Indefinite Integration

1 Fundamental Definition of Indefinite Integration

If f and F and functions such that $\dfrac{d}{dx}(F(x)) = f(x)$ then F is an anti-derivative of f w.r.t. x
Symbolically,

$$\int f(x)\,dx = F(x) + C$$

where C is the constant of Integration

2 Anti-Derivatives of Some Standard Functions

i. $\displaystyle\int k \cdot f(x)\,dx = k \cdot \int f(x)\,dx$

ii. $\displaystyle\int [f_1(x) \pm f_2(x) \pm f_3(x) \pm \ldots \pm f_n(x)]\ dx = \int f_1(x)\,dx \pm \int f_2(x)\,dx \pm \int f_3(x)\,dx \pm \ldots \int f_n(x)\,dx$

iii. $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad n \neq -1$

iv. $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$

v. $\displaystyle\int e^x\,dx = e^x + C$

vi. $\displaystyle\int a^x\,dx = \frac{a^x}{\ln a} + C$

vii. $\displaystyle\int \sin x\,dx = -\cos x + C$

viii. $\displaystyle\int \cos x\,dx = \sin x + C$

ix. $\int \sec^2 x \, dx = \tan x + C$

x. $\int \csc^2 x \, dx = -\cot x + C$

xi. $\int \sec x \tan x \, dx = \sec x + C$

xii. $\int \csc x \cot x \, dx = -\csc x + C$

xiii. $\int \cot x \, dx = \ln|\sin x| + C$

xiv. $\int \tan x \, dx = -\ln|\cos x| + C$

xv. $\int \sec x \, dx = \ln|\sec x + \tan x| + C$

xvi. $\int \csc x \, dx = \ln|\csc x - \cot x| + C$

xvii. $\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + C$

xviii. $\int \frac{-1}{\sqrt{1-x^2}} \, dx = \cos^{-1} x + C$

xix. $\int \frac{1}{1+x^2} \, dx = \tan^{-1} x + C$

xx. $\int \frac{-1}{1+x^2} \, dx = \cot^{-1} x + C$

xxi. $\int \frac{1}{x\sqrt{x^2-1}} \, dx = \sec^{-1} x + C$

xxii. $\int \frac{-1}{x\sqrt{x^2-1}} \, dx = \csc^{-1} x + C$

xxiii. $\int \sqrt{x} \, dx = \frac{2}{3} x^{3/2} + C$

xxiv. $\int \frac{1}{\sqrt{x}} \, dx = 2\sqrt{x} + C$

xxv. $\int \frac{dx}{x^2-1} = \ln\left|\frac{x-1}{x+1}\right| + C$

xxvi. $\displaystyle\int \frac{dx}{\sqrt{1+x^2}} = \ln\left|x+\sqrt{x^2+1}\right| + C$

xxvii. $\displaystyle\int \frac{dx}{\sqrt{x^2-1}} = \ln\left|x+\sqrt{x^2-1}\right| + C$

xxviii. $\displaystyle\int \sqrt{1-x^2}\,dx = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x + C$

xxix. $\displaystyle\int \sqrt{x^2-1}\,dx = \frac{1}{2}x\sqrt{x^2-1} - \frac{1}{2}\ln\left|x+\sqrt{x^2-1}\right| + C$

xxx. $\displaystyle\int \sqrt{x^2+1}\,dx = \frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\ln\left|x+\sqrt{x^2+1}\right| + C$

xxxi. $\displaystyle\int \frac{x^2+1}{x^4+1}\,dx = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-{}^1\!/\!_x}{\sqrt{2}}\right) + C$

xxxii. $\displaystyle\int \frac{x^2-1}{x^4+1}\,dx = \frac{1}{2\sqrt{2}}\ln\left|\frac{x+{}^1\!/\!_x-\sqrt{2}}{x+{}^1\!/\!_x+\sqrt{2}}\right| + C$

Important Results

1. If $F_1(x)$ and $F_2(x)$ are two anti-derivatives of a function $f(x)$, then $F_1(x)$ and $F_2(x)$ only differ by a constant, i.e.

$$F_1(x) - F_2(x) = C$$

   where, C is a $\mathbb{R}$ constant.

2. If $f(x)$ is continuous $\forall\, x \in D_f$ and, $\displaystyle\int f(x)\,dx = F(x) + C$, then $F(x)$ always exists and is continuous.

