Matrices

1 Types of Matrices

1.1 Row Matrix or Row Vector

A matrix which contains only **One Row**.

$$A = \left[a_{ij}\right]_{1\times n}$$

1.2 Column Matrix or Column Vector

A matrix which contains only **One Column**.

$$A = \left[a_{ij}\right]_{n\times 1}$$

1.3 Rectangular Matrix

A matrix in which **no. of rows is not equal to no. of columns**.

$$A = \left[a_{ij}\right]_{m\times n}, m \neq n$$

1.4 Square Matrix

A matrix in which **no. of rows is equal to no. of columns**.

$$A = \left[a_{ij}\right]_{n\times n}$$

1.5 Diagonal Matrix

A **square matrix** in which **all non-diagonal entries are zero**.

$$A = \left[a_{ij}\right]_{n\times n}, a_{ij} = 0 \ \forall \ i \neq j$$

1.6 Scalar Matrix

A **Diagonal Matrix** in which **all diagonal entries are equal**.

$$A = \left[a_{ij}\right]_{n\times n}, a_{ij} = \begin{cases} a & i = j, a \in \mathbb{C} \\ 0 & i \neq j \end{cases}$$

1.7 Identity or Unit Matrix

A **Diagonal Matrix** in which **all diagonal entries equal 1**.

$$A = \left[a_{ij}\right]_{n\times n}, a_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

1.8 Singleton Matrix

A matrix which consists of **only one element**.

$$A = \left[a_{ij}\right]_{1\times 1}$$

1.9 Triangular Matrix

1. **Upper Triangular Matrix**

   A **Square Matrix** in which **all elements below principal diagonal are zero**.

   $$A = \left[a_{ij}\right]_{n\times n}, a_{ij} = 0 \ \forall \ i > j$$

2. **Lower Triangular Matrix**

   A **Square Matrix** in which **all elements above principal diagonal are zero**.

   $$A = \left[a_{ij}\right]_{n\times n}, a_{ij} = 0 \ \forall \ i < j$$

1.10 Horizontal Matrix

A matrix in which **no. of rows is less than no. of columns**

$$A = \left[a_{ij}\right]_{m\times n}, m < n$$

1.11 Vertical Matrix

A matrix in which **no. of rows is greater than no. of columns**.

$$A = \left[a_{ij}\right]_{m\times n}, m > n$$

1.12 Null or Zero Matrix

A matrix in which **all elements equal zero.**

$$O = \left[a_{ij}\right]_{m\times n}, a_{ij} = 0 \ \forall \ i, j$$

1.13 Sub-Matrix

A matrix which is obtained from a given matrix **by deleting any no. of rows and no. of columns**.

e.g. $\begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}_{2\times 2}$ is a sub-matrix of $\begin{bmatrix} 8 & 9 & 5 \\ 2 & 3 & 4 \\ 3 & -2 & 5 \end{bmatrix}_{3\times 3}$

1.14 Comparable Matrices

Two Matrices $A = \left[a_{ij}\right]_{m\times n}$ and $B = \left[b_{ij}\right]_{p\times q}$ are **comparable**,
if $m = p \wedge n = q$

1.15 Idempotent Matrix

A square matrix whose square equals itself.

$$A = \left[a_{ij}\right]_{n\times n}, A^2 = A$$

- For a Idempotent Matrix, $A^n = A \qquad \forall \ n \in \mathbb{Z}^+$

1.16 Periodic Matrix

A square matrix A is called periodic,
If

$$A^{k+1} = A, k \in \mathbb{Z}^+$$

- The least value of all such k is called the period of matrix A.
- Period of an Idempotent Matrix is 1.

1.17 Nilpotent Matrix

A square matrix $A = \left[a_{ij}\right]_{n\times n}$ which
satisfies

$$A^k = O, A^{k-1} \neq O$$

1.18 Involutory Matrix

A square matrix A which
satisfies,

$$A^2 = I$$

- For Involutory Matrix, $A = A^{-1}$

1.19 Symmetric Matrix

A square matrix $A = \left[a_{ij}\right]_{n\times n}$ in which $A = A^T$.

