Indefinite Integration

1 Fundamental Definition of Indefinite Integration

If f and F and functions such that $\frac{d}{dx}(F(x)) = f(x)$ then F is anti-derivative of f w.r.t. x symbolically,

$$\int f(x)\,dx = F(x) + C$$

where C is the constant of Integration

2 Anti-Derivatives of Some Standard Functions

i. $\int k \cdot f(x)\,dx = k \cdot \int f(x)\,dx$

ii. $\int [f_1(x) \pm f_2(x) \pm f_3(x) \pm \ldots \pm f_n(x)]\,dx \ = \int f_1(x)\,dx \pm \int f_2(x)\,dx \pm \int f_3(x)\,dx \pm \ldots \int f_n(x)\,dx$

iii. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad n \neq -1$

iv. $\int \frac{1}{x}\,dx = \ln|x| + C$

v. $\int e^x\,dx = e^x + C$

vi. $\int a^x\,dx = \frac{a^x}{\ln a} + C$

vii. $\int \sin x\,dx = -\cos x + C$

viii. $\int \cos x\,dx = \sin x + C$

ix. $\displaystyle\int \sec^2 x\,dx = \tan x + C$

x. $\displaystyle\int \csc^2 x\,dx = -\cot x + C$

xi. $\displaystyle\int \sec x \tan x\,dx = \sec x + C$

xii. $\displaystyle\int \csc x \cot x\,dx = -\csc x + C$

xiii. $\displaystyle\int \cot x\,dx = \ln|\sin x| + C$

xiv. $\displaystyle\int \tan x\,dx = -\ln|\cos x| + C$

xv. $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x| + C$

xvi. $\displaystyle\int \csc x\,dx = \ln|\csc x - \cot x| + C$

xvii. $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$

xviii. $\displaystyle\int \frac{-1}{\sqrt{1-x^2}}\,dx = \cos^{-1} x + C$

xix. $\displaystyle\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$

xx. $\displaystyle\int \frac{-1}{1+x^2}\,dx = \cot^{-1} x + C$

xxi. $\displaystyle\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1} x + C$

xxii. $\displaystyle\int \frac{-1}{x\sqrt{x^2-1}}\,dx = \csc^{-1} x + C$

xxiii. $\displaystyle\int \sqrt{x}\,dx = \frac{2x\sqrt{x}}{3} + C$

xxiv. $\displaystyle\int \frac{dx}{x^2-1} = \ln\left|\frac{x-1}{x+1}\right| + C$

xxv. $\displaystyle\int \frac{dx}{\sqrt{1+x^2}} = \ln\left|x + \sqrt{x^2+1}\right| + C$

xxvi. $\displaystyle\int \frac{dx}{\sqrt{x^2-1}} = \ln\left|x+\sqrt{x^2-1}\right| + C$

xxvii. $\displaystyle\int \sqrt{1-x^2}\,dx = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x + C$

xxviii. $\displaystyle\int \sqrt{x^2-1}\,dx = \frac{1}{2}x\sqrt{x^2-1} - \frac{1}{2}\ln\left|x+\sqrt{x^2-1}\right| + C$

Important Results

1. If $F_1(x)$ and $F_2(x)$ are two anti-derivatives of a function $f(x)$, then $F_1(x)$ and $F_2(x)$ only differ by a constant, i.e.

$$F_1(x) - F_2(x) = C$$

   where, C is a $\mathbb{R}$ constant.

2. If $f(x)$ is continuous $\forall\, x \in D_f$ and,
   $\displaystyle\int f(x)\,dx = F(x) + C$, then $F(x)$ always exists and is continuous.

3. If $f(x)$ is discontinuous at $x = x_1$, then its anti-derivative can be continuous at $x = x_1$.

4. Anti-derivative of a periodic function may not be periodic.

3 Methods of Integration

3.1 u substitution

Integrals of form,

$$I = \int f(g(x)) \cdot g'(x)\,dx$$

Can be solved by, the substitution,

$$u = g(x)$$

Differentiating both sides w.r.t. x,

$$du = g'(x)dx$$

Now,

$$I = \int f(u)\,du$$

3.2 Integrals of form

$$\int \frac{dx}{ax^2+bx+c}, \int \frac{dx}{\sqrt{ax^2+bx+c}}, \int \sqrt{ax^2+bx+c}\,dx$$

Using completing the square,

$$ax^2+bx+c = a\left[\left(x+\frac{b}{2a}\right)+\frac{c}{a}-\frac{b^2}{4a}\right]$$

Now, using u sub, let

$$u = x+\frac{b}{2a}$$

The transformed integral can be integrated using previous methods.

