Session Types

Logical Foundations of Concurrent Computation

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(Part 1.2)

Outline

Linear Logic

- Introduced by Jean-Yves Girard in 1987 (famous paper in the TCS journal)
- Interest/applications in proof theory and computer science
- A "resource-aware" logic:
  - $A \vdash B$: given A-resources, a way to obtain B-resources
  - Proof theoretically: substructural logic
- In this course: intuitionistic variant, dubbed ILL

Linear Logic: Propositions

'A is true' vs 'I have A-resources'

$A \wedge B$	both A and B are true
$A \rightarrow B$	if A is true, then B is true

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E.g. $\mathit{euro} \otimes \mathit{euro} \multimap \mathit{pizza}, \quad \mathit{dough} \otimes (\mathit{sauce} \otimes \mathit{toppings}) \multimap \mathit{pizza}$

Linear Logic: Linearity

$$A \rightarrow A \wedge A \qquad\qquad A \not\multimap A \otimes A$$

$$A \wedge B \rightarrow A \qquad\qquad A \otimes B \not\multimap A$$

Linear Logic: Linearity

$A \to A \wedge A$	$A \not\multimap A \otimes A$
$A \wedge B \to A$	$A \otimes B \not\multimap A$
$A \wedge (A \to B) \to B$	$A \otimes (A \multimap B) \multimap B$

Linear Logic: Linearity

$$A \to A \wedge A \qquad A \not\multimap A \otimes A$$

$$A \wedge B \to A \qquad A \otimes B \not\multimap A$$

$$A \wedge (A \to B) \to B \qquad A \otimes (A \multimap B) \multimap B$$

$$A \wedge (A \to B) \to A \wedge B \qquad A \otimes (A \multimap B) \not\multimap A \otimes B$$

Linear Logic is **substructural**, resource-aware.

Proof Theory of $\wedge$

Sequent calculus: $\Gamma \vdash A$

$$A \vdash A$$

$$\wedge R \quad \frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \wedge B}$$

$$\wedge L \quad \frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}$$

$$\text{Weakn} \quad \frac{\Gamma \vdash C}{\Gamma, A \vdash C}$$

$$\text{Contr} \quad \frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C}$$

$$\text{eXchg} \quad \frac{\Gamma, A, B, \Gamma' \vdash C}{\Gamma, B, A, \Gamma' \vdash C}$$

Proof Theory of $\wedge$

Assertion A holds under the list of hypotheses Γ.
Intuitively, $\bigwedge \Gamma \implies A$

Sequent calculus: $\Gamma \vdash A$

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Proof Theory of $\wedge$

Axiom rule

Sequent calculus: $\Gamma \vdash A$

Right and left rules for $\wedge$

$$A \vdash A$$

$\wedge$R
$$\frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \wedge B}$$

$\wedge$L
$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}$$

Weakn
$$\frac{\Gamma \vdash C}{\Gamma, A \vdash C}$$

Contr
$$\frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C}$$

eXchg
$$\frac{\Gamma, A, B, \Gamma' \vdash C}{\Gamma, B, A, \Gamma' \vdash C}$$

Proof Theory of $\wedge$

Sequent calculus: $\Gamma \vdash A$

$$A \vdash A \qquad \frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \wedge B}\,\wedge\mathsf{R} \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}\,\wedge\mathsf{L}$$

$$\underbrace{\frac{\Gamma \vdash C}{\Gamma, A \vdash C}\,\mathsf{Weakn} \qquad \frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C}\,\mathsf{Contr} \qquad \frac{\Gamma, A, B, \Gamma' \vdash C}{\Gamma, B, A, \Gamma' \vdash C}\,\mathsf{eXchg}}_{\text{Structural rules}}$$

