# Centrality

**Ec142 Applied Econometrics**

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# Centrality

Removal of which agent would reduce crime the most in a criminal network?

“Where” should a policy-maker introduce new technologies & innovations?

How do agent-specific shocks percolate through a network?

Merger analysis?

For many policy questions a measure of agent “centrality” is useful.

## Directed Networks (Recap)

For what follows it will be useful to extend our setup to accommodate directed networks or *digraphs*.

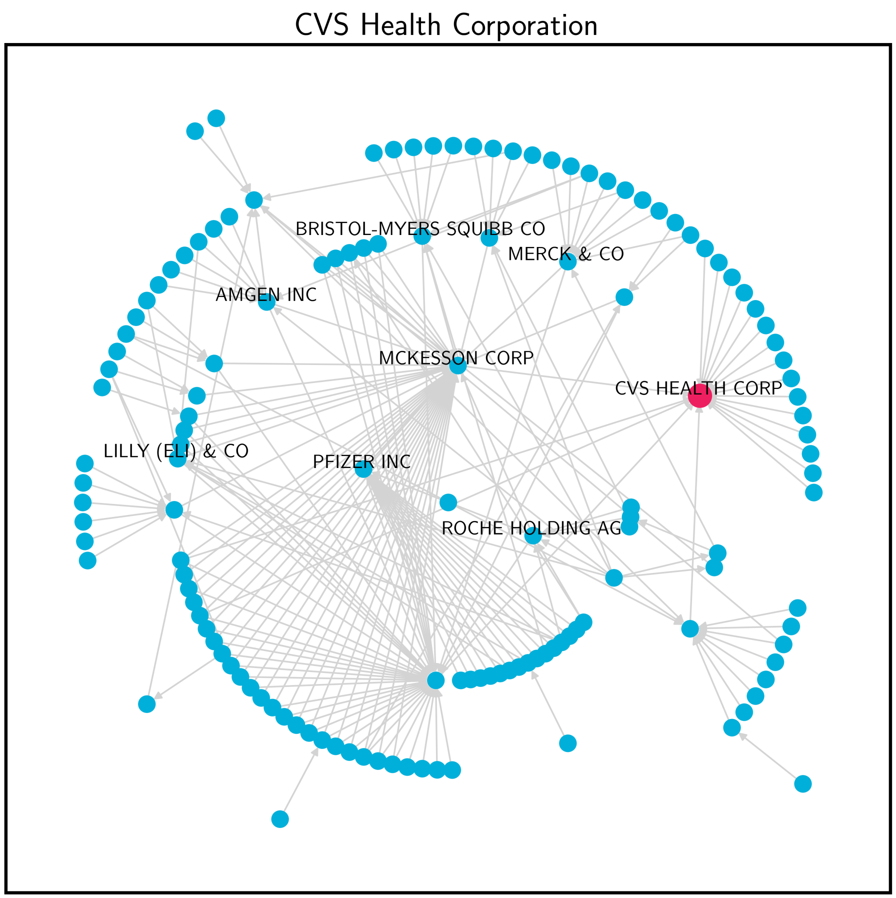
In directed networks all links have a specific *direction*.

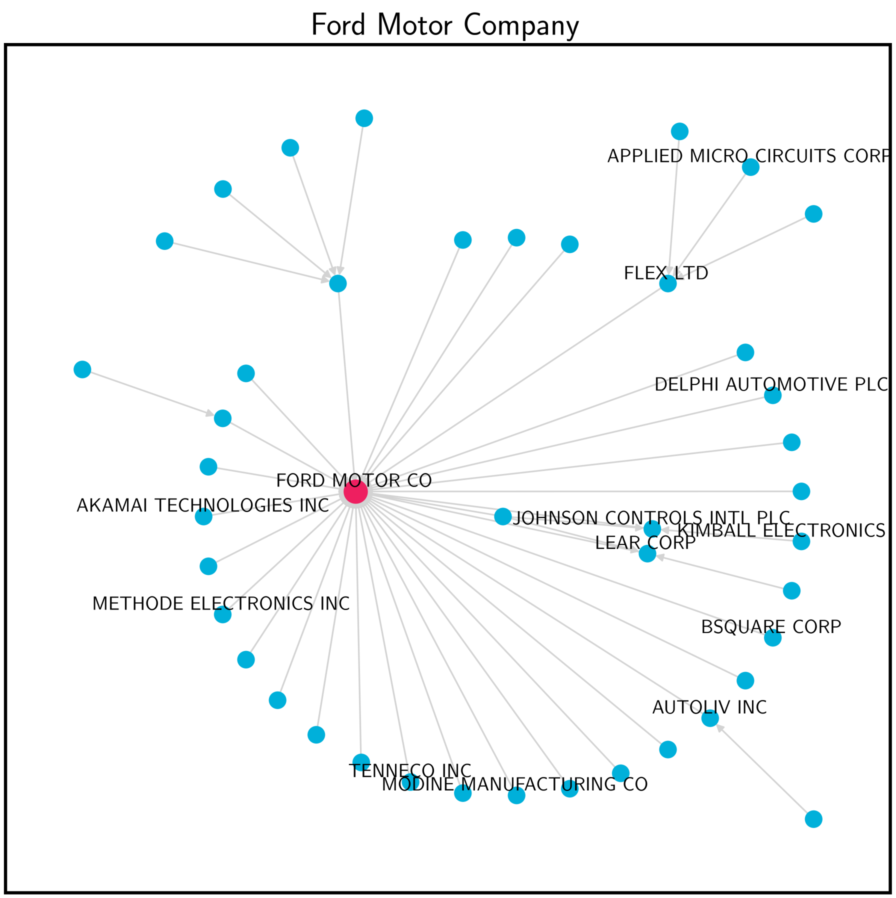
$i$ sends a link to $j$ and $j$ may or may not *reciprocate* by sending a link to $i$.

