ABSTRACT

The “Perfect” Tuning System

Jessica Nickell

Director: David Ryden, Ph.D.

Mathematicians and musicians alike have done vast research into the musical tuning systems of Western music, but a “perfect” tuning system has yet to be developed. We will delve into the development and shortcomings of four major tuning systems in Western music. In Chapter 1 we will delve into Pythagorean Tuning, Chapter 2 will be about Just Intonation, Chapter 3 will be about Meantone Temperament and Equal Temperament, and Chapter 4 will discuss what a “perfect” tuning system requires. Through this analysis of the benefits and imperfections of these tuning systems, we will compose a list of qualitative principles as well as quantitative criteria that a tuning system must uphold to be “perfect,” in the sense that it has all of the advantages of each of these four systems. We will then propose three mathematical propositions and proofs to ultimately prove mathematically that a “perfect” tuning system cannot exist due to an inconsistency that arises between two of our criteria. Broadening the scope, we will also use the qualitative principles to argue that we will never be able to develop a perfect tuning system outside of Heaven due to human imperfection and the finitude of human reasoning, as God’s ways are higher than our own.

APPROVED BY DIRECTOR OF HONORS THESIS:

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Dr. David Ryden, Department of Mathematics

APPROVED BY THE HONORS PROGRAM:

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Dr. Elizabeth Corey, Director

DATE: ________________________

THE “PERFECT” TUNING SYSTEM

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By

Jessica Nickell

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TABLE OF CONTENTS

CHAPTER ONE

The Pythagorean Tuning System

One of the core systems of Western music was Pythagorean Tuning. Although there are other preferred systems today, Pythagorean Tuning was a foundational building block for the tuning systems more commonly used today. There are still some uses for Pythagorean Tuning, but its shortcomings, that were discovered later, keep it from being used today. This opened the door for other tuning systems and temperaments to take the forefront. We will delve into how Pythagoras created his tuning system, his discovery of the relationship between musical intervals and mathematical ratios, and why his system lost its prevalence in the modern music world.

Consonance vs. Dissonance

Prior to breaking down Pythagorean Tuning, we will define consonance and dissonance. Every scale is based on intervals that are deemed consonant and intervals that are deemed dissonant. A consonant interval is one that is harmonious and pleasing to the ear. Boethius defined it as "when two strings, one of which is lower, are stretched and struck at the same time, and they produce, so to speak, an intermingled and sweet sound, and the two pitches coalesce into one as if linked together" (Boethius et al. 47). Whereas dissonance is defined by Boethius to be when two strings "are struck at the same time and each desires to go its own way, and they do not bring together a sweet sound in the ear, a single sound composed of two" (Boethius et al. 47). Therefore, consonance is when two sounds blend together into one and dissonance is when two sounds when put together can still be separated by the ear and are distinct from each other. As can be seen through these definitions, much of what is deemed consonant or dissonant is decided by the human ear and what the pitches sound like (this is especially prevalent in the time of Pythagoras and later Boethius, when the theory of tonality had not been developed with the precision we have today to measure sound waves).

In Pythagorean Tuning specifically, the major third (M3) and the major sixth (M6), which are widely recognized as consonant intervals in the Equal Temperament tuning system used today, are not very harmonious and thus were not deemed consonant. This is an effect of the main goal of Pythagorean Tuning: to perfectly tune the octave, the fifth, and the fourth. When this is done, the third and the sixth are negatively impacted and are not as harmonious as in other tuning systems and temperaments. Today, we see this most prevalently in string instruments which tune by a fifth or a fourth. Since they tune to the fifth/fourth, string instruments tend towards Pythagorean Tuning causing them to have to adjust when non-string instruments are accompanying them. Additionally, in modern string instruments, many do not have frets and thus the fingerboard is a “continuous” plane, leaving the notes to be modified by the musician's finger placement. This is what allows string instruments to tune to the fifth, but still blend together harmoniously on the other intervals that are not consonant in Pythagorean Tuning.

Pythagorean Tuning

In general, there is a specific approach or goal that each mathematician/musician used when creating their tuning system. Just as great research starts with a question that needs to be answered, each tuning system/scale starts with a specific “problem” that it is addressing, or with a specific goal in mind. These goals also widely varied between mathematicians and musicians and were constantly changing with time. Some, like Pythagoras, started with a specific interval ratio and built a scale from there; whereas others built one based on a specific instrument and how it sounds, or how multiple instruments sound together. Thus, from the very beginning of their creation the scales and systems created differed in some way from each other and musicians were left with the task of bridging the gap and choosing which to use.

In determining his tuning system, Pythagoras experimented with the monochord, which is a single stringed wooden instrument that could be divided with a bridge to alter the length, and thus the pitch of the string. He figured out that "ratios of small numbers produced harmonious, pleasant combinations of sounds - consonances - whereas ratios of large numbers produced dissonances" (Maor). He accomplished this by plucking the single string and then, using the bridge, he could divide the string in half or into some other ratio and pluck again to hear the difference. To get the octave for example, you would divide the string exactly in half, but to get the fifth you could divide the string into 3 equal parts and put the bridge after the second part, creating the ratio 3:2. Thus, Pythagoras' approach to the musical intervals was very mathematical, because he focused on building each interval through "a series of perfect fifths (or fourths) and octaves" (Chalmers and Polansky 7). Essentially, he added and subtracted perfect octaves (P8), perfect fourths (P4), and perfect fifths (P5) to get the 8 notes in a typical diatonic scale. For instance, C + P8 = C, C + P5 = G, G + P5 = D, D + P5 = A, A + P5 = E, E + P5 = B. Then to get the F, he went down a P5 from C by inverting the ratio 3:2 to get $(3/2)^{-1}$: C - P5 = F. Thus, the basis for his tuning system is the ratio 3:2 for the perfect fifth, and the rest of his scale was powers of 3:2 only (Maor 16–17). This discovery led Pythagoras to the conviction that the numbers and their ratios were the key to understanding reality.

Since Pythagoras's tuning system was very mathematical in approach, each interval corresponds to a whole number ratio and has a specific name that varies from what we call them today. We will now consider the ratios that Pythagoras assigned to each interval, so that we can see the exact mathematical differences between the different systems and why one might sound more harmonious than the other. First, we will look at the intervals in the diatonic scale which has the tetrachord (semitone + tone + tone). The modern diatonic scale we use today has 8 notes in the octave, and as one moves up the scale it is of the following pattern of tones and semitones: 2 tones, 1 semitone, 3 tones, 1 semitone. There are other types of scales and tetrachords like enharmonic and chromatic which do not follow this pattern. The diatonic scale is the standard scale that is most familiar to musicians and non-musicians alike. We will also use the diatonic scale as the basis of our explanation, because "Greek Musical Theory used the tetrachord as a building block or module from which scales and systems could be constructed"(Chalmers and Polansky 2). Boethius broke down the Pythagorean ratios and the names of the intervals in his *Foundations of Music*, which is what we will use to explain Pythagorean Tuning. In the list below, the classical name of the interval will be listed first, followed by its mathematical ratio, and the relative interval it corresponds to today (despite the sometimes varying ratios we use for them today). The intervals are:

- Diapente (Sesquilater) - 3:2 (Perfect Fifth)
- Diapason (Duple) - 2:1 (Perfect Octave)
- Diatessaron (Sesquitertian) - 4:3 (Perfect Fourth)

- Tone (Sesquioctave) - 9:8 (Whole Step/Tone)

Pythagoras got these ratios by building off the perfect fifth and octave ratios, as described previously. For example, to get a Tone, he could add two fifths together and then subtract an octave to get a whole step (9:8). This would mathematically be $(3/2 * 3/2)/(2/1) = 9/8$.

