Application of Derivatives

1 Derivative as Rate of Change

If a variable quantity y is some function of time t, i.e. $y = f(t)$, then a small change in time Δt has a corresponding change Δy in y.

$$\therefore \textit{ Average Rate Change } = \frac{\Delta y}{\Delta t}$$

As $\Delta t \to 0$ the rate of change becomes *Instantaneous*

$$\therefore \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \frac{dy}{dt}$$

The Rate of Change of any variable with respect to some another variable is the derivative of the first variable with respect to another variable.

2 Errors and Approximations

Approximations

Let $y = f(x)$,

Let Δx denote a small change in x and let Δy be the corresponding change in y.

$$\Delta y \approx \frac{dy}{dx} \cdot \Delta x$$

$$\implies f(x + \Delta x) - f(x) \approx f'(x) \cdot \Delta x$$

Errors

Absolute Error

Δx or dx is called Absolute Error in x.

Relative Error

$\dfrac{\Delta x}{x}$ or $\dfrac{dx}{x}$ is called Relative Error in x.

Percentage Error

$\left(\frac{\Delta x}{x}\right) \cdot 100$ or $\left(\frac{dx}{x}\right) \cdot 100$ is called Percentage Error in x

3 Tangent and Normal

Let $y = f(x)$ be a continuous curve and let $P\,(x_1, y_1)$ be a point on it.

3.1 Slope of Tangent

$\left.\frac{dy}{dx}\right|_{(x_1,y_1)}$ is the slope of tangent to the curve $y = f(x)$ at point (x_1, y_1)

3.2 Slope of Normal

$$\begin{aligned} \text{Slope of Normal at P} &= -\frac{1}{\text{Slope of Tangent at P}} \\ &= -\left.\frac{dx}{dy}\right|_{(x_1,y_1)} \end{aligned}$$

3.3 Equation of Tangent

If $m_T = \left.\frac{dy}{dx}\right|_{(x_1,y_1)}$ then, equation of Tangent T at point P is

$$T \equiv y - y_1 = m_T\,(x - x_1)$$

Equation of Tangent from External Point

If a point $P\,(a, b)$ does not lie on the curve $y = f(x)$, then let $Q(h, f(h))$ be a point on the curve $y = f(x)$ such that PQ is tangent to the curve.

$\implies f'(h) = \frac{f(h) - b}{h - a}$

Solving this equation for h we may get multiple values of h. The corresponding equation of Tangent T from P becomes,

$$T \equiv y - b = f'(h)\,(x - a)$$

3.4 Equation of Normal

Equation of Normal N at point P is,

$$N \equiv y - y_1 = -\frac{1}{m_T}\,(x - x_1)$$

3.5 Some Important Points regarding Tangent and Normal

1. If a curve passes through the origin, then the equation of the tangent at the origin can be directly written by equating the lowest degree terms appearing in the equation of the curve to zero.

2. $x^3 + y^3 - 3xy = 0$ is the *Folium of Descartes* and the coordinate axes are tangent to it at the origin.

3. Some common parametric coordinates on a curve.

   i. $x^{2/3} + y^{2/3} = a^{2/3} \to \begin{cases} x = a\cos^3\theta \\ y = a\sin^3\theta \end{cases}$

   ii. $\sqrt{x} + \sqrt{y} = \sqrt{a} \to \begin{cases} x = a\cos^4\theta \\ y = a\sin^4\theta \end{cases}$

   iii. $\dfrac{x^n}{a^n} + \dfrac{y^n}{b^n} = 1 \to \begin{cases} x = a\left(\cos\theta\right)^{2/n} \\ y = a\left(\sin\theta\right)^{2/n} \end{cases}$

   iv. $c^2\left(x^2 + y^2\right) = x^2y^2 \to \begin{cases} x = c\sec\theta \\ y = c\csc\theta \end{cases}$

   v. $y^2 = x^3 \to \begin{cases} x = t^2 \\ y = t^3 \end{cases}$

4 Angle of Intersection of Two Curves

The angle of intersection of two curves is defined as the acute angle between the tangents of the two curves at their point of intersection.

Let $S_1 \equiv y_1 = f(x)$ and $S_2 \equiv y_2 = g(x)$.
Now, let the point of intersection be P and θ be the angle between them.

$$\tan\theta = \left| \frac{\left.\dfrac{dy_1}{dx}\right|_P - \left.\dfrac{dy_2}{dx}\right|_P}{1 + \left.\dfrac{dy_1}{dx}\right|_P \cdot \left.\dfrac{dy_2}{dx}\right|_P} \right|$$

Orthogonal Curves

If the Angle of Intersection of Two Curves is a right angle, then the two curves are considered Orthogonal.

$$\left.\frac{dy_1}{dx}\right|_P \cdot \left.\frac{dy_2}{dx}\right|_P = -1$$

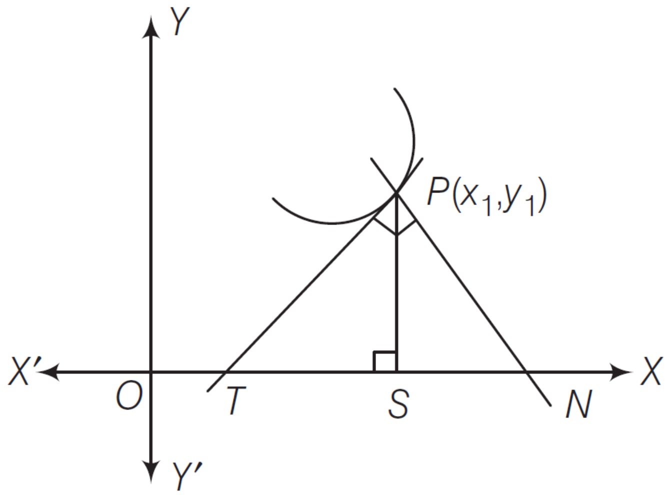
Conditon for Two Curves to Touch

$\tan\theta = 0$

$$\left.\frac{dy_1}{dx}\right|_P = \left.\frac{dy_2}{dx}\right|_P$$

5 Length of Tangent, Length of Normal, Subtangent, Subnormal

Let $y = f(x)$ be a curve and $P\,(x_1, y_1)$ be a point on it.



5.1 Length of Tangent

The portion of the tangent intercepted between the point of contact and x axis.

$$\textbf{Len(Tangent)} = PT = \left| y_1 \sqrt{1 + \left(\left.\frac{dx}{dy}\right|_P \right)^2} \right|$$

5.2 Length of Normal

The portion of the normal intercepted between the point of contact and x axis.

$$\textbf{Len(Normal)} = PN = \left| y_1 \sqrt{1 + \left(\left.\frac{dy}{dx}\right|_P \right)^2} \right|$$

5.3 Subtangent

The projection on the x axis of the Length of Tangent.

$$\textbf{Subtangent} = ST = \left| y_1 \cdot \frac{dx}{dy}\bigg|_P \right|$$

5.4 Subnormal

The projection on the x axis of the Length of Normal.

$$\textbf{Subnormal} = SN = \left| y_1 \cdot \frac{dy}{dx}\bigg|_P \right|$$