Indefinite Integration

1 Fundamental Definition of Indefinite Integration

If f and F and functions such that $\dfrac{d}{dx}(F(x)) = f(x)$ then F is anti-derivative of f w.r.t. x symbolically,

$$\int f(x)\,dx = F(x) + C$$

where C is the constant of Integration

2 Anti-Derivatives of Some Standard Functions

i. $\displaystyle\int k \cdot f(x)\,dx = k \cdot \int f(x)\,dx$

ii. $\displaystyle\int [f_1(x) \pm f_2(x) \pm f_3(x) \pm \ldots \pm f_n(x)]\,dx \;=\; \int f_1(x)\,dx \pm \int f_2(x)\,dx \pm \int f_3(x)\,dx \pm \ldots \int f_n(x)\,dx$

iii. $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad n \neq 1$

iv. $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$

v. $\displaystyle\int e^x\,dx = e^x + C$

vi. $\displaystyle\int a^x\,dx = \frac{a^x}{\ln a} + C$

vii. $\displaystyle\int \sin x\,dx = \cos x + C$

viii. $\displaystyle\int \cos x\,dx = \sin x + C$

ix. $\displaystyle\int \sec^2 x\,dx = \tan x + C$

x. $\displaystyle\int \csc^2 x\,dx = -\cot x + C$

xi. $\displaystyle\int \sec x \tan x\,dx = \sec x + C$

xii. $\displaystyle\int \csc x \cot x\,dx = -\csc x + C$

xiii. $\displaystyle\int \cot x\,dx = \ln|\sin x| + C$

xiv. $\displaystyle\int \tan x\,dx = -\ln|\cos x| + C$

xv. $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x| + C$

xvi. $\displaystyle\int \csc x\,dx = \ln|\csc x - \cot x| + C$

xvii. $\displaystyle\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$

xviii. $\displaystyle\int \frac{-1}{\sqrt{a^2 - x^2}}\,dx = \cos^{-1}\left(\frac{x}{a}\right) + C$

xix. $\displaystyle\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

xx. $\displaystyle\int \frac{-1}{a^2 + x^2}\,dx = \frac{1}{a}\cot^{-1}\left(\frac{x}{a}\right) + C$

xxi. $\displaystyle\int \frac{1}{x\sqrt{x^2 - a^2}}\,dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$

xxii. $\displaystyle\int \frac{-1}{x\sqrt{x^2 - a^2}}\,dx = \frac{1}{a}\csc^{-1}\left(\frac{x}{a}\right) + C$

xxiii. $\displaystyle\int \sqrt{x}\,dx = \frac{2x\sqrt{x}}{3} + C$

xxiv. $\displaystyle\int \frac{dx}{x^2 - 1} = \ln\left|\frac{x-1}{x+1}\right| + C$

xxv. $\displaystyle\int \frac{dx}{\sqrt{1 + x^2}} = \ln\left|x + \sqrt{x^2 + 1}\right| + C$

xxvi. $\displaystyle\int \frac{dx}{\sqrt{x^2-1}} = \ln\left|x+\sqrt{x^2-1}\right| + C$

xxvii. $\displaystyle\int \sqrt{1-x^2}\,dx = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x + C$

xxviii. $\displaystyle\int \sqrt{x^2-1}\,dx = \frac{1}{2}x\sqrt{x^2-1} - \frac{1}{2}\ln\left|x+\sqrt{x^2-1}\right| + C$

Important Results

1. If $F_1(x)$ and $F_2(x)$ are two anti-derivatives of a function $f(x)$, then $F_1(x)$ and $F_2(x)$ only differ by a constant, i.e.

$$F_1(x) - F_2(x) = C$$

where, C is a $\mathbb{R}$ constant.

2. If $f(x)$ is continuous $\forall\, x \in D_f$ and, $\displaystyle\int f(x)\,dx = F(x) + C$, then $F(x)$ always exists and is continuous.

3. If $f(x)$ is discontinuous at $x = x_1$, then its anti-derivative can be continuous at $x = x_1$.

4. Anti-derivative of a periodic function may not be periodic.

3 Methods of Integration

3.1 u substitution

Integrals of form,

$$I = \int f(g(x)) \cdot g'(x)\,dx$$

Can be solved by, the substitution,

$$u = g(x)$$

Differentiating both sides w.r.t. x,

$$du = g'(x)dx$$

Now,

$$I = \int f(u)\,du$$

3.2 Integrals of form

$$\int \frac{dx}{ax^2+bx+c}, \int \frac{dx}{\sqrt{ax^2+bx+c}}, \int \sqrt{ax^2+bx+c}\,dx$$

Using completing the square,

$$ax^2+bx+c = a\left[\left(x+\frac{b}{2a}\right)+\frac{c}{a}-\frac{b^2}{4a}\right]$$

Now, using $u\ sub$, let

$$u = x+\frac{b}{2a}$$

The transformed integral can be integrated using previous methods.

