Algorithms for Longest Common Extensions

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The LCE Problem and the DirectComp algorithm

Input

- $s = abbababba$
- $(i, j) = (4, 6)$

The DirectComp algorithm

$s =$	a	b	b	a	b	a	b	b	a
				↑		↑			
				4		6			

The LCE Problem and the DirectComp algorithm

Input

- $s = abbababba$
- $(i, j) = (4, 6)$

The DirectComp algorithm

$s =$	a	b	b	a	b	a	b	b	a
				4		6			

1 match

The LCE Problem and the DirectComp algorithm

Input

- $s = abbababba$
- $(i, j) = (4, 6)$

The DirectComp algorithm

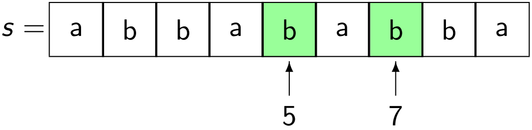
$s =$	a	b	b	a	b	a	b	b	a	1 match
					↑		↑			
					5		7			

The LCE Problem and the DirectComp algorithm

Input

- $s = abbababba$
- $(i, j) = (4, 6)$

The DirectComp algorithm



2 matches

The LCE Problem and the DIRECTCOMP algorithm

Input

- $s = abbababba$
- $(i, j) = (4, 6)$

The DIRECTCOMP algorithm

$s =$	a	b	b	a	b	a	b	b	a
						↑		↑	
						6		8	

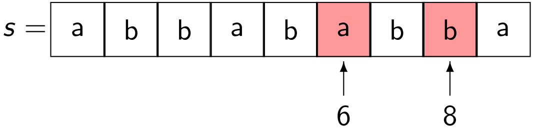
2 matches

The LCE Problem and the DirectComp algorithm

Input

- $s = abbababba$
- $(i, j) = (4, 6)$

The DirectComp algorithm



2 matches

The LCE Problem and the DirectComp algorithm

Input

- $s = abbababba$
- $(i, j) = (4, 6)$

The DirectComp algorithm

$s =$	a	b	b	a	b	a	b	b	a	2 matches

Result

$LCE_s(4, 6) = 2$

LCE problem

Efficiently perform multiple queries (i, j) on a static string s

Existing Algorithm: DirectComp

Preprocessing	$O(1)$
Space	$O(1)$
Query	$O(\lvert LCE(i,j)\rvert) = O(n)$
Average query	$O(1)$
Query I/O	$O\left(\frac{\lvert LCE(i,j)\rvert}{B}\right) = O\left(\frac{n}{B}\right)$

For a string length n and alphabet size σ, the average LCE value over all n^{σ} strings and n^2 query pairs is $O(1)$.

Existing Algorithm: LcpRmq

$$s = abbababba$$
$$s[2..n] = bbababba$$
$$s[3..n] = bababba$$
$$LCE_s(2,3) = 1$$

LCP[2..n]	$suff_{SA[1..n]}$
	a
1	ababba
2	abba
4	abbababba
0	ba
2	bababba
3	babba
1	bba
3	bbababba

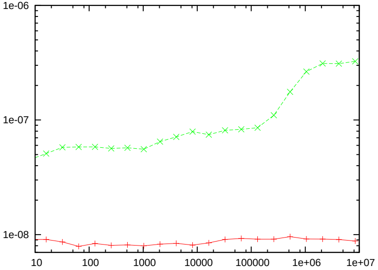
$$LCE(i,j) = LCP[RMQ_{LCP}(SA^{-1}[i]+1, SA^{-1}[j])]$$
where $SA^{-1}[i] < SA^{-1}[j]$

Preprocessing	$O(sort(n,\sigma))$
Space	$O(n)$
Query	$O(1)$
Average query	$O(1)$
Query I/O	$O(1)$

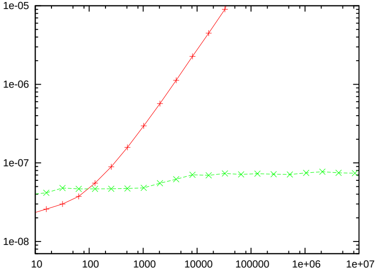
Existing Algorithms: Practical Results

Query times of DirectComp and LcpRmq by string length

Average case



Worst case



The $\textsc{Fingerprint}_k$ Algorithm: Data Structure

- For a string $s[1\,..\,n]$, the t-length fingerprints $F_t[1\,..\,n]$ are natural numbers, such that $F_t[i] = F_t[j]$ if and only if $s[i\,..\,i+t-1] = s[j\,..\,j+t-1]$.
- k levels, $1 \le k \le \lceil \log n \rceil$
- For each level, $\ell = 0\,..\,k-1$:
  - $t_\ell = \Theta(n^{\ell/k})$, $t_0 = 1$
  - $H_\ell = F_{t_\ell}$

