Sample questions

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The midterm will have 6 questions. All questions will be marked and your grade will be computed from the 5 best questions. You will have access to an A4 (letter) sized note sheet, front and back. Each question is worth 20 marks. Some questions may have a part marked with a star – it is intended to indicate that this part may be somewhat more challenging and less rewarding in terms of credit. The total points of such parts will be less than 10.

1. Probability.

   Alice has two coins: coin A, which is a fair coin, and coin B, which is a double-headed coin.

   (a) (2 points) Let C_A be the event that Alice chooses coin A, and C_B the event that Alice chooses coin B.

   What is $\mathrm{P}(C_A) + \mathrm{P}(C_B)$ (no justification needed).

   Solution: As Alice has to choose either coin A or coin B, we must have that $\mathrm{P}(C_A)+\mathrm{P}(C_B) = 1$.

   (b) (3 points) Suppose that Alice picks a coin and tosses it, and let H_1 be the event that the coin lands on heads.

   Write down $\mathrm{P}(H_1 \mid C_A)$ and $\mathrm{P}(H_1 \mid C_B)$ from the problem description.

   Solution: Coin A is a fair coin, so by definition we have $\mathrm{P}(H_1 \mid C_A) = 0.5$. On the other hand, coin B is double-headed, so we have that $\mathrm{P}(H_1 \mid C_B) = 1$.

   (c) (6 points) Suppose that Alice picked the coin at random, that is, $\mathrm{P}(C_A) = 0.5$. Compute $\mathrm{P}(H_1)$.

   Solution: We use the law of total probability to compute:

$$\begin{aligned}\mathrm{P}(H_1) &= \mathrm{P}(H_1 \mid C_A)\,\mathrm{P}(C_A) + \mathrm{P}(H_1 \mid C_B)\,\mathrm{P}(C_B) \\ &= \frac{1}{2} \times \frac{1}{2} + 1 \times \frac{1}{2} \\ &= \frac{3}{4}\end{aligned}$$

   (d) (6 points) Now, suppose that the coin lands on heads. Compute the probability $\mathrm{P}(C_A \mid H_1)$ that Alice chose the coin A.

Solution: We use Bayes formula to obtain (reading off the values from the previous subparts):

$$\begin{aligned}
\mathrm{P}(C_A \mid H_1) &= \frac{\mathrm{P}(H_1 \mid C_A)\,\mathrm{P}(C_A)}{\mathrm{P}(H_1)} \\
&= \frac{\frac{1}{2} \times \frac{1}{2}}{3/4} \\
&= \frac{1}{3}
\end{aligned}$$

(e)* (3 points) Suppose that Alice uses the same coin, and tosses it again. Let H_2 be the event that it lands on heads. Compute $\mathrm{P}(H_2 \mid H_1)$.

Solution: This part is a bit more challenging, and we will work from the definition.

$$\mathrm{P}(H_2 \mid H_1) = \frac{\mathrm{P}(H_2 \cap H_1)}{\mathrm{P}(H_1)} \tag{1}$$

We have already computed the denominator, let us think about the numerator. As the choice of the coin seems crucial, we will use the law of total probability to obtain:

$$\mathrm{P}(H_2 \cap H_1) = \mathrm{P}(H_2 \cap H_1 \mid C_A)\,\mathrm{P}(C_A) + \mathrm{P}(H_2 \cap H_1 \mid C_B)\,\mathrm{P}(C_B) \tag{2}$$

The event $H_2 \cap H_1$ corresponds to that of obtaining two heads in a row. If we have chosen coin A, the fair coin, this happens with probability $1/4$. On the other hand, if we have chosen coin B, the double-headed coin, this happens with probability 1. Hence we have that:

$$\mathrm{P}(H_2 \cap H_1) = \frac{1}{4} \times \frac{1}{2} + 1 \times \frac{1}{2} = \frac{5}{8} \tag{3}$$

Hence we deduce that:

$$\mathrm{P}(H_2 \mid H_1) = \frac{5/8}{3/4} = \frac{5}{6} \tag{4}$$

2. Jointly distributed random variable.

Let X and Y be continuous random variables with $0 \leq X \leq 2$ and $0 \leq Y \leq 2$. Suppose that X and Y have joint p.d.f.

$$f_{XY}(x, y) = C(x + y^2) \tag{5}$$

where C is some real number.

(a) (3 points) Compute the value of C.