A canonical directed network is a *buyer-supplier network*.

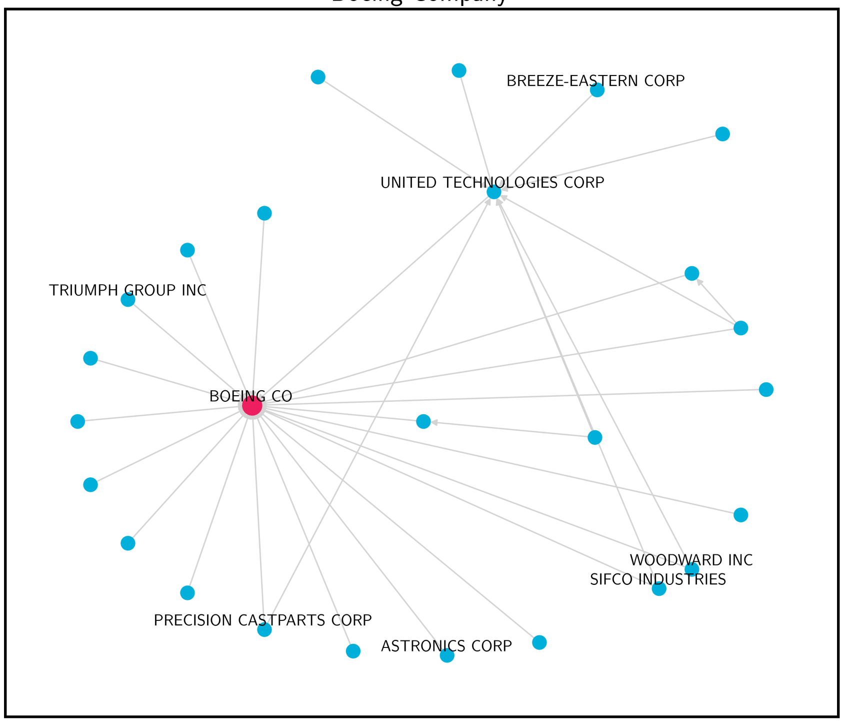
Firms (suppliers) sell inputs to other firms (buyers).

For example Hasbro sells to Walmart.

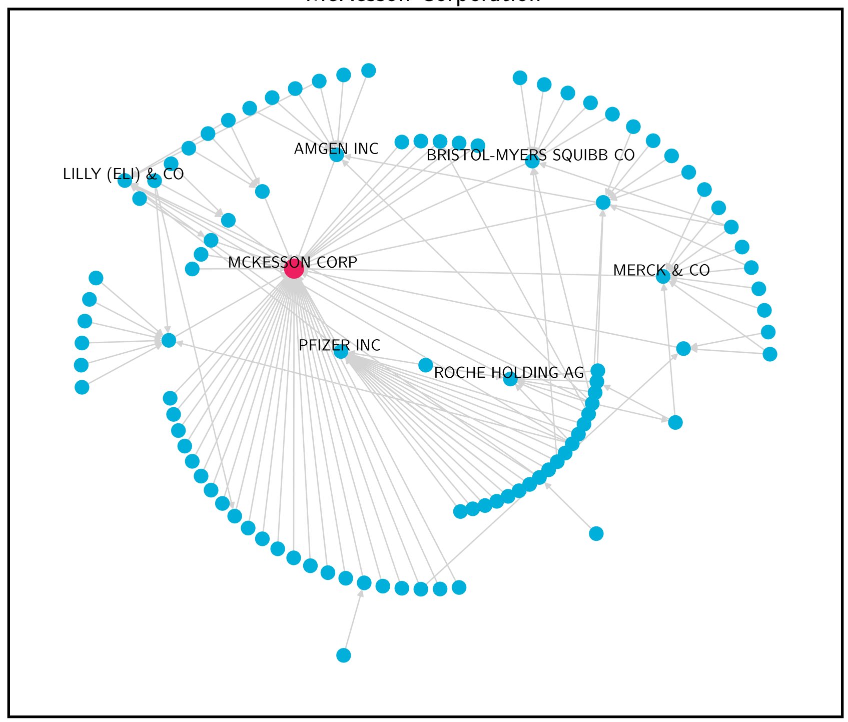

CVS Health Corporation
BRISTOL-MYERS SQUIBB CO
MERCK & CO
AMGEN INC
MCKESSON CORP
CVS HEALTH CORP
LILLY (ELI) & CO
PFIZER INC
ROCHE HOLDING AG


