Coalgebraic Modal Logic over Set

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Helle Hvid Hansen

Bernoulli Institute for Maths, CS and AI, University of Groningen, NL

Introduction

Previous lectures:

Universal Coalgebra: a theory of systems $X \to F(X)$

- developed uniformly, parametric in functor F (system type);
- captures observable system behaviour;
- canonical notion of behavioural equivalence and bisimulation.

Coalgebraic modal logic: framework for specifying coalgebra behaviours

Modal logics are coalgebraic: $\dfrac{\text{Equational Logic}}{\text{Algebra}} = \dfrac{\text{Coalgebraic Modal Logic}}{\text{Coalgebra}}$

At the level of universal coalgebra:

- What is a modality?
- Which properties do we want to express? Which ones can be expressed?
- Which concrete results (e.g. Hennessy-Milner theorem) hold more generally?
- ... and under which conditions?

This lecture: Set-based coalgebras $X \to F(X)$ (i.e., X is a set, $F\colon \mathsf{Set} \to \mathsf{Set}$).
(intuitions, covers many examples relevant for applications)

Next lecture: Beyond Set (more abstract setting)

Overview and Learning Goals

Overview of lecture

- Modal semantics as predicate liftings
- Invariance and expressiveness results
- (Modal semantics via relation lifting)
- Concluding remarks

Focus on definitions, results examples.

After this lecture...

- You can explain the basic concepts of coalgebraic modal logic;
- You can explain some basic invariance and expressiveness results of coalgebraic modal logic;
- You can apply the basic theory of coalgebraic modal logic to concrete system types.

Modalities as Predicate Liftings

Labelled transition systems are $\mathcal{P}^A$-Coalgebras

Covariant powerset $\mathcal{P}\colon \mathsf{Set} \to \mathsf{Set}$	Exponentiation $(-)^A\colon \mathsf{Set} \to \mathsf{Set}$
$\mathcal{P}(X) = \{U \mid U \subseteq X\}$	$X^A = \{g\colon X \to A\}$
$\mathcal{P}(f\colon X \to Y) = f[-]\colon \mathcal{P}(X) \to \mathcal{P}(Y)$ (direct image)	$(f\colon X \to Y)^A = f \circ (-)\colon X^A \to Y^A$ (pre-composition)

LTS is coalgebra $R\colon X \to \mathcal{P}(X)^A$: $R(x)(a) = \{y \in X \mid x \xrightarrow{a} y\} =: R_a(x)$

Semantics of modal formulas:

Let $[\![\varphi]\!] = \{x \in X \mid \mathbb{M}, x \models \varphi\}$ (truth-set of φ).

$$(X, R), x \models [a]\varphi \quad \text{iff} \quad R_a(x) \subseteq [\![\varphi]\!] \quad \text{iff} \quad R(x) \in \{g \in \mathcal{P}(X)^A \mid g(a) \subseteq [\![\varphi]\!]\}$$
$$(X, R), x \models \langle a\rangle\varphi \quad \text{iff} \quad R_a(x) \cap [\![\varphi]\!] \neq \emptyset \quad \text{iff} \quad R(x) \in \{g \in \mathcal{P}(X)^A \mid g(a) \cap [\![\varphi]\!] \neq \emptyset\}$$

Markov chains are $\mathcal{D}_\omega$-Coalgebras

Finitary distribution functor $\mathcal{D}_\omega \colon \mathsf{Set} \to \mathsf{Set}$

$$\mathcal{D}_\omega(X) = \{\mu \mid X \to [0,1] \mid \mathsf{supp}(\mu) \text{ finite}, \sum_{x \in X} \mu(x) = 1\}$$

$$\mathcal{D}_\omega(f \colon X \to Y) = \mathcal{D}_\omega(f) \colon \mathcal{D}_\omega(X) \to \mathcal{D}_\omega(Y)$$

$$\mathcal{D}_\omega(f)(\mu)(y) = \sum_{x \in X, f(x) = y} \mu(x)$$

Markov chain is coalgebra $\gamma \colon X \to \mathcal{D}_\omega(X)$: $\quad \gamma(x)(x') = p \quad \text{iff} \quad x \xrightarrow{p} x'$

Semantics of modal formulas:

Let $[\![\varphi]\!] = \{x \in X \mid (X,\gamma), x \models \varphi\}$ (truth-set of φ).