Solution: The value of C is given by the fact that the p.d.f. must integrate to 1. Hence we compute (being mindful of the integration bounds):

$$\int_0^2 \int_0^2 (x + y^2)\,dx\,dy = \int_0^2 \int_0^2 (2 + 2y^2)\,dy = 4 + \frac{16}{3} = 28/3 \tag{6}$$

Thus we must have $C = 3/28$.

(b) (5 points) What is the marginal p.d.f. f_X of X?

Solution: The marginal p.d.f. of X is given by:

$$\begin{aligned} f_X(x) &= \int_0^2 f_{XY}(x,y)\,dy \\ &= \int_0^2 \frac{3}{28}(x+y^2)\,dy \\ &= \frac{3}{28}\left(2x+\frac{8}{3}\right) \\ &= \frac{3x+4}{14} \end{aligned}$$

(c) (6 points) Compute $\mathbb{E}\,X^2$

Solution: We can use the marginal p.d.f. of X to compute this expectation:

$$\begin{aligned} \mathbb{E}\,X^2 &= \int_0^2 x^2 f_X(x)\,dx \\ &= \int_0^2 \frac{3x^3+4x^2}{14}\,dx \\ &= \frac{8}{7} \end{aligned}$$

(d) (6 points) Compute $\mathbb{E}\,XY$

Solution: We use again the formula for expectation to obtain:

$$\begin{aligned} \mathbb{E}\,XY &= \frac{3}{28}\int_0^2\int_0^2 xy(x+y^2)\,dx\,dy \\ &= \frac{3}{28}\int_0^2\int_0^2 (x^2y+xy^3)\,dx\,dy \\ &= \frac{3}{28}\int_0^2 \frac{8y}{3}+2y^3\,dy \\ &= \frac{3}{28}(\frac{8}{3}\times 2+2\times 4)\,dy \\ &= \frac{10}{7} \end{aligned}$$

3. Discrete random variable

Let X have binomial distribution with parameters $n=16$ and $p=1/4$. Let Y have binomial distribution with parameters $n=16$ and $p=3/4$. Suppose that X and Y are independent.

Hint: there is no need to use the p.m.f. of the binomial ditribution for the questions below.

(a) (2 points) Compute $\mathbb{E}\,X$, $\mathbb{E}\,Y$ and $\mathbb{E}\,X+Y$

Solution: We use the existing results for the binomial distribution to obtain: $\mathbb{E}\,X = 16/4 = 4$, and $\mathbb{E}\,Y = 16\times 3/4 = 12$. We can also use the linearity of the expectation to obtain: $\mathbb{E}\,X+Y = 16$.

(b) (2 points) What is the maximal value of $X + Y$?

Solution: A binomial random variable has maximum value n. Hence X has maximal value 16, and so does Y. Thus $X + Y$ has maximum value 32.

(c) (4 points) What is $\mathbb{E}\, XY$?

Solution: As X and Y are independent, we have that $\mathbb{E}\, XY = \mathbb{E}\, X\, \mathbb{E}\, Y$. Thus we compute $\mathbb{E}\, XY = 4 \times 12 = 48$.

(d) (8 points) Compute $\mathbb{E}(X + Y)^2$

Solution: By the algebra of the expectation, we have that:

$$\mathbb{E}(X + Y)^2 = \mathbb{E}\, X^2 + \mathbb{E}\, Y^2 + 2\, \mathbb{E}\, XY \tag{7}$$

We know from the alternative definition of the variance that:

$$\begin{aligned} \mathbb{E}\, X^2 &= \mu_X^2 + \sigma_X^2 = 4^2 + 3 = 19 \\ \mathbb{E}\, Y^2 &= \mu_Y^2 + \sigma_Y^2 = 12^2 + 3 = 147 \end{aligned}$$

Where we have used the properties of the binomial for the variance.
Hence we have finally that:

$$\mathbb{E}(X + Y)^2 = 19 + 147 + 2 \times 48 = 262 \tag{8}$$

(e) (4 points) Compute $\mathbb{E}(Y - 12)(X - 4)$.

Solution: We may see that this quantity is simply the covariance of X and Y, and hence is 0 as X and Y are independent. Otherwise, we may again expand and compute:

$$\begin{aligned} \mathbb{E}(Y - 12)(X - 4) &= \mathbb{E}\, XY - 12\, \mathbb{E}\, X - 4\, \mathbb{E}\, Y + 48 \\ &= 48 - 48 - 48 + 48 \\ &= 0 \end{aligned}$$