Ford Motor Company
APPLIED MICRO CIRCUITS CORP
FLEX LTD
DELPHI AUTOMOTIVE PLC
FORD MOTOR CO
AKAMAI TECHNOLOGIES INC
JOHNSON CONTROLS INTL PLC
KIMBALL ELECTRONICS
LEAR CORP
METHODE ELECTRONICS INC
BSQUARE CORP
AUTOLIV INC
TENNECO INC
MODINE MANUFACTURING CO

Boeing Company




McKesson Corporation

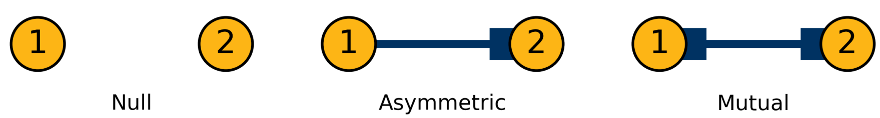


# <u>Directed Networks (continued)</u>

If United Technologies Corporation supplies inputs to Boeing Corporation then there exists a *directed edge*  *from* United Technologies *to* Boeing.

1. The supplying firm (left node) is called the *tail* of the edge.

2. The buying firm (right node) is its *head.*



## Directed Networks (continued)

Define $N \times N$ *adjacency matrix* $\mathbf{D} = \left[ D_{ij} \right]$ where

$$D_{ij} = \begin{cases} 1, & \{i, j\} \in \mathcal{E}(G) \\ 0, & \text{otherwise} \end{cases} . \tag{1}$$

Here $D_{ij} = 1$ if agent $i$ "sends" or "directs" a link to agent $j$ (and zero otherwise)...

...while $D_{ji} = 1$ if agent $j$ directs a link to $i$.

The adjacency matrix need not be symmetric...

...but self-links, or loops, are ruled-out, such that $D_{ii} = 0$ for all $i = 1, \ldots, N$.

## Directed Networks (continued)

If $i$ directs an edge to $j$...

...and $j$ likewise directs and edge back to $i$, we say the link is *reciprocated*.

## Directed Networks: Paths

In digraphs paths are directional (think of edges as one way or, under reciprocity, two way streets).

It may be possible to 'drive' from $i$ to $j$, but not vice-versa.

If there is a path from $i$ to $j$ *or* from $j$ to $i$ we say that $i$ and $j$ are *weakly connected.*

If paths are present in both directions, then they are *strongly connected.*

## Directed Networks: Paths

If all pairs of agents in a digraph are weakly (strongly) connected, then we say the digraph is weakly (strongly) connected.

Real world directed networks are rarely strongly connected, but typically include a giant weakly connected component.

For example the US Buyer-Supplier network consists of a giant weakly connect component which contains about 85 percent of all firms.

## Directed Networks: Centrality

There are two natural notions of centrality in a directed network.

1. Agents with high *indegree* are more central (prestige, popularity, buyers)

2. Agents with high *outdegree* are more central (extroverts, diffusers, suppliers)

## Indegree and Outdegree

The *indegree* of agent $i$ equals the number of arcs directed toward her, while her *outdegree* equals the number of arcs she directs toward other agents.

Indegree: $D_{+i} = \sum_j D_{ji}$, (column sums of $\mathbf{D}$)

Outdegree: $D_{i+} = \sum_j D_{ij}$, (row sums of $\mathbf{D}$)

## Top Buying Firms by Indegree, 2015

| Firm | Number of Suppliers |
|---|---|
| Walmart Stores Inc. | 115 |
| Royal Dutch Shell plc | 48 |
| McKesson Corp. | 41 |
| Cardinal Health Inc. | 40 |
| Home Depot Inc. | 37 |
| AmerisourceBergen Corp. | 35 |
| Ford Motor Co. | 31 |
| General Motors Co. | 28 |
| Target Corp. | 26 |
| AT&T Inc. | 22 |

## <u>**Indegree: Limitations**</u>

Imagine two firms, both with ten suppliers.

For the *first*, each of its suppliers has only <u>*one*</u> upstream supplier each.

Firm 1 has ten direct, and ten indirect suppliers.

For the *second*, each of its suppliers has <u>*ten*</u> upstreams supplier each.

Firm 2 has ten direct, and one hundred indirect suppliers.

Which firm is a more ‘important’ buyer?

## Indegree: Limitations (continued)

Many generalizations of indegree and outdegree centrality designed to address above limitation.

I will focus on indegree extensions first.

The generalization to outdegree-type measures then follows easily.

# Eigenvector Centrality

Bonacich (1972), building on Katz (1953), recursively defined an agent's **centrality,** power, or importance within a network, $c_i^{\mathsf{EC}}(\mathbf{D}, \phi)$, to be proportional to the sum of her links to other agents, weighted by their own centralities.