These ratios do not use irrational numbers because of Pythagoras's conviction. Due to the Pythagorean doctrine, the early ancient Greeks thought they could understand the whole of reality with whole numbers and their ratios, which is why those at the time did not see a need even to look for irrational numbers. However, some of the intervals that we deem as consonant today do use irrational numbers and were not present in Pythagorean Tuning. Some of the intervals in Pythagoras's scale were "out of tune with the natural sequence of harmonics, or overtones, generated by practically all musical instruments" (Maor 18). This fault in the tuning system made way for many other tuning systems and temperaments to be developed that preserve more of what we recognize as consonant today.

Moreover, the Pythagorean Tone cannot be divided into two equal semitones using Pythagorean principles, so there ends up being a major semitone and a minor semitone, where the larger semitone is the ratio 1:16 and the smaller semitone is 1:17. When the Pythagorean Diatonic Scale is put together, it uses the tone ratio for each tone in the diatonic pattern, however, the semitone ratios incorporated into the scale end up being less than either of these semitones, in order to get a correct octave. The smaller semitone has the ratio 256:243. This fact is exhibited today in the instruments that can use Pythagorean Tuning, like string instruments. When playing a scale on a string instrument, the string player often makes the semitone intervals in a diatonic scale slightly smaller by slightly raising the seventh pitch in the scale, which is often referred to as the “leading” tone. This is because it is said to “lead” to the next note and resolve the melodic line. It is most often used when leading into an octave, where the Major Seventh is raised slightly to increase the dissonance before it resolves into the consonance of the Perfect Octave. This is an example of how Pythagorean Tuning concepts are still in use today.

Shortcomings of Pythagorean Tuning

There is no single reason for why musical composition and theory as a whole drifted away from Pythagorean Tuning. Rather, Pythagorean Tuning slowly lost its appeal when additional mathematical and scientific discoveries were made that impacted what we knew and believed about music and its intervals, as well as when different types of musical composition arose and adapted to the time periods and the instruments used in that time period. Furthermore, there are some shortcomings that arise in any tuning system when one tries to strictly measure out mathematical ratios that correspond to musical intervals. These shortcomings will be described for Pythagorean Tuning, but hold for other tuning systems as well.

Time Period Shortcomings

Firstly, the amount of mathematical and scientific knowledge attained at the time a tuning system was developed heavily affected how it was created and the reasoning behind it. While the Ancient Greeks made many advancements in the realms of mathematics and science, there are still things they did not use or did not know that we have since discovered or modified. For instance, in Pythagoras's time, the Greeks did not have irrational numbers since, in their view, whole numbers and ratios that used whole numbers were sufficient. It is due to this assumption/belief that one of the widely used temperaments today, Equal Temperament, would never have been used or even thought of as a valid solution in the time of Pythagoras because it uses $\sqrt[12]{2}$ for its semitone, which is an irrational number. This one example raises the following question: what other personal or generational restrictions, caused by the level of knowledge of the time period, have been placed on tuning systems? Have these restrictions kept us from looking into a realm of mathematics with a better or more universal (across different instruments) tuning system? It also begs the question: due to these possible restrictions like the avoidance of irrational numbers, how might the progression of musical discoveries be impacted had, for instance, the Greeks used irrational numbers? These questions remain unanswered, and suggest that there is still more to be discovered in the realm of music theory, possibly an even better tuning system.

Along this same line of thought, the tuning systems created during the time of Ancient Greece were before modern technology and research on sound waves and frequencies, as mentioned above. Their tuning systems were reliant on the human ear and what it deems as consonant and dissonant. This leaves much of what we know about tuning systems and much of what was discovered in that time up to the human ear, which is very unreliable. The human ear can only differentiate up to about 5-10 cents pitch depending on the ear and how refined it is. A cent is defined as 1/1200 of an octave, which is very minuscule and shows how precise the human ear can be. However, this range also means that there is a small area in which humans physically cannot differentiate between two sounds, and that area is different from person to person. This area is small in the grand scheme of things, as a semitone differs by around 100 cents depending on how big or small it is, but it can have a strong impact on how something sounds if more than one voice or instrument is added. The once small difference of smaller than 5 cents can be compounded and multiplied with each voice added and become audible to the human ear. Additionally, a musician who actively practices an instrument or sings is also more likely to have a more refined ear and hear a difference as low as 5 cents than someone who does not have musical training or any practice in differentiating notes and chords. Therefore, the tuning systems of Ancient Greece, which were more reliant on the human ear, like Pythagorean Tuning, are not as reliable in what they deem consonant or dissonant, due to the perception of consonance being subjective. This was more or less addressed when frequency technology was invented, as we can now mathematically deduce if soundwaves are consonant by how they fit together rather than on the human ear. However, consonance and dissonance is ultimately decided by the listener.

Tuning systems, although varying due to the goal of the mathematician researching and developing it, can also vary due to the types of instruments being used to create them, or due to the types of instruments they are developed for. Pythagoras's tuning system worked best for the solo voice or instrument, and the dissonances it contains are apparent when more than one instrument or voice join together in harmony. This can possibly be attributed to his use of the monochord to create his tuning system. Since he based his intervals and their ratios off of the sound of one string, the consonances and intervals he decided upon are not maintained when other instruments or voices are added to it. When a harmony is added to a melodic line, each note in the line is then measured against that of the harmony and this can cause the notes to be adjusted to fit the interval and thus break out of the ratio used to get the original note. Thus, the more instruments or voices added, the more adjusting that might occur causing the ratios not to be maintained in light of the other voices. This may not have been a problem in Ancient Greece, as the modern orchestra, with its extensive use of harmony, did not exist, and they may not have realized this shortcoming. This adds to how the general musical trends of the time period could impact the type of system needed. Either way, this inconsistency between solo instrument and multiple instruments made a way for other tuning systems to be experimented with that work on a larger scale. Not to mention that "the Pythagorean tuning per se was used only for the diatonic genus, and was modified in the chromatic and enharmonic genera"(Barbour 1).

Mathematical Shortcomings

In addition to these limitations, there was also a discrepancy in the mathematics of Pythagorean Tuning. As mentioned previously, Pythagoras built his diatonic scale through the addition of Perfect Fifths and the subsequent subtracting of octaves. Similarly, "the circle of fifths is a sequence of notes in which a starting note is played and followed by successive fifths until the starting note is reached again albeit several octaves higher"(DuBose-Schmitt 10). It is through this process of adding Perfect Fifths until you reach the starting note several octaves higher that Pythagorean Tuning falls short. Since Pythagoras established the octave as being a ratio of 2:1, "the final note of the circle of fifths should be a power of 2" (DuBose-Schmitt 11). Additionally, the final note "should also be a power of 3:2 since that is the frequency ratio of the fifth" (DuBose-Schmitt 11). Hence the frequency of the final note would be both a power of 2 and a power of 3:2, but this cannot be the case. When calculated using powers of 2, the final note in the circle of fifths is $2^7 = 128$, but when calculated using the Pythagorean ratio of 3:2 for the Perfect Fifth, the final note in the circle of fifths is $(3/2)^{12} \cong 129.75$. "This discrepancy in the frequency ratios is known as the Pythagorean Comma, and its numerical value is $129.75/128 \cong 1.013643$ *Hz*" (DuBose-Schmitt 12). This discrepancy gradually accumulates over the span of six octaves (like in the circle of fifths), so that when one plays a G on an instrument tuned to C major, it is slightly lower than a G on an instrument tuned to F major. This discrepancy caused musicians to have to re-tune their instrument every time they changed keys, which is not ideal and is a contributing cause to the eventual shift to other tuning systems.