3.3 Integrals of form

$$\int \frac{px+q}{ax^2+bx+c}\,dx, \int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx, \int (px+q)\sqrt{ax^2+bx+c}\,dx$$

$$px+q = \lambda\frac{d}{dx}(ax^2+bx+c)+\mu$$

Now, after finding λ, μ,
For the 1st part, use u sub,
Let

$$u = ax^2+bx+c$$

2nd part of the integral can be integrated using previous methods.

3.4 Integrals of form

$$\int \frac{K(x)}{ax^2+bx+c}\,dx$$

where $deg(K(x)) \geq 2$

By polynomial long division

$$\frac{K(x)}{ax^2+bx+c} = Q(x)+\frac{R(x)}{ax^2+bx+c}$$

$deg(R(x)) \leq 1$,
Now, the integral

$$\int \frac{R(x)}{ax^2+bx+c}\,dx$$

can be integrated using previous methods.

3.5 Integrals of form

$$\int \frac{ax^2+bx+c}{px^2+qx+r}\,dx, \int \frac{ax^2+bx+c}{\sqrt{px^2+qx+r}}\,dx, \int \left(ax^2+bx+c\right)\sqrt{px^2+qx+r}\,dx$$

on hold

$$ax^2+bx+c = \lambda\left(px^2+qx+r\right)+\mu\frac{d}{dx}\left(px^2+qx+r\right)+\gamma$$

3.6 Trig. Integrals

3.6.1 Integrals of form

$$\int \frac{dx}{a\cos^2 x + b\sin^2 x}, \int \frac{dx}{a+b\sin^2 x}, \int \frac{dx}{a+b\cos^2 x}$$
$$\int \frac{dx}{(a\sin x + b\cos x)^2}, \int \frac{dx}{a+b\sin^2 x + c\cos^2 x}$$

Steps -

1. Multiply numerator and denominator by $\sec^2 x$
2. Replace $\sec^2 x$ (if any) by $1+\tan^2 x$ except the one multiplied in step 1.
3. Let $u = \tan x$, then $du = \sec^2 x\,dx$

Now, the transformed integral can be integrated using previous methods.

3.6.2 Integrals of form

$$\int \frac{dx}{a\sin x + b\cos x}, \int \frac{dx}{a+b\sin x}, \int \frac{dx}{a+b\cos x}, \int \frac{dx}{a\sin x + b\cos x + c}$$

Steps -

1. Replace $\sin x = \dfrac{2\tan {}^{x}\!/_{2}}{1+\tan^2 {}^{x}\!/_{2}}$ and $\cos x = \dfrac{1-\tan^2 {}^{x}\!/_{2}}{1+\tan^2 {}^{x}\!/_{2}}$
2. Let $u = \tan {}^{x}\!/_{2}$, $du = \dfrac{1}{2}\sec^2 {}^{x}\!/_{2}\,dx$ is already present in the numerator.

Now, the transformed integral can be integrated using previous methods.

3.6.2.1 Alternative Method to Integrate

$$I = \int \frac{dx}{a \sin x + b \cos x}$$

$$a \sin x + b \cos x = \sqrt{a^2 + b^2} \sin\left(x + \tan^{-1}\left(\frac{b}{a}\right)\right)$$
$$I = \frac{1}{\sqrt{a^2 + b^2}} \int \csc\left(x + \tan^{-1}\left(\frac{b}{a}\right)\right) dx$$
$$I = \frac{1}{\sqrt{a^2 + b^2}} \ln\left|\tan\left(\frac{x}{2} + \frac{1}{2}\tan^{-1}\left(\frac{b}{a}\right)\right)\right| + C$$

3.6.3 Integrals of form

$$\int \frac{p \cos x + q \sin x + r}{a \cos x + b \sin x + c}\, dx, \int \frac{p \cos x + q \sin x}{a \cos x + b \sin x}\, dx$$

Steps for (i),

1. Express *Numerator* $= \lambda$ *Denominator* $+ \mu$ *Derivative of denominator* $+ \gamma$

Now, the transformed integral can be integrated using previous methods.

