

Matrices

1 Types of Matrices

1.1 Row Matrix or Row Vector

A matrix which contains only **One Row**.

$$A = [a_{ij}]_{1 \times n}$$

1.2 Column Matrix or Column Vector

A matrix which contains only **One Column**.

$$A = [a_{ij}]_{n \times 1}$$

1.3 Rectangular Matrix

A matrix in which **no. of rows is not equal to no. of columns**.

$$A = [a_{ij}]_{m \times n}, m \neq n$$

1.4 Square Matrix

A matrix in which **no. of rows is equal to no. of columns**.

$$A = [a_{ij}]_{n \times n}$$

1.5 Diagonal Matrix

A **square matrix** in which **all non-diagonal entries are zero**.

$$A = [a_{ij}]_{n \times n}, a_{ij} = 0 \quad \forall \quad i \neq j$$

1.6 Scalar Matrix

A **Diagonal Matrix** in which **all diagonal entries are equal**.

$$A = [a_{ij}]_{n \times n}, a_{ij} = \begin{cases} a & i = j, a \in \mathbb{C} \\ 0 & i \neq j \end{cases}$$

1.7 Identity or Unit Matrix

A **Diagonal Matrix** in which **all diagonal entries equal 1**.

$$A = [a_{ij}]_{n \times n}, a_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

1.8 Singleton Matrix

A matrix which consists of **only one element**.

$$A = [a_{ij}]_{1 \times 1}$$

1.9 Triangular Matrix

1. Upper Triangular Matrix

A **Square Matrix** in which **all elements below principal diagonal are zero**.

$$A = [a_{ij}]_{n \times n}, a_{ij} = 0 \quad \forall \quad i > j$$

2. Lower Triangular Matrix

A **Square Matrix** in which **all elements above principal diagonal are zero**.

$$A = [a_{ij}]_{n \times n}, a_{ij} = 0 \quad \forall \quad i < j$$

1.10 Horizontal Matrix

A matrix in which **no. of rows is less than no. of columns**

$$A = [a_{ij}]_{m \times n}, m < n$$

1.11 Vertical Matrix

A matrix in which **no. of rows is greater than no. of columns**.

$$A = [a_{ij}]_{m \times n}, m > n$$

1.12 Null or Zero Matrix

A matrix in which **all elements equal zero**.

$$O = [a_{ij}]_{m \times n}, a_{ij} = 0 \quad \forall \quad i, j$$

1.13 Sub-Matrix

A matrix which is obtained from a given matrix **by deleting any no. of rows and no. of columns**.

e.g. $\begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}_{2 \times 2}$ is a sub-matrix of $\begin{bmatrix} 8 & 9 & 5 \\ 2 & 3 & 4 \\ 3 & -2 & 5 \end{bmatrix}_{3 \times 3}$

1.14 Comparable Matrices

Two Matrices $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{p \times q}$ are **comparable**, if $m = p \wedge n = q$

1.15 Idempotent Matrix

A square matrix whose square equals itself.

$$A = [a_{ij}]_{n \times n}, A^2 = A$$

- For a Idempotent Matrix, $A^n = A \quad \forall \quad n \in \mathbb{Z}^+$

1.16 Periodic Matrix

A square matrix A is called periodic,

If

$$A^{k+1} = A, k \in \mathbb{Z}^+$$

- The least value of all such k is called the period of matrix A .
- Period of an Idempotent Matrix is 1.

1.17 Nilpotent Matrix

A square matrix $A = [a_{ij}]_{n \times n}$ which satisfies

$$A^k = O, A^{k-1} \neq O$$

1.18 Involutory Matrix

A square matrix A which satisfies,

$$A^2 = I$$

- For Involutory Matrix, $A = A^{-1}$

1.19 Symmetric Matrix

A square matrix $A = [a_{ij}]_{n \times n}$ in which $A = A^T$.

- In a Symmetric Matrix, elements situated at equal distance from the diagonal are equal.
- For any Square Matrix A with $a_{ij} \in \mathbb{R}$, then $A + A^T$ is symmetric.
- If A is a Square Matrix, then AA^T and $A^T A$ are symmetric matrices.
- All positive integral powers of a symmetric matrix are symmetric.

$$\text{If } A = [a_{ij}]_{n \times n}, a_{ij} = a_{ji} \text{ Then } A^n = [\alpha_{ij}]_{n \times n}, \alpha_{ij} = \alpha_{ji}$$

