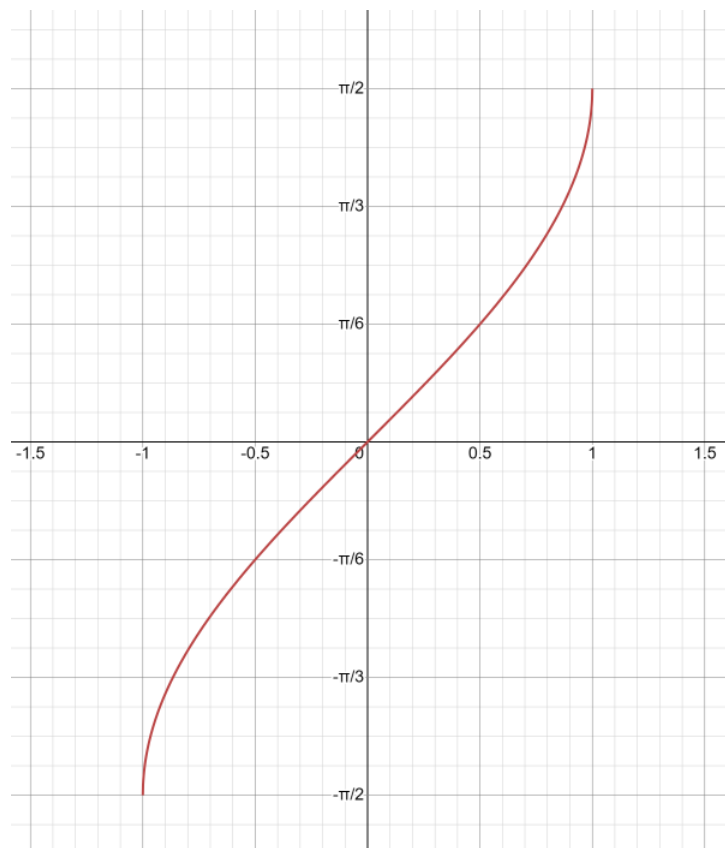


Inverse Trigonometric Functions

1 Domain and Range of ITFs

1.1 Sine Inverse

$$f : [-1, 1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], f(x) = \sin^{-1} x = \arcsin x$$

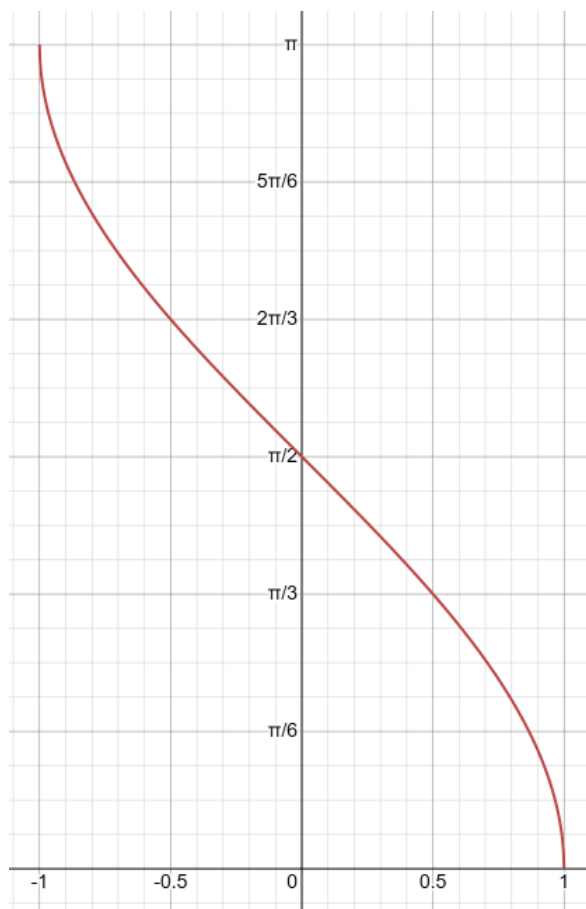


- The branch with range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is called **principal value branch**.
- The numerically least angle is called the **principal value branch**.

