

# Indefinite Integration

## 1 Fundamental Definition of Indefinite Integration

If  $f$  and  $F$  are functions such that  $\frac{d}{dx}(F(x)) = f(x)$  then  $F$  is anti-derivative of  $f$  w.r.t.  $x$  symbolically,

$$\int f(x) dx = F(x) + C$$

where  $C$  is the constant of Integration

## 2 Anti-Derivatives of Some Standard Functions

i.  $\int k \cdot f(x) dx = k \cdot \int f(x) dx$

ii.  $\int [f_1(x) \pm f_2(x) \pm f_3(x) \pm \dots \pm f_n(x)] dx = \int f_1(x) dx \pm \int f_2(x) dx \pm \int f_3(x) dx \pm \dots \pm \int f_n(x) dx$

iii.  $\int x^n dx = \frac{x^{n+1}}{n+1} + C$   $n \neq -1$

iv.  $\int \frac{1}{x} dx = \ln |x| + C$

v.  $\int e^x dx = e^x + C$

vi.  $\int a^x dx = \frac{a^x}{\ln a} + C$

vii.  $\int \sin x dx = -\cos x + C$

viii.  $\int \cos x dx = \sin x + C$

$$\text{ix. } \int \sec^2 x \, dx = \tan x + C$$

$$\text{x. } \int \csc^2 x \, dx = -\cot x + C$$

$$\text{xi. } \int \sec x \tan x \, dx = \sec x + C$$

$$\text{xii. } \int \csc x \cot x \, dx = -\csc x + C$$

$$\text{xiii. } \int \cot x \, dx = \ln |\sin x| + C$$

$$\text{xiv. } \int \tan x \, dx = -\ln |\cos x| + C$$

$$\text{xv. } \int \sec x \, dx = \ln |\sec x + \tan x| + C$$

$$\text{xvi. } \int \csc x \, dx = \ln |\csc x - \cot x| + C$$

$$\text{xvii. } \int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + C$$

$$\text{xviii. } \int \frac{-1}{\sqrt{1-x^2}} \, dx = \cos^{-1} x + C$$

$$\text{xix. } \int \frac{1}{1+x^2} \, dx = \tan^{-1} x + C$$

$$\text{xx. } \int \frac{-1}{1+x^2} \, dx = \cot^{-1} x + C$$

$$\text{xxi. } \int \frac{1}{x\sqrt{x^2-1}} \, dx = \sec^{-1} x + C$$

$$\text{xxii. } \int \frac{-1}{x\sqrt{x^2-1}} \, dx = \csc^{-1} x + C$$

$$\text{xxiii. } \int \sqrt{x} \, dx = \frac{2x\sqrt{x}}{3} + C$$

$$\text{xxiv. } \int \frac{dx}{x^2-1} = \ln \left| \frac{x-1}{x+1} \right| + C$$

$$\text{xxv. } \int \frac{dx}{\sqrt{1+x^2}} = \ln \left| x + \sqrt{x^2+1} \right| + C$$

$$\text{xxvi. } \int \frac{dx}{\sqrt{x^2-1}} = \ln \left| x + \sqrt{x^2-1} \right| + C$$

$$\text{xxvii. } \int \sqrt{1-x^2} dx = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x + C$$

$$\text{xxviii. } \int \sqrt{x^2-1} dx = \frac{1}{2}x\sqrt{x^2-1} - \frac{1}{2}\ln \left| x + \sqrt{x^2-1} \right| + C$$

### Important Results

1. If  $F_1(x)$  and  $F_2(x)$  are two anti-derivatives of a function  $f(x)$ , then  $F_1(x)$  and  $F_2(x)$  only differ by a constant, i.e.

$$F_1(x) - F_2(x) = C$$

where,  $C$  is a  $\mathbb{R}$  constant.

2. If  $f(x)$  is continuous  $\forall x \in D_f$  and,  

$$\int f(x) dx = F(x) + C,$$
then  $F(x)$  always exists and is continuous.
3. If  $f(x)$  is discontinuous at  $x = x_1$ , then its anti-derivative can be continuous at  $x = x_1$ .
4. Anti-derivative of a periodic function may not be periodic.