Overall Shortcomings

Finally, there lies the more abstract shortcomings of Pythagorean Tuning and any tuning system at that. Pythagoras's main approach, like many others, focused on the mathematical side of intervals and gave every interval a mathematical ratio. He explained music in a structured way, giving a definition for everything; however, music itself does not fit solely in one structured box. It varies each time it is played, and it varies from player to player and singer to singer. No two people will play or sing a piece exactly the same, and that is what makes music so compelling and unique. That is also what makes it a creative outlet for one to express their emotions. As Richard Parncutt and Graham Hair stated, "musicians are not aware of ratios as they perform melodies" (Parncutt and Hair

475). Musical intervals are one of the ways in which music and math have to compromise, but at the end of the day, music is whatever the musician playing or singing it feels it to be. Pythagoras tried to fix the intervals, but no intervals in music are fixed unless the musician playing them measures each one out on a fixed scale. However, that would be akin to pulling out a ruler or a protractor each time you drew a line and measuring it exactly. That does not work in the context of a musical piece for many reasons, one of which being that sometimes an interval is changed/modified to add tension or "lead" to a resolution like with the "leading" tone, mentioned above. Therefore, music is beyond a literal definition and is, in its practiced use, partially subjective. This makes it hard for any tuning system or temperament to fully define every interval and musical aspect of a piece.

Conclusion

While Pythagoras took a major step in musical history through his tuning system, these are a few reasons why we have since created and widely used other systems and temperaments. Furthermore, it is this revolving door of tuning systems and temperaments that makes it hard to define what the "perfect" system would be. There is not a single tuning system that has withstood the test of time, since the world of music is constantly growing, changing, and reinventing itself. Thus, we move down our timeline to the next major system: Just Intonation.

CHAPTER 2

Just Intonation

Bridging the Gap

There were some significant changes between the time when Pythagorean Tuning was established and when Just Intonation came about. Pythagorean Tuning was established around 500 B.C., and Just Intonation was developed in the 16th century AD. This is a major jump in history, so we will first bridge the gap between the time of Pythagoras and the Renaissance by addressing some of the major developments in both musical history and mathematical history that influenced the development of Just Intonation and the subsequent temperaments that followed shortly thereafter.

One development in music was with the type of instruments used. Some of the instruments we know of today were already in common use prior to Just Intonation. For instance, the harp, lyre, drums, trumpets, fiddles, etc. were commonly used throughout history, although their modern renditions have evolved over the years. A major shift took place in music from 800 to 1100 AD, where there was a "tendency towards polyphony," which is the use of more than one melodic line in a piece creating "polyphonic harmony"(Sachs 269). The phenomena of consonance and dissonance that occurred when multiple pitches were played simultaneously "demanded exactitude of pitch and intervals" (Sachs 269). Prior to this polyphonic shift, there had not been a need for "any mathematically regulated scale," but "in polyphonic harmony, music could not exist without rule and system" (Sachs 269). This is because the introduction of harmonies highlighted the dissonance when intervals and pitches didn't coalign, uncovering the need for a standard scale. Additionally, “medieval men felt no confidence in their senses; the musical ear proved to be unsufficiently reliable,” so there was a struggle in establishing a standardized scale (Sachs 269).

In this time frame, there was also the introduction of bowed instruments. According to Sachs, “the first evidence of a bowed instrument is found in Spanish manuscripts of the tenth and eleventh centuries” (Sachs 275). Thus, what we picture in the modern-day violin family did not come about until several centuries after the polyphonic shift. The introduction of bowed instruments had an impact on the temperaments that were developed, as “after the unfretted violins became the backbone of the 17th century orchestra, their flexibility of intonation made” the problem of dissonance in polyphonic music “less pressing” (Barbour 8). However, this solution to the polyphonic problem was a setback in establishing a standardized tuning system or temperament, because unfretted bowed instruments could adjust to the instruments they were accompanying. This was a setback because it had been discovered that fretted instruments and keyboard instruments had dissonance when played together, and the unfretted instruments’ ability to quickly adjust to whomever they were accompanying inhibited the bridging of this dissonant gap.

Another shift in music history took place during the Renaissance (1400 - 1600 AD). In musical development, the social setting music is performed in impacts how the instrument is used; thus, it wasn’t until the Renaissance that instruments were used for solo work and not just for accompanying vocal performances. This change in the social setting of instruments introduced new combinations of instruments that didn’t have the flexibility of intonation, which is what contributed to the polyphonic dilemma of there being dissonance when multiple instruments were put together. As discussed previously, Pythagorean Tuning fell short when used outside of a solo voice or instrument, and it was in the Middle Ages and the Renaissance that this became apparent. This shift into instrumental music is also what led to the modern day orchestra and what we think of as classical music today, because orchestration was first considered during the 16th century (Sachs 298).

Although there were few advances in the realm of mathematics between the time of the Greeks and the 16th century, some significant discoveries were made which applied to music. The Greeks sought to discover mathematical facts, as they believed that eternal abstract objects were not created but discovered. They "identified mathematics with the reality of the physical world and saw in mathematics the ultimate truth about the structure and design of the universe" (Kline 172). This focus on concrete proofs and axioms and explaining their reality with them greatly shaped modern mathematics, but was also a limitation to what they could discover. For example, since they presumed to explain their reality with whole numbers, the Greeks did not recognize irrational numbers as actual numbers. Nevertheless, when they discovered that the diagonal of a square of side length 1 could not be described with ratios of whole numbers, this led to the Greeks turning away from arithmetic and to geometry "because geometrical thinking avoided explicit confrontation of the irrational as a number" (Kline 173). This geometric approach persisted "as late as the seventeenth and eighteenth centuries, when algebra and the calculus had already become extensive" (Kline 176).

Although the Arabs and Hindus made great discoveries in the realm of algebra and arithmetic, it took a few centuries for Europe to learn of and use their discoveries.

The geometric approach of the Greeks, the fall of the Roman Empire, and the delay of Europe learning the discoveries made by the Arabs and Hindus contributed to the stagnation of mathematics in Europe from about 400 to 1100 AD. Additionally, the spread of Christianity throughout Europe led to an increased focus on spiritual matters and life after death. The Catholic church was heavily involved in operating schools and monasteries throughout Europe, which contributed to the growth of academic institutions and discoveries, but focused on spiritual matters first, followed later by other matters, like mathematics. However, the rediscovery of the Greek works, along with the eventual discovery of what the Arabs and Hindus had developed, reignited the focus on mathematics around 1100. This sparked further mathematical discoveries, and brings us up to Just Intonation in our timeline.

