Algorithms for Longest Common Extensions

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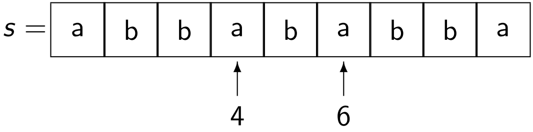
Contents

The DIRECTCOMP algorithm

Input

- $s = abbababba$
- $(i, j) = (4, 6)$

The DIRECTCOMP algorithm



The DIRECTCOMP algorithm

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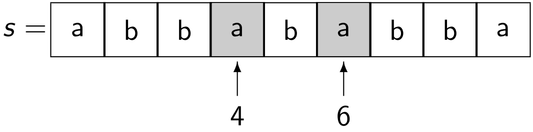
- $(i, j) = (4, 6)$

The DIRECTCOMP algorithm



The DIRECTCOMP algorithm

Input

- $s = abbababba$
- $(i, j) = (4, 6)$

The DIRECTCOMP algorithm

$s =$	a	b	b	a	b	a	b	b	a
				4		6			

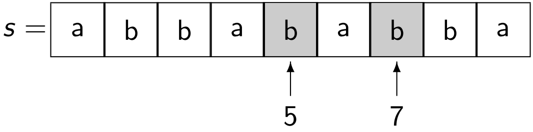
1 match

The DirectComp algorithm

Input

- $s = abbababba$
- $(i, j) = (4, 6)$

The DirectComp algorithm



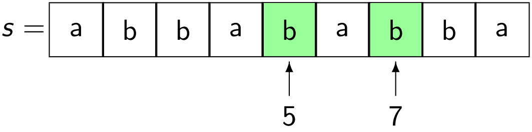
1 match

The DIRECTCOMP algorithm

Input

- $s = abbababba$
- $(i, j) = (4, 6)$

The DIRECTCOMP algorithm



2 matches

The DirectComp algorithm

Input

- $s = abbababba$
- $(i, j) = (4, 6)$

The DirectComp algorithm

$s =$	a	b	b	a	b	a	b	b	a
						6		8	

2 matches

The DirectComp algorithm

Input

- $s = abbababba$
- $(i, j) = (4, 6)$

The DirectComp algorithm

$s =$	a	b	b	a	b	a	b	b	a
						6		8	

2 matches

Result

$LCE_s(4, 6) = 2$

The LCE Problem

LCE value $LCE_s(i, j)$ is the length of the longest common prefix of the two suffixes of s starting at index i and j

LCE problem Efficiently query multiple LCE values on a static string s and varying pairs (i, j)

Existing Algorithm: DirectComp

Preprocessing	None
Space	$O(1) + \|s\|$
Query	$O(\|LCE(i,j)\|) = O(n)$
Average query	$O(1)$

For a string length n and alphabet size σ, the average LCE value over all n^σ strings and n^2 query pairs is $O(1)$.

References

L. Ilie, G. Navarro, and L. Tinta. The longest common extension problem revisited and applications to approximate string searching. *J. Disc. Alg.*, 8(4):418-428, 2010.

Existing Algorithms: SuffixNca and LcpRmq

Two algorithms with best known bounds:

SuffixNca	Nearest common ancestor queries on a suffix tree
LcpRmq	Range minimum queries on a longest common prefix array

Preprocessing	$O(sort(n, \sigma))$
Space	$O(n)$
Query	$O(1)$
Average query	$O(1)$

References

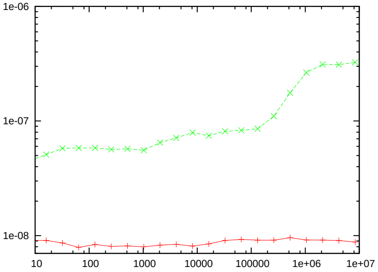
J. Fischer, and V. Heun. Theoretical and Practical Improvements on the RMQ-Problem, with Applications to LCA and LCE. In *Proc. 17th CPM*, pages 36-48, 2006.

