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Entanglement in four qubit states: Polynomial invariant of degree 2, genuine multipartite concurrence and one-tangle

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ABSTRACT

Characterization of the multipartite mixed state entanglement is still a challenging problem. This is due to the fact that the entanglement for the mixed states, in general, is defined by a convex-roof extension. That is the entanglement measure of a mixed state ρ of a quantum system can be defined as the minimum average entanglement of an ensemble of pure states. In this paper, we show that polynomial entanglement measures of degree 2 of even-N qubits X states is in the full agreement with the genuine multipartite (GM) concurrence. Then, we plot the hierarchy of entanglement classification for four qubit pure states and then using new invariants, we classify the four qubit pure states. We focus on the convex combination of the classes whose at most the one of the invariants is non-zero and find the relationship between entanglement measures consist of non-zero-invariant, GM concurrence and one-tangle. We show that in many entanglement classes of four qubit states, GM concurrence is equal to the square root of one-tangle.

1. Introduction

A fundamental element of quantum information science is quantum entanglement. Quantum entanglement is a physical phenomenon that occurs when pairs or groups of particles interact in ways such that the quantum state of each particle cannot be described independently of the others. Such phenomena were the subject of a 1935 paper by Einstein, Podolsky, and Rosen [2] and several papers by Schrödinger shortly thereafter like [3] describing what came to be known as the EPR paradox. Quantum entangled states are crucial resource and play key roles in the quantum information processing such as quantum teleportation [4], quantum cryptography [5] and quantum computation [6]. An entangled system is defined to be one whose quantum state cannot be factored as a product of states of its local constituents. If a pure state $|\psi\rangle \in H_1 \otimes H_2 \otimes \ldots \otimes H_n$ – H_i is the Hilbert space of the i-th subsystem – can be written in the form $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \ldots \otimes |\psi_n\rangle$ where $|\psi_i\rangle$ is a pure state of the i-th subsystem, it is said to be separable, otherwise it is called entangled. When a system is in an entangled pure state, it is not possible to assign states to its subsystems. This will be true, in the appropriate sense, for the mixed state case as well. A mixed state of the composite system is described by a density matrix ρ acting on $H_1 \otimes H_2 \otimes \ldots \otimes H_n$. ρ is separable if there exist $p_k \geq 0$, $\{\rho_1^k\}, \{\rho_2^k\}, \ldots, \{\rho_n^k\}$ which are mixed states of the respective subsystems such that $\rho = \sum_k p_k \rho_1^k \otimes \rho_2^k \otimes \ldots \otimes \rho_n^k$ where $\sum_k p_k = 1$.

So far, different measures of the entanglement for the mixed quantum states is introduced like the entanglement of formation [7], the entanglement cost [8] and the distillable entanglement [7]. All of these measures are common in the some properties: They arrive to zero for each separable state, are invariant under local unitary transformations, and are never increasing on average by local operations and classical communication (LOCC) [9–11]. The latter property meaning that the entanglement measure is a so-called entanglement monotone. One of the ways to determining the entanglement of a pure quantum states is a polynomial function in the coefficients of states which are invariant under stochastic LOCC (SLOCC) and play a critical role in the investigation of entanglement measures. The polynomial function P of degree l of a system of m qudits is defined as [12]:

$$P(\kappa L|\psi\rangle) = \kappa^l P(|\psi\rangle),$$

for a constant $\kappa > 0$ and an invertible linear operator $L \in SL(l, C)^{\times m}$ representing the SLOCC transformation. The absolute value of any such polynomial with $l \leq 4$ defines in fact an entan-

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glement monotone [13]. For two and three qubits, the concurrence and the three-tangle are polynomial invariants of degrees 2 and 4, respectively [1,14]. Many efforts have been done over the last decade on the study of polynomial invariants for four or more qubits [15–18]. Trying to express of the entanglement measures in terms of mean values of physical quantities because of unphysical operations like complex conjugation for the definition of the concurrence is an interesting and experimental problem in quantum mechanical science. In [19], the authors showed that the amount of concurrence of two-qubit pure state can be obtained by measurement of mean values of the basic observables given by Pauli operators. For example, in [20], the multipartite concurrence of pure states was expressed in terms of only one single factorizable observable that has utility on trapped-ion and entangled photon experiments, the authors in [21], presented a scheme to directly measure the genuine tripartite entanglement for $2\otimes 2\otimes n$-dimensional pure quantum states, also, in [22], a method was proposed to obtain lower bounds for the genuine multipartite concurrence that these lower bounds can be easily obtained from the expectation value of a single observable, namely the fidelity of a pure state.

A polynomial invariant E is extended to the mixed states by means of the convex roof that is the largest convex function on the set of mixed states which corresponds to E on the pure states, given for a mixed state matrix ρ by:

$$E(\rho)=min_{\{p_i,|\psi_i\rangle\}}\sum_i p_i E(\psi_i),$$

where $\{p_i,|\psi_i\rangle\}$ is the ensemble of a pure state for the given density matrix ρ such that $\sum_i p_i=1$. The ensemble that minimizes $E(\rho)$ is called optimal. Any convex hull of all pure states that is vanished E, called zero-polytope and in the outside of this convex hull, it will never vanish.

In [1], Wootters find an explicit formula for the entanglement of formation of a pair of qubits as a function of their density matrix. Similar to that, we use the concept of polynomial invariant of degree 2 for even-N qubits of pure and mixed states. The absolute value of this invariant is entanglement monotones. Then we compare this polynomial invariant with genuine multipartite concurrence for even-N qubits X states.

Another important subject in quantum information is the classification of entanglement by means of mathematical equivalence. If two states can be obtained from each other by means of the local operations and classical communication with nonzero probability (SLOCC), then the two states have the same kind of entanglement. These two states belong to a same SLOCC entanglement class. Mathematically, two states are in a same class if and only if they are connected by an invertible local operation:

$$|\psi\rangle,|\phi\rangle\in same\quad class\Leftrightarrow|\psi\rangle=O_1\otimes\ldots\otimes O_N|\phi\rangle$$

that O_i $(i=1,..,N)$ represent the invertible local operators. Buniy and co-workers in [23] used the algebraic invariants that distinguish and classify entangled states. They used 19 invariants to distinguish the four qubit system and found the 82 classes for this system that regardless of the permutations of qubits, they obtained 27 different classes. In this paper, we use the 19 invariants to plot the hierarchy of the entanglement classes of four qubit pure states. We gain the same 27 classes of four qubit states by means of polynomial invariant of degree 2 and some another invariants. Among 27 classes, there are classes that at most the one of the invariants are non-zero. We study the convex combination of these classes and find the relation between non-zero invariant, genuine multipartite concurrence and one-tangle.

2. Polynomial invariant of degree 2

A function that quantify the entanglement of quantum states must be non-increasing (on average) under stochastic local operations and classical communication (SLOCC) where is so-called an entanglement monotone [24].

Here we investigate the polynomial invariant of degree 2 as an entanglement measure for any even-N qubit quantum states. Generic pure N-qubit states are of the form

$$|\psi_{A_1A_2\ldots A_N}\rangle=\sum_{i_1,i_2,\ldots,i_N=0}^{1}\psi_{i_1i_2\ldots i_N}|i_1i_2\ldots i_N\rangle, \tag{1}$$

where, due to normalization, $\sum_{i_1,i_2,\ldots,i_N=0}^{1}|\psi_{i_1i_2\ldots i_N}|^2=1$. For any even-N qubits pure quantum state, the polynomial invariant of degree 2 is defined as:

$$C(|\psi_{A_1\ldots A_N}\rangle)=|\varepsilon_{i_1j_1}\varepsilon_{i_2j_2}\ldots\varepsilon_{i_Nj_N}\psi_{i_1i_2\ldots i_N}\psi_{j_1j_2\ldots j_N}|, \tag{2}$$

where summation is over the repeated indices that values are 0 and 1, and ε is the SL(2,C)-invariant alternating tensor

$$\varepsilon:=\begin{pmatrix}0&1\\-1&0\end{pmatrix}.$$

Note that for any odd-qubits pure quantum state, this invariant is zero.