Let $r \in [0,1]$. Consider modality L_r ("probability at least r")

$$(X,\gamma), x \models L_r\varphi \quad \text{iff} \quad \sum_{x \in [\![\varphi]\!]} \gamma(x) \geq r \quad \text{iff} \quad \gamma(x) \in \{\mu \in \mathcal{D}_\omega(X) \mid \sum_{x \in [\![\varphi]\!]} \mu(x) \geq r\}$$

Neighbourhood Structures are Coalgebras

- Contravariant powerset functor $\mathcal{Q}\colon \mathsf{Set}^{\mathrm{op}} \to \mathsf{Set}$

$$\begin{aligned} \mathcal{Q}(X) &= \mathcal{P}(X)\ (= 2^X) \\ \mathcal{Q}(f) &= f^{-1}[-]\colon\ \mathcal{Q}(Y) \to \mathcal{Q}(X) \quad \text{(inverse image)} \end{aligned}$$

- Neighbourhood frames are $\mathcal{N}$-coalgebras $X \to \mathcal{N}(X)$ where

$$\begin{aligned} \mathcal{N}(X) &= (\mathcal{Q} \circ \mathcal{Q}^{\mathrm{op}})(X) \\ \mathcal{N}(f) &= (f^{-1})^{-1}[-]\colon \mathcal{N}(X) \to \mathcal{N}(Y) \quad \text{(double inverse image)} \\ & \ U \in \mathcal{N}(f)(H) \ \text{ iff } \ f^{-1}[U] \in H \end{aligned}$$

- Monotonic neighbourhood frames are coalgebras $X \to \mathcal{M}(X)$ for

$$\begin{aligned} \mathcal{M}(X) &= \{H \in \mathcal{N}(X) \mid H \text{ closed under supersets}\} \\ \mathcal{M}(f) &= (f^{-1})^{-1}[-]\colon \mathcal{M}(X) \to \mathcal{M}(Y) \quad \text{(double inverse image)} \end{aligned}$$

Remark: $\mathcal{N}$ and $\mathcal{M}$ do not preserve weak pullbacks, hence behav.equiv. $\not\Rightarrow$ bisimilarity.

Neighbourhood Modalities

In neighbourhood model $\mathbb{M} = (X, N \colon X \to \mathcal{N}(X), v \colon X \to \mathcal{P}(\mathsf{Prop}))$:

$$\mathbb{M}, x \models \Box\varphi \quad \text{iff} \quad [\![\varphi]\!] \in N(x) \quad \text{iff} \quad N(x) \in \{H \in \mathcal{N}(X) \mid [\![\varphi]\!] \in H\}$$

If $\mathbb{M} = (X, N \colon X \to \mathcal{M}(X), v \colon X \to \mathcal{P}(\mathsf{Prop}))$ monotonic, then equivalent with:

$$\begin{aligned} \mathbb{M}, x \models \Box\varphi \quad &\text{iff} \quad \exists U \in N(x) : U \subseteq [\![\varphi]\!] \\ &\text{iff} \quad N(x) \in \{H \in \mathcal{M}(X) \mid \exists U \in N(x) : U \subseteq [\![\varphi]\!]\} \end{aligned}$$

General picture:

In coalgebraic model $\mathbb{X} = (X, \gamma\colon X \to F(X), v\colon X \to \mathcal{P}(\mathsf{Prop}))$

$$\mathbb{X}, x \models \heartsuit\varphi \qquad \text{iff} \qquad \gamma(x) \in \{f \in F(X) \mid \text{condition involving } [\![\varphi]\!]\}$$

Predicate $[\![\varphi]\!]$ over state space X is lifted to predicate $F(X)$.

A n-ary predicate lifting for F is a natural transformation $\lambda\colon\ \mathcal{Q}^n \Rightarrow \mathcal{Q} \circ F$ (recall $\mathcal{Q}$ is contravariant powerset), that is, λ is a family of set-indexed functions $\lambda_X\colon (2^X)^n \to 2^{F(X)}$ such that:

$$\begin{tikzcd} X \arrow[d, "\forall f"'] & (2^X)^n \arrow[r, "\lambda_X"] & 2^{F(X)} \\ Y & (2^Y)^n \arrow[u, "(f^{-1})^n"] \arrow[r, "\lambda_Y"] & 2^{F(Y)} \arrow[u, "F(f)^{-1}"'] \end{tikzcd}$$

Coalgebraic Modal Logic via Predicate Liftings

Coalgebraic modal logic means coalgebraic semantics of modal languages.

