Definite Integration

1 Definite Integral

Let $f(x)$ be a function defined in the closed interval $[a, b]$ and $F(x)$ be its anti-derivative, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

is called the definite integral of the function $f(x)$ over the interval $[a, b]$, a and b are called limits of integration, lower and upper limit respectively.

2 Geometrical Interpretation of Definite Integral

If $f(x) > 0 \ \forall \ x \in [a, b]$, then $\int_a^b f(x)\,dx$ is numerically equal to the area bounded by the curves $y = f(x), y = 0, x = a, x = b$

In general, $\int_a^b f(x)\,dx$ represents the net signed area (or algebraic sum of areas) i.e area below the axis of x is counted as $-ve$ and that above is counted as $+ve$

3 Definite Integration by *u-sub*

To evaluate definite integral of type,

$$I = \int_a^b f(x)g'(x)\,dx$$

Let

$$u = g(x) \implies du = g'(x)\,dx$$

Now, I trasforms to,

$$I = \int_{g(a)}^{g(b)} f(u)\,du$$

Important Note

- For the substitution to be valid, it must be continuous in the interval of integration, i.e. If $u = g(x)$, then $g(x)$ must be continuous in $[a, b]$.

4 Properties of Definite Integration

4.1 Definite Integration is independent of the change of variable

$$\int_a^b f(x)\,dx = \int_a^b f(u)\,du$$

4.2 If limits of definite integral are flipped, then its value only differs in sign

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

5 King's Rule

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

6 Integration of Piecewise Functions

6.1

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$c \in \mathbb{R}$$

6.2

$$\int_0^a f(x)\,dx = \int_0^{a/2} f(x)\,dx + \int_0^{a/2} f(a-x)\,dx$$

7 Integration of Even, Odd Functions

$$\int_{-a}^a f(x)\,dx = \begin{cases} 0 & f(x) = -f(-x) \text{ Odd symmetric about } x = 0 \\ 2\int_0^a f(x)\,dx & f(x) = f(-x) \text{ Even symmetric about } x = 0 \end{cases}$$

8 Integration in case of Even, Odd Symmetries

$$\int_a^b f(x)\,dx = \begin{cases} 0 & f(a+x) = -f(b-x) \quad \text{Odd symmetric} \\ & \text{or } f\left(\frac{a+b}{2} - x\right) = -f\left(\frac{a+b}{2} + x\right) \text{ about } x = \frac{a+b}{2} \\ 2\int_{(a\,+\,b)/2}^{b} f(x)\,dx & f(a+x) = f(b-x) \quad \text{Even symmetric} \\ & \text{or } f\left(\frac{a+b}{2} - x\right) = f\left(\frac{a+b}{2} + x\right) \text{ about } x = \frac{a+b}{2} \end{cases}$$

9 Change of Limits property

$$\int_a^b f(x)\,dx = (b-a)\int_0^1 f[(b-a)x + a]\,dx$$

Now, we can add any two (or more) integrals as they all will have same limits 0 and 1.

10 Even and Odd Integral Functions

10.1 Even Integral Functions

If $O(x)$ is Odd Symmetric about $x = 0$

then, $f(x) = \displaystyle\int_a^x O(x)\,dx$ is Even Symmetric about $x = 0$

10.2 Odd Integral Functions

If $E(x)$ is Even Symmetric about $x = 0$ and $\displaystyle\int_0^a E(x)\,dx = 0$

then, $f(x) = \displaystyle\int_a^x E(x)\,dx$ is Odd symmetric about $x = 0$

11 Integration of Periodic Functions

Let $f(x)$ be a function with fundamental period T

11.1 Area under a periodic curve repeats with same periodicity

$$\int_0^{nT} f(x)\,dx = n\int_0^T f(x)\,dx$$

11.2 Area under a periodic curve over an interval with length same as that of the period is same as the area enclosed in first period

