Exercise session on Fourier Series

November 20, 2020

Exercise 1

Compute the discrete Fourier coefficients of the functions

- $f(x) = \sin(4\pi x/L)$
- $f(x) = \exp(-((x-L/2)/(L/10))^2)$
- $f(x) = \exp(-|(x-L/2)/(L/20)|)$
- $f(x) = \exp(-((x-L/2)/(L/10))^2) + \epsilon_{0.001}$, where $\epsilon_{0.001}$ is a Gaussian random variable with zero mean and variance 0.001 (use the function `randn`)
- $f(x) = \exp(-((x-L/2)/(L/10))^2) + \epsilon_{0.1}$, where $\epsilon_{0.1}$ is a Gaussian random variable with zero mean and variance 0.1 (use the function `randn`)

on the interval $[0, L]$ with $L = 10$. Use the `fft` command after sampling the function on a constant step grid of $N = 100$ points. Plot the spectrum of each function. Repeat the computation with $N = 10, 20, 40, 50$ points.

Exercise 2

For all functions of exercise 1, using the values sampled on a grid with $N = 100$ points, compute the corresponding Fourier series with $m = 10, 20, 40, 50$ Fourier modes. In each case, compute the absolute approximation error in the l_2 and l_∞ and estimate empirically the convergence order.

Exercise 3

Compute the discrete Fourier coefficients of the function $f(x) = \exp(-|x-5|^{5/2})$ on $[0, 20]$ using the command `fft` after sampling the function on a grid with $N = 100$ points. Using the fact that $\hat{f}_k^{(N)} = \hat{f}_{k+N}^{(N)}$, determine the minimum number of modes m for which

$$\sqrt{\frac{\sum_{k=-m}^{m} |\hat{f}_k^{(N)}|^2}{\|f\|_2^2}} \geq 0.97.$$

Exercise 4

Consider the function $y(x) = \frac{x}{L} \sin\left(\frac{10\pi}{L} x\right)$, whose first derivative is

$$y'(x) = \frac{1}{L} \sin\left(\frac{10\pi}{L} x\right) + \frac{10\pi x}{L^2} \cos\left(\frac{10\pi}{L} x\right)$$

on the interval $[0, L]$ with $L = 8$. Use the `fft` command to compute the discrete Fourier coefficients of y after sampling the function on a constant step grid of $N = 160$ points. Compute the Fourier series of y, y' using $m = 10, 20, 40, 80$ Fourier modes. In each case, compute the absolute approximation error in the l_2 and l_∞ and estimate empirically the convergence order.

Exercise 5

Repeat the previous exercise for $y(x) = (2x^3 - x)\cos(x^2)$, whose first derivative is

$$y'(x) = (6x^2 - 1)\cos(x^2) + 2x(x - 2x^3)\sin(x^2),$$

on the interval $[0, 4\pi]$ using $N = 200$ points.