Homework 2

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May 29, 2017

You may submit handwritten or typed answer. Write your answers clearly and justify all your computations. You may cite any result shown in class.

1. (1 point) Independent events.

   Let A and B be independent events. Prove that A and B^C are independent events.

2. (2 points) The Monty hall problem.

   The Monty Hall problem was made famous in Vos Savant's "Ask Marilyn" column in *Parade* magazine:

   Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?

   (a) Let C_1, C_2, C_3 be the events that the car is behind door 1, 2 and 3 respectively. Suppose that the player always chooses door 1. Suppose that the host never opens a door with the car behind it or that the player has chosen, and otherwise picks at random. Compute the probability that the hosts opens door 3, for each possible scenario of the car. That is, let H_3 be the event that the host opens door 3, compute $\mathrm{P}(H_3 \mid C_1)$, $\mathrm{P}(H_3 \mid C_2)$ and $\mathrm{P}(H_3 \mid C_3)$.

   (b) Write down Bayes's rule for $\mathrm{P}(C_2 \mid H_3)$, the probability that the car is behind door 2 given that the player was shown door 3. Compute its value. Conclude whether it is advantageous to switch your choice.

3. (2 points) Discrete random variable.

   Let X be a discrete random variable on the integers, with $1 \leq X \leq n$, for some n integer given. The p.m.f. of X is given as:

   $$\mathrm{P}(X = k) = \frac{2k}{n(n+1)}. \tag{1}$$

   Hint: for this question, you may find it useful to recall the formulas for the sum of the n first integers and the n first squares:

   $$\sum_{k=1}^{n} k = n(n+1)/2,$$
   $$\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6.$$

   (a) Compute $\mathrm{P}(X \geq 50)$. Hint: you may find it useful to compute $\mathrm{P}(X \leq 49)$ instead and use the complement rule.

   (b) Compute $\mathbb{E}\, X$.

4. (2 points) Continuous random variable.

   Let X be a continuous random variable on $[0, 1]$, with p.d.f. f_X given by:

$$f_X = C(x-1)^2, \tag{2}$$

   for some C real.

   (a) What is the value of C? (Hint: recall that a p.d.f. must be normalized).

   (b) Compute the expectation $\mathbb{E}\, X$.

   (c) What is $\mathrm{P}(X \geq \frac{1}{2})$?

5. (3 points) The memoryless property of the exponential distribution.

   Let $\lambda > 0$, and let $X \sim \mathrm{Exp}(\lambda)$.

   (a) Compute the cumulative distribution $F_X(x)$ of X. Hint: recall that $F_X(x) = \mathrm{P}(X \leq x)$, and that $X \geq 0$.

   (b) Let $a > 0$ be a real number. Compute the probability $\mathrm{P}(X \geq a)$. Use this result and the previous result to compute the conditional cumulative distribution G of $X - a \mid X \geq a$ defined as:

$$G(t) = \mathrm{P}(X - a \leq t \mid X \geq a). \tag{3}$$

   Hint: first compute $\mathrm{P}(X - a \geq t \mid X \geq a)$, and use the rule for the complement, that:

$$\mathrm{P}(X - a \leq t \mid X \geq a) = 1 - \mathrm{P}(X - a \geq t \mid X \geq a) \tag{4}$$

   (c) Differentiate G as a function of t to obtain the density of $X - a$ conditional on $X \geq a$. What is the distribution of $X - a$?