Jointly distributed random variables

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May 30, 2017

We will often want to understand several random variables at the same time.

Example

For example, consider the height X and the weight Y of a person. As they are related, need to understand both simultaneously.

Joint distribution

We will express the random behavior of these two random variables using a joint distribution.

Joint p.m.f.

Suppose both X and Y are discrete. The joint p.m.f. is

$$f_{XY}(x, y) = \mathrm{P}(X = x, Y = y) \tag{1}$$

Joint p.d.f

If both X and Y are continuous, we may write the joint p.d.f. f_{XY}. It can be interpreted as the infinitesimal version of the joint p.m.f.

Joint distribution

Example: die roll

Suppose that we roll a single die, and let X be the random variable such that $X = 1$ if the roll is even, and $X = 0$ otherwise. Let Y be the random variable such that $Y = 1$ if the roll is prime, and $Y = 0$ otherwise.

Die	1	2	3	4	5	6
X	0	1	0	1	0	1
Y	0	1	1	0	1	0

Joint distribution

Example: die roll

Hence the joint p.m.f. is given by

$$\begin{aligned}
f_{XY}(0,0) &= \mathrm{P}(X=0, Y=0) = 1/6 \\
f_{XY}(1,0) &= \mathrm{P}(X=1, Y=0) = 2/6 \\
f_{XY}(0,1) &= \mathrm{P}(X=0, Y=1) = 2/6 \\
f_{XY}(1,1) &= \mathrm{P}(X=1, Y=1) = 1/6
\end{aligned}$$

Marginal distribution

If we have two variables X and Y with a joint density f_{XY}, we can always "forget" about Y and only consider X.
What is the distribution of X?

Marginal distribution

If we have two variables X and Y with a joint density f_{XY}, we can always "forget" about Y and only consider X.
What is the distribution of X?
We have that if X, Y discrete:

$$f_X(x) = \sum_y f_{XY}(x, y) \tag{2}$$

and if X, Y continuous:

$$f_X(x) = \int f_{XY}(x, y)\, dy \tag{3}$$

Marginal distribution

Example: die roll

What is the marginal distribution of X?

$$
\begin{aligned}
\mathrm{P}(X=0) = f_X(0) = f_{XY}(0,0) + f_{XY}(0,1) = 1/6 + 2/6 = \frac{1}{2} \\
\mathrm{P}(X=1) = f_X(1) = f_{XY}(1,0) + f_{XY}(1,1) = 2/6 + 1/6 = \frac{1}{2}
\end{aligned}
$$

Joint distribution and independence

Suppose that the event $X = x$ and $Y = y$ are independent. Then, we have that:

$$\mathrm{P}(X = x, Y = y) = \mathrm{P}(X = x)\,\mathrm{P}(Y = y) \tag{4}$$

In other words, we have

$$f_{XY}(x, y) = f_X(x) f_Y(y) \tag{5}$$

Joint distribution and independence

If X and Y are such that their p.m.f. or p.d.f. verifies:

$$f_{XY}(x, y) = f_X(x) f_Y(y) \tag{6}$$

we say X and Y are independent.

Joint distribution and independence

Example

Suppose that X is uniform on $[0, 1]$, and Y is an independent random variable uniform on $[0, 1]$.
The joint density is given by the product

$$f_{XY}(x, y) = f_X(x)f_Y(y) = 1 \tag{7}$$

for $0 \leq x, y \leq 1$.

Joint distribution and conditioning

If X and Y are discrete random variables with joint p.m.f. f_{XY}, we can compute the probability of:

$$P(X = x \mid Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)} = \frac{f_{XY}(x, y)}{f_Y(y)} \tag{8}$$

We write the conditional p.m.f. / p.d.f. as:

$$f_{X|Y}(x \mid y) = \frac{f_{XY}(x, y)}{f_Y(y)} \tag{9}$$

Joint distribution and conditioning

Example

Let X and Y have joint p.d.f.

$$f_{XY}(x, y) = 6(x - y)^2 \text{ for } 0 \le x, y \le 1. \tag{10}$$

Let us compute:

$$f_{X|Y}(x \mid y) = \frac{f_{XY}(x, y)}{f_Y(y)} \tag{11}$$

Joint distribution and probability

The joint p.d.f / p.m.f. can be used to compute probabilities.
For example, suppose that X and Y have joint p.d.f.

$$f_{XY}(x,y) = 6(x-y)^2 \text{ for } 0 \leq x, y \leq 1. \tag{12}$$

Let us compute $\mathrm{P}(X > Y/2)$.
Need to integrate:

$$\iint_{\{x,y:x>y/2\}} 6(x-y)^2 \, dx \, dy \tag{13}$$

Expectation of joint distributions

The joint p.d.f. / p.m.f. can be used to compute expectations of functions of two variables.
Let $g(X, Y)$ be a function of two variables, then we have

$$\mathbb{E}\, g(X, Y) = \sum_{x,y} g(x, y) f_{XY}(x, y) \tag{14}$$

for X, Y discrete, and

$$\mathbb{E}\, g(X, Y) = \iint g(x, y)\, dx\, dy \tag{15}$$

Expectation of joint distributions

Let X and Y be random variables on $[0,1]$ with joint p.d.f. $6(x-y)^2$.
Let us compute $\mathbb{E}\, g(X,Y)$ for $g(x,y)=xy$. We need to compute the integral:

$$\begin{aligned}\mathbb{E}\, XY &= \iint_{[0,1]^2} xyf_{XY}(x,y)\,dx\,dy\\ &= \int_0^1\int_0^1 xy\times 6(x-y)^2\,dx\,dy\end{aligned}$$