Just Intonation

As mentioned previously, in Pythagorean Tuning the only intervals that were considered consonant intervals were that of the Perfect Fourth, Fifth, and the Octave. It wasn't until the Middle Ages that musicians started to recognize that the intervals of the Major Third and the Minor Third "sounded just as pleasing as the previously-allowed consonances" (Maor 70). Thus the Major and Minor Third, and their corresponding inverses the Major and Minor Sixth, began to be considered as consonances. This redefining of consonant and dissonant intervals was "embraced by composers," but it also meant that Pythagorean Tuning would no longer be the ideal tuning system for the Middle Ages (Maor 71). Additionally, as mentioned in the Pythagorean shortcomings, the Pythagorean comma which forced musicians to retune their instrument whenever they changed keys left a problem that needed to be fixed. This led to the invention of the Just Intonation Scale by Gioseffo Zarlino in 1558.

Just Intonation "is built using the octave, major third, and fifth, which mimics the modern-day arpeggio" (DuBose-Schmitt 14). Similar to Pythagorean Tuning, the Just Intonation scale is very mathematical in nature. Each interval ratio is found using the arithmetic mean or the harmonic mean, starting with the octave or perfect eighth which has the same ratio as in Pythagorean Tuning, 2:1. The mean of 2 and 1 (the octave) is the fraction 3/2 which is the ratio for the Perfect Fifth, similar to the Pythagorean Scale. Similarly, the mean of 1 (the root) and 3/2 (the fifth) is 5/4 which is the ratio for the major third. The rest of the intervals are found through a similar process of finding the arithmetic or harmonic mean of the different interval ratios and are listed below:

- Root - 1
- Whole Tone - 9:8 (Arithmetic Mean of 1 and 5/4) or 10:9 (Harmonic Mean of 1 and 5/4)
- Major Third - 5:4 (Arithmetic Mean of 1 and 3/2)
- Perfect Fourth - 4:3 (Harmonic Mean of 1 and 2)
- Perfect Fifth - 3:2 (Arithmetic Mean of 1 and 2)
- Major Sixth - 5:3 (Arithmetic Mean of 4/3 and 2)
- Major Seventh - 15:8 (Arithmetic Mean of 7/4 and 2)
- Perfect Octave - 2:1

*Note: These are the formulas for the Arithmetic and Harmonic Means:

- Arithmetic Mean of a and b = $(a + b)/2$
- Harmonic Mean of a and b = $(2ab)/(a + b)$

Additionally, it is worth noting that there are two intervals listed for the whole tone. This is because "when we divide each member of the sequence by its predecessor, we get the intervals *between* the notes" and the whole tone that appears between the notes is represented by both the ratio 9:8 and 10:9 in different parts of the scale (Maor 71). This discrepancy around the whole tone is one of the shortcomings of Just Intonation.

Shortcomings of Just Intonation

Despite the short-lived use of Just Intonation, "the just-intonation scale had the advantage of being based on the first six members of the harmonic series, the progression of overtones produced by nearly all musical instruments," which caused it to "conform more closely to the laws of acoustics" than Pythagorean Tuning (Maor 71). These overtones, also known as harmonics, can be isolated "on a single string on the monochord by dividing the string in whole number divisions" (*The Harmonic Series, Musical Ratios & Intervals – Microtonal Guitar*). The result is the harmonic series, "a series of numbers related by whole-number ratios" where the "harmonics are equally-spaced" (*The Harmonic Series, Musical Ratios & Intervals – Microtonal Guitar*). The first 6 harmonics from the harmonic series are as follows:

- Unison - 1:1
- Octave - 2:1
- Perfect Fifth - 3:2
- Octave - 4:2 (2:1)
- Major Third - 5:4
- Perfect Fifth - 6:4 (3:2)

These intervals align with the corresponding intervals in Just Intonation. This benefit of Just Intonation may have been the reason why "some musicians and scholars argue that just intonation produces the purest sound" (DuBose-Schmitt 15). However, there was a major discrepancy that caused musicians and mathematicians alike to look elsewhere for a better tuning system.

As mentioned previously, there was a discrepancy in the Just Intonation scale over the ratio of the whole tone. This prevalence of two different whole tones made it difficult to transpose or modulate the key of a piece, similar to the error that occurs in Pythagorean Tuning. And, like with Pythagorean Tuning, instruments with a continuous range of notes, like a string instrument, could easily fix this problem while playing, but that does not fix the overall problem. Different routes were taken to address this problem, for example, "harpsichords of the Baroque period were often built with up to three keyboards, each tuned to a different key" (Maor 72). However, this did not fix the problem for every possible key and was more difficult to play. Thus, the journey for a "perfect" tuning system, that was able to switch between different keys, continued.

CHAPTER THREE

Meantone and Equal Temperament

Overview

Meantone Temperament was developed around the 16th century, shortly after Just Intonation, and was followed by Equal Temperament in the 17th century. Today Equal Temperament is the main system used in music. This progression from Just Intonation to Meantone Temperament occurred much quicker than the progression from Pythagorean Tuning to Just Intonation. This was in part due to the sudden development in both the field of mathematics and music, but also in part due to the shortcomings of Just Intonation discussed previously. In this section, we will break down what Meantone and Equal Temperament are and their shortcomings.

Temperaments vs. Tuning System

First, we must distinguish between a tuning system and a temperament. Although very similar in how they are used, there is a slight difference between a tuning system and a temperament. A tuning system, as defined by J. Murray Barbour in *Tuning and Temperament,* refers to a system like Pythagorean Tuning or Just Intonation, "in which all intervals may be expressed as the ratio of 2 integers" (Barbour 5). Conversely, temperament refers to a "modification of a tuning" where "some or all of [its] intervals cannot be expressed in rational numbers" (Barbour 5, xii). Therefore, a temperament will use irrational numbers in at least one of its intervals. For example, a perfect 5th is expressed with a ratio of 3:2 in Pythagorean Tuning and in Just Intonation, but in Equal Temperament it is expressed with a ratio of $2^{7/12}$:1. This distinction sets temperaments apart from tuning systems, and shows why temperaments came about after the first tuning systems in Western music (due to the Ancient Greeks not making use of irrational numbers). It also hints at an advancement in mathematical and scientific discoveries, as they had to be able to measure these intervals with irrational numbers which required different tools than were available or readily used in the time period of the Pythagoreans or during the early Renaissance period.

Meantone Temperament

Meantone Temperament came about a few centuries earlier than Equal Temperament. It is a “system of tuning keyboard instruments” that was mainly used in the 16th - 18th centuries (*Meantone Temperament -- Britannica Academic*). Unlike Pythagorean Tuning, which was based off of the P8 and the P5, Meantone Temperament was based off of major thirds (M3), which is the interval with 4 semitones. This was done by making the interval of the major third equal on both sides of a pitch, meaning that the difference between the root of a chord and the third was the same as the difference between the third and the fifth. This resulted in a P5 that was smaller than a Pythagorean or Equal Temperament fifth (*Meantone Temperament -- Britannica Academic*). Meantone Temperament was intended as an alternative to Just Intonation, described previously, because Meantone Temperament “substituted a single, mean whole tone” rather than whole tones of different sizes, like in Just Intonation and similar to Pythagorean Tuning which had a small and a large semitone.

Shortcomings of Meantone Temperament

Meantone Temperament, because of its “pleasing sonority for triads” from tuning the third equally on both sides, was commonly used to tune keyboard instruments. However, the black keys, which share enharmonic pitches, were not the same in Meantone Temperament. This resulted in dissonance in certain keys that required an alternative note than “the usual choices being C#, E-flat, F#, G#, and B-flat” (*Meantone Temperament -- Britannica Academic*). Thus, “if an instrument was played in a key requiring an alternative note, say A-flat instead of G#, a strong dissonance … resulted” (*Meantone Temperament -- Britannica Academic*). This dissonance was referred to as the “wolf” and is what led to the switch to Equal Temperament, which was commonly used by fretted instruments.

