Computational Physics Project A
Modelling Single & Double Pendulum Systems Computationally

Joshua Maxey

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The Euler, Leapfrog and Runge-Kutta (4th order) numerical methods are used to computationally model a single pendulum system. By varying system and method parameters limits are found for the stabiity of these methods ad their suitability for oscillatory problems. The conclusions of this preliminary investigation feed into the more complex double pendulum system. THIS ISN'T OVER!

Contents

1 Introduction

Computational models of physical systems require us to discretise continuous variables (often, but not exclusively, time). Numerical methods approximate derivatives to provide a means to solving differential equaitons computationally. The three numerical methods investigated in this project are Euler's method, the Leapfrog method and the 4th order Runge-Kutta method.

2 The Single Pendulum

2.1 Theory

System Description & Parameters

The idealised single pendulum system comprises a fixed mass m attached to the end of a massless rod of length l suspended from a pivot which applies a friction that is characterised by the damping constant γ. It is depicted in figure 1.

Equations of Motion

The motion of the pendulum is defined by equation 1 where θ is the angle the pendulum makes with the vertical.

$$ml\frac{d^2\theta}{dt^2} = -mg\sin(\theta) - \gamma\frac{d\theta}{dt} \tag{1}$$

By applying the small angle approximation, choosing dimensionless units of time to yeild reduced time, $\widetilde{t}$, such that

$$\widetilde{t} = t\sqrt{\frac{l}{g}} \quad \text{and choosing} \quad \beta = \frac{\gamma}{m\sqrt{gl}} \tag{2}$$

equation 1 reduces to

$$\frac{d^2\theta}{d\widetilde{t}^2} = \theta - \beta\frac{d\theta}{d\widetilde{t}} \tag{3}$$

This can be expressed as a pair of coupled first order differential equations where ω is the rate of change of θ with respect to reduced time. They are presented in equation 4.

$$\frac{d}{d\widetilde{t}}\begin{pmatrix}\theta \\ \omega\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -\beta \end{pmatrix}\begin{pmatrix}\theta \\ \omega\end{pmatrix} \tag{4}$$
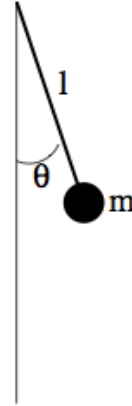
Figure 1: The single pendulum notation used throughout this report.

A Computer Model for the Single Pendulum System

Euler's Method

Euler's method is the simplest numerical method employed. It is obtained by Taylor expanding our function at an advanced time step $t + h$ and truncating the result after the first two terms. As such, it is accurate to $O(h^2)$.

Defining $\underline{L}$ as the 2x2 matrix from 4 we can create the update matrix $\underline{T}$ where $\underline{T}$ is defined as $\underline{I} + h\,\underline{L}$. To apply Euler's method we simply multiply the position vector at the current time step by this update matrix to find the values of θ and ω at the next time step. Hence,

$$
\begin{aligned}
\theta_{n+1} &= \theta_n + h\omega_n \\
\omega_{n+1} &= -h\theta_n + (1 - h\beta)\omega_n
\end{aligned}
\tag{5}
$$

The Leapfrog Method

The Leapfrog method is the simplest finite difference method that requires the storing of system variables at a previous timestep. It produces a gradient estimate from y_{n-1} and y_n which it uses to calculate y_{n+1}. Using our notation, the Leapfrog method can be expressed as

$$
\begin{pmatrix} \theta_{n+1} \\ \omega_{n+1} \end{pmatrix} = \begin{pmatrix} \theta_{n-1} \\ \omega_{n-1} \end{pmatrix} + 2h\underline{L} \cdot \begin{pmatrix} \theta_n \\ \omega_n \end{pmatrix}
\tag{6}
$$

These are manifest in code as equations 7

$$
\begin{aligned}
\theta_{n+1} &= \theta_{n-1} + 2h\omega_n \\
\omega_{n+1} &= \omega_{n-1} - 2h(\theta_n + \beta\omega)
\end{aligned}
\tag{7}
$$

4th Order Runge-Kutta Method

The Runge-Kutta method provide a generalisation for taking any number of gradients and producing a weighted average of them to predict the next value of y. This investigation employs the 4th order Runge-Kutta method (RK4). It is the highest order (and therefore most accurate) Runge-Kutta method where the number of evaulations required per step is less than or equal to the order of error. It is fourth order accurate; $O(h^4)$.

