

Indefinite Integration

1 Fundamental Definition of Indefinite Integration

If f and F are functions such that $\frac{d}{dx}(F(x)) = f(x)$ then F is an anti-derivative of f w.r.t. x

Symbolically,

$$\int f(x) dx = F(x) + C$$

where C is the constant of Integration

2 Anti-Derivatives of Some Standard Functions

i. $\int k \cdot f(x) dx = k \cdot \int f(x) dx$

ii. $\int [f_1(x) \pm f_2(x) \pm f_3(x) \pm \dots \pm f_n(x)] dx = \int f_1(x) dx \pm \int f_2(x) dx \pm \int f_3(x) dx \pm \dots \pm \int f_n(x) dx$

iii. $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ $n \neq -1$

iv. $\int \frac{1}{x} dx = \ln |x| + C$

v. $\int e^x dx = e^x + C$

vi. $\int a^x dx = \frac{a^x}{\ln a} + C$

vii. $\int \sin x dx = -\cos x + C$

viii. $\int \cos x dx = \sin x + C$

$$\text{ix. } \int \sec^2 x \, dx = \tan x + C$$

$$\text{x. } \int \csc^2 x \, dx = -\cot x + C$$

$$\text{xi. } \int \sec x \tan x \, dx = \sec x + C$$

$$\text{xii. } \int \csc x \cot x \, dx = -\csc x + C$$

$$\text{xiii. } \int \cot x \, dx = \ln |\sin x| + C$$

$$\text{xiv. } \int \tan x \, dx = -\ln |\cos x| + C$$

$$\text{xv. } \int \sec x \, dx = \ln |\sec x + \tan x| + C$$

$$\text{xvi. } \int \csc x \, dx = \ln |\csc x - \cot x| + C$$

$$\text{xvii. } \int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + C$$

$$\text{xviii. } \int \frac{-1}{\sqrt{1-x^2}} \, dx = \cos^{-1} x + C$$

$$\text{xix. } \int \frac{1}{1+x^2} \, dx = \tan^{-1} x + C$$

$$\text{xx. } \int \frac{-1}{1+x^2} \, dx = \cot^{-1} x + C$$

$$\text{xxi. } \int \frac{1}{x\sqrt{x^2-1}} \, dx = \sec^{-1} x + C$$

$$\text{xxii. } \int \frac{-1}{x\sqrt{x^2-1}} \, dx = \csc^{-1} x + C$$

$$\text{xxiii. } \int \sqrt{x} \, dx = \frac{2}{3} x^{3/2} + C$$

$$\text{xxiv. } \int \frac{1}{\sqrt{x}} \, dx = 2\sqrt{x} + C$$

$$\text{xxv. } \int \frac{dx}{x^2-1} = \ln \left| \frac{x-1}{x+1} \right| + C$$

$$\text{xxvi. } \int \frac{dx}{\sqrt{1+x^2}} = \ln \left| x + \sqrt{x^2+1} \right| + C$$

$$\text{xxvii. } \int \frac{dx}{\sqrt{x^2-1}} = \ln \left| x + \sqrt{x^2-1} \right| + C$$

$$\text{xxviii. } \int \sqrt{1-x^2} dx = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x + C$$

$$\text{xxix. } \int \sqrt{x^2-1} dx = \frac{1}{2}x\sqrt{x^2-1} - \frac{1}{2}\ln \left| x + \sqrt{x^2-1} \right| + C$$

$$\text{xxx. } \int \sqrt{x^2+1} dx = \frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\ln \left| x + \sqrt{x^2+1} \right| + C$$

$$\text{xxxi. } \int \frac{x^2+1}{x^4+1} dx = \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x-1/x}{\sqrt{2}} \right) + C$$

$$\text{xxxii. } \int \frac{x^2-1}{x^4+1} dx = \frac{1}{2\sqrt{2}} \ln \left| \frac{x+1/x-\sqrt{2}}{x+1/x+\sqrt{2}} \right| + C$$

Important Results

1. If $F_1(x)$ and $F_2(x)$ are two anti-derivatives of a function $f(x)$, then $F_1(x)$ and $F_2(x)$ only differ by a constant, i.e.

$$F_1(x) - F_2(x) = C$$

where, C is a \mathbb{R} constant.

2. If $f(x)$ is continuous $\forall x \in D_f$ and,
 $\int f(x) dx = F(x) + C$, then $F(x)$ always exists and is continuous.
3. If $f(x)$ is discontinuous at $x = x_1$, then its anti-derivative can be continuous at $x = x_1$.
4. Anti-derivative of a periodic function may not be periodic.

