Differentiation

1 Fundamental Definition of a Derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to x} \frac{f(h) - f(x)}{h - x}$$

2 Derivatives of Standard Functions

i $\frac{d}{dx}(constant) = 0$

ii $\frac{d}{dx}(x^n) = nx^{n-1}$

iii $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$

iv $\frac{d}{dx}(|x|) = \frac{x}{|x|}$

v $\frac{d}{dx}(e^x) = e^x$

vi $\frac{d}{dx}(a^x) = a^x \cdot \ln a$

vii $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$

viii $\frac{d}{dx}(\log_a|x|) = \frac{1}{x \ln a}$

ix $\frac{d}{dx}(\sin x) = \cos x$

x $\frac{d}{dx}(\cos x) = -\sin x$

xi $\frac{d}{dx}(\tan x) = \sec^2 x$

xii $\frac{d}{dx}(\cot x) = -\csc^2 x$

xiii $\frac{d}{dx}(\sec x) = \sec x \tan x$

xiv $\frac{d}{dx}(\csc x) = -\csc x \cot x$

xv $\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$

xvi $\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$

xvii $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$

xviii $\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{|x|\sqrt{x^2-1}}$

xix $\frac{d}{dx}\left(\cot^{-1} x\right) = -\frac{1}{1+x^2}$

xx $\frac{d}{dx}\left(\csc^{-1} x\right) = -\frac{1}{|x|\sqrt{x^2-1}}$

xxi $\frac{d}{dx}(x^x) = x^x(1 + \ln x)$

3 Rules of Differentiation

- $\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)$
- $\frac{d}{dx}(k \cdot f(x)) = k \cdot f(x)$
- $\frac{d}{dx}(f(x) \cdot g(x)) = f(x)g'(x) + g(x)f'(x)$ 1st function derivative of 2nd + 2nd function derivative of 1st
- $$\begin{aligned}\frac{d}{dx}(f_1(x)f_2(x)f_3(x)\ldots f_n(x)) &= [f_1'(x)f_2(x)f_3(x)\ldots f_n(x)] \\ &+ [f_1(x)f_2'(x)f_3(x)\ldots f_n(x)] \\ &+ [f_1(x)f_2(x)f_3'(x)\ldots f_n(x)] \\ &+ \ldots + [f_1(x)f_2(x)f_3(x)\ldots f_n'(x)]\end{aligned}$$
- $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$ Low DHigh - High DLow square the bottom and the way we go
- $\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$

4 Implicit Differentiation

$\frac{d}{dx}(f(x,y)) = -\frac{\frac{\partial}{\partial x}(f(x,y))}{\frac{\partial}{\partial y}(f(x,y))}$ OR Differentiate both sides w.r.t. x of the given relation, then solve for $\frac{dy}{dx}$

5 Inverse Trigonometric Substitutions

Expression	**Substitution**
$\sqrt{a^2 - x^2}$	$x = a \sin \theta$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta$ or $a \cot \theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta$ or $a \csc \theta$
$\sqrt{\frac{a+x}{a-x}}$ or $\sqrt{\frac{a-x}{a+x}}$	$x = a \cos \theta$ or $a \cos 2\theta$
$\sqrt{(a-x)(x-b)}$ or $\sqrt{\frac{a-x}{x-b}}$ or $\sqrt{\frac{x-b}{a-x}}$	$x = a \cos^2 \theta - b \sin^2 \theta$
$\sqrt{2ax - x^2}$	$x = a(1 - \cos \theta)$

6 Parametric Derivatives

- Let $x = g(t)$, $y = f(t)$, If possible, eliminate t and obtain the relation between y and x. Now differentiate this relation implicitly.
- If it is not possible to eliminate t then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{f'(t)}{g'(t)}$$

7 Logarithmic Differentiation

If $y = u^v$ where, $u = f(x), v = g(x)$
then take ln on both sides $\implies \ln y = v \ln u$. Now differentiate this realtion implicitly.

OR

$$\frac{dy}{dx} = \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}$$

8 Derivative of functions in product

To differentiate, $\phi(x) = f_1(x) \cdot f_2(x) \cdot f_3(x) \cdot \ldots \cdot f_n(x)$, take ln both sides then differentiate implicitly to get,

$$\frac{d}{dx}(\phi(x)) = (f_1(x) \cdot f_2(x) \cdot f_3(x) \cdot \ldots \cdot f_n(x)) \left(\frac{f_1'(x)}{f_1(x)} + \frac{f_2'(x)}{f_2(x)} + \ldots + \frac{f_n'(x)}{f_n(x)} \right)$$

OR

$$\frac{d}{dx}(\phi(x)) = \prod_{i=1}^{n} f_i(x) \cdot \sum_{i=1}^{n} \frac{f_i'(x)}{f_i(x)}$$

9 Differentiation w.r.t. Another Function

Let $u = f(x)$, $v = g(x)$ to find $\frac{dv}{du}$ or $\frac{du}{dv}$.
First, find $\frac{d}{dx}(u)$ and $\frac{d}{dx}(v)$.
Now, $\frac{du}{dv} = \frac{du}{dx} \cdot \frac{dx}{dv} \implies \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{u'}{v'}$

OR

$$\frac{df(x)}{dg(x)} = \frac{f'(x)}{g'(x)}$$

10 Higher Order Derivatives

nth derivative of $f(x)$ w.r.t. x is denoted by $\frac{d^n}{dx^n}(f(x))$
$\frac{d^n}{dx^n}(f(x)) = \underbrace{\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx}\left(\dots\frac{d}{dx}(f(x))\right)\right)\right)}_{\text{n times}}$

10.1 Higher Order Parametric Derivatives

Let $x = \phi(t), y = \psi(x)$ then $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{\psi'(x)}{\phi'(x)}\right) = \frac{d}{dt}\left(\frac{\psi'(x)}{\phi'(x)}\right) \cdot \frac{dt}{dx}$

11 Derivative of a Determinant

Let $\Delta = \begin{vmatrix} u(x) & v(x) & w(x) \\ p(x) & q(x) & r(x) \\ \lambda(x) & \mu(x) & \eta(x) \end{vmatrix}$
then,

$$\frac{d\Delta}{dx} = \begin{vmatrix} u'(x) & v'(x) & w'(x) \\ p(x) & q(x) & r(x) \\ \lambda(x) & \mu(x) & \eta(x) \end{vmatrix} + \begin{vmatrix} u(x) & v(x) & w(x) \\ p'(x) & q'(x) & r'(x) \\ \lambda(x) & \mu(x) & \eta(x) \end{vmatrix} + \begin{vmatrix} u(x) & v(x) & w(x) \\ p(x) & q(x) & r(x) \\ \lambda'(x) & \mu'(x) & \eta'(x) \end{vmatrix}$$

12 Derivative of Inverse Functions

Let inverse of $y = f(x)$ be $f^{-1}(x)$

$$\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{\frac{d}{dx}(f(x))}$$

OR

$$\left(f^{-1}(y)\right)' = \frac{1}{f'(x)}$$