- In a Symmetric Matrix, elements situated at equal distance from the diagonal are equal.
- For any Square Matrix A with $a_{ij} \in \mathbb{R}$, then $A + A^T$ is symmetric.
- If A is a Square Matrix, then AA^T and A^TA are symmetric matrices.
- All positive integral powers of a symmetric matrix are symmetric.

$$\textit{If } A = \left[a_{ij}\right]_{n\times n}, a_{ij} = a_{ji} \ \textit{Then } A^n = \left[\alpha_{ij}\right]_{n\times n}, \alpha_{ij} = \alpha_{ji}$$

- If A be a symmetric matrix and B be a square matrix of order that of A, then $-A, kA, A^T, A^{-1}, A^n, B^TAB$ are also symmetric. $n \in \mathbb{N}, k \in \mathbb{C}$
- Let A and B be two symmetric matrices of same order, then

  a. $A \pm B, AB + BA$ are symmetric.
  b. $AB - BA$ is skew-symmetric.
  c. AB is symmetric iff, $AB = BA$

1.20 Skew-Symmetric Matrix

A Square Matrix $A = \left[a_{ij}\right]_{n\times n}$ in which $a_{ij} = -a_{ji} \ \ \forall \ \ i, j$

- In a Skew-Symmetric Matrix all diagonal entries are **zero.**
- In a Skew-Symmetric Matrix, the elements situated at equal distance from the diagonal are equal in magnitude but opposite in sign.
- If A is a Skew-Symmetric Matrix, then $Tr(A) = 0$
- For any Square Matrix A with $a_{ij} \in \mathbb{R}$, then $A - A^T$ is a Skew-Symmetric Matrix.
- Every Square Matrix can be uniquely expressed as the sum of a symmetric and skew-symmetric matrix.

$$A = \frac{1}{2}\left(A + A^T\right) + \frac{1}{2}\left(A - A^T\right)$$

- All positive odd integral powers of a skew-symmetric matrix are skew-symmetric.

$$\text{If } A = \left[a_{ij}\right]_{m\times m}, a_{ij} = -a_{ji} \text{ Then } A^{2n+1} = \left[\alpha_{ij}\right]_{m\times m}, \alpha_{ij} = -\alpha_{ji}, n \in \mathbb{N}$$

- All positive even integral powers of a skew-symmetric matrix are symmetric.

$$\text{If } A = \left[a_{ij}\right]_{m\times m}, a_{ij} = -a_{ji} \text{ Then } A^{2n} = \left[\alpha_{ij}\right]_{m\times m}, \alpha_{ij} = \alpha_{ji}, n \in \mathbb{N}$$

- If A be a skew-symmetric matrix and B be a square matrix of order that of A then, $kA, B^T AB$ are also skew-symmetric.
- Let A and B be two skew-symmetric matrices of same order, then
  a. $A \pm B, AB - BA$ are skew-symmetric.
  b. $AB + BA$ is symmetric
- Let A be skew-symmetric matrix and C be column vector, then $C^T AC = O$

1.21 Orthogonal Matrix

A square matrix whose transpose is its inverse.

$$AA^T = I$$

- $A^{-1} = A^T$
- Let A and B be Orthogonal, then AB is also Orthogonal.
- If A is orthogonal, then A^{-1}, A^T are also Orthogonal.

