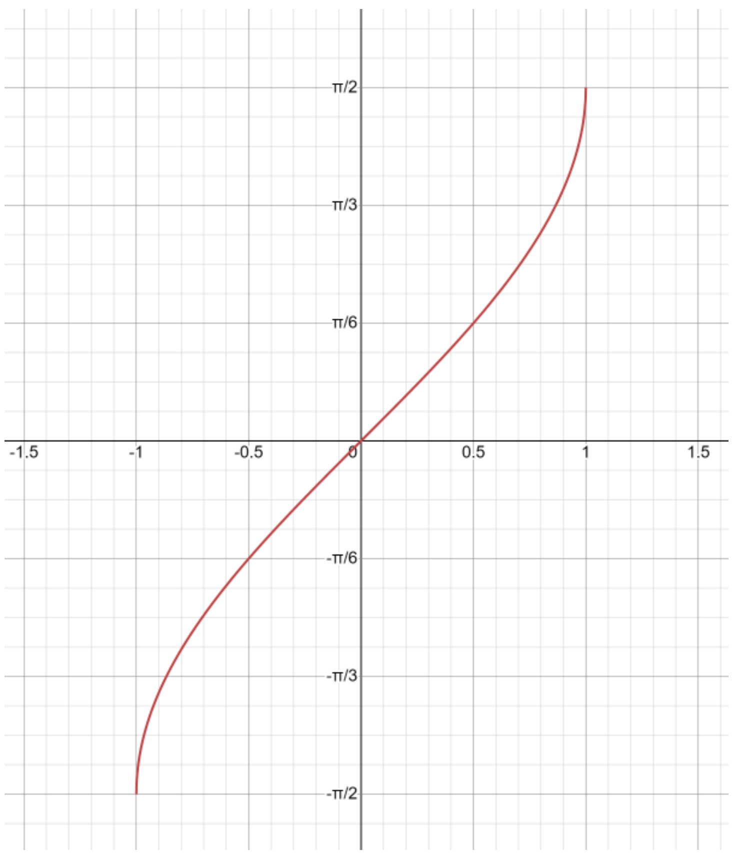
Inverse Trigonometric Functions

1 Domain and Range of ITFs

1.1 Sine Inverse

$f : [-1, 1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], f(x) = \sin^{-1} x = \arcsin x$

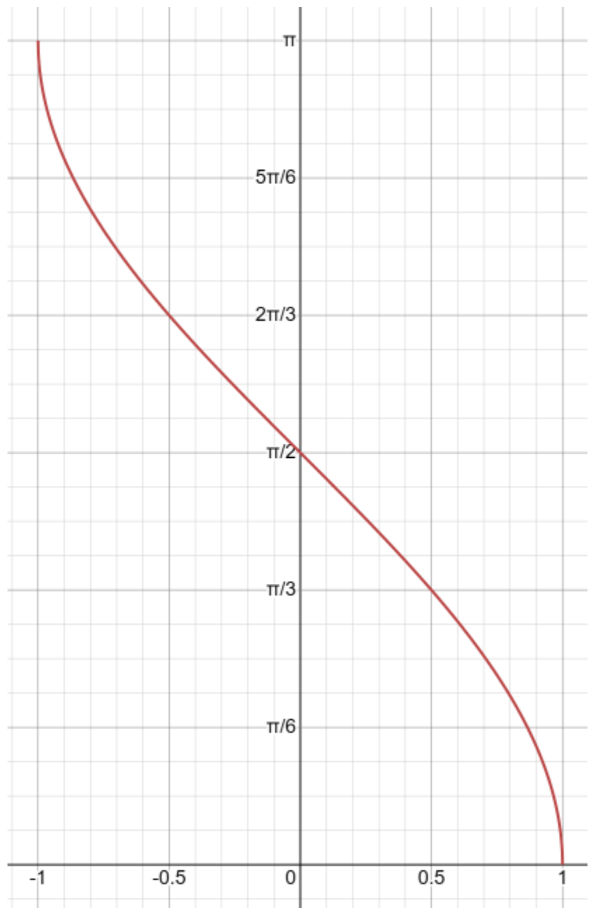


- The branch with range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is called **principal value branch**.
- The numerically least angle is called the **principal value branch**.