Letting $\mathbf{c}^{\mathsf{EC}}(\mathbf{D}, \phi)$ be the $N$ vector of centrality measures this gives

$$c_i^{\mathsf{EC}}(\mathbf{D}, \phi) = \phi \sum_j c_j^{\mathsf{EC}}(\mathbf{D}, \phi)\, D_{ji}.$$

$$\underbrace{\mathbf{c}^{\mathsf{EC}}(\mathbf{D}, \phi)}_{1\times N} = \phi \mathbf{c}^{\mathsf{EC}}(\mathbf{D}, \phi)\mathbf{D}$$

## Eigenvector Centrality (continued)

Typically $\phi = 1/\lambda_{\text{max}}$, with $\lambda_{\text{max}}$ the largest eigenvalue of $\mathbf{D}$, is used for normalization.

This choice ensures a solution w/ positive values *when* the network is strongly connected (Perron-Frobenius Theorem).

Since $\mathbf{c}^{\text{EC}}(\mathbf{D}, \phi)$ is the solution to $\mathbf{c}^{\text{EC}}(\mathbf{D}, \phi)\left[\frac{1}{\phi}I_N - \mathbf{D}\right] = 0$, it corresponds to the left eigenvector associated with the largest eigenvalue of $\mathbf{D}$.

## Row Normalization

Katz (1953) suggested an alternative approach to normalization.

The *row normalized* adjacency matrix is

$$\mathbf{G} = \text{diag}\left\{\text{max}\left(1, D_{1+}\right), \ldots, \text{max}\left(1, D_{N+}\right)\right\}^{-1} \times \mathbf{D}$$

The $i^{th}$ row of $\mathbf{G}$ sums to either zero (if agent $i$ has an outdegree of zero) or one (if agent $i$ has positive outdegree).

If all agents have positive outdegree, then $\mathbf{G}$ will be a row-stochastic matrix.

## Row Normalization (continued)

Katz (1953) suggested a centrality measure of

$$c_i^{\mathsf{K}}(\mathbf{D}) = \sum_j c_j^{\mathsf{K}}(\mathbf{D})\, G_{ji}$$
$$\mathbf{c}^{\mathsf{K}}(\mathbf{D}) = \mathbf{c}^{\mathsf{K}}(\mathbf{D})\, \mathbf{G}$$

Row normalization ensures that the largest eigenvalue of $\mathbf{G}$ is one and hence that $\mathbf{c}^{\mathsf{K}}(\mathbf{D})$ is well-defined (see below).

## Markov Chain Interpretation

If $\mathbf{G}$ is row stochastic, then $\mathbf{c}^{\mathsf{K}}(\mathbf{D})$ corresponds to a stationary vector of a Markov chain with transition matrix $\mathbf{G}$.

If the matrix $\mathbf{G}$ is irreducible, then this stationary vector is unique.

Irreducibility holds if, and only if, the network is strongly connected.

## Markov Chain Interpretation

Assume strong connectivity.

*Traveling saleswoman process*:

1. Saleswoman begins at any node.

2. She chooses a buyer at random from the set of buyers of her current supplier/node (at random) and moves *downstream* to the selected buyer/node.

3. Repeat *Step 2* many times...

## Markov Chain Interpretation (continued)

In the long run the elements of $\mathbf{c}^{\mathsf{K}}(\mathbf{D})$ equal the proportions of time our saleswoman will spend at each node.

Our saleswoman will spend more time at important 'buyer' nodes.

Such nodes will be chosen more frequently at step 2 of the traveling saleswoman process.

## Dangling Nodes

Few real world social and economic (directed) networks are strongly connected

Not only does strong connectivity typically fail, but many directed networks have "dangling nodes" (agents with zero indegree).

${c_i}^{\mathsf{K}}(\mathbf{G})$ will equal zero for such agents.

This will also be the case for all agents with incoming links solely from dangling nodes and so on.

## PageRank

The problem of dangling nodes, as well as the failure of strong connectivity, motivated Sergey Brin and Lawrence Page, at the time graduate students in computer science at Stanford University, to develop the PageRank centrality measure, now used by Google to rank web-search results.

## PageRank (continued)

Brin and Page made two changes to the Katz (1953) measure:

1. Regularize the (row normalized) adjacency matrix so that all rows, including those associated with dangling nodes, sum to one.

2. As in Bonacich (1987), endow each agent with a small amount of exogenous centrality.

## PageRank (continued)

Modification #1

Brin and Page defined the *Google Matrix* $\mathbf{H} = \left[ H_{ij} \right]$ with elements

$$\boxed{H_{ij} = \begin{cases} \phi G_{ij} + \frac{(1-\phi)}{N} & \text{if } D_{i+} > 0 \\ \frac{1}{N} & \text{otherwise} \end{cases}}$$

Observe that $\mathbf{H}$ is both row stochastic and irreducible.

## PageRank (continued)

Modification #2

Each agent has a small amount of exogenous centrality:

$$\mathbf{c}^{\mathsf{PR}}(\mathbf{D},\phi) = \phi\mathbf{c}^{\mathsf{PR}}(\mathbf{D},\phi)\,\mathbf{H} + \left(\frac{1-\phi}{N}\right)\iota_N'$$

A typical value for $\phi$, at least in web search, is 0.85.

