Homework 2

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You may submit handwritten or typed answer. Write your answers clearly and justify all your computations. You may cite any result shown in class.

1. (1 point) Independent events.

   Let A and B be independent events. Prove that A and B^C are independent events.

   Solution: We need to show that $\mathrm{P}(A \cap B^C) = \mathrm{P}(A)\,\mathrm{P}(B^C)$.

   First, we have that $(A \cap B^C) \cup (A \cap B) = A$, and the two events are disjoint, so we have by the additivity rule that:
   $$\mathrm{P}(A \cap B^C) + \mathrm{P}(A \cap B) = \mathrm{P}(A) \tag{1}$$
   This implies that we have:
   $$\mathrm{P}(A \cap B^C) = \mathrm{P}(A) - \mathrm{P}(A \cap B) = \mathrm{P}(A) - \mathrm{P}(A)\,\mathrm{P}(B) \tag{2}$$
   where we have used $\mathrm{P}(A \cap B) = \mathrm{P}(A)\,\mathrm{P}(B)$ by independence of A and B.

   Now, we have by the rule for complements that:
   $$\mathrm{P}(B^C) = 1 - \mathrm{P}(B) \tag{3}$$
   hence in particular this gives $\mathrm{P}(A)\,\mathrm{P}(B^C) = \mathrm{P}(A)(1 - \mathrm{P}(B))$.

   Hence we see that the right-hand side and the left-hand side are equal, completing the proof.

2. (2 points) The Monty hall problem.

   The Monty Hall problem was made famous in Vos Savant's "Ask Marilyn" column in *Parade* magazine:

   Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?

   (a) Let C_1, C_2, C_3 be the events that the car is behind door 1, 2 and 3 respectively. Suppose that the player always chooses door 1. Suppose that the host never opens a door with the car behind it or that the player has chosen, and otherwise picks at random. Compute the probability that the hosts opens door 3, for each possible scenario of the car. That is, let H_3 be the event that the host opens door 3, compute $\mathrm{P}(H_3 \mid C_1)$, $\mathrm{P}(H_3 \mid C_2)$ and $\mathrm{P}(H_3 \mid C_3)$.

   Solution: We may write these probabilities down from the problem statement. First, consider $\mathrm{P}(H_3 \mid C_1)$, the probability of opening door 3 given the car is behind door 1. As per the rules, both door 2 and 3 could be opened, so the host chooses at random. Thus we have $\mathrm{P}(H_3 \mid C_1) = 0.5$

For $\mathrm{P}(H_3 \mid C_2)$, the only possible choice is door 3 (as the car is behind door 2), hence $\mathrm{P}(H_3 \mid C_2) = 1$. For $\mathrm{P}(H_3 \mid C_3)$, the host is not allowed to open door 3 with the car behind door 3, henc $\mathrm{P}(H_3 \mid C_3) = 0$.

(b) Write down Bayes's rule for $\mathrm{P}(C_2 \mid H_3)$, the probability that the car is behind door 2 given that the player was shown door 3. Compute its value. Conclude whether it is advantageous to switch your choice.

Solution: The Bayes rule to compute the probability that the car is behind door 2 is given by:

$$\mathrm{P}(C_2 \mid H_3) = \frac{\mathrm{P}(H_3 \mid C_2)\,\mathrm{P}(C_2)}{\mathrm{P}(H_3)} \tag{4}$$

Now, we are given $\mathrm{P}(H_3 \mid C_2) = 1$, and we have that $\mathrm{P}(C_2) = 1/3$ (the car is placed at random). We only need to compute $\mathrm{P}(H_3)$.

To compute $\mathrm{P}(H_3)$, we may use the law of total probability, which gives:

$$\begin{aligned}
\mathrm{P}(H_3) &= \mathrm{P}(H_3 \mid C_1)\,\mathrm{P}(C_1) + \mathrm{P}(H_3 \mid C_2)\,\mathrm{P}(C_2) + \mathrm{P}(H_3 \mid C_3)\,\mathrm{P}(C_3) \\
&= \frac{1}{2} \times \frac{1}{3} + 1 \times \frac{1}{3} + 0 \times \frac{1}{3} \\
&= \frac{1}{2}
\end{aligned}$$

Hence putting everything together gives:

$$\mathrm{P}(C_2 \mid H_3) = \frac{1 \times \frac{1}{3}}{1/2} = \frac{2}{3}. \tag{5}$$

Hence by switching we have a two-thirds chance of getting the car.

3. (2 points) Discrete random variable.

Let X be a discrete random variable on the integers, with $1 \leq X \leq n$, for some n integer given. The p.m.f. of X is given as:

$$\mathrm{P}(X = k) = \frac{2k}{n(n+1)}. \tag{6}$$

Hint: for this question, you may find it useful to recall the formulas for the sum of the n first integers and the n first squares:

$$\begin{aligned}
\sum_{k=1}^{n} k &= n(n+1)/2, \\
\sum_{k=1}^{n} k^2 &= n(n+1)(2n+1)/6.
\end{aligned}$$

(a) Compute $\mathrm{P}(X \geq 50)$. Hint: you may find it useful to compute $\mathrm{P}(X \leq 49)$ instead and use the complement rule.