- If A be a symmetric matrix and B be a square matrix of order that of A , then $-A, kA, A^T, A^{-1}, A^n, B^T A B$ are also symmetric. $n \in \mathbb{N}, k \in \mathbb{C}$
- Let A and B be two symmetric matrices of same order, then
 - a. $A \pm B, AB + BA$ are symmetric.
 - b. $AB - BA$ is skew-symmetric.
 - c. AB is symmetric iff, $AB = BA$

1.20 Skew-Symmetric Matrix

A Square Matrix $A = [a_{ij}]_{n \times n}$ in which $a_{ij} = -a_{ji} \quad \forall \quad i, j$

- In a Skew-Symmetric Matrix all diagonal entries are **zero**.
- In a Skew-Symmetric Matrix, the elements situated at equal distance from the diagonal are equal in magnitude but opposite in sign.
- If A is a Skew-Symmetric Matrix, then $Tr(A) = 0$
- For any Square Matrix A with $a_{ij} \in \mathbb{R}$, then $A - A^T$ is a Skew-Symmetric Matrix.
- Every Square Matrix can be uniquely expressed as the sum of a symmetric and skew-symmetric matrix.

$$A = \frac{1}{2} (A + A^T) + \frac{1}{2} (A - A^T)$$

- All positive odd integral powers of a skew-symmetric matrix are skew-symmetric.

$$\text{If } A = [a_{ij}]_{m \times m}, a_{ij} = -a_{ji} \text{ Then } A^{2n+1} = [\alpha_{ij}]_{m \times m}, \alpha_{ij} = -\alpha_{ji}, n \in \mathbb{N}$$

- All positive even integral powers of a skew-symmetric matrix are symmetric.

$$\text{If } A = [a_{ij}]_{m \times m}, a_{ij} = -a_{ji} \text{ Then } A^{2n} = [\alpha_{ij}]_{m \times m}, \alpha_{ij} = \alpha_{ji}, n \in \mathbb{N}$$

- If A be a skew-symmetric matrix and B be a square matrix of order that of A then, $kA, B^T A B$ are also skew-symmetric.
- Let A and B be two skew-symmetric matrices of same order, then
 - a. $A \pm B, AB - BA$ are skew-symmetric.
 - b. $AB + BA$ is symmetric
- Let A be skew-symmetric matrix and C be column vector, then $C^T A C = 0$

1.21 Orthogonal Matrix

A square matrix whose transpose is its inverse.

$$A A^T = I$$

- $A^{-1} = A^T$
- Let A and B be Orthogonal, then AB is also Orthogonal.
- If A is orthogonal, then A^{-1}, A^T are also Orthogonal.

1.22 Hermitian Matrix

A square matrix which is equal to its conjugate transpose matrix.

$$\text{If } A_{n \times n} = A_{n \times n}^\theta \quad \vee \quad A = [a_{ij}]_{n \times n}, a_{ij} = \overline{a_{ji}}, \text{ Then } A \text{ is Hermitian}$$

- The diagonal entries of a Hermitian Matrix are \mathbb{R}
- Elements situated at equal distance from diagonal are complex conjugate of each other.
- Let A be any square matrix, then $A + A^\theta$ is a Hermitian Matrix.
- If A be a square matrix, then $AA^\theta, A^\theta A$ are Hermitian.
- If A is Hermitian Matrix, then

- a. iA is Skew-Hermitian Matrix. $i^2 = -1$
- b. \bar{A} is also a Hermitian Matrix.
- c. kA is Hermitian Matrix. $k \in \mathbb{R}$
- d. If A, B are Hermitian Matrices of same order, then
 - a. $k_1A + k_2B$ is also Hermitian. $k_1, k_2 \in \mathbb{R}$
 - b. AB is also Hermitian if $AB = BA$
 - c. $AB + BA$ is Hermitian.
 - d. $AB - BA$ is Skew-Hermitian.

1.23 Skew-Hermitian Matrix

A square matrix which is equal to negative of its conjugate transpose matrix.

If $A_{n \times n} = -A_{n \times n}^\theta \quad \vee \quad A = [a_{ij}]_{n \times n}, a_{ij} = -\bar{a}_{ji}$, Then A is Skew-Hermitian

- All diagonal entries of a Skew-Hermitian Matrix are purely imaginary or zero.
- In a Skew-Hermitian Matrix elements situated at equal distance from the diagonal only differ in the sign of real part.
- Let A be any square matrix, then $A - A^\theta$ is a Skew-Hermitian Matrix.
- Every square matrix can be uniquely expressed as the sum of a Hermitian and a Skew-Hermitian Matrix.

$$A_{n \times n} = \frac{1}{2} (A + A^\theta) + \frac{1}{2} (A - A^\theta)$$

- If A is a Skew-Hermitian Matrix, then
 - a. iA is a Hermitian Matrix. $i^2 = -1$
 - b. \bar{A} is also Skew-Hermitian.
 - c. kA is Skew-Hermitian. $k \in \mathbb{R}$
- If A, B are two Skew-Hermitian Matrices of same order, then $k_1A + k_2B$ is also Skew-Hermitian.