1.22 Hermitian Matrix

A square matrix which is equal to its conjugate transpose matrix.

$$\text{If } A_{n\times n} = A^{\theta}_{n\times n} \ \vee A = \left[a_{ij}\right]_{n\times n}, a_{ij} = \overline{a_{ji}}, \text{ Then } A \text{ is Hermitian}$$

- The diagonal entries of a Hermitian Matrix are $\mathbb{R}$
- Elements situated at equal distance from diagonal are complex conjugate of each other.
- Let A be any square matrix, then $A + A^{\theta}$ is a Hermitian Matrix.
- If A be a square matrix, then $AA^{\theta}, A^{\theta}A$ are Hermitian.
- If A is Hermitian Matrix, then

a. iA is Skew-Hermitian Matrix. $i^2 = -1$

b. $\overline{A}$ is also a Hermitian Matrix.

c. kA is Hermitian Matrix. $k \in \mathbb{R}$

d. If A, B are Hermitian Matrices of same order, then

   a. $k_1 A + k_2 B$ is also Hermitian. $k_1, k_2 \in \mathbb{R}$

   b. AB is also Hermitian if $AB = BA$

   c. $AB + BA$ is Hermitian.

   d. $AB - BA$ is Skew-Hermitian.

1.23 Skew-Hermitian Matrix

A square matrix which is equal to negative of its conjugate transpose matrix.

$$\textit{If } A_{n\times n} = -A^{\theta}_{n\times n} \;\; \vee \;\; A = \left[a_{ij}\right]_{n\times n}, a_{ij} = -\overline{a_{ji}}, \; \textit{Then } A \textit{ is Skew-Hermitian}$$

- All diagonal entries of a Skew-Hermitian Matrix are purely imaginary or zero.
- In a Skew-Hermitian Matrix elements situated at equal distance from the diagonal only differ in the sign of real part.
- Let A be any square matrix, then $A - A^{\theta}$ is a Skew-Hermitian Matrix.
- Every square matrix can be uniquely expressed as the sum of a Hermitian and a Skew-Hermitian Matrix.

$$A_{n\times n} = \frac{1}{2}\left(A + A^{\theta}\right) + \frac{1}{2}\left(A - A^{\theta}\right)$$

- If A is a Skew-Hermitian Matrix, then

   a. iA is a Hermitian Matrix. $i^2 = -1$

   b. $\overline{A}$ is also Skew-Hermitian.

   c. kA is Skew-Hermitian. $k \in \mathbb{R}$

- If A, B are two Skew-Hermitian Matrices of same order, then $k_1 A + k_2 B$ is also Skew-Hermitian.

1.24 Unitary Matrix

A Square Matrix whose inverse is its conjugate transpose matrix.

$$AA^{\theta} = I \implies A^{-1} = A^{\theta}$$

- If A and B are unitary, then AB is also unitary.
- If A is unitary, then A^{-1}, A^{T} are also unitary.

1.25 Equivalent Matrices

Two matrices are said to be equivalent if one is obtained from the other by elementary operations.

$$A \sim B, A \textit{ is equivalent to } B$$

- If $A \sim B \ \exists$ invertible matrices P and Q such that $B = PAQ$
- Every invertible square matrix can be expressed as the product of elementary matrices.

2 Trace of a Matrix

The **sum of all diagonal entries** of a **square matrix** is called its **Trace**.

$$A = \left[a_{ij}\right]_{n \times n}, Tr(A) = \sum_{i=1}^{n} a_{ii}$$

2.1 Properties of Trace of a Matrix

Let $A = \left[a_{ij}\right]_{n \times n}, B = \left[b_{ij}\right]_{n \times n}, k \in \mathbb{C}$

i. $Tr(kA) = k \cdot Tl(A)$

ii. $Tr(A \pm B) = Tr(A) \pm Tr(B)$

iii. $Tr(AB) = Tr(BA)$

iv. $Tr(A) = Tr(A')$

v. $Tr(I_n) = n$

vi. $Tr(AB) \neq Tr(A) \cdot Tr(B)$

vii. $Tr(A) = Tr(CAC^{-1})C$ is a non-singular square matrix of order n.

