Discretized ODE PINN Solver

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Abstract

In this article, I tried to make a PINN [1] solver for discretized ode problem.

1 state the problem

The general autonomous system can written in the following form.

$$\dot{x} = f(x)$$

where generally $x \in \mathbb{R}^n$

This is a dynamical system where the underlying Space is $\mathbb{R}^n$. If denote $\Phi(x,t) = \Phi_t(x)$, then the group is $t \in \mathbb{R}$, equiped with the group operation $\Phi_{t_2}(\Phi_{t_1}(x)) = \Phi_{t_1+t_2}(x)$.

The discretized by a constant time version of this problem is by setting a constant time interval and turn this continuous problem on $\mathbb{R}$ to $\mathbb{Z}$. If the discretized time is Δ then the group of this system is $\mathbb{Z}$ and the group operation is $\Phi_{m\Delta}(\Phi_{n\Delta}(x)) = \Phi_{(n+m)\Delta}(x)$.

So the only thing we need to find for this discretized problem is $\Phi(x, \Delta)$ which is the unit of the group $\mathbb{Z}$

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2 derive the pde relations

For the time interval $[0, \Delta]$, The continuous problem will have partial derivatives for the solution $\Phi(x, t)$.

Which is as follow.

$$\frac{\partial \Phi}{\partial t} = f(\Phi)$$

and

$$\frac{\partial^2 \Phi}{\partial x \partial t} = f'(\Phi)\frac{\partial \Phi}{\partial x}$$

3 road map

The basic idea is to solve the pde in the previous section on $(0, \Delta)$, to get the $\Phi(x, \Delta)$.

For more continuous solution $\Phi((x, t))$, we just need to find the largest n such that $n\Delta \leq t$, then use $\Phi_\Delta^n(x) = \Phi(x, n\Delta)$ and then for the system is a autonomous system we can just apply one more transform, $\Phi(x, t) = \Phi_{t-n\Delta}(\Phi_{n\Delta}(x))$. The function $\Phi(x, t-n\Delta), t-n\Delta \in [0, \Delta)$ is already solved at the first time.

Separation——————

There are 10 drones and fly on the sky obeys Newton's second law of motion. which is

$$\vec{F} = m\vec{a}$$
$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{x}}{dt^2}$$

And I mean the policy by, we need a function of force depending on some communication between drones to decide the $\vec{F}$

$$\vec{F} = f(thecurrentinformation)$$

And then we want the following dynamic system

$$\begin{bmatrix} \frac{d\vec{d}}{dt} \\ \frac{d\vec{v}}{dt} \end{bmatrix} = \begin{bmatrix} \vec{v} \\ \vec{a} = f/m \end{bmatrix}$$

has some Self-organized emergent phenomena, to automatically emergent a circle rounding pattern.

4 Jinrui Zhang's prompt

4.1 a simple prompt

To be more clear of the notations we use, we have $i \in \{1, 2, ..., 10\} = N$ And the drones are ignored of its flying height, which the position vector can be a 2d vector note it as $\vec{d_i}$ And so the velocity and acceleration we denote as $\vec{v_i} = \frac{d\vec{d_i}}{dt}$ and $\vec{a_i} = \frac{d\vec{v_i}}{dt}$ I want to prompt a f so that it can form a cirle.

$$\vec{f_i} = m_i\left(\sum_{\forall k \neq i, \|\vec{d_i} - \vec{d_k}\| \leq R} \left(\frac{\vec{d_i} - \vec{d_k}}{\|\vec{d_i} - \vec{d_k}\|^3}\right) + \left(\frac{\vec{d_{t(i)}} - \vec{d_i}}{\|\vec{d_{t(i)}} - \vec{d_i}\|} - v_i\right)\right)$$

This model is easy to explain, the first term is just a inverse square propell force, the second term is make the velocity quickly approch a set direction the $t(i)$ is just a randomly choosed target drone other than i that is $t(i) \in N, t(i) \neq i$

This formula can be rewrite without physical term as follow.

$$\vec{a_i} = \sum_{\forall k \neq i, \|\vec{d_i} - \vec{d_k}\| \leq R} \left(\frac{\vec{d_i} - \vec{d_k}}{\|\vec{d_i} - \vec{d_k}\|^3}\right) + \left(\frac{\vec{d_{t(i)}} - \vec{d_i}}{\|\vec{d_{t(i)}} - \vec{d_i}\|} - v_i\right)$$

separately view this is combined by two independent force

$$(\vec{a_i})_{target} = \left(\frac{\vec{d_{t(i)}} - \vec{d_i}}{\|\vec{d_{t(i)}} - \vec{d_i}\|} - v_i\right)$$

$$(\vec{a_i})_{propell} = \sum_{\forall k \neq i, \|\vec{d_i} - \vec{d_k}\| \leq R} \left(\frac{\vec{d_i} - \vec{d_k}}{\|\vec{d_i} - \vec{d_k}\|^3}\right)$$

4.2 a simple prompt:simulation

4.2.1 Four drone case:derivation

This case just choose $N = \{1, 2, 3, 4\}$ and don't allow $t(t(i)) = i$ which definitely form a three element loop and a dangling drone.