$$\int_a^{a+T} f(x)\,dx = \int_0^T f(x)\,dx$$

Also,

i.
$$\int_a^{a+nT} f(x)\,dx = \int_0^{nT} f(x)\,dx = n\int_0^T f(x)\,dx$$

ii.
$$\int_{a+nT}^{b+nT} f(x)\,dx = \int_a^b f(x)\,dx$$

iii.
$$\int_a^{b+nT} f(x)\,dx = \int_a^b f(x)\,dx + n\int_0^T f(x)\,dx$$

iv.
$$\int_{n_1T}^{n_2T} f(x)\,dx = (n_2 - n_1)\int_0^T f(x)\,dx$$

v.
$$\int_{a+n_1T}^{b+n_2T} f(x)\,dx = \int_a^b f(x)\,dx + (n_2 - n_1)\int_0^T f(x)\,dx$$

12 Newton-Leibnitz's Rule for Differentiation under Integral sign

12.1 General Case

If $\Phi(x)$ and $\Psi(x)$ are defined on $[a,b]$and are diffenetiable on (a,b) and $f(x,t)$ is continuous, then

$$\frac{d}{dx}\left[\int_{\Phi(x)}^{\Psi(x)} f(x,t)\,dt\right] = \int_{\Phi(x)}^{\Psi(x)} \frac{\partial}{\partial x} f(x,t)\,dt\ + \Psi'(x) f(x,\Psi(x)) + \Phi'(x) f(x,\Phi(x))$$

Corollary 1

$$\frac{d}{dx}\left[\int_{\Phi(x)}^{\Psi(x)} f(t)\,dt\right] = \Psi'(x) f(\Psi(x)) + \Phi'(x) f(\Phi(x))$$

Corollary 2

$$\frac{d}{dx}\left[\int_a^b f(x,t)\,dt\right] = \int_a^b \frac{\partial}{\partial x} f(x,t)\,dt$$

13 Fundamental Definition of Definite Integration, *Integration as limit of a sum*

13.1 Definite Integral as limit of Riemann Sum

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

- The interval $[a, b]$ is divided into n partitions.
- $a = x_0 < x_1 < x_2 < x_3 < \ldots < x_{n-1} < x_n = b$
- x_i^* is a point lying in the interval (x_{i-1}, x_i) $\qquad i \in \mathbb{Z}^+$
- $f(x_i^*)$ gives height of formed rectangle, $\Delta x_i = x_i - x_{i-1}$ is its width.

13.1.1 Definite Integral as limit of Upper Riemann Sum

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

$f(x_i^*) = max\,\{f(x_{i-1}), f(x_i)\}$

13.1.2 Definite Integral as limit of Lower Riemann Sum

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

$f(x_i^*) = min\,\{f(x_{i-1}), f(x_i)\}$

13.1.3 Definite Integral as limit of Left Endpoint Riemann Sum

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

i. $x_i^* = x_{i-1} = \dfrac{(b-a)}{n}(i-1)$

ii. $x_0 = x_1 = x_2 = \ldots = x_{n-1} = x_n$

iii. $\Delta x = \dfrac{b-a}{n}$

13.1.4 Definite Integral as limit of Midpoint Riemann Sum

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

i. $x_i^* = \dfrac{x_{i-1}+x_i}{2} = \dfrac{(b-a)}{n}(2i-1)$

ii. $x_0 = x_1 = x_2 = \ldots = x_{n-1} = x_n$

iii. $\Delta x = \dfrac{b-a}{n}$

13.1.5 Definite Integral as limit of Right Endpoint Riemann Sum

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

i. $x_i^* = x_i = \dfrac{(b-a)}{n} i$

ii. $x_0 = x_1 = x_2 = \ldots = x_{n-1} = x_n$

iii. $\Delta x = \dfrac{b-a}{n}$

13.2 $\dfrac{r}{n}$ form

$$\lim_{n\to\infty} \frac{1}{n} \sum_{r=\Phi(n)}^{\Psi(n)} f\left(\frac{r}{n}\right) = \int_a^b f(x)\,dx$$

$a = \lim_{n\to\infty} \dfrac{\Phi(n)}{n}, b = \lim_{n\to\infty} \dfrac{\Psi(n)}{n}$