- $\sin^{-1} x$ is Bounded, Odd, Increasing, Aperiodic, Max (at $x = 1$) $= \dfrac{\pi}{2}$, Min (at $x = -1$) $= -\dfrac{\pi}{2}$

1.2 Cosine Inverse

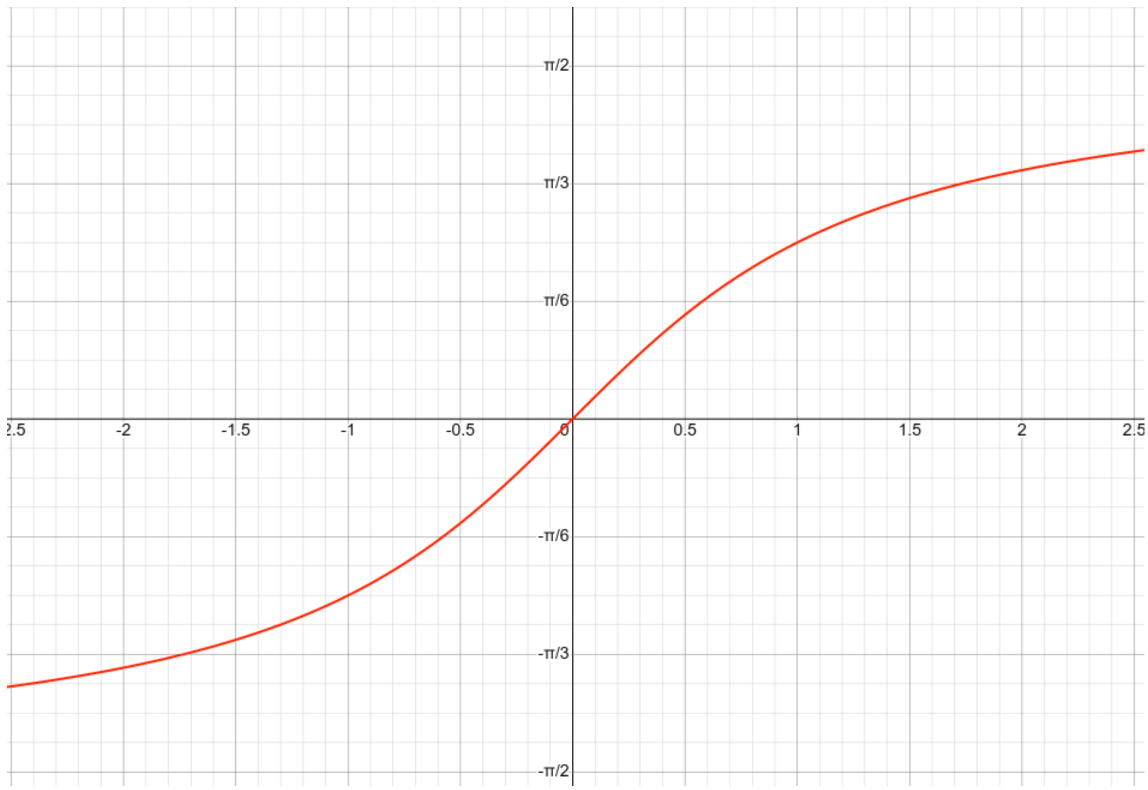
$f : [-1, 1] \longrightarrow [0, \pi]\,, f(x) = \cos^{-1} x = \arccos x$



- Principal Value Branch for $\cos^{-1} x$ is $[0, \pi]$
- $\cos^{-1} x$ is Bounded, Decreasing, Aperiodic, Max (at $x = -1$) $= \pi$, Min (at $x = 0$) $= 0$

1.3 Tangent Inverse

$f : \mathbb{R} \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], f(x) = \tan^{-1} x = \arctan x$



- Principal Value Branch for $\tan^{-1} x$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
- $\tan^{-1} x$ is Bounded, Odd, Increasing, Aperiodic,
  Horizontal Asymptote $y = \pm\frac{\pi}{2}$

