

Discretized ODE PINN Solver

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Abstract

In this article, I tried to make a PINN [1] solver for discretized ode problem.

1 state the problem

The general autonomous system can written in the following form.

$$\dot{x} = f(x)$$

where generally $x \in \mathbb{R}^n$

This is a dynamical system where the underlying Space is \mathbb{R}^n . If denote $\Phi(x, t) = \Phi_t(x)$, then the group is $t \in \mathbb{R}$, equiped with the group operation $\Phi_{t_2}(\Phi_{t_1}(x)) = \Phi_{t_1+t_2}(x)$.

The discretized by a constant time version of this problem is by setting a constant time interval and turn this continuous problem on \mathbb{R} to \mathbb{Z} . If the discretized time is Δ then the group of this system is \mathbb{Z} and the group operation is $\Phi_{m\Delta}(\Phi_{n\Delta}(x)) = \Phi_{(n+m)\Delta}(x)$.

So the only thing we need to find for this discretized problem is $\Phi(x, \Delta)$ which is the unit of the group \mathbb{Z}

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2 derive the pde relations

For the time interval $[0, \Delta]$, The continuous problem will have partial derivatives for the solution $\Phi(x, t)$.

Which is as follow.

$$\frac{\partial \Phi}{\partial t} = f(\Phi)$$

and

$$\frac{\partial^2 \Phi}{\partial x \partial t} = f'(\Phi) \frac{\partial \Phi}{\partial x}$$

3 road map

The basic idea is to solve the pde in the previous section on $(0, \Delta)$, to get the $\Phi(x, \Delta)$.

For more continuous solution $\Phi((x, t))$, we just need to find the largest n such that $n\Delta \leq t$, then use $\Phi_{\Delta}^n(x) = \Phi(x, n\Delta)$ and then for the system is a autonomous system we can just apply one more transform, $\Phi(x, t) = \Phi_{t-n\Delta}(\Phi_{n\Delta}(x))$. The function $\Phi(x, t-n\Delta), t-n\Delta \in [0, \Delta)$ is already solved at the first time.

We can then obtain $\Phi_{2^k\Delta}$ by repeatly take the function composition, and for each k we can continue to train $\Phi_{2^k\Delta}$ base on Φ_{Δ} to get more accuracy.

Separation—————

There are 10 drones and fly on the sky obeys Newton's second law of motion. which is

$$\begin{aligned}\vec{F} &= m\vec{a} \\ \vec{a} &= \frac{d\vec{v}}{dt} = \frac{d^2\vec{x}}{dt^2}\end{aligned}$$

And I mean the policy by, we need a function of force depending on some communication between drones to decide the \vec{F}

$$\vec{F} = f(\text{thecurrentinformation})$$

And then we want the following dynamic system

$$\begin{bmatrix} \frac{d\vec{x}}{dt} \\ \frac{d\vec{v}}{dt} \end{bmatrix} = \begin{bmatrix} \vec{v} \\ \vec{a} = f/m \end{bmatrix}$$

has some Self-organized emergent phenomena, to automatically emergent a circle rounding pattern.

4 Jinrui Zhang's prompt

4.1 a simple prompt

To be more clear of the notations we use, we have $i \in \{1, 2, \dots, 10\} = N$. And the drones are ignored of its flying height, which the position vector can be a 2d vector note it as \vec{d}_i . And so the velocity and acceleration we denote as $\vec{v}_i = \frac{d\vec{d}_i}{dt}$ and $\vec{a}_i = \frac{d\vec{v}_i}{dt}$. I want to prompt a f so that it can form a circle.

$$\vec{f}_i = m_i \left(\sum_{\forall k \neq i, \|\vec{d}_i - \vec{d}_k\| \leq R} \left(\frac{\vec{d}_i - \vec{d}_k}{\|\vec{d}_i - \vec{d}_k\|^3} \right) + \left(\frac{\vec{d}_{t(i)} - \vec{d}_i}{\|\vec{d}_{t(i)} - \vec{d}_i\|} - v_i \right) \right)$$

This model is easy to explain, the first term is just a inverse square propell force, the second term is make the velocity quickly approach a set direction the $t(i)$ is just a randomly choosed target drone other than i that is $t(i) \in N, t(i) \neq i$.