Proof Theory of $\wedge$

$\Gamma : \mathit{Frml} \to \mathbb{N}$ is a multiset

Sequent calculus: $\Gamma \vdash A$

$$A \vdash A \qquad \wedge\mathrm{R}\ \frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \wedge B} \qquad \wedge\mathrm{L}\ \frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}$$

$$\mathrm{Weakn}\ \frac{\Gamma \vdash C}{\Gamma, A \vdash C} \qquad \mathrm{Contr}\ \frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C} \qquad \mathrm{eXchg}\ \frac{\Gamma, A, B, \Gamma' \vdash C}{\Gamma, B, A, \Gamma' \vdash C}$$

We can omit the exchange rule by treating Γ as a multiset

Generalized Structural Rules

$$\frac{\Gamma, \Gamma \vdash C}{\Gamma \vdash C} \qquad \frac{\Gamma \vdash C}{\Gamma, \Delta \vdash C}$$

Obtained from repeatedly using the usual rules (and exchange).

Using Structural Rules

$$\frac{}{A \wedge B \vdash A} \qquad \frac{}{A \vdash A \wedge A}$$

Using Structural Rules

$$\dfrac{\overline{A, B \vdash A}}{A \land B \vdash A} \qquad\qquad \dfrac{}{A \vdash A \land A}$$

Using Structural Rules

$$\dfrac{\dfrac{A \vdash A}{A, B \vdash A}}{A \wedge B \vdash A}$$

$$\dfrac{}{A \vdash A \wedge A}$$

Using Structural Rules

$$\dfrac{\dfrac{A \vdash A}{A, B \vdash A}}{A \land B \vdash A} \qquad\qquad \dfrac{\overline{A, A \vdash A \land A}}{A \vdash A \land A}$$

Using Structural Rules

$$\dfrac{\dfrac{A \vdash A}{A, B \vdash A}}{A \land B \vdash A}$$

$$\dfrac{\dfrac{A \vdash A \qquad A \vdash A}{A, A \vdash A \land A}}{A \vdash A \land A}$$

Using Structural Rules

$$\dfrac{\dfrac{A \vdash A}{A, B \vdash A}}{A \wedge B \vdash A} \qquad \dfrac{\dfrac{A \vdash A \qquad A \vdash A}{A, A \vdash A \wedge A}}{A \vdash A \wedge A}$$

Crucial is the usage of the contraction and weakening rules

Associativity

$$\frac{}{A \land (B \land C) \vdash (A \land B) \land C}$$

Associativity

$$\dfrac{\overline{A, B, C \vdash (A \wedge B) \wedge C}}{A \wedge (B \wedge C) \vdash (A \wedge B) \wedge C}$$

Associativity

$$\dfrac{\dfrac{}{A, B \vdash A \wedge B} \qquad C \vdash C}{\dfrac{A, B, C \vdash (A \wedge B) \wedge C}{A \wedge (B \wedge C) \vdash (A \wedge B) \wedge C}}$$

Associativity

$$\dfrac{\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A, B \vdash A \wedge B} \qquad C \vdash C}{A, B, C \vdash (A \wedge B) \wedge C}}{A \wedge (B \wedge C) \vdash (A \wedge B) \wedge C}$$

Alternative Rules for $\wedge$

We can replace $\wedge L$ with the following two rules:

$$\wedge\mathrm{L}\ \frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C} \quad \longrightarrow \quad \wedge\mathrm{L1}\ \frac{\Gamma, A \vdash C}{\Gamma, A \wedge B \vdash C} \qquad \wedge\mathrm{L2}\ \frac{\Gamma, B \vdash C}{\Gamma, A \wedge B \vdash C}$$

These rules **internalize weakening**.

Alternative Rules for $\wedge$

We can replace $\wedge R$ with the following rule:

$$\frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \wedge B} \qquad \longrightarrow \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

Outline

Motivation

From Intuitionistic to Linear Logic

Examples

From Intuitionistic to Linear Logic

- Main idea: treat $\otimes$ the same as $\wedge$, minus contraction and weakening rules.
- The context Γ then externalizes $\otimes$ on the meta-level:

$$\Gamma \vdash A \qquad \longrightarrow \qquad \bigotimes \Gamma \implies A$$

- We will recover contraction and weakening later with the ! modality.