D. Harel, R. E. Tarjan. Fast Algorithms for Finding Nearest Common Ancestors. *SIAM J. Comput.*, 13(2):338-355, 1984.

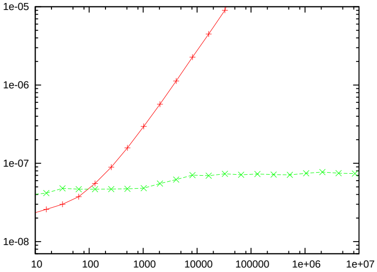
Existing Algorithms: Practical Results

Query times of DirectComp and LcpRmq by string length

Average case



Worst case



The FINGERPRINT$_k$ Algorithm: Data Structure

- For a string $s[1\mathinner{.\,.}n]$, the t-length fingerprints $F_t[1\mathinner{.\,.}n]$ are natural numbers, such that $F_t[i] = F_t[j]$ if and only if $s[i\mathinner{.\,.}i+t-1] = s[j\mathinner{.\,.}j+t-1]$.
- k levels, $1 \le k \le \lceil \log n \rceil$
- For each level, $\ell = 0\mathinner{.\,.}k-1$:
  - $t_\ell = \Theta(n^{\ell/k})$, $t_0 = 1$
  - $H_\ell = F_{t_\ell}$

$H_2[i]$	1	2	3	4	5	6	7	8	9	1	2	3	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
$H_1[i]$	1	2	3	4	1	2	5	6	5	1	2	3	4	1	2	5	6	5	6	3	4	6	5	6	7	8	9
$s = H_0[i]$	a	b	b	a	a	b	b	a	b	a	b	b	a	a	b	b	a	b	a	b	a	a	b	a	b	a	$
i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27

Space $O(k \cdot n)$

The FINGERPRINT$_k$ Algorithm: Query

1. As long as $H_\ell[i+v] = H_\ell[j+v]$, increment v by t_ℓ, increment ℓ by one, and repeat this step unless and $\ell = k-1$.
2. As long as $H_\ell[i+v] = H_\ell[j+v]$, increment v by t_ℓ and repeat this step.
3. Stop and return v when $\ell = 0$, otherwise decrement ℓ by one and go to step two.

$H_2[i+v]$	1	2	3	4	5	6	7	8	9	1	2	3	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
$H_1[i+v]$	1	2	3	4	1	2	5	6	5	1	2	3	4	1	2	5	6	5	6	3	4	6	5	6	7	8	9
$H_0[i+v]$	a	b	b	a	a	b	b	a	b	a	b	b	a	a	b	b	a	b	a	b	a	a	b	a	b	a	$
$i+v$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27

$H_2[j+v]$	1	2	3	4	5	6	7	8	9	1	2	3	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
$H_1[j+v]$	1	2	3	4	1	2	5	6	5	1	2	3	4	1	2	5	6	5	6	3	4	6	5	6	7	8	9
$H_0[j+v]$	a	b	b	a	a	b	b	a	b	a	b	b	a	a	b	b	a	b	a	b	a	a	b	a	b	a	$
$j+v$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27

$$LCE(3, 12) = 9$$

Query	$O(k \cdot n^{1/k})$
Average query	$O(1)$

The FINGERPRINT$_k$ Algorithm: I/O

- Original:
  - Data structure: $H_\ell[i] = F_{t_\ell}[i]$
  - Size: $|H_\ell| = n$
  - I/O: $O\big(k \cdot n^{1/k}\big)$
- Cache optimized:
  - Data structure:
    $H_\ell[((i-1) \mod t_\ell) \cdot \lceil n/t_\ell \rceil + \lfloor (i-1)/t_\ell \rfloor + 1] = F_{t_\ell}[i]$
  - Size: $|H_\ell| = n + t_\ell$
  - I/O: $O\Big(k \cdot \Big(\frac{n^{1/k}}{B} + 1\Big)\Big)$
    - Best when k is small $\implies$ $n^{1/k}$ is large.

$H_2[i+v]$	1	2	3	4	5	6	7	8	9	1	2	3	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
$H_1[i+v]$	1	2	3	4	1	2	5	6	5	1	2	3	4	1	2	5	6	5	6	3	4	6	5	6	7	8	9
$H_0[i+v]$	a	b	b	a	a	b	b	a	b	a	b	b	a	a	b	b	a	b	a	b	a	a	b	a	b	a	$
$i+v$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27

$H_2[j+v]$	1	2	3	4	5	6	7	8	9	1	2	3	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
$H_1[j+v]$	1	2	3	4	1	2	5	6	5	1	2	3	4	1	2	5	6	5	6	3	4	6	5	6	7	8	9
$H_0[j+v]$	a	b	b	a	a	b	b	a	b	a	b	b	a	a	b	b	a	b	a	b	a	a	b	a	b	a	$
$j+v$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27