Although it requires significantly more evaluations per time step, the RK4 method is more computationally efficient due to its

ability to remain accurate for much larger values of h. The set of equations 8 show the four evaluations required before one can calculate the variables at the next time step.

$$k_1 = h\underline{L} \cdot \underline{y}_n$$
$$k_2 = h\underline{L} \cdot (\underline{y}_n + \frac{1}{2}\underline{k}_1)$$
$$k_3 = h\underline{L} \cdot (\underline{y}_n + \frac{1}{2}\underline{k}_2)$$
$$k_4 = h\underline{L} \cdot (\underline{y}_n + \underline{k}_3) \qquad (8)$$
$$\underline{y}_{n+1} = \underline{y}_n + \frac{1}{6}(\underline{k}_1 + 2\underline{k}_2 + 2\underline{k}_3 + \underline{k}_4)$$
$$\text{where} \quad \underline{y}_n = \begin{pmatrix} \theta_n \\ \omega_n \end{pmatrix}$$

2.2 Method

2.3 Results & Discussion

3 The Double Pendulum

3.1 Theory

System Description & Parameters

The idealised double pendulum system comprises the single pendulum setup as before with an added bob of mass M attached by a massless rod (also of length l) to the first mass which draws an angle from the vertical of ψ as depicted in figure 3.

Equations of Motion

A Computer Model for the Single Pendulum System
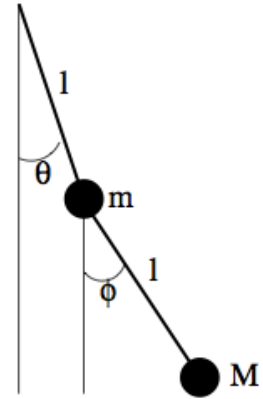
3.2 Method

3.3 Results & Discussion

Figure 3: The single pendulum notation used throughout this report.
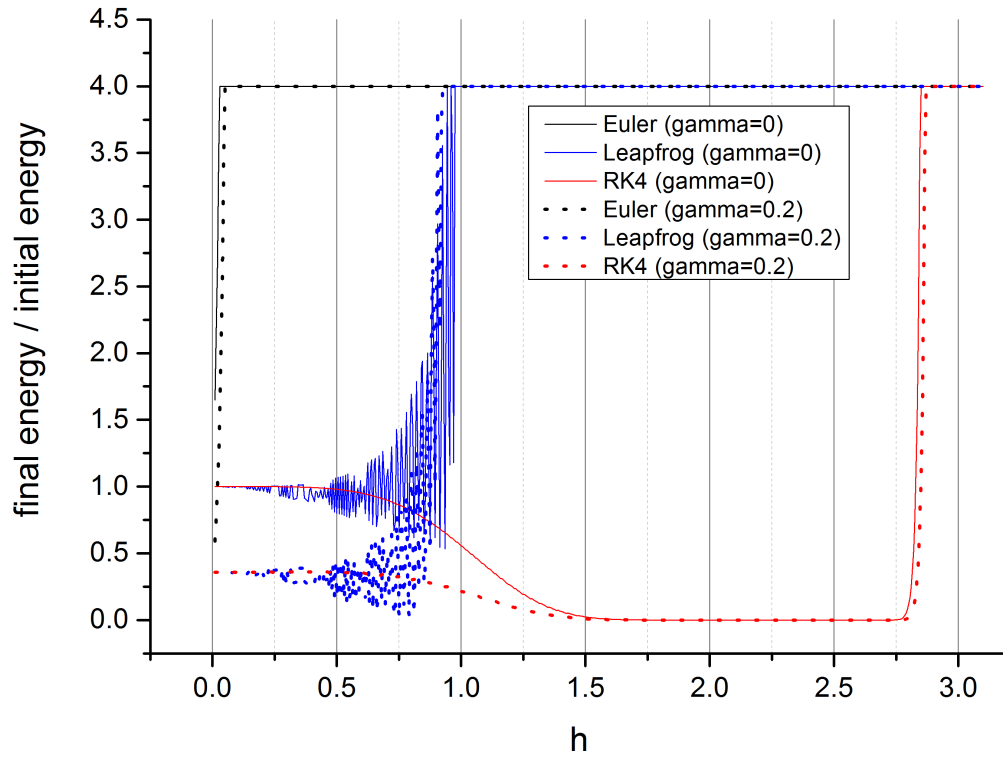
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Figure 2: The ratio of final pendulum energy ($E = $ kinetic + potential) and intial E versus the step size h for the three numerical methods in the undamped case and a lightly damped case ($\gamma = 0.2$).