3 Methods of Integration

3.1 u substitution

Integrals of form,

$$I = \int f(g(x)) \cdot g'(x) dx$$

Can be solved by, the substitution,

$$u = g(x)$$

Differentiating both sides w.r.t. x ,

$$du = g'(x)dx$$

Now,

$$I = \int f(u) du$$

3.2 Integrals of form

$$\int \frac{dx}{ax^2 + bx + c}, \int \frac{dx}{\sqrt{ax^2 + bx + c}}, \int \sqrt{ax^2 + bx + c} dx$$

Using completing the square,

$$ax^2 + bx + c = a \left[\left(x + \frac{b}{2a} \right)^2 + \frac{c}{a} - \frac{b^2}{4a} \right]$$

Now, using u sub, let

$$u = x + \frac{b}{2a}$$

The transformed integral can be integrated using previous methods.

3.3 Integrals of form

$$\int \frac{px + q}{ax^2 + bx + c} dx, \int \frac{px + q}{\sqrt{ax^2 + bx + c}} dx, \int (px + q) \sqrt{ax^2 + bx + c} dx$$

$$px + q = \lambda \frac{d}{dx} (ax^2 + bx + c) + \mu$$

After finding λ, μ ,

For the 1st part, use u sub,

Let

$$u = ax^2 + bx + c$$

2nd part of the integral can be integrated using previous methods.

3.4 Integrals of form

$$\int \frac{K(x)}{ax^2 + bx + c} dx$$

where $\deg(K(x)) \geq 2$

By polynomial long division

$$\frac{K(x)}{ax^2 + bx + c} = Q(x) + \frac{R(x)}{ax^2 + bx + c}$$

$\deg(R(x)) \leq 1$,

Now, the integral

$$\int \frac{R(x)}{ax^2 + bx + c} dx$$

can be integrated using previous methods.

3.5 Integrals of form

$$\int \frac{ax^2 + bx + c}{px^2 + qx + r} dx, \int \frac{ax^2 + bx + c}{\sqrt{px^2 + qx + r}} dx, \int (ax^2 + bx + c) \sqrt{px^2 + qx + r} dx$$

$$ax^2 + bx + c = \lambda (px^2 + qx + r) + \mu \frac{d}{dx} (px^2 + qx + r) + \gamma$$

For the 3rd integral, in the 1st part use Trig sub. and Integration by Parts, 2nd part use *u-sub* and 3rd part can be integrated using previous methods.

3.6 Trig. Integrals

3.6.1 Integrals of form

$$\int \frac{dx}{a \cos^2 x + b \sin^2 x}, \int \frac{dx}{a + b \sin^2 x}, \int \frac{dx}{a + b \cos^2 x}$$

$$\int \frac{dx}{(a \sin x + b \cos x)^2}, \int \frac{dx}{a + b \sin^2 x + c \cos^2 x}$$

Steps -

1. Multiply numerator and denominator by $\sec^2 x$
2. Replace $\sec^2 x$ (if any) by $1 + \tan^2 x$ except the one multiplied in step 1.
3. Let $u = \tan x$, then $du = \sec^2 x dx$

Now, the transformed integral can be integrated using previous methods.

3.6.2 Integrals of form

$$\int \frac{dx}{a \sin x + b \cos x}, \int \frac{dx}{a + b \sin x}, \int \frac{dx}{a + b \cos x}, \int \frac{dx}{a \sin x + b \cos x + c}$$

Steps -

1. Replace $\sin x = \frac{2 \tan x/2}{1 + \tan^2 x/2}$ and $\cos x = \frac{1 - \tan^2 x/2}{1 + \tan^2 x/2}$
2. Let $u = \tan x/2 \implies du = \frac{1}{2} \sec^2 x/2 dx$ is already present in the numerator.

Now, the transformed integral can be integrated using previous methods.

3.6.2.1 Alternative Method to Integrate

$$I = \int \frac{dx}{a \sin x + b \cos x}$$

$$a \sin x + b \cos x = \sqrt{a^2 + b^2} \sin \left(x + \tan^{-1} \left(\frac{b}{a} \right) \right)$$

$$I = \frac{1}{\sqrt{a^2 + b^2}} \int \csc \left(x + \tan^{-1} \left(\frac{b}{a} \right) \right) dx$$

$$I = \frac{1}{\sqrt{a^2 + b^2}} \ln \left| \tan \left(\frac{x}{2} + \frac{1}{2} \tan^{-1} \left(\frac{b}{a} \right) \right) \right| + C$$

3.6.3 Integrals of form

$$\int \frac{p \cos x + q \sin x + r}{a \cos x + b \sin x + c} dx, \int \frac{p \cos x + q \sin x}{a \cos x + b \sin x} dx$$

Steps for (i),

1. Express $Numerator = \lambda Denominator + \mu Derivative of denominator + \gamma$

Now, the transformed integral can be integrated using previous methods.