## 3 Methods of Integration

### 3.1 $u$ substitution

Integrals of form,

$$I = \int f(g(x)) \cdot g'(x) dx$$

Can be solved by, the substitution,

$$u = g(x)$$

Differentiating both sides w.r.t.  $x$ ,

$$du = g'(x)dx$$

Now,

$$I = \int f(u) du$$

### 3.2 Integrals of form

$$\int \frac{dx}{ax^2 + bx + c}, \int \frac{dx}{\sqrt{ax^2 + bx + c}}, \int \sqrt{ax^2 + bx + c} dx$$

Using completing the square,

$$ax^2 + bx + c = a \left[ \left( x + \frac{b}{2a} \right)^2 + \frac{c}{a} - \frac{b^2}{4a} \right]$$

Now, using *u sub*, let

$$u = x + \frac{b}{2a}$$

The transformed integral can be integrated using previous methods.

### 3.3 Integrals of form

$$\int \frac{px + q}{ax^2 + bx + c} dx, \int \frac{px + q}{\sqrt{ax^2 + bx + c}} dx, \int (px + q) \sqrt{ax^2 + bx + c} dx$$

$$px + q = \lambda \frac{d}{dx} (ax^2 + bx + c) + \mu$$

Now, after finding  $\lambda, \mu$ ,

For the 1st part, use *u sub*,

Let

$$u = ax^2 + bx + c$$

2nd part of the integral can be integrated using previous methods.

### 3.4 Integrals of form

$$\int \frac{K(x)}{ax^2 + bx + c} dx$$

**where**  $\deg(K(x)) \geq 2$

By polynomial long division

$$\frac{K(x)}{ax^2 + bx + c} = Q(x) + \frac{R(x)}{ax^2 + bx + c}$$

$\deg(R(x)) \leq 1$ ,

Now, the integral

$$\int \frac{R(x)}{ax^2 + bx + c} dx$$

can be integrated using previous methods.

### 3.5 Integrals of form

$$\int \frac{ax^2 + bx + c}{px^2 + qx + r} dx, \int \frac{ax^2 + bx + c}{\sqrt{px^2 + qx + r}} dx, \int (ax^2 + bx + c) \sqrt{px^2 + qx + r} dx$$

on hold

$$ax^2 + bx + c = \lambda (px^2 + qx + r) + \mu \frac{d}{dx} (px^2 + qx + r) + \gamma$$

### 3.6 Trig. Integrals

#### 3.6.1 Integrals of form

$$\int \frac{dx}{a \cos^2 x + b \sin^2 x}, \int \frac{dx}{a + b \sin^2 x}, \int \frac{dx}{a + b \cos^2 x}$$

$$\int \frac{dx}{(a \sin x + b \cos x)^2}, \int \frac{dx}{a + b \sin^2 x + c \cos^2 x}$$

Steps -

1. Multiply numerator and denominator by  $\sec^2 x$
2. Replace  $\sec^2 x$  (if any) by  $1 + \tan^2 x$  except the one multiplied in step 1.
3. Let  $u = \tan x$ , then  $du = \sec^2 x dx$

Now, the transformed integral can be integrated using previous methods.

#### 3.6.2 Integrals of form

$$\int \frac{dx}{a \sin x + b \cos x}, \int \frac{dx}{a + b \sin x}, \int \frac{dx}{a + b \cos x}, \int \frac{dx}{a \sin x + b \cos x + c}$$

Steps -

1. Replace  $\sin x = \frac{2 \tan x/2}{1 + \tan^2 x/2}$  and  $\cos x = \frac{1 - \tan^2 x/2}{1 + \tan^2 x/2}$
2. Let  $u = \tan x/2$ ,  $du = \frac{1}{2} \sec^2 x/2 dx$  is already present in the numerator.

Now, the transformed integral can be integrated using previous methods.

### 3.6.2.1 Alternative Method to Integrate

$$I = \int \frac{dx}{a \sin x + b \cos x}$$

$$a \sin x + b \cos x = \sqrt{a^2 + b^2} \sin \left( x + \tan^{-1} \left( \frac{b}{a} \right) \right)$$

$$I = \frac{1}{\sqrt{a^2 + b^2}} \int \csc \left( x + \tan^{-1} \left( \frac{b}{a} \right) \right) dx$$

$$I = \frac{1}{\sqrt{a^2 + b^2}} \ln \left| \tan \left( \frac{x}{2} + \frac{1}{2} \tan^{-1} \left( \frac{b}{a} \right) \right) \right| + C$$

### 3.6.3 Integrals of form

$$\int \frac{p \cos x + q \sin x + r}{a \cos x + b \sin x + c} dx, \int \frac{p \cos x + q \sin x}{a \cos x + b \sin x} dx$$

Steps for (i),

1. Express  $Numerator = \lambda Denominator + \mu Derivative of denominator + \gamma$

Now, the transformed integral can be integrated using previous methods.