Proof Theory of $\otimes$

Sequent calculus: $\Gamma \vdash A$

$$A \vdash A$$

$\otimes$R
$$\frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \otimes B}$$

$\otimes$L
$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}$$

~~Weakn~~
$$\cancel{\frac{\Gamma \vdash C}{\Gamma, A \vdash C}}$$

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eXchg
$$\frac{\Gamma, A, B, \Gamma' \vdash C}{\Gamma, B, A, \Gamma' \vdash C}$$

Proof Theory of $\otimes$

Left and right rules for $\otimes$

Sequent calculus: $\Gamma \vdash A$

$$A \vdash A \qquad \otimes\mathrm{R}\ \frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \otimes B} \qquad \otimes\mathrm{L}\ \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}$$

$$\cancel{\mathrm{Weakn}\ \frac{\Gamma \vdash C}{\Gamma, A \vdash C}} \qquad \cancel{\mathrm{Contr}\ \frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C}} \qquad \mathrm{eXchg}\ \frac{\Gamma, A, B, \Gamma' \vdash C}{\Gamma, B, A, \Gamma' \vdash C}$$

Proof Theory of $\otimes$

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$$A \vdash A \qquad \otimes\mathrm{R}\ \frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \otimes B} \qquad \otimes\mathrm{L}\ \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}$$

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No weakening or contraction

Adding the Unit

A proof theory with just $\otimes$ is not fun.
We need multiplicative unit (1) to signify absence of resources:

$$\emptyset \vdash 1 \qquad\qquad \frac{\Gamma \vdash C}{\Gamma, 1 \vdash C}$$

Adding implication

A proof theory with just $\otimes, 1$ is not fun. We need linear implication ($\multimap$, *lollipop*) to form linear hypotheticals:

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B} \qquad\qquad \frac{}{\Gamma_1, \Gamma_2, A \multimap B \vdash C}$$

Adding implication

A proof theory with just $\otimes, 1$ is not fun. We need linear implication ($\multimap$, *lollipop*) to form linear hypotheticals:

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B} \qquad\qquad \frac{\Gamma_1 \vdash A \qquad \Gamma_2, B \vdash C}{\Gamma_1, \Gamma_2, A \multimap B \vdash C}$$

Exercise: try to derive $(A \otimes B) \multimap C \dashv\vdash A \multimap (B \multimap C)$

Cut rule

$$\dfrac{\Gamma \vdash A \qquad \Gamma', A \vdash B}{\Gamma, \Gamma' \vdash B}\;\text{Cut}$$

What is so special about this rule?

Each cut rule introduces a piece of “complexity” as a new formula A

Cut rule

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What is so special about this rule?
Each cut rule introduces a piece of "complexity" as a new formula A

Theorem

Every proof of $\Gamma \vdash A$ that uses cut can be converted into a proof that does not use cut

MILL

$$A, B \quad ::= \quad 1 \mid A \otimes B \mid A \multimap B$$

$$A \vdash A \qquad\qquad \emptyset \vdash 1 \qquad\qquad \otimes\mathsf{R}\ \frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \otimes B} \qquad\qquad \otimes\mathsf{L}\ \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}$$

$$\mathsf{Cut}\ \frac{\Gamma \vdash A \qquad \Gamma', A \vdash B}{\Gamma, \Gamma' \vdash B} \qquad\qquad \multimap\mathsf{R}\ \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B} \qquad\qquad \multimap\mathsf{L}\ \frac{\Gamma_1 \vdash A \qquad \Gamma_2, B \vdash C}{\Gamma_1, \Gamma_2, A \multimap B \vdash C}$$

Adding &

Recall the different versions of left/right rules for $\wedge$

$$\wedge\mathsf{R}\ \frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \wedge B}$$

$$\wedge\mathsf{R'}\ \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

$$\wedge\mathsf{L}\ \frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}$$

$$\wedge\mathsf{L}_i\ \frac{\Gamma, A_i \vdash C}{\Gamma, A_1 \wedge A_2 \vdash C}$$