3 Determinant of a Square Matrix

Let $A = \left[a_{ij}\right]_{n \times n}$. The determinant formed by the elements of A is said to be the determinant of matrix $A = det(A) = |A|$

Important Points

a. $det\left(\prod_{r=1}^{n} A_r\right) = \prod_{r=1}^{n} det(A_r)$

b. $det(kA) = k^n det(A)$ $\quad k \in \mathbb{C}, n =$ order of A

c. $det(A)$ exists $\iff$ A is a square matrix.

d. $det(A) = det(A^T)$

e. If A is Orthogonal Matrix, then $det(A) = \pm 1$

f. If A is Skew-Symmetric Matrix of odd order, then $det(A) = 0$

g. If A is Skew-Symmetric Matrix of even order, then $det(A)$ is a perfect square.

h. If $A = diag(a_1, a_2, \ldots, a_n)$, then $det(A) = \prod_{r=1}^{n} a_r$

i. $det(A^{-1}) = \dfrac{1}{det(A)}$

Singular (Invertible) and non-singular (Non-Invertible) Matrix

If for $A = \left[a_{ij}\right]_{n\times n}, det(A) = 0$ then A is called a singular matrix.
If $det(A) \neq 0$ then it is called **non-singular** matrix.

4 Equal Matrices

Two matrices are equal, iff

i. They are of same order.

ii. The elements in the corresponding positions of the two matrices are equal.
Let $A = \left[a_{ij}\right]_{m\times n}$ and $B = \left[b_{ij}\right]_{p\times q}$
Now, $A = B$ iff, $m = p, n = q, a_{ij} = b_{ij} \;\; \forall \;\; i, j$

5 Algebra of Matrices

5.1 Addition

If $A = \left[a_{ij}\right]_{m\times n}$ and $B = \left[b_{ij}\right]_{m\times n}$

Then,

$$A + B = \left[a_{ij} + b_{ij}\right]_{m\times n}$$

Properties of Matrix Addition

1. Addition of Matrices is **commutative**.

$$A + B = B + A$$

2. Addition of Matrices is **associative**.

$$(A + B) + C = A + (B + C)$$

3. Additive Identity is O (Null Matrix)

$$A + O = A$$

4. Additive Inverse is $-A$

$$A + (-A) = O$$

5. Cancellation law

   $A + B = A + C \implies B = C$

5.2 Multiplication

5.2.1 Scalar Multiplication

The Matrix obtained by **multiplying every element of a matrix with a scalar** is called the scalar multiple of that matrix.

Let $A = \left[a_{ij}\right]_{m \times n}, k \in \mathbb{C}$

Now,

$$kA = \left[k \cdot a_{ij}\right]_{m \times n}$$

5.2.2 Matrix Multiplication

Conformable for Multiplication

Two matrices $A = \left[a_{ij}\right]_{m \times n}, B = \left[b_{ij}\right]_{p \times q}$ are conformable for the product AB, if the no. of columns in the pre-factor (A) is equal to the no. of rows in the post-factor (B)

$$AB \textit{ exists iff } n = p$$

Multiplication of Matrices

Let $A = \left[a_{ij}\right]_{m \times n}, B = \left[b_{ij}\right]_{n \times q}, C = \left[c_{ij}\right]_{m \times q}$ *s.t.* $AB = C$

$\therefore c_{ij} = \sum_{r=1}^{n} a_{ir} \cdot b_{rj}$ *Row by Column Multiplication*

5.2.3 Properties of Matrix Multiplication

i. Matrix Multiplication is not commutative.

$$AB \neq BA$$

- If $AB = BA$ then matrices A, B are said to commute.
- If $AB = -BA$ then matrices A, B are said to anti-commute.
- Diagonal Matrices of same order always commute.

ii. Matrix Multiplication is associative.

$$A(BC) = (AB)C$$

iii. If $A = \left[a_{ij}\right]_{m\times n}$, then $I_m\,A = A = AI_n$

iv. Matrix Multiplication is distributive.