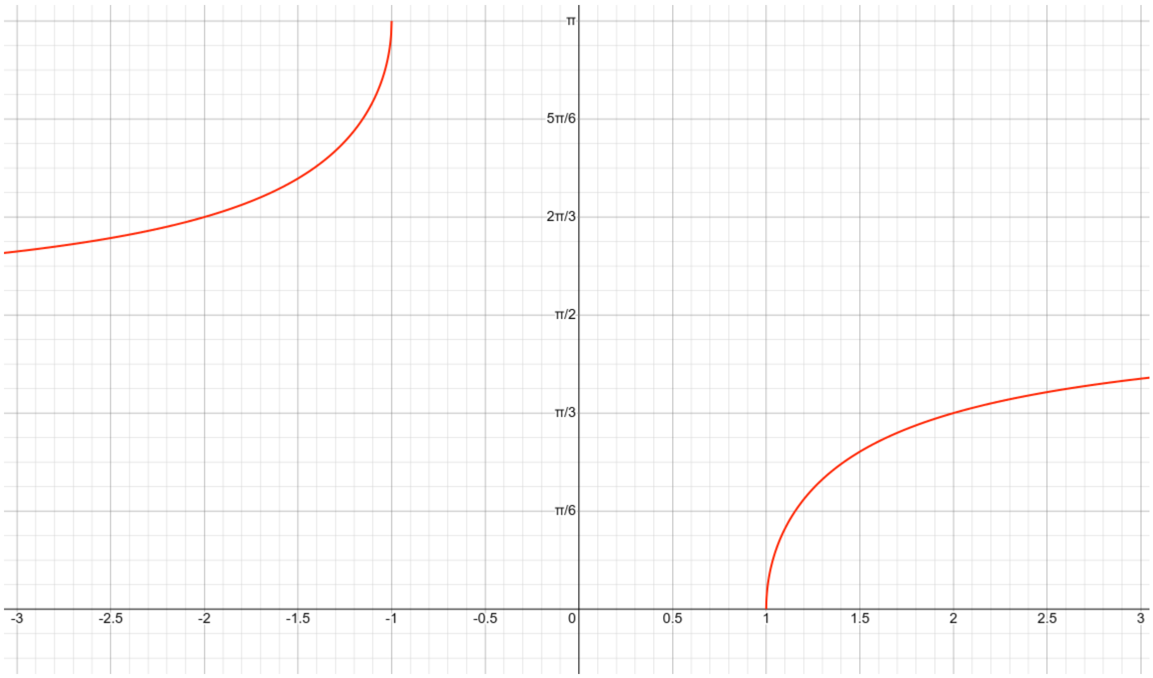
1.4 Cotangent Inverse

$f : \mathbb{R} \longrightarrow [0, \pi]\,, f(x) = \cot^{-1} x = \operatorname{arccot} x$

- Principal Value Branch for $\cot^{-1} x$ is $[0, \pi]$
- $\cot^{-1} x$ is Bounded, Decreasing, Aperiodic,
  Horizontal Asymptote $y = 0, y = \pi$

1.5 Secant Inverse

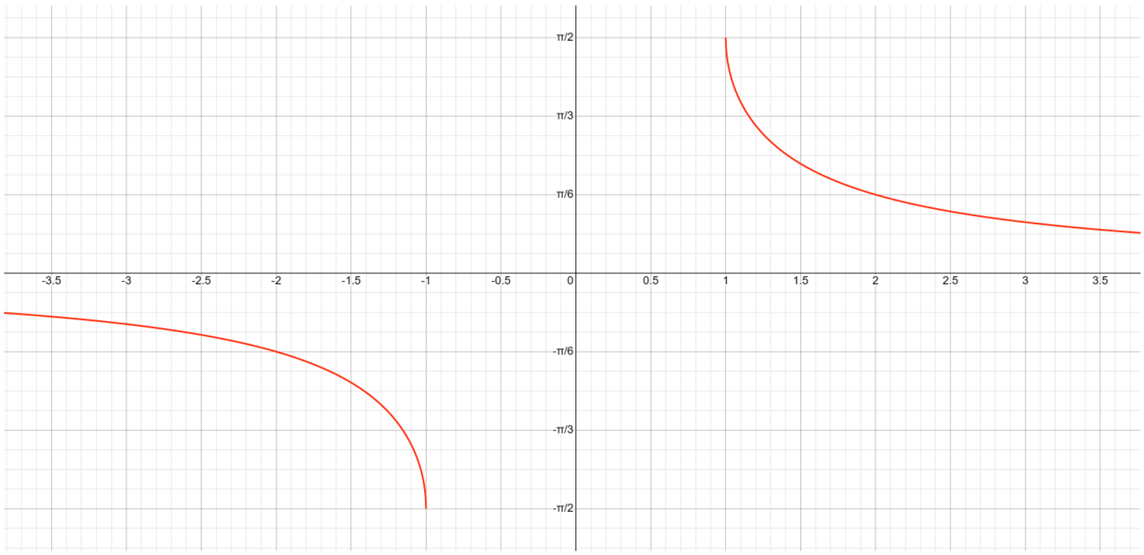
$f : \mathbb{R} - (-1, 1) \longrightarrow [0, \pi] - \left\{\dfrac{\pi}{2}\right\}, f(x) = \sec^{-1} x = \operatorname{arcsec} x$



- Principal Value Branch for $\sec^{-1} x$ is $[0, \pi] - \left\{\dfrac{\pi}{2}\right\}$
- $\sec^{-1} x$ is Bounded, Increasing, Aperiodic, Max (at $x = -1) = \pi$, Min (at $x = 1) = 0$, Horizontal Asymptote $y = \dfrac{\pi}{2}$

1.6 Cosecant Inverse

$f:\mathbb{R}-(-1,1)\longrightarrow\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}\,,f(x)=\csc^{-1}x=\operatorname{arccsc}x$



- Principal Value Branch for $\csc^{-1}x$ is $(0,\pi)$
- $\csc^{-1}x$ is Bounded, Odd, Decreasing, Aperiodic, Max (at $x=1$) $=\frac{\pi}{2}$, Min (at $x=-1$) $=-\frac{\pi}{2}$, Horizontal Asymptote $y=0$