$H_2[i]$	1	2	3	4	5	6	7	8	9	1	2	3	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
$H_1[i]$	1	2	3	4	1	2	5	6	5	1	2	3	4	1	2	5	6	5	6	3	4	6	5	6	7	8	9
$s = H_0[i]$	a	b	b	a	a	b	b	a	b	a	b	b	a	a	b	b	a	b	a	b	a	a	b	a	b	a	$
i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27

Space $\quad O(k \cdot n)$

The FINGERPRINT$_k$ Algorithm: Query

1. As long as $H_\ell[i+v] = H_\ell[j+v]$, increment v by t_ℓ, increment ℓ by one, and repeat this step unless and $\ell = k-1$.
2. As long as $H_\ell[i+v] = H_\ell[j+v]$, increment v by t_ℓ and repeat this step.
3. Stop and return v when $\ell = 0$, otherwise decrement ℓ by one and go to step two.

$H_2[i+v]$	1	2	3	4	5	6	7	8	9	1	2	3	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
$H_1[i+v]$	1	2	3	4	1	2	5	6	5	1	2	3	4	1	2	5	6	5	6	3	4	6	5	6	7	8	9
$H_0[i+v]$	a	b	b	a	a	b	b	a	b	a	b	b	a	a	b	b	a	b	a	b	a	a	b	a	b	a	$
$i+v$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27

$H_2[j+v]$	1	2	3	4	5	6	7	8	9	1	2	3	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
$H_1[j+v]$	1	2	3	4	1	2	5	6	5	1	2	3	4	1	2	5	6	5	6	3	4	6	5	6	7	8	9
$H_0[j+v]$	a	b	b	a	a	b	b	a	b	a	b	b	a	a	b	b	a	b	a	b	a	a	b	a	b	a	$
$j+v$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27

$$LCE(3,12) = 9$$

Query	$O(k \cdot n^{1/k})$
Average query	$O(1)$

The FINGERPRINT$_k$ Algorithm: Preprocessing

- For each level ℓ
  - For each t_ℓ-length substring in lexicographically sorted order
    - If the current substring $s[SA[i] \,..\, SA[i] + t_\ell - 1]$ is equal to the previous substring, give it the same fingerprint as the previous substring, otherwise give it a new unused fingerprint. The two substrings are equal when $LCE[i] \geq t_\ell$.

$s = $ abbababba

For $t_\ell = 3$:

$H_\ell = [3, 8, 6, 2, 6, 3, 8, 5, 1]$

Subst.	$H_l[i]$	i
a	1	9
aba	2	4
abb	3	6
abb	3	1
ba	5	8
bab	6	3
bab	6	5
bba	8	7
bba	8	2

Preprocessing $\quad O(k \cdot n + sort(n, \sigma))$

The $\textsc{Fingerprint}_k$ Algorithm: I/O

- Original:
  - Data structure: $H_\ell[i] = F_{t_\ell}[i]$
  - Size: $|H_\ell| = n$
  - I/O: $O\left(k \cdot n^{1/k}\right)$
- Cache optimized:
  - Data structure:
    $H_\ell[((i-1) \mod t_\ell) \cdot \lceil n/t_\ell \rceil + \lfloor (i-1)/t_\ell \rfloor + 1] = F_{t_\ell}[i]$
  - Size: $|H_\ell| = n + t_\ell$
  - I/O: $O\left(k \cdot \left(\frac{n^{1/k}}{B} + 1\right)\right)$
    - Best when k is small $\implies$ $n^{1/k}$ is large.

$H_2[i+v]$	1	2	3	4	5	6	7	8	9	1	2	3	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
$H_1[i+v]$	1	2	3	4	1	2	5	6	5	1	2	3	4	1	2	5	6	5	6	3	4	6	5	6	7	8	9
$H_0[i+v]$	a	b	b	a	a	b	b	a	b	a	b	b	a	a	b	b	a	b	a	b	a	a	b	a	b	a	$
$i+v$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27

$H_2[j+v]$	1	2	3	4	5	6	7	8	9	1	2	3	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
$H_1[j+v]$	1	2	3	4	1	2	5	6	5	1	2	3	4	1	2	5	6	5	6	3	4	6	5	6	7	8	9
$H_0[j+v]$	a	b	b	a	a	b	b	a	b	a	b	b	a	a	b	b	a	b	a	b	a	a	b	a	b	a	$
$j+v$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27

