

# Exercise session on finite difference approximations of parabolic PDEs, I

November 27, 2020

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In all exercises, compute the  $l^2$  norm using the formula

$$\|g\|_2 = \sqrt{\sum_{i=0}^{N_x} |g_i|^2 \Delta x}.$$

## Exercise 1

Consider the homogeneous heat equation

$$\frac{\partial c}{\partial t} = \nu \frac{\partial^2 c}{\partial x^2}$$

with periodic boundary conditions on the spatial domain  $[0, L]$  and on the time interval  $[0, T]$ . Assume  $L = 10$ ,  $T = 5$ ,  $\nu = 0.05$ . Assume as initial datum

$$c_0(x) = 10 \exp \left\{ - \left( \frac{x - L/2}{L/10} \right)^2 \right\}.$$

- Compute the exact solution by separation of variables on a uniform mesh of  $N = 100$  points. ['safhe 79 lecture note'](#) [We used forier series in order to find exact solution of C\(x,T\)](#)
- Compute on the same mesh a numerical solution by the explicit Euler method using  $M = 200$  time steps. ['safhe 154 lecture note'](#) [We used explicit Euler in order to find numerical solution of C\(x,T\)](#)
- Estimate empirically the convergence rate by repeating the computation using  $N = 200$  time steps and  $M = 400$  time steps.

Plot the absolute value of the difference between the numerical and the exact solution, and compute the  $l^2$  and  $l^\infty$  relative errors.

## Exercise 2

Consider the non homogeneous heat equation

$$\frac{\partial c}{\partial t} = \nu \frac{\partial^2 c}{\partial x^2} + \exp \left\{ - \left( \frac{x - L/3}{L/20} \right)^2 \right\}$$

with periodic boundary conditions on the spatial domain  $[0, L]$  and on the time interval  $[0, T]$ . Assume  $L = 4$ ,  $T = 5$ ,  $\nu = 0.1$ . Assume as initial datum  $c_0(x) = 0$ .

- Compute the exact solution by separation of variables on a uniform mesh of  $N = 50$  points.
- Compute on the same mesh a numerical solution by the explicit Euler method using  $M = 500$  time steps.
- Estimate empirically the convergence rate by repeating the computations on a mesh of  $N = 100$  points and using  $M = 1000$  time steps.

In all cases, plot the absolute value of the difference between the numerical and the exact solution, and compute the  $l^2$  and  $l^\infty$  relative errors.

### Exercise 3

Consider the non homogeneous, variable coefficients diffusion equation with Dirichlet boundary conditions

$$\begin{cases} \frac{\partial y}{\partial t} = \frac{\partial}{\partial x} \left[ \left( \frac{1+x}{1+t} \right) \frac{\partial y}{\partial x} \right] + s(x, t) & x \in (0, 2\pi), t \in (0, 1), \\ y(x, 0) = y_0(x) = \sin(8x) & x \in (0, 2\pi), \\ y(0, t) = 0 \quad y(2\pi, t) = 0 & t \in (0, 1) \end{cases}$$

with

$$s(x, t) = -\sin(8x) \exp(-t) - 8 \frac{\cos(8x) \exp(-t)}{t+1} + 64 \frac{\sin(8x) \exp(-t)(x+1)}{t+1}.$$

Notice that the exact solution is given by  $y(x, t) = \exp(-t) \sin(8x)$ .

- Compute a numerical solution on a uniform mesh of  $N = 50$  intervals by the explicit Euler method using  $M = 4000$  time steps.
- Estimate empirically the convergence rate by repeating the computations on a mesh of  $N = 100$  points and using  $M = 8000$  time steps.