Exercise session on finite difference approximations of the advection equation, II

December 10, 2020

Exercise 1

Consider the homogeneous advection equation

$$\frac{\partial c}{\partial t} = -a\frac{\partial c}{\partial x}$$

with periodic boundary conditions on the spatial domain $[0, L]$ and on the time interval $[0, T]$. Assume $L = 10$, $T = 2$, $a = 1$. Assume as initial datum

$$c_0(x) = 10\exp\Big\{ -\Big(\frac{x - L/2}{L/10}\Big)^2\Big\}.$$

- Compute the exact solution $c_0(x - aT)$ on a uniform mesh of $N = 100$ points.
- Compute on the same mesh a numerical solution by the implicit upwind method using $M = 100$ time steps. Compare the results with those using $M = 50$ time steps and $M = 10$ time steps.
- Compute on the same mesh a numerical solution by the Crank-Nicolson method in time and centered finite differences in space, using $M = 100$ time steps. Compare the results with those using $M = 50$ time steps and $M = 10$ time steps.

For all numerical methods, plot the absolute value of the difference between the numerical and the exact solution, and compute the l^2 and l^∞ relative errors. Furthermore, repeat the computation with $N = 200$ points and $M = 200$ time steps and compute the empirical convergence orders of each method. For the l^2 norm use the formula

$$\|g\|_2 = \sqrt{\sum_{i=0}^{N_x} |g_i|^2 \Delta x}.$$

Exercise 2

Consider the homogeneous advection equation

$$\frac{\partial c}{\partial t} = -a\frac{\partial c}{\partial x}$$

with periodic boundary conditions on the spatial domain $[0, L]$ and on the time interval $[0, T]$. Assume $L = 2\pi$, $T = 1$, $a = -1$. Assume as initial datum

$$c_0(x) = \frac{1 - \exp(100 \cos x)}{1 + \exp(100 \cos x)}.$$

- Compute the exact solution $c_0(x - aT)$ on a uniform mesh of $N = 100$ points.
- Compute on the same mesh a numerical solution by the implicit upwind method using $M = 100$ time steps.
- Compute on the same mesh a numerical solution by the Crank-Nicolson method in time and centered finite differences in space, using $M = 100$ time steps.

For all numerical methods, plot the absolute value of the difference between the numerical and the exact solution, and compute the l^2 and l^∞ relative errors. Furthermore, repeat the computation with $N = 200$ points and $M = 200$ time steps and compute the empirical convergence orders of each method. For the l^2 norm use the formula

$$\|g\|_2 = \sqrt{\sum_{i=0}^{N_x} |g_i|^2 \Delta x}.$$

Exercise 3

Consider the homogeneous advection equation with Dirichlet boundary condition

$$\begin{cases} \dfrac{\partial y}{\partial t} = -\, a\dfrac{\partial y}{\partial x} & x \in (0, 2\pi), t \in (0, 2), \\ y(x, 0) = y_0(x) = 1 & x \in (0, 2\pi), \\ y(0, t) = 1 + \sin t & t \in (0, 2) \end{cases}$$

with $a = 3$.

- Compute the exact solution at time $T = 2$ on a uniform mesh of $N = 100$ intervals.
- Compute a numerical solution on a uniform mesh of $N = 100$ intervals by the explicit upwind method using the minimum number of time steps required by the Courant Friedrichs Lewy stability condition.
- Compute on the same mesh a numerical solution by the implicit upwind method using half the number of time steps used in the first part of the previous point.
- Compute on the same mesh a numerical solution by the Crank-Nicolson method in time and centered finite differences in space, using the same number of time steps as in the previous point. Use the value $c(2\pi, t) = 1$ for the boundary condition at $x = 2\pi$ required by the Crank-Nicolson method.

Exercise 4

Consider the homogeneous advection diffusion equation

$$\frac{\partial c}{\partial t} = -a\frac{\partial c}{\partial x} + \nu\frac{\partial^2 c}{\partial x^2}$$

with periodic boundary conditions on the spatial domain $[0, L]$ and on the time interval $[0, T]$. Assume $L = 5$, $T = 5$, $a = 1$, $\nu = 0.05$. Assume as initial datum $c_0(x) = \exp\Big\{-\Big(\frac{x-L/5}{L/20}\Big)^2\Big\}$.

- Compute the exact solution by separation of variables on a uniform mesh of $N = 50$ points.
- Compute on the same mesh a numerical solution by the explicit method combining an upwind finite difference for the advection term and the second centered finite difference for the diffusion term, using $M = 100$ time steps. Estimate empirically the convergence rate by repeating the computation on a mesh of $N = 100$ points and using $M = 200$ time steps.
- Compute on the same mesh a numerical solution by combining the Lax-Wendroff method for the advection term and the second centered finite difference for the diffusion term, using $M = 100$ time steps. Estimate empirically the convergence rate by repeating the computation on a mesh of $N = 100$ points and using $M = 200$ time steps.
- Compute on the same mesh a numerical solution by combining the centered finite difference method for the advection term and the second centered finite difference for the diffusion term, using $M = 100$ time steps. Estimate empirically the convergence rate by repeating the computation on a mesh of $N = 100$ points and using $M = 200$ time steps.

Exercise 5

Repeat the previous exercise in the case $\nu = 1$, employing centered finite differences to approximate the advection term, second centered finite difference for the diffusion term and 1) the backward Euler method 2) the Crank-Nicolson method for time discretization.

Exercise 6

Consider the non homogeneous advection diffusion equation

$$\frac{\partial c}{\partial t} = -a\frac{\partial c}{\partial x} + \nu\frac{\partial^2 c}{\partial x^2} + \exp\Big\{-\Big(\frac{x-L/4}{L/20}\Big)^2\Big\}$$

with periodic boundary conditions on the spatial domain $[0, L]$ and on the time interval $[0, T]$. Assume $L = 4$, $T = 5$, $a = 2$, $\nu = 1/100$. Assume as initial datum

$$c_0(x) = 0.$$

- Compute the exact solution by separation of variables on a uniform mesh of $N = 50$ points.

- Compute on the same mesh a numerical solution by the explicit upwind method for the advection term and the second order centered differences for the diffusion term using $M = 100$ time steps. Estimate empirically the convergence rate by repeating the computation on a mesh of $N = 100$ points and using $M = 200$ time steps.

- Compute on the mesh of $N = 50$ points a numerical solution using the Lax-Wendroff method for the advection term with $M = 100$ time steps. Estimate empirically the convergence rate by repeating the computation on a mesh of $N = 100$ points and using $M = 200$ time steps.

In all cases, plot the absolute value of the difference between the numerical and the exact solution, and compute the l^2 and l^∞ relative errors.

Exercise 7

Repeat the previous exercise in the case $\nu = 1$, employing centered finite differences to approximate the advection term, second centered finite difference for the diffusion term and 1) the backward Euler method 2) the Crank-Nicolson method for time discretization.