We have a (4,2)-tensor $\vec{d_i}$ and two other (4,2)-tensor $\vec{v_i}$ and $\vec{a_i}$ The initial points are randomly choosed in Uniformly $[0, 1] \times [0, 1]$

choose a time increment dt and the simulation update formula is simple to write

just as follow

$$\begin{bmatrix} \vec{d_{n+1}}_i \\ \vec{v_{n+1}}_i \end{bmatrix} = \begin{bmatrix} \vec{d_n}_i + \vec{v_n}_i dt \\ \vec{v_n}_i + (\sum_{\forall k \neq i, \|\vec{d_n}_i - \vec{d_n}_k\| \leq R} (\frac{\vec{d_n}_i - \vec{d_n}_k}{\|\vec{d_n}_i - \vec{d_n}_k\|^3}) + (\frac{\vec{d_n}_{t(i)} - \vec{d_n}_i}{\|\vec{d_n}_{t(i)} - \vec{d_n}_i\|} - v_{n\,i}))dt \end{bmatrix}$$

simple Euler method.

4.2.2 Four drone case:code & result

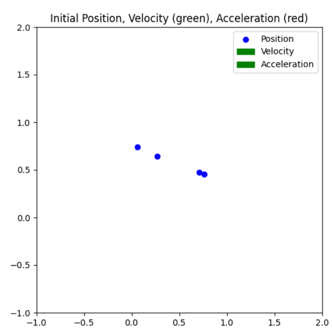

图 1: sample1 randomly initial position

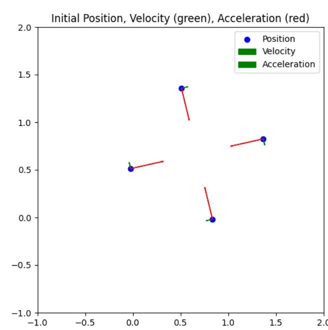

图 2: sample1 after a period of time

4.2.3 Ten drone case:derivation

undergoing

4.2.4 Ten drone case:result

undergoing

4.3 some analysis why it will have a stability property

4.3.1 the terminate radius R

undergoing

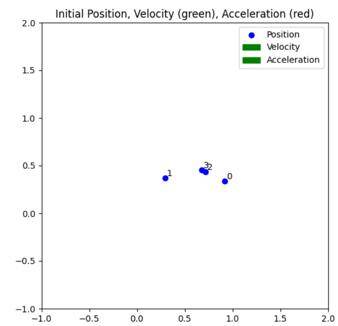

图 3: sample1 randomly initial position

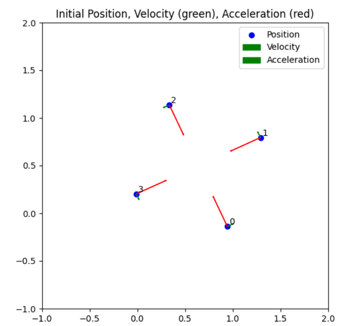

图 4: sample2 after a period of time

4.3.2 the terminate center O

undergoing

4.3.3 graph theory part

The $t(i)$ forms a graph which have n points and n oriented edges, this forms a tree with a extra edges, and this case It will obviously form a Unicyclic Graph.

Which is a tree if we treat all the point on the loop as the same point.

4.4 target distance method

undergoing

4.5 target distance method:simulation

undergoing

5 Zinan Su's approach

5.1 notations & equations

safe collide radius is d_s

$$\sigma = 2d_s$$

NUM is the total number of the drones. And then we want the following dynamic system

$$\begin{bmatrix} \frac{d\vec{p_i}}{dt} \\ \frac{d\vec{v_i}}{dt} \end{bmatrix} = \begin{bmatrix} \vec{v_i} \\ \vec{a_i} \end{bmatrix}$$

circle origin is a function

$$c = \frac{1}{NUM} \sum_{k=1}^{NUM} p_k$$

Four constants.