2 Properties of ITFs

2.1 Inverse Composition Property

i. $\sin^{-1}(\sin x)=x \qquad \forall\ x\in\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$

ii. $\cos^{-1}(\cos x)=x \qquad \forall\ x\in[0,\pi]$

iii. $\tan^{-1}(\tan x)=x \qquad \forall\ x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

iv. $\csc^{-1}(\csc x)=x \qquad \forall\ x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)-\{0\}$

v. $\sec^{-1}(\sec x)=x \qquad \forall\ x\in[0,\pi]-\left\{\frac{\pi}{2}\right\}$

vi. $\cot^{-1}(\cot x)=x \qquad \forall\ x\in(0,\pi)$

2.2 Negative Argument Property

i. $\sin^{-1}(-x) = -\sin^{-1} x$

ii. $\tan^{-1}(-x) = -\tan^{-1} x$

iii. $\csc^{-1}(-x) = -\csc^{-1} x$

iv. $\cos^{-1}(-x) = \pi - \cos^{-1} x$

v. $\cot^{-1}(-x) = \pi - \cot^{-1} x$

vi. $\sec^{-1}(-x) = \pi - \sec^{-1} x$

2.3 Trig. Composition Property

i. $\sin\big(\sin^{-1} x\big) = x \qquad \forall\ x \in [-1, 1]$

ii. $\cos\big(\cos^{-1} x\big) = x \qquad \forall\ x \in [-1, 1]$

iii. $\tan\big(\tan^{-1} x\big) = x \qquad \forall\ x \in \mathbb{R}$

iv. $\csc\big(\csc^{-1} x\big) = x \qquad \forall\ x \in \mathbb{R} - (-1, 1)$

v. $\sec\big(\sec^{-1} x\big) = x \qquad \forall\ x \in \mathbb{R} - (-1, 1)$

vi. $\cot\big(\cot^{-1} x\big) = x \qquad \forall\ x \in \mathbb{R}$

2.4 Argument Reciprocal Property

i. $\csc^{-1} x = \sin^{-1}\left(\dfrac{1}{x}\right) \qquad \forall\ x \in \mathbb{R} - (-1, 1)$

ii. $\sin^{-1} x = \csc^{-1}\left(\dfrac{1}{x}\right) \qquad \forall\ x \in [-1, 1] - \{0\}$

iii. $\sec^{-1} x = \cos^{-1}\left(\dfrac{1}{x}\right) \qquad \forall\ x \in \mathbb{R} - (-1, 1)$

iv. $\cos^{-1} x = \sec^{-1}\left(\dfrac{1}{x}\right) \qquad \forall\ x \in [-1, 1] - \{0\}$

v. $\cot^{-1} x = \begin{cases} \tan^{-1}\left(\dfrac{1}{x}\right) & x > 0 \\ \pi + \tan^{-1}\left(\dfrac{1}{x}\right) & x < 0 \end{cases}$

vi. $\tan^{-1} x = \begin{cases} \cot^{-1}\left(\dfrac{1}{x}\right) & x > 0 \\ -\pi + \cot^{-1}\left(\dfrac{1}{x}\right) & x < 0 \end{cases}$

2.5 Complementary Angle Property

i. $\sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2} \qquad \forall\ x \in [-1, 1]$

ii. $\tan^{-1} x + \cot^{-1} x = \dfrac{\pi}{2} \qquad \forall\ x \in \mathbb{R}$

iii. $\sec^{-1} x + \csc^{-1} x = \dfrac{\pi}{2} \qquad \forall\ x \in \mathbb{R} - (-1, 1)$

2.6 Sum Property

i. $$\tan^{-1} x + \tan^{-1} y = \begin{cases} \tan^{-1}\left(\dfrac{x+y}{1-xy}\right) & xy < 1 \\ \pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right) & x > 0, y > 0, xy < 1 \\ -\pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right) & x < 0, y < 0, xy > 1 \end{cases}$$

ii. $$\tan^{-1} x - \tan^{-1} y = \begin{cases} \tan^{-1}\left(\dfrac{x-y}{1+xy}\right) & xy > -1 \\ \pi + \tan^{-1}\left(\dfrac{x-y}{1+xy}\right) & x > 0, y < 0, xy < -1 \\ -\pi + \tan^{-1}\left(\dfrac{x-y}{1+xy}\right) & x < 0, y > 0, xy < -1 \end{cases}$$

iii. $$\tan^{-1} x_1 + \tan^{-1} x_2 + \ldots + \tan^{-1} x_n = \tan^{-1}\left(\frac{S_1 - S_3 + S_5 - \ldots}{1 - S_2 + S_4 - S_6 + \ldots}\right)$$
$$\forall\ x_i \in \mathbb{R}, i \in \mathbb{N}$$