Equal Temperament

Equal Temperament is the system more widely used today, as most keyboard instruments are tuned with Equal Temperament. Its development was in response to the shortcomings of Just Intonation paired with the growing complexity of music and its composition. There were also many mathematical discoveries related to frequencies that came about in the 16th and 17th centuries that contributed to the development of the Equal Temperament scale.

Despite there being several mathematicians around the world who had stepped into the realm of Equal Temperament, Marin Mersenne “was the first to give us a full account of the equal-tempered scale, complete with detailed numerical calculations”

(Maor 73). Mersenne was the first person to did this by measuring the frequency of notes. Similar to Pythagoras, Mersenne used the monochord to measure the frequency, but instead of shortening the string length to figure out interval ratios, Mersenne doubled the string length until its vibrations were slow enough to count. "Because the initial and final notes were separated by an exact number of octaves, Mersenne was able to determine the frequency of the higher note, and from this the frequency of all other notes of the scale, using the known ratios between them," which will be defined later (Maor 30).

Furthermore, Mersenne's work with measuring the frequency of notes by counting individual vibrations "led to the main method of relative tuning (adjusting the tuning relative to the notes being sounded) - beat elimination" (DuBose-Schmitt 16). When two notes are sounded at the same time, there is a "pulsing rhythm" that rings out as they sound that indicates they are out of tune with each other (DuBose-Schmitt 16). The faster the beats or pulses, the more out of tune the interval is. This method of relative tuning is used by string players, who tune their A string first and tune the rest of their strings relatively to that A. This other discovery from Mersenne's work adds to the reasoning for why Pythagorean Tuning and Just Intonation are not the ideal tuning systems, as "they did not have a perfect octave" and the beats would never be eliminated completely in their octave ratio (DuBose-Schmitt 18). However, Just Intonation had the advantage in the other intervals that were not the octave, because it was based on the harmonic series and its other intervals were more consonant than those of Equal Temperament, as discussed previously. Furthermore, from this discovery of beats and relative tuning came much more innovation in the realm of music theory that added to our understanding of how consonance and harmony work.

In Pythagorean Tuning and Just Intonation scales, the octave was made up of 2 whole tones followed by a semitone followed by 3 whole tones and then 1 semitone. In Equal Temperament, each whole tone was split into two semitones making 12 semitones in one octave and then the octave ratio of 2:1 was divided equally among the 12 semitones, resulting in each semitone being of the ratio $\sqrt[12]{2}$: 1. This solution seems apparent, but would have never been considered by the Pythagoreans, because the $\sqrt[12]{2}$ is an irrational number. Each semitone being the same ratio led to the octave being perfectly in tune, unlike in Pythagorean Tuning or Just Intonation which were affected by the Pythagorean Comma. However this solution resulted in every other interval being "slightly out of tune" by "hid[ing] the Pythagorean comma in between every interval" (DuBose-Schmitt 19). The benefit of this was the easy modulations of keys, because Equal Temperament fixed the problem of the octave ending out of tune in the circle of fifths and now the circle of fifths ends on the same pitch it started on, only six octaves higher. Additionally, Equal Temperament was also found to be more ideal and functional for keyboard instruments. This is because the functionality of a keyboard that had 12 semitones in an octave was more practical and easier to play on then tuning systems with greater than 12 semitones. Thus, Equal Temperament is the main tuning system used today.

Shortcomings of Equal Temperament

Despite the many benefits of Equal Temperament, there were still some compromises made and corners cut that contribute to the imperfections of Equal Temperament. As mentioned previously, in order to make the octave in tune, the other intervals were made slightly out of tune on purpose. This causes the beats in each interval of Equal Temperament to never fully be eliminated, resulting in chords that are not perfectly tuned. This makes Equal Temperament “not quite as harmonious as the just intonation or Pythagorean scale” (DuBose-Schmitt 20).

As mentioned previously, string players tune their instruments using relative tuning by playing two strings at a time and working to eliminate any audible beats or pulses. Thus, their strings are not necessarily tuned exactly to the Equal Temperament ratios. Additionally, since Equal Temperament is not as harmonious as other tuning systems like Just Intonation, some singers also prefer or are inclined to sing using a different tuning system when not playing alongside fretted or keyboard instruments. This preference is sometimes done consciously, but can also be an unconscious decision made by the musician while performing. This follows most probably from the use of the ear in tuning both strings and in singing musical intervals. The musician listens for consonance and harmony in those intervals, and as revealed, that does not always fall under Equal Temperament. Consequently, string players and vocalists often have to adjust their tuning and the pitches they play/sing when they are accompanying other instruments like the piano in order to not sound out of tune. Hence, string players, vocalists and instrumentalists with a continuous range of notes play with a compromise between the harmonious parts of Equal Temperament (the perfectly tuned octave), and that of Just Intonation (the more harmonious intervals excluding the octave).

Conclusion

Even though Equal Temperament fixed many of the errors in Pythagorean Tuning and Just Intonation, it is ultimately “an acceptable compromise between the dictates of musical harmony and the practicality of playing a piece on the keyboard” (Maor 74). The adjustments made by certain musicians and the use of other tuning systems corroborates the belief that Equal Temperament, although the most prevalent tuning system of the past four centuries, is not a “perfect” tuning system and still leaves much to be desired and hoped for from a better tuning system that has yet to be discovered. This begs the question, “what must a tuning system and scale require to be “perfect” both mathematically and musically?”

CHAPTER FOUR

The "Perfect" Tuning System

Overview

Throughout this work, we have seen the development of four of the main tuning systems in Western music, and we have noted what imperfections they possess. This was done with one goal in mind: to prove more or less that there is no tuning system that can satisfy all expectations and conditions required of a "perfect" tuning system. Thus, we will compose a list of principles that a tuning system should possess from the shortcomings we have observed. Then we will demonstrate how no tuning system can achieve all of the principles without certain imperfections arising.

First we must define perfection and what we mean when we theorize about a "perfect" tuning system. The definition of perfection is "satisfying all requirements" ("Perfect" Merriam-Webster). Additionally, a tuning system, as we have seen, is a system for defining tones and pitches that can be used when playing music. A general assumption about a perfect tuning system would be one that sounds pleasing or "perfect" to the ear, however, we have shown that some of what makes a tuning system sound "good" is subjective and varies based on the listener. We will show more concrete reasons for the imperfections in tuning systems by compiling a list of requirements for a "perfect" tuning system using our knowledge of four major tuning systems of Western music: Pythagorean Tuning, Just Intonation, Meantone Temperament, and Equal Temperament.

We will first separate our principles between those of a more concrete and physical nature and those that are more subjective and undefinable. This will be done with the main goal of proving in a more real sense that a "perfect" tuning system has yet to be discovered, and in a more philosophical/theological sense to ultimately argue that there can never be a "perfect" tuning system outside of Heaven, because we do not possess the understanding to develop one since God's ways are higher than our own and we are too narrow-minded. Before we breakdown what each principle means and how it relates to our main goal, we will give a general list below of each principle:

- *Physical Principles*
    - A tuning system must work in every musical key … (a)
    - A tuning system must work in different instrumental contexts … (b)
    - A tuning system must work across different countries and cultures … (c)
    - A tuning system must be independent of the time period it is created in … (d)
- *Abstract Principles*
    - A tuning system must be able to hold in practice … (e)
    - A tuning system must be "agreeable" to the human ear … (f)

*Note: We are operating under the assumption that a tuning system will have a perfect octave with 12 semitones in an octave.