Syntax
Given a modal signature Λ (i.e., collection of modal operators with arities), and a set Prop of propositional variables, the set $\mathcal{L}(\Lambda, \mathsf{Prop})$ of formulas over Λ and Prop is:

$$\mathcal{L}(\Lambda, \mathsf{Prop}) \ni \varphi ::= p \in \mathsf{Prop} \mid \top \mid \neg\varphi \mid \varphi \wedge \varphi \mid \heartsuit(\varphi_1, \ldots, \varphi_n), \quad \heartsuit \in \Lambda,\ n\text{-ary}$$

Coalgebraic semantics: We want to interpret formulas in F-coalgebra model
$\mathbb{X} = (X, \gamma \colon X \to F(X), v \colon \mathsf{Prop} \to \mathcal{P}(X))$
which corresponds to $F \times \mathcal{P}(\mathsf{Prop})$-coalgebra $\langle \gamma : X \to F(X), \hat{v} : X \to \mathcal{P}(\mathsf{Prop}) \rangle$.
(We can take atomic props to be part of the functor.)

Coalgebraic semantics of $\mathcal{L}(\Lambda, \mathsf{Prop})$ consists of:

- a functor $F\colon \mathsf{Set} \to \mathsf{Set}$
- for every n-ary modal operator $\heartsuit \in \Lambda$, a natural transformation

$$\lambda^{\heartsuit} : \mathcal{Q}^n \Rightarrow \mathcal{Q}\, F.$$

Truth in F-model $\mathbb{X} = (X, \gamma : X \to F(X), v : X \to \mathcal{P}(\mathsf{Prop}))$

$$\begin{array}{lll} \mathbb{X}, x \models p & \text{iff} & x \in v(p) \qquad \text{for } p \in \mathsf{Prop} \\ & \vdots & \\ \mathbb{X}, x \models \heartsuit(\varphi_1, \ldots, \varphi_n) & \text{iff} & \gamma(x) \in \lambda^{\heartsuit}_X([\![\varphi_1]\!], \ldots, [\![\varphi_n]\!]) \end{array}$$

Predicate Lifting Examples

HML box:

$$\lambda_X^{[a]}(U) = \{g \in \mathcal{P}(X)^A \mid g(a) \subseteq U\}$$

$$(X, R), x \models [a]\varphi \iff R(x) \in \lambda_X^{[a]}(\llbracket\varphi\rrbracket)$$

HML diamond:

$$\lambda_X^{\langle a\rangle}(U) = \{g \in \mathcal{P}(X)^A \mid g(a) \cap U \neq \emptyset\}$$

$$(X, R), x \models \langle a\rangle\varphi \iff R(x) \in \lambda_X^{\langle a\rangle}(\llbracket\varphi\rrbracket)$$

Probabilistic modality:

$$\lambda_X^{L_r}(U) = \{\mu \in \mathcal{D}_\omega(X) \mid \textstyle\sum_{x \in U} \mu(x) \ \geq r\}$$

$$(X, \gamma), x \models L_r\varphi \iff \gamma(x) \in \lambda_X^{L_r}(\llbracket\varphi\rrbracket)$$

Neighbourhood modality:

$$\lambda_X^{\Box}(U) = \{H \in \mathcal{N}(X) \mid U \in H\}$$

$$(X, N), x \models \Box\varphi \iff N(x) \in \lambda_X^{\Box}(\llbracket\varphi\rrbracket)$$

Exercise: Check naturality condition for the above.

Modalities as "Allowed Patterns", cf. [Schröder'08],[Gumm]

Let $C_\heartsuit \subseteq F(2^n)$ specify which 0-1 patterns of F-structures are "allowed" by $\heartsuit$.
(For $U \subseteq X$, characteristic function $\chi_U : X \to 2$)

$$\begin{array}{ccccc} X & \xrightarrow{\langle \chi_{[\![\varphi_1]\!]},\dots,\chi_{[\![\varphi_n]\!]}\rangle} & 2^n & & \\ {\scriptstyle\gamma}\downarrow & & & & \\ F(X) & \xrightarrow{F\langle \chi_{[\![\varphi]\!]}\dots,\chi_{[\![\varphi_n]\!]}\rangle} & F(2^n) & \xrightarrow{\chi_{C_\heartsuit}} & 2 \end{array}$$

Define: $(X,\gamma), x \models \heartsuit(\varphi_1,\dots,\varphi_n) \iff F(\chi_{[\![\varphi_1]\!]},\dots,\chi_{[\![\varphi_n]\!]})(\gamma(x)) \in C_\heartsuit$

