

Limits

1 Indeterminate Forms

7 indeterminate forms are,

$$\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 1^\infty, 0^0, \infty^0, \infty \cdot 0$$

2 L'Hôpital's Rule

$$L = \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} \iff \frac{f(a)}{g(a)} = \frac{0}{0} \vee \frac{\infty}{\infty}$$

3 Taylor Expansions

$$1. e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \dots \infty$$

$$2. a^x = e^{x \ln a} = \sum_{n=0}^{\infty} \frac{(x \ln a)^n}{n!} = 1 + x \ln a + \frac{(x \ln a)^2}{2!} + \dots \infty \quad a \in \mathbb{R}^+$$

$$3. (1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \dots \infty$$

$$4. \ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \infty \quad x \in (-1, 1]$$

$$5. \frac{x^n - a^n}{x - a} = \sum_{r=0}^{n-1} a^r \cdot x^{n-r-1} = x^{n-1} + ax^{n-2} + a^2x^{n-3} + \dots + a^{n-1}$$

$$6. (1+x)^{\frac{1}{x}} = e \left(1 - \frac{x}{2} + \frac{11x^2}{24} + \dots \infty \right)$$

$$7. \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \infty$$

$$8. \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \infty$$

$$9. \tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \dots \infty$$

$$10. \sin^{-1} x = x + \frac{1^2 x^3}{3!} + \frac{1^2 \cdot 3^2 x^5}{5!} + \frac{1^2 \cdot 3^2 \cdot 5^2 x^7}{7!} + \dots \infty$$

$$11. \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} + \dots \infty$$

$$12. \sec^{-1} x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + \dots \infty$$

$$13. (\sin^{-1} x)^2 = \frac{2x^2}{2!} + \frac{2 \cdot 2^2 x^4}{4!} + \frac{2 \cdot 2^2 \cdot 4^2 x^6}{6!} + \dots \infty$$

$$14. x \cot x = 1 - \frac{x^3}{3} + \frac{x^4}{45} - \frac{2x^6}{945} + \dots \infty$$

$$15. \sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \dots \infty$$

$$16. x \csc x = 1 + \frac{x^2}{6} + \frac{7x^4}{360} + \frac{31x^6}{15120} + \dots \infty$$

4 Standard Limits

$$1. \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = n \cdot a^{n-1} \quad a \in \mathbb{R}$$

$$2. \lim_{x \rightarrow \infty} \frac{a_m x^m + a_{m-1} x^{m-1} + \dots + a_1 x + a_0}{b_n x^n + b_{n-1} x^{n-1} + \dots + b_1 x + b_0} = \begin{cases} 0 & m < n \\ \frac{a_m}{b_n} & m = n \\ +\infty & m > n \wedge a_m \cdot b_n > 0 \\ -\infty & m > n \wedge a_m \cdot b_n < 0 \end{cases}$$

$$3. \text{ If } A, B \in \{\sin x, \tan x, \sin^{-1} x, \tan^{-1} x, x\},$$

$$\text{then } \lim_{x \rightarrow 0} \frac{A}{B} = 1$$

$$4. \lim_{x \rightarrow 0} \frac{\sin x^\circ}{x} = \frac{\pi}{180^\circ}$$

$$5. \lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1$$

$$6. \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$$

$$7. \lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \ln a$$

$$8. \lim_{x \rightarrow \infty} \left(1 + \frac{n}{x}\right)^x = e^n$$

$$9. \lim_{x \rightarrow 0} (1 + nx)^{\frac{1}{x}} = e^n$$

5 Algebra of Limits

If $\lim_{x \rightarrow a} f(x) = L \wedge \lim_{x \rightarrow a} g(x) = M$ where $L, M \in \mathbb{R}$ and are finite then

- $\lim_{x \rightarrow a} (f(x) \pm g(x)) = L \pm M$
- $\lim_{x \rightarrow a} (f(x) \cdot g(x)) = L \cdot M$
- $\lim_{x \rightarrow a} \left(\frac{f(x)}{g(x)} \right) = \frac{L}{M} \quad M \neq 0$
- $\lim_{x \rightarrow a} k \cdot f(x) = k \cdot L \quad k \in \mathbb{R}$