Principles

Physical Principles

Our first principle is that a tuning system must work in every musical key. This follows from the shortcomings for Just Intonation and Pythagorean Tuning. Both systems fell short when they would change keys, forcing the instrumentalist to re-tune their instrument to that key. In Pythagorean Tuning this was a result of the Pythagorean Comma, a small discrepancy in the frequency ratios that gradually accumulated over the span of the six octaves in the circle of fifths. In Just Intonation this was caused by the discrepancy of having two different whole tones which made it difficult to transpose or modulate pieces. There were different solutions offered for this problem, but it was ultimately solved through the creation of Equal Temperament which provided 12 equal semitones in an octave. Thus naturally it follows that a “perfect” tuning system must allow for easy modulations in key without the instrumentalist having to adjust or re-tune. It is also this need for a perfect octave in order to change keys that caused a perfect octave to be an assumption for a “perfect” tuning system.

The second principle we will define is that a tuning system must work in different instrumental contexts. Through our breakdown of four major tuning systems, we have found that a tuning system’s shortcomings are often revealed when used among different instrument groupings. For example, as mentioned previously, one of Equal Temperament's few shortcomings is that it is not the preferred tuning system for some musicians who use string instruments or voice, but is the tuning system used among keyboard and fretted instruments. This causes string instruments and vocalists to tune/sing to slightly different pitches when performing with an instrument like the piano than they would performing the same piece alone. The reasoning for string instrumentalists and vocalists to not always perform in Equal Temperament was described previously and falls under principle (f). Despite this, a “perfect” tuning system would be used and preferred across varying instrument types and would not cause dissonance in an ensemble if left unaccounted for. Thus, the individual ratios in said tuning system should be the same, no matter the instrument or musician playing/singing them. Each of the tuning systems described in this work do not meet this principle and it was often one of the errors that caused people to keep searching for another tuning system. It is also important to note that although Equal Temperament is not preferred across instrument types, since it meets principle (a), it is easier to fix when the need arises. This is why Equal Temperament is the main tuning system used today and is objectively one of the more”perfect” tuning systems that has been developed.

The third physical principle that a tuning system must have is that it must work across different countries and cultures. By this we mean that a tuning system would need to work beyond Western music into other realms of music outside the scope of this paper. Many other cultures make use of different instruments not regularly used in Western music that could potentially clash with Western musical concepts and tuning (like what intervals are consonant and their corresponding ratios). Additionally, other cultures have different musical tendencies, just like different genres in Western music have different tendencies. This is in part because “music is not in fact an international language” (*East Asian Arts - Traditional, Instruments, Melodies | Britannica*). Many different cultures and musical styles clash with one another because they are built from different *closed* tuning systems, so “the musical facets mesh perfectly within a given system, but they often may prove difficult or impossible to transfer to another system” (*East Asian Arts - Traditional, Instruments, Melodies | Britannica*). This is a result of different countries or cultures having their own distinct historical developments and relationships with music set apart from one another by geographical location. Differences in how music was established and used varies by culture and resulted in variation in musical sound and what is perceived as logical and pleasing to the ear. Therefore, a "perfect" tuning system would be one that could transcend cultural boundaries and musical styles and be accepted by all, meaning that its individual ratios for each musical interval would be the same across these cultural boundaries. This principle is one that has not been met by any tuning system, despite the similarities of some systems, due to this closed manner in which they were developed and in the variation in musical styles throughout history.

This third principle leads us into our fourth principle, that is that a tuning system must be independent of the time period it is created in. Much of the variation in music history and development in different countries or cultures described above is related to the limitations of their time period. We have seen this limitation in some of our tuning systems from Western music. For example, in Pythagorean Tuning we noted how the musical trends of the time period caused the Pythagoreans and many others not to notice or not care that their tuning system was dissonant when multiple instruments were played together, like in an orchestra. Since polyphonic music and harmony was not as popular in that time, this dissonance did not become a problem needing a solution until the Renaissance when instruments started being used for solo and ensemble work and not just to accompany vocalists. Similarly, many tuning systems created before modern sound wave/frequency technology and research were reliant on the human ear and what was deemed subjective by the listener, which was also a limitation of the time period they were in. These limitations naturally lead us to our principle that a "perfect" tuning system must be independent of such time period limitations, however, it is impossible to say if we have achieved this as we do not fully know what we have yet to discover. Thus, any tuning system that we would try to create today could still fail this principle on the grounds that we do not know if it is being restricted by our current knowledge or use of instruments and music theory. Despite this limitation of the present, we can still conclude that a "perfect" tuning system would need to work in the future when more has been discovered, meaning that the interval ratios would remain the same no matter the time period in which they are played.

Abstract Principles

As mentioned above, our abstract principles are more subjective and undefinable. Our first abstract principle is that a tuning system must hold in practice. By this we mean to say that a tuning system's ratios and intervals must be able to be played exactly every time. This follows from the idea that if we are to set a discrete ratio for an interval, then any ratio that is not exactly equal to that set ratio is incorrect and "out of tune". For instruments like the piano, errors in tuning are due to hitting a wrong note rather than improperly tuning the interval. This is because the pianist doesn't set the ratios and intervals on a piano, but rather the strings are tuned to a specific tuning system that produces consistent interval ratios (excluding the fact that instruments wear over time and fall out of tune). However, this does not hold across instrument types, as any instrument that relies on the musician to place a finger in the proper place or blow with the proper amount of air, does not hold in practice. Such instruments are dependent on humans for the tuning of intervals and will not be produced exactly when played in practice. This was described in the overall shortcomings of Pythagorean Tuning, and is also an applicable shortcoming to any tuning system. In that section we described how instruments that are reliant on musicians for exact interval tuning, will not be perfectly in tune to the set tuning system ratio every time they are played because musicians are not exactly measuring each note before it is played. Rather they adjust based on how it sounds and sometimes move too quickly to properly adjust even by ear. Thus it follows that a "perfect" tuning system would be able to have the exact same interval ratio played every time the corresponding pitches are played. This would mean that the tuning system could not be dependent on humans to measure out the intervals, as humans are imperfect and will not play the exact ratio every time.

However, one might argue that this dependence on humans is what makes music so compelling, as it isn't the same every time. No performance is the same due to human imperfection and variation. Thus, it is up to the reader to decide if human imperfection takes away from or adds to music and whether that imperfection makes the tuning system itself imperfect. In regards to our endeavor of defining a "perfect" tuning system, however, the set intervals and ratios must be able to be played consistently and reliably every time or they are deemed out of tune of that tuning system.

This leads us into our last abstract principle, that a tuning system must be "agreeable" to the human ear. In our description of consonance and dissonance, we noted that scales and tuning systems are founded on a mixture of intervals that are deemed consonant and intervals deemed dissonant. However, as mentioned previously, this idea of consonance and dissonance has a subjective element that varies by person. Despite this, the goal of creating a tuning system is to find a system that works both mathematically and musically. To work musically, we know it must be pleasing to the ear, but there is a way to define "pleasing to the ear" mathematically. As discussed in Chapter 2, when a note is played there are certain overtones or harmonics that can sometimes be heard as the pitch is played. It is this sequence of harmonics/overtones that helps us dictate what sounds "agreeable" and pleasing to the ear. This is a result of beat elimination, which is "the main method of tuning between instruments" that was described previously (DuBose-Schmitt 33). Since the goal of this tuning method is "to match wavelengths of [the two pitches'] individual Fourier Series," the result is a tuning system that follows the harmonic/overtone series (DuBose-Schmitt 33). Thus a "perfect" tuning system would agree with the natural harmonics that can be heard when a pitch is played and would line up these harmonic overtones when an interval of two pitches is played. We must acknowledge that this mathematical definition could still be deemed unpleasant to the ear by a listener since they are the foremost judge on what is pleasing, but due to the nature of harmonic overtones and the method of beat elimination tuning we can still conclude that a "perfect" tuning system would most likely follow the harmonic/overtone series.