## PageRank (continued)

For $|\phi| < 1$ we can solve for the PageRank vector as

$$\boxed{\mathbf{c}^{\mathsf{PR}}(\mathbf{D}, \phi) = \left(\frac{1-\phi}{N}\right) \iota_N' \left(I_N - \phi \mathbf{H}\right)^{-1}}$$

## PageRank (continued)

Modified traveling saleswoman process:

1. Saleswoman begins at any node.

2. She chooses a buyer at random...

   (a) ...with probability $\phi$ from the set of buyers of her current supplier/node (at random)

   (b) ...with probability $1 - \phi$ from the set of all firms (at random)

## PageRank (continued)

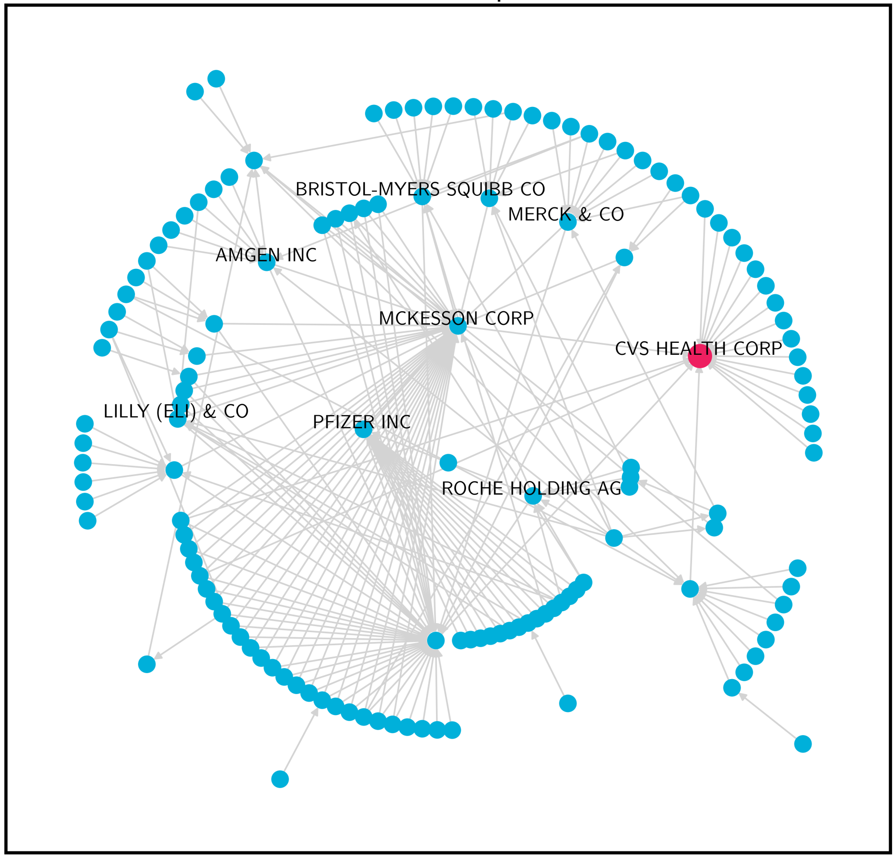
3. She moves *downstream* to the node selected in *Step 2*.