$$A(B + C) = AB + AC$$

Note: $A(B + C) \neq AB + CA$

v. If product of two matrices is a null matrix then it is not necessary that one of the matrices is also a null matrix.

$$AB = O \not\Longrightarrow A = O \vee B = O$$

If $AB = O$ then A, B are divisors of O

$AB = O \implies det(AB) = 0 \implies det(A) \cdot det(B) = 0$

vi. $AO = O$

vii. $I^m = I$ $\qquad \forall\ m \in \mathbb{Z}^+$

viii. if $A = \left[a_{ij}\right]_{n\times n}$

then $\underbrace{A \cdot A \cdot A \ldots A}_{m \text{ times}} = A^m$ $\qquad \forall\ m \in \mathbb{Z}^+$

6 Pre-Multiplication and Post-Multiplication of Matrices

The Matrix AB is the matrix B **pre-multiplied** by A and the matrix BA is the matrix B **post-multiplied** by A.

7 Transpose of a Matrix

The matrix obtained by **interchanging rows and columns**.

$$A = [a_{ij}]_{m\times n}, A^T = [a_{ji}]_{n\times m}$$

- $A^T = A' = A^t =$ Transpose of Matrix A

7.1 Properties of Transpose Matrices

i. $(A^T)^T = A$

ii. $(A \pm B)^T = A^T \pm B^T$

iii. $(kA)^T = k \cdot A^T$

iv. $(AB)^T = B^T A^T$, In General, $\left(\prod_{r=1}^{n} A_r\right)^T = \prod_{r=1}^{n} A^T_{n+1-r}$

v. $I^T = I$

8 Complex Conjugate of a Matrix

The Matrix obtained by replacing each element by its complex conjugate.

$$A = [a_{ij}]_{m\times n}, \overline{A} = [\overline{a_{ij}}]_{m\times n}$$

- If all elements are $\mathbb{R}$, then $A = \overline{A}$
- $\overline{\overline{A}} = A$
- $\overline{A+B} = \overline{A} + \overline{B}$
- $\overline{kA} = \overline{k} \cdot \overline{A}$ $\qquad k \in \mathbb{C}$
- $\overline{AB} = \overline{A}\,\overline{B}$

9 Conjugate Transpose of a Matrix

The complex conjugate of the transpose of a matrix.

$$A = [a_{ij}]_{m\times n}, A^\theta = \overline{A^T} = [\overline{a_{ji}}]_{n\times m}$$

If A and B are two matrices of same order, then

i. $\overline{A^T} = \left(\overline{A}\right)^T$

ii. $\left(A^{\theta}\right)^{\theta} = A$

iii. $(A+B)^{\theta} = A^{\theta} + B^{\theta}$

iv. $(kA)^{\theta} = \overline{k}A^{\theta}$ $\qquad k \in \mathbb{C}$

v. $(AB)^{\theta} = B^{\theta}A^{9}$

10 Adjoint of A Matrix

The Transpose of Square Matrix formed by its co-factors is its Adjoint.

Let $A = \left[a_{ij}\right]_{n\times n}$

then

$$adj(A) = \left[C_{ji}\right]_{n\times n}$$

10.1 Properties of Adjoint Matrix

i. $A \cdot adj(A) = adj(A) \cdot A = det(A)I$

ii. $det(adj(A)) = (det(A))^{n-1}$

iii. $det(\underbrace{adj(adj(adj(\ldots \qquad))))}_{m \text{ times } adj}) = det(A)^{(n-1)^m}$

iv. $(adj(A))^T = adj(A^T)$

v. $adj(adj(A)) = (det(A))^{n-2}A$

vi. $adj(kA) = k^{n-1}adj(A)$ $\qquad k \in \mathbb{C}$

vii. $adj(A^m) = (adj(A))^m$ $\qquad m \in \mathbb{N}$

viii. $det(A \cdot adj(A) + kI_n) = (det(A) + k)^n$

ix. Adjoint of a Diagonal Matrix is a Diagonal Matrix.