1.24 Unitary Matrix

A Square Matrix whose inverse is its conjugate transpose matrix.

$$AA^\theta = I \implies A^{-1} = A^\theta$$

- If A and B are unitary, then AB is also unitary.
- If A is unitary, then A^{-1}, A^T are also unitary.

1.25 Equivalent Matrices

Two matrices are said to be equivalent if one is obtained from the other by elementary operations.

$$A \sim B, A \text{ is equivalent to } B$$

- If $A \sim B \exists$ invertible matrices P and Q such that $B = PAQ$
- Every invertible square matrix can be expressed as the product of elementary matrices.

2 Trace of a Matrix

The **sum of all diagonal entries** of a **square matrix** is called its **Trace**.

$$A = [a_{ij}]_{n \times n}, \text{Tr}(A) = \sum_{i=1}^n a_{ii}$$

2.1 Properties of Trace of a Matrix

Let $A = [a_{ij}]_{n \times n}, B = [b_{ij}]_{n \times n}, k \in \mathbb{C}$

- $\text{Tr}(kA) = k \cdot \text{Tr}(A)$
- $\text{Tr}(A \pm B) = \text{Tr}(A) \pm \text{Tr}(B)$
- $\text{Tr}(AB) = \text{Tr}(BA)$
- $\text{Tr}(A) = \text{Tr}(A')$
- $\text{Tr}(I_n) = n$
- $\text{Tr}(AB) \neq \text{Tr}(A) \cdot \text{Tr}(B)$
- $\text{Tr}(A) = \text{Tr}(CAC^{-1})$ C is a non-singular square matrix of order n .

3 Determinant of a Square Matrix

Let $A = [a_{ij}]_{n \times n}$. The determinant formed by the elements of A is said to be the determinant of matrix $A = \det(A) = |A|$

Important Points

- a. $\det\left(\prod_{r=1}^n A_r\right) = \prod_{r=1}^n \det(A_r)$
- b. $\det(kA) = k^n \det(A)$ $k \in \mathbb{C}, n = \text{order of } A$
- c. $\det(A)$ exists $\iff A$ is a square matrix.
- d. $\det(A) = \det(A^T)$
- e. If A is Orthogonal Matrix, then $\det(A) = \pm 1$
- f. If A is Skew-Symmetric Matrix of odd order, then $\det(A) = 0$
- g. If A is Skew-Symmetric Matrix of even order, then $\det(A)$ is a perfect square.
- h. If $A = \text{diag}(a_1, a_2, \dots, a_n)$, then $\det(A) = \prod_{r=1}^n a_r$
- i. $\det(A^{-1}) = \frac{1}{\det(A)}$

Singular (Invertible) and non-singular (Non-Invertible) Matrix

If for $A = [a_{ij}]_{n \times n}$, $\det(A) = 0$ then A is called a singular matrix.
If $\det(A) \neq 0$ then it is called **non-singular** matrix.

4 Equal Matrices

Two matrices are equal, iff

- i. They are of same order.
- ii. The elements in the corresponding positions of the two matrices are equal.

Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{p \times q}$

Now, $A = B$ iff, $m = p, n = q, a_{ij} = b_{ij} \ \forall \ i, j$

5 Algebra of Matrices

5.1 Addition

If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$

Then,

$$A + B = [a_{ij} + b_{ij}]_{m \times n}$$

Properties of Matrix Addition

1. Addition of Matrices is **commutative**.

$$A + B = B + A$$

2. Addition of Matrices is **associative**.

$$(A + B) + C = A + (B + C)$$

3. Additive Identity is O (Null Matrix)

$$A + O = A$$

4. Additive Inverse is $-A$

$$A + (-A) = O$$

5. Cancellation law

$$A + B = A + C \implies B = C$$

5.2 Multiplication

5.2.1 Scalar Multiplication

The Matrix obtained by **multiplying every element of a matrix with a scalar** is called the scalar multiple of that matrix.