4. Repeat *Steps 2 & 3* many times...

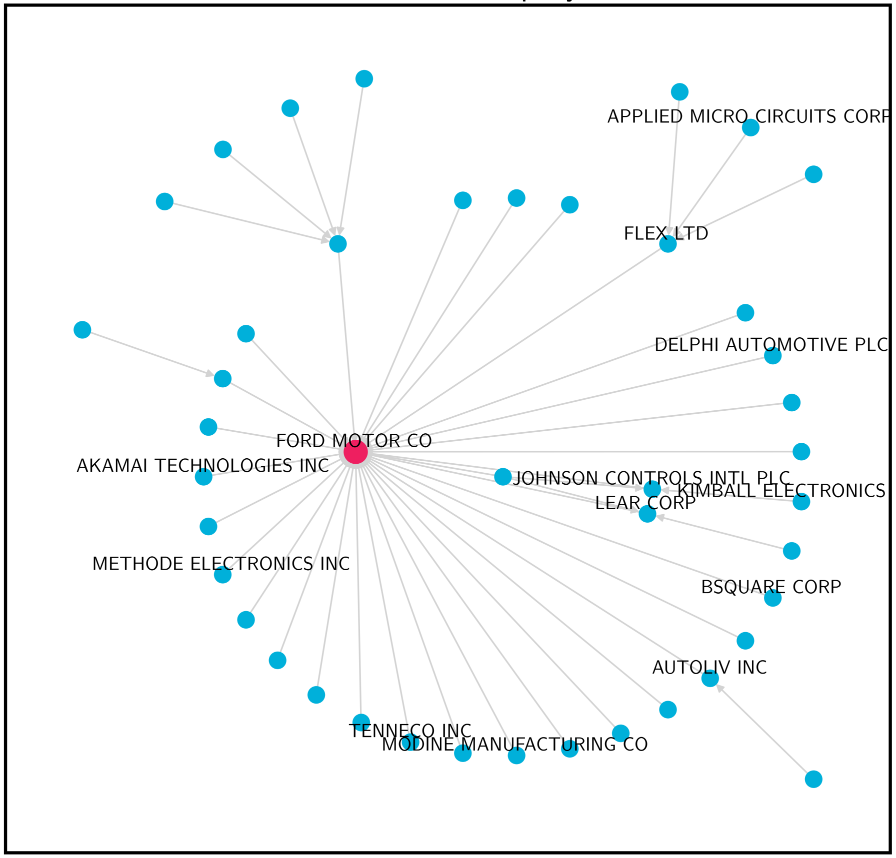
## Top Buyers by PageRank, 2015

| Firm | Buyer's PageRank | CumSum |
|---|---|---|
| Walmart Stores Inc. | 0.0272 | 0.0272 |
| CVS Health Corp. | 0.0198 | 0.0470 |
| Royal Dutch Shell plc | 0.0124 | 0.0594 |
| AmerisourceBergen Corp. | 0.0094 | 0.0688 |
| McKesson Corp. | 0.0086 | 0.0774 |
| Cardinal Health Inc. | 0.0081 | 0.0855 |
| Home Depot Inc. | 0.0006 | 0.0914 |
| Walgreens Boots Alliance Inc. | 0.0006 | 0.0974 |
| HP Inc. | 0.0056 | 0.1030 |
| Express Scripts Holding Co. | 0.0050 | 0.1080 |

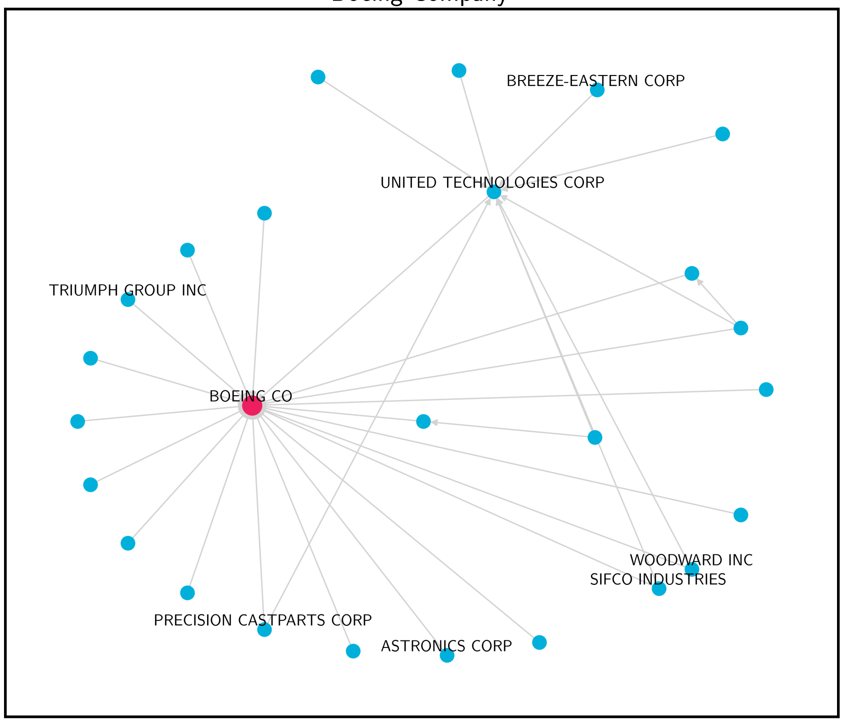
CVS Health Corporation



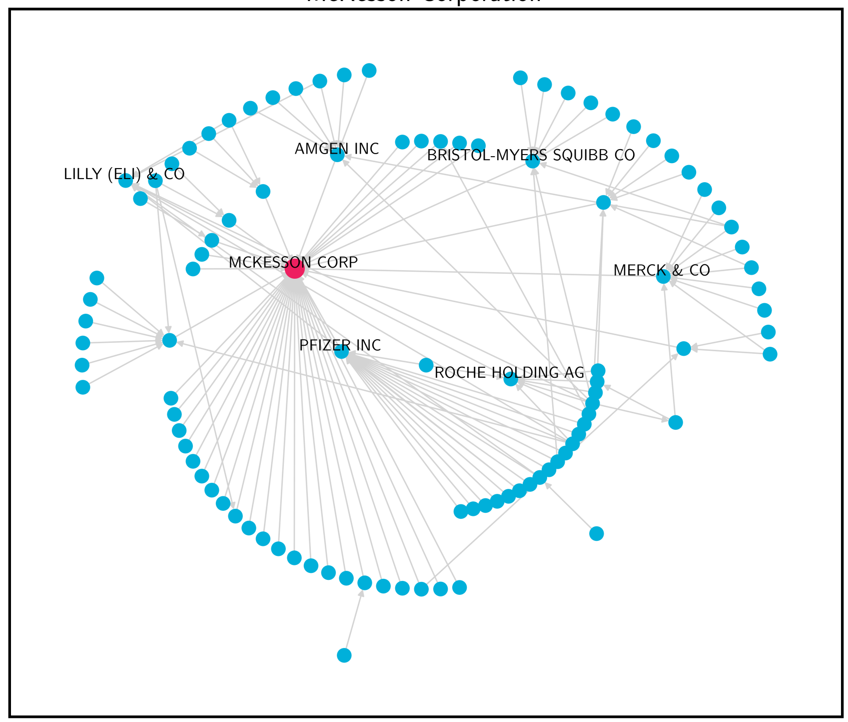

Ford Motor Company



Boeing Company




McKesson Corporation



## Centrality & Multipliers

Concept of a *social multiplier* a key theme in economics at least since the publication of Manski (1993).