Steps for (ii),

1. Express $Numerator = \lambda Denominator + \mu Derivative of Denominator$

Now, the transformed integral can be integrated using previous methods.

3.7 Integration by parts

$$\int uv dx = u \int v dx - \int \left(u' \int v dx \right) dx$$

u is the function which has to be differentiated (D), v is the function which has to be integrated (I)

3.8 Integrals of form

$$I = \int e^{g(x)} (f(x)g'(x) + f'(x)) dx$$

$$I = e^{g(x)} \cdot f(x) + C$$

If $g(x) = x$,

Then,

$$\int e^x (f(x) + f'(x)) dx = e^x \cdot f(x) + C$$

3.9 Integrals of form

$$S = \int e^{ax} \sin bx \, dx, C = \int e^{ax} \cos bx \, dx$$

$$S = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) + C_0, C = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) + C_{00}$$

3.10 Integration by Partial Fraction Decomposition

Let $f(x) = \sum_{i=0}^n a_i x^i, g(x) = \sum_{i=0}^m b_i x^i$.

We define a rational function $h(x) = \frac{f(x)}{g(x)}$,

$$h(x) \text{ is } \begin{cases} \text{Proper Rational Function} & m > n \\ \text{Improper Rational Function} & m \leq n \end{cases}$$

If $h(x)$ is Improper, we make it Proper by polynomial long division, i.e.

$$h(x) = Q(x) + \frac{r(x)}{g(x)}$$

Clearly, $\frac{r(x)}{g(x)}$ is Proper.

Now, assuming $h(x)$ is Proper, following cases arise -

3.10.1 $g(x)$ is the product of non-repeating linear factors

Let

$$g(x) = L_1(x) \cdot L_2(x) \cdot \dots \cdot L_m(x)$$

where $L_i(x)$ are linear functions.

Then, we can expand $\frac{f(x)}{g(x)}$ in terms of partial fractions as,

$$\frac{f(x)}{g(x)} = \frac{A_1}{L_1} + \frac{A_2}{L_2(x)} + \dots + \frac{A_m}{L_m(x)}$$

where, $A_i \in \mathbb{R}$ constants.

3.10.2 $g(x)$ is the product of non-repeating linear factors, but a particular factor is repeated k times

Let

$$g(x) = L_1^k(x) \cdot L_2(x) \cdot \dots \cdot L_\eta(x)$$

Then, we can expand $\frac{f(x)}{g(x)}$ in terms of partial fractions as,

$$\frac{f(x)}{g(x)} = \frac{A_1}{L_1(x)} + \frac{A_2}{L_1^2(x)} + \frac{A_3}{L_1^3(x)} + \dots + \frac{A_k}{L_1^k(x)} + \frac{B_2}{L_2(x)} \dots + \frac{B_\eta}{L_\eta(x)}$$

3.10.3 $g(x)$ contains some non-repeating linear as well as quadratic factors

Let

$$g(x) = \prod_i L_i(x) \cdot \prod_j Q_j(x)$$

where, $Q_j(x)$ are quadratic factors.

Then, we can expand $\frac{f(x)}{g(x)}$ in terms of partial fractions as,

$$\frac{f(x)}{g(x)} = \sum_i \frac{A_i}{L_i(x)} + \sum_j \frac{x B_j + C_j}{Q_j(x)}$$

3.10.4 $g(x)$ contains some non-repeating linear and repeating quadratic factors

Let

$$g(x) = \prod_i L_i(x) \prod_j Q_j(x) \prod_{\omega} \aleph_{\omega}^k(x)$$

Where, $\aleph_{\omega}(x)$ are repeating quadratic factors.

