Exercise session on numerical solution of ODEs, 1

October 29, 2020

Exercise 1

Consider the Cauchy problem

$$y' = -3t^2 y \qquad y(0) = 3 \quad t \in [0, 2],$$

whose exact solution is $y(t) = 3\exp(-t^3)$.

(a) Compute an approximate solution using the forward Euler method method with $h = 0.01$. Compute the absolute and relative errors at the final time. Compute the absolute and relative error in the maximum norm on the whole time interval and discuss the possible differences with the errors at the final time.

(b) Repeat the previous point using the Heun method.

(c) Repeat the previous point using the leapfrog method. Use the forward Euler method to compute the auxiliary initial condition.

(d) Repeat the previous point using the leapfrog method, using the Heun method to compute the auxiliary initial condition.

(e) Say which of the computed solutions is the most accurate in this case.

Exercise 2

Consider the differential equation

$$y' = \frac{\alpha}{2}\cos(t)\left(1 - y^2\right) \qquad t \in [0\,,\, 6\pi]$$

with initial datum $y(0) = 0$, where α is a positive parameter. Notice that its exact solution is given by

$$y(t) = \frac{e^{\alpha\sin(t)} - 1}{e^{\alpha\sin(t)} + 1}.$$

(a) Setting $\alpha = 1$, compute an approximate solution using the forward Euler method with time step $h = 0.1, 0.05$. For each value of the time step compute the absolute and relative error in the maximum norm on the whole time interval and estimate empirically the convergence order of the method.

(b) Repeat the previous point using the modified Euler method.

(c) Repeat the previous point using the leapfrog method, using the modified Euler method to provide the auxiliary initial condition.

(d) Compare the results and say which of the methods is the most accurate for this problem.

(e) Repeat the previous points after setting $\alpha = 15$. Explain the difference of the results with respect to the previous case.

Exercise 3

Consider the initial value problem

$$y' = -y + \exp t \sin t \qquad t \in [0,\, 5]$$

$$y(0) = 10,$$

whose exact solution is

$$y_{ex}(t) = \frac{51}{5}\exp(-t) - \frac{1}{5}\exp(t)(\cos(t) - 2\sin(t)).$$

(a) Compute an approximate solution using the Heun method with time steps $h = 0.01, 0.005$. For each value of the time step compute the absolute and relative error in the maximum norm on the whole time interval and estimate empirically the convergence order of the method.

(b) Repeat the previous point using the modified Euler method.

(c) Repeat the previous point using the leapfrog method, using the modified Euler method to provide the auxiliary initial condition.

(d) Compare the results and say which of the methods is the most accurate for this problem.

(e) Repeat the previous points with time steps $h = 0.5,\, 0.25$. Explain the difference of the results with respect to the previous case.

Exercise 4

Repeat the previous two exercises computing numerical solutions of the initial value problems using the MATLAB solver `ode45`. Impose a relative error tolerance of 10^{-7}. Use a maximum time step $h = 0.01$ and use the same time step for the output of the results. Compute the absolute and relative error in the maximum norm and at the end of the interval. Repeat the computation using the time step $h = 0.01$ only for the output of the results.

Exercise 5

Consider the Cauchy problem

$$y' = -2ty \quad y(0) = 1/2 \;\; t \in [0, 2]$$

whose exact solution is $y(t) = \exp(-t^2)/2$.

(a) Compute a numerical solution using the MATLAB solver `ode45`, imposing a relative error tolerance of 10^{-10}. Use a maximum time step $h = 0.01$ and use the same time step for the output of the results. Compute the absolute and relative error in the maximum norm and at the end of the interval.

(b) Repeat the previous point with the solvers `ode23`, `ode113`, `ode15s`, `ode23tb`, imposing the same relative error tolerance but without imposing a maximum time step value. Compute the absolute and relative errors of the four methods in the maximum norm and at the end of the interval and say which method is the most accurate in this case.

(c) Repeat the previous point using a maximum time step $h = 0.1$ and using the same time step for the output of the results.

Exercise 6

Consider the initial value problem

$$y' = \frac{y^2}{1+y^2} \exp\left(-10\sin(t)\right) \qquad t \in [0,\ 4]$$

$$y(0) = 1.$$

(a) Compute a reference numerical solution using the MATLAB solver `ode45`, imposing a relative error tolerance of 10^{-11} and an absolute error tolerance of 10^{-12}. Use a maximum time step $h = 0.01$ and use the same time step for the output of the results.

(b) Using the value $h = 0.01$ for the time step, compute a numerical solution with the Heun method, the modified Euler method and the leapfrog method, using the reference solution to provide the supplementary initial condition.

(c) Compute the absolute and relative errors over the whole interval in the infinity norm and say which method is the most accurate in this case.