^2}$.

(16) $\dfrac{5}{x+4}-\dfrac{32}{(x+4)^2}+\dfrac{36}{(x+4)^3}$.

(17) $\dfrac{7}{9(3x-2)^2}+\dfrac{55}{9(3x-2)^3}+\dfrac{73}{9(3x-2)^4}$.

(18) $\dfrac{1}{6(x-2)}+\dfrac{1}{3(x-2)^2}-\dfrac{x}{6(x^2+2x+4)}$.

Exercises XII. (p. 153.)

(1) $ab(\epsilon^{ax} + \epsilon^{-ax})$. (2) $2at + \dfrac{2}{t}$. (3) $\log_\epsilon n$.

(5) npv^{n-1}. (6) $\dfrac{n}{x}$. (7) $\dfrac{3\epsilon^{-\frac{x}{x-1}}}{(x-1)^2}$.

(8) $6x\epsilon^{-5x} - 5(3x^2+1)\epsilon^{-5x}$. (9) $\dfrac{ax^{a-1}}{x^a + a}$.

(10) $\left(\dfrac{6x}{3x^2-1} + \dfrac{1}{2\left(\sqrt{x}+x\right)}\right)\left(3x^2-1\right)\left(\sqrt{x}+1\right)$.

(11) $\dfrac{1 - \log_\epsilon (x+3)}{(x+3)^2}$.

(12) $a^x\left(ax^{a-1} + x^a \log_\epsilon a\right)$. (14) Min.: $y = 0.7$ for $x = 0.694$.

(15) $\dfrac{1+x}{x}$. (16) $\dfrac{3}{x}(\log_\epsilon ax)^2$.

Exercises XIII. (p. 163.)

(1) Let $\dfrac{t}{T} = x$ ($\therefore t = 8x$), and use the Table on page 160.

(2) $T = 34.627$; 159.46 minutes.

(3) Take $2t = x$; and use the Table on page 160.

(5) (a) $x^x\left(1 + \log_\epsilon x\right)$; (b) $2x(\epsilon^x)^x$; (c) $\epsilon^{x^x} \times x^x\left(1 + \log_\epsilon x\right)$.

(6) 0.14 second.

(7) (a) 1.642; (b) 15.58.

(8) $\mu = 0.00037$, $31^m\frac{1}{4}$.

(9) i is 63.4% of i_0, 220 kilometres.

(10) 0.133, 0.145, 0.155, mean 0.144; −10.2%, −0.9%, +77.2%.

(11) Min. for $x = \dfrac{1}{\epsilon}$.

(12) Max. for $x = \epsilon$.

(13) Min. for $x = \log_\epsilon a$.

Exercises XIV. (p. 173.)

(1) (i) $\dfrac{dy}{d\theta} = A\cos\left(\theta - \dfrac{\pi}{2}\right)$;

(ii) $\dfrac{dy}{d\theta} = 2\sin\theta\cos\theta = \sin 2\theta$ and $\dfrac{dy}{d\theta} = 2\cos 2\theta$;

(iii) $\dfrac{dy}{d\theta} = 3\sin^2\theta\cos\theta$ and $\dfrac{dy}{d\theta} = 3\cos 3\theta$.

(2) $\theta = 45^\circ$ or $\dfrac{\pi}{4}$ radians.

(3) $\dfrac{dy}{dt} = -n\sin 2\pi nt$.

(4) $a^x \log_\epsilon a \cos a^x$.

(5) $\dfrac{\cos x}{\sin x} = \operatorname{cotan} x$

(6) $18.2\cos(x + 26^\circ)$.

(7) The slope is $\dfrac{dy}{d\theta} = 100\cos(\theta - 15^\circ)$, which is a maximum when $(\theta - 15^\circ) = 0$, or $\theta = 15^\circ$; the value of the slope being then $= 100$. When $\theta = 75^\circ$ the slope is $100\cos(75^\circ - 15^\circ) = 100\cos 60^\circ = 100 \times \frac{1}{2} = 50$.

(8) $$\cos\theta\sin 2\theta + 2\cos 2\theta\sin\theta = 2\sin\theta\left(\cos^2\theta + \cos 2\theta\right) = 2\sin\theta\left(3\cos^2\theta - 1\right).$$

(9) $amn\theta^{n-1}\tan^{m-1}(\theta^n)\sec^2\theta^n$.