- $\sin^{-1} x$ is Bounded, Odd, Increasing, Aperiodic, Max (at $x = 1$) = $\frac{\pi}{2}$,
Min (at $x = -1$) = $-\frac{\pi}{2}$

1.2 Cosine Inverse

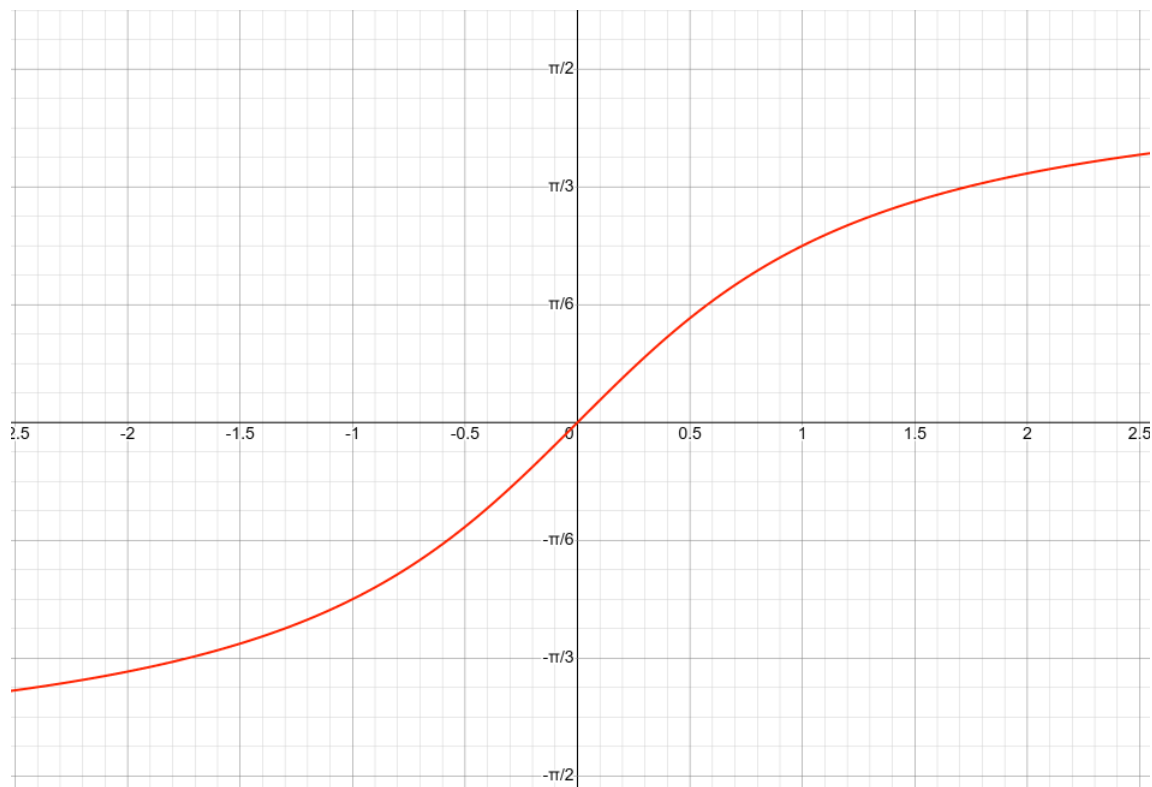
$$f : [-1, 1] \longrightarrow [0, \pi], f(x) = \cos^{-1} x = \arccos x$$



- Principal Value Branch for $\cos^{-1} x$ is $[0, \pi]$
- $\cos^{-1} x$ is Bounded, Decreasing, Aperiodic, Max (at $x = -1$) = π ,
Min (at $x = 1$) = 0