As the same way, given a density matrix ρ of the quantum systems with N-qubits that N is even, consider all ensembles of pure states $|\psi_{i_1,i_2,\ldots,i_N}\rangle$ with probabilities $p_{i_1,i_2,\ldots,i_N}$, such that

$$\rho=\sum_{i_1,i_2,\ldots,i_N=0}^{1}p_{i_1,i_2,\ldots,i_N}|\psi_{i_1,i_2,\ldots,i_N}\rangle\langle\psi_{i_1,i_2,\ldots,i_N}|. \tag{3}$$

The polynomial invariant of degree 2 for this mixed state ρ is obtained by the convex roof concept. In other words, it is defined first on the set of pure states and then extended to the set of all mixed states by minimizing its average value over all possible convex decompositions of the given state ρ into pure states [7]:

$$C(\rho)=min_{\{p_{i_1,i_2,\ldots,i_N},|\psi_{i_1,i_2,\ldots,i_N}\rangle\}}\times\sum_{i_1,i_2,\ldots,i_N=0}^{1}p_{i_1,i_2,\ldots,i_N}C(\psi_{i_1,i_2,\ldots,i_N}), \tag{4}$$

where for any pure state, the polynomial invariant of degree 2, $C(\psi_{i_1,i_2,\ldots,i_N})$, is defined as the Eq. (2). The decomposition(s) $\{p_{i_1,i_2,\ldots,i_N},|\psi_{i_1,i_2,\ldots,i_N}\rangle\}$ realising the minimum value of Eq. (4), is (are) called optimal. Wootters in [1] showed how to find the optimal decompositions for the most simple bipartite cases where enables us to compute the concurrence, analytically for the arbitrary two-qubits mixed states. In [25], the problem of determining the amount of entanglement of rank-2 state with any polynomial entanglement measure is seen as a geometric problem on the corresponding Bloch sphere. In [26], Osterloh and co-workers provided a non-trivial lower bound for the convex roof by means of the two concepts: zero-polytope and convex characteristic curve. In [27–31] found the explicit expressions for the three-tangle and optimal decompositions for three-qubit mixed states of rank-n (n = 2, 3, ..., 8) examples. In the next section of this letter, we give an exact formula of the polynomial invariant of degree 2 for any even qubit mixed state.

Let us first consider a pure state $|\psi\rangle$ of even-N qubit state. The polynomial invariant of degree 2 ($C(\psi)$) of this state is defined to be $C(\psi)=|\langle\psi|\widetilde{\psi}\rangle|$, where the tilde represents the "spin-flip" operation $|\widetilde{\psi}\rangle=(\underbrace{\sigma_y\otimes\sigma_y\otimes...\otimes\sigma_y}_{N\text{-times}})|\psi^*\rangle$. Here $|\psi^*\rangle$ is the complex conjugate of $|\psi\rangle$ in the standard basis $\{|0...0\rangle,...,|1...1\rangle\}$, and σ_y is the Pauli operator $\begin{pmatrix}0&-i\\i&0\end{pmatrix}$. Similarly, for a general state ρ of even-N qubits, the spin-flipped state is

$$\widetilde{\rho}=(\underbrace{\sigma_y\otimes\sigma_y\otimes...\otimes\sigma_y}_{N\text{-times}})\rho^*(\underbrace{\sigma_y\otimes\sigma_y\otimes...\otimes\sigma_y}_{N\text{-times}}), \tag{5}$$

where again the complex conjugate is taken in the standard basis. Then, the exact formula for the polynomial entanglement measure of degree-2 for any even-N qubits mixed state ρ in the binary form is given by:

$$C(\rho)=max\{0,\lambda_{0,0,...,0}-\sum_{i_1,i_2,...,i_N=0}^{1}\lambda_{i_1,i_2,...,i_N}\}, \tag{6}$$

where the $\lambda_{0,0,...,0}$ and $\lambda_{i_1,i_2,...,i_N}$ are the eigenvalues, in decreasing order, of the Hermitian matrix as:

$$R=\sqrt{\sqrt{\rho}\widetilde{\rho}\sqrt{\rho}}, \tag{7}$$

and we consider that $\lambda_{0,0,...,0}$ is larger than all of the $\lambda_{i_1,i_2,...,i_N}$s. Note that each $\lambda_{i_1,i_2,...,i_N}$ is a non-negative real number and that the all of $i_1,...,i_N$ cannot be zero simultaneously.

3. Genuine multipartite entanglement

A system consisting of N qubits is said to have genuine multipartite (GM) entanglement if each qubit is entangled with all of the other qubits and not only to some of them. A pure N-qubit state $|\psi\rangle$, is called biseparable if it is separable under some bipartition. In other words if the pure state $|\psi\rangle$ can be written as $|\psi\rangle=|\phi_1\rangle\otimes|\phi_2\rangle$, the state is biseparable. We denote this biseparable pure state as $|\psi^{B.S}\rangle$. A mixed state is biseparable if it can be written as $\rho^{B.S}=\sum_k p_k|\psi_k^{B.S}\rangle\langle\psi_k^{B.S}|$, that $|\psi_k^{B.S}\rangle$ might be biseparable. One of the measures to compute the GM entanglement is GM concurrence [22,32,33] that for a pure state $|\psi\rangle$ is defined as:

$$C_{GM}(|\psi\rangle)=min_{\lambda\in\tau}\sqrt{2}\sqrt{1-Tr(\rho_{A_\lambda}^2)}, \tag{8}$$

where τ represents the set of all possible bi-partitions $\{A_\lambda|B_\lambda\}$, and ρ_{A_λ} is the reduced density matrix: $\rho_{A_\lambda}=Tr_{B_\lambda}(|\psi\rangle\langle\psi|)$. For the mixed state $\rho=\sum_i p_i|\psi^i\rangle\langle\psi^i|$, this entanglement measure is defined as the average pure-state GM concurrence:

$$C_{GM}(\rho)=inf_{\{p_i,|\psi_i\rangle\}}\sum_i p_i C_{GM}(|\psi^i\rangle), \tag{9}$$

where minimization is over all possible pure state decompositions of ρ. In this paper, we compute this measure for the convex combinations of pure states and compare with another measures.

4. Hierarchy of the four qubit pure states and the role of the polynomial invariant of degree 2 for the classification of four qubit pure states

An important subject in quantum information is the classification of entanglement by means of mathematical equivalence. If two states can be obtained from each other by means of local operations and classical communication with nonzero probability (SLOCC), then the two states have the same kind of entanglement. These two states belong to a same SLOCC entanglement class. Mathematically, two states are in a same class if and only if they are connected together as the following form:

$$|\psi\rangle,|\varphi\rangle\in same\quad class\Leftrightarrow|\psi\rangle=O_1\otimes...\otimes O_N|\varphi\rangle$$

that in this relation, O_i $(i=1,..,N)$ display the invertible local operators.

Buniy and co-worker in [23] classified the pure four qubit quantum states. Their work was based on how the quantum system is decomposed into its subsystems. The brevity of their method for the classification of entanglement is given in Appendix. They use 19 invariants (n_{Q_i}, $i=1,...,19$) to classify the four qubit pure states in 82 class that by regardless of permutation of qubits, remain the 27 class shown in Table 4. In this section, we compare the n_{Q_i}, $i=1,...,19$, related to classes. The result is plotted in Fig. 1. In this figure, both classes that are comparable to each other, are connected by a line. Furthermore, if all of the n_{Q_i}, $i=1,...,19$ of one class be greater than the other class, then the first one is placed under the second. To illustrate how to draw this figure, we give an example here:

Consider the class C_9 whose dimensions of kernels are (0, 0, 0, 0, 0, 0, 0, 0, 0, 9, 0, 0, 9, 0, 0, 0, 0, 0, 9). This class is comparable to classes C_3, C_1 and C_0 whose dimensions of kernels are written in Table 4. On the other hand, all of the 19 dimensions belong to C_9 are less than the related dimensions in C_3, C_1 and C_0, so is placed on the these three classes.

In the rest of this section, we classify the four qubit pure states by means of new invariants. To do this, consider the arbitrary of four qubits:

$$v=\sum_{j_1=1}^{2}...\sum_{j_4=1}^{2}v_{j_1,...,j_4}e_{1,j_1}\otimes...\otimes e_{4,j_4},$$

and the related four independent classical invariants [23]:

$$\begin{aligned}h_1=&v_{1,1,1,1}v_{2,2,2,2}-v_{1,1,1,2}v_{2,2,2,1}-v_{1,1,2,1}v_{2,2,1,2}\\&+v_{1,1,2,2}v_{2,2,1,1}-v_{1,2,1,1}v_{2,1,2,2}+v_{1,2,1,2}v_{2,1,2,1}\\&+v_{1,2,2,1}v_{2,1,1,2}-v_{1,2,2,2}v_{2,1,1,1}\end{aligned} \tag{10}$$

$$h_2=\begin{vmatrix}v_{1,1,1,1}&v_{1,2,1,1}&v_{2,1,1,1}&v_{2,2,1,1}\\v_{1,1,1,2}&v_{1,2,1,2}&v_{2,1,1,2}&v_{2,2,1,2}\\v_{1,1,2,1}&v_{1,2,2,1}&v_{2,1,2,1}&v_{2,2,2,1}\\v_{1,1,2,2}&v_{1,2,2,2}&v_{2,1,2,2}&v_{2,2,2,2}\end{vmatrix} \tag{11}$$

$$h_3=\begin{vmatrix}v_{1,1,1,1}&v_{2,1,1,1}&v_{1,1,2,1}&v_{2,1,2,1}\\v_{1,1,1,2}&v_{2,1,1,2}&v_{1,1,2,2}&v_{2,1,2,2}\\v_{1,2,1,1}&v_{2,2,1,1}&v_{1,2,2,1}&v_{2,2,2,1}\\v_{1,2,1,2}&v_{2,2,1,2}&v_{1,2,2,2}&v_{2,2,2,2}\end{vmatrix} \tag{12}$$

