Exercise session on constant coefficient PDEs

November 26, 2020

Exercise 1

Consider the following advection diffusion reaction equation for $c = c(x,t)$ with constant coefficients u, ν, γ and source term $s(x)$:

$$\begin{cases} \dfrac{\partial c}{\partial t} = -u\dfrac{\partial c}{\partial x} + \nu\dfrac{\partial^2 c}{\partial x^2} - \gamma c + s(x) & x \in (0,L), t \in (0,T), \\ c(x,0) = c_0(x) & x \in (0,L), \\ c(0,t) = c(L,t) & t \in (0,T), \end{cases}$$

with $L = 10$, $T = 5$. Consider also the functions

$$c_{0,1}(x) = \exp\left\{-\left(\frac{x - L/2}{L/10}\right)^2\right\}, \quad c_{0,2}(x) = \frac{e^{100\sin(t-2)} - 1}{e^{100\sin(t-2)} + 1}.$$

Compute the exact solution of this equation in the following cases, using Fourier series and sseparation of variables on a mesh with $N = 200$ nodes and Fourier series expansions with $m = 100$ modes:

(a) Pure advection (i.e. $\nu = 0$, $\gamma = 0$) for $u = 0.2, u = 0.5$, and $u = -0.5$. Use $s = 0$ and initial value $c_{0,1}$.

(b) Pure diffusion (i.e. $u = 0$, $\gamma = 0$) for $\nu = 0.01, \nu = 0.1$, and $\nu = -0.01$. Use $s = 0$ and initial value $c_{0,1}$.

(c) Pure advection for $u = 0.5$. Use $s = 0$ and initial value $c_{0,2}$.

(d) Pure diffusion for $\nu = 0.01$. Use $s = 0$ and initial value $c_{0,2}$.

(e) Pure advection with $u = 0.5$, and $u = -0.5$. Use a zero initial datum and the source term

$$s_1(x) = \exp\left\{-\left(\frac{x - L/3}{L/20}\right)^2\right\}.$$

(f) Pure diffusion with $\nu = 0.05$. Use a zero initial data and the source term

$$s_2(x) = \exp\left\{-\left(\frac{x - L/2}{L/20}\right)^2\right\}.$$

(g) Advection diffusion reaction with $u = 0.2$, $\nu = 0.01$, $\gamma = -0.01$ with initial datum $c_{0,1}$ and source term s_1.