1.3 Tangent Inverse

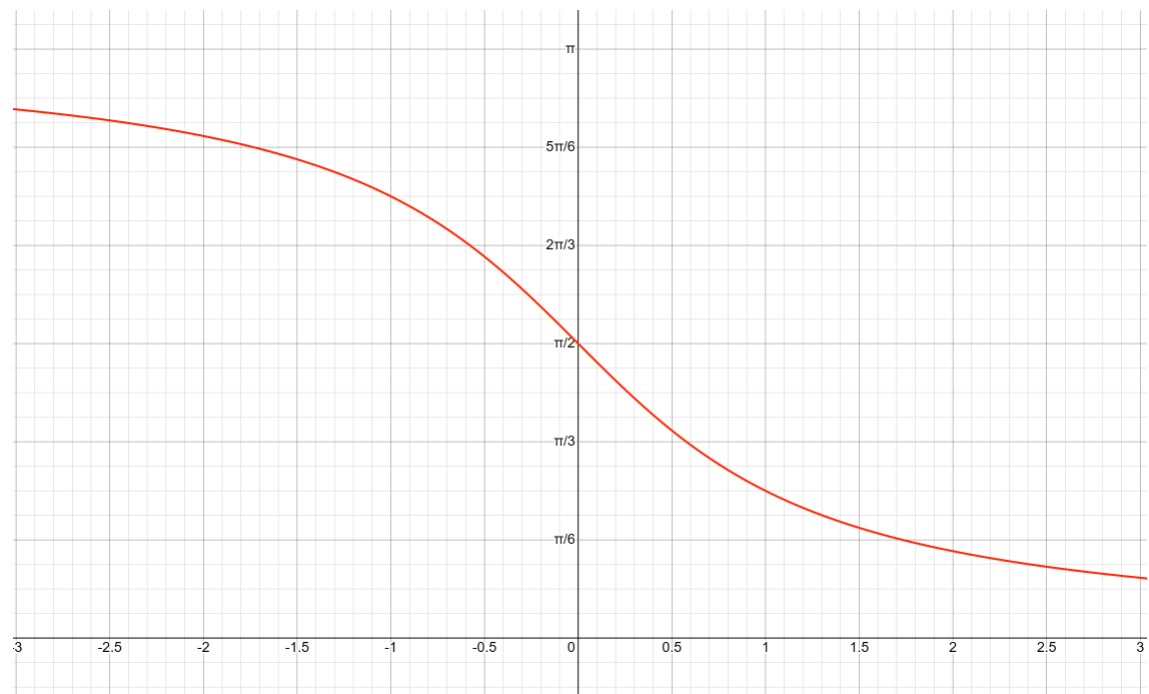
$$f : \mathbb{R} \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], f(x) = \tan^{-1} x = \arctan x$$



- Principal Value Branch for $\tan^{-1} x$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
- $\tan^{-1} x$ is Bounded, Odd, Increasing, Aperiodic,
Horizontal Asymptote $y = \pm \frac{\pi}{2}$

1.4 Cotangent Inverse

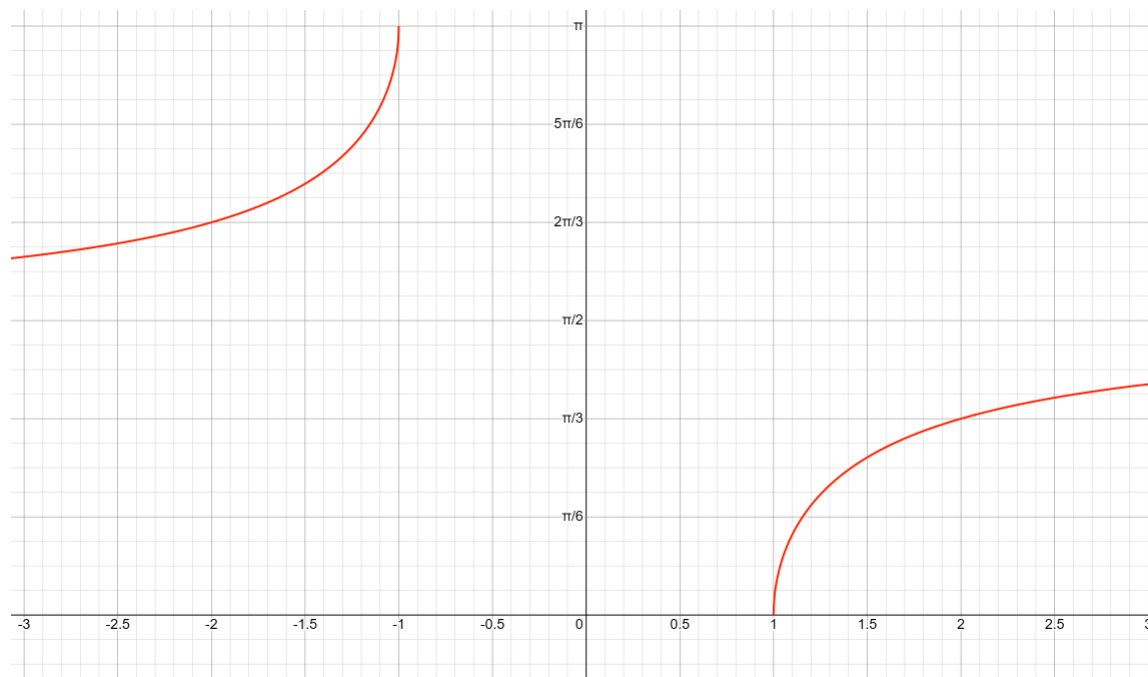
$$f : \mathbb{R} \longrightarrow [0, \pi], f(x) = \cot^{-1} x = \operatorname{arccot} x$$