The $\textsc{Fingerprint}_k$ Algorithm

Preprocessing	$O(k \cdot n + sort(n, \sigma))$
Space	$O(k \cdot n)$
Query	$O(k \cdot n^{1/k})$
Average query	$O(1)$
Query I/O	$O\left(k \cdot \left(\frac{n^{1/k}}{B} + 1\right)\right)$

$k = 1$:

Same as $\textsc{DirectComp}$

$k = 2$:

Preprocessing	$O(sort(n, \sigma))$
Space	$O(n)$
Query	$O(\sqrt{n})$
Average query	$O(1)$
Query I/O	$O\left(\frac{\sqrt{n}}{B}\right)$

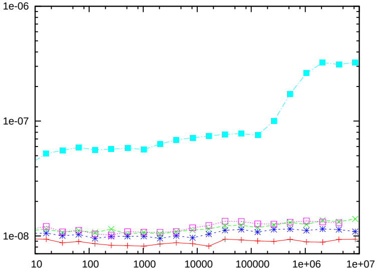
$k = \lceil \log n \rceil$:

Preprocessing	$O(n \log n)$
Space	$O(n \log n)$
Query	$O(\log n)$
Average query	$O(1)$
Query I/O	$O(\log n)$

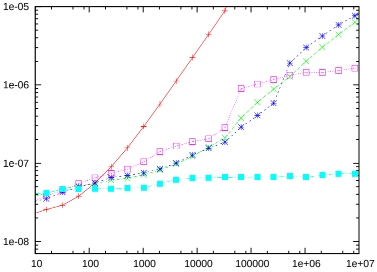
Practical Results

Query times of DirectComp, Fingerprint$_2$ (cache opt.), Fingerprint$_3$ (not cache opt.), Fingerprint$_{\lceil \log n \rceil}$ (not cache opt.) and LcpRmq by string length

Average case

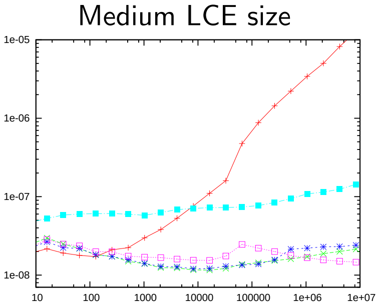


Worst case



Practical Results

Query times of DIRECTCOMP, FINGERPRINT$_2$ (cache opt.), FINGERPRINT$_3$ (not cache opt.), FINGERPRINT$_{\lceil \log n \rceil}$ (not cache opt.) and LCPRMQ by string length



Cache Optimization, Practical Results

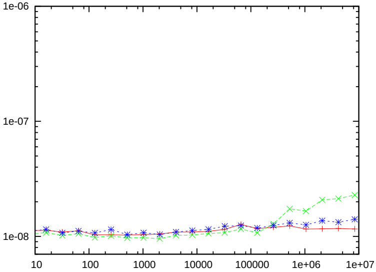
Is I/O optimization good in practice?

- Pro: better cache efficiency
  - Best for small k, no change for $k = \lceil \log n \rceil$
- Con: Calculating memory addresses is more complicated
  - $((i-1) \mod t_\ell) \cdot \lceil n/t_\ell \rceil + \lfloor (i-1)/t_\ell \rfloor + 1$ vs. i

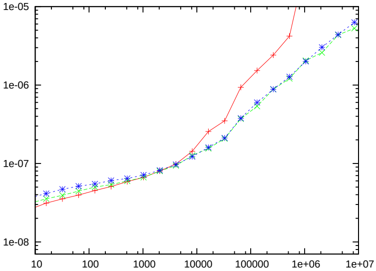
The FINGERPRINT$_k$ Algorithm: Practical Results, I/O

Query times of FINGERPRINT$_2$ without cache optimization and with cache optimization using shift operations vs. multiplication and division

Average case



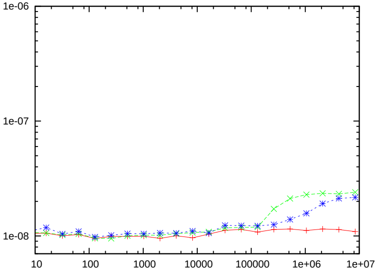
Worst case



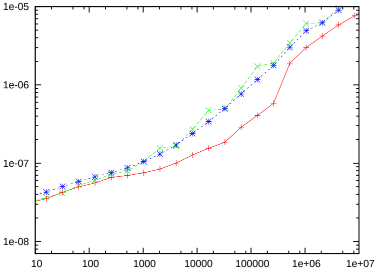
The FINGERPRINT$_k$ Algorithm: Practical Results, I/O