This formula can be rewrite without physical term as follow.

$$\vec{a}_i = \sum_{\forall k \neq i, \|\vec{d}_i - \vec{d}_k\| \leq R} \left(\frac{\vec{d}_i - \vec{d}_k}{\|\vec{d}_i - \vec{d}_k\|^3} \right) + \left(\frac{\vec{d}_{t(i)} - \vec{d}_i}{\|\vec{d}_{t(i)} - \vec{d}_i\|} - v_i \right)$$

separately view this is combined by two independent force

$$\begin{aligned} (\vec{a}_i)_{target} &= \left(\frac{\vec{d}_{t(i)} - \vec{d}_i}{\|\vec{d}_{t(i)} - \vec{d}_i\|} - v_i \right) \\ (\vec{a}_i)_{propell} &= \sum_{\forall k \neq i, \|\vec{d}_i - \vec{d}_k\| \leq R} \left(\frac{\vec{d}_i - \vec{d}_k}{\|\vec{d}_i - \vec{d}_k\|^3} \right) \end{aligned}$$

4.2 a simple prompt:simulation

4.2.1 Four drone case:derivation

This case just choose $N = \{1, 2, 3, 4\}$ and don't allow $t(t(i)) = i$ which definitely form a three element loop and a dangling drone.

We have a (4,2)-tensor \vec{d}_i and two other (4,2)-tensor \vec{v}_i and \vec{a}_i The initial points are randomly choosed in Uniformly $[0, 1] \times [0, 1]$

choose a time increment dt and the simulation update formula is simple to write

just as follow

$$\begin{bmatrix} \vec{d}_{n+1_i} \\ \vec{v}_{n+1_i} \end{bmatrix} = \begin{bmatrix} \vec{d}_{n_i} + \vec{v}_{n_i} dt \\ \vec{v}_{n_i} + (\sum_{\forall k \neq i, \|\vec{d}_{n_i} - \vec{d}_{n_k}\| \leq R} (\frac{\vec{d}_{n_i} - \vec{d}_{n_k}}{\|\vec{d}_{n_i} - \vec{d}_{n_k}\|^3}) + (\frac{\vec{d}_{n_{t(i)}} - \vec{d}_{n_i}}{\|\vec{d}_{n_{t(i)}} - \vec{d}_{n_i}\|} - \vec{v}_{n_i})) dt \end{bmatrix}$$

simple Euler method.

4.2.2 Four drone case:code & result

4.2.3 Ten drone case:derivation

undergoing

4.2.4 Ten drone case:result

undergoing

4.3 some analysis why it will have a stability property

4.3.1 the terminate radius R

undergoing

4.3.2 the terminate center O

undergoing

4.3.3 graph theory part

The $t(i)$ forms a graph which have n points and n oriented edges, this forms a tree with a extra edges, and this case It will obviously form a Unicyclic Graph.

Which is a tree if we treat all the point on the loop as the same point.

4.4 target distance method

undergoing

4.5 target distance method:simulation

undergoing

5 Zinan Su's approach

5.1 notations & equations

safe collide radius is d_s

$$\sigma = 2d_s$$

NUM is the total number of the drones. And then we want the following dynamic system

$$\begin{bmatrix} \frac{d\vec{p}_i}{dt} \\ \frac{d\vec{v}_i}{dt} \end{bmatrix} = \begin{bmatrix} \vec{v}_i \\ \vec{a}_i \end{bmatrix}$$

circle origin is a function

$$c = \frac{1}{NUM} \sum_{k=1}^{NUM} p_k$$

Four constants.