Then, we can expand $\frac{f(x)}{g(x)}$ in terms of partial fractions as,

$$\frac{f(x)}{g(x)} = \sum_i \frac{A_i}{L_i(x)} + \sum_j \frac{x B_j + C_j}{Q_j(x)} + \sum_{\omega} \sum_r \frac{x D_r + E_r}{\aleph_{\omega}^r(x)}$$

3.11 Integrals of form

3.11.1

$$I = \int f \left(x + \frac{1}{x} \right) \left(1 - \frac{1}{x^2} \right) dx$$

$$\text{Let, } u = x + \frac{1}{x} \implies du = \left(1 - \frac{1}{x^2} \right) dx$$

$$\text{Now, } I = \int f(u) du$$

3.11.2 Integrals of form

$$\int \frac{x^2 + 1}{x^4 + kx^2 + 1} dx$$

Divide numerator and denominator by x^2

Now, the transformed integral can be integrated using previous methods.

3.12 Integration of Special Irrational Functions

3.12.1 Integrals of form

$$\int \frac{1}{(ax+b)\sqrt{cx+d}} dx$$

Using *u-sub*,

Let

$$u^2 = cx + d$$

Now, the transformed integral can be integrated using previous methods.

3.12.2 Integrals of form

$$\int \frac{1}{(ax^2+bx+c)\sqrt{px+q}} dx$$

Using *u-sub*,

Let

$$u^2 = px + q$$

Now, the transformed integral can be integrated using previous methods.

3.12.3 Integrals of form

$$\int \frac{1}{(ax+b)(\sqrt{px^2+qx+r})} dx$$

Using *u-sub*,

Let

$$\frac{1}{u} = ax + b$$

Now, the transformed integral can be integrated using previous methods.

3.12.4 Integrals of form

$$\int \frac{1}{(ax^2+b)\sqrt{cx^2+d}} dx$$

Using *u-sub*,

Let

$$\frac{1}{\sqrt{u}} = x$$

Now, the transformed integral can be integrated using previous methods.

3.12.5 Integrals of form

$$\int \frac{1}{(x-k)^r \sqrt{ax^2+bx+c}} dx, r \geq 2$$

Using *u-sub*,

Let

$$\frac{1}{u} = x - k$$

Now, the transformed integral can be integrated using previous methods.

3.12.6 Integrals of form

$$\int \frac{ax^2+bx+c}{(\alpha x+\beta)\sqrt{px^2+qx+r}} dx$$

Express,

$$ax^2+bx+c = \lambda(\alpha x+\beta) \left[\frac{d}{dx}(px^2+qx+r) \right] + \mu(\alpha x+\beta) + \gamma$$

Now, the transformed integral can be integrated using previous methods.

3.13 integrals of form

$$\int \sin^m x \cdot \cos^n x dx$$

3.13.1 If one of m or n is odd, $m, n \in \mathbb{N}$

Then, we *u-sub* the term with even power, i.e.

If $m = 2k+1, n = 2p$ then, $u = \cos x$

If $m = 2p, n = 2k+1$ then, $u = \sin x$

3.13.2 If both m and n are odd, $m, n \in \mathbb{N}$

Then, *u-sub* any of $\cos x$ or $\sin x$.

3.13.3 If both m and n are even, $m, n \in \mathbb{N}$

Use Trig. identities

3.13.4 If $\frac{m+n-2}{2} \in \mathbb{Z}^-, m, n \in \mathbb{Q}$

Then, *u-sub*, $u = \tan x$ or $u = \cot x$

3.14 Integrals of form

$$\int x^m (a+bx^n)^p dx$$

3.14.1 If $P \in \mathbb{N}$

Use binomial expansion and then integrate.