Solution: As per the hint, we compute $\mathrm{P}(X \leq 49)$ first. We are given

$$\begin{aligned}
\mathrm{P}(X \leq 49) &= \sum_{k=1}^{49} \frac{2k}{n(n+1)} \\
&= \frac{2}{n(n+1)} \sum_{k=1}^{49} k \\
&= \frac{2}{n(n+1)} 1225 \\
&= \frac{2450}{n(n+1)}
\end{aligned}$$

Hence the required probability is given by:

$$\mathrm{P}(X \geq 50) = 1 - \mathrm{P}(X \leq 49) = 1 - \frac{2450}{n(n+1)} \tag{7}$$

(b) Compute $\mathbb{E}\, X$.

Solution: The expectation is defined as:

$$\begin{aligned}
\mathbb{E}\, X &= \sum_{k=1}^{n} k \frac{2k}{n(n+1)} \\
&= \frac{2}{n(n+1)} \sum_{k=1}^{n} k^2 \\
&= \frac{2}{n(n+1)} \frac{n(n+1)(2n+1)}{6} \\
&= \frac{2n+1}{3}
\end{aligned}$$

4. (2 points) Continuous random variable.

Let X be a continuous random variable on $[0, 1]$, with p.d.f. f_X given by:

$$f_X = C(x-1)^2, \tag{8}$$

for some C real.

(a) What is the value of C? (Hint: recall that a p.d.f. must be normalized).

Solution: As f_X must be normalized, we may compute:

$$C^{-1} = \int_0^1 (x-1)^2 \, dx = \frac{1}{3}. \tag{9}$$

Hence we deduce that $C = 3$.

(b) Compute the expectation $\mathbb{E}\, X$.

Solution: By definition, the expectation is given by:

$$\begin{aligned}\mathbb{E}\, X &= \int_0^1 x f_X(x)\, dx \\ &= \int_0^1 xC(x-1)^2\, dx \\ &= C\int_0^1 x(x-1)^2\, dx \\ &= \frac{1}{4}\end{aligned}$$

where we have put $C = 3$ from the previous part.

(c) What is $\mathrm{P}(X \geq \frac{1}{2})$?

Solution: We may compute the probability as:

$$\begin{aligned}\mathrm{P}(X \geq \frac{1}{2}) &= \int_{1/2}^1 f_(X)\, dx \\ &= \int_{0.5}^1 3(x-1)^2\, dx \\ &= \frac{1}{24}\end{aligned}$$

5. (3 points) The memoryless property of the exponential distribution.

Let $\lambda > 0$, and let $X \sim \mathrm{Exp}(\lambda)$.

(a) Compute the cumulative distribution $F_X(x)$ of X. Hint: recall that $F_X(x) = \mathrm{P}(X \leq x)$, and that $X \geq 0$.

Solution: The cumulative distribution function is given as:

$$\mathrm{P}(X \leq x) = \int_0^x f_X(t)\, dt = \int_0^x \lambda e^{-\lambda t}\, dt = 1 - e^{\lambda x} \tag{10}$$

(b) Let $a > 0$ be a real number. Compute the probability $\mathrm{P}(X \geq a)$. Use this result and the previous result to compute the conditional cumulative distribution G of $X - a \mid X \geq a$ defined as:

$$G(t) = \mathrm{P}(X - a \leq t \mid X \geq a). \tag{11}$$

Hint: first compute $\mathrm{P}(X - a \geq t \mid X \geq a)$, and use the rule for the complement, that:

$$\mathrm{P}(X - a \leq t \mid X \geq a) = 1 - \mathrm{P}(X - a \geq t \mid X \geq a) \tag{12}$$

Solution: First, we compute $\mathrm{P}(X \geq a)$.

$$\mathrm{P}(X \geq a) = 1 - \mathrm{P}(X \leq a) = 1 - (1 - e^{-\lambda x}) = e^{-\lambda x} \tag{13}$$

where we have used the c.d.f. we computed in part 1.
Now, we have by definition that:

$$P(X - a \geq t \mid X \geq a) = \frac{P(X - a \geq t, X \geq a)}{P(X \geq a)} \tag{14}$$

Now, note that for $t \geq 0$, we have $X - a \geq t$ implies $X \geq a$, hence we have that

$$P(X - a \geq t, X \geq a) = P(X - a \geq t) \tag{15}$$

as the two events are equal.
Now, we have $P(X - a \geq t) = P(X \geq t + a) = e^{-\lambda(t+a)}$
Hence finally we have

$$P(X - a \geq t \mid X \geq a) = \frac{e^{-\lambda(t+a)}}{e^{-\lambda a}} = e^{-\lambda t} \tag{16}$$

and thus $G(t) = 1 - e^{-\lambda t}$.

(c) Differentiate G as a function of t to obtain the density of $X - a$ conditional on $X \geq a$. What is the distribution of $X - a$?

Solution: The density of $X - a$ is the derivative $G'(t)$. We have:

$$G'(t) = \lambda e^{-\lambda t} \tag{17}$$

hence we see that $X - a$ given $X \geq a$ has the distribution of an exponential random variable with parameter λ.