Additionally, through this principle we can prove that Equal Temperament is by definition not "perfect" as it is not the most "agreeable" to the ear. If we have 12 semitones in an octave that are equal to each other, then it is by nature irrational as the $\sqrt[12]{2}$ is irrational. Thus, the irrational intervals contained in Equal Temperament are not equal to the rational intervals in the harmonic/overtone series. Therefore, Equal Temperament is not "agreeable" to the human ear and is not a "perfect" tuning system.

Is There a "Perfect" Tuning System?

Thus far, we have used our knowledge of how the major tuning systems of Western music were developed and why they are not used or preferred to create a list of principles that one would presumably desire in a "perfect" tuning system. Having now defined each of our principles, we will prove that there is no "perfect" tuning system that has been created and show how no tuning system can meet all of these principles due to the limitations of human knowledge and ability and the subjectivity of how music is perceived.

What Physical Principles Do the Major Tuning Systems Follow?

Since our principles were derived from the shortcomings of Pythagorean Tuning, Just Intonation, Meantone Temperament, and Equal Temperament, we know that each of these systems fails to meet at least one of the principles. In order to show how we have yet to find a "perfect" tuning system, we will first detail which of the physical principles these four major tuning systems do follow.

Pythagorean Tuning. For Pythagorean Tuning, we find that it fails to meet all of our physical principles. We can definitively say that Pythagorean Tuning does not work in different instrumental contexts in part due to the fact that it was not independent of its time period (principles (b) and (d)). The dissonance that was revealed when multiple different instruments were played together under Pythagorean Tuning showed that it did not hold across instrument types without creating a dissonance, because the interval ratios were not equal for every instrument type. This was revealed many centuries later when the social role for musical ensembles changed and instrumentalists started playing together rather than only accompanying vocalists. We also know that Pythagorean tuning does not work in every musical key, due to the Pythagorean Comma (principle (a)). The Pythagorean Comma revealed a dissonance in the interval of the Perfect Octave, and thus also failed our assumption that a "perfect" tuning system would have a perfect octave. Furthermore, it is hard to say whether Pythagorean Tuning works across different countries and cultures, but due to the presence of the Pythagorean Comma and the fact that it is not regularly used in Western music today much less outside of Western music, it is implied that Pythagorean Tuning does not hold under principle (c) either. Thus, we can say that Pythagorean Tuning is not a "perfect" tuning system.

Just Intonation. Although Just Intonation remains in use today by some musicians, we can also see how it does not follow all of our physical principles and is consequently not "perfect." Just Intonation does seem to hold in different instrumental contexts, assuming one were to develop a keyboard instrument that follows the Just Intonation ratios; however, keyboard instruments like the piano which rely on an equal whole tone and semitone for functionality of playing would not work under Just Intonation, so Just Intonation does not meet principle (b). Just Intonation does not work in every musical key due to the prevalence of two different sized whole tones (principle (a)). Additionally, Just Intonation appears to be independent of its time period (principle (d) since it follows the harmonic series (principle (f)). However, since it is not primarily used across different countries and cultures (or even predominantly in Western music) it does not appear to follow principle (c). Thus, despite Just Intonation being an improvement on Pythagorean Tuning, as it follows one of the four physical principles and even one of the abstract principles, it does not meet all of them and is thus not "perfect."

Meantone Temperament. Although not a major tuning system in Western music, Meantone Temperament was a stepping stone in music history connecting Just Intonation and Equal Temperament. Due to its dissonance between certain enharmonic pitches, Meantone Temperament does not meet our principle of working in every musical key (principle (a)). It also does not work in different instrumental contexts as it was mainly used for keyboard instruments and not among other instrument types despite the fact that it had equal whole tones and thus the potential to work better in different instrument groupings (principle (b)). Furthermore, it follows from its short time of being in use and its enharmonic dissonance, that Meantone Temperament would also not work across different countries and cultures or outside of its time period otherwise it would still be in use today (principle (c) and (d)). It may appear from Meantone Temperament's failure to meet any of our physical principles that it was a step back from Just Intonation, but Meantone Temperament was a solution to the two different sized whole tones in Just Intonation. Despite this advantage, Meantone Temperament is still an imperfect tuning system since it fails to meet our physical principles.

Equal Temperament. Lastly, we have Equal Temperament which meets a couple of our principles. Since Equal Temperament is the most recent on our timeline and is still heavily used today, it is arguably one of the better tuning systems we have established for music. This is due to its use of a perfect octave split equally by 12 semitones (which falls under our assumption for a “perfect” tuning system). The 12 equal semitones structure, as described previously, fixed many dissonance issues present in our other systems: it fixed the Pythagorean Comma by creating a perfect octave that was equally divided; it solved some of the issues in Just Intonation with equal whole tones and semitones; and it fixed the dissonance present between certain enharmonic notes that occurred in Meantone Temperament. Thus, Equal Temperament works in every musical key (principle (a)). It also appears to work across countries and cultures, at least more so than any other tuning system created, as it is the most widely used tuning system today (principle (c)). One could also argue that it is independent of the time period, as it is not based solely off of today’s technology, but rather is an equal division of the octave. However, since it uses irrational numbers, one could argue the reverse, that it would not be accepted in Ancient Greece, and is thus not independent of time period. For our purposes, we shall follow the line of thought that a tuning system must be independent of time going forward, because it is not logical to assume a system must work in a time period before its development, since that time period is restricted by what hasn’t been discovered yet. For example, the Ancient Greeks not accepting irrational numbers should not impact if Equal Temperament is independent of its time period, as we have now acknowledged and seen the use of irrational numbers. Furthermore, Equal Temperament is not independent of the instrumental context in which it is used, and thus fails to meet principle (b). This is because instrumentalists who use relative tuning and follow Just Intonation more closely in solo practice have to adjust when playing with an instrument tuned to Equal Temperament. From these physical principles, we can see how Equal Temperament is not a “perfect” tuning system. We also noted this in our description of principle (f), as Equal Temperament does not follow the harmonic series.

Has A “Perfect” Tuning System Already Been Developed?

In summary, we have shown that each of the major tuning systems used throughout the history of Western music are not “perfect” as defined at the beginning of our discourse, because they fail to meet our requirements of a “perfect” tuning system. Even Equal Temperament, which is the foremost tuning system used globally (especially in regards to Western music), does not meet each of the four physical principles that we deduced from our analysis of these major tuning systems. Furthermore, the prevalence of a more perfect tuning system outside of the systems seen in Western Music is beyond the scope of this paper, but one can assume that if there was a more perfect system that has been developed outside of Western music then it would be adopted globally. In any case, a future area of research in this topic would be to see if our list of principles would be modified or added to by looking into the history of music and the different tuning systems outside of Western music. Based on the scope of this paper, we can currently conclude that a “perfect” tuning system has yet to be developed.