Steps for (ii),

1. Express  $Numerator = \lambda Denominator + \mu Derivative of Denominator$

Now, the transformed integral can be integrated using previous methods.

### 3.7 Integration by parts

$$\int uv dx = u \int v dx - \int \left( u' \int v dx \right) dx$$

$u$  is the function which has to be differentiated ( $D$ ),  $v$  is the function which has to be integrated ( $I$ )

### 3.8 Integral of form

$$I = \int e^{g(x)} (f(x)g'(x) + f'(x)) dx$$

$$I = e^{g(x)} \cdot f(x) + C$$

### 3.9 Integrals of form

$$S = \int e^{ax} \sin bx dx, C = \int e^{ax} \cos bx dx$$

$$S = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) + C_0, C = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) + C_{00}$$

## 4 Partial Fraction Decomposition

Let  $f(x) = \sum_{i=0}^n a_i x^i, g(x) = \sum_{i=0}^m b_i x^i$ .

We define a rational function  $h(x) = \frac{f(x)}{g(x)}$ ,

$h(x)$  is  $\begin{cases} \text{Proper Rational Function} & m > n \\ \text{Improper Rational Function} & m \leq n \end{cases}$

If  $h(x)$  is Improper, we make it Proper by polynomial long division, i.e.

$$h(x) = Q(x) + \frac{r(x)}{g(x)}$$

Clearly,  $\frac{r(x)}{g(x)}$  is Proper.

Assuming,  $h(x)$  is proper,

### 4.1 $g(x)$ is the product of non-repeating linear factors

Let

$$g(x) = L_1(x) \cdot L_2(x) \cdot \dots \cdot L_m(x)$$

where  $L_i(x)$  are linear functions. Then, we can expand  $\frac{f(x)}{g(x)}$  in terms of partial fractions as,

$$\frac{f(x)}{g(x)} = \frac{A_1}{L_1} + \frac{A_2}{L_2(x)} + \dots + \frac{A_m}{L_m(x)}$$

where,  $A_i \in \mathbb{R}$  constants.

### 4.2 $g(x)$ is the product of non-repeating linear factors, but a particular factor is repeated $k$ times

Let

$$g(x) = L_1^k(x) \cdot L_2(x) \cdot \dots \cdot L_\eta(x)$$

Then, we can expand  $\frac{f(x)}{g(x)}$  in terms of partial fractions as,

$$\frac{f(x)}{g(x)} = \frac{A_1}{L_1(x)} + \frac{A_2}{L_1^2(x)} + \frac{A_3}{L_1^3(x)} + \dots + \frac{A_k}{L_1^k(x)} + \frac{B_2}{L_2(x)} \dots + \frac{B_\eta}{L_\eta(x)}$$

### 4.3 $g(x)$ contains some non-repeating linear as well as quadratic factors

Let

$$g(x) = \prod_i L_i(x) \cdot \prod_j Q_j(x)$$

where,  $Q_j(x)$  are quadratic factors.

Then, we can expand  $\frac{f(x)}{g(x)}$  in terms of partial fractions as,

$$\frac{f(x)}{g(x)} = \sum_i \frac{A_i}{L_i(x)} + \sum_j \frac{x B_j + C_j}{Q_j(x)}$$

#### 4.4 $g(x)$ contains some non-repeating linear and repeating quadratic factors

Let

$$g(x) = \prod_i L_i(x) \prod_j Q_j(x) \prod_{\omega} Q_{\omega}^k(x)$$

where,  $Q_{\omega}^k(x)$  are repeating quadratic factors. Then, we can expand  $\frac{f(x)}{g(x)}$  in terms of partial fractions as,

$$\frac{f(x)}{g(x)} = \sum_i \frac{A_i}{L_i(x)} + \sum_j \frac{x B_j + C_j}{Q_j(x)} + \sum_{\omega} \sum_r \frac{x D_r + E_r}{Q_{\omega}^r(x)}$$