Matrices

1   Types of Matrices

1.1   Row Matrix or Row Vector

A matrix which contains only **One Row**.

$$A = \left[a_{ij}\right]_{1 \times n}$$

1.2   Column Matrix or Column Vector

A matrix which contains only **One Column**.

$$A = \left[a_{ij}\right]_{n \times 1}$$

1.3   Rectangular Matrix

A matrix in which **no. of rows is not equal to no. of columns**.

$$A = \left[a_{ij}\right]_{m \times n}, m \neq n$$

1.4   Square Matrix

A matrix in which **no. of rows is equal to no. of columns**.

$$A = \left[a_{ij}\right]_{n \times n}$$

1.5   Diagonal Matrix

A **square matrix** in which **all non-diagonal entries are zero**.

$$A = \left[a_{ij}\right]_{n \times n}, a_{ij} = 0 \ \forall \ i \neq j, a_{ij} \neq 0 \ \forall \ i = j$$

1.6 Scalar Matrix

A **Diagonal Matrix** in which **all diagonal entries are equal**.

$$A = \left[a_{ij}\right]_{n \times n}, a_{ij} = \begin{cases} a & i = j, a \in \mathbb{C} \\ 0 & i \neq j \end{cases}$$

1.7 Identity or Unit Matrix

A **Diagonal Matrix** in which **all diagonal entries equal 1**.

$$A = \left[a_{ij}\right]_{n \times n}, a_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

1.8 Singleton Matrix

A matrix which consists of **only one element**.

$$A = \left[a_{ij}\right]_{1 \times 1}$$

1.9 Triangular Matrix

1. **Upper Triangular Matrix**

   A **Square Matrix** in which **all elements below principal diagonal are zero**.

   $$A = \left[a_{ij}\right]_{n \times n}, a_{ij} = 0 \ \forall \ i > j$$

2. **Lower Triangular Matrix**

   A **Square Matrix** in which **all elements above principal diagonal are zero**.

   $$A = \left[a_{ij}\right]_{n \times n}, a_{ij} = 0 \ \forall \ i < j$$

1.10 Horizontal Matrix

A matrix in which **no. of rows is less than no. of columns**

$$A = \left[a_{ij}\right]_{m \times n}, m < n$$

1.11 Vertical Matrix

A matrix in which **no. of rows is greater than no. of columns**.

$$A = \left[a_{ij}\right]_{m \times n}, m > n$$

1.12　Null or Zero Matrix

A matrix in which **all elements equal zero.**

$$O = \left[a_{ij}\right]_{m \times n}, a_{ij} = 0 \ \forall \ i, j$$

1.13　Sub-Matrix

A matrix which is obtained from a given matrix **by deleting any no. of rows and no. of columns**.

e.g. $\begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}_{2 \times 2}$ is a sub-matrix of $\begin{bmatrix} 8 & 9 & 5 \\ 2 & 3 & 4 \\ 3 & -2 & 5 \end{bmatrix}_{3 \times 3}$

1.14　Comparable Matrices

Two Matrices $A = \left[a_{ij}\right]_{m \times n}$ and $B = \left[b_{ij}\right]_{p \times q}$ are **comparable**, if $m = p \wedge n = q$

1.15　Idempotent Matrix

A square matrix whose square equals itself.

$$A = \left[a_{ij}\right]_{n \times n}, A^2 = A$$

- For a Idempotent Matrix, $A^n = A$ $\hspace{4cm} \forall \ n \in \mathbb{Z}^+$

1.16　Periodic Matrix

A square matrix A is called periodic,
　　If
$$A^{k+1} = A, k \in \mathbb{Z}^+$$

- The least value of all such k is called the period of matrix A.

- Period of an Idempotent Matrix is 1.

1.17　Nilpotent Matrix

A square matrix $A = \left[a_{ij}\right]_{n \times n}$ which
　　satisfies
$$A^k = O, A^{k-1} \neq O$$

1.18 Involutory Matrix

A square matrix A which satisfies,

$$A^2 = I$$

- For Involutory Matrix, $A = A^{-1}$

1.19 Symmetric Matrix

A square matrix $A = \left[a_{ij}\right]_{n \times n}$ in which $A = A^T$.

