

# Matrices

## 1 Types of Matrices

### 1.1 Row Matrix or Row Vector

A matrix which contains only **One Row**.

$$A = [a_{ij}]_{1 \times n}$$

### 1.2 Column Matrix or Column Vector

A matrix which contains only **One Column**.

$$A = [a_{ij}]_{n \times 1}$$

### 1.3 Rectangular Matrix

A matrix in which **no. of rows is not equal to no. of columns**.

$$A = [a_{ij}]_{m \times n}, m \neq n$$

### 1.4 Square Matrix

A matrix in which **no. of rows is equal to no. of columns**.

$$A = [a_{ij}]_{n \times n}$$

### 1.5 Diagonal Matrix

A **square matrix** in which **all non-diagonal entries are zero**.

$$A = [a_{ij}]_{n \times n}, a_{ij} = 0 \quad \forall \quad i \neq j$$

## 1.6 Scalar Matrix

A **Diagonal Matrix** in which **all diagonal entries are equal**.

$$A = [a_{ij}]_{n \times n}, a_{ij} = \begin{cases} a & i = j, a \in \mathbb{C} \\ 0 & i \neq j \end{cases}$$

## 1.7 Identity or Unit Matrix

A **Diagonal Matrix** in which **all diagonal entries equal 1**.

$$A = [a_{ij}]_{n \times n}, a_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

## 1.8 Singleton Matrix

A matrix which consists of **only one element**.

$$A = [a_{ij}]_{1 \times 1}$$

## 1.9 Triangular Matrix

### 1. Upper Triangular Matrix

A **Square Matrix** in which **all elements below principal diagonal are zero**.

$$A = [a_{ij}]_{n \times n}, a_{ij} = 0 \quad \forall \quad i > j$$

### 2. Lower Triangular Matrix

A **Square Matrix** in which **all elements above principal diagonal are zero**.

$$A = [a_{ij}]_{n \times n}, a_{ij} = 0 \quad \forall \quad i < j$$

## 1.10 Horizontal Matrix

A matrix in which **no. of rows is less than no. of columns**

$$A = [a_{ij}]_{m \times n}, m < n$$

## 1.11 Vertical Matrix

A matrix in which **no. of rows is greater than no. of columns**.

$$A = [a_{ij}]_{m \times n}, m > n$$

### 1.12 Null or Zero Matrix

A matrix in which **all elements equal zero**.

$$O = [a_{ij}]_{m \times n}, a_{ij} = 0 \quad \forall \quad i, j$$

### 1.13 Sub-Matrix

A matrix which is obtained from a given matrix **by deleting any no. of rows and no. of columns**.

e.g.  $\begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}_{2 \times 2}$  is a sub-matrix of  $\begin{bmatrix} 8 & 9 & 5 \\ 2 & 3 & 4 \\ 3 & -2 & 5 \end{bmatrix}_{3 \times 3}$

### 1.14 Comparable Matrices

Two Matrices  $A = [a_{ij}]_{m \times n}$  and  $B = [b_{ij}]_{p \times q}$  are **comparable**, if  $m = p \wedge n = q$

### 1.15 Idempotent Matrix

A square matrix whose square equals itself.

$$A = [a_{ij}]_{n \times n}, A^2 = A$$

- For a Idempotent Matrix,  $A^n = A \quad \forall \quad n \in \mathbb{Z}^+$

### 1.16 Periodic Matrix

A square matrix  $A$  is called periodic,

If

$$A^{k+1} = A, k \in \mathbb{Z}^+$$

- The least value of all such  $k$  is called the period of matrix  $A$ .
- Period of an Idempotent Matrix is 1.

### 1.17 Nilpotent Matrix

A square matrix  $A = [a_{ij}]_{n \times n}$  which satisfies

$$A^k = O, A^{k-1} \neq O$$

### 1.18 Involutory Matrix

A square matrix  $A$  which satisfies,

$$A^2 = I$$

- For Involutory Matrix,  $A = A^{-1}$

### 1.19 Symmetric Matrix

A square matrix  $A = [a_{ij}]_{n \times n}$  in which  $A = A^T$ .