3.3 Integrals of form

$$\int \frac{px+q}{ax^2+bx+c}\,dx, \int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx, \int (px+q)\sqrt{ax^2+bx+c}\,dx$$

$$px+q = \lambda\frac{d}{dx}(ax^2+bx+c)+\mu$$

Now, after finding λ, μ,
For the 1st part, use $u\ sub$,
Let

$$u = ax^2+bx+c$$

2nd part of the integral can be integrated using previous methods.

3.4 Integrals of form

$$\int \frac{K(x)}{ax^2+bx+c}\,dx$$

where $deg(K(x)) \geq 2$

By polynomial long division

$$\frac{K(x)}{ax^2+bx+c} = Q(x)+\frac{R(x)}{ax^2+bx+c}$$

Now, $deg(R(x)) \leq 1$, Now, the integral

$$\int \frac{R(x)}{ax^2+bx+c}\,dx$$

can be integrated using previous methods.

3.5 Integrals of form

$$\int \frac{ax^2+bx+c}{px^2+qx+r}\,dx, \int \frac{ax^2+bx+c}{\sqrt{px^2+qx+r}}\,dx, \int \left(ax^2+bx+c\right)\sqrt{px^2+qx+r}\,dx$$

$$ax^2+bx+c = \lambda\left(px^2+qx+r\right) + \mu\frac{d}{dx}\left(px^2+qx+r\right) + \gamma$$

3.6 Trig. Integrals

3.6.1 Integrals of form

$$\int \frac{dx}{a\cos^2 x + b\sin^2 x}, \int \frac{dx}{a+b\sin^2 x}, \int \frac{dx}{a+b\cos^2 x}$$
$$\int \frac{dx}{\left(a\sin x + b\cos x\right)^2}, \int \frac{dx}{a+b\sin^2 x + c\cos^2 x}$$

Steps -

1. Multiply numerator and denominator by $\sec^2 x$
2. Replace $\sec^2 x$ (if any) by $1+\tan^2 x$ except the one multiplied in step 1.
3. Let $u = \tan x$, then $du = \sec^2 x\,dx$

Now, the transformed integral can be integrated using previous methods.

3.6.2 Integrals of form

$$\int \frac{dx}{a\sin x + b\cos x}, \int \frac{dx}{a+b\sin x}, \int \frac{dx}{a+b\cos x}, \int \frac{dx}{a\sin x + b\cos x + c}$$

Steps -

1. Replace $\sin x = \dfrac{2\tan {}^x\!/_2}{1+\tan^2 {}^x\!/_2}$ and $\cos x = \dfrac{1-\tan^2 {}^x\!/_2}{1+\tan^2 {}^x\!/_2}$
2. Let $u = \tan {}^x\!/_2$, $du = \dfrac{1}{2}\sec^2 {}^x\!/_2\,dx$ is already present in the numerator.

Now, the transformed integral can be integrated using previous methods.

3.6.2.1 Alternative Method to Integrate

$$I = \int \frac{dx}{a\sin x + b\cos x}$$

$$a\sin x + b\cos x = \sqrt{a^2+b^2}\sin\left(x + \tan^{-1}\left(\frac{b}{a}\right)\right)$$

$$I = \frac{1}{\sqrt{a^2+b^2}} \int \csc\left(x + \tan^{-1}\left(\frac{b}{a}\right)\right)\, dx$$
$$I = \frac{1}{\sqrt{a^2+b^2}} \ln\left|\tan\left(\frac{x}{2} + \frac{1}{2}\tan^{-1}\left(\frac{b}{a}\right)\right)\right| + C$$

3.6.3 Integrals of form

$$\int \frac{p\cos x + q\sin x + r}{a\cos x + b\sin x + c}\, dx, \int \frac{p\cos x + q\sin x}{a\cos x + b\sin x}\, dx$$

Steps for (i),

1. Express *Numerator* $= \lambda$ *Denominator* $+ \mu$ *Derivative of denominator* $+ \gamma$

Now, the transformed integral can be integrated using previous methods.

Steps for (ii),

1. Express *Numerator* $= \lambda$ *Denominator* $+ \mu$ *Derivative of Denominator*

Now, the transformed integral can be integrated using previous methods.

3.7 Integration by parts