3. If $f(x)$ is discontinuous at $x = x_1$, then its anti-derivative can be continuous at $x = x_1$.

4. Anti-derivative of a periodic function may not be periodic.

3 Methods of Integration

3.1 u substitution

Integrals of form,

$$I = \int f(g(x)) \cdot g'(x)\,dx$$

Can be solved by, the substitution,

$$u = g(x)$$

Differentiating both sides w.r.t. x,

$$du = g'(x)dx$$

Now,

$$I = \int f(u)\, du$$

3.2 Integrals of form

$$\int \frac{dx}{ax^2+bx+c}, \int \frac{dx}{\sqrt{ax^2+bx+c}}, \int \sqrt{ax^2+bx+c}\, dx$$

Using completing the square,

$$ax^2+bx+c = a\left[\left(x+\frac{b}{2a}\right)+\frac{c}{a}-\frac{b^2}{4a}\right]$$

Now, using u *sub*, let

$$u = x + \frac{b}{2a}$$

The transformed integral can be integrated using previous methods.

3.3 Integrals of form

$$\int \frac{px+q}{ax^2+bx+c}\, dx, \int \frac{px+q}{\sqrt{ax^2+bx+c}}\, dx, \int (px+q)\sqrt{ax^2+bx+c}\, dx$$

$$px+q = \lambda\frac{d}{dx}\left(ax^2+bx+c\right)+\mu$$

After finding λ, μ,
For the 1st part, use u *sub*,
Let

$$u = ax^2+bx+c$$

2nd part of the integral can be integrated using previous methods.

3.4 Integrals of form

$$\int \frac{K(x)}{ax^2+bx+c}\, dx$$

where $deg(K(x)) \geq 2$

By polynomial long division

$$\frac{K(x)}{ax^2+bx+c} = Q(x) + \frac{R(x)}{ax^2+bx+c}$$

$deg(R(x)) \leq 1$,
Now, the integral

$$\int \frac{R(x)}{ax^2 + bx + c}\, dx$$

can be integrated using previous methods.

3.5 Integrals of form

$$\int \frac{ax^2 + bx + c}{px^2 + qx + r}\, dx, \int \frac{ax^2 + bx + c}{\sqrt{px^2 + qx + r}}\, dx, \int \left(ax^2 + bx + c\right) \sqrt{px^2 + qx + r}\, dx$$

$$ax^2 + bx + c = \lambda \left(px^2 + qx + r\right) + \mu \frac{d}{dx} \left(px^2 + qx + r\right) + \gamma$$

For the 3rd integral, in the 1st part use Trig sub. and Integration by Parts, 2nd part use *u-sub* and 3rd part can be integrated using previous methods.

3.6 Trig. Integrals

3.6.1 Integrals of form

$$\int \frac{dx}{a\cos^2 x + b\sin^2 x}, \int \frac{dx}{a + b\sin^2 x}, \int \frac{dx}{a + b\cos^2 x}$$
$$\int \frac{dx}{(a\sin x + b\cos x)^2}, \int \frac{dx}{a + b\sin^2 x + c\cos^2 x}$$

Steps -

1. Multiply numerator and denominator by $\sec^2 x$
2. Replace $\sec^2 x$ (if any) by $1 + \tan^2 x$ except the one multiplied in step 1.
3. Let $u = \tan x$, then $du = \sec^2 x\, dx$

Now, the transformed integral can be integrated using previous methods.

3.6.2 Integrals of form

$$\int \frac{dx}{a\sin x + b\cos x}, \int \frac{dx}{a + b\sin x}, \int \frac{dx}{a + b\cos x}, \int \frac{dx}{a\sin x + b\cos x + c}$$

Steps -

1. Replace $\sin x = \dfrac{2\tan {}^x\!/_2}{1 + \tan^2 {}^x\!/_2}$ and $\cos x = \dfrac{1 - \tan^2 {}^x\!/_2}{1 + \tan^2 {}^x\!/_2}$
2. Let $u = \tan {}^x\!/_2 \implies du = \dfrac{1}{2} \sec^2 {}^x\!/_2\, dx$ is already present in the numerator.

Now, the transformed integral can be integrated using previous methods.

3.6.2.1 Alternative Method to Integrate

$$I = \int \frac{dx}{a\sin x + b\cos x}$$

$$\begin{aligned}
a\sin x + b\cos x &= \sqrt{a^2+b^2}\sin\left(x + \tan^{-1}\left(\frac{b}{a}\right)\right)\\
I &= \frac{1}{\sqrt{a^2+b^2}}\int \csc\left(x + \tan^{-1}\left(\frac{b}{a}\right)\right)\,dx\\
I &= \frac{1}{\sqrt{a^2+b^2}}\ln\left|\tan\left(\frac{x}{2} + \frac{1}{2}\tan^{-1}\left(\frac{b}{a}\right)\right)\right| + C
\end{aligned}$$

3.6.3 Integrals of form

$$\int \frac{p\cos x + q\sin x + r}{a\cos x + b\sin x + c}\,dx, \int \frac{p\cos x + q\sin x}{a\cos x + b\sin x}\,dx$$

Steps for (i),

1. Express $Numerator = \lambda\ Denominator\ + \mu\ Derivative\ of\ denominator\ + \gamma$

Now, the transformed integral can be integrated using previous methods.