$$h_4=det(h_{4,i,j})_{1\le i\le 3;1\le j\le 3},$$

$$h_{4,1,1}=-v_{1,1,1,2}v_{1,1,2,1}+v_{1,1,1,1}v_{1,1,2,2},$$

$$\begin{aligned}h_{4,1,2}=&v_{1,1,2,2}v_{1,2,1,1}-v_{1,1,2,1}v_{1,2,1,2}\\&-v_{1,1,1,2}v_{1,2,2,1}+v_{1,1,1,1}v_{1,2,2,2},\end{aligned}$$

$$h_{4,1,3}=-v_{1,2,1,2}v_{1,2,2,1}+v_{1,2,1,1}v_{1,2,2,2},$$

$$\begin{aligned}h_{4,2,1}=&v_{1,1,2,2}v_{2,1,1,1}-v_{1,1,2,1}v_{2,1,1,2}\\&-v_{1,1,1,2}v_{2,1,2,1}+v_{1,1,1,1}v_{2,1,2,2},\end{aligned}$$

$$\begin{aligned}h_{4,2,2}=&v_{1,2,2,2}v_{2,1,1,1}-v_{1,2,2,1}v_{2,1,1,2}\\&-v_{1,2,1,2}v_{2,1,2,1}+v_{1,2,1,1}v_{2,1,2,2}\\&+v_{1,1,2,2}v_{2,2,1,1}-v_{1,1,2,1}v_{2,2,1,2}\\&-v_{1,1,1,2}v_{2,2,2,1}+v_{1,1,1,1}v_{2,2,2,2},\end{aligned}$$

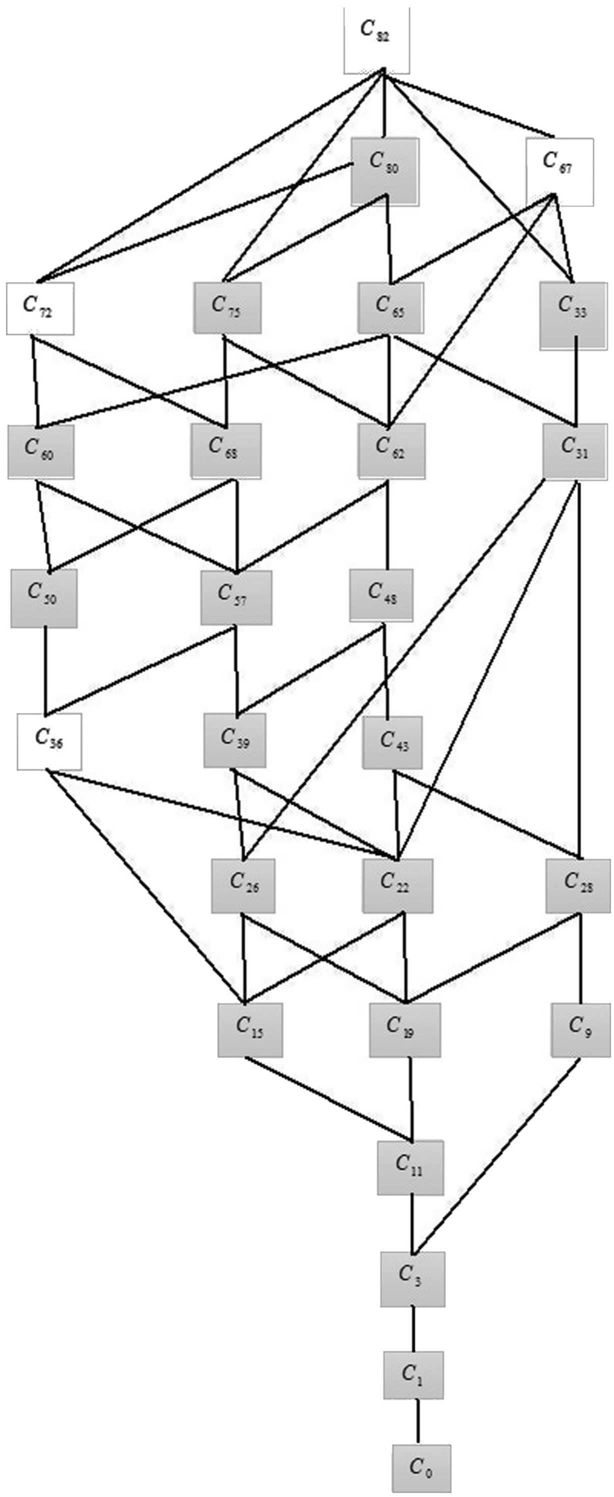

Fig. 1. Hierarchy of quantum entanglement of four qubit states.

$$
\begin{aligned}
h_{4,2,3} &= v_{1,2,2,2}v_{2,2,1,1} - v_{1,2,2,1}v_{2,2,1,2} \\
&- v_{1,2,1,2}v_{2,2,2,1} + v_{1,2,1,1}v_{2,2,2,2}, \\
h_{4,3,1} &= -v_{2,1,1,2}v_{2,1,2,1} + v_{2,1,1,1}v_{2,1,2,2}, \\
h_{4,3,2} &= v_{2,1,2,2}v_{2,2,1,1} - v_{2,1,2,1}v_{2,2,1,2} \\
&- v_{2,1,1,2}v_{2,2,2,1} + v_{2,1,1,1}v_{2,2,2,2}, \\
h_{4,3,3} &= -v_{2,2,1,2}v_{2,2,2,1} + v_{2,2,1,1}v_{2,2,2,2}. \qquad (13)
\end{aligned}
$$

Note that h_1 is the same polynomial invariant of degree two with factor 2. We use these invariants and also Wootters concurrence of all possible two qubit reduce density matrices of the four qubit states ψ_{ABCD} consisted of $C_{AB}, C_{AC}, C_{AD}, C_{BC}, C_{BD}, C_{CD}$ to classify the entanglement classes without considering permutation of particles. The results are in Table 1. This table (regardless of the last column) shows that all of the classes can be classified by means of 10 invariants, expect that for the sets of $\{C_1, C_{15}, C_{50}\}$ and $\{C_9, C_{43}\}$, these invariants are the same. To classify these sets, we use the GM concurrence added in the last column of Table 1. So, by considering this invariant, only the set of $\{C_1, C_{15}\}$ is not classified. In order to complete the entanglement classification, we compute the three tangle of reduce density matrices for both of classes. For a pure three qubit state $|\psi\rangle = \sum_{i,j,k=0}^{1} a_{ijk} \mid ijk\rangle$, its three tangle is defined by:

$$\tau_3(|\psi\rangle) = 4 \mid d_1 - 2d_2 + 4d_3 \mid, \qquad (14)$$

where

$$
\begin{aligned}
d_1 &= a_{000}^2 a_{111}^2 + a_{001}^2 a_{110}^2 + a_{010}^2 a_{101}^2 + a_{100}^2 a_{011}^2, \\
d_2 &= a_{000}a_{111}a_{011}a_{100} + a_{000}a_{111}a_{101}a_{010} \\
&+ a_{000}a_{111}a_{110}a_{001} + a_{011}a_{100}a_{101}a_{010} \\
&+ a_{011}a_{100}a_{110}a_{001} + a_{101}a_{010}a_{110}a_{001}, \\
d_3 &= a_{000}a_{110}a_{101}a_{011} + a_{111}a_{001}a_{010}a_{100}
\end{aligned}
$$

Now, we compute the three tangle of C_{15}:

$$|C_{15}\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1110\rangle)$$

$$\rho_{ABC} = tr_D(|C_{15}\rangle\langle C_{15}|) = |GHZ\rangle\langle GHZ|$$

$$\tau_3(ABC) = 1$$

and similarly, it can easily be shown that

$$\tau_3(ABD) = 0; \tau_3(ACD) = 0; \tau_3(BCD) = 0$$

but C_1 is separable state, so all of its possible three tangles are zero, and then the classification of four qubit pure states complete.

5. Investigation of the convex combinations of some classes with particular property

Some of classes in Table 1 have certain property. In these classes only one of the h_i and the GM concurrence are nonzero. We rewrite Table 1 by adding one-tangle. Entanglement of a specific qubit with the remaining qubits is an important issue in studying monogamy of N-qubit entanglement [34]. This kind of measure is called one-tangle. The one-tangle of a pure N-qubit system $|\psi_{ijk..n}\rangle$ is defined as $\tau_{i|jk...N} = 4Det(\rho_i)$ which $\rho_i = tr_{jk..n}(|\psi_{ijk..n}\rangle\langle\psi_{ijk..n}|)$. In this section, we minimize over all possible one-tangles of a system and compute this measure for the classes of four qubit states and compare the results with the genuine entanglement measure. The results are shown in Table 2. This table shows that in many classes, the one-tangle is equal to square of genuine entanglement measure.

In the following, we focus our attention in Table 1 on classes in which at most one of the invariants is zero, and investigate the relationship between the h_i, GM concurrence and one-tangle in the mixed states of these classes.

