

Operations on random variables and limit theorems

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Transformations of a random variable

Suppose that we are given a distribution on X , and a function $g(x)$.
What is the distribution of $g(X)$?

If $Y = g(X)$, then the density of Y is given by:

$$f_Y(y) = \frac{f_X(x)}{|g'(x)|} \quad (1)$$

Transformations of a random variable

Example

Suppose X is standard normal, and $Y = e^X$.

What is f_Y ?

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} \frac{1}{\log y} e^{-\frac{1}{2}(\log y)^2} \quad (2)$$

Sums of random variables

Suppose X and Y are discrete random variables.

Can we characterise $X + Y$?

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If X and Y are continuous (and independent):

$$f_{X+Y}(z) = \int_{-\infty}^{+\infty} f_X(t) f_Y(z - t) dt \quad (5)$$

Sums of random variables

Example: Poisson

Suppose $X \sim \text{Poisson}(\lambda_1)$ and $Y \sim \text{Poisson}(\lambda_2)$.

Let us compute the distribution of $X + Y$.

We will see $X + Y \sim \text{Poisson}(\lambda_1 + \lambda_2)$.

Sums of random variables

Example: Uniform

Suppose $X \sim \mathcal{U}([0, 1])$ and $Y \sim \mathcal{U}([0, 1])$
Let us compute the distribution of $X + Y$.

Algebraic properties of expectation

Compute expectation of complex quantities from those of simpler quantities.

Expectation of a constant

$$\mathbb{E} 1 = 1 \tag{6}$$

related to the fact that a distribution is **normalized**.

Expectations of sums

For any two random variables X and Y , we can derive the distribution of $X + Y$.

Can also compute $\mathbb{E} X + Y$ directly:

$$\mathbb{E}[X + Y] = \mathbb{E} X + \mathbb{E} Y \quad (7)$$

X and Y **need not** be independent!

Linearity of expectation

In general, if a is real constant, and X , Y are random variables, have:

$$\mathbb{E}[aX + Y] = a\mathbb{E}X + \mathbb{E}Y \quad (8)$$

Linearity of expectation

Example: variance

Let us obtain an alternative formula for the variance.

$$\sigma^2 = \mathbb{E}(X - \mathbb{E} X)^2 = \mathbb{E} X^2 - (\mathbb{E} X)^2 \quad (9)$$

Expectation of products

Let X and Y be independent random variables. We have that:

$$\mathbb{E} XY = \mathbb{E} X \mathbb{E} Y \quad (10)$$

Expectation of products

Example: variance of sum

Let X and Y be random variables. Let us compute the variance of $X + Y$. Have:

$$\sigma_{X+Y}^2 = \sigma_X^2 + \sigma_Y^2 + 2\sigma_{XY} \quad (11)$$

Covariance

The covariance of X and Y is given by:

$$\sigma_{XY} = \mathbb{E}(X - \mathbb{E} X)(Y - \mathbb{E} Y) \quad (12)$$

If X and Y be independent random variables, then:

$$\sigma_{XY} = 0 \quad (13)$$

However, the converse **does not** hold.

Limit theorems

Want to understand the behaviour of averages:

$$\frac{1}{n} \sum_{i=1}^n X_i \tag{14}$$

where X_i is a sample of i.i.d. random variables.

Law of large numbers

As $n \rightarrow \infty$, we expect to get the “right answer”.

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n X_i = \mathbb{E} X \quad (15)$$

Central limit theorem

Quantify deviations of the sample mean.

$$\lim_{n \rightarrow \infty} \sqrt{n} \left(\frac{1}{n} \sum_{i=1}^n X_i - \mathbb{E} X \right) = \mathcal{N}(0, \sigma^2) \quad (16)$$

where σ^2 is the variance of X .

Central limit theorem

Example

Suppose that a store has 200 customers per day on average. We wish to compute the probability that there are more than 6150 customers in 30 days.

Let X_i be the number of customers on day i , suppose $X_i \sim \text{Poisson}(200)$.

Wish to compute:

$$P\left(\sum_i X_i \geq 6150\right) \tag{17}$$