Steps for (ii),

1. Express $Numerator = \lambda\ Denominator\ + \mu\ Derivative\ of\ Denominator$

Now, the transformed integral can be integrated using previous methods.

3.7 Integration by parts

$$\int uv\,dx = u\int v\,dx - \int\left(u'\int v\,dx\right)\,dx$$

u is the function which has to be differentiated (D), v is the function which has to be integrated (I)

3.8 Integrals of form

$$I = \int e^{g(x)}\left(f(x)g'(x) + f'(x)\right)\,dx$$

$I = e^{g(x)}\cdot f(x) + C$
If $g(x) = x$,
Then,

$$\int e^{x}\left(f(x) + f'(x)\right)\,dx = e^{x}\cdot f(x) + C$$

3.9 Integrals of form

$$S = \int e^{ax} \sin bx \, dx, C = \int e^{ax} \cos bx \, dx$$

$$S = \frac{e^{ax}}{a^2 + b^2} \left(a \sin bx - b \cos bx \right) + C_0, C = \frac{e^{ax}}{a^2 + b^2} \left(a \cos bx + b \sin bx \right) + C_{00}$$

3.10 Integration by Partial Fraction Decomposition

Let $f(x) = \sum_{i=0}^{n} a_i x^i, g(x) = \sum_{i=0}^{m} b_i x^i$.

We define a rational function $h(x) = \frac{f(x)}{g(x)}$,

$h(x)$ is $\begin{cases} \textit{Proper Rational Function} & m > n \\ \textit{Improper Rational Function} & m \leq n \end{cases}$

If $h(x)$ is Improper, we make it Proper by polynomial long division, i.e.

$$h(x) = Q(x) + \frac{r(x)}{g(x)}$$

Clearly, $\frac{r(x)}{g(x)}$ is Proper.

Now, assuming $h(x)$ is Proper, following cases arise -

3.10.1 $g(x)$ is the product of non-repeating linear factors

Let

$$g(x) = L_1(x) \cdot L_2(x) \cdot \ldots \cdot L_m(x)$$

where $L_i(x)$ are linear functions.

Then, we can expand $\frac{f(x)}{g(x)}$ in terms of partial fractions as,

$$\frac{f(x)}{g(x)} = \frac{A_1}{L_1} + \frac{A_2}{L_2(x)} + \ldots + \frac{A_m}{L_m(x)}$$

where, $A_i \in \mathbb{R}$ constants.

3.10.2 $g(x)$ is the product of non-repeating linear factors, but a particular factor is repeated k times

Let

$$g(x) = L_1^k(x) \cdot L_2(x) \cdot \ldots \cdot L_\eta(x)$$

Then, we can expand $\frac{f(x)}{g(x)}$ in terms of partial fractions as,

$$\frac{f(x)}{g(x)} = \frac{A_1}{L_1(x)} + \frac{A_2}{L_1^2(x)} + \frac{A_3}{L_1^3(x)} + \ldots + \frac{A_k}{L_1^k(x)} + \frac{B_2}{L_2(x)} \ldots + \frac{B_\eta}{L_\eta(x)}$$

3.10.3 $g(x)$ contains some non-repeating linear as well as quadratic factors

Let

$$g(x) = \prod_i L_i(x) \cdot \prod_j Q_j(x)$$

where, $Q_j(x)$ are quadratic factors.

Then, we can expand $\dfrac{f(x)}{g(x)}$ in terms of partial fractions as,

$$\frac{f(x)}{g(x)} = \sum_i \frac{A_i}{L_i(x)} + \sum_j \frac{xB_j + C_j}{Q_j(x)}$$

3.10.4 $g(x)$ contains some non-repeating linear and repeating quadratic factors

Let

$$g(x) = \prod_i L_i(x) \prod_j Q_j(x) \prod_\omega \aleph_\omega^k(x)$$

Where, $\aleph_\omega(x)$ are repeating quadratic factors.