- In a Symmetric Matrix, elements situated at equal distance from the diagonal are equal.

- For any Square Matrix A with $a_{ij} \in \mathbb{R}$, then $A + A^T$ is symmetric.

- If A is a Square Matrix, then AA^T and $A^T A$ are symmetric matrices.

- All positive integral powers of a symmetric matrix are symmetric.

$$\text{If } A = \left[a_{ij}\right]_{n \times n}, a_{ij} = a_{ji} \text{ Then } A^n = \left[\alpha_{ij}\right]_{n \times n}, \alpha_{ij} = \alpha_{ji}$$

- If A be a symmetric matrix and B be a square matrix of order that of A, then $-A, kA, A^T, A^{-1}, A^n, B^T AB$ are also symmetric. $n \in \mathbb{N}, k \in \mathbb{C}$

- Let A and B be two symmetric matrices of same order, then

  a. $A \pm B, AB + BA$ are symmetric.
  b. $AB - BA$ is skew-symmetric.
  c. AB is symmetric iff, $AB = BA$

1.20 Skew-Symmetric Matrix

A Square Matrix $A = \left[a_{ij}\right]_{n \times n}$ in which $a_{ij} = -a_{ji} \ \forall \ i, j$

- In a Skew-Symmetric Matrix all diagonal entries are **zero.**

- In a Skew-Symmetric Matrix, the elements situated at equal distance from the diagonal are equal in magnitude but opposite in sign.

- If A is a Skew-Symmetric Matrix, then $Tr(A) = 0$

- For any Square Matrix A with $a_{ij} \in \mathbb{R}$, then $A - A^T$ is a Skew-Symmetric Matrix.

- Every Square Matrix can be uniquely expressed as the sum of a symmetric and skew-symmetric matrix.

$$A = \frac{1}{2}\left(A + A^T\right) + \frac{1}{2}\left(A - A^T\right)$$

- All positive odd integral powers of a skew-symmetric matrix are skew-symmetric.

$$\text{If } A = \left[a_{ij}\right]_{m \times m}, a_{ij} = -a_{ji} \text{ Then } A^{2n+1} = \left[\alpha_{ij}\right]_{m \times m}, \alpha_{ij} = -\alpha_{ji}, n \in \mathbb{N}$$

- All positive even integral powers of a skew-symmetric matrix are symmetric.

$$\text{If } A = \left[a_{ij}\right]_{m \times m}, a_{ij} = -a_{ji} \text{ Then } A^{2n} = \left[\alpha_{ij}\right]_{m \times m}, \alpha_{ij} = \alpha_{ji}, n \in \mathbb{N}$$

- If A be a skew-symmetric matrix and B be a square matrix of order that of A then, $kA, B^T A B$ are also skew-symmetric.

- Let A and B be two skew-symmetric matrices of same order, then

    a. $A \pm B, AB - BA$ are skew-symmetric.
    b. $AB + BA$ is symmetric

- Let A be skew-symmetric matrix and C be column vector, then $C^T A C = O$

1.21 Orthogonal Matrix

A square matrix whose transpose is its inverse.

$$AA^T = I$$

- $A^{-1} = A^T$

- Let A and B be Orthogonal, then AB is also Orthogonal.

- If A is orthogonal, then A^{-1}, A^T are also Orthogonal.

1.22 Hermitian Matrix

A square matrix which is equal to its conjugate transpose matrix.

$$\text{If } A_{n \times n} = A^{\theta}_{n \times n} \quad \vee A = \left[a_{ij}\right]_{n \times n}, a_{ij} = \overline{a_{ji}}, \text{ Then } A \text{ is Hermitian}$$

- The diagonal entries of a Hermitian Matrix are $\mathbb{R}$

- Elements situated at equal distance from diagonal are complex conjugate of each other.

- Let A be any square matrix, then $A + A^{\theta}$ is a Hermitian Matrix.

- If A be a square matrix, then $AA^{\theta}, A^{\theta} A$ are Hermitian.

