Perfect Reconstruction of Discrete Framelet Transforms

Chapter 1.1.2 Summary

Based on *Framelets and Wavelets*
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Sequence Spaces

- $\ell(\mathbb{Z}) = \big\{ v = \{v(k)\}_{k\in\mathbb{Z}} \mid v(k) \in \mathbb{C} \big\}$
- $\ell_0(\mathbb{Z}) = \big\{ u \in \ell(\mathbb{Z}) \mid \text{only finitely many } u(k) \neq 0 \big\}$
- filter bank: $\{u_0, \ldots, u_s\} \subset \ell_0(\mathbb{Z})$

Operators

- **Subdivision:**
$$[S_u v](n) = 2 \sum_{k \in \mathbb{Z}} v(k)\, u(n-2k)$$
- **Transition:**
$$[T_u v](n) = 2 \sum_{k \in \mathbb{Z}} v(k)\, \overline{u(k-2n)}$$
- **Discrete Framelet Transform (DFrT):**
$$\frac{1}{2} \sum_{\ell=0}^{s} S_{u_\ell} T_{\tilde{u}_\ell} v$$

Perfect Reconstruction Condition

- the transform satisfies perfect reconstruction if

$$\frac{1}{2}\sum_{\ell=0}^{s} S_{u_\ell} T_{\tilde{u}_\ell} v = v \quad \forall v \in \ell(\mathbb{Z}).$$

Main Theorem – Four Equivalent Statements

The following are equivalent:

1. The filter bank $(\{\tilde{u}_\ell\}, \{u_\ell\})$ has PR.
2. The identity $\frac{1}{2}\sum_{\ell=0}^{s} S_{u_\ell} T_{\tilde{u}_\ell} v = v$ holds for all $v \in \ell_0(\mathbb{Z})$.
3. The identity $\frac{1}{2}\sum_{\ell=0}^{s} S_{u_\ell} T_{\tilde{u}_\ell} v = v$ holds for $v = \delta$ and $v = \delta(\cdot - 1)$.
4. Frequency-domain identities for all $\omega \in \mathbb{R}$:

$$\sum_{\ell=0}^{s} \overline{\hat{\tilde{u}}_\ell(\omega)}\hat{u}_\ell(\omega) = 1,$$

$$\sum_{\ell=0}^{s} \overline{\hat{\tilde{u}}_\ell(\omega)}\hat{u}_\ell(\omega + \pi) = 0.$$

(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii)

Trivial inclusions:

$$\ell_0(\mathbb{Z}) \subseteq \ell(\mathbb{Z}) \quad \text{and} \quad \delta, \delta(\cdot - 1) \in \ell_0(\mathbb{Z}).$$

(iii) $\Rightarrow$ (iv)

1. **Frequency Domain Conversion:** Using Fourier transforms of $T_{\tilde{u}_\ell}$ and S_{u_ℓ}:

$$\widehat{T_{\tilde{u}_\ell} v}(\omega) = \hat{v}(\omega/2)\overline{\hat{\tilde{u}}_\ell(\omega/2)} + \hat{v}(\omega/2+\pi)\overline{\hat{\tilde{u}}_\ell(\omega/2+\pi)},$$

$$\widehat{S_{u_\ell} w}(\omega) = \hat{w}(2\omega)\hat{u}_\ell(\omega).$$

2. **Substitute Basis Signals:** Plugging $v = \delta$ ($\hat{v} = 1$) and $v = \delta(\cdot - 1)$ ($\hat{v} = e^{-i\omega}$) into the PR condition yields the system:

$$\sum_{\ell=0}^{s} \overline{\hat{\tilde{u}}_\ell(\omega)}\hat{u}_\ell(\omega) + \sum_{\ell=0}^{s} \overline{\hat{\tilde{u}}_\ell(\omega+\pi)}\hat{u}_\ell(\omega+\pi) = 1,$$

$$\sum_{\ell=0}^{s} \overline{\hat{\tilde{u}}_\ell(\omega)}\hat{u}_\ell(\omega) - \sum_{\ell=0}^{s} \overline{\hat{\tilde{u}}_\ell(\omega+\pi)}\hat{u}_\ell(\omega+\pi) = 1.$$

3. **Solve System:** Adding and subtracting these equations gives condition (iv).

(iv) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i)

- (iv) $\Rightarrow$ (ii): Condition (iv) implies the frequency identity holds for all $v \in \ell_0(\mathbb{Z})$ by linearity of Fourier transform.
- (ii) $\Rightarrow$ (i): Localization Argument: For any $v \in \ell(\mathbb{Z})$, truncate to local signal $v_n \in \ell_0(\mathbb{Z})$ using finite support of filters. Apply (ii) to show:

$$v(n) = \frac{1}{2}\sum_{\ell=0}^{s}[\mathcal{S}_{u_\ell}\mathcal{T}_{\tilde{u}_\ell}v](n).$$