Then, we can expand $\dfrac{f(x)}{g(x)}$ in terms of partial fractions as,

$$\frac{f(x)}{g(x)} = \sum_i \frac{A_i}{L_i(x)} + \sum_j \frac{xB_j + C_j}{Q_j(x)} + \sum_\omega \sum_r \frac{xD_r + E_r}{\aleph_\omega^r(x)}$$

3.11 Integrals of form

3.11.1

$$I = \int f\left(x + \frac{1}{x}\right)\left(1 - \frac{1}{x^2}\right) dx$$

Let, $u = x + \dfrac{1}{x} \implies du = \left(1 - \dfrac{1}{x^2}\right) dx$

Now, $I = \displaystyle\int f(u)\, du$

3.11.2 Integrals of form

$$\int \frac{x^2 + 1}{x^4 + kx^2 + 1}\, dx$$

Divide numerator and denominator by x^2

Now, the transformed integral can be integrated using previous methods.

3.12 Integration of Special Irrational Functions

3.12.1 Integrals of form

$$\int \frac{1}{(ax+b)\sqrt{cx+d}}\,dx$$

Using *u-sub*,
Let

$$u^2 = cx + d$$

Now, the transformed integral can be integrated using previous methods.

3.12.2 Integrals of form

$$\int \frac{1}{(ax^2+bx+c)\sqrt{px+q}}\,dx$$

Using *u-sub*,
Let

$$u^2 = px + q$$

Now, the transformed integral can be integrated using previous methods.

3.12.3 Integrals of form

$$\int \frac{1}{(ax+b)(\sqrt{px^2+qx+r})}\,dx$$

Using *u-sub*,
Let

$$\frac{1}{u} = ax + b$$

Now, the transformed integral can be integrated using previous methods.

3.12.4 Integrals of form

$$\int \frac{1}{(ax^2+b)\sqrt{cx^2+d}}\,dx$$

Using *u-sub*,
Let

$$\frac{1}{\sqrt{u}} = x$$

Now, the transformed integral can be integrated using previous methods.

3.12.5 Integrals of form

$$\int \frac{1}{(x-k)^r\sqrt{ax^2+bx+c}}\,dx, r \geq 2$$

Using *u-sub*,
Let

$$\frac{1}{u} = x - k$$

Now, the transformed integral can be integrated using previous methods.

3.12.6 Integrals of form

$$\int \frac{ax^2+bx+c}{(\alpha x+\beta)\sqrt{px^2+qx+r}}\,dx$$

Express,

$$ax^2+bx+c = \lambda(\alpha x+\beta)\left[\frac{d}{dx}\big(px^2+qx+r\big)\right] + \mu(\alpha x+\beta) + \gamma$$

Now, the transformed integral can be integrated using previous methods.

3.13 integrals of form

$$\int \sin^m x \cdot \cos^n x\,dx$$

3.13.1 If one of m or n is odd, $\quad m, n \in \mathbb{N}$

Then, we *u-sub* the term with even power, i.e.
If $m = 2k+1, n = 2p$ then, $u = \cos x$
If $m = 2p, n = 2k+1$ then, $u = \sin x$

3.13.2 If both m and n are odd, $\quad m, n \in \mathbb{N}$

Then, *u-sub* any of $\cos x$ or $\sin x$.

3.13.3 If both m and n are even, $\quad m, n \in \mathbb{N}$

Use Trig. indentities

3.13.4 If $\frac{m+n-2}{2} \in \mathbb{Z}^-$, $\quad m, n \in \mathbb{Q}$

Then, *u-sub*, $u = \tan x$ or $u = \cot x$

3.14 Integrals of form

$$\int x^m(a+bx^n)^p\,dx$$

3.14.1 If $P \in \mathbb{N}$

Use binomial expansion and then integrate.