Query times of FINGERPRINT$_3$ without cache optimization and with cache optimization using shift operations vs. multiplication and division

Average case

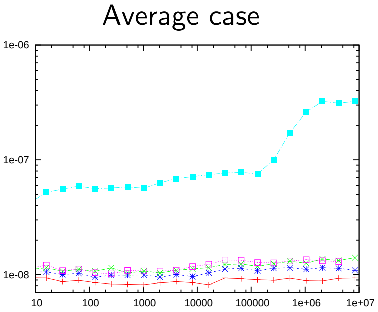


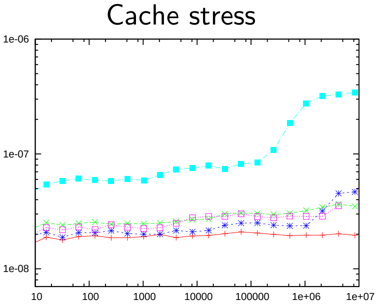
Worst case



The FINGERPRINT_k Algorithm: Practical Results, I/O

Query times of DIRECTCOMP, FINGERPRINT_2 (cache opt.), FINGERPRINT_3 (not cache opt.), $\text{FINGERPRINT}_{\lceil \log n \rceil}$ (not cache opt.) and LCPRMQ by string length





LCE on Compressed Strings

Goal

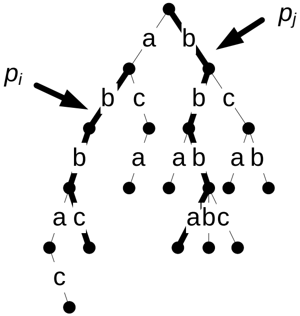
- Allow LCE queries without decompressing the string
- Using Ziv-Lempel compression (LZ)

How

- LZ compression represents the string as a tree
- An LCE query on a LZ compressed string is a number of LCE queries on a tree

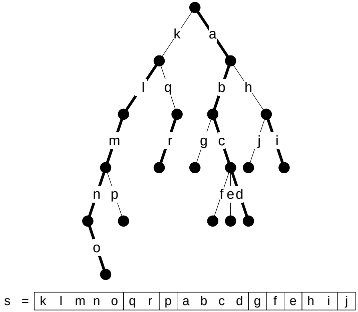
LCE on Trees

- Trees:
  - One character on each edge
  - LCE is the length of the longest common prefix of two strings along two paths
- $p_i = bbc$ and $p_j = bbba$ gives $LCE(p_i, p_j) = 2$



Constant Time String LCE on Heavy Paths

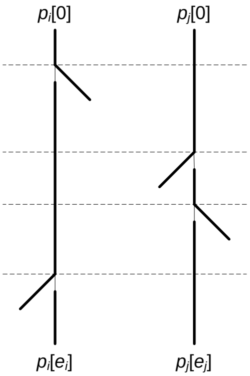
- Data structure:
  - Construct a heavy tree decomposition
  - For each heavy path, store characters as a substring
- Query:
  - Use constant time string LCE on each heavy path



Preprocessing	$O(sort(n, \sigma))$
Space	$O(n)$
Query	$O(\log n)$

Constant Time String LCE on Heavy Paths

- How to find the indexes (i, j) in the string:
  - Store an index at each node
- How to know then the heavy path splits from the queried path:
  - Store a pointer to the end of the heavy path at each node
  - Find NCA of the end of the heavy path and the end of the queried path
- How to find a node on the queried path:
  - Use level ancestor



Summary

	DirectComp	LcpRmq / SuffixNca	Fingerprint$_k$
Preprocessing	$O(1)$	$O(sort(n,\sigma))$	$O(k \cdot n + sort(n,\sigma))$
Space	$O(1)$	$O(n)$	$O(k \cdot n)$
Query	$O(n)$	$O(1)$	$O(k \cdot n^{1/k})$
Average query	$O(1)$	$O(1)$	$O(1)$
Query I/O	$O\left(\frac{n}{B}\right)$	$O(1)$	$O\left(k \cdot \left(\frac{n^{1/k}}{B} + 1\right)\right)$

- In practice, the Fingerprint$_k$ algorithm is...
  - ...almost as good as DirectComp and significantly better than LcpRmq in average case
  - ...significantly better than DirectComp but worse than LcpRmq in worst case
- Cache optimization of Fingerprint$_k$ improves query times at $k = 2$ and worsens query times at $k \geq 3$

Kommentarer til rapporten

- Hvordan jeg fandt frem til $r = 0.73n^{0.42}$
- Der står $\textsc{Fingerprint}_3$ nogle steder i cache-afsnittet hvor der skal stå $\textsc{Fingerprint}_2$