Table 1
Classification of four qubit pure states.

Class	h_1	h_2	h_3	h_4	C_{AB}	C_{AC}	C_{AD}	C_{BC}	C_{BD}	C_{CD}	C_{GM}
C_1	0	0	0	0	0	0	0	0	0	0	0
C_3	0	0	0	0	0	$\neq 0$	0	0	0	0	0
C_9	$\neq 0$	$\neq 0$	0	0	0	$\neq 0$	0	0	$\neq 0$	0	0
C_{11}	0	0	0	0	$\neq 0$	$\neq 0$	0	$\neq 0$	0	0	0
C_{15}	0	0	0	0	0	0	0	0	0	0	0
C_{19}	0	0	0	0	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$
C_{22}	0	0	0	0	0	0	0	0	$\neq 0$	0	$\neq 0$
C_{26}	$\neq 0$	0	0	0	0	0	0	0	0	0	$\neq 0$
C_{28}	$\neq 0$	$\neq 0$	0	0	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$
C_{31}	$\neq 0$	$\neq 0$	0	$\neq 0$	$\neq 0$	0	$\neq 0$	$\neq 0$	0	$\neq 0$	$\neq 0$
C_{33}	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	0	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	0	$\neq 0$
C_{36}	0	0	0	0	0	0	$\neq 0$	0	$\neq 0$	0	$\neq 0$
C_{39}	$\neq 0$	0	0	0	0	$\neq 0$	0	0	$\neq 0$	0	$\neq 0$
C_{43}	$\neq 0$	$\neq 0$	0	0	0	$\neq 0$	0	0	$\neq 0$	0	$\neq 0$
C_{48}	0	$\neq 0$	0	0	0	0	0	0	0	0	$\neq 0$
C_{50}	0	0	0	0	0	0	0	0	0	0	$\neq 0$
C_{57}	$\neq 0$	0	0	0	$\neq 0$	$\neq 0$	0	0	$\neq 0$	$\neq 0$	$\neq 0$
C_{60}	$\neq 0$	0	0	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$
C_{62}	0	$\neq 0$	0	0	$\neq 0$	0	$\neq 0$	$\neq 0$	0	$\neq 0$	$\neq 0$
C_{65}	$\neq 0$	$\neq 0$	0	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$
C_{67}	$\neq 0$	$\neq 0$	$\neq 0$	0	0	$\neq 0$	0	0	$\neq 0$	0	$\neq 0$
C_{68}	$\neq 0$	0	0	0	$\neq 0$	$\neq 0$	0	$\neq 0$	0	0	$\neq 0$
C_{72}	0	0	0	$\neq 0$	0	0	0	0	0	0	$\neq 0$
C_{75}	0	$\neq 0$	0	0	0	$\neq 0$	$\neq 0$	0	0	$\neq 0$	$\neq 0$
C_{80}	0	$\neq 0$	0	$\neq 0$	0	$\neq 0$	0	0	0	0	$\neq 0$
C_{82}	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	0	0	$\neq 0$	$\neq 0$	$\neq 0$

Table 2
Comparison of one-tangle and GM concurrence.

Classes	One-tangle	$C_{GM}(\|\psi\rangle)$
C_1	0	0
C_3	0	0
C_9	1	0
C_{11}	0	0
C_{15}	0	0
C_{19}	3/4	$\sqrt{3}/2$
C_{22}	8/9	$2\sqrt{2}/3$
C_{26}	1	1
C_{28}	24/25	4/5
C_{31}	1	$\sqrt{5/6}$
C_{33}	1	$\min\left[1, \sqrt{\frac{3(1+c)^2(1+c^2)}{2(1+c+c^2)^2}}, \sqrt{\frac{3+6c(1+c)}{2(1+c+c^2)^2}}, \sqrt{\frac{3c^2(2+c(2+c))}{2(1+c+c^2)^2}}\right]$
C_{36}	3/4	$\sqrt{3}/2$
C_{39}	8/9	2/3
C_{43}	24/25	$2\sqrt{2}/5$
C_{48}	1	1
C_{50}	3/4	$\sqrt{3}/2$
C_{57}	3/4	$\sqrt{3}/2$
C_{60}	1	1
C_{62}	7/9	$\sqrt{7}/3$
C_{65}	16/25	4/5
C_{67}	8/9	8/9
C_{68}	16/25	4/5
C_{72}	24/25	$2\sqrt{6}/5$
C_{75}	4/5	$2/\sqrt{5}$
C_{80}	8/9	$2\sqrt{2}/3$
C_{82}	44/49	$2\sqrt{11}/7$

5.1. Polynomial entanglement measure of degree 2 for even qubits X states

In this subsection we compare the polynomial invariant of degree 2 and the GM concurrence [35] for the even N-qubit X density matrices. In [35], Rafsanjani and co-workers found an algebraic formula for the N-partite concurrence of N qubits in an X matrix. The general form of X density matrix is given by:

$$\rho = \sum_{i_1,\ldots,i_N=0}^{1} (p^+_{i_1\ldots i_N}|\psi^+_{i_1\ldots i_N}\rangle\langle\psi^+_{i_1\ldots i_N}| + p^-_{i_1\ldots i_N}|\psi^-_{i_1\ldots i_N}\rangle\langle\psi^-_{i_1\ldots i_N}|) \tag{15}$$

where:

$$|\psi^+_{i_1\ldots i_N}\rangle = \cos(\tfrac{\theta_{i_1\ldots i_N}}{2})|i_1\ldots i_N\rangle + e^{i\varphi_{i_1\ldots i_N}}\sin(\tfrac{\theta_{i_1\ldots i_N}}{2})|\bar{i}_1\ldots\bar{i}_N\rangle$$

$$|\psi^-_{i_1\ldots i_N}\rangle = -\sin(\tfrac{\theta_{i_1\ldots i_N}}{2})|i_1\ldots i_N\rangle + e^{i\varphi_{i_1\ldots i_N}}\cos(\tfrac{\theta_{i_1\ldots i_N}}{2})|\bar{i}_1\ldots\bar{i}_N\rangle$$

that $\bar{i} = i+1$ in modulo 2 arithmetic. Here, we consider any even-N qubits X density matrix and compute $C(\rho)$ by means of Eq. (6). The result is as the following form:

$$C(\rho) = \max\{0, |z_{i_1\ldots i_N}| - \sum_{j\neq i}\sqrt{a_{j_1\ldots j_N}b_{j_1\ldots j_N}}\}, \tag{16}$$

where $a_{j_1\ldots j_N}$ and $b_{j_1\ldots j_N}$ are 2^{N-1} diagonal elements of density matrix, and $z_{i_1\ldots i_N}$ are 2^{N-1} off-diagonal elements of this density matrix:

$$a_{j_1\ldots j_n} = p^+_{j_1\ldots j_N}\cos^2(\tfrac{\theta_{j_1\ldots j_N}}{2}) + p^-_{j_1\ldots j_N}\sin^2(\tfrac{\theta_{j_1\ldots j_N}}{2})$$

$$b_{j_1\ldots j_n} = p^-_{j_1\ldots j_N}\cos^2(\tfrac{\theta_{j_1\ldots j_N}}{2}) + p^+_{j_1\ldots j_N}\sin^2(\tfrac{\theta_{j_1\ldots j_N}}{2})$$

$$z_{i_1\ldots i_n} = \tfrac{1}{2}e^{-i\varphi_{i_1\ldots i_N}}\sin(\theta_{i_1\ldots i_N})(p^+_{i_1\ldots i_N} - p^-_{i_1\ldots i_N})$$

Eq. (16) is in full agreement with the C_{GM} in [35]. Note that in four qubit case, the $|\psi^+_{i_1\ldots i_N}\rangle$ and $|\psi^-_{i_1\ldots i_N}\rangle$ belong to the class C_{26} in which all of the h_i are zero except h_1 and GM concurrence. In this case, the h_1 (polynomial invariant degree 2) and multipartite entanglement measure are equal in both of the pure states and convex combinations of them.