- In a Symmetric Matrix, elements situated at equal distance from the diagonal are equal.
- For any Square Matrix  $A$  with  $a_{ij} \in \mathbb{R}$ , then  $A + A^T$  is symmetric.
- If  $A$  is a Square Matrix, then  $AA^T$  and  $A^T A$  are symmetric matrices.
- All positive integral powers of a symmetric matrix are symmetric.

$$\text{If } A = [a_{ij}]_{n \times n}, a_{ij} = a_{ji} \text{ Then } A^n = [\alpha_{ij}]_{n \times n}, \alpha_{ij} = \alpha_{ji}$$

- If  $A$  be a symmetric matrix and  $B$  be a square matrix of order that of  $A$ , then  $-A, kA, A^T, A^{-1}, A^n, B^T A B$  are also symmetric.  $n \in \mathbb{N}, k \in \mathbb{C}$
- Let  $A$  and  $B$  be two symmetric matrices of same order, then
  - a.  $A \pm B, AB + BA$  are symmetric.
  - b.  $AB - BA$  is skew-symmetric.
  - c.  $AB$  is symmetric iff,  $AB = BA$

### 1.20 Skew-Symmetric Matrix

A Square Matrix  $A = [a_{ij}]_{n \times n}$  in which  $a_{ij} = -a_{ji} \quad \forall \quad i, j$

- In a Skew-Symmetric Matrix all diagonal entries are **zero**.
- In a Skew-Symmetric Matrix, the elements situated at equal distance from the diagonal are equal in magnitude but opposite in sign.
- If  $A$  is a Skew-Symmetric Matrix, then  $Tr(A) = 0$
- For any Square Matrix  $A$  with  $a_{ij} \in \mathbb{R}$ , then  $A - A^T$  is a Skew-Symmetric Matrix.
- Every Square Matrix can be uniquely expressed as the sum of a symmetric and skew-symmetric matrix.

$$A = \frac{1}{2} (A + A^T) + \frac{1}{2} (A - A^T)$$

- All positive odd integral powers of a skew-symmetric matrix are skew-symmetric.

$$\text{If } A = [a_{ij}]_{m \times m}, a_{ij} = -a_{ji} \text{ Then } A^{2n+1} = [\alpha_{ij}]_{m \times m}, \alpha_{ij} = -\alpha_{ji}, n \in \mathbb{N}$$

- All positive even integral powers of a skew-symmetric matrix are symmetric.

$$\text{If } A = [a_{ij}]_{m \times m}, a_{ij} = -a_{ji} \text{ Then } A^{2n} = [\alpha_{ij}]_{m \times m}, \alpha_{ij} = \alpha_{ji}, n \in \mathbb{N}$$

- If  $A$  be a skew-symmetric matrix and  $B$  be a square matrix of order that of  $A$  then,  $kA, B^T A B$  are also skew-symmetric.
- Let  $A$  and  $B$  be two skew-symmetric matrices of same order, then
  - a.  $A \pm B, AB - BA$  are skew-symmetric.
  - b.  $AB + BA$  is symmetric
- Let  $A$  be skew-symmetric matrix and  $C$  be column vector, then  $C^T A C = 0$

## 1.21 Orthogonal Matrix

A square matrix whose transpose is its inverse.

$$A A^T = I$$

- $A^{-1} = A^T$
- Let  $A$  and  $B$  be Orthogonal, then  $AB$  is also Orthogonal.
- If  $A$  is orthogonal, then  $A^{-1}, A^T$  are also Orthogonal.

## 1.22 Hermitian Matrix

A square matrix which is equal to its conjugate transpose matrix.

$$\text{If } A_{n \times n} = A_{n \times n}^\theta \vee A = [a_{ij}]_{n \times n}, a_{ij} = \overline{a_{ji}}, \text{ Then } A \text{ is Hermitian}$$

- The diagonal entries of a Hermitian Matrix are  $\mathbb{R}$
- Elements situated at equal distance from diagonal are complex conjugate of each other.
- Let  $A$  be any square matrix, then  $A + A^\theta$  is a Hermitian Matrix.
- If  $A$  be a square matrix, then  $A A^\theta, A^\theta A$  are Hermitian.
- If  $A$  is Hermitian Matrix, then