3.14.2 If $P \in \mathbb{Z}^-$

Use *u-sub*,
Let

$$u^k = x, \quad k = LCM(m, n)$$

3.14.3 If $\frac{m+1}{n} \in \mathbb{Z}$ and $P \in \mathbb{Q}$

Use *u-sub*,
Let

$$u^k = a + bx^n$$

where, if $P = \frac{a}{b}, HCF(a, b) = 1$ then $k = b$

3.14.4 $\frac{m+1}{n} + P \in \mathbb{Z}$ and $P \in \mathbb{Q}$

Use *u-sub*,
Let

$$u^k x^n = a + bx^n$$

where, if $P = \frac{a}{b}, HCF(a, b) = 1$ then $k = b$

3.15 Euler's Substitutions

Integrals of Form

$$\int R(x, \sqrt{ax^2 + bx + c})\, dx$$

where, integrand is a rational function of x and $\sqrt{ax^2 + bx + c}$

3.15.1 If $a > 0$

Then,
Let

$$\sqrt{ax^2 + bx + c} = u \pm x\sqrt{a}$$

3.15.2 If $c > 0$

Then,
Let

$$\sqrt{ax^2 + bx + c} = ux + \sqrt{c}$$

3.15.3 If real roots α, β of the equation $ax^2 + bx + c = 0$

Then,
Let

$$\sqrt{ax^2 + bx + c} = (x - \alpha)u$$

3.16 Reduction Formulae

3.16.1 Reduction Formula for

$$I_n = \int \sin^n x \, dx$$

$$nI_n = -\sin^{n-1} x \cdot \cos x + (n-1)I_{n-2}$$

3.16.2 Reduction Formula for

$$I_n = \int \cos^n \, dx$$

$$nI_n = \sin x \cdot \cos^{n-1} x + (n-1)I_{n-2}$$

3.16.3 Reduction Formula for

$$I_n = \int \tan^n x \, dx$$

$$I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$$

3.16.4 Reduction Formula for

$$I_n = \int \csc^n x \, dx$$

$$(n-1)I_n = -\csc^{n-2} x \cot x + (n-2)I_{n-2}$$

3.16.5 Reduction Formula for

$$I_n = \int \sec^n x \, dx$$

$$(n-1)I_n = \sec^{n-2} x \tan x + (n-2)I_{n-2}$$

3.16.6 Reduction Formula for

$$I_n = \int \cot^n x \, dx$$

$$I_n = -\frac{\cot^{n-1} x}{n-1} - I_{n-2}$$

3.16.7 Reduction Formula for

$$I_{m,n} = \int \sin^m x \cos^n x \, dx$$

$$\begin{aligned}
I_{m,n} &= -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} I_{m-2,n} \\
&= \frac{\sin^{m+1} x \cos^{n+1} x}{m+n} + \frac{n-1}{m+n} I_{m,n-2} \\
&= \frac{\sin^{m+1} x \cos^{n+1} x}{m+1} + \frac{m+n+2}{m+1} I_{m+2,n} \\
&= \frac{\sin^{m+1} x \cos^{n+1} x}{n+1} + \frac{m+n+2}{n+1} I_{m,n+2} \\
&= \frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1} I_{m-2,n+2} \\
&= \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1} I_{m+2,n-2}
\end{aligned}$$

3.16.8 Reduction Formula for

$$I_{m,n} = \int \cos^m x \sin nx \, dx$$

$$I_{m,n} = -\frac{\cos^m x \cos nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1}$$

3.16.9 Reduction Formula for

$$I_{m,n} = \int \cos^m x \cos nx \, dx$$

$$I_{m,n} = \frac{\cos^m x \sin nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1}$$

3.16.10 Reduction Formula for

$$I_{m,n} = \int \sin^m x \sin nx \, dx$$

$$I_{m,n} = \frac{n \sin^m x \cos nx}{m^2 - n^2} - \frac{m \sin^{m-1} x \cos x \cos nx}{m^2 - n^2} + \frac{m(m-1)}{m^2 - n^2} I_{m-2,n}$$

3.16.11 Reduction Formula for

$$I_{m,n} = \int \sin^m x \cos nx \, dx$$

$$I_{m,n} = \frac{n \sin^m x \sin nx}{m^2 - n^2} - \frac{m \sin^{m-1} x \cos x \cos nx}{m^2 - n^2} + \frac{m(m-1)}{m^2 - n^2} I_{m-2,n}$$

3.17 Integration Using Differentiation

Integrals of form

$$\int \frac{1}{(a+b\cos x)^2}\,dx, \int \frac{1}{(a+b\sin x)^2}\,dx, \int \frac{1}{(\sin x+a\sec x)^2}\,dx$$

$$\int \frac{a+b\sin x}{(b+a\sin x)^2}\,dx$$

To evaluate integral of above form, follow the steps-

i. Let $A = \dfrac{\sin x}{a+b\cos x}$ or $\dfrac{\cos x}{a+b\sin x}$ according to the integral to be evaluated is of the form $\displaystyle\int \frac{1}{(a+b\cos x)^2}\,dx$ or $\displaystyle\int \frac{1}{(a+b\sin x)^2}\,dx$

ii. Find $\dfrac{dA}{dx}$ and express it in terms of $\dfrac{1}{a+b\cos x}$ or $\dfrac{1}{a+b\sin x}$ as the case may be.

iii. Integrate both sides of the expression obtained in step (ii) to obtain the value of the required integral.

3.18 Integral of Inverse Functions

If $\displaystyle\int f(x)\,dx = F(x)$

Then,

$$\int f^{-1}(x)\,dx = xf^{-1}(x) - F(f^{-1}(x))$$