- If A is Hermitian Matrix, then

a. iA is Skew-Hermitian Matrix. $\qquad\qquad\qquad i^2 = -1$

b. $\overline{A}$ is also a Hermitian Matrix.

c. kA is Hermitian Matrix. $\qquad\qquad\qquad\qquad k \in \mathbb{R}$

d. If A, B are Hermitian Matrices of same order, then

    a. $k_1 A + k_2 B$ is also Hermitian. $\qquad\qquad k_1, k_2 \in \mathbb{R}$

    b. AB is also Hermitian if $AB = BA$

    c. $AB + BA$ is Hermitian.

    d. $AB - BA$ is Skew-Hermitian.

1.23 Skew-Hermitian Matrix

A square matrix which is equal to negative of its conjugate transpose matrix.

$$\text{If } A_{n\times n} = -A^\theta_{n\times n} \quad \vee \quad A = \left[a_{ij}\right]_{n\times n}, a_{ij} = -\overline{a_{ji}}, \quad \text{Then } A \text{ is Skew-Hermitian}$$

- All diagonal entries of a Skew-Hermitian Matrix are purely imaginary or zero.

- In a Skew-Hermitian Matrix elements situated at equal distance from the diagonal only differ in the sign of real part.

- Let A be any square matrix, then $A - A^\theta$ is a Skew-Hermitian Matrix.

- Every square matrix can be uniquely expressed as the sum of a Hermitian and a Skew-Hermitian Matrix.

$$A_{n\times n} = \frac{1}{2}\left(A + A^\theta\right) + \frac{1}{2}\left(A - A^\theta\right)$$

- If A is a Skew-Hermitian Matrix, then

    a. iA is a Hermitian Matrix. $\qquad\qquad\qquad i^2 = -1$

    b. $\overline{A}$ is also Skew-Hermitian.

    c. kA is Skew-Hermitian. $\qquad\qquad\qquad\qquad k \in \mathbb{R}$

- If A, B are two Skew-Hermitian Matrices of same order, then $k_1 A + k_2 B$ is also Skew-Hermitian.

1.24 Unitary Matrix

A Square Matrix whose inverse is its conjugate transpose matrix.

$$AA^\theta = I \implies A^{-1} = A^\theta$$

- If A and B are unitary, then AB is also unitary.

- If A is unitary, then A^{-1}, A^T are also unitary.

2 Trace of a Matrix

The **sum of all diagonal entries** of a **square matrix** is called its **Trace**.

$$A = \left[a_{ij}\right]_{n \times n}, Tr(A) = \sum_{i=1}^{n} a_{ii}$$

2.1 Properties of Trace of a Matrix

Let $A = \left[a_{ij}\right]_{n \times n}, B = \left[b_{ij}\right]_{n \times n}, k \in \mathbb{C}$

  i. $Tr(kA) = k \cdot Tl(A)$

  ii. $Tr(A \pm B) = Tr(A) \pm Tr(B)$

  iii. $Tr(AB) = Tr(BA)$

  iv. $Tr(A) = Tr(A')$

  v. $Tr(I_n) = n$

  vi. $Tr(AB) \neq Tr(A) \cdot Tr(B)$

  vii. $Tr(A) = Tr(CAC^{-1})C$ is a non-singular square matrix of order n.

3 Determinant of a Square Matrix

Let $A = \left[a_{ij}\right]_{n \times n}$. The determinant formed by the elements of A is said to be the determinant of matrix $A = det(A) = |A|$

Important Points

a. $det \left(\prod_{r=1}^{n} A_r \right) = \prod_{r=1}^{n} det(A_r)$

b. $det(kA) = k^n det(A)$ $\hspace{4cm}$ $k \in \mathbb{C}, n = $ order of A

c. $det(A)$ exists $\iff A$ is a square matrix.

d. $det(A) = det(A^T)$

e. If A is Orthogonal Matrix, then $det(A) = \pm 1$

f. If A is Skew-Symmetric Matrix of odd order, then $det(A) = 0$

g. If A is Skew-Symmetric Matrix of even order, then $det(A)$ is a perfect square.

h. If $A = diag(a_1, a_2, \ldots, a_n)$, then $det(A) = \prod_{r=1}^{n} a_r$

7

i. $det(A^{-1}) = \dfrac{1}{det(A)}$

Singular (Invertible) and non-singular (Non-Invertible) Matrix

If for $A = \begin{bmatrix} a_{ij} \end{bmatrix}_{n \times n}, det(A) = 0$ then A is called a singular matrix.
If $det(A) \neq 0$ then it is called **non-singular** matrix.

