Exercise session on finite difference approximations of parabolic PDEs, II

December 3, 2020

In all exercises, compute the l^2 norm using the formula

$$\|g\|_2 = \sqrt{\sum_{i=0}^{N_x} |g_i|^2 \Delta x}.$$

Exercise 1

Consider the homogeneous heat equation

$$\frac{\partial c}{\partial t} = \nu \frac{\partial^2 c}{\partial x^2}$$

with periodic boundary conditions on the spatial domain $[0, L]$ and on the time interval $[0, T]$. Assume $L = 10$, $T = 5$, $\nu = 0.1$. Assume as initial datum

$$c_0(x) = 10 \exp\Big\{ - \Big(\frac{x - L/2}{L/10}\Big)^2 \Big\}.$$

- Compute the exact solution by separation of variables on a uniform mesh of $N = 100$ points.
- Compute on the same mesh a numerical solution by the implicit Euler method using $M = 20$ time steps. Plot the absolute value of the difference between the numerical and the exact solution, and compute the l^2 and l^∞ relative errors.
- Estimate empirically the convergence rate by repeating the computation using $N = 200$ time steps and $M = 40$ time steps.
- Compare the accuracy of the results with those of the explicit Euler method applied with a time step such that $\nu\Delta t/\Delta x^2 = 1/5$.

Exercise 2

Repeat the previous exercise using the Crank-Nicolson method. Compare the accuracy of the results with those of the implicit Euler method.

Exercise 3

Consider the non homogeneous, variable coefficients diffusion equation

$$\begin{cases} \dfrac{\partial c}{\partial t} = \dfrac{\partial}{\partial x}\left[\dfrac{x^2}{100}\dfrac{\partial c}{\partial x}\right] + s(x,t) & x \in (0,2), t \in (0,2), \\ y(x,0) = y_0(x) = x^3 & x \in (0,2), \end{cases}$$

with $s(x,t) = 2tx^3 - 3x^3(t^2+1)/25$. Notice that the exact solution is given by $c(x,t) = x^3(t^2+1)$. Solve the equation on a mesh of $N = 200$ nodes and $M = 50$ time steps with the Crank Nicolson method, considering

- Dirichlet boundary conditions $g_0(t) = 0$, $g_L(t) = 8(t^2+1)$;
- Neumann boundary conditions $g_0(t) = 0$, $g_L(t) = 12(t^2+1)$;
- flux boundary conditions $g_0(t) = 0$, $g_L(t) = 12(t^2+1)/25$.

Plot the absolute value of the difference between the numerical and the exact solution, and compute the l^2 and l^∞ relative errors.