- a.  $iA$  is Skew-Hermitian Matrix.  $i^2 = -1$
- b.  $\overline{A}$  is also a Hermitian Matrix.
- c.  $kA$  is Hermitian Matrix.  $k \in \mathbb{R}$
- d. If  $A, B$  are Hermitian Matrices of same order, then
  - a.  $k_1A + k_2B$  is also Hermitian.  $k_1, k_2 \in \mathbb{R}$
  - b.  $AB$  is also Hermitian if  $AB = BA$
  - c.  $AB + BA$  is Hermitian.
  - d.  $AB - BA$  is Skew-Hermitian.

### 1.23 Skew-Hermitian Matrix

A square matrix which is equal to negative of its conjugate transpose matrix.

If  $A_{n \times n} = -A_{n \times n}^\theta \quad \vee \quad A = [a_{ij}]_{n \times n}, a_{ij} = -\overline{a_{ji}},$  Then  $A$  is Skew-Hermitian

- All diagonal entries of a Skew-Hermitian Matrix are purely imaginary or zero.
- In a Skew-Hermitian Matrix elements situated at equal distance from the diagonal only differ in the sign of real part.
- Let  $A$  be any square matrix, then  $A - A^\theta$  is a Skew-Hermitian Matrix.
- Every square matrix can be uniquely expressed as the sum of a Hermitian and a Skew-Hermitian Matrix.

$$A_{n \times n} = \frac{1}{2} (A + A^\theta) + \frac{1}{2} (A - A^\theta)$$

- If  $A$  is a Skew-Hermitian Matrix, then
  - a.  $iA$  is a Hermitian Matrix.  $i^2 = -1$
  - b.  $\overline{A}$  is also Skew-Hermitian.
  - c.  $kA$  is Skew-Hermitian.  $k \in \mathbb{R}$
- If  $A, B$  are two Skew-Hermitian Matrices of same order, then  $k_1A + k_2B$  is also Skew-Hermitian.

### 1.24 Unitary Matrix

A Square Matrix whose inverse is its conjugate transpose matrix.

$$AA^\theta = I \implies A^{-1} = A^\theta$$

- If  $A$  and  $B$  are unitary, then  $AB$  is also unitary.
- If  $A$  is unitary, then  $A^{-1}, A^T$  are also unitary.

### 1.25 Equivalent Matrices

Two matrices are said to be equivalent if one is obtained from the other by elementary operations.

$$A \sim B, A \text{ is equivalent to } B$$

- If  $A \sim B \exists$  invertible matrices  $P$  and  $Q$  such that  $B = PAQ$
- Every invertible square matrix can be expressed as the product of elementary matrices.

## 2 Trace of a Matrix

The **sum of all diagonal entries** of a **square matrix** is called its **Trace**.

$$A = [a_{ij}]_{n \times n}, \text{Tr}(A) = \sum_{i=1}^n a_{ii}$$

### 2.1 Properties of Trace of a Matrix

Let  $A = [a_{ij}]_{n \times n}, B = [b_{ij}]_{n \times n}, k \in \mathbb{C}$

- $\text{Tr}(kA) = k \cdot \text{Tr}(A)$
- $\text{Tr}(A \pm B) = \text{Tr}(A) \pm \text{Tr}(B)$
- $\text{Tr}(AB) = \text{Tr}(BA)$
- $\text{Tr}(A) = \text{Tr}(A')$
- $\text{Tr}(I_n) = n$
- $\text{Tr}(AB) \neq \text{Tr}(A) \cdot \text{Tr}(B)$
- $\text{Tr}(A) = \text{Tr}(CAC^{-1})$   $C$  is a non-singular square matrix of order  $n$ .