Exercise: Let $F(X) = X \times \{a,b\} \times X$ (labelled binary trees).
What are the possible unary patterns ($n = 1$)?
Which $C \subseteq F(2)$ corresponds to:
$\heartsuit\varphi$: "the label is a and φ is true at some child";
$C_\heartsuit = \{(1,a,0),(0,a,1),(1,a,1)\}$
$\clubsuit\varphi$: "if the label is b then φ is true at both children";
$C_\clubsuit = \{(1,b,1),(0,a,0),(1,a,0),(0,a,1),(1,a,1)\}$

Exercise:

For $\mathcal{P}$, the unary 0-1 patterns are subsets $C \subseteq \mathcal{P}(2) = \mathcal{P}(\{0,1\})$.
The "singleton patterns" (i.e., $c \in \mathcal{P}(2)$) are:

$\emptyset$	No successors
$\{0\}$	Some successors and φ is false at all
$\{1\}$	Some successors and φ is true at all
$\{0,1\}$	Some successors where φ is false, and some where φ is true

Describe informally the modality represented by each singleton pattern.
Which subsets $C \subseteq \mathcal{P}(2)$ correspond to $\Box\varphi$ and $\Diamond\varphi$?

Correspondence and Counting Modalities

Via Yoneda Lemma, 1-1 correspondence:

$$\frac{\text{predicate liftings } \lambda^{\heartsuit} \colon (2^{-})^{n} \Rightarrow 2^{F-}}{\text{subsets } C_{\heartsuit} \in 2^{F(2^{n})}}$$

It also tells us how many predicate liftings, there are.
E.g. Number of distinct unary modalities/predicate liftings for $\mathcal{P}$ is $2^{\mathcal{P}(2)} = 16$

Exercise. The correspondence can be proven directly.

- For $C \subseteq F(2^{n})$, let $\lambda^{C}_{X}(U_1, \dots, U_n) := F(\chi_{U_1}, \dots, \chi_{U_n})^{-1}[C]$.
- For n-ary λ, let $C_{\lambda} := \lambda_{2^{n}}(\pi_1^{-1}[\{1\}], \dots, \pi_n^{-1}[\{1\}])$ where $\pi_i \colon 2^{n} \to 2$ is i-th projection.

Verify the correspondence.

QUESTIONS?

Invariance and Expressiveness

Theorem (Invariance for morphisms)

For all F-coalgebra morphisms $f\colon (X,\gamma) \to (Y,\delta)$ and all φ: $x \models \varphi$ iff $f(x) \models \varphi$.
(Equivalently: $[\![\varphi]\!]_X = f^{-1}[[\![\varphi]\!]_Y]$.

Proof By structural induction on φ. Induction step, modal case, use that (writing 2^X for $\mathcal{Q}X$):

$$\begin{tikzcd}
2^X & 2^Y \arrow[l, "2^f"'] \\
2^{FX} \arrow[u, "2^\gamma"] & 2^{FY} \arrow[l, "2^{Ff}"'] \arrow[u, "2^\delta"'] \\
(2^X)^n \arrow[u, "\lambda_X^\heartsuit"] & (2^Y)^n \arrow[l, "(2^f)^n"'] \arrow[u, "\lambda_Y^\heartsuit"']
\end{tikzcd}$$

which says:
for all $x \in X$, and all $\langle U_1, \ldots, U_n\rangle \in (2^Y)^n$:

$$\gamma(x) \in \lambda_X^\heartsuit(f^{-1}[U_1], \ldots, f^{-1}[U_n])$$
iff
$$\delta(f(x)) \in \lambda_Y^\heartsuit(U_1, \ldots, U_n)$$

Corollary: Truth is invariant for bisimulations and behavioural equivalence:

Expressive Logics and Separating Liftings

Def. A logic $\mathcal{L}(\Lambda)$ is expressive if $\mathbb{X}, x \equiv \mathbb{Y}, y$ implies $\mathbb{X}, x \sim \mathbb{Y}, y$.

Def. A collection $[\![\Lambda]\!] = \{\lambda^{\heartsuit}\}_{\heartsuit\in\Lambda}$ of predicate liftings for F is separating if for all $t_1 \neq t_2$ in $F(X)$ there is a $\heartsuit \in \Lambda$ (n-ary) and $(A_1, \ldots, A_n) \in (\mathcal{Q}X)^n$ such that $t_1 \in \lambda^{\heartsuit}_X(A_1, \ldots, A_n)$ and $t_2 \notin \lambda^{\heartsuit}_X(A_1, \ldots, A_n)$, or vice versa. [Pattinson'04]

Informally: points in $F(X)$ can be distinguished with lifted predicates.