(10) $\epsilon^x\left(\sin^2 x + \sin 2x\right)$; $\quad \epsilon^x\left(\sin^2 x + 2\sin 2x + 2\cos 2x\right)$.

(11) $(i)\ \dfrac{dy}{dx} = \dfrac{ab}{(x+b)^2}$; (ii) $\dfrac{a}{b}\epsilon^{-\frac{x}{b}}$; (iii) $\dfrac{1}{90}^\circ \times \dfrac{ab}{(b^2 + x^2)}$.

(12) (i) $\dfrac{dy}{dx} = \sec x\tan x$;

(ii) $\dfrac{dy}{dx} = -\dfrac{1}{\sqrt{1-x^2}}$;

(iii) $\dfrac{dy}{dx} = \dfrac{1}{1+x^2}$;

(iv) $\dfrac{dy}{dx} = \dfrac{1}{x\sqrt{x^2-1}}$;

(v) $\dfrac{dy}{dx} = \dfrac{\sqrt{3\sec x}\,(3\sec^2 x - 1)}{2}$.

(13) $\dfrac{dy}{d\theta} = 4.6\,(2\theta + 3)^{1.3}\cos(2\theta + 3)^{2.3}$.

(14) $\dfrac{dy}{d\theta} = 3\theta^2 + 3\cos(\theta + 3) - \log_\epsilon 3\left(\cos\theta \times 3^{\sin\theta} + 3\theta\right)$.

(15) $\theta = \cot\theta; \theta = \pm 0.86$; is max. for $+\theta$, min. for $-\theta$.

Exercises XV. (p. 180.)

(1) $x^3 - 6x^2y - 2y^2$; $\quad \frac{1}{3} - 2x^3 - 4xy$.

(2) $2xyz + y^2z + z^2y + 2xy^2z^2$;

$2xyz + x^2z + xz^2 + 2x^2yz^2$;

$2xyz + x^2y + xy^2 + 2x^2y^2z$.

(3) $\dfrac{1}{r}\{(x-a) + (y-b) + (z-c)\} = \dfrac{(x+y+z)-(a+b+c)}{r}$; $\dfrac{3}{r}$.

(4) $dy = vu^{v-1}\,du + u^v \log_\epsilon u\,dv$.

(5) $dy = 3\sin vu^2\,du + u^3\cos v\,dv$,

$dy = u\sin x^{u-1}\cos x\,dx + (\sin x)^u \log_\epsilon \sin x du$,

$dy = \dfrac{1}{v}\,\dfrac{1}{u}\,du - \log_\epsilon u \dfrac{1}{v^2}\,dv$.

(7) Minimum for $x = y = -\frac{1}{2}$.

(8) (a) Length 2 feet, width = depth = 1 foot, vol. = 2 cubic feet.

(b) Radius $= \dfrac{2}{\pi}$ feet = 7.46 in., length = 2 feet, vol. = 2.54.

(9) All three parts equal; the product is maximum.

(10) Minimum for $x = y = 1$.

(11) Min.: $x = \frac{1}{2}$ and $y = 2$.

(12) Angle at apex = $90°$; equal sides = length = $\sqrt[3]{2V}$.

Exercises XVI. (p. 190.)

(1) $1\frac{1}{3}$. (2) 0.6344. (3) 0.2624.

(4) (a) $y = \frac{1}{8}x^2 + C$; (b) $y = \sin x + C$.

(5) $y = x^2 + 3x + C$.

Exercises XVII. (p. 205.)

(1) $\dfrac{4\sqrt{a}x^{\frac{3}{2}}}{3} + C$. (2) $-\dfrac{1}{x^3} + C$. (3) $\dfrac{x^4}{4a} + C$.

(4) $\frac{1}{3}x^3 + ax + C$. (5) $-2x^{-\frac{5}{2}} + C$.

(6) $x^4 + x^3 + x^2 + x + C$. (7) $\dfrac{ax^2}{4} + \dfrac{bx^3}{9} + \dfrac{cx^4}{16} + C$.