5.2. The convex combination of the class C_{39}

The class C_{39} is defined as:

$$|C_{39}\rangle = \frac{1}{\sqrt{3}}(|0000\rangle + |0101\rangle + |1010\rangle).$$

The polynomial invariant of degree 2 for this class is equal to $h_1 = \frac{2}{3}$. The others h_is are zero for this class. The GM concurrence and one tangle for this class are $\frac{2}{3}$ and $\frac{8}{9}$, respectively. So, in this case the polynomial invariant of degree 2 and GM concurrence are equal. Now, the following convex combination is considered:

$$\rho_{C_{39}} = p \mid C_{39}\rangle\langle C_{39} \mid +(1-p) \mid C_{39_2}\rangle\langle C_{39_2} \mid$$

which

$$\mid C_{39_2}\rangle = \frac{1}{\sqrt{3}}(\mid 1000\rangle + \mid 1101\rangle + \mid 0010\rangle)$$

By considering the pure state $\mid \psi_{C_{39}}\rangle = \sqrt{p} \mid C_{39}\rangle + e^{i\varphi}\sqrt{1-p} \mid C_{39_2}\rangle$ and by means of the convex roof construction and minimization over the φ, the following results for $\rho_{C_{39}}$ are obtained:

$$one\text{-}tangle(\rho_{C_{39}}) = \frac{8}{9}$$

$$h_1(\rho_{C_{39}}) = \frac{2}{3}(2p-1)$$

$$C_{GM}(\rho_{C_{39}}) = \frac{2}{3}.$$

In this case, the measures of one-tangle and GM concurrence are constant and independent of p. Also, in this case, the relationship between polynomial invariant degree 2 and C_{GM} is as follows:

$$h_1(\rho_{C_{39}}) = (2p-1)C_{GM}(\rho_{C_{39}})$$

5.3. The convex combination of the class C_{57}

The class C_{57} is defined as:

$$\mid C_{57}\rangle = \frac{1}{2}(\mid 0000\rangle + \mid 0011\rangle + \mid 0110\rangle + \mid 1100\rangle)$$

The polynomial invariant of degree 2 for this class is equal to $h_1 = \frac{1}{2}$. The others h_is are zero for this class. The GM concurrence and one tangle for this class are $\frac{\sqrt{3}}{2}$ and $\frac{3}{4}$, respectively. So, in this case the one tangle and square of the GM concurrence are equal. Now, the following convex combination is considered:

$$\rho_{C_{57}} = p \mid C_{57}\rangle\langle C_{57} \mid +(1-p) \mid C_{57_2}\rangle\langle C_{57_2} \mid$$

which

$$\mid C_{57_2}\rangle = \frac{1}{2}(\mid 1000\rangle + \mid 1011\rangle + \mid 1110\rangle + \mid 0100\rangle)$$

By considering the pure state $\mid \psi_{C_{57}}\rangle = \sqrt{p} \mid C_{57}\rangle + e^{i\varphi}\sqrt{1-p} \mid C_{57_2}\rangle$ and by means of convex roof construction, the following results for $\rho_{C_{57}}$ are obtained:

$$one\text{-}tangle(\rho_{C_{57}}) = \frac{3}{4}$$

$$h_1(\rho_{C_{57}}) = \frac{1}{2}(2p-1)$$

$$C_{GM}(\rho_{C_{57}}) = \frac{\sqrt{3}}{2}$$

Similar to C_{39}, in this class the one-tangle and C_{GM} are constant, and also we have:

$$one\text{-}tangle(\rho_{C_{57}}) = C^2_{GM}(\rho_{C_{57}})$$

5.4. The convex combination of the class C_{68}

The class C_{68} is defined as:

$$\mid C_{68}\rangle = \frac{1}{\sqrt{5}}(\mid 0000\rangle + \mid 0010\rangle + \mid 0100\rangle + \mid 1000\rangle + \mid 1111\rangle)$$

The polynomial invariant of degree 2 for this class is equal to $h_1 = \frac{2}{5}$. The others h_is are zero for this class. The GM concurrence and one tangle for this class are $\frac{4}{5}$ and $\frac{16}{25}$, respectively. So, in this case the one tangle and square of the GM concurrence are equal. Now, the following convex combination is considered:

$$\rho_{C_{68}} = p \mid C_{68}\rangle\langle C_{68} \mid +(1-p) \mid C_{68_2}\rangle\langle C_{68_2} \mid$$

which

$$\mid C_{68_2}\rangle = \frac{1}{\sqrt{5}}(\mid 1110\rangle + \mid 1100\rangle + \mid 1010\rangle + \mid 0110\rangle + \mid 0001\rangle)$$

By considering the pure state $\mid \psi_{C_{68}}\rangle = \sqrt{p} \mid C_{68}\rangle + e^{i\varphi}\sqrt{1-p} \mid C_{68_2}\rangle$ and by means of convex roof construction, the following results for $\rho_{C_{68}}$ are obtained:

$$one\text{-}tangle(\rho_{C_{68}}) = \frac{16}{25}$$

$$h_1(\rho_{C_{68}}) = \frac{2}{5}(2p-1)$$

$$C_{GM}(\rho_{C_{68}}) = \frac{4}{5}$$

So:

$$one\text{-}tangle(\rho_{C_{68}}) = C^2_{GM}(\rho_{C_{68}})$$

5.5. The convex combination of the class C_{48}

The class C_{48} is defined as:

$$\mid C_{48}\rangle = \frac{1}{2}(\mid 0000\rangle + \mid 0111\rangle + \mid 1010\rangle + \mid 1101\rangle)$$

In this case, among the four invariants, the classical invariant h_2 is non-zero and equal to $h_2 = \frac{1}{16}$. The GM concurrence and one tangle for this class is equal to one. Now, we consider the following convex combination:

$$\rho_{C_{48}} = p \mid C_{48}\rangle\langle C_{48} \mid +(1-p) \mid C_{48_2}\rangle\langle C_{48_2} \mid$$

which

$$\mid C_{48_2}\rangle = \frac{1}{2}(\mid 1111\rangle + \mid 1000\rangle + \mid 0101\rangle + \mid 0010\rangle)$$

By considering the pure state $\mid \psi_{C_{48}}\rangle = \sqrt{p} \mid C_{48}\rangle + e^{i\varphi}\sqrt{1-p} \mid C_{48_2}\rangle$:

$$C_{GM}(\mid \psi_{C_{48}}) = min[\frac{3}{2} - 2p(1-p)\cos^2\varphi, 1 - 4p(1-p)\cos^2\varphi]$$

now, by minimizing over φ ($\varphi = 0, \pi$), the following result for $\rho_{C_{48}}$ is obtained:

$$C_{GM}(\rho_{C_{48}}) = |2p-1|$$

and at the same way, one can obtain:

$$one\text{-}tangle(\rho_{C_{48}}) = (2p-1)^2$$

So:

$$one\text{-}tangle(\rho_{C_{48}}) = C^2_{GM}(\rho_{C_{48}})$$

The similar calculations for the classes C_{62}, C_{75} show that the invariant h_2 is non-zero, and for the convex combination of them, the one-tangle and square of GM concurrence are equal.

5.6. The convex combination of the class C_{50}

Looking at Table 1, we get an interesting point. All of the invariants are zero for class C_{50} except the GM concurrence that its value is equal to $\frac{\sqrt{3}}{2}$. This class is as the following form:

$$|\psi_1\rangle \equiv |C_{50}\rangle = \frac{1}{2}(|0000\rangle + |0111\rangle + |1010\rangle + |1100\rangle).$$

Since the local unitary transformation does not change the class, so we apply $\sigma_x \otimes \sigma_x \otimes \sigma_x \otimes \sigma_x$ transformation, that σ_x is the Pauli matrix, to $|\psi_1\rangle$ and get:

$$|\psi_2\rangle = \frac{1}{2}(|1111\rangle + |1000\rangle + |0101\rangle + |0011\rangle).$$

Now, we are interested in studying the following convex combination:

$$\rho = p\,|\psi_1\rangle\langle\psi_1| + (1-p)\,|\psi_2\rangle\langle\psi_2| \tag{17}$$

Given that the polynomial invariant of degree 2 (h_1) is zero for both of $|\psi_1\rangle$ and $|\psi_2\rangle$, then any convex combinations of them is zero. Now, we compute the GM concurrence of ρ by means of convex roof construction [36]:

To find the convex roof of a GM concurrence, it is useful to study the pure states that are superpositions of the eigenstates of ρ

$$|p,\varphi\rangle = \sqrt{p}\,|\psi_1\rangle + \sqrt{1-p}e^{i\varphi}\,|\psi_2\rangle \tag{18}$$

The GM concurrence of these states is

$$C_{GM}(p) = min\{\sqrt{p^2-p+1}, \frac{1}{2}\sqrt{3+4p(1-p)}\}, \tag{19}$$

which is independent of the phase. The first term of Eq. (19) is convex, but not the second term. As the GM concurrence must be convex, the characteristic curve needs to be convexified. In order this, we find the intersection points of two curves:

$$\sqrt{p^2-p+1} = \frac{1}{2}\sqrt{3+4p(1-p)} \Rightarrow$$
$$p_0 = \frac{1}{4}(2-\sqrt{2}),$$
$$p_1 = \frac{1}{4}(2+\sqrt{2})$$

We divide the interval $[0,1]$ into three parts:

In the first interval $[0,p_0]$, the GM concurrence is concave, so good optimal decomposition [36] in this interval is

$$\rho(p) = \alpha\,|\psi_2\rangle\langle\psi_2| + \frac{1-\alpha}{2}\sum_{\varphi=0,\pi}|p_0,\varphi\rangle\langle p_0,\varphi| \tag{20}$$

By comparing two density matrices Eq. (17) and Eq. (20), we can find:

$$\alpha = \frac{p_0-p}{p_0},$$

so in the interval $[0,p_0]$, the optimal decomposition is as the following form:

$$\rho(p) = \frac{p_0-p}{p_0}\,|\psi_2\rangle\langle\psi_2| + \frac{p}{2p_0}\sum_{\varphi=0,\pi}|p_0,\varphi\rangle\langle p_0,\varphi| \tag{21}$$

and the average GM concurrence for this decomposition is

$$C_{GM}(p,p_0) = \frac{p_0-p}{p_0}C_{GM}(|\psi_2\rangle) + \frac{p}{p_0}C_{GM}(p_0) \tag{22}$$

that leads to:

$$C_{GM}(p,p_0) = \frac{p_0-p}{p_0}\frac{\sqrt{3}}{2} + 0.935414\frac{p}{p_0} \tag{23}$$

In the second interval $[p_0,p_1]$, the term $\sqrt{p^2-p+1}$ is convex.