Steps for (ii),

1. Express *Numerator* $= \lambda$ *Denominator* $+ \mu$ *Derivative of Denominator*

Now, the transformed integral can be integrated using previous methods.

3.7 Integration by parts

$$\int uv\, dx = u \int v\, dx - \int \left(u' \int v\, dx\right) dx$$

u is the function which has to be differentiated (D), v is the function which has to be integrated (I)

3.8 Integral of form

$$I = \int e^{g(x)} \left(f(x)g'(x) + f'(x)\right) dx$$

$I = e^{g(x)} \cdot f(x) + C$

3.9 Integrals of form

$$S = \int e^{ax} \sin bx\, dx, C = \int e^{ax} \cos bx\, dx$$

$$S = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) + C_0, C = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) + C_{00}$$

4 Partial Fraction Decomposition

Let $f(x) = \sum_{i=0}^{n} a_i x^i, g(x) = \sum_{i=0}^{m} b_i x^i$.

We define a rational function $h(x) = \frac{f(x)}{g(x)}$,

$h(x)$ is $\begin{cases} \textit{Proper Rational Funciton} & m > n \\ \textit{Improper Rational Function} & m \leq n \end{cases}$

If $h(x)$ is Improper, we make it Proper by polynomial long division, i.e.

$$h(x) = Q(x) + \frac{r(x)}{g(x)}$$

Clearly, $\frac{r(x)}{g(x)}$ is Proper.

Assuming, $h(x)$ is proper,

4.1 $g(x)$ is the product of non-repeating linear factors

Let

$$g(x) = L_1(x) \cdot L_2(x) \cdot \ldots \cdot L_m(x)$$

where $L_i(x)$ are linear functions. Then, we can expand $\frac{f(x)}{g(x)}$ in terms of partial fractions as,

$$\frac{f(x)}{g(x)} = \frac{A_1}{L_1} + \frac{A_2}{L_2(x)} + \ldots + \frac{A_m}{L_m(x)}$$

where, $A_i \in \mathbb{R}$ constants.

4.2 $g(x)$ is the product of non-repeating linear factors, but a particular factor is repeated k times

Let

$$g(x) = L_1^k(x) \cdot L_2(x) \cdot \ldots \cdot L_\eta(x)$$

Then, we can expand $\frac{f(x)}{g(x)}$ in terms of partial fractions as,

$$\frac{f(x)}{g(x)} = \frac{A_1}{L_1(x)} + \frac{A_2}{L_1^2(x)} + \frac{A_3}{L_1^3(x)} + \ldots + \frac{A_k}{L_1^k(x)} + \frac{B_2}{L_2(x)} \ldots + \frac{B_\eta}{L_\eta(x)}$$

4.3 $g(x)$ contains some non-repeating linear as well as quadratic factors

Let

$$g(x) = \prod_i L_i(x) \cdot \prod_j Q_j(x)$$

where, $Q_j(x)$ are quadratic factors.

Then, we can expand $\dfrac{f(x)}{g(x)}$ in terms of partial fractions as,

$$\frac{f(x)}{g(x)} = \sum_i \frac{A_i}{L_i(x)} + \sum_j \frac{xB_j + C_j}{Q_j(x)}$$

4.4 $g(x)$ contains some non-repeating linear and repeating quadratic factors

Let

$$g(x) = \prod_i L_i(x) \prod_j Q_j(x) \prod_\omega Q_\omega^k(x)$$

where, $Q^k_{omega}(x)$ are repeating quadratic factors. Then, we can expand $\dfrac{f(x)}{g(x)}$ in terms of partial fractions as,

$$\frac{f(x)}{g(x)} = \sum_i \frac{A_i}{L_i(x)} + \sum_j \frac{xB_j + C_j}{Q_j(x)} + \sum_\omega \sum_r \frac{xD_r + E_r}{Q^r_\omega(x)}$$