$$k_p =$$

$$k_d =$$

$$k_v =$$

$$k_r =$$

$$R^* = \frac{1}{NUM} \sum_{k=1}^{NUM} p_k(0) - c(0)$$

$$v_d = \frac{1}{NUM} \sum_{k=1}^{NUM} \|v_k(0)\|$$

and

$$r_i = p_i - c$$

$$\begin{aligned}
d_i &= \|r_i\| \\
\hat{r}_i &= \frac{r_i}{d_i} \\
\hat{\theta}_i &= \mathbb{M}_\theta r_i \\
\mathbb{M}_\theta &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\
v_{i\parallel} &= \hat{r}_i \cdot v_i \\
v_{i\perp} &= \hat{\theta}_i \cdot v_i \\
U(r) &= k_r e^{-\frac{r}{2\sigma^2}} \\
U_{ij} &= U(\|p_i - p_j\|) \\
\vec{u}_{i1} &= [-k_p(d_i - R^*) - k_d v_{i\parallel}] \hat{r}_i \\
\vec{u}_{i2} &= [-k_v(v_{i\perp} - v_d)] \hat{\theta}_i \\
\vec{u}_{i3} &= \sum_{\forall k \neq i} (-\nabla_{p_i} U_{ij}) \\
\vec{u}_i &= \vec{u}_{i1} + \vec{u}_{i2} + \vec{u}_{i3}
\end{aligned}$$

5.2 analysis

考虑带阻力的动力方程

$$m \frac{d^2 \mathbf{p}_i}{dt^2} + c \left\| \frac{d\mathbf{p}_i}{dt} \right\| \frac{d\mathbf{p}_i}{dt} = \mathbf{u}_i,$$

其中 m 为质量, c 为阻力系数, 即

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X}), \quad (1)$$

其中

$$\mathbf{X} = \begin{pmatrix} \mathbf{p} \\ \mathbf{v} \end{pmatrix} \in \mathbb{R}^{40}, \quad \mathbf{p} = \begin{pmatrix} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_{10} \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_{10} \end{pmatrix}, \quad \mathbf{F}(\mathbf{X}) = \begin{pmatrix} \mathbf{v} \\ m^{-1}(\mathbf{u} - c\|\mathbf{v}\|\mathbf{v}) \end{pmatrix}.$$

易知质心 $\mathbf{c}(\mathbf{X})$, $\|r_i\|$, \hat{r}_i , Φ 均为 Lipschitz 连续函数, 且初值条件满足

$$\min_{i \neq j} \|\mathbf{p}_i(0) - \mathbf{p}_j(0)\| > d_s > 0.$$

由 Picard 定理可知方程 (1) 的解存在且唯一, 解为

$$\mathbf{X}(t) = \mathbf{X}(0) + \int_0^t \mathbf{F}(\mathbf{X}(s))ds,$$

即

$$\begin{aligned} \mathbf{v}_i(t) &= \mathbf{v}_i(0) + \int_0^t \left[m^{-1} \mathbf{u}_i(\mathbf{X}(s)) - \frac{c}{m} \|\mathbf{v}_i(s)\| \mathbf{v}_i(s) \right] ds, \\ \mathbf{p}_i(t) &= \mathbf{p}_i(0) + \int_0^t \mathbf{v}_i(s) ds. \end{aligned}$$

下面证明防撞性. 定义

$$\begin{aligned} \Psi(d) &= k_r \exp \left\{ -\frac{(d - d_s)^2}{2\sigma^2} \right\}, \\ \Phi(d) &= -\frac{d\Psi}{dd} = \frac{k_r}{\sigma^2} \exp \left\{ -\frac{(d - d_s)^2}{2\sigma^2} \right\} (d - d_s), \\ E_{ij}(t) &= \frac{1}{2} \left(\frac{dd_{ij}}{dt} \right)^2 + \psi(d_{ij}(t)), \end{aligned}$$

其中 $d_{ij}(t) = \|\mathbf{p}_i(t) - \mathbf{p}_j(t)\|$. 则系统能量为

$$\mathcal{E}(t) = \sum_{1 \leq i < j \leq N} E_{ij}(t).$$

而

$$\begin{aligned} \frac{dE_{ij}}{dt} &= \frac{dd_{ij}}{dt} \cdot \frac{d^2d_{ij}}{dt^2} + \Phi(d_{ij}) \frac{dd_{ij}}{dt}, \\ \frac{d^2d_{ij}}{dt^2} &= \frac{1}{d_{ij}} \left[\|\mathbf{v}_i - \mathbf{v}_j\|^2 + (\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{u}_i - \mathbf{u}_j) - \left(\frac{dd_{ij}}{dt} \right)^2 \right], \end{aligned} \quad (2)$$

