Wavelet Transform for Image Processing

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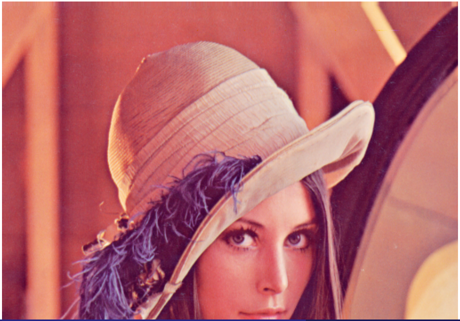
1. Image Loading and Grayscale Conversion

Step Explanation

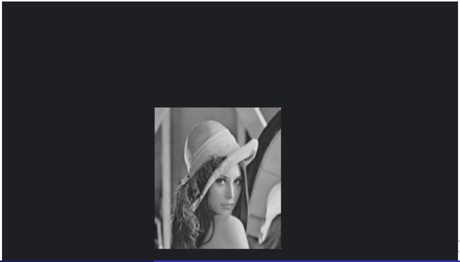
The image is loaded using a Python imaging library and converted to grayscale.
This reduces data complexity and focuses analysis on structural content.



2. Rescaling and 3. Centered Cropping

Rescaling with Aspect Ratio Preservation

The image is resized using high-fidelity interpolation to ensure one side reaches the target length (2^N), without distortion.
The resized image is cropped to $2^N \times 2^N$, ensuring input uniformity without compromising important visual features.



4. Matrix Output

Step Explanation

The final image is converted into a matrix format suitable for:

- Mathematical operations (e.g., wavelet transforms)
- Storage and machine learning integration

128	135	142
130	138	145
125	132	139

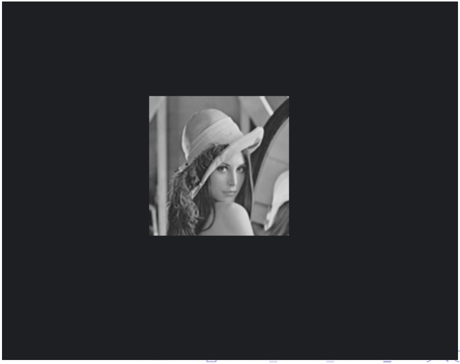
Example of 3x3 image matrix (simplified)

5. Convert Matrix Information into an Image

Concept

Each matrix element represents the gray value of a pixel. Using this matrix, we can reconstruct the grayscale image.

```
[[162 161 162 ... 117 165 167]
 [160 160 160 ... 127 130 101]
 [157 156 157 ... 105  53  40]
 ...
 [ 54  56  58 ...  57  52  61]
 [ 50  53  52 ...  57  70  88]
 [ 48  53  49 ...  67  93 103]]
矩阵形状：(128, 128)
图像尺寸：(128, 128)
```



Standard Haar Decomposition

- sdlfkj
- sDF
- sdf

sfdfg

- sdlfkj
- sDF
- sdf

sadsgkjh

- sdlfkj
- sDF
- sdf

sdf

- sdlfkj
- sDF
- sdf

sadsgkjh

- sdlfkj
- sDF
- sdf

Basic Idea

- $$\sum_{k=1}^{n} \hat{l}_k^4 \geq (\sum_{k=1}^{n} \hat{l}_k^2)^2 = (\sum_{k=1}^{n} l_k^2)^2$$
- $$\min_{\hat{l}_k \in l^2 st. \sum_{k=1}^{n} \hat{l}_k^2 = \sum_{k=1}^{n} l_k^2} (-\sum_{k=1}^{n} \hat{l}_k^4)$$

Basic Idea

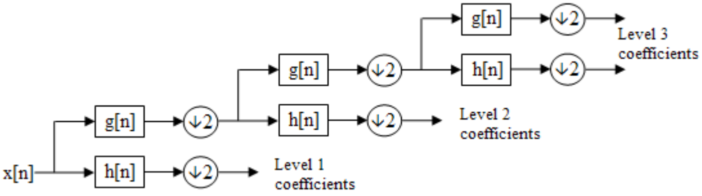
- 

Figure: Cascading

- $$\min_{\Phi}(-\sum_{k=1}^{n} \hat{l}_k^4)$$

- Condense the energy to as less coefficients as possible.

Results

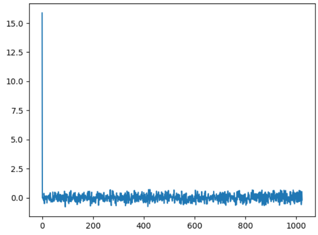
- 

Figure: Haar 2x2 filter bank random input. Compression Rate 0.2568

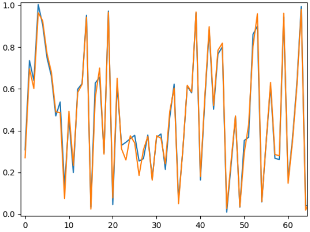


Figure: Haar 2x2 filter bank random input.Total average energy loss 0.0009

Results

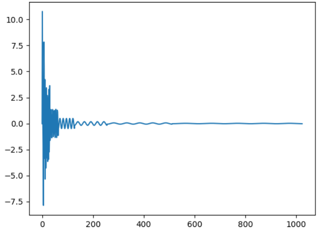
- 

Figure: Haar 2x2 filter bank sin input frequency. Compression Rate 0.9287

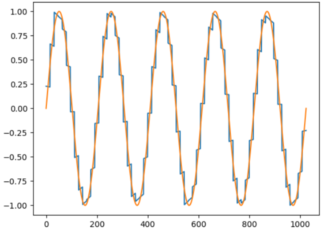


Figure: Haar 2x2 filter bank sin input reconstruction.Total average energy loss 0.0104