(8) $\dfrac{x^2+a}{x+a} = x - a + \dfrac{a^2+a}{x+a}$ by division. Therefore the answer is $\dfrac{x^2}{2} - ax + (a^2+a)\log_\epsilon(x+a) + C$. (See pages 199 and 201.)

(9) $\dfrac{x^4}{4} + 3x^3 + \dfrac{27}{2}x^2 + 27x + C$. (10) $\dfrac{x^3}{3} + \dfrac{2-a}{2}x^2 - 2ax + C$.

(11) $a^2(2x^{\frac{3}{2}} + \frac{9}{4}x^{\frac{4}{3}}) + C$. (12) $-\frac{1}{3}\cos\theta - \frac{1}{6}\theta + C$.

(13) $\dfrac{\theta}{2} + \dfrac{\sin 2a\theta}{4a} + C$. (14) $\dfrac{\theta}{2} - \dfrac{\sin 2\theta}{4} + C$.

(15) $\dfrac{\theta}{2} - \dfrac{\sin 2a\theta}{4a} + C.$

(16) $\frac{1}{3}\epsilon^{3x}.$

(17) $\log(1+x) + C.$

(18) $-\log_\epsilon(1-x) + C.$

Exercises XVIII. (p. 224.)

(1) Area $= 60$; mean ordinate $= 10$.

(2) Area $= \frac{2}{3}$ of $a \times 2a\sqrt{a}$.

(3) Area $= 2$; mean ordinate $= \dfrac{2}{\pi} = 0.637$.

(4) Area $= 1.57$; mean ordinate $= 0.5$.

(5) 0.572, 0.0476.

(6) Volume $= \pi r^2 \dfrac{h}{3}$.

(7) 1.25.

(8) 79.4.

(9) Volume $= 4.9348$; area of surface $= 12.57$ (from 0 to π).

(10) $a\log_\epsilon a$, $\quad \dfrac{a}{a-1}\log_\epsilon a$.

(12) Arithmetical mean $= 9.5$; quadratic mean $= 10.85$.

(13) Quadratic mean $= \dfrac{1}{\sqrt{2}}\sqrt{A_1^2 + A_3^2}$; arithmetical mean $= 0$.

The first involves a somewhat difficult integral, and may be stated thus: By definition the quadratic mean will be

$$\sqrt{\frac{1}{2\pi}\int_0^{2\pi} (A_1 \sin x + A_3 \sin 3x)^2\, dx.}$$

Now the integration indicated by

$$\int (A_1^2 \sin^2 x + 2A_1 A_3 \sin x \sin 3x + A_3^2 \sin^2 3x)\, dx$$

is more readily obtained if for $\sin^2 x$ we write

$$\frac{1 - \cos 2x}{2}.$$

For $2 \sin x \sin 3x$ we write $\cos 2x - \cos 4x$; and, for $\sin^2 3x$,

$$\frac{1 - \cos 6x}{2}.$$

Making these substitutions, and integrating, we get (see p. 201)

$$\frac{A_1^2}{2}\left(x - \frac{\sin 2x}{2}\right) + A_1 A_3 \left(\frac{\sin 2x}{2} - \frac{\sin 4x}{4}\right) + \frac{A_3^2}{2}\left(x - \frac{\sin 6x}{6}\right).$$

At the lower limit the substitution of 0 for x causes all this to vanish, whilst at the upper limit the substitution of 2π for x gives $A_1^2\pi + A_3^2\pi$. And hence the answer follows.

(14) Area is 62.6 square units. Mean ordinate is 10.42.

(16) 436.3. (This solid is pear shaped.)

Exercises XIX. (p. 234.)

(1) $\dfrac{x\sqrt{a^2 - x^2}}{2} + \dfrac{a^2}{2} \sin^{-1} \dfrac{x}{a} + C$. (2) $\dfrac{x^2}{2}(\log_\epsilon x - \frac{1}{2}) + C$.

