Limits

1 Indeterminate Forms

7 indeterminate forms are,
$\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 1^{\infty}, 0^{0}, \infty^{0}, \infty \cdot 0$

2 L'Hôpital's Rule

$L = \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \iff \frac{f(a)}{g(a)} = \frac{0}{0} \quad \vee \frac{\infty}{\infty}$

3 Taylor Expansions

1. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \ldots \infty$
2. $a^x = e^{xlna} = \sum_{n=0}^{\infty} \frac{(x \ln a)^n}{n!} = 1 + x \ln a + \frac{(x \ln a)^2}{2!} + \ldots \infty \qquad a \in \mathbb{R}^+$
3. $(1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \ldots \infty$
4. $\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots \infty \qquad x \in (-1, 1]$
5. $\frac{x^n - a^n}{x-a} = \sum_{r=0}^{n-1} a^r \cdot x^{n-r-1} = x^{n-1} + ax^{n-2} + a^2 x^{n-3} + \ldots + a^{n-1}$
6. $(1+x)^{\frac{1}{x}} = e\left(1 - \frac{x}{2} + \frac{11x^2}{24} + \ldots \infty\right)$
7. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \infty$
8. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots \infty$
9. $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \ldots \infty$

10. $\sin^{-1} x = x + \frac{1^2 x^3}{3!} + \frac{1^2 \cdot 3^2 x^5}{5!} + \frac{1^2 \cdot 3^2 \cdot 5^2 x^7}{7!} + \ldots \infty$
11. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} + \ldots \infty$
12. $\sec^{-1} x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + \ldots \infty$
13. $\left(\sin^{-1} x\right)^2 = \frac{2x^2}{2!} + \frac{2 \cdot 2^2 x^4}{4!} + \frac{2 \cdot 2^2 \cdot 4^2 x^6}{6!} + \ldots \infty$
14. $x \cot x = 1 - \frac{x^3}{3} + \frac{x^4}{45} - \frac{2x^6}{945} + \ldots \infty$
15. $\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \ldots \infty$
16. $x \csc x = 1 + \frac{x^2}{6} + \frac{7x^4}{360} + \frac{31x^6}{15120} + \ldots \infty$

4 Standard Limits

1. $\lim\limits_{x \to a} \frac{x^n - a^n}{x - a} = n \cdot a^{n-1} \qquad a \in \mathbb{R}$
2. $\lim\limits_{x \to \infty} \frac{a_m x^m + a_{m-1} x^{m-1} + \ldots + a_1 x + a_0}{b_n x^n + b_{n-1} x^{n-1} + \ldots + b_1 x + b_0} = \begin{cases} 0 & m < n \\ \frac{a_m}{b_n} & m = n \\ +\infty & m > n \ \wedge \ a_m \cdot b_n > 0 \\ -\infty & m > n \ \wedge \ a_m \cdot b_n < 0 \end{cases}$
3. If $A, B \in \{\sin x, \tan x, \sin^{-1} x, \tan^{-1} x, x\}$,

   then $\lim\limits_{x \to 0} \frac{A}{B} = 1$
4. $\lim\limits_{x \to 0} \frac{\sin x^\circ}{x} = \frac{\pi}{180^\circ}$
5. $\lim\limits_{x \to 0} \frac{\ln(1+x)}{x} = 1$
6. $\lim\limits_{x \to 0} \frac{e^x - 1}{x} = 1$
7. $\lim\limits_{x \to 0} \frac{a^x - 1}{x} = \ln a$
8. $\lim\limits_{x \to \infty} \left(1 + \frac{n}{x}\right)^x = e^n$
9. $\lim\limits_{x \to 0} (1 + nx)^{\frac{1}{x}} = e^n$

5 Algebra of Limits

If $\lim_{x \to a} f(x) = L \quad \wedge \quad \lim_{x \to a} g(x) = M$ where $L, M \in \mathbb{R}$ and are finite then

- $\lim_{x \to a} (f(x) \pm g(x)) = L \pm M$
- $\lim_{x \to a} (f(x) \cdot g(x)) = L \cdot M$
- $\lim_{x \to a} \left(\frac{f(x)}{g(x)} \right) = \frac{L}{M} \quad M \neq 0$
- $\lim_{x \to a} k \cdot f(x) = k \cdot L \quad k \in \mathbb{R}$