Results

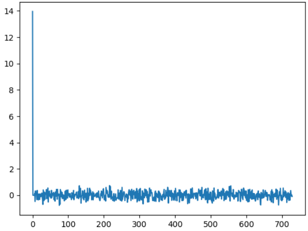
- 

Figure: Haar 3x3 filter bank random input. Compression Rate 0.1906

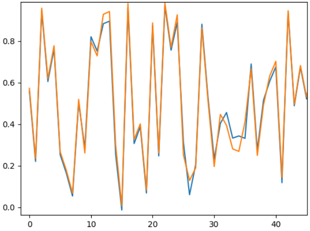


Figure: Haar 3x3 filter bank random input.Total average energy loss 0.0008

Results

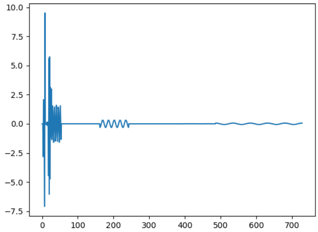


Figure: Haar 3x3 filter bank sin input frequency. Compression Rate 0.7750

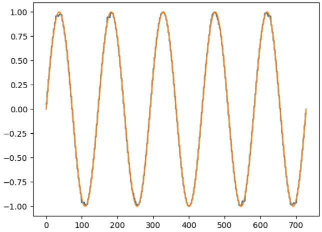


Figure: Haar 3x3 filter bank sin input reconstruction.Total average energy loss 0.0007

Main Theorem – Four Equivalent Statements

The following are equivalent:

1. The filter bank $(\{\tilde{u}_\ell\}, \{u_\ell\})$ has PR.
2. $\dfrac{1}{2}\displaystyle\sum_{\ell=0}^{s} S_{u_\ell} T_{\tilde{u}_\ell} v = v$ for all $v \in \ell(\mathbb{Z})$.
3. The identity holds for $v = \delta$ and $v = \delta(\cdot - 1)$.
4. Frequency-domain identities for all $\omega \in \mathbb{R}$:

$$\sum_{\ell=0}^{s} \overline{\hat{\tilde{u}}_\ell(\omega)}\, \hat{u}_\ell(\omega) = 1,$$

$$\sum_{\ell=0}^{s} \overline{\hat{\tilde{u}}_\ell(\omega)}\, \hat{u}_\ell(\omega + \pi) = 0.$$

(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii)

Trivial inclusions:

$$\ell_0(\mathbb{Z}) \subseteq \ell(\mathbb{Z}) \quad \text{and} \quad \delta, \delta(\cdot - 1) \in \ell_0(\mathbb{Z}).$$

(iii) $\Rightarrow$ (iv)

1. **Frequency Domain Conversion:** Using Fourier transforms of $T_{\tilde{u}_\ell}$ and S_{u_ℓ}:

$$\widehat{T_{\tilde{u}_\ell} v}(\omega) = \hat{v}(\omega/2)\overline{\hat{\tilde{u}}_\ell(\omega/2)} + \hat{v}(\omega/2+\pi)\overline{\hat{\tilde{u}}_\ell(\omega/2+\pi)},$$

$$\widehat{S_{u_\ell} w}(\omega) = \hat{w}(2\omega)\hat{u}_\ell(\omega).$$

2. **Substitute Basis Signals:** Plugging $v = \delta$ ($\hat{v} = 1$) and $v = \delta(\cdot - 1)$ ($\hat{v} = e^{-i\omega}$) into the PR condition yields the system:

$$\sum_{\ell=0}^{s} \overline{\hat{\tilde{u}}_\ell(\omega)}\hat{u}_\ell(\omega) + \sum_{\ell=0}^{s} \overline{\hat{\tilde{u}}_\ell(\omega+\pi)}\hat{u}_\ell(\omega+\pi) = 2,$$

$$\sum_{\ell=0}^{s} \overline{\hat{\tilde{u}}_\ell(\omega)}\hat{u}_\ell(\omega) - \sum_{\ell=0}^{s} \overline{\hat{\tilde{u}}_\ell(\omega+\pi)}\hat{u}_\ell(\omega+\pi) = 0.$$

3. **Solve System:** Adding and subtracting these equations gives condition (iv).

(iv) ⇒ (ii) and (ii) ⇒ (i)

- (iv) ⇒ (ii): Condition (iv) implies the frequency identity holds for all $v \in \ell_0(\mathbb{Z})$ by linearity of Fourier transform.
- (ii) ⇒ (i): Localization Argument: For any $v \in \ell(\mathbb{Z})$, truncate to local signal $v_n \in \ell_0(\mathbb{Z})$ using finite support of filters. Apply (ii) to show:

$$v(n) = \frac{1}{2} \sum_{\ell=0}^{s} [S_{u_\ell} T_{\tilde{u}_\ell} v](n).$$