- Principal Value Branch for $\cot^{-1} x$ is $[0, \pi]$
- $\cot^{-1} x$ is Bounded, Decreasing, Aperiodic, Horizontal Asymptote $y = 0, y = \pi$

1.5 Secant Inverse

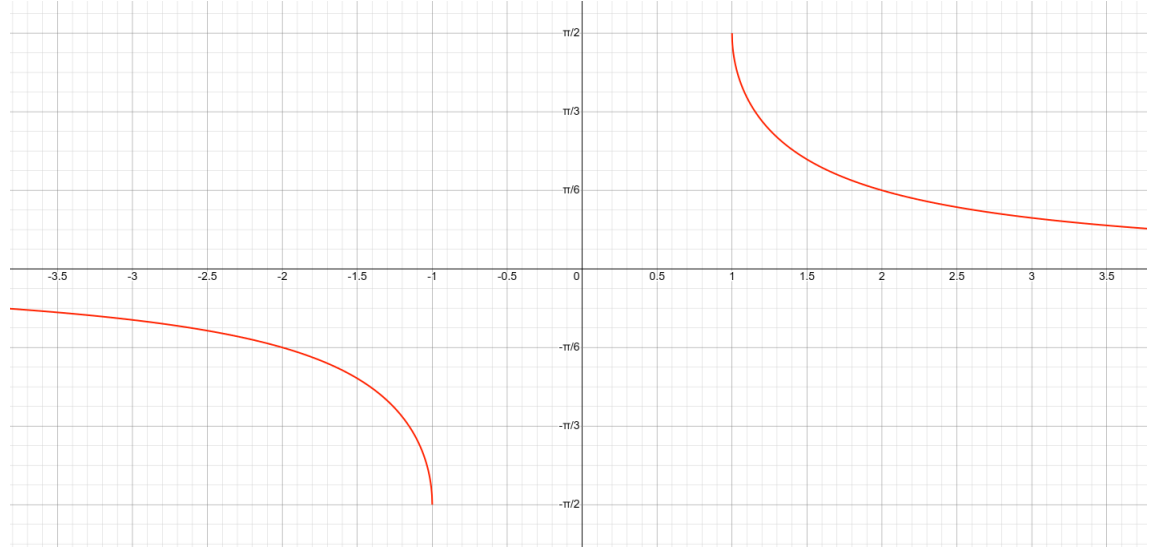
$$f : \mathbb{R} - (-1, 1) \longrightarrow [0, \pi] - \left\{ \frac{\pi}{2} \right\}, f(x) = \sec^{-1} x = \operatorname{arcsec} x$$



- Principal Value Branch for $\sec^{-1} x$ is $[0, \pi] - \left\{ \frac{\pi}{2} \right\}$
- $\sec^{-1} x$ is Bounded, Increasing, Aperiodic, Max (at $x = -1$) = π ,
Min (at $x = 1$) = 0, Horizontal Asymptote $y = \frac{\pi}{2}$

1.6 Cosecant Inverse

$$f : \mathbb{R} - (-1, 1) \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}, f(x) = \csc^{-1} x = \operatorname{arccsc} x$$



- Principal Value Branch for $\csc^{-1} x$ is $(0, \pi)$
- $\csc^{-1} x$ is Bounded, Odd, Decreasing, Aperiodic, Max (at $x = 1$) = $\frac{\pi}{2}$,
Min (at $x = -1$) = $-\frac{\pi}{2}$, Horizontal Asymptote $y = 0$

2 Properties of ITFs

2.1 Inverse Composition Property

- | | |
|------------------------------|--|
| i. $\sin^{-1}(\sin x) = x$ | $\forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ |
| ii. $\cos^{-1}(\cos x) = x$ | $\forall x \in [0, \pi]$ |
| iii. $\tan^{-1}(\tan x) = x$ | $\forall x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ |
| iv. $\csc^{-1}(\csc x) = x$ | $\forall x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) - \{0\}$ |
| v. $\sec^{-1}(\sec x) = x$ | $\forall x \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$ |
| vi. $\cot^{-1}(\cot x) = x$ | $\forall x \in (0, \pi)$ |