If $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, then $adj(A) = \begin{bmatrix} bc & 0 & 0 \\ 0 & ca & 0 \\ 0 & 0 & ab \end{bmatrix}$

x. $adj(I_n) = I_n$

xi. $adj(adj(A)) = (det(A))^{n-2}A$

11 Inverse of a Matrix

A square matrix of is said to be Invertible, iff its determinant is non zero.

$$det(A) \neq 0$$

$$A^{-1} = \frac{adj(A)}{det(A)}$$

11.1 Properties of Inverse Matrix

i. Every invertible matrix possesses a unique inverse.

ii. $(AB)^{-1} = B^{-1}A^{-1}$

iii. $(A^T)^{-1} = (A^{-1})^T$

iv. $(A^k)^{-1} = (A^{-1})^k = A^{-k}$ $\qquad k \in \mathbb{N}$

v. $(A^{-1})^{-1} = A$

vi. $I^{-1} = I$

vii. $det(A^{-1}) = \dfrac{1}{det(A)}$

viii. Inverse of a digonal matrix is a digonal matrix.

If $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ then $A^{-1} = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix}$

12 Elementary Row Operations

i. The interchange of ith row and jth row.

$$R_i \leftrightarrow R_j$$

ii. Multiplication of ith row by a non zero constant k.

$$R_i \rightarrow k \cdot R_i$$

iii. Addition of ith row to the elements of jth row multiplied by a non zero constant k.

$$R_i \rightarrow R_i + k \cdot R_j$$

13 Gauss Jordan Method for Computing Inverses

Let A be an invertible matrix. To find inverse of a using Gauss Jordan Method,

1. Create a matrix $G = \begin{bmatrix} A & | & I \end{bmatrix}$
2. Now using elementary operations on G transform A to I.
3. In the process of trasforming A to I, I transforms to A^{-1}

14 Matrix Polynomial

$$f(A) = a_n A^n + a_{n-1} A^{n-1} + a_{n-2} A^{n-2} + \ldots + a_1 A + a_0 I$$

For $n \geq 2, A$ should be a square matrix.

- If $det(A) \neq 0, a_0 \neq 0, a_0 I + a_1 A + a_2 A^2 + \ldots + a_n A^n = O$, then

$$A^{-1} = \frac{1}{a_0}\left(a_1 I + a_2 A + a_3 A^2 + \ldots + a_n A^{n-1}\right)$$

15 Linear System of Equations

Consider,

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3n}x_n &= b_3 \\
\vdots \quad\quad \vdots \quad\quad \vdots \quad\quad \ddots \quad \vdots \quad\quad &\vdots \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n &= b_n
\end{aligned}$$

If $b_i = 0, i \in \{x : x \leq n, x \in \mathbb{N}\}$, then the above system is called Homogenous. If not then it is Non-Homogenous.

In Matrix form,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$$

$$AX = B$$

$$X = A^{-1}B = \frac{adj(A)B}{det(A)}$$

15.1 Types of Equations

I. If system is Non-Homogenous

   i. If $det(A) \neq 0$, then system possesses a unique solution.
   ii. If $det(A) = 0, adj(A)B \neq O$, then system possesses no solution.
   iii. If $det(A) = 0, adj(A)B = O$, then system possesses ∞ solutions.

II. If system is Homogenous

   i. If $det(A) \neq 0$, then system possesses only one trivial solution.
   ii. If $det(A) = 0$, then system possesses non-trivial solutions and ∞ solutions.
   iii. If no. of equations < no. of unknowns, then it has non-trivial solution.

16 Echelon Form of a Matrix

A matrix A is said to be in echelon form, if

i. The first non-zero element in each row is 1.

ii. Every non-zero row is A preceds eveny zero row (If any).

iii. The number of zeroes before the first non-zero element in 1st, 2nd, ... rows should be in increasing order.

