Operations on random variables and limit theorems

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Transformations of a random variable

Suppose that we are given a distribution on X, and a function $g(x)$.
What is the distribution of $g(X)$?
If $Y = g(X)$, then the density of Y is given by:

$$f_Y(y) = \frac{f_X(x)}{|g'(x)|} \tag{1}$$

Transformations of a random variable

Example

Suppose X is standard normal, and $Y = e^X$.
What is f_Y?

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} \frac{1}{\log y} e^{-\frac{1}{2}(\log y)^2} \tag{2}$$

Sums of random variables

Suppose X and Y are discrete random variables.
Can we characterise $X + Y$?

$$\mathrm{P}(X + Y = k) = \sum_{l} \mathrm{P}(X = l, Y = k - l) \tag{3}$$

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If X and Y are continuous (and independent):

$$f_{X+Y}(z)=\int_{-\infty}^{+\infty} f_X(t) f_Y(z-t)\, dt \tag{5}$$

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Sums of random variables

Example: Poisson

Suppose $X \sim \text{Poisson}(\lambda_1)$ and $Y \sim \text{Poisson}(\lambda_2)$.
Let us compute the distribution of $X + Y$.
We will see $X + Y \sim \text{Poisson}(\lambda_1 + \lambda_2)$.

Sums of random variables

Example: Uniform

Suppose $X \sim \mathcal{U}([0,1])$ and $Y \sim \mathcal{U}([0,1])$
Let us compute the distribution of $X + Y$.

Algebraic properties of expectation

Compute expectation of complex quantities from those of simpler quantities.

Expectation of a constant

$$\mathbb{E}\,1 = 1 \tag{6}$$

related to the fact that a distribution is normalized.

Expectations of sums

For any two random variables X and Y, we can derive the distribution of $X + Y$.
Can also compute $\mathbb{E}\, X + Y$ directly:

$$\mathbb{E}[X + Y] = \mathbb{E}\, X + \mathbb{E}\, Y \tag{7}$$

X and Y need not be independent!

Linearity of expectation

In general, if a is real constant, and X, Y are random variables, have:

$$\mathbb{E}[aX + Y] = a\,\mathbb{E}\,X + \mathbb{E}\,Y \tag{8}$$

Linearity of expectation

Example: variance

Let us obtain an alternative formula for the variance.

$$\sigma^2 = \mathbb{E}(X - \mathbb{E}\,X)^2 = \mathbb{E}\,X^2 - (\mathbb{E}\,X)^2 \tag{9}$$

Expectation of products

Let X and Y be independent random variables. We have that:

$$\mathbb{E}\,XY = \mathbb{E}\,X\,\mathbb{E}\,Y \tag{10}$$

Expectation of products

Example: variance of sum

Let X and Y be random variables. Let us compute the variance of $X + Y$. Have:

$$\sigma^2_{X+Y} = \sigma^2_X + \sigma^2_Y + 2\sigma_{XY} \tag{11}$$

Covariance

The covariance of X and Y is given by:

$$\sigma_{XY} = \mathbb{E}(X - \mathbb{E}\,X)(Y - \mathbb{E}\,Y) \tag{12}$$

If X and Y be independent random variables, then:

$$\sigma_{XY} = 0 \tag{13}$$

However, the converse does not hold.

Limit theorems

Want to understand the behaviour of averages:

$$\frac{1}{n}\sum_{i=1}^{n} X_i \tag{14}$$

where X_i is a sample of i.i.d. random variables.

Law of large numbers

As $n \to \infty$, we expect to get the "right answer".

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i = \mathbb{E}\, X \tag{15}$$

Central limit theorem

Quantify deviations of the sample mean.

$$\lim_{n\to\infty} \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} X_i - \mathbb{E}\,X\right) = \mathcal{N}(0,\sigma^2) \tag{16}$$

where σ^2 is the variance of X.

Central limit theorem

Example

Suppose that a store has 200 customers per day on average. We wish to compute the probability that there are more than 6150 customers in 30 days.

Let X_i be the number of customers on day i, suppose $X_i \sim \text{Poisson}(200)$.

Wish to compute:

$$\mathrm{P}(\sum_i X_i \geq 6150) \tag{17}$$