## 3 Determinant of a Square Matrix

Let  $A = [a_{ij}]_{n \times n}$ . The determinant formed by the elements of  $A$  is said to be the determinant of matrix  $A = \det(A) = |A|$

## Important Points

- a.  $\det\left(\prod_{r=1}^n A_r\right) = \prod_{r=1}^n \det(A_r)$
- b.  $\det(kA) = k^n \det(A)$   $k \in \mathbb{C}, n = \text{order of } A$
- c.  $\det(A)$  exists  $\iff A$  is a square matrix.
- d.  $\det(A) = \det(A^T)$
- e. If  $A$  is Orthogonal Matrix, then  $\det(A) = \pm 1$
- f. If  $A$  is Skew-Symmetric Matrix of odd order, then  $\det(A) = 0$
- g. If  $A$  is Skew-Symmetric Matrix of even order, then  $\det(A)$  is a perfect square.
- h. If  $A = \text{diag}(a_1, a_2, \dots, a_n)$ , then  $\det(A) = \prod_{r=1}^n a_r$
- i.  $\det(A^{-1}) = \frac{1}{\det(A)}$

## Singular (Invertible) and non-singular (Non-Invertible) Matrix

If for  $A = [a_{ij}]_{n \times n}$ ,  $\det(A) = 0$  then  $A$  is called a singular matrix.  
If  $\det(A) \neq 0$  then it is called **non-singular** matrix.

## 4 Equal Matrices

Two matrices are equal, iff

- i. They are of same order.
- ii. The elements in the corresponding positions of the two matrices are equal.

Let  $A = [a_{ij}]_{m \times n}$  and  $B = [b_{ij}]_{p \times q}$

Now,  $A = B$  iff,  $m = p, n = q, a_{ij} = b_{ij} \ \forall \ i, j$

## 5 Algebra of Matrices

### 5.1 Addition

If  $A = [a_{ij}]_{m \times n}$  and  $B = [b_{ij}]_{m \times n}$

Then,

$$A + B = [a_{ij} + b_{ij}]_{m \times n}$$



## Properties of Matrix Addition

1. Addition of Matrices is **commutative**.

$$A + B = B + A$$

2. Addition of Matrices is **associative**.

$$(A + B) + C = A + (B + C)$$

3. Additive Identity is  $O$  (Null Matrix)

$$A + O = A$$

4. Additive Inverse is  $-A$

$$A + (-A) = O$$

5. Cancellation law

$$A + B = A + C \implies B = C$$

## 5.2 Multiplication

### 5.2.1 Scalar Multiplication

The Matrix obtained by **multiplying every element of a matrix with a scalar** is called the scalar multiple of that matrix.

Let  $A = [a_{ij}]_{m \times n}$ ,  $k \in \mathbb{C}$

Now,

$$kA = [k \cdot a_{ij}]_{m \times n}$$

### 5.2.2 Matrix Multiplication

#### Conformable for Multiplication

Two matrices  $A = [a_{ij}]_{m \times n}$ ,  $B = [b_{ij}]_{p \times q}$  are conformable for the product  $AB$ , if the no. of columns in the pre-factor ( $A$ ) is equal to the no. of rows in the post-factor ( $B$ )

$$AB \text{ exists iff } n = p$$

#### Multiplication of Matrices

Let  $A = [a_{ij}]_{m \times n}$ ,  $B = [b_{ij}]_{n \times q}$ ,  $C = [c_{ij}]_{m \times q}$  s.t.  $AB = C$

$$\therefore c_{ij} = \sum_{r=1}^n a_{ir} \cdot b_{rj} \quad \text{Row by Column Multiplication}$$

### 5.2.3 Properties of Matrix Multiplication

- i. Matrix Multiplication is not commutative.

$$AB \neq BA$$

- If  $AB = BA$  then matrices  $A, B$  are said to commute.
- If  $AB = -BA$  then matrices  $A, B$  are said to anti-commute.
- Diagonal Matrices of same order always commute.

- ii. Matrix Multiplication is associative.

$$A(BC) = (AB)C$$

- iii. If  $A = [a_{ij}]_{m \times n}$ , then  $I_m A = A = AI_n$

- iv. Matrix Multiplication is distributive.

$$A(B + C) = AB + AC$$

Note:  $A(B + C) \neq AB + CA$

- v. If product of two matrices is a null matrix then it is not necessary that one of the matrices is also a null matrix.