Closely related concepts appear in Leonteif's work on Input-Output models in the 1940s.

A game-theoretic definition of *social multiplier centrality* provides additional insight into PageRank.

## Social Multiplier Centrality

Quadratic complementarity game (e.g., Jackson and Zenou, 2015).

Let $Y_i$ be some continuously-valued action chosen by agent $i = 1, \ldots, N$.

Let $\mathbf{Y}$ be the $N \times 1$ vector of all agents' actions.

Let $\mathbf{G}$ be the row-normalized network adjacency matrix.

Observe that

$$\mathbf{G}_i \mathbf{y} = \sum_{j \neq i} G_{ij} y_j \overset{def}{\equiv} \bar{y}_{n(i)}$$

equals the average action of player $i$'s direct peers.

Assume that the network is strongly connected.

## Social Multiplier Centrality (continued)

The utility agent $i$ receives from action profile $\mathbf{y}$ given the network structure is

$$\begin{aligned} u_i(\mathbf{y};\mathbf{D}) &= (\alpha_0 + U_i)\, y_i - \frac{1}{2} y_i^2 + \beta_0 \bar{y}_{n(i)} y_i \\ &= (\alpha_0 + U_i)\, y_i - \frac{1}{2} y_i^2 + \beta_0 \mathbf{G}_i \mathbf{y} y_i \end{aligned}$$

with $0 < |\beta_0| < 1$ and $\mathbb{E}[U_i] = 0$.

Here $U_i$ captures heterogeneity in agents' preferences for action.

Holding peers' actions fixed, there are diminishing returns to additional action.

## Social Multiplier Centrality (continued)

The marginal utility associated with an increase in $y_i$ is increasing in the average action of one's peers, $\bar{y}_{n(i)}$:

$$\frac{\partial^2 u_i(\mathbf{y}, \mathbf{D})}{\partial y_i \partial \bar{y}_{n(i)}} = \beta_0.$$

Own- and peer-action are complements.

The magnitude of $\beta_0$ indexes the strength of any *endogenous social interactions* (Manski, 1993).

## Social Multiplier Centrality (continued)

The observed action $\mathbf{Y}$ corresponds to a Nash equilibrium.

Agents observe $\mathbf{D}$, the network structure, and $\mathbf{U}$, the $N \times 1$ vector of individual-level heterogeneity terms.

The best response function is:

$$y_i = \alpha_0 + \beta_0 \bar{y}_{n(i)} + U_i$$

for $i = 1, \ldots, N$.

Special case of *linear-in-means* model of social interactions.

## Social Multiplier Centrality (continued)

The best response functions define an $N \times 1$ system of simultaneous equations.

Writing the system in matrix form gives:

$$\mathbf{Y} = \alpha_0 \iota_N + \beta_0 \mathbf{GY} + \mathbf{U}$$

## Social Multiplier Centrality (continued)

For $|\beta_0| < 1$, solving for the equilibrium action vector, $\mathbf{Y}$, as a function of $\mathbf{D}$ and $\mathbf{U}$ alone, yields the reduced form

$$\mathbf{Y} = \alpha_0 \left(I_N - \beta_0 \mathbf{G}\right)^{-1} \iota_N + \left(I_N - \beta_0 \mathbf{G}\right)^{-1} \mathbf{U}.$$

Using a series representation:

$$\mathbf{Y} = \frac{\alpha_0}{1 - \beta_0} \iota_N + \left[\sum_{k=0}^{\infty} \beta_0^k \mathbf{G}^k\right] \mathbf{U}.$$

## Social Multiplier Centrality (continued)

The infinite series representation provides insight into the social multiplier.

Consider a policy which increases the $i^{th}$ agent's value of $U_i$ by $\Delta$.

We can conceptualize the full effect of this increase on the network's distribution of outcomes as occurring in "waves".

In the initial wave only agent $i$'s outcome increases. The change in the entire action vector is therefore

$$\triangle \mathbf{e}_i,$$

where $\mathbf{e}_i$ is an $N$-vector with a one in its $i^{th}$ element and zeros elsewhere.

## Social Multiplier Centrality (continued)

In the second wave all of agent $i$'s peers experience outcome increases.

Their best reply actions change in response to the increase in agent $i's$ action in the initial wave.