Exercise: Show that

- $\Lambda = \{\lambda^{\Box}\}$ is separating for $\mathcal{P}$ where $\lambda^{\Box}(U)_X = \{V \in \mathcal{P}(X) \mid V \subseteq U\}$.
- $\Lambda = \{\lambda^{L_q} \mid q \in [0,1] \cap \mathbb{Q}\}$ is separating for $\mathcal{D}_\omega$ where $\lambda^{L_q}_X(U) = \{\mu \in \mathcal{D}_\omega(X) \mid \sum_{x\in U} \mu(x) \geq q\}$.

Finitary Functors

Recall: Over *image-finite* LTSs, Hennessy-Milner Logic is expressive.
(But not over arbitrary LTSs.)

A functor $F\colon \mathsf{Set} \to \mathsf{Set}$ is finitary if

$$\text{for all sets } X\text{:} \quad F(X) = \bigcup\{(Fi)[F(A)] \mid A \subseteq_\omega X, i : A \hookrightarrow X\}$$

If F preserves inclusions, i.e., $X \subseteq Y \Rightarrow F(X) \subseteq F(Y)$, then F is finitary if

$$\text{for all sets } X\text{:} \quad F(X) = \bigcup\{F(A) \mid A \subseteq_\omega X\}$$

(Every Set-functor can be modified in a non-essential way to preserve inclusions.)

Examples: $\mathcal{P}_\omega$, $\mathcal{D}_\omega$, $(-)^A$ for finite A, bag functor, integer-bag, ...
Finitary functors are closed under composition.

Theorem 1 If F is finitary and $[\![\Lambda]\!]$ is separating, then $\mathcal{L}(\Lambda)$ is expressive.
[Pattinson'04][Schröder'08]

Do Separating $[\![\Lambda]\!]$ Exist?

Yes, but not necessarily of unary predicate liftings.

Definition
Let $\alpha \in \mathbb{N} \cup \{\omega\}$. We call Λ α-bounded if for all $\lambda \in \Lambda$, λ has arity $k < \alpha$.

Lemma
There exists an α-bounded separating set of predicate liftings for F
$\iff \{F(f) \colon F(X) \to F(2^k) \mid f \colon X \to 2^k, k < \alpha\}$ is jointly injective for all X.

Exercise: Prove the Lemma. Hint: Use Yoneda correspondence.

Theorem 2 [Schröder'08]
If F is finitary, then there is a separating set of finite-arity predicate liftings for F (and hence an expressive modal logic).

Exercise: Show that $\mathcal{P}_\omega \circ \mathcal{P}_\omega$ is finitary, but not n-separating for any $n < \omega$. Hence, modalities with unbounded arity needed. Cf. Instantial neighbourhood logic: $\Box(\varphi_1, \ldots, \varphi_n; \psi)$ for all $n < \omega$.

QUESTIONS?

Coalgebraic Modal Logic via Relation Lifting

Coalgebraic Modal Logic via Relation Lifing

Introduced by [Moss'00]. Basic idea:

- Language has one "canonical" modality ∇ that takes elements from $F(\mathcal{L})$ as argument (where $\mathcal{L}$ is the set of formulas).
- Semantics of ∇ via lifting of satisfaction relation $\models \subseteq X \times \mathcal{L}$: For $\alpha \in F(\mathcal{L})$,

$$(X, \gamma), x \models \nabla\alpha \quad \text{iff} \quad (\gamma(x), \alpha) \in \overline{F}(\models)$$

  where $\overline{F}$ is the so-called *Barr lifting*:

$$\overline{F}(R) = \{\langle F(\pi_1)(u), F(\pi_2)(u)\rangle \mid u \in F(R)\} \quad \subseteq F(X_1) \times F(X_2)$$

Remarks:

- To show adequacy, F needs to preserve weak pullbacks.
- ∇-logic is always expressive.
- Canonical language, but non-standard.