$$AB = O \not\Rightarrow A = O \vee B = O$$

If  $AB = O$  then  $A, B$  are divisors of  $O$

$$AB = O \implies \det(AB) = 0 \implies \det(A) \cdot \det(B) = 0$$

- vi.  $AO = O$

- vii.  $I^m = I \quad \forall m \in \mathbb{Z}^+$

- viii. if  $A = [a_{ij}]_{n \times n}$

$$\text{then } \underbrace{A \cdot A \cdot A \dots A}_{m \text{ times}} = A^m \quad \forall m \in \mathbb{Z}^+$$

## 6 Pre-Multiplication and Post-Multiplication of Matrices

The Matrix  $AB$  is the matrix  $B$  **pre-multiplied** by  $A$  and the matrix  $BA$  is the matrix  $B$  **post-multiplied** by  $A$ .

## 7 Transpose of a Matrix

The matrix obtained by **interchanging rows and columns**.

$$A = [a_{ij}]_{m \times n}, A^T = [a_{ji}]_{n \times m}$$

- $A^T = A' = A^t = \text{Transpose of Matrix } A$

### 7.1 Properties of Transpose Matrices

- $(A^T)^T = A$
- $(A \pm B)^T = A^T \pm B^T$
- $(kA)^T = k \cdot A^T$
- $(AB)^T = B^T A^T$ , In General,  $\left(\prod_{r=1}^n A_r\right)^T = \prod_{r=1}^n A_{n+1-r}^T$
- $I^T = I$

## 8 Complex Conjugate of a Matrix

The Matrix obtained by replacing each element by its complex conjugate.

$$A = [a_{ij}]_{m \times n}, \bar{A} = [\bar{a}_{ij}]_{m \times n}$$

- If all elements are  $\mathbb{R}$ , then  $A = \bar{A}$
- $\overline{\bar{A}} = A$
- $\overline{A + B} = \bar{A} + \bar{B}$
- $\overline{kA} = \bar{k} \cdot \bar{A}$   $k \in \mathbb{C}$
- $\overline{AB} = \bar{A} \bar{B}$

## 9 Conjugate Transpose of a Matrix

The complex conjugate of the transpose of a matrix.

$$A = [a_{ij}]_{m \times n}, A^\theta = \overline{A^T} = [\bar{a}_{ji}]_{n \times m}$$

If  $A$  and  $B$  are two matrices of same order, then

- $\overline{A^T} = (\bar{A})^T$

- ii.  $(A^\theta)^\theta = A$
- iii.  $(A + B)^\theta = A^\theta + B^\theta$
- iv.  $(kA)^\theta = \bar{k}A^\theta$   $k \in \mathbb{C}$
- v.  $(AB)^\theta = B^\theta A^\theta$

## 10 Adjoint of A Matrix

The Transpose of Square Matrix formed by its co-factors is its Adjoint.

Let  $A = [a_{ij}]_{n \times n}$

then

$$\text{adj}(A) = [C_{ji}]_{n \times n}$$

### 10.1 Properties of Adjoint Matrix

- i.  $A \cdot \text{adj}(A) = \text{adj}(A) \cdot A = \det(A)I$
- ii.  $\det(\text{adj}(A)) = (\det(A))^{n-1}$
- iii.  $\det(\underbrace{\text{adj}(\text{adj}(\text{adj}(\dots)))}_{m \text{ times adj}})) = \det(A)^{(n-1)^m}$
- iv.  $(\text{adj}(A))^T = \text{adj}(A^T)$
- v.  $\text{adj}(\text{adj}(A)) = (\det(A))^{n-2} A$
- vi.  $\text{adj}(kA) = k^{n-1} \text{adj}(A)$   $k \in \mathbb{C}$
- vii.  $\text{adj}(A^m) = (\text{adj}(A))^m$   $m \in \mathbb{N}$
- viii.  $\det(A \cdot \text{adj}(A) + kI_n) = (\det(A) + k)^n$
- ix. Adjoint of a Diagonal Matrix is a Diagonal Matrix.  
 If  $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ , then  $\text{adj}(A) = \begin{bmatrix} bc & 0 & 0 \\ 0 & ca & 0 \\ 0 & 0 & ab \end{bmatrix}$
- x.  $\text{adj}(I_n) = I_n$
- xi.  $\text{adj}(\text{adj}(A)) = (\det(A))^{n-2} A$

