

Jointly distributed random variables

Wenda Zhou

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We will often want to understand several random variables at the same time.

Example

For example, consider the height X and the weight Y of a person. As they are related, need to understand both simultaneously.

Joint distribution

We will express the random behavior of these two random variables using a **joint distribution**.

Joint p.m.f.

Suppose both X and Y are discrete. The joint p.m.f. is

$$f_{XY}(x, y) = P(X = x, Y = y) \quad (1)$$

Joint p.d.f

If both X and Y are continuous, we may write the joint p.d.f. f_{XY} . It can be interpreted as the infinitesimal version of the joint p.m.f.

Joint distribution

Example: die roll

Suppose that we roll a single die, and let X be the random variable such that $X = 1$ if the roll is even, and $X = 0$ otherwise. Let Y be the random variable such that $Y = 1$ if the roll is prime, and $Y = 0$ otherwise.

Die	1	2	3	4	5	6
X	0	1	0	1	0	1
Y	0	1	1	0	1	0

Joint distribution

Example: die roll

Hence the joint p.m.f. is given by

$$f_{XY}(0, 0) = P(X = 0, Y = 0) = 1/6$$

$$f_{XY}(1, 0) = P(X = 1, Y = 0) = 2/6$$

$$f_{XY}(0, 1) = P(X = 0, Y = 1) = 2/6$$

$$f_{XY}(1, 1) = P(X = 1, Y = 1) = 1/6$$

Marginal distribution

If we have two variables X and Y with a joint density f_{XY} , we can always “forget” about Y and only consider X .

What is the distribution of X ?

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What is the distribution of X ?

We have that if X, Y discrete:

$$f_X(x) = \sum_y f_{XY}(x, y) \quad (2)$$

and if X, Y continuous:

$$f_X(x) = \int f_{XY}(x, y) dy \quad (3)$$

Marginal distribution

Example: die roll

What is the marginal distribution of X ?

$$P(X = 0) = f_X(0) = f_{XY}(0, 0) + f_{XY}(0, 1) = 1/6 + 2/6 = \frac{1}{2}$$

$$P(X = 1) = f_X(1) = f_{XY}(1, 0) + f_{XY}(1, 1) = 2/6 + 1/6 = \frac{1}{2}$$

Joint distribution and independence

Suppose that the event $X = x$ and $Y = y$ are independent. Then, we have that:

$$P(X = x, Y = y) = P(X = x) P(Y = y) \quad (4)$$

In other words, we have

$$f_{XY}(x, y) = f_X(x) f_Y(y) \quad (5)$$

Joint distribution and independence

If X and Y are such that their p.m.f. or p.d.f. verifies:

$$f_{XY}(x, y) = f_X(x)f_Y(y) \quad (6)$$

we say X and Y are **independent**.

Joint distribution and independence

Example

Suppose that X is uniform on $[0, 1]$, and Y is an independent random variable uniform on $[0, 1]$.

The joint density is given by the product

$$f_{XY}(x, y) = f_X(x)f_Y(y) = 1 \quad (7)$$

for $0 \leq x, y \leq 1$.

Joint distribution and conditioning

If X and Y are discrete random variables with joint p.m.f. f_{XY} , we can compute the probability of:

$$P(X = x \mid Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)} = \frac{f_{XY}(x, y)}{f_Y(y)} \quad (8)$$

We write the conditional p.m.f. / p.d.f. as:

$$f_{X|Y}(x \mid y) = \frac{f_{XY}(x, y)}{f_Y(y)} \quad (9)$$

Joint distribution and conditioning

Example

Let X and Y have joint p.d.f.

$$f_{XY}(x, y) = 6(x - y)^2 \text{ for } 0 \leq x, y \leq 1. \quad (10)$$

Let us compute:

$$f_{X|Y}(x | y) = \frac{f_{XY}(x, y)}{f_Y(y)} \quad (11)$$

Joint distribution and probability

The joint p.d.f / p.m.f. can be used to compute probabilities.
For example, suppose that X and Y have joint p.d.f.

$$f_{XY}(x, y) = 6(x - y)^2 \text{ for } 0 \leq x, y \leq 1. \quad (12)$$

Let us compute $P(X > Y/2)$.

Need to integrate:

$$\iint_{\{x, y: x > y/2\}} 6(x - y)^2 dx dy \quad (13)$$

Expectation of joint distributions

The joint p.d.f. / p.m.f. can be used to compute expectations of functions of two variables.

Let $g(X, Y)$ be a function of two variables, then we have

$$\mathbb{E} g(X, Y) = \sum_{x,y} g(x, y) f_{XY}(x, y) \quad (14)$$

for X, Y discrete, and

$$\mathbb{E} g(X, Y) = \iint g(x, y) f_{XY}(x, y) dx dy \quad (15)$$

Expectation of joint distributions

Let X and Y be random variables on $[0, 1]$ with joint p.d.f. $6(x - y)^2$.

Let us compute $\mathbb{E} g(X, Y)$ for $g(x, y) = xy$. We need to compute the integral:

$$\begin{aligned}\mathbb{E} XY &= \iint_{[0,1]^2} xy f_{XY}(x, y) \, dx \, dy \\ &= \int_0^1 \int_0^1 xy \times 6(x - y)^2 \, dx \, dy\end{aligned}$$