$$k_p =$$

$$k_d =$$

$$k_v =$$

$$k_r =$$

$$R^* = \frac{1}{NUM} \sum_{k=1}^{NUM} p_k(0) - c(0)$$

$$v_d = \frac{1}{NUM} \sum_{k=1}^{NUM} \|v_k(0)\|$$

and

$$r_i = p_i - c$$

$$d_i = \|r_i\|$$

$$\hat{r_i} = \frac{r_i}{d_i}$$

$$\hat{\theta}_i = \mathbb{M}_\theta r_i$$

$$\mathbb{M}_\theta = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$v_{i\|} = \hat{r_i} \cdot v_i$$

$$v_{i\perp} = \hat{\theta}_i \cdot v_i$$

$$U(r) = k_r e^{-\frac{r}{2\sigma^2}}$$

$$U_{ij} = U(\|p_i - p_j\|)$$

$$\vec{u_{i1}} = [-k_p(d_i - R^*) - k_d v_{i\|}]\hat{r}_i$$

$$\vec{u_{i2}} = [-k_v(v_{i\perp} - v_d)]\hat{\theta}_i$$

$$\vec{u_{i3}} = \sum_{\forall k \neq i}(-\nabla_{p_i} U_{ij})$$

$$\vec{u_i} = \vec{u_{i1}} + \vec{u_{i2}} + \vec{u_{i3}}$$

5.2 analysis

考虑带阻力的动力方程

$$m\frac{\mathrm{d}^2\boldsymbol{p}_i}{\mathrm{d}t^2} + c\left\|\frac{\mathrm{d}\boldsymbol{p}_i}{\mathrm{d}t}\right\|\frac{\mathrm{d}\boldsymbol{p}_i}{\mathrm{d}t} = \boldsymbol{u}_i,$$

其中 m 为质量, c 为阻力系数, 即

$$\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t} = \boldsymbol{F}(\boldsymbol{X}), \tag{1}$$

其中

$$\boldsymbol{X} = \begin{pmatrix}\boldsymbol{p}\\ \boldsymbol{v}\end{pmatrix} \in \mathbb{R}^{40},\ \boldsymbol{p} = \begin{pmatrix}\boldsymbol{p}_1\\ \vdots\\ \boldsymbol{p}_{10}\end{pmatrix},\ \boldsymbol{v} = \begin{pmatrix}\boldsymbol{v}_1\\ \vdots\\ \boldsymbol{v}_{10}\end{pmatrix},\ \boldsymbol{F}(\boldsymbol{X}) = \begin{pmatrix}\boldsymbol{v}\\ m^{-1}(\boldsymbol{u} - c\|\boldsymbol{v}\|\boldsymbol{v})\end{pmatrix}.$$

易知质心 $\boldsymbol{c}(\boldsymbol{X}), \|\boldsymbol{r}_i\|, \widehat{\boldsymbol{r}}_i, \mathit{\Phi}$ 均为 Lipschitz 连续函数, 且初值条件满足

$$\min_{i\neq j}\|\boldsymbol{p}_i(0) - \boldsymbol{p}_j(0)\| > d_s > 0.$$

由 Picard 定理可知方程 (1) 的解存在且唯一, 解为

$$\boldsymbol{X}(t) = \boldsymbol{X}(0) + \int_0^t \boldsymbol{F}(\boldsymbol{X}(s))\mathrm{d}s,$$

即

$$\begin{aligned}\boldsymbol{v}_i(t) &= \boldsymbol{v}_i(0) + \int_0^t \left[m^{-1}\boldsymbol{u}_i(\boldsymbol{X}(s)) - \frac{c}{m}\|\boldsymbol{v}_i(s)\|\boldsymbol{v}_i(s)\right]\mathrm{d}s,\\ \boldsymbol{p}_i(t) &= \boldsymbol{p}_i(0) + \int_0^t \boldsymbol{v}_i(s)\mathrm{d}s.\end{aligned}$$

下面证明防撞性. 定义

$$\begin{aligned}
\Psi(d) &= k_r \exp\left\{-\frac{(d-d_s)^2}{2\sigma^2}\right\}, \\
\Phi(d) &= -\frac{\mathrm{d}\Psi}{\mathrm{d}d} = \frac{k_r}{\sigma^2}\exp\left\{-\frac{(d-d_s)^2}{2\sigma^2}\right\}(d-d_s), \\
E_{ij}(t) &= \frac{1}{2}\left(\frac{\mathrm{d}d_{ij}}{\mathrm{d}t}\right)^2 + \psi(d_{ij}(t)),
\end{aligned}$$

其中 $d_{ij}(t) = \|\boldsymbol{p}_i(t) - \boldsymbol{p}_j(t)\|$. 则系统能量为

$$\mathcal{E}(t) = \sum_{1\leqslant i<j\leqslant N} E_{ij}(t).$$

而

$$\frac{\mathrm{d}E_{ij}}{\mathrm{d}t} = \frac{\mathrm{d}d_{ij}}{\mathrm{d}t}\cdot\frac{\mathrm{d}^2 d_{ij}}{\mathrm{d}t^2} + \Phi(d_{ij})\frac{\mathrm{d}d_{ij}}{\mathrm{d}t},$$

$$\frac{\mathrm{d}^2 d_{ij}}{\mathrm{d}t^2} = \frac{1}{d_{ij}}\left[\|\boldsymbol{v}_i - \boldsymbol{v}_j\|^2 + (\boldsymbol{p}_i - \boldsymbol{p}_j)\cdot(\boldsymbol{u}_i - \boldsymbol{u}_j) - \left(\frac{\mathrm{d}d_{ij}}{\mathrm{d}t}\right)^2\right], \qquad (2)$$

