2dof Equations of State, Basic Overview

References:

$$\theta = \text{tilt about the y axis}, \quad \phi = \text{tilt about the x axis}$$

$$r_{cm} = \text{length to the center of mass}, \quad l = \text{length of pendulum}$$

$$L = T - U = KE - PE$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = Q$$

Kinematics:

$$x = r_{cm} \sin\theta \cos\phi, \quad y = r_{cm} \sin\theta, \quad z = r_{cm} \cos\theta \cos\phi$$

$$\dot{x} = r_{cm}\dot{\theta} \cos\theta \cos\phi - r_{cm}\dot{\phi} \sin\theta \sin\phi$$

$$\dot{y} = r_{cm}\dot{\phi} \cos\phi$$

$$\dot{z} = -r_{cm}\dot{\theta} \sin\theta \cos\phi - r_{cm}\dot{\phi} \cos\theta \sin\phi$$

$$T = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m(\dot{x} + \dot{y} + \dot{z}) = \tfrac{1}{2}mr_{cm}^2(\dot{\phi}^2 + \dot{\theta}^2 \cos^2\phi)$$

$$U = mgz = mgr_{cm} \cos\theta \cos\phi$$

Lagrangian

$$L = T - U = \frac{1}{2}mr_{cm}^2\dot{\phi}^2 + \frac{1}{2}mr_{cm}^2\dot{\theta}^2 \cos^2\phi - mgr_{cm} \cos\theta \cos\phi$$

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Equations of Motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \tau_\theta$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = \tau_\phi$$

Thus:

$$mr_{cm}^2\ddot{\theta}\cos^2\phi - mr_{cm}^2\dot{\theta}\dot{\phi}\sin(2\phi) - mgr_{cm}\sin\theta\cos\phi = \tau_\theta = F_x l$$

$$mr_{cm}\ddot{\phi} + \frac{1}{2}mr_{cm}^2\dot{\theta}^2\sin(2\phi) - mgr_{cm}\cos\theta\sin\phi = \tau_\phi = F_y l$$

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State-Space Equations

Define states:

$$x_1 = \theta, \quad x_2 = \dot{\theta}, \quad y_1 = \phi, \quad y_2 = \dot{\phi}$$

Dynamics:

$$\dot{x}_1 = x_2 = \dot{\theta}$$
$$\dot{x}_2 = \ddot{\theta} = \frac{F_x l}{mr_{cm}\cos^2\phi} + \frac{\dot{\theta}\dot{\phi}\sin(2\phi)}{\cos^2\phi} + \frac{g\sin\theta}{r_{cm}\cos\phi}$$
$$\dot{y}_1 = y_2 = \dot{\phi}$$
$$\dot{y}_2 = \ddot{\phi} = \frac{F_x l}{mr_{cm}^2} - \frac{\dot{\theta}^2\sin(2\phi)}{2} + \frac{g\cos\theta\sin\phi}{r_{cm}}$$