Determinants

1 Definition of Determinants

Let

$$\begin{cases} a_1x + b_1y = 0 \\ a_2x + b_2y = 0 \end{cases}$$

Solving for $\dfrac{y}{x}$ we get,

$$\frac{y}{x} = -\frac{a_1}{b_1} = -\frac{a_2}{b_2} \implies \frac{a_1}{b_1} = \frac{a_2}{b_2}$$

The quantity $a_1b_2 - a_2b_1$ is represented by $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

Also, for

$$\begin{cases} a_1x + b_1y + c_1z = 0 \\ a_2x + b_2y + c_2z = 0 \\ a_3x + b_3y + c_3z = 0 \end{cases}$$

$$\therefore \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\left(b_2c_3 - b_3c_2\right) - b_1\left(a_2c_3 - c_2a_3\right) + c_1\left(c_3a_2 - c_2a_3\right)$$

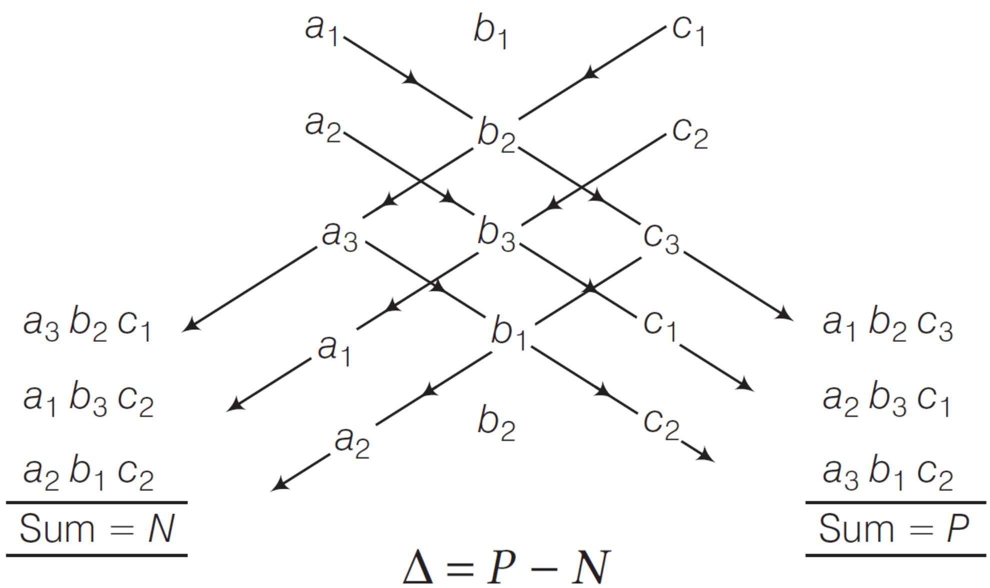
- A determinant of nth order consists of $n!$ terms in its expansion.
- Shape of Every Determinant is a square.

2 Sarrus Rule of Determinant Expansion

Let,

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

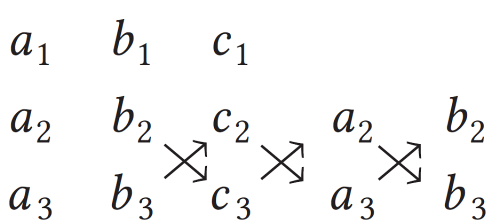
Write down the three rows of Δ and rewrite the first two rows.



3 Window Rule

Consider,

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$



$\Delta = a_1 (b_2c_3 - b_3c_2) - b_1 (a_2c_3 - c_2a_3) + c_1 (c_3a_2 - c_2a_3)$

4 Minors and Cofactors

Let

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix} \qquad \forall\ n \in \mathbb{N} - \{1\}$$

The determinant of order $n-1$ obtained from Δ after deleting *ith* row and *jth* column is called the *Minor of* a_{ij} (Denoted by M_{ij})

Cofactor of $a_{ij} = C_{ij} = (-1)^{i+j} \cdot M_{ij}$

4.1 Important Results for Cofactors

1. $\Delta = a_{11}C_{11} + a_{12}C_{12} + \ldots + a_{nn}C_{nn} = \displaystyle\sum_{r=1}^{n} a_{\omega r}C_{\omega r} \qquad \omega \in \mathbb{N}$

2. The sum of the product of an element of any row (or column) with corresponding cofactors of another row (or column) is equal to zero.

   $$a_{11}C_{21} + a_{12}C_{22} + \ldots + a_{1n}C_{2n} = \sum_{r=1}^{n} a_{\omega r}C_{(\omega+1)r} = 0 \qquad \omega \in \mathbb{N} - \{n\}$$

3. $\Delta^c = \Delta^{n-1}, \Delta^c$ is determinant made by cofactors of elements in Δ

   For 3rd order determinant $\Delta^c = \Delta^2$

5 Use of Determinants in Coordinate Geometry

1. Area of Triangle whose vertices are $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ is

   $$\frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|$$

2. If $a_ix + b_iy + c_i = 0, i \in \{1, 2, 3\}$ are the sides of a triangle, then the area of triangle is

   $$\frac{1}{|2C_1C_2C_3|}\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2$$