In the third interval $[p_1,1]$, the GM concurrence is concave, so similar to first interval, the good optimal decomposition is

$$\rho(p) = \beta\,|\psi_1\rangle\langle\psi_1| + \frac{1-\beta}{2}\sum_{\varphi=0,\pi}|p_0,\varphi\rangle\langle p_1,\varphi| \tag{24}$$

where

$$\beta = \frac{p-p_1}{1-p_1},$$

so in the interval $[p_0,p_1]$, the optimal decomposition is as the following form:

$$\rho(p) = \frac{p-p_1}{1-p_1}\,|\psi_1\rangle\langle\psi_1| + \frac{1-p}{2(1-p_1)}\sum_{\varphi=0,\pi}|p_1,\varphi\rangle\langle p_1,\varphi| \tag{25}$$

and the average GM concurrence for this decomposition is

$$C_{GM}(p,p_1) = \frac{p-p_1}{1-p_1}\frac{\sqrt{3}}{2} + 0.935414\frac{1-p}{1-p_1} \tag{26}$$

So in general form, the optimal decomposition and the GM concurrence is as the following form:

$$\rho = \begin{cases} \frac{p_0-p}{p_0}|\psi_2\rangle\langle\psi_2| + \frac{p}{p_0}\rho_\Delta(\mathrm{p_0}) & 0 \le \mathrm{p} < \mathrm{p_0} \\ \rho_\Delta(\mathrm{p}) & \mathrm{p_0} \le \mathrm{p} < \mathrm{p_1} \\ \frac{p-p_1}{1-p_1}|\psi_1\rangle\langle\psi_1| + \frac{1-p}{1-p_1}\rho_\Delta(\mathrm{p_1}) & \mathrm{p_1} \le \mathrm{p} \le 1 \end{cases} \tag{27}$$

which

$$\rho_\Delta(\mathrm{p}) = \sum_{\varphi=0,\pi}|\psi(p,\varphi)\rangle\langle\psi(p,\varphi)|$$

and

$$C_{GM}(\rho) = \begin{cases} \frac{p_0-p}{p_0}\frac{\sqrt{3}}{2} + 0.935414\frac{p}{p_0} & 0 \le \mathrm{p} < \mathrm{p_0} \\ \sqrt{p^2-p+1} & \mathrm{p_0} \le p < p_1 \\ \frac{p-p_1}{1-p_1}\frac{\sqrt{3}}{2} + 0.935414\frac{1-p}{(1-p_1)} & \mathrm{p_1} \le \mathrm{p} < 1 \end{cases} \tag{28}$$

The function $C_{GM}(\rho)$ is not convex in general. This function is shown in Fig. 2 by black curve. Its convex hull, i.e., the convex characteristic curve [26] for $C_{GM}(\rho)$ is equal to $\frac{\sqrt{3}}{2}$ shown in Fig. 2 by red dashed line.

Also, the similar computation of genuine entanglement of mixed state Eq. (17), for one-tangle gives the following result:

$$one\text{-}tangle(\rho) = \begin{cases} \frac{p_0-p}{p_0}\frac{3}{4} + 0.875\frac{p}{p_0} & 0 \le \mathrm{p} < \mathrm{p_0} \\ p^2-p+1 & \mathrm{p_0} \le p < p_1 \\ \frac{p-p_1}{1-p_1}\frac{3}{4} + 0.875\frac{1-p}{(1-p_1)} & \mathrm{p_1} \le \mathrm{p} < 1 \end{cases} \tag{29}$$

again, since the above equation is not convex, its convex hull results $\frac{3}{4}$. So, in this class, also we have:

$$one\text{-}tangle(\rho) = C^2_{GM}(\rho)$$

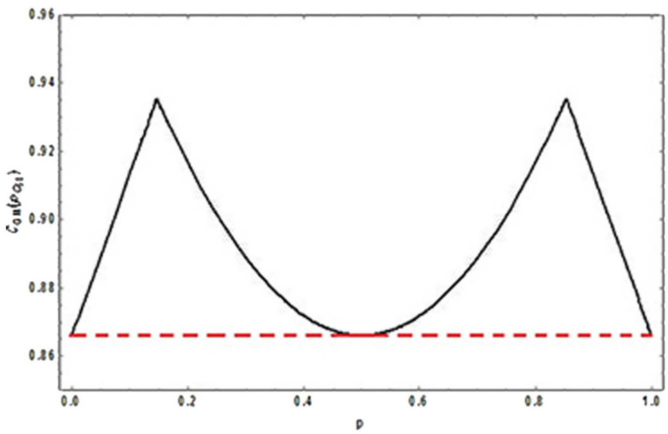

Fig. 2. GM concurrence curve for the mixed state ρ (black curve) and its convex hull (red dashed line). (For interpretation of the colors in the figure, the reader is referred to the web version of this article.)

5.7. The convex combination of the class C_{72}

The class C_{72} is defined as:

$$| C_{72}\rangle = \frac{1}{\sqrt{5}}(| 0000\rangle + | 0111\rangle + | 1001\rangle + | 1010\rangle + | 1100\rangle)$$

The invariant $h_4 = \frac{1}{125}$. The others h_is are zero for this class. The GM concurrence and one tangle for this class are $\frac{2\sqrt{6}}{5}$ and $\frac{24}{25}$, respectively. So, in this case the one tangle and square of the GM concurrence are equal. Now, the following convex combination is considered:

$$\rho_{C_{72}} = p \mid C_{72}\rangle\langle C_{72} \mid +(1-p) \mid C_{72_2}\rangle\langle C_{72_2} \mid$$

which

$$| C_{72_2}\rangle = \frac{1}{\sqrt{5}}(| 1111\rangle + | 1000\rangle + | 0110\rangle + | 0101\rangle + | 0011\rangle)$$

By considering the pure state $| \psi_{C_{72}} = \sqrt{p} \mid C_{72}\rangle + e^{i\varphi}\sqrt{1-p} \mid C_{72_2}\rangle$ and by means of convex roof construction, the following result for $\rho_{C_{72}}$ are obtained:

$$\textit{one-tangle} = \frac{12}{25}(2 - p(1-p))$$

$$C_{GM} = \sqrt{\frac{12}{25}}(\sqrt{2 - p(1-p)})$$

So,

$$\textit{one-tangle}(\rho_{C_{72}}) = C^2_{GM}(\rho_{C_{72}})$$

5.8. The convex combination of the class $C_{19} \equiv W$

Consider the set of four qubit W states that are belonging to C_{19}:

$$| W_1\rangle = \frac{1}{2}(| 0111\rangle + | 1011\rangle + | 1101\rangle + | 1110\rangle)$$

$$| W_2\rangle = \frac{1}{2}(| 0111\rangle + i \mid 1011\rangle - | 1101\rangle - i \mid 1110\rangle)$$

$$| W_3\rangle = \frac{1}{2}(| 0111\rangle - | 1011\rangle - i \mid 1101\rangle + i \mid 1110\rangle)$$

$$| W_4\rangle = \frac{1}{2}(| 0111\rangle - i \mid 1011\rangle + i \mid 1101\rangle - | 1110\rangle)$$

$$| W_5\rangle = \frac{1}{2}(| 1000\rangle + | 0100\rangle + | 0010\rangle + | 0001\rangle)$$

$$| W_6\rangle = \frac{1}{2}(| 1000\rangle + i \mid 0100\rangle - | 0010\rangle - i \mid 0001\rangle)$$

$$| W_7\rangle = \frac{1}{2}(| 1000\rangle - | 0100\rangle - i \mid 0010\rangle + i \mid 0001\rangle)$$

$$| W_8\rangle = \frac{1}{2}(| 1000\rangle - i \mid 0100\rangle + i \mid 0010\rangle - | 0001\rangle)$$