The $\textsc{Fingerprint}_k$ Algorithm

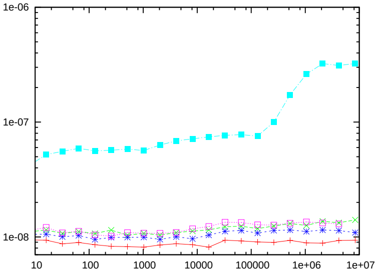
	$1 \leq k \leq \lceil \log n \rceil$
Preprocessing	$O(k \cdot n + sort(n, \sigma))$
Space	$O(k \cdot n)$
Query	$O(k \cdot n^{1/k})$
Average query	$O(1)$
Query I/O	$O\left(k \cdot \left(\frac{n^{1/k}}{B} + 1\right)\right)$

	$k = 1$	$k = 2$	$k = \lceil \log n \rceil$
Preprocessing	$O(sort(n, \sigma))$	$O(sort(n, \sigma))$	$O(n \log n)$
Space	$O(n)$	$O(n)$	$O(n \log n)$
Query	$O(n)$	$O(\sqrt{n})$	$O(\log n)$
Average query	$O(1)$	$O(1)$	$O(1)$
Query I/O	$O\left(\frac{n}{B}\right)$	$O\left(\frac{\sqrt{n}}{B}\right)$	$O(\log n)$

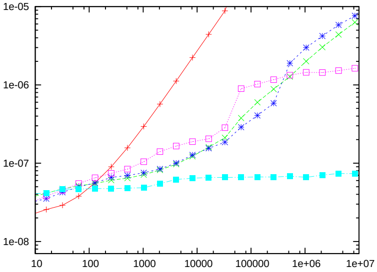
Practical Results

Query times of DirectComp, Fingerprint$_2$ (cache opt.), Fingerprint$_3$ (not cache opt.), Fingerprint$_{\lceil \log n \rceil}$ (not cache opt.) and LcpRmq by string length

Average case

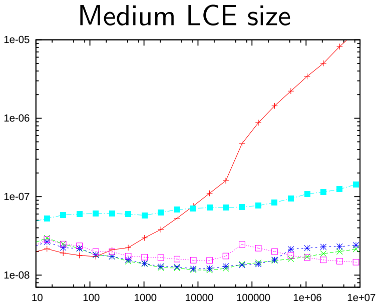


Worst case



Practical Results

Query times of DIRECTCOMP, FINGERPRINT$_2$ (cache opt.), FINGERPRINT$_3$ (not cache opt.), FINGERPRINT$_{\lceil \log n \rceil}$ (not cache opt.) and LCPRMQ by string length



Cache Optimization, Practical Results

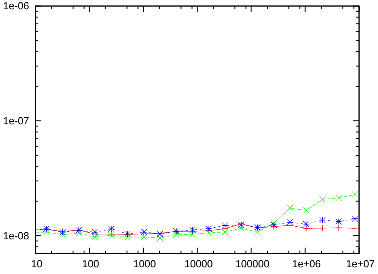
Is I/O optimization good in practice?

- Pro: better cache efficiency
  - Best for small k, no change for $k = \lceil \log n \rceil$
- Con: Calculating memory addresses is more complicated
  - $((i-1) \mod t_\ell) \cdot \lceil n/t_\ell \rceil + \lfloor (i-1)/t_\ell \rfloor + 1$ vs. i

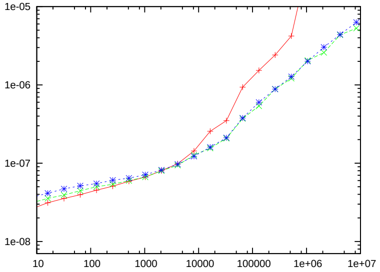
The $\textsc{Fingerprint}_k$ Algorithm: Practical Results, I/O

Query times of $\textsc{Fingerprint}_2$ without cache optimization and with cache optimization using shift operations vs. multiplication, division and modulo

Average case



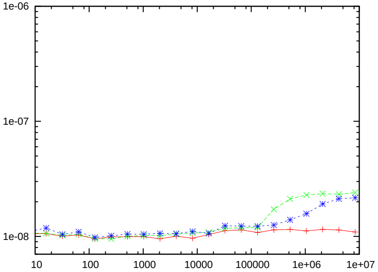
Worst case



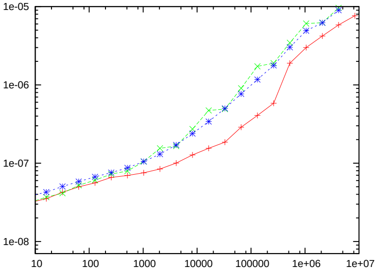
The FINGERPRINT_k Algorithm: Practical Results, I/O

