

Exercise session on finite difference approximations of the advection equation, I

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Exercise 1

Consider the homogeneous advection equation

$$\frac{\partial c}{\partial t} = -a \frac{\partial c}{\partial x}$$

with periodic boundary conditions on the spatial domain $[0, L]$ and on the time interval $[0, T]$. Assume $L = 10$, $T = 2$, $a = 1$. Assume as initial datum

$$c_0(x) = 10 \exp \left\{ - \left(\frac{x - L/2}{L/10} \right)^2 \right\}.$$

- Compute the exact solution $c_0(x - aT)$ on a uniform mesh of $N = 100$ points.
- Compute on the same mesh a numerical solution by the explicit upwind method using $M = 100$ time steps. Compare the results with those using $M = 50$ time steps and $M = 10$ time steps.
- Compute on the same mesh a numerical solution by the Lax-Wendroff method using $M = 100$ time steps. Compare the results with those using $M = 50$ time steps and $M = 10$ time steps.

For all numerical methods, plot the absolute value of the difference between the numerical and the exact solution, and compute the l^2 and l^∞ relative errors. Furthermore, repeat the computation with $N = 200$ points and $M = 200$ time steps and compute the empirical convergence orders of each method. For the l^2 norm use the formula

$$\|g\|_2 = \sqrt{\sum_{i=0}^{N_x} |g_i|^2 \Delta x}.$$

Exercise 2

Consider the homogeneous advection equation

$$\frac{\partial c}{\partial t} = -a \frac{\partial c}{\partial x}$$

with periodic boundary conditions on the spatial domain $[0, L]$ and on the time interval $[0, T]$. Assume $L = 2\pi$, $T = 1$, $a = -1$. Assume as initial datum

$$c_0(x) = \frac{1 - \exp(100 \cos x)}{1 + \exp(100 \cos x)}.$$

- Compute the exact solution $c_0(x - aT)$ on a uniform mesh of $N = 100$ points.
- Compute on the same mesh a numerical solution by the explicit upwind method using $M = 100$ time steps.
- Compute on the same mesh a numerical solution by the Lax-Wendroff method using $M = 100$ time steps.

For all numerical methods, plot the absolute value of the difference between the numerical and the exact solution, and compute the l^2 and l^∞ relative errors. Furthermore, repeat the computation with $N = 200$ points and $M = 200$ time steps and compute the empirical convergence orders of each method. For the l^2 norm use the formula

$$\|g\|_2 = \sqrt{\sum_{i=0}^{N_x} |g_i|^2 \Delta x}.$$

Exercise 3

Consider the homogeneous advection equation with Dirichlet boundary condition

$$\begin{cases} \frac{\partial y}{\partial t} = -a \frac{\partial y}{\partial x} & x \in (0, 2\pi), t \in (0, 2), \\ y(x, 0) = y_0(x) = 1 & x \in (0, 2\pi), \\ y(0, t) = 1 + \sin t & t \in (0, 2) \end{cases}$$

with $a = 3$.

- Compute the exact solution at time $T = 2$ on a uniform mesh of $N = 100$ intervals.
- Compute a numerical solution on a uniform mesh of $N = 100$ intervals by the explicit upwind method using the minimum number of time steps required by the Courant Friedrichs Lewy stability condition.

Exercise 4

Consider the non homogeneous equation

$$\frac{\partial c}{\partial t} = -a \frac{\partial c}{\partial x} + \exp \left\{ - \left(\frac{x - L/3}{L/20} \right)^2 \right\}$$

with periodic boundary conditions on the spatial domain $[0, L]$ and on the time interval $[0, T]$. Assume $L = 4$, $T = 5$, $a = 1$. Assume as initial datum $c_0(x) = 0$.

- Compute the exact solution by separation of variables on a uniform mesh of $N = 50$ points.
- Compute on the same mesh a numerical solution by the explicit upwind method using $M = 100$ time steps. Estimate empirically the convergence rate by repeating the computation on a mesh of $N = 100$ points and using $M = 200$ time steps.
- Compute on the mesh of $N = 50$ points a numerical solution by the Lax-Wendroff method using $M = 100$ time steps. Estimate empirically the convergence rate by repeating the computation on a mesh of $N = 100$ points and using $M = 200$ time steps.

In all cases, plot the absolute value of the difference between the numerical and the exact solution, and compute the l^2 and l^∞ relative errors.