For all of the above states, four invariants h_i are zero. So, for any convex combination of these states, the h_is are zero. Also, similar to previous computations. The GM concurrence and one-tangle of the following convex combination is given by: For the following convex combinations:

$$\rho_{ij} = p \mid W_i\rangle\langle W_i \mid +(1-p) \mid W_j\rangle\langle W_j \mid$$

which $(i, j) = \{(1, 2), (1, 3), (1, 4), (5, 6), (5, 7), (5, 8)\}$, GM concurrence and one-tangle are:

$$C^2_{GM}(\rho_{ij}) = \textit{one-tangle}(\rho_{ij}) = \frac{3}{4} - p(1-p) - \sqrt{p(1-p)}$$

and for the following convex combinations:

$$\rho_{kl} = p \mid W_k\rangle\langle W_k \mid +(1-p) \mid W_l\rangle\langle W_l \mid$$

which $(k, l) = \{(1, 5), (1, 6), (1, 7), (1, 8), (2, 5), (2, 6), (2, 7), (2, 8), (3, 5), (3, 6), (3, 7), (3, 8), (4, 5), (4, 6), (4, 7), (4, 8)\}$, GM concurrence and one-tangle are:

$$C^2_{GM}(\rho_{kl}) = \textit{one-tangle}(\rho_{ij}) = \frac{3}{4}$$

and for the following convex combinations:

$$\rho_{mn} = p \mid W_m\rangle\langle W_m \mid +(1-p) \mid W_n\rangle\langle W_n \mid$$

which $(m, n) = \{(2, 3), (2, 4), (3, 4), (6, 7), (6, 8), (7, 8)\}$, GM concurrence and one-tangle are:

$$C_{GM}(\rho_{mn}) = \begin{cases} \mathrm{C(p)} & 0 \le \mathrm{p} \le 0.050146 \\ 0 & 0.050146 \le p \le 0.949854 \\ \mathrm{C(p)} & 0.949854 \le p \le 1 \end{cases}$$

$$C(p) = \sqrt{\tfrac{3}{4} - 2p(1-p) - 3\sqrt{p(1-p)}}$$

and

$$\textit{one-tangle}(\rho_{mn}) = \frac{3}{4} + p(1-p) - 2\sqrt{p(1-p)}$$

6. Conclusion

The study of the entanglement measures in the mixed states and entanglement classification are the important subjects in quantum information theory. In this paper, we have plotted the hierarchy of the four qubit pure states by means of 19 invariants presented by [23] and then have used new invariants to classify the entanglement states of four qubit pure states. In these classes, we have found the classes with the certain property in which at most one of the invariants are non-zero. We have studied the convex combination of these classes and computed the nonzero invariant, GM concurrence and one-tangle of them. We have shown that for any even-N qubits X density matrices, the GM concurrence and the polynomial invariant of degree 2 are equal. Also, we have found that in many of the classes with only one non-zero invariant, the GM concurrence is equal to the square root of the one-tangle.

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Appendix A. Classification of four qubit pure states

Consider S be a system that corresponding state space is denoted by V. This system consists of subsystems $S_1, \ldots, S_n$ that their state spaces are $V_1, .., V_n$, respectively, so that $V = V_1 \otimes \ldots \otimes V_n$. If $v \in V$, then the method that v is formed from elements of $V_1, \ldots, V_n$, characterize the entanglement properties. For example, when v can be written as $v = v_1 \otimes \ldots \otimes v_n$, which $v_i \in V_I$, then this state is called separable state, otherwise, is entangled. Buniy and co-worker in [23] use the dimension of a linear subspace V to characterize the entanglement classification of the pure four qubit states. They chose kernel or image of a linear map as linear subspace of V. Let W and W' be two vector space, and $f : W \to W'$ be linear map. So, the kernel of this map is defined as:

$$kerf = \{w \in W : f(w) = 0\}.$$

They considered the linear map as follows:

$$f(v) : W \to W' : \qquad f(v)(w) = v \otimes w^*$$

that $W \otimes W' = V$ and w^* is the dual of w. Then, they used $K(v) = kerf(v)$ and $k(v) = dimK(v)$ to describe the entanglement property of v associated with a particular choice of (W, W').

For an arbitrary vector $v \in V_1 \otimes V_2 \otimes V_3 \otimes V_4$ in its general form:

$$v = \sum_{i,j,k,l=1}^{2} v_{ijkl} e_{1,i} \otimes e_{2,j} \otimes e_{3,k} \otimes e_{4,l},$$

which e_i are the arbitrary basis of the corresponding space, the linear maps $f(v)$ are given by

$$f_1(v)(w) = \sum_{i,j,k,l=1}^{2} v_{ijkl} w_i e_{2,j} \otimes e_{3,k} \otimes e_{4,l}, \quad w \in V_1$$

$$f_2(v)(w) = \sum_{i,j,k,l=1}^{2} v_{ijkl} w_j e_{1,i} \otimes e_{3,k} \otimes e_{4,l}, \quad w \in V_2$$

$$f_3(v)(w) = \sum_{i,j,k,l=1}^{2} v_{ijkl} w_k e_{1,i} \otimes e_{2,j} \otimes e_{4,l}, \quad w \in V_3$$

$$f_4(v)(w) = \sum_{i,j,k,l=1}^{2} v_{ijkl} w_l e_{1,i} \otimes e_{2,j} \otimes e_{3,k}, \quad w \in V_4$$

$$f_{1,2}(v)(w) = \sum_{i,j,k,l=1}^{2} v_{ijkl} w_{ij} e_{3,k} \otimes e_{4,l}, \quad w \in V_1 \otimes V_2$$

$$f_{1,3}(v)(w) = \sum_{i,j,k,l=1}^{2} v_{ijkl} w_{ik} e_{2,j} \otimes e_{4,l}, \quad w \in V_1 \otimes V_3$$

$$f_{1,4}(v)(w) = \sum_{i,j,k,l=1}^{2} v_{ijkl} w_{il} e_{2,j} \otimes e_{3,k}, \quad w \in V_1 \otimes V_4$$

$$f_{2,3}(v)(w) = \sum_{i,j,k,l=1}^{2} v_{ijkl} w_{jk} e_{1,i} \otimes e_{4,l}, \quad w \in V_2 \otimes V_3$$

$$f_{2,4}(v)(w) = \sum_{i,j,k,l=1}^{2} v_{ijkl} w_{jl} e_{1,i} \otimes e_{4,l}, \quad w \in V_2 \otimes V_4$$

$$f_{3,4}(v)(w) = \sum_{i,j,k,l=1}^{2} v_{ijkl} w_{kl} e_{3,k} \otimes e_{4,l}, \quad w \in V_3 \otimes V_4$$

$$f_{1,2,3}(v)(w) = \sum_{i,j,k,l=1}^{2} v_{ijkl} w_{ijk} e_{4,l}, \quad w \in V_1 \otimes V_2 \otimes V_3$$

$$f_{1,2,4}(v)(w) = \sum_{i,j,k,l=1}^{2} v_{ijkl} w_{ijl} e_{3,k}, \quad w \in V_1 \otimes V_2 \otimes V_4$$

$$f_{1,3,4}(v)(w) = \sum_{i,j,k,l=1}^{2} v_{ijkl} w_{ikl} e_{2,j}, \quad w \in V_1 \otimes V_3 \otimes V_4$$

$$f_{2,3,4}(v)(w) = \sum_{i,j,k,l=1}^{2} v_{ijkl} w_{jkl} e_{1,i}, \quad w \in V_2 \otimes V_3 \otimes V_4$$

and the corresponding Kernels are as the following forms:

$$K_1(v) = \{w \in V_1 : \sum_{i=1}^{2} v_{ijkl} w_i = 0, \quad j,k,l \in \{1,2\}\}$$

$$K_2(v) = \{w \in V_2 : \sum_{i=1}^{2} v_{jikl} w_i = 0, \quad j,k,l \in \{1,2\}\}$$

$$K_3(v) = \{w \in V_3 : \sum_{i=1}^{2} v_{jkil} w_i = 0, \quad j,k,l \in \{1,2\}\}$$

$$K_4(v) = \{w \in V_4 : \sum_{i=1}^{2} v_{jkli} w_i = 0, \quad j,k,l \in \{1,2\}\}$$

$$K_{12}(v) = \{w \in V_1 \otimes V_2 \otimes V_3 : \sum_{i,j=1}^{2} v_{ijkl} w_{ijm} = 0\}$$

$$K_{13}(v) = \{w \in V_1 \otimes V_2 \otimes V_3 : \sum_{i,j=1}^{2} v_{ikjl} w_{ijm} = 0\}$$

$$K_{14}(v) = \{w \in V_1 \otimes V_2 \otimes V_3 : \sum_{i,j=1}^{2} v_{iklj} w_{ijm} = 0\}$$

$$K_{23}(v) = \{w \in V_1 \otimes V_2 \otimes V_3 : \sum_{i,j=1}^{2} v_{kijl} w_{ijm} = 0\}$$

$$K_{24}(v) = \{w \in V_1 \otimes V_2 \otimes V_3 : \sum_{i,j=1}^{2} v_{kilj} w_{ijm} = 0\}$$

$$K_{34}(v) = \{w \in V_1 \otimes V_2 \otimes V_3 : \sum_{i,j=1}^{2} v_{klij} w_{ijm} = 0\}$$

$$K_{123}(v) = \{w \in V_1 \otimes V_2 \otimes V_3 \otimes V_4 : \sum_{i,j,k=1}^{2} v_{ijkl} w_{ijkm} = 0\}$$

$$K_{124}(v) = \{w \in V_1 \otimes V_2 \otimes V_3 \otimes V_4 : \sum_{i,j,k=1}^{2} v_{ijlk} w_{ijmk} = 0\}$$

$$K_{134}(v) = \{w \in V_1 \otimes V_2 \otimes V_3 \otimes V_4 : \sum_{i,j,k=1}^{2} v_{iljk} w_{imjk} = 0\}$$

Table 3
The sets of Q_i related to partitioning the system into two and three subsystems in 4 qubit states.

