Continuity and Differentiability

Continuity

1 Continuity of a Function at a Point

A function $f(x)$ is said to be continuous at $x = a$; where $a \in$ domain of $f(x)$
Iff

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$$

2 Continuity at End Points

Let $f(x)$ be defined on $[a, b]$ then
$f(x)$ is continuous at $x = a$
Iff

$$\lim_{x\to a^+} f(x) = f(a)$$

$f(x)$ is continuous at $x = b$
Iff

$$\lim_{x\to b^-} f(x) = f(b)$$

3 Discontinuity of a Function

A function f is discontinuous if it has any kind of "Breaks".

3.1 Removable Discontinuity

$\lim_{x\to a} f(x)$ exists but it is not equal to $f(a)$ or $f(a)$ is not defined.

3.1.1 Missing Point Discontinuity

$\lim_{x\to a} f(x)$ exists but $f(a)$ is not defined.

3.1.2 Isolated Point Discontinuity

$\lim_{x \to a} f(x)$ exists but it is not equal to $f(a)$

3.2 Non-Removable Discontinuity

$\lim_{x \to a} f(x)$ does not exist.

3.2.1 Finite Discontinuity (Jump Discontinuity)

$\lim_{x \to a^-} f(x) = L_1$, $\lim_{x \to a^+} f(x) = L_2$, $L_1 \neq L_2, L_1$ and L_2 are finite.

3.2.1.1 Jump of Discontinuity

$|L_1 - L_2|$ is the Jump of Discontinuity

3.2.1.2 Piecewise Continuous or Sectionally continuous Function

A function having a finite number of jumps in a given interval.

3.2.2 Infinite Discontinuity

$\lim_{x \to a^-} f(x) = L_1$ and $\lim_{x \to a^+} f(x) = L_2$. Either L_1 or L_2 is $\pm\infty$

For graph of $y = f(x)$, if at $x = a$ there is a vertical Asymptote, then there is a Infinite Discontinuity at $x = a$

3.2.3 Oscillatory Discontinuity

$\lim_{x \to a} f(x)$ doesn't exist but oscillates between two finite quantites.
e.g. $f(x) = \sin \frac{1}{x}$

4 Theorems on Continuity

4.1 Theorem 1

Sum, difference and product of two continuous functions is always a continuous function.
However, quotient $h(x) = \frac{f(x)}{g(x)}$ is continuous at $x = a$ only if $g(a) \neq 0$.

4.2 Theorem 2

If $f(x)$ is continuous and $g(x)$ is discontinuous at $x = a$, then the product function $\phi(x) = f(x) \cdot g(x)$is not necessarily be discontinuous at $x = a$.

4.3 Theorem 3

If $f(x)$ and $g(x)$ both are discontinuous at $x = a$, then the product function $\phi(x) = f(x) \cdot g(x)$ is not necessarily be discontinuous at $x = a$.

5 Continuity of Composite Functions

Let $f(x)$ and $g(x)$ be two discontinuous functions and $g(x)$ is discontinuous $\forall\ x \in [\alpha, \beta]$ $\{\}$

$$On$$

$$Hold$$

6 Intermediate Value Theorem

If $f(x)$ is continuous $\forall\ x \in [\alpha, \beta]$ and $f(a) \neq f(b)$, then for any value $L \in (f(a), f(b))$, there exists atleast one number $c \in (a, b)$ for which $f(c) = L$

7 Continuity of Rational and Irrational Functions

Consider the function, $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} f_1(x) & x \in \mathbb{Q} \\ f_2(x) & x \notin \mathbb{Q} \end{cases}$$

then f is continous at $x \in A$ where $A = \{c : f_1(c) = f_2(c)\}$

Differentiability

8 Meaning of a Derivative

- The instantaneous rate of change of a function w.r.t. the independent variable is called the *Derivative*.
- Derivative also represents the slope of tangent line on a curve.
- Derivative of a function f is generally denoted by

$$f'(x), \frac{d}{dx}(f(x)), \frac{df(x)}{dx}, \frac{d}{dx}f(x)$$

- Fundamental Definition of a Derivative

$$f(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

- Evalution of Derivative at a point (Say a) is denoted by

$$f'(a), \frac{d}{dx}(f(x))\bigg|_{x=a}$$

9 Existance of a Derivative

The derivative of a function f exists at $x = a$,
iff,

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

exists and is finite
Derivative doesn't exist at sharp points.

9.1 Right Hand Derivative

The Right Hand Derivative of $f(x)$ at $x = a$ is

$$f'(a^+) = \lim_{h \to 0^+} \frac{f(a + h) - f(a)}{h}$$

9.2 Left Hand Derivative

The Left Hand Derivative of $f(x)$ at $x = a$ is

$$f'(a^-) = \lim_{h \to 0^-} \frac{f(a + h) - f(a)}{h}$$

10 Relation between Continuity and Differentiability

If a function is differentiable at a point, it is necessarily continuous at that point. But the converse is not necessarily true.

11 Differentiability in an Interval

- A function f defined in an open interval (a, b) is said to be differentiable in open interval (a, b), if it is differentiable at each point in (a, b).
- A function f defined in a close interval $[a, b]$ is said to differentiable at end points a and b, if RHD at a and LHD at b both exist and are finite.

12 Theorems on Differentiability

12.1 Theorem 1

If $f(x)$ and $g(x)$ are both differentiable at $x = a$, $f(x) \pm g(x), f(x) \cdot g(x)$ will also be differentiable at $x = a$ but $\dfrac{f(x)}{g(x)}$ is differentiable only at $x = a$ if $g(a) \neq 0$.

12.2 Theorem 2

If $f(x)$ is differentiable at $x = a$ and $g(x)$ is not differentiable at $x = a$, then $f(x) \pm g(x)$ will not be differentiable at $x = a$,

However nothing can be said about the product function $f(x) \cdot g(x)$.

12.3 Theorem 3

If both $f(x)$ and $g(x)$ are not differentiable at $x = a$, then nothing can be said about the sum, difference, product function.

12.4 Theorem 4

If $f(x)$ is differentiable at $x = a$ and $f(a)$ and $g(x)$ is continous at $x = a$

Then, the product function $f(x) \cdot g(x)$ will be differentiable at $x = a$

12.5 Theorem 5

Derivative of a continous function need not be a continous