4   Equal Matrices

Two matrices are equal, iff

  i. They are of same order.

  ii. The elements in the corresponding positions of the two matrices are equal.

  Let $A = \begin{bmatrix} a_{ij} \end{bmatrix}_{m \times n}$ and $B = \begin{bmatrix} b_{ij} \end{bmatrix}_{p \times q}$

  Now, $A = B$ iff, $m = p, n = q, a_{ij} = b_{ij} \ \forall \ i, j$

5   Algebra of Matrices

5.1   Addition

If $A = \begin{bmatrix} a_{ij} \end{bmatrix}_{m \times n}$ and $B = \begin{bmatrix} b_{ij} \end{bmatrix}_{m \times n}$

Then,
$$A + B = \begin{bmatrix} a_{ij} + b_{ij} \end{bmatrix}_{m \times n}$$

Properties of Matrix Addition

1. Addition of Matrices is **commutative**.
$$A + B = B + A$$

2. Addition of Matrices is **associative**.
$$(A + B) + C = A + (B + C)$$

3. Additive Identity is O (Null Matrix)

$$A + O = A$$

4. Additive Inverse is $-A$

$$A + (-A) = O$$

5. Cancellation law
$$A + B = A + C \implies B = C$$

5.2 Multiplication

5.2.1 Scalar Multiplication

The Matrix obtained by **multiplying every element of a matrix with a scalar** is called the scalar multiple of that matrix.

Let $A = \left[a_{ij}\right]_{m \times n}, k \in \mathbb{C}$

Now,

$$kA = \left[k \cdot a_{ij}\right]_{m \times n}$$

5.2.2 Matrix Multiplication

Conformable for Multiplication

Two matrices $A = \left[a_{ij}\right]_{m \times n}, B = \left[b_{ij}\right]_{p \times q}$ are conformable for the product AB, if the no. of columns in the pre-factor (A) is equal to the no. of rows in the post-factor (B)

$$AB \ exists \ iff \ n = p$$

Multiplication of Matrices

Let $A = \left[a_{ij}\right]_{m \times n}, B = \left[b_{ij}\right]_{n \times q}, C = \left[c_{ij}\right]_{m \times q} \ s.t. \ AB = C$

$$\therefore c_{ij} = \sum_{r=1}^{n} a_{ir} \cdot b_{rj} \qquad\qquad Row \ by \ Column \ Multiplication$$

5.2.3 Properties of Matrix Multiplication

i. Matrix Multiplication is not commutative.

$$AB \neq BA$$

- If $AB = BA$ then matrices A, B are said to commute.
- If $AB = -BA$ then matrices A, B are said to anti-commute.
- Diagonal Matrices of same order always commute.

ii. Matrix Multiplication is associative.

$$A(BC) = (AB)C$$

iii. If $A = \left[a_{ij}\right]_{m \times n}$, then $I_m \, A = A = AI_n$

iv. Matrix Multiplication is distributive.

$$A(B + C) = AB + AC$$

Note: $A(B + C) \neq AB + CA$

v. If product of two matrices is a null matrix then it is not necessary that one of the matrices is also a null matrix.

$$AB = O \not\Longrightarrow A = O \vee B = O$$

If $AB = O$ then A, B are divisors of O

$AB = O \implies det(AB) = 0 \implies det(A) \cdot det(B) = 0$

vi. $AO = O$

vii. $I^m = I$ $\hspace{3cm} \forall\ m \in \mathbb{Z}^+$

viii. if $A = \left[a_{ij}\right]_{n \times n}$

then $\underbrace{A \cdot A \cdot A \dots A}_{m \text{ times}} = A^m$ $\hspace{2cm} \forall\ m \in \mathbb{Z}^+$

6 Pre-Multiplication and Post-Multiplication of Matrices

The Matrix AB is the matrix B **pre-multiplied** by A and the matrix BA is the matrix B **post-multiplied** by A.