$Q_1 = \{\{1\}\}$
$Q_2 = \{\{2\}\}$
$Q_3 = \{\{3\}\}$
$Q_4 = \{\{4\}\}$
$Q_5 = \{\{1,2\},\{1,3\},\{2,3\}\}$
$Q_6 = \{\{1,2\},\{1,4\},\{2,4\}\}$
$Q_7 = \{\{1,3\},\{1,4\},\{3,4\}\}$
$Q_8 = \{\{2,3\},\{2,4\},\{3,4\}\}$
$Q_9 = \{\{1,2\},\{1,3,4\},\{2,3,4\}\}$
$Q_{10} = \{\{1,3\},\{1,2,4\},\{2,3,4\}\}$
$Q_{11} = \{\{1,4\},\{1,2,3\},\{2,3,4\}\}$
$Q_{12} = \{\{2,3\},\{1,2,4\},\{1,3,4\}\}$
$Q_{13} = \{\{2,4\},\{1,2,3\},\{1,3,4\}\}$
$Q_{14} = \{\{3,4\},\{1,2,3\},\{1,2,4\}\}$
$Q_{15} = \{\{1,2\},\{1,3\},\{1,4\},\{2,3,4\}\}$
$Q_{16} = \{\{1,2\},\{2,3\},\{2,4\},\{1,3,4\}\}$
$Q_{17} = \{\{1,3\},\{2,3\},\{3,4\},\{1,2,4\}\}$
$Q_{18} = \{\{1,4\},\{2,4\},\{3,4\},\{1,2,3\}\}$
$Q_{19} = \{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\}$

$$K_{234}(v) = \{w \in V_1 \otimes V_2 \otimes V_3 \otimes V_4 : \sum_{i,j,k=1}^{2} v_{lijk} w_{mijk} = 0\}$$

Note that in the above equalities $l, m \in \{1,2\}$. In [23], the authors considered the following generating set:

$$N''(v) = (n_{Q_1}, n_{Q_2}, \ldots, n_{Q_{19}}) \tag{A.1}$$

where Q_i are shown in Table 3 that the sets $Q_1, \ldots, Q_4$ and $Q_5, \ldots, Q_8$ are related to partitioning the system into two and three subsystems, respectively. Partitions into four subsystems are given by the $Q_9, \ldots, Q_{19}$. The parameters n_{Q_i} are the dimensions of the following kernels:

$$n_{Q_i} = \dim K_i(v), \quad i = 1,2,3,4$$
$$n_{Q_5} = \dim (K_{12}(v) \bigcup K_{13}(v) \bigcup K_{23}(v))$$
$$n_{Q_6} = \dim (K_{12}(v) \bigcup K_{14}(v) \bigcup K_{24}(v))$$
$$n_{Q_7} = \dim (K_{13}(v) \bigcup K_{14}(v) \bigcup K_{34}(v))$$
$$n_{Q_8} = \dim (K_{23}(v) \bigcup K_{24}(v) \bigcup K_{34}(v))$$
$$n_{Q_9} = \dim (K_{12}(v) \bigcup K_{134}(v) \bigcup K_{234}(v))$$
$$n_{Q_{10}} = \dim (K_{13}(v) \bigcup K_{124}(v) \bigcup K_{234}(v))$$
$$n_{Q_{11}} = \dim (K_{14}(v) \bigcup K_{123}(v) \bigcup K_{234}(v))$$
$$n_{Q_{12}} = \dim (K_{23}(v) \bigcup K_{124}(v) \bigcup K_{134}(v))$$
$$n_{Q_{13}} = \dim (K_{24}(v) \bigcup K_{123}(v) \bigcup K_{134}(v))$$
$$n_{Q_{14}} = \dim (K_{34}(v) \bigcup K_{123}(v) \bigcup K_{124}(v))$$
$$n_{Q_{15}} = \dim (K_{12}(v) \bigcup K_{13}(v) \bigcup K_{14}(v) \bigcup K_{234}(v))$$
$$n_{Q_{16}} = \dim (K_{12}(v) \bigcup K_{23}(v) \bigcup K_{24}(v) \bigcup K_{134}(v))$$
$$n_{Q_{17}} = \dim (K_{13}(v) \bigcup K_{23}(v) \bigcup K_{34}(v) \bigcup K_{124}(v))$$
$$n_{Q_{18}} = \dim (K_{14}(v) \bigcup K_{24}(v) \bigcup K_{34}(v) \bigcup K_{123}(v))$$
$$n_{Q_{19}} = \dim (K_{123}(v) \bigcup K_{124}(v) \bigcup K_{134}(v) \bigcup K_{234}(v))$$

Table 4
27 different entanglement classes in four qubit.

		$N''(v) = (n_{Q_1}, n_{Q_2}, \ldots, n_{Q_{19}})$
A_1	C_0	(2, 2, 2, 2, 8, 8, 8, 8, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16)
A_2	C_1	(1, 1, 1, 1, 4, 4, 4, 4, 10, 10, 10, 10, 10, 10, 8, 8, 8, 8, 11)
A_3	C_3	(0, 1, 0, 1, 3, 2, 3, 2, 7, 9, 7, 7, 10, 7, 3, 7, 3, 7, 10)
A_4	C_9	(0, 0, 0, 0, 0, 0, 0, 0, 0, 9, 0, 0, 9, 0, 0, 0, 0, 0, 9)
A_5	C_{11}	(0, 0, 0, 1, 1, 2, 2, 2, 5, 5, 7, 5, 7, 7, 2, 2, 2, 7, 8)
A_6	C_{15}	(0, 0, 0, 1, 0, 2, 2, 2, 4, 4, 7, 4, 7, 7, 2, 2, 2, 7, 7)
A_7	C_{19}	(0, 0, 0, 0, 1, 1, 1, 1, 5, 5, 5, 5, 5, 5, 2, 2, 2, 2, 7)
A_8	C_{22}	(0, 0, 0, 0, 0, 1, 0, 1, 2, 4, 2, 2, 5, 2, 1, 1, 1, 1, 6)
A_9	C_{26}	(0, 0, 0, 0, 0, 0, 0, 0, 4, 4, 4, 4, 4, 4, 0, 0, 0, 0, 6)
A_{10}	C_{28}	(0, 0, 0, 0, 0, 0, 0, 0, 0, 5, 0, 0, 5, 0, 0, 0, 0, 0, 6)
A_{11}	C_{31}	(0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 6)
A_{12}	C_{33}	(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 6)
A_{13}	C_{36}	(0, 0, 0, 0, 0, 1, 0, 0, 2, 2, 2, 2, 2, 2, 1, 1, 0, 1, 5)
A_{14}	C_{39}	(0, 0, 0, 0, 0, 0, 0, 0, 2, 4, 2, 2, 4, 2, 0, 0, 0, 0, 5)
A_{15}	C_{43}	(0, 0, 0, 0, 0, 0, 0, 0, 0, 4, 0, 0, 5, 0, 0, 0, 0, 0, 5)
A_{16}	C_{48}	(0, 0, 0, 0, 0, 0, 0, 0, 0, 4, 0, 0, 4, 0, 0, 0, 0, 0, 5)
A_{17}	C_{50}	(0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 2, 1, 2, 2, 0, 0, 0, 1, 4)
A_{18}	C_{57}	(0, 0, 0, 0, 0, 0, 0, 0, 2, 2, 2, 1, 2, 2, 0, 0, 0, 0, 4)
A_{19}	C_{60}	(0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 4)
A_{20}	C_{62}	(0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 0, 0, 2, 0, 0, 0, 0, 0, 4)
A_{21}	C_{65}	(0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 4)
A_{22}	C_{67}	(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 4)
A_{23}	C_{68}	(0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 2, 1, 2, 2, 0, 0, 0, 0, 3)
A_{24}	C_{72}	(0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 3)
A_{25}	C_{75}	(0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 2, 0, 0, 0, 0, 0, 3)
A_{26}	C_{80}	(0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 3)
A_{27}	C_{82}	(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3)

By using these n_{Q_i}, they classified the four qubit systems in 82 classes. Some of these classes have the same amount of entanglement, for example the classes C_{15}, C_{16}, C_{17} and C_{18}, have the same amount of entanglement, which are in different classes in Table 8 of [14]. These classes are related together with the permutation of qubits. So, regardless of permutations of qubits, there are 27 classes that are shown in Table 4. In this table C_1 is related to full separable state that has the maximum values of n_{Q_i}, $i = 1, \ldots, 19$ (except for C_0 which is zero state). After that, the dimension of kernels (n_{Q_i}, $i = 1, \ldots, 19$) are gradually decreasing.

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