14 Inequalities involving Definite Integrals

14.1

$$\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx$$

Equality holds when $f(x)$ is of same sign $\forall\, x \in [a,b]$

14.2

$$\left| \int_a^b f(x)g(x)\,dx \right| \leq \sqrt{\left(\int_a^b f^2(x)\,dx \right) \left(\int_a^b g^2(x)\,dx \right)}$$

14.3

If $f(x)$ is continuous on $[a, b]$ and $f_l(x)$ and $f_h(x)$ are also continuous on $[a, b]$ such that,

$$f_l(x) \leq f(x) \leq f_h(x) \qquad \forall\, x \in [a, b]$$

then,

$$\int_a^b f_l(x)\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b f_h(x)\,dx$$

14.4

If m and M be global minimum and global maximum of $f(x)$ respectively in $[a, b]$, then

$$m(b - a) \leq \int_a^b f(x)\,dx \leq M(b - a)$$

15 Gamma Function

Let $s > 0$, then the improper integral

$$\begin{aligned} \Gamma(s) &= \int_0^\infty e^{-t}\, t^{s-1}\, dt \\ &= \int_0^1 \left(\ln\left(\frac{1}{t}\right) \right)^{s-1} dt \end{aligned}$$

is defined to be the Gamma Function.

15.1 Properties of Gamma Function

i. $\Gamma(s + 1) = s \cdot \Gamma(s)$

ii. $\Gamma(n + 1) = n! \qquad n \in \mathbb{N}_0$

iii. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

iv. $\displaystyle\int_0^{\pi/2} \sin^m x \cdot \cos^n x\,dx = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$

v. $\Gamma(s) \cdot \Gamma(1-s) = \dfrac{\pi}{\sin s\pi}$ $\qquad s \in (0,1)$

vi. $\Gamma(s) \cdot \Gamma\left(s + \dfrac{1}{2}\right) = \dfrac{\sqrt{\pi}}{2^{2s-1}} \cdot \Gamma(2s)$

vii. $\displaystyle\prod_{r=1}^{n-1} \Gamma\left(\frac{r}{n}\right) = \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}}$

16 Beta Function

Let $m, n > 0$, then the integral

$$\begin{aligned} B(m,n) &= \int_0^1 t^{m-1}(1-t)^{n-1}\, dt \\ &= \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}\, dt \end{aligned}$$

is defined to be the Beta Function.

16.1 Properties of Beta Function

i. $B(m,n) = B(n,m)$

ii. $B(m,n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$

17 Walli's Formula

17.1

To evaluate integrals of form,

$$\int_0^{\pi/2} \sin^m x \cdot \cos^n x\, dx$$

where, $m, n \in \mathbb{Z}^+$

$$\int_0^{\pi/2} \sin^m x \cdot \cos^n x \, dx = \int_0^{\pi/2} \sin^n x \cdot \cos^m x \, dx$$

$$= \frac{\prod\limits_{r\geq 0}(m-2r-1)\cdot\prod\limits_{r\geq 0}(n-2r-1)}{\prod\limits_{r\geq 0}(m+n-2r)}\cdot\frac{\pi}{2} \qquad (m,n\in\{2k:k\in\mathbb{N}\})$$

$$= \frac{\prod\limits_{r\geq 0}(m-2r-1)\cdot\prod\limits_{r\geq 0}(n-2r-1)}{\prod\limits_{r\geq 0}(m+n-2r)} \qquad (\text{Any one of } m,n \text{ is odd})$$

17.2

$$\int_0^{\pi/2} \sin^n x \, dx = \int_0^{\pi/2} \cos^n x \, dx$$

$$= \frac{\pi}{2}\cdot\prod_{r\geq 0}\frac{n-2r-1}{n-2r} \qquad n\in\{2k:k\in\mathbb{N}\}$$

$$= \prod_{r\geq 0}\frac{n-2r-1}{n-2r} \qquad n\in\{2k-1:k\in\mathbb{N}\}$$