Let $A = [a_{ij}]_{m \times n}$, $k \in \mathbb{C}$

Now,

$$kA = [k \cdot a_{ij}]_{m \times n}$$

5.2.2 Matrix Multiplication

Conformable for Multiplication

Two matrices $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{p \times q}$ are conformable for the product AB , if the no. of columns in the pre-factor (A) is equal to the no. of rows in the post-factor (B)

$$AB \text{ exists iff } n = p$$

Multiplication of Matrices

Let $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{n \times q}$, $C = [c_{ij}]_{m \times q}$ s.t. $AB = C$

$$\therefore c_{ij} = \sum_{r=1}^n a_{ir} \cdot b_{rj} \quad \text{Row by Column Multiplication}$$

5.2.3 Properties of Matrix Multiplication

- i. Matrix Multiplication is not commutative.

$$AB \neq BA$$

- If $AB = BA$ then matrices A, B are said to commute.
- If $AB = -BA$ then matrices A, B are said to anti-commute.
- Diagonal Matrices of same order always commute.

- ii. Matrix Multiplication is associative.

$$A(BC) = (AB)C$$

- iii. If $A = [a_{ij}]_{m \times n}$, then $I_m A = A = AI_n$

- iv. Matrix Multiplication is distributive.

$$A(B + C) = AB + AC$$

Note: $A(B + C) \neq AB + CA$

- v. If product of two matrices is a null matrix then it is not necessary that one of the matrices is also a null matrix.

$$AB = O \not\Rightarrow A = O \vee B = O$$

If $AB = O$ then A, B are divisors of O

$$AB = O \implies \det(AB) = 0 \implies \det(A) \cdot \det(B) = 0$$

- vi. $AO = O$

- vii. $I^m = I \quad \forall m \in \mathbb{Z}^+$

- viii. if $A = [a_{ij}]_{n \times n}$

$$\text{then } \underbrace{A \cdot A \cdot A \dots A}_{m \text{ times}} = A^m \quad \forall m \in \mathbb{Z}^+$$

6 Pre-Multiplication and Post-Multiplication of Matrices

The Matrix AB is the matrix B **pre-multiplied** by A and the matrix BA is the matrix B **post-multiplied** by A .

7 Transpose of a Matrix

The matrix obtained by **interchanging rows and columns**.

$$A = [a_{ij}]_{m \times n}, A^T = [a_{ji}]_{n \times m}$$

- $A^T = A' = A^t = \text{Transpose of Matrix } A$

7.1 Properties of Transpose Matrices

- $(A^T)^T = A$
- $(A \pm B)^T = A^T \pm B^T$
- $(kA)^T = k \cdot A^T$
- $(AB)^T = B^T A^T$, In General, $\left(\prod_{r=1}^n A_r\right)^T = \prod_{r=1}^n A_{n+1-r}^T$
- $I^T = I$

8 Complex Conjugate of a Matrix

The Matrix obtained by replacing each element by its complex conjugate.

$$A = [a_{ij}]_{m \times n}, \bar{A} = [\bar{a}_{ij}]_{m \times n}$$

- If all elements are \mathbb{R} , then $A = \bar{A}$
- $\overline{\bar{A}} = A$
- $\overline{A + B} = \bar{A} + \bar{B}$
- $\overline{kA} = \bar{k} \cdot \bar{A}$ $k \in \mathbb{C}$
- $\overline{AB} = \bar{A} \bar{B}$

9 Conjugate Transpose of a Matrix

The complex conjugate of the transpose of a matrix.

$$A = [a_{ij}]_{m \times n}, A^\theta = \overline{A^T} = [\bar{a}_{ji}]_{n \times m}$$

If A and B are two matrices of same order, then

- $\overline{A^T} = (\bar{A})^T$

- ii. $(A^\theta)^\theta = A$
- iii. $(A + B)^\theta = A^\theta + B^\theta$
- iv. $(kA)^\theta = \bar{k}A^\theta$ $k \in \mathbb{C}$
- v. $(AB)^\theta = B^\theta A^\theta$

10 Adjoint of A Matrix

The Transpose of Square Matrix formed by its co-factors is its Adjoint.

Let $A = [a_{ij}]_{n \times n}$

then

$$\text{adj}(A) = [C_{ji}]_{n \times n}$$

10.1 Properties of Adjoint Matrix

- i. $A \cdot \text{adj}(A) = \text{adj}(A) \cdot A = \det(A)I$
- ii. $\det(\text{adj}(A)) = (\det(A))^{n-1}$
- iii. $\det(\underbrace{\text{adj}(\text{adj}(\text{adj}(\dots)))}_{m \text{ times adj}})) = \det(A)^{(n-1)^m}$
- iv. $(\text{adj}(A))^T = \text{adj}(A^T)$
- v. $\text{adj}(\text{adj}(A)) = (\det(A))^{n-2}A$
- vi. $\text{adj}(kA) = k^{n-1}\text{adj}(A)$ $k \in \mathbb{C}$
- vii. $\text{adj}(A^m) = (\text{adj}(A))^m$ $m \in \mathbb{N}$
- viii. $\det(A \cdot \text{adj}(A) + kI_n) = (\det(A) + k)^n$
- ix. Adjoint of a Diagonal Matrix is a Diagonal Matrix.
 If $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, then $\text{adj}(A) = \begin{bmatrix} bc & 0 & 0 \\ 0 & ca & 0 \\ 0 & 0 & ab \end{bmatrix}$
- x. $\text{adj}(I_n) = I_n$