## 11 Inverse of a Matrix

A square matrix is said to be Invertible, iff its determinant is non zero.

$$\det(A) \neq 0$$

$$A^{-1} = \frac{\text{adj}(A)}{\det(A)}$$

### 11.1 Properties of Inverse Matrix

- i. Every invertible matrix possesses a unique inverse.
- ii.  $(AB)^{-1} = B^{-1}A^{-1}$
- iii.  $(A^T)^{-1} = (A^{-1})^T$
- iv.  $(A^k)^{-1} = (A^{-1})^k = A^{-k}$   $k \in \mathbb{N}$
- v.  $(A^{-1})^{-1} = A$
- vi.  $I^{-1} = I$
- vii.  $\det(A^{-1}) = \frac{1}{\det(A)}$
- viii. Inverse of a diagonal matrix is a diagonal matrix.

$$\text{If } A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \text{ then } A^{-1} = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix}$$

## 12 Elementary Row Operations

- i. The interchange of  $i$ th row and  $j$ th row.

$$R_i \leftrightarrow R_j$$

- ii. Multiplication of  $i$ th row by a non zero constant  $k$ .

$$R_i \rightarrow k \cdot R_i$$

- iii. Addition of  $i$ th row to the elements of  $j$ th row multiplied by a non zero constant  $k$ .

$$R_i \rightarrow R_i + k \cdot R_j$$

## 13 Gauss Jordan Method for Computing Inverses

Let  $A$  be an invertible matrix. To find inverse of  $A$  using Gauss Jordan Method,

1. Create a matrix  $G = [A \mid I]$
2. Now using elementary operations on  $G$  transform  $A$  to  $I$ .
3. In the process of transforming  $A$  to  $I$ ,  $I$  transforms to  $A^{-1}$

## 14 Matrix Polynomial

$$f(A) = a_n A^n + a_{n-1} A^{n-1} + a_{n-2} A^{n-2} + \dots + a_1 A + a_0 I$$

For  $n \geq 2$ ,  $A$  should be a square matrix.

- If  $\det(A) \neq 0$ ,  $a_0 \neq 0$ ,  $a_0 I + a_1 A + a_2 A^2 + \dots + a_n A^n = O$ ,  
then

$$A^{-1} = \frac{1}{a_0} (a_1 I + a_2 A + a_3 A^2 + \dots + a_n A^{n-1})$$

## 15 Linear System of Equations

Consider,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$$

$$\vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \ddots \quad \quad \quad \vdots \quad \quad \quad \vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots + a_{nn}x_n = b_n$$

If  $b_i = 0$ ,  $i \in \{x : x \leq n, x \in \mathbb{N}\}$ , then the above system is called Homogenous.  
If not then it is Non-Homogenous.

In Matrix form,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$$

$$AX = B$$

$$X = A^{-1}B = \frac{\text{adj}(A)B}{\det(A)}$$

## 15.1 Types of Equations

I. If system is Non-Homogenous

- i. If  $\det(A) \neq 0$ , then system possesses a unique solution.
- ii. If  $\det(A) = 0, \text{adj}(A)B \neq O$ , then system possesses no solution.
- iii. If  $\det(A) = 0, \text{adj}(A)B = O$ , then system possesses  $\infty$  solutions.

II. If system is Homogenous

- i. If  $\det(A) \neq 0$ , then system possesses only one trivial solution.
- ii. If  $\det(A) = 0$ , then system possesses non-trivial solutions and  $\infty$  solutions.
- iii. If no. of equations  $<$  no. of unknowns, then it has non-trivial solution.

## 16 Echelon Form of a Matrix

A matrix  $A$  is said to be in echelon form, if

- i. The first non-zero element in each row is 1.
- ii. Every non-zero row is  $A$  preceds every zero row (If any).
- iii. The number of zeroes before the first non-zero element in 1st, 2nd, ... rows should be in increasing order.