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Recall the different versions of left/right rules for $\wedge$

$$\otimes\mathrm{R}\ \frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \otimes B}$$

$$\&\mathrm{R}\ \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \,\&\, B}$$

$$\otimes\mathrm{L}\ \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}$$

$$\&\mathrm{L}_i\ \frac{\Gamma, A_i \vdash C}{\Gamma, A_1 \,\&\, A_2 \vdash C}$$

Interplay of $\otimes$ and $\&$

- $A \,\&\, B \vdash A$, and $A \,\&\, B \vdash B$
- $A \,\&\, B \not\dashv\vdash A \otimes B$
- $(A \multimap B) \,\&\, (A \multimap C) \vdash A \multimap B \,\&\, C$
- $(A \,\&\, B) \multimap C \nvdash A \multimap (B \multimap C)$

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Examples

Menu at the Linear Logic Cafe

Summer Menu! €15 p.p.
Starter: Greek Salad, or Soup
Main: Chicken Schnitzel, Tofu, or Salmon (for additional €2)
(all main dishes come with a side of fries)
Desert: Ice Cream, or Cheese Platter
Drinks: Leffe Tripel (for additional €3)

$$E^{15} \multimap \left(\qquad \right)$$

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$$E^{15} \multimap \left(\begin{array}{l} (Sal \mathbin{\&} Soup) \otimes \\ \big((Schn \mathbin{\&} Tofu \mathbin{\&} (E \otimes E \multimap Fish)) \\ \\ \\ \\ \end{array} \right)$$

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Grandma's Linear Logic Pizza recipe

LL Pizza Ingredients:

- Yeast and flour,
- Salt and sugar
- Tomatoes
- Sausage or paprika

$$\left(\qquad \qquad \qquad \right) \multimap Pizza$$

Grandma's Linear Logic Pizza recipe

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- Tomatoes
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$$\left(\mathit{Yeast} \otimes \mathit{Flour} \otimes \qquad \right) \multimap \mathit{Pizza}$$

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- Sausage or paprika

$$\left(\begin{array}{l} \mathit{Yeast} \otimes \mathit{Flour} \otimes \\ \\ \mathit{Salt} \otimes \mathit{Sugar} \otimes \\ \\ \mathit{Tomatoes} \otimes \\ \\ \end{array}\right) \multimap \mathit{Pizza}$$

Grandma's Linear Logic Pizza recipe

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- Yeast and flour,
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- Tomatoes
- Sausage or paprika

$$\begin{pmatrix} \mathit{Yeast} \otimes \mathit{Flour} \otimes \\ \\ \mathit{Salt} \otimes \mathit{Sugar} \otimes \\ \\ \mathit{Tomatoes} \otimes \\ \\ (\mathit{Sausage}\ \textcolor{red}{\text{or}}\ \mathit{Paprika}) \end{pmatrix}$$
$$\multimap \mathit{Pizza}$$

Grandma's Linear Logic Pizza recipe

LL Pizza Ingredients:

- Yeast and flour,
- Salt and sugar
- Tomatoes
- Sausage or paprika

$$\begin{pmatrix} \mathit{Yeast} \otimes \mathit{Flour} \otimes \\ \\ \mathit{Salt} \otimes \mathit{Sugar} \otimes \\ \\ \mathit{Tomatoes} \otimes \\ \\ (\mathit{Sausage} \oplus \mathit{Paprika}) \end{pmatrix} \multimap \mathit{Pizza}$$

Adding $\oplus$

$$\oplus\mathsf{R}_1 \quad \frac{\Gamma \vdash A_1}{\Gamma \vdash A_1 \oplus A_2} \qquad\qquad \oplus\mathsf{R}_2 \quad \frac{\Gamma \vdash A_2}{\Gamma \vdash A_1 \oplus A_2}$$

$$\oplus\mathsf{L} \quad \frac{\Gamma, A_1 \vdash C \qquad \Gamma, A_2 \vdash C}{\Gamma, A_1 \oplus A_2 \vdash C}$$

$\oplus$ vs $\&$

The rules are the same as for $\vee$ in intuitionistic sequent calculus.

$\oplus$ vs $\&$

The rules are the same as for $\vee$ in intuitionistic sequent calculus.