(3) $\dfrac{x^{a+1}}{a+1}\left(\log_\epsilon x - \dfrac{1}{a+1}\right) + C.$ (4) $\sin\epsilon^x + C.$

(5) $\sin(\log_\epsilon x) + C.$ (6) $\epsilon^x(x^2 - 2x + 2) + C.$

(7) $\dfrac{1}{a+1}(\log_\epsilon x)^{a+1} + C.$ (8) $\log_\epsilon(\log_\epsilon x) + C.$

(9) $2\log_\epsilon(x-1) + 3\log_\epsilon(x+2) + C.$

(10) $\frac{1}{2}\log_\epsilon(x-1) + \frac{1}{5}\log_\epsilon(x-2) + \frac{3}{10}\log_\epsilon(x+3) + C.$

(11) $\dfrac{b}{2a}\log_\epsilon\dfrac{x-a}{x+a} + C.$ (12) $\log_\epsilon\dfrac{x^2-1}{x^2+1} + C.$

(13) $\frac{1}{4}\log_\epsilon\dfrac{1+x}{1-x} + \frac{1}{2}\arctan x + C.$

(14) $\dfrac{1}{\sqrt{a}}\log_\epsilon\dfrac{\sqrt{a}-\sqrt{a-bx^2}}{x\sqrt{a}}.$ (Let $\dfrac{1}{x} = v$; then, in the result, let $\sqrt{v^2 - \dfrac{b}{a}} = v - u$.)

You had better differentiate now the answer and work back to the given expression as a check.

Every earnest student is exhorted to manufacture more examples for himself at every stage, so as to test his powers. When integrating he can always test his answer by differentiating it, to see whether he gets back the expression from which he started.

There are lots of books which give examples for practice. It will suffice here to name two: R. G. Blaine's *The Calculus and its Applications*, and F. M. Saxelby's *A Course in Practical Mathematics*.

GLASGOW: PRINTED AT THE UNIVERSITY PRESS BY ROBERT MACLEHOSE AND CO. LTD.

Differential and Integral Calculus for Technical Schools and Colleges. By P. A. LAMBERT, M.A. 7s. 6d.

Differential and Integral Calculus. With Applications. By Sir A. G. GREENHILL, F.R.S. 10s. 6d.

A Treatise on the Integral Calculus and its Applications. By I. TODHUNTER, F.R.S. 10s. 6d. Key. By H. ST. J. HUNTER, M.A. 10s. 6d.

A Treatise on the Differential Calculus and the Elements of the Integral Calculus. With numerous Examples. By I. TODHUNTER, F.R.S. 10s. 6d. Key. By H. ST. J. HUNTER, M.A. 10s. 6d.

Ordinary Differential Equations. An Elementary Text-book. By JAMES MORRIS PAGE, Ph.D. 6s. 6d.

An Introduction to the Modern Theory of Equations. By Prof. F. CAJORI, Ph.D. 7s. 6d. net.

A Treatise on Differential Equations. By ANDREW RUSSELL FORSYTH, Sc.D., LL.D. Fourth Edition. 14s. net.

A Short Course on Differential Equations. By Prof. DONALD F. CAMPBELL, Ph.D. 4s. net.

A Manual of Quaternions. By C. J. JOLY, M.A., D.Sc., F.R.S. 10s. net.

The Theory of Determinants in the Historical Order of Development. Vol. I. Part I. General Determinants, up to 1841. Part II. Special Determinants, up to 1841. 17s. net. Vol. II. The Period 1841 to 1860. 17s. net. By T. MUIR, M.A., LL.D., F.R.S.

An Introduction to the Theory of Infinite Series. By T. J. I'a Bromwich, M.A., F.R.S. 15s. net.

Introduction to the Theory of Fourier's Series and Integrals, and the Mathematical Theory of the Conduction of Heat. By Prof. H. S. Carslaw, M.A., D.Sc., F.R.S.E. 14s. net.

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TRANSCRIBER'S NOTE

The diagrams have been re-created, using accompanying formulas or descriptions from the text where possible.

In Capítulo XIV, pages 135–162, numerical values of $\left(1 + \frac{1}{n}\right)^n$, ϵ^x, and related quantities of British currency have been verified and rounded to the nearest digit.

On page 145 (page 146 in the original), the graphs of the natural logarithm and exponential functions, Figuras 38 and 39, have been interchanged to match the surrounding text.

The vertical dashed lines in the natural logarithm graph, Figura 39 (Figura 38 in the original), have been moved to match the data in the corresponding table.

On page 167 (page 167 in the original), the graphs of the sine and cosine functions, Figuras 44 and 45, have been interchanged to match the surrounding text.