其中 $\boldsymbol{u}_i = \dot{\boldsymbol{v}}_i$. 对于 $\boldsymbol{u}_i$, 有

$$\boldsymbol{u}_i = \frac{1}{m}[-k_p(d_i - R^*)\widehat{\boldsymbol{r}}_i - k_d v_{r,i}\widehat{\boldsymbol{r}}_i - k_v(v_{\theta,i} - v_d)\hat{\boldsymbol{\theta}}_{\boldsymbol{i}}] + \frac{1}{m}\sum_{k\neq i}\Phi(d_{ik})(\boldsymbol{p}_i - \boldsymbol{p}_k) =: \boldsymbol{u}_{i_1} + \boldsymbol{u}_{i_2}.$$

设 $\|\boldsymbol{u}_{i_1} - \boldsymbol{u}_{j_1}\| \leqslant L$. 取

$$k_r > L\sigma^2 \mathrm{e}^{\frac{1}{2}} \max\left\{\frac{1}{d_s}, \frac{1}{\min\limits_{k\neq l} d_{kl}(0)}\right\},$$

则当 $d_{ij} \leqslant d_s + \sigma$ 时, 有

$$\|\boldsymbol{u}_{i_2} - \boldsymbol{u}_{j_2}\| > 2L.$$

于是

$$(\boldsymbol{p}_i - \boldsymbol{p}_j)\cdot(\boldsymbol{u}_i - \boldsymbol{u}_j) \geqslant \|\boldsymbol{u}_{i_2} - \boldsymbol{u}_{j_2}\| d_{ij} - L d_{ij} > L d_{ij}.$$

代入 (2) 式有

$$\frac{\mathrm{d}E_{ij}}{\mathrm{d}t} \geqslant \frac{\mathrm{d}d_{ij}}{\mathrm{d}t}(L + \Phi(d_{ij})) > 0.$$

由此即知

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} \geqslant -\kappa\mathcal{E}(t),$$

其中 $\kappa > 0$ 为常数. 而

$$\begin{aligned}
&\mathcal{E}(0) \geqslant \sum_{i<j} \Psi(d_{ij}(0)) > \psi(d_s+\sigma)\cdot\binom{N}{2},\\
&\mathcal{E}(t) \geqslant \mathcal{E}(0)\mathrm{e}^{-\kappa t} > 0,\\
&\Psi(d_{ij}(t)) \leqslant E_{ij}(t) \leqslant \mathcal{E}(t).
\end{aligned}$$

由 Ψ 严格单调递减可知

$$d_{ij}(t) \geqslant \Psi^{-1}(\mathcal{E}(t)) > \Psi^{-1}(\mathcal{E}(0)\mathrm{e}^{-\kappa t}).$$

令 $t \to \infty$, 有

$$\varliminf_{t\to\infty} d_{ij}(t) \geqslant \Psi^{-1}(0) = d_s.$$

故总是不会相撞.

下面讨论收敛性, 即讨论系统收敛至

$$\mathcal{S} = \{\|\boldsymbol{r}_i\| = R, \boldsymbol{v}_i \cdot \widehat{\boldsymbol{r}}_i = 0, \|\boldsymbol{v}_i\| = v_d\}.$$

构造 Lyapunov 函数

$$V = \frac{1}{2}\sum_{i=1}^{N}[k_p(d_i-R)^2 + \|\boldsymbol{v}_i - v_d\widehat{\boldsymbol{\theta}}_i\|^2] + \sum_{i<j}\Psi(d_{ij}),$$

则

$$\dot{V} = -\sum_i k_d v_{r,i}^2 - \sum_i k_v(v_{\theta,i}-v_d)^2 - \sum_i c\|\boldsymbol{v}_i\|^3 \leqslant 0,$$

故方程渐进收敛至 $\mathcal{S}$.

5.3 some constants calculation

undergoing c is air resistance constant.

$$m\frac{d^2\vec{p_i}}{dt^2} + c\|\frac{d\vec{p_i}}{dt}\|\frac{d\vec{p_i}}{dt} = u_i$$

References

[1] Maziar Raissi, Paris Perdikaris, and George Em Karniadakis. Physics informed deep learning (part i): Data-driven solutions of nonlinear partial differential equations. *arXiv preprint arXiv:1711.10561*, 2017.