However,

$$A \mathbin{\&} (B \oplus C) \nvdash (A \mathbin{\&} B) \oplus (A \mathbin{\&} C)$$

- A = one euro;
- B = tea, C = coffee;
- $(B \oplus C)$ = either tea or coffee, but do not know which;
- $A \mathbin{\&} (B \oplus C)$ = I can either have a euro or a beverage

$\oplus$ vs $\&$

$$tea \oplus coffee \vdash awake$$

$$tea \mathbin{\&} coffee \vdash awake$$

$\oplus$ vs $\&$

$tea \oplus coffee \vdash awake$

Given either tea or coffee (I don't care which one), I can drink it and get awake

$tea \,\&\, coffee \vdash awake$

$\oplus$ vs $\&$

$tea \oplus coffee \vdash awake$

Given either tea or coffee (I don't care which one), I can drink it and get awake

$tea \mathbin{\&} coffee \vdash awake$

Given a choice of tea and coffee (for example at a hotel breakfast), I can drink a hot beverage and get awake

$\oplus$ vs $\&$

$$tea \oplus coffee \vdash awake$$

Given either tea or coffee (I don't care which one), I can drink it and get awake

$$tea \mathbin{\&} coffee \vdash awake$$

Given a choice of tea and coffee (for example at a hotel breakfast), I can drink a hot beverage and get awake

$$tea \mathbin{\&} coffee \vdash tea \oplus coffee$$

$\oplus$ vs &

$tea \oplus coffee \vdash awake$

Given either tea or coffee (I don't care which one), I can drink it and get awake

$tea \,\&\, coffee \vdash awake$

Given a choice of tea and coffee (for example at a hotel breakfast), I can drink a hot beverage and get awake

$tea \,\&\, coffee \vdash tea \oplus coffee$

$A \,\&\, B \vdash A \oplus B$ $\quad\quad$ $A \oplus B \nvdash A \,\&\, B$

MAILL

$$A, B \quad ::= \; 1 \mid A \otimes B \mid A \multimap B \mid A \,\&\, B \mid A \oplus B$$

$$A \vdash A \qquad \emptyset \vdash 1 \qquad \frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \otimes B}\,{\otimes}\mathsf{R} \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\,{\otimes}\mathsf{L}$$

$$\frac{\Gamma \vdash A \qquad \Gamma', A \vdash B}{\Gamma, \Gamma' \vdash B}\,\mathsf{Cut} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\,{\multimap}\mathsf{R} \qquad \frac{\Gamma_1 \vdash A \qquad \Gamma_2, B \vdash C}{\Gamma_1, \Gamma_2, A \multimap B \vdash C}\,{\multimap}\mathsf{L} \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \,\&\, B}\,\&\mathsf{R}$$

$$\frac{\Gamma, A_i \vdash C}{\Gamma, A_1 \,\&\, A_2 \vdash C}\,\&\mathsf{L}_i \qquad \frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \oplus A_2}\,{\oplus}\mathsf{R}_i \qquad \frac{\Gamma, A_1 \vdash C \qquad \Gamma, A_2 \vdash C}{\Gamma, A_1 \oplus A_2 \vdash C}\,{\oplus}\mathsf{L}$$

Related Logics

- Classical Linear Logic

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- Relevance logics
  - Contraction but no weakening
  - Intuition: all hypothesis are relevant and must be used

Related Logics

- Classical Linear Logic
- Relevance logics
  - Contraction but no weakening
  - Intuition: all hypothesis are relevant and must be used
- Logic of Bunched Implications
  - Better known as the kernel of Separation Logic
  - Popular in deductive program verification

Taking Stock

Up to here:

- Correctness for communicating programs
- Sessions as protocol specifications
- A formal syntax for session types
- Example: A two-buyer protocol
- Protocol compatibility in terms of duality for session types
- **Intuitionistic Linear Logic (MAILL)**

Tomorrow:

- Interpreting ILL as session types for message-passing processes

Session Types

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Dutch Winter School 2026
(Part 1.2)