2.2 Negative Argument Property

- i. $\sin^{-1}(-x) = -\sin^{-1} x$
- ii. $\tan^{-1}(-x) = -\tan^{-1} x$
- iii. $\csc^{-1}(-x) = -\csc^{-1} x$
- iv. $\cos^{-1}(-x) = \pi - \cos^{-1} x$
- v. $\cot^{-1}(-x) = \pi - \cot^{-1} x$
- vi. $\sec^{-1}(-x) = \pi - \sec^{-1} x$

2.3 Trig. Composition Property

- i. $\sin(\sin^{-1} x) = x$ $\forall x \in [-1, 1]$
- ii. $\cos(\cos^{-1} x) = x$ $\forall x \in [-1, 1]$
- iii. $\tan(\tan^{-1} x) = x$ $\forall x \in \mathbb{R}$
- iv. $\csc(\csc^{-1} x) = x$ $\forall x \in \mathbb{R} - (-1, 1)$
- v. $\sec(\sec^{-1} x) = x$ $\forall x \in \mathbb{R} - (-1, 1)$
- vi. $\cot(\cot^{-1} x) = x$ $\forall x \in \mathbb{R}$

2.4 Argument Reciprocal Property

- i. $\csc^{-1} x = \sin^{-1} \left(\frac{1}{x} \right)$ $\forall x \in \mathbb{R} - (-1, 1)$
- ii. $\sin^{-1} x = \csc^{-1} \left(\frac{1}{x} \right)$ $\forall x \in [-1, 1] - \{0\}$
- iii. $\sec^{-1} x = \cos^{-1} \left(\frac{1}{x} \right)$ $\forall x \in \mathbb{R} - (-1, 1)$
- iv. $\cos^{-1} x = \sec^{-1} \left(\frac{1}{x} \right)$ $\forall x \in [-1, 1] - \{0\}$
- v. $\cot^{-1} x = \begin{cases} \tan^{-1} \left(\frac{1}{x} \right) & x > 0 \\ \pi + \tan^{-1} \left(\frac{1}{x} \right) & x < 0 \end{cases}$
- vi. $\tan^{-1} x = \begin{cases} \cot^{-1} \left(\frac{1}{x} \right) & x > 0 \\ -\pi + \cot^{-1} \left(\frac{1}{x} \right) & x < 0 \end{cases}$

2.5 Complementary Angle Property

$$\text{i. } \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} \quad \forall x \in [-1, 1]$$

$$\text{ii. } \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2} \quad \forall x \in \mathbb{R}$$

$$\text{iii. } \sec^{-1} x + \csc^{-1} x = \frac{\pi}{2} \quad \forall x \in \mathbb{R} - (-1, 1)$$

2.6 Sum Property

$$\text{i. } \tan^{-1} x + \tan^{-1} y = \begin{cases} \tan^{-1} \left(\frac{x+y}{1-xy} \right) & xy < 1 \\ \pi + \tan^{-1} \left(\frac{x+y}{1-xy} \right) & x > 0, y > 0, xy > 1 \\ -\pi + \tan^{-1} \left(\frac{x+y}{1-xy} \right) & x < 0, y < 0, xy > 1 \end{cases}$$

$$\text{ii. } \tan^{-1} x - \tan^{-1} y = \begin{cases} \tan^{-1} \left(\frac{x-y}{1+xy} \right) & xy > -1 \\ \pi + \tan^{-1} \left(\frac{x-y}{1+xy} \right) & x > 0, y < 0, xy < -1 \\ -\pi + \tan^{-1} \left(\frac{x-y}{1+xy} \right) & x < 0, y > 0, xy < -1 \end{cases}$$

$$\text{iii. } \tan^{-1} x_1 + \tan^{-1} x_2 + \dots + \tan^{-1} x_n = \tan^{-1} \left(\frac{S_1 - S_3 + S_5 - \dots}{1 - S_2 + S_4 - S_6 + \dots} \right) \\ \forall x_i \in \mathbb{R}, i \in \mathbb{N}$$

$$\text{iv. } \sin^{-1} x + \sin^{-1} y = \begin{cases} \sin^{-1} (x + \sqrt{1-y^2} + y\sqrt{1-x^2}) & x \geq -1, y \leq 1, x^2 + y^2 \leq 1 \\ \pi - \sin^{-1} (x\sqrt{1-y^2} + y\sqrt{1-x^2}) & x > 0, y \leq 1, x^2 + y^2 > 1 \\ -\pi - \sin^{-1} (x\sqrt{1-y^2} + y\sqrt{1-x^2}) & x \geq -1, y < 0, x^2 + y^2 > 1 \end{cases}$$

v.