7 Transpose of a Matrix

The matrix obtained by **interchanging rows and columns**.

$$A = \left[a_{ij}\right]_{m \times n}, A^T = \left[a_{ji}\right]_{n \times m}$$

- $A^T = A' = A^t =$ Transpose of Matrix A

7.1 Properties of Transpose Matrices

i. $(A^T)^T = A$

ii. $(A \pm B)^T = A^T \pm B^T$

iii. $(kA)^T = k \cdot A^T$

iv. $(AB)^T = B^T A^T$, In General, $\left(\prod_{r=1}^{n} A_r\right)^T = \prod_{r=1}^{n} A_{n+1-r}^T$

v. $I^T = I$

8 Complex Conjugate of a Matrix

The Matrix obtained by replacing each element by its complex conjugate.

$$A = \left[a_{ij}\right]_{m \times n}, \overline{A} = \left[\overline{a_{ij}}\right]_{m \times n}$$

- If all elements are $\mathbb{R}$, then $A = \overline{A}$

- $\overline{\overline{A}} = A$

- $\overline{A + B} = \overline{A} + \overline{B}$

- $\overline{kA} = \overline{k} \cdot \overline{A}$ $k \in \mathbb{C}$

- $\overline{AB} = \overline{A}\,\overline{B}$

9 Conjugate Transpose of a Matrix

The complex conjugate of the transpose of a matrix.

$$A = \left[a_{ij}\right]_{m \times n}, A^{\theta} = \overline{A^{T}} = \left[\overline{a_{ji}}\right]_{n \times m}$$

If A and B are two matrices of same order, then

i. $\overline{A^{T}} = \left(\overline{A}\right)^{T}$

ii. $\left(A^{\theta}\right)^{\theta} = A$

iii. $(A + B)^{\theta} = A^{\theta} + B^{\theta}$

iv. $(kA)^{\theta} = \overline{k}A^{\theta}$ $k \in \mathbb{C}$

v. $(AB)^{\theta} = B^{\theta}A^{9}$

10 Adjoint of A Matrix

The Transpose of Square Matrix formed by its co-factors is its Adjoint.

Let $A = \left[a_{ij}\right]_{n \times n}$

then

$$adj(A) = \left[C_{ji}\right]_{n \times n}$$

10.1   Properties of Adjoint Matrix

i. $A \cdot adj(A) = adj(A) \cdot A = det(A)I$

ii. $det(adj(A)) = (det(A))^{n-1}$

iii. $det(\underbrace{adj(adj(adj(\ldots \qquad))))}_{m \text{ times } adj} = det(A)^{(n-1)^m}$

iv. $(adj(A))^T = adj(A^T)$

v. $adj(adj(A)) = (det(A))^{n-2}A$

vi. $adj(kA) = k^{n-1}adj(A)$ $\hspace{4cm} k \in \mathbb{C}$

vii. $adj(A^m) = (adj(A))^m$ $\hspace{4cm} m \in \mathbb{N}$

viii. $det(A \cdot adj(A) + kI_n) = (det(A) + k)^n$

ix. Adjoint of a Diagonal Matrix is a Diagonal Matrix.

$\text{If } A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}, \text{ then } adj(A) = \begin{bmatrix} bc & 0 & 0 \\ 0 & ca & 0 \\ 0 & 0 & ab \end{bmatrix}$

x. $adj(I_n) = I_n$

11   Inverse of a Matrix

A square matrix of is said to be Invertible, iff its determinant is non zero.

$$det(A) \neq 0$$

$$A^{-1} = \frac{adj(A)}{det(A)}$$

11.1   Properties of Inverse Matrix

i. Every invertible matrix possesses a unique inverse.

ii. $(AB)^{-1} = B^{-1}A^{-1}$

iii. $(A^T)^{-1} = (A^{-1})^T$

iv. $(A^k)^{-1} = (A^{-1})^k = A^{-k}$ $\hspace{4cm} k \in \mathbb{N}$

v. $(A^{-1})^{-1} = A$

vi. $I^{-1} = I$

vii. $det(A^{-1}) = \dfrac{1}{det(A)}$

viii. Inverse of a digonal matrix is a digonal matrix.

$$\text{If } A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \text{ then } A^{-1} = \begin{bmatrix} \dfrac{1}{a} & 0 & 0 \\ 0 & \dfrac{1}{b} & 0 \\ 0 & 0 & \dfrac{1}{c} \end{bmatrix}$$