## 17 Rank of a Matrix

Rank of a Matrix  $A$ ,  $\rho(A) = \text{Number of non-zero rows in Echelon form of } A$

### 17.1 Properties of Rank of Matrices

Let  $A = [a_{ij}]_{m \times n}, B = [b_{ij}]_{m \times n}$

- i.  $\rho(A + B) \leq \rho(A) + \rho(B)$
- ii.  $\rho(AB) \leq \rho(A) \wedge \rho(AB) \leq \rho(B)$
- iii.  $\rho(A) = \rho(A^T)$

## 18 Solutions to Linear System of Equations Using Rank Method

Consider,

$$\begin{aligned}
 a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n &= b_1 \\
 a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n &= b_2 \\
 a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n &= b_3 \\
 \vdots & \quad \quad \quad \vdots \quad \quad \quad \ddots \quad \quad \quad \vdots \quad \quad \quad \vdots \\
 a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n &= b_m
 \end{aligned}$$

In Matrix form,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}$$

$$AX = B$$

Now,

$$C = [A \ B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} & b_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} & b_m \end{bmatrix}$$

$C$  is called augmented matrix.

### 18.1 Types of Equations

- I. Consistent System, If  $\rho(A) = \rho(C)$ 
  - i. Unique Solution, If  $\rho(A) = \rho(C) = n = \text{no. of unknowns}$
  - ii.  $\infty$  Solutions, If  $\rho(A) = \rho(C) = r < n$
- II. Inconsistent System, If  $\rho(A) \neq \rho(C) \implies \text{no solution}$



## 19 Linear Transformations

### 19.1 Reflection Matrices

#### 19.1.1 Reflection through axis of $x$

The Matrix  $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$  describes the reflection of any point  $P(x, y)$  through the axis of  $x$ .

#### 19.1.2 Reflection through the axis of $y$

The Matrix  $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$  describes the reflection of any point  $P(x, y)$  through the axis of  $y$ .

#### 19.1.3 Reflection through the origin

The Matrix  $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$  describes the reflection of any point  $P(x, y)$  through the origin.

#### 19.1.4 Reflection through line $y = x$

The matrix  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$  describes the reflection of any point  $P(x, y)$  through the line  $y = x$

#### 19.1.5 Reflection through the line $y = x \tan \theta$

The matrix  $\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$  describes the reflection of any point  $P(x, y)$  through the line  $y = x \tan \theta$

#### 19.1.6 Rotation Through Angle $\theta$

The Matrix  $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$  describes a rotation of a vector through an angle  $\theta$ .

## 20 Eigen Vectors and Eigen Values

Let  $\nu$  be any non-zero vector satisfying,

$$A\nu = \lambda\nu$$

This scalar  $\lambda$  is called the Eigen Value (Characteristic Root) and the vector  $\nu$  is the Eigen vector (Characteristic Vector).

## 20.1 Important Properties of Eigen Values

- i. Any square matrix  $A$  and its transpose  $A^T$  have the same Eigen Values.
- ii. The sum of Eigen Values of a matrix is equal to its trace.
- iii. The product of Eigen Values of a matrix  $A$  is equal to its determinant.
- iv. If  $\lambda_i$  are the Eigen Value of  $A$ , then the Eigen Values of
  - a.  $kA$  are  $k\lambda_i$
  - b.  $A^m$  are  $\lambda_i^m$
  - c.  $A^{-1}$  are  $\lambda_i^{-1}$
- v. All Eigen Values of a real symmetric matrix are  $\mathbb{R}$  and the eigen vectors corresponding to two distinct eigen values are orthogonal.
- vi. All Eigen Values of a real skew-symmetric matrix are purely imaginary or zero.
- vii. An odd order skew-symmetric matrix is singular and hence has zero as an Eigen value.

## 21 Characteristic Equation

$$\det(A - \lambda I) = 0$$

is called characteristic equation of matrix  $A_{n \times n}$ .

The above equation reduces to a  $n$ th degree polynomial equation in  $\lambda$ .

## 22 Cayley Hamilton Theorem

Every square matrix  $A$  satisfies its characteristic equation  $\det(A - \lambda I) = 0$

$$a_0 + a_1\lambda + \dots + a_n\lambda^n = 0$$

By Cayley Hamilton Theorem,  $\lambda \rightarrow A$

$$a_0I + a_1A + a_2A^2 + \dots + a_nA^n = O$$