Exercise session on polynomial interpolation

October 8, 2020

Exercise 1

Compute the following Lagrange polynomial interpolations using the functions `polyfit` and `polyval` with appropriate values of the arguments:

(a) $f(x) = e^x$ on the nodes $[-2, -1, 0, 1, 2, 3]$,

(b) $f(x) = x \cdot \sin(x)$ on the nodes $[0, 1.5, 2, 2.5, 4.5, 5, 5.5, 7]$,

(c) $f(x) = x^4 + 3 \cdot x^3 - 2x + 4$ with degree $n = 4$ on uniformly spaced nodes on $[-3, 3]$,

(d) $f(x) = x^4 + 3 \cdot x^3 - 2x + 4$ with degree $n = 2$ on uniformly spaced nodes on $[-3, 3]$,

(e) $f(x) = x^4 + 3 \cdot x^3 - 2x + 4$ with degree $n = 5$ on uniformly spaced nodes on $[-3, 3]$.

In all cases, evaluate the interpolating polynomial on a uniform mesh of step $h = 0.001$ on the given interval. Compute the relative and absolute errors with respect to the exact solution computed on the same mesh in the $||\cdot||_\infty$ and $||\cdot||_2$ norms. Plot the absolute error $|f(x) - P_n(x)|$ in case (b) and study the spatial distribution of the error. Plot the results of (c),(d),(e) on the same figure and compare the results. In (c) and (e), change the interpolation nodes, by choosing $[-3, -2.9, -2.8, 2.9, 3]$ and $[-3, -2.9, -2.8, 2.8, 2.9, 3]$, respectively. Explain the differences between the two different approximations.

Exercise 2

Consider the problem of interpolating the value of $\cos(x)$ from its values at six equispaced nodes on the interval $[-2, 4]$.

(a) Compute the value on a mesh of step 0.01 on $[-2, 4]$ using the functions `polyfit` and `polyval`. Compute the absolute error in the infinity norm on $[-2, 4]$.

(b) Repeat the computation using six equispaced nodes on the interval $[-1, 2]$. Use again a mesh of step 0.01 on $[-1, 2]$ to compute the interpolating polynomial. Compute the empirical estimate of the convergence order, saying if the results are coherent with the theory and why.

Exercise 3

Compute the Lagrange polynomial interpolation of degree $n = 5$ of the function

$$f(x) = \frac{e^x}{x^5}$$

on the interval $[3, 15]$ using the functions `polyfit` and `polyval` with appropriate values of the arguments and uniformly spaced nodes. Plot on the same graph the function $f(x)$ and the interpolation polynomial obtained before on the interval $[1, 17]$. Explain the behaviour outside the interpolation interval $[3, 15]$.

Exercise 4

Consider function $f(x) = \exp(-x^2)$ over the interval $[-0.7, 4]$.

(a) Using the functions `polyfit` and `polyval`, compute the polynomial $p(x)$ with degree 8 that interpolates $f(x)$ in the nodes $[-0.7, -0.2, 0.0, 0.05, 0.5, 0.6, 2, 3.5, 4]$, using a uniform mesh over this interval with increment 0.001. Compute the relative error in norm $||\cdot||_2$ and $||\cdot||_\infty$ over the whole interval.

(b) Repeat the computations of point (a) using the function $f(x) = \exp(-|x|^{\frac{1}{7}})$.

(c) Explain which theorem can be used to explain the difference between the results in the two cases and why.

Exercise 5

Consider the function $f(x) = (1 - x)^{\frac{1}{5}}$ on the interval $[0, 1]$.

(a) Using functions `polyfit` and `polyval`, determine the Lagrange interpolating polynomial built from the nodes $[0, 0.1, 0.2, 0.5, 0.7, 0.8, 1]$. Compute the relative error of the approximation of f in the 2 and infinity norm on a mesh of constant step $h = 0.01$.

(b) Repeat the previous point for the Lagrange interpolating polynomial built from the nodes $[0, 0.1, 0.2, 0.5, 0.6, 0.7, 0.8]$, explaining possible differences between the results obtained with respect to those with the first set of nodes.