3.14.2 If $P \in \mathbb{Z}^-$

Use u -sub,

Let

$$u^k = x, \quad k = LCM(m, n)$$

3.14.3 If $\frac{m+1}{n} \in \mathbb{Z}$ and $P \in \mathbb{Q}$

Use u -sub,

Let

$$u^k = a + bx^n$$

where, if $P = \frac{a}{b}$, $HCF(a, b) = 1$ then $k = b$

3.14.4 $\frac{m+1}{n} + P \in \mathbb{Z}$ and $P \in \mathbb{Q}$

Use u -sub,

Let

$$u^k x^n = a + bx^n$$

where, if $P = \frac{a}{b}$, $HCF(a, b) = 1$ then $k = b$

3.15 Euler's Substitutions**Integrals of Form**

$$\int R(x, \sqrt{ax^2 + bx + c}) dx$$

where, integrand is a rational function of x and $\sqrt{ax^2 + bx + c}$

3.15.1 If $a > 0$

Then,

Let

$$\sqrt{ax^2 + bx + c} = u \pm x\sqrt{a}$$

3.15.2 If $c > 0$

Then,

Let

$$\sqrt{ax^2 + bx + c} = ux + \sqrt{c}$$

3.15.3 If real roots α, β of the equation $ax^2 + bx + c = 0$

Then,

Let

$$\sqrt{ax^2 + bx + c} = (x - \alpha)u$$

3.16 Reduction Formulae

3.16.1 Reduction Formula for

$$I_n = \int \sin^n x \, dx$$

$$nI_n = -\sin^{n-1} x \cdot \cos x + (n-1)I_{n-2}$$

3.16.2 Reduction Formula for

$$I_n = \int \cos^n x \, dx$$

$$nI_n = \sin x \cdot \cos^{n-1} x + (n-1)I_{n-2}$$

3.16.3 Reduction Formula for

$$I_n = \int \tan^n x \, dx$$

$$I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$$

3.16.4 Reduction Formula for

$$I_n = \int \csc^n x \, dx$$

$$(n-1)I_n = -\csc^{n-2} x \cot x + (n-2)I_{n-2}$$

3.16.5 Reduction Formula for

$$I_n = \int \sec^n x \, dx$$

$$(n-1)I_n = \sec^{n-2} x \tan x + (n-2)I_{n-2}$$

3.16.6 Reduction Formula for

$$I_n = \int \cot^n x \, dx$$

$$I_n = -\frac{\cot^{n-1} x}{n-1} - I_{n-2}$$

3.16.7 Reduction Formula for

$$\begin{aligned}
I_{m,n} &= \int \sin^m x \cos^n x \, dx \\
I_{m,n} &= -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} I_{m-2,n} \\
&= \frac{\sin^{m+1} x \cos^{n+1} x}{m+n} + \frac{n-1}{m+n} I_{m,n-2} \\
&= \frac{\sin^{m+1} x \cos^{n+1} x}{m+1} + \frac{m+n+2}{m+1} I_{m+2,n} \\
&= \frac{\sin^{m+1} x \cos^{n+1} x}{n+1} + \frac{m+n+2}{n+1} I_{m,n+2} \\
&= \frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1} I_{m-2,n+2} \\
&= \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1} I_{m+2,n-2}
\end{aligned}$$

3.16.8 Reduction Formula for

$$\begin{aligned}
I_{m,n} &= \int \cos^m x \sin nx \, dx \\
I_{m,n} &= -\frac{\cos^m x \cos nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1}
\end{aligned}$$

3.16.9 Reduction Formula for

$$\begin{aligned}
I_{m,n} &= \int \cos^m x \cos nx \, dx \\
I_{m,n} &= \frac{\cos^m x \sin nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1}
\end{aligned}$$

3.16.10 Reduction Formula for

$$\begin{aligned}
I_{m,n} &= \int \sin^m x \sin nx \, dx \\
I_{m,n} &= \frac{n \sin^m x \cos nx}{m^2 - n^2} - \frac{m \sin^{m-1} x \cos x \cos nx}{m^2 - n^2} + \frac{m(m-1)}{m^2 - n^2} I_{m-2,n}
\end{aligned}$$

3.16.11 Reduction Formula for

$$\begin{aligned}
I_{m,n} &= \int \sin^m x \cos nx \, dx \\
I_{m,n} &= \frac{n \sin^m x \sin nx}{m^2 - n^2} - \frac{m \sin^{m-1} x \cos x \cos nx}{m^2 - n^2} + \frac{m(m-1)}{m^2 - n^2} I_{m-2,n}
\end{aligned}$$

3.17 Integration Using Differentiation

Integrals of form

$$\int \frac{1}{(a + b \cos x)^2} dx, \int \frac{1}{(a + b \sin x)^2} dx, \int \frac{1}{(\sin x + a \sec x)^2} dx$$
$$\int \frac{a + b \sin x}{(b + a \sin x)^2} dx$$

To evaluate integral of above form, follow the steps-

- i. Let $A = \frac{\sin x}{a + b \cos x}$ or $\frac{\cos x}{a + b \sin x}$ according to the integral to be evaluated
is of the form $\int \frac{1}{(a + b \cos x)^2} dx$ or $\int \frac{1}{(a + b \sin x)^2} dx$
- ii. Find $\frac{dA}{dx}$ and express it in terms of $\frac{1}{a + b \cos x}$ or $\frac{1}{a + b \sin x}$ as the case may be.
- iii. Integrate both sides of the expression obtained in step (ii) to obtain the value of the required integral.

3.18 Integral of Inverse Functions

If $\int f(x) dx = F(x)$

Then,

$$\int f^{-1}(x) dx = x f^{-1}(x) - F(f^{-1}(x))$$