Query times of FINGERPRINT_3 without cache optimization and with cache optimization using shift operations vs. multiplication, division and modulo

Average case



Worst case



LCE on Compressed Strings

Goal

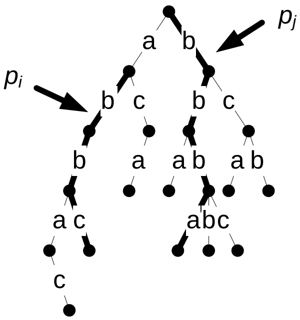
- Allow LCE queries without decompressing the string
- Using Ziv-Lempel compression (LZ)

How

- LZ compression represents the string as a tree
- An LCE query on a LZ compressed string is a number of LCE queries on a tree

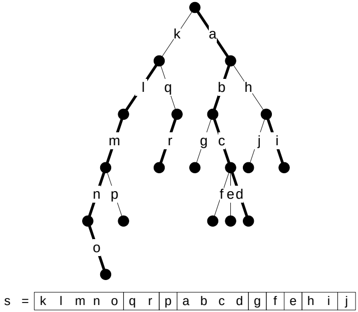
LCE on Trees

- Trees:
  - One character on each edge
  - LCE is the length of the longest common prefix of two strings along two paths
- $p_i = bbc$ and $p_j = bbba$ gives $LCE(p_i, p_j) = 2$



Constant Time String LCE on Heavy Paths

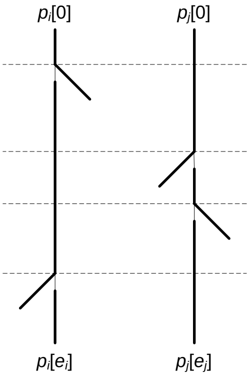
- Data structure:
  - Construct a heavy path decomposition
  - For each heavy path, store characters as a substring of *s*
- Query:
  - Use constant time string LCE on *s* for each heavy path



Preprocessing	$O(sort(n, \sigma))$
Space	$O(n)$
Query	$O(\log n)$

Constant Time String LCE on Heavy Paths

- How to find the indexes (i, j) in the string:
  - Store an index at each node
- How to know then the heavy path splits from the queried path:
  - Store a pointer to the end of the heavy path at each node
  - Find NCA of the end of the heavy path and the end of the queried path
- How to find a node on the queried path:
  - Use level ancestor



References

Michael A. Bendera, Martín Farach-Colton. The Level Ancestor Problem simplified. Theoretical Computer Science 321, 2004, pages 5-12.

Constant Time String LCE on Heavy Paths

Data structure:

- A **heavy path decomposition**
- A string s storing labels of each heavy path
  - Constant time **string LCE** on s
- **Pointers** from each node to
  - its index in s
  - the end of its heavy path
- Constant time **nearest common ancestor**
- Constant time **level ancestor**

Preprocessing	$O(sort(n, \sigma))$
Space	$O(n)$
Query	$O(\log n)$

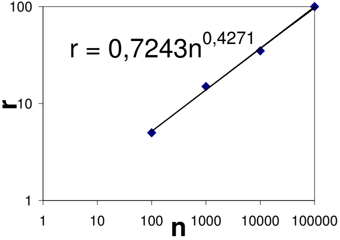
Summary

	DirectComp	LcpRmq / SuffixNca	Fingerprint$_k$
Preprocessing	$O(1)$	$O(sort(n,\sigma))$	$O(k \cdot n + sort(n,\sigma))$
Space	$O(1)$	$O(n)$	$O(k \cdot n)$
Query	$O(n)$	$O(1)$	$O(k \cdot n^{1/k})$
Average query	$O(1)$	$O(1)$	$O(1)$
Query I/O	$O\left(\frac{n}{B}\right)$	$O(1)$	$O\left(k \cdot \left(\frac{n^{1/k}}{B} + 1\right)\right)$

- In practice, the Fingerprint$_k$ algorithm is...
  - ...almost as good as DirectComp and significantly better than LcpRmq in average case
  - ...significantly better than DirectComp but worse than LcpRmq in worst case
- Cache optimization of Fingerprint$_k$ improves query times at $k = 2$ and worsens query times at $k \geq 3$
- We can generalize LCE to use on trees with $O(\log n)$ query time in linear space

Kommentarer og rettelser til rapporten

- Hvor kommer $r = 0.73n^{0.42}$ fra?



- Afsnit 4.5.3 Cache Optimization: $\textsc{Fingerprint}_3$ skal være $\textsc{Fingerprint}_2$