The action vector in wave two therefore changes by

$$\triangle\beta_0\mathbf{G}\mathbf{e}_i.$$

## Social Multiplier Centrality (continued)

In the third wave the outcomes of agent $i$'s friends' friends change (this may include a direct feedback effect back onto agent $i$ if some of her links are reciprocated).

In wave three we get a further change in the action vector of

$$\Delta\beta_0^2\mathbf{G}^2\mathbf{e}_i.$$

## Social Multiplier Centrality (continued)

In the $k^{th}$ wave we have a change in the action vector of

$$\triangle \beta_0^{k-1} \mathbf{G}^{k-1} \mathbf{e}_i.$$

Observing the pattern of geometric decay, the "long-run" effect of a $\Delta$ change in $U_i$ on the entire distribution of outcomes is given by

$$\triangle \left( I_N - \beta_0 \mathbf{G} \right)^{-1} \mathbf{e}_i$$

## Social Multiplier Centrality (continued)

The effect of perturbing $U_i$ by $\triangle$ on the equilibrium action vector coincides with the $i^{th}$ column of the matrix $\triangle (I_N - \beta_0 \mathbf{G})^{-1}$.

Hence the row vector

$$\boxed{\mathbf{c}^{\mathsf{SM}} (\mathbf{D}, \beta) = \iota_N' (I_N - \beta \mathbf{G})^{-1}}$$

equals *social multiplier centrality.*

## Social Multiplier Centrality (continued)

Social multiplier centrality is greater than or equal to one for $\beta_0 \geq 0$.

If $c_i^{\mathsf{SM}}(\mathbf{D}, \beta) = 2$, then the effect of intervening to increase $U_i$ by $\boldsymbol{\Delta}$ on the aggregate action $\sum_{i=1}^{N} Y_i$ is twice the initial direct effect of $\triangle$.

Averaging over all agents we get

$$\frac{1}{N} \sum_{i=1}^{N} c_i^{\mathsf{SM}}(\mathbf{D}, \beta) = \frac{1}{1-\beta}$$

This is the form of the social multiplier in the linear-in-means model.

## Social Multiplier Centrality (continued)

In the presence of non-trivial network structure, the full effect of an intervention will, unlike in Manski (1993), vary heterogeneously across agents.

Shocks to central agents will have larger aggregate affects than equally-sized shocks to less central agents.

If we multiply the elements of $\mathbf{c}^{\mathsf{SM}}(\mathbf{D}, \beta)$ by $(1-\beta)/N$ we recover PageRank (w/o regularization).

## Katz-Bonacich Centrality

This measure is increasing in the number of direct friends and indirect friends, with weights discounted according to the degree of separation.

The $1 \times N$ vector of centrality measures for each agent is:

$$
\begin{aligned}
\mathbf{c}^{\mathsf{KB}}(\mathbf{D},\phi) &= \phi\iota_N'\mathbf{D} + \phi^2\iota_N'\mathbf{D}^2 + \phi^3\iota_N'\mathbf{D}^3 + \cdots \\
&= \left(\phi\iota_N'\mathbf{D}\right)\left(I_N + \phi\mathbf{D} + \phi^2\mathbf{D}^2 + \cdots\right) \\
&= \left(\phi\iota_N'\mathbf{D}\right)\left[\sum_{k=0}^{\infty}\phi^k\mathbf{D}^k\right].
\end{aligned}
$$

## Katz-Bonacich Centrality (continued)

For $\phi < 1/\lambda_{\text{max}}$ the sequence converges so that:

$$\mathbf{c}^{\text{KB}}(\mathbf{D}, \phi) = \left(\phi \iota_N' \mathbf{D}\right) (I_N - \phi \mathbf{D})^{-1}.$$

For $\phi \to 1/\lambda_{\text{max}}$ from below $\mathbf{c}^{\text{KB}}(\mathbf{D}, \phi) \to \mathbf{c}^{\text{EC}}(\mathbf{D}, \phi)$.

Related to equilibrium effort in quadratic complementarity games on networks (e.g., Jackson and Zenou, 2015).

See Calvó-Armengol, Patacchini and Zenou (2009) for a nice example.

## Generalizations of Outdegree

We can define outdegree-based versions of all of the centrality measures defined above by replacing $\mathbf{D}$ in their definitions with $\mathbf{D}'$.

Consider eigenvector centrality:

$$\underbrace{\mathbf{c}^{\mathsf{EC}}\left(\mathbf{D}',\phi\right)}_{1\times N} = \phi\mathbf{c}^{\mathsf{EC}}\left(\mathbf{D}',\phi\right)\mathbf{D}'$$

$$c_i^{\mathsf{EC}}\left(\mathbf{D},\phi\right) = \phi\sum_j c_j^{\mathsf{EC}}\left(\mathbf{D},\phi\right) D_{ij}$$

Agent $i$'s centrality depends on the centrality of those agents to whom *she directs* links and so on.

## Wrapping-Up

Identifying central agents in networks has a long history.

Recent measures, like PageRank, generalize earlier recursive definitions which go back (at least) to the work of Leontief.

Many potential empirical applications.

We will study one such application, due to Acemoglu and various co-authors, in the next lecture.