Example: ∇ for $\mathcal{P}$-coalgebras

Example: For $F = \mathcal{P}$, $\overline{\mathcal{P}}$ is also known as the Egli-Milner lifting

$$\begin{aligned}\overline{\mathcal{P}}(R) \quad = \quad & \{(U,V) \in \mathcal{P}X \times \mathcal{P}Y \mid \forall u \in U \exists v \in V : (u,v) \in R\} \cap \\ & \{(U,V) \in \mathcal{P}X \times \mathcal{P}Y \mid \forall v \in V \exists u \in U : (u,v) \in R\}\end{aligned}$$

That means, for a set $\Phi \in \mathcal{P}(\mathcal{L})$ of formulas

$x \models \nabla\Phi$ iff

- all R-successors of x satisfy some $\varphi \in \Phi$, and
- all $\varphi \in \Phi$ are satisfied by some R-successor of x.

In other words,

$$\nabla\Phi \quad \text{equiv. with} \quad \Box \bigvee_{\varphi\in\Phi} \varphi \wedge \bigwedge_{\varphi\in\Phi} \Diamond\varphi$$

In general, ∇ can be expressed by predicate liftings and vice versa. [Leal & Kurz]

Concluding

Coalgebraic modal logic

- uniform framework for studying modal logics for coalgebras.
- encompasses many known logics: basic ML, HML, probabilistic ML, graded ML, ...
- is adequate for reasoning about coalgebraic behaviours: behavioural equivalence implies logical equivalence.
- finitary functors F, expressive logics exist. (Note: property of functor!)

Framework identifies abstract conditions (e.g. natural transf., separating, finitary,...) that suffice for certain results to hold. Avoids reproving from scratch.

Framework allows exploring limits for logics for coalgebras.

Extensions

Extensions of basic coalgebraic modal logic:

- with fixpoints: coalgebraic μ-calculus (both ∇ and pred.lifts) [Venema, Kupke, Fontaine, Enqvist, Seifan,...]
- with temporal operators [Cirstea]
- coalgebraic dynamic logic (PDL) [H, Kupke]
- coalgebraic predicate logic [Litak, Sano, Pattinson, Schröder]

(Uniform) Coalgebraic Theorems

Some coalgebraic generalisations of classic results

- Hennessy-Milner thm (Schröder),
- Van Benthem Characterisation thm: CML = FOL/ $\sim$ (Pattinson, Schröder, Litak, Sano)
- Janin-Walukiewics thm: μ-CML = MSOL/ $\sim$ (Enqvist,Seifan,Venema)
- Goldblatt-Thomason thm: modal analogue of Birkhoff Variety thm. (Kurz, Rosický)
- Completeness
    - coalgebraic canonical model construction (Pattinson, Schröder),
    - ∇-logic (Kupke, Kurz, Venema),
    - coalgebraic dynamic logics (H, Kupke)
- Decidability in PSPACE (Schröder, Pattinson)
- Uniform Interpolation (Marti, Enqvist, Seifan, Venema)

Modal Logic via Dual Adjunctions

Stone-type duality:

$$\begin{tikzcd}
\mathsf{Coalg}(F) \arrow[r, shift left, "\overline{Q}"] \arrow[d] & \mathsf{Alg}(L)^{\mathsf{op}} \arrow[l, shift left, "\overline{\mathrm{Uf}}"] \arrow[d] \\
\mathsf{Set} \arrow[loop left, "F"] \arrow[r, bend left, "Q"] \arrow[r, phantom, "\perp"] & \mathsf{BA}^{\mathsf{op}} \arrow[loop right, "L^{\mathsf{op}}"] \arrow[l, bend left, "\mathrm{Uf}"]
\end{tikzcd}$$

Generalise to non-classical base logic and other base categories

$$\begin{tikzcd}
\mathsf{Coalg}(F) \arrow[r, shift left, "\overline{P}"] \arrow[d] & \mathsf{Alg}(L)^{\mathsf{op}} \arrow[l, shift left, "\overline{S}"] \arrow[d] \\
\mathsf{C} \arrow[loop left, "F"] \arrow[r, bend left, "P"] \arrow[r, phantom, "\perp"] & \mathsf{D}^{\mathsf{op}} \arrow[loop right, "L^{\mathsf{op}}"] \arrow[l, bend left, "S"]
\end{tikzcd}$$

Functor L specifies type of "modal algebras". Modal semantics given by natural transformation $\delta \colon LP \Rightarrow PF$.

cf. [Kupke et al'04], [Bonsangue & Kurz'05], [Klin'07], [Jacobs & Sokolova'10], [Klin & Rot'16], [de Groot et al.'20] m.m.

The End

QUESTIONS?

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