11 Inverse of a Matrix

A square matrix is said to be Invertible, iff its determinant is non zero.

$$\det(A) \neq 0$$

$$A^{-1} = \frac{\text{adj}(A)}{\det(A)}$$

11.1 Properties of Inverse Matrix

- i. Every invertible matrix possesses a unique inverse.
- ii. $(AB)^{-1} = B^{-1}A^{-1}$
- iii. $(A^T)^{-1} = (A^{-1})^T$
- iv. $(A^k)^{-1} = (A^{-1})^k = A^{-k}$ $k \in \mathbb{N}$
- v. $(A^{-1})^{-1} = A$
- vi. $I^{-1} = I$
- vii. $\det(A^{-1}) = \frac{1}{\det(A)}$
- viii. Inverse of a diagonal matrix is a diagonal matrix.

$$\text{If } A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \text{ then } A^{-1} = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix}$$

12 Elementary Row Operations

- i. The interchange of i th row and j th row.

$$R_i \leftrightarrow R_j$$

- ii. Multiplication of i th row by a non zero constant k .

$$R_i \rightarrow k \cdot R_i$$

- iii. Addition of i th row to the elements of j th row multiplied by a non zero constant k .

$$R_i \rightarrow R_i + k \cdot R_j$$

13 Gauss Jordan Method for Computing Inverses

Let A be an invertible matrix. To find inverse of A using Gauss Jordan Method,

1. Create a matrix $G = [A \mid I]$
2. Now using elementary operations on G transform A to I .
3. In the process of transforming A to I , I transforms to A^{-1}

14 Matrix Polynomial

$$f(A) = a_n A^n + a_{n-1} A^{n-1} + a_{n-2} A^{n-2} + \dots + a_1 A + a_0 I$$

For $n \geq 2$, A should be a square matrix.

- If $\det(A) \neq 0$, $a_0 \neq 0$, $a_0 I + a_1 A + a_2 A^2 + \dots + a_n A^n = O$,
then

$$A^{-1} = \frac{1}{a_0} (a_1 I + a_2 A + a_3 A^2 + \dots + a_n A^{n-1})$$

15 Linear System of Equations

Consider,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$$

$$\vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \ddots \quad \quad \quad \vdots \quad \quad \quad \vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots + a_{nn}x_n = b_n$$

If $b_i = 0$, $i \in \{x : x \leq n, x \in \mathbb{N}\}$, then the above system is called Homogenous.
If not then it is Non-Homogenous.

In Matrix form,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$$

$$AX = B$$

$$X = A^{-1}B = \frac{\text{adj}(A)B}{\det(A)}$$

15.1 Types of Equations

I. If system is Non-Homogenous

- i. If $\det(A) \neq 0$, then system possesses a unique solution.
- ii. If $\det(A) = 0, \text{adj}(A)B \neq O$, then system possesses no solution.
- iii. If $\det(A) = 0, \text{adj}(A)B = O$, then system possesses ∞ solutions.

II. If system is Homogenous

- i. If $\det(A) \neq 0$, then system possesses only one trivial solution.
- ii. If $\det(A) = 0$, then system possesses non-trivial solutions and ∞ solutions.
- iii. If no. of equations $<$ no. of unknowns, then it has non-trivial solution.

16 Echelon Form of a Matrix

A matrix A is said to be in echelon form, if

- i. The first non-zero element in each row is 1.
- ii. Every non-zero row is A preceds every zero row (If any).
- iii. The number of zeroes before the first non-zero element in 1st, 2nd, ... rows should be in increasing order.

17 Rank of a Matrix

Rank of a Matrix A , $\rho(A) = \text{Number of non-zero rows in Echelon form of } A$

17.1 Properties of Rank of Matrices

Let $A = [a_{ij}]_{m \times n}, B = [b_{ij}]_{m \times n}$

- i. $\rho(A + B) \leq \rho(A) + \rho(B)$
- ii. $\rho(AB) \leq \rho(A) \wedge \rho(AB) \leq \rho(B)$
- iii. $\rho(A) = \rho(A^T)$