其中 $\mathbf{u}_i = \dot{\mathbf{v}}_i$. 对于 \mathbf{u}_i , 有

$$\mathbf{u}_i = \frac{1}{m} [-k_p(d_i - R^*)\hat{\mathbf{r}}_i - k_d v_{r,i} \hat{\mathbf{r}}_i - k_v(v_{\theta,i} - v_d)\hat{\boldsymbol{\theta}}_i] + \frac{1}{m} \sum_{k \neq i} \Phi(d_{ik})(\mathbf{p}_i - \mathbf{p}_k) =: \mathbf{u}_{i_1} + \mathbf{u}_{i_2}.$$

设 $\|\mathbf{u}_{i_1} - \mathbf{u}_{j_1}\| \leq L$. 取

$$k_r > L\sigma^2 e^{\frac{1}{2}} \max \left\{ \frac{1}{d_s}, \frac{1}{\min_{k \neq l} d_{kl}(0)} \right\},$$

则当 $d_{ij} \leq d_s + \sigma$ 时, 有

$$\|\mathbf{u}_{i_2} - \mathbf{u}_{j_2}\| > 2L.$$

于是

$$(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{u}_i - \mathbf{u}_j) \geq \|\mathbf{u}_{i_2} - \mathbf{u}_{j_2}\| d_{ij} - L d_{ij} > L d_{ij}.$$

代入 (2) 式有

$$\frac{dE_{ij}}{dt} \geq \frac{dd_{ij}}{dt} (L + \Phi(d_{ij})) > 0.$$

由此即知

$$\frac{d\mathcal{E}}{dt} \geq -\kappa \mathcal{E}(t),$$

其中 $\kappa > 0$ 为常数. 而

$$\mathcal{E}(0) \geq \sum_{i < j} \Psi(d_{ij}(0)) > \psi(d_s + \sigma) \cdot \binom{N}{2},$$

$$\mathcal{E}(t) \geq \mathcal{E}(0)e^{-\kappa t} > 0,$$

$$\Psi(d_{ij}(t)) \leq E_{ij}(t) \leq \mathcal{E}(t).$$

由 Ψ 严格单调递减可知

$$d_{ij}(t) \geq \Psi^{-1}(\mathcal{E}(t)) > \Psi^{-1}(\mathcal{E}(0)e^{-\kappa t}).$$

令 $t \rightarrow \infty$, 有

$$\lim_{t \rightarrow \infty} d_{ij}(t) \geq \Psi^{-1}(0) = d_s.$$

故总是不会相撞.

下面讨论收敛性, 即讨论系统收敛至

$$\mathcal{S} = \{\|\mathbf{r}_i\| = R, \mathbf{v}_i \cdot \hat{\mathbf{r}}_i = 0, \|\mathbf{v}_i\| = v_d\}.$$

构造 Lyapunov 函数

$$V = \frac{1}{2} \sum_{i=1}^N [k_p(d_i - R)^2 + \|\mathbf{v}_i - v_d \hat{\boldsymbol{\theta}}_i\|^2] + \sum_{i < j} \Psi(d_{ij}),$$

则

$$\dot{V} = - \sum_i k_d v_{r,i}^2 - \sum_i k_v (v_{\theta,i} - v_d)^2 - \sum_i c \|\mathbf{v}_i\|^3 \leq 0,$$

故方程渐进收敛至 \mathcal{S} .

5.3 some constants calculation

undergoing c is air resistance constant.

$$m \frac{d^2 \vec{p}_i}{dt^2} + c \left\| \frac{d\vec{p}_i}{dt} \right\| \frac{d\vec{p}_i}{dt} = \mathbf{u}_i$$

References

- [1] Maziar Raissi, Paris Perdikaris, and George Em Karniadakis. Physics informed deep learning (part i): Data-driven solutions of nonlinear partial differential equations. *arXiv preprint arXiv:1711.10561*, 2017.