17 Rank of a Matrix

Rank of a Matrix A, $\rho(A) = \textit{Number of non-zero rows in Echelon form of } A$

17.1 Properties of Rank of Matrices

Let $A = \left[a_{ij}\right]_{m\times n}, B = \left[b_{ij}\right]_{m\times n}$

i. $\rho(A+B) \leq \rho(A) + \rho(B)$

ii. $\rho(AB) \leq \rho(A) \wedge \rho(AB) \leq \rho(B)$

iii. $\rho(A) = \rho(A^T)$

18 Solutions to Linear System of Equations Using Rank Method

Consider,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3n}x_n &= b_3 \\ \vdots \qquad \vdots \qquad \vdots \qquad \ddots \qquad \vdots \qquad &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

In Matrix form,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}$$

$$AX = B$$

Now,

$$C = \begin{bmatrix} A & B \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \ldots & a_{3n} & b_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mn} & bm \end{bmatrix}$$

C is called augmented matrix.

18.1 Types of Equations

I. Consistent System, If $\rho(A) = \rho(C)$

  i. Unique Solution, If $\rho(A) = \rho(C) = n = no.\ of\ unknowns$

  ii. ∞ Solutions, If $\rho(A) = \rho(C) = r < n$

II. Inconsistent System, If $\rho(A) \neq \rho(C) \implies no\ solution$

19 Linear Transformations

19.1 Reflection Matrices

19.1.1 Reflection through axis of x

The Matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ describes the reflection of any point $P(x, y)$ through the axis of x.

19.1.2 Reflection through the axis of y

The Matrix $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ describes the reflection of any point $P(x, y)$ through the axis of y.

19.1.3 Reflection through the origin

The Matrix $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ describes the reflection of any point $P(x, y)$ through the origin.

19.1.4 Reflection through line $y = x$

The matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ describes the reflection of any point $P(x, y)$ through the line $y = x$

19.1.5 Reflection through the line $y = x \tan\theta$

The matrix $\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$ describes the reflection of any point $P(x, y)$ through the line $y = x \tan\theta$

19.1.6 Rotation Through Angle θ

The Matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ describes a rotation of a vector through an angle θ.

20 Eigen Vectors and Eigen Values

Let ν be any non-zero vector satisfying,

$$A\nu = \lambda\nu$$

This scalar λ is called the Eigen Value (Characteristic Root) and the vector ν is the Eigen vector (Characteristic Vector).

20.1 Important Properties of Eigen Values

i. Any square matrix A and its transpose A^T have the same Eigen Values.

ii. The sum of Eigen Values of a matrix is equal to its trace.

iii. The product of Eigen Values is of a matrix A is equal to its determinant.

iv. If λ_i are the Eigen Value of A, then the Eigen Values of

   a. kA are $k\lambda_i$
   b. A^m are ${\lambda_i}^m$
   c. A^{-1} are λ_i^{-1}

v. All Eigen Values of a real symmetric matrix are $\mathbb{R}$ and the eigen vectors corresponding to two distinct eigen values are orthogonal.

vi. All Eigen Values of a real skew-symmetric matrix are purely imaginary or zero.

vii. An odd order skew-symmetric matrix is singular and hence has zero as an Eigen value.

viii. If sum of entries in every row or column is a constant, then that constant value is an eigen value of that matrix.

21 Characteristic Equation

$$det(A - \lambda I) = 0$$

is called characteristic equation of matrix $A_{n \times n}$.

The adove equation reduces to a nth degree polynomial equation in λ.

22 Cayley Hamilton Theorem

Every square matrix A satisfies its characteristic equation $det(A - \lambda I) = 0$

$$a_0 + a_1\lambda + \ldots a_n\lambda^n = 0$$

By Cayley Hamilton Theorem, $\lambda \to A$

$$a_0 I + a_1 A + a_2 A^2 + \ldots + a_n A^n = O$$