Can there be a “Perfect” Tuning System?

Mathematical Reasoning

Since we have established that none of the major tuning systems of Western music are “perfect” (and presumably neither are any outside of Western music), it begs the question if such a tuning system is in the realm of possibility. To answer this question we have derived three mathematical criteria and several propositions from our physical and abstract principles that we will use to prove that no such tuning system can exist due to the mathematical inconsistencies that arise between two of the criteria.

We are assuming that there are 12 semitones in an octave and that a perfect tuning system will have a perfect octave. Let $r_1, r_2, \ldots, r_{12}$ be the ratios between consecutive pitches of the tuning system such that if x is the frequency for the tonic of a scale with ratios $r_1, r_2, \ldots, r_{12}$, then the pitches of the chromatic scale are $x, xr_1, xr_1r_2, \ldots, xr_1 \ldots r_{12}$. Note that not all semitones are equal and will not be unless all r_n are equal. Through the shortcomings we have discussed and the assumptions we are making, we have derived the following criteria, propositions and corollaries.

- Criterion I: $r_1r_2 \ldots r_{12} = 2$.
  - It follows from this criterion that $xr_1 \ldots r_{12}$ is a perfect octave above x. This is derived from principles (a) and (f), as a perfect octave is needed for a tuning system to work in every musical key and for it to sound "agreeable" to the human ear.
- Criterion II: $xr_1 \ldots r_{n+m} = (xr_1 \ldots r_n)(r_1 \ldots r_m)$ for all n, m.
  - The Left-hand side of the equation is a pitch that is $n + m$ semitones above x. The Right-hand side of the equation is a pitch that is m semitones above the pitch that is n semitones above x. This shows how $n + m$ semitones above x is the same as n semitones above x multiplied by m semitones. This comes from principle (a), that a tuning system must work in every musical key

and is what leads into our later proposition that supports Equal Temperament.

- Criterion III: $r_1 r_2 \ldots r_{12}$ satisfy the following.
    - $r_1 r_2 = 9/8$
    - $r_1 r_2 r_3 r_4 = 5/4$
    - $r_1 r_2 r_3 r_4 r_5 = 4/3$
    - $r_1 r_2 r_3 r_4 r_5 r_6 r_7 = 3/2$
    - $r_1 r_2 \ldots r_{12} = 2$

        - These ratios correspond to the ratios in the overtone sequence. This comes from principle (f), that a tuning system must be "agreeable" to the human ear.

These 3 mathematical criteria exhibit the main requirements of a tuning system that we have found to be needed in order to meet our physical and abstract principles. Criteria I and II align well with one another and are both met in Equal Temperament. They require that there be a perfect octave and that each ratio in the octave is equal, as noted below. Contrarily, Criterion III establishes that certain products of ratios must equal the corresponding overtone sequence, which is found in Just Intonation. Although there is certainly room for more research and possibly more principles, it is these core criteria that we think a tuning system must uphold in order to be deemed "perfect." However, these criteria cannot be upheld simultaneously.

Consequences of Criterion II

Proposition A: Assume Criterion II is true. Then $r_n = r_1$ for $n = 1, 2, \ldots, n$.

Proof. Trivial for $n = 1$. If $n > 1$, then

$xr_1 \ldots r_n = (xr_1 \ldots r_{n-1})r_1$

$(xr_1 \ldots r_{n-1})r_n = (xr_1 \ldots r_{n-1})r_1$

$r_n = r_1$. ■

Corollary: $r_{n+1} \ldots r_{n+m} = r_1^m$ for all $m, n \in \mathbb{N}$.

Note: $xr_{n+1} \ldots r_{n+m} = xr_1^m$ is m semitones above x.

- The ratio associated with the interval of m semitones is the same between any two pitches in the scale that are separated by m semitones

Consequences of Criteria I and II

Proposition B: Assuming Criteria I and II are true, then $r_n = \sqrt[12]{2}$ for each n.

Proof. Let $r_1 r_2 \ldots r_{12} = 2$. From Proposition A, we know that $r_n = r_1$ for $n = 1, 2, \ldots, n$. Hence, $r_n^{12} = 2$. Therefore, $r_n = \sqrt[12]{2}$. ■

Underlying Mathematical Discrepancy

Proposition C: No tuning system satisfies both Criteria II and III.

Proof. Assume that Criterion III is true. Then $r_1 r_2 = 9/8$ and $r_1 r_2 r_3 r_4 = 5/4$. Assume for the sake of establishing contradiction that Criterion II is also true, then from the corollary to Proposition A, $r_1 r_2 = r_1^2$ and $r_1 r_2 r_3 r_4 = r_1^4$. Then it follows that $r_1^4 = (r_1^2)^2 = (9/8)^2 = 81/64 \neq 5/4 = r_1 r_2 r_3 r_4$ under the conditions of Criterion III. Hence, Criterion III cannot be true if Criterion II is true. ■

Through our derivation of the three criteria, as well as the subsequent propositions and corollaries, we have been able to show that a "perfect" tuning system cannot exist mathematically. This is because it is mathematically impossible for a tuning system to both have equal interval ratios (Criterion II) and to follow the harmonic overtone sequence (Criterion III).

Beyond Mathematical Reasoning

Furthermore, there is also a level of impossibility that lies outside of the realm of mathematics, and that is the fallibility and inconsistency of human beings. As discussed previously, since humans are imperfect and inconsistent, we cannot play the same piece of music the same every time. We will mess up different notes each time whether by playing a wrong note or playing a note slightly out of tune. Thus, any tuning system which tries to assign an exact mathematical measurement to interval ratios will fail to be upheld when it comes to the execution on an instrument that relies on the musician to set the interval ratio. This is because humans are incapable of playing something perfectly as we are imperfect. Due to our imperfections, the execution of musical intervals has been impacted and no tuning system will be able to be played exactly every time on every instrument type. Even if we were to develop one that is mathematically perfect and fulfills some set of mathematical criteria, it would still fall short in its execution.

Consequently, even if a person were able to play the exact intervals laid out in a tuning system, it could still fall short of our last abstract principle (principle (f)). That is, a tuning system must be “agreeable” to the human ear. It is this principle that has been the driving force behind many different tuning systems and preferences about tuning systems. As we have discussed previously, most tuning systems are a product of someone having a specific goal in mind or a specific idea of what intervals are consonant and how they should sound. For example, Pythagorean Tuning was based off of the octave and the fifth, but Meantone Temperament used the consonance of the major third to create its fifth. It is this disagreement over what is deemed consonant that ultimately inhibits the creation of a “perfect” tuning system. This is in part due to the nature of the human ear, and how no person perceives music the same way. Thus, something deemed dissonant to one person can sound consonant to another. Additionally, how someone perceives music is impacted by what they are used to hearing and their musical background. This is seen primarily between people from different musical backgrounds and cultures. For example in Western music, the tritone - the interval of 3 whole steps - is deemed dissonant, but someone accustomed to Asian music rather than Western music might think it is consonant and pleasing to hear. However, we were able to conclude that for a “perfect” tuning system to be pleasing to the ear it must follow the harmonic series, since the sequence of overtones that can be heard when a pitch is played are the same across cultures and instruments. Even with this definition, as we noted before, if the listener were to think that an interval following the harmonic series were not pleasing then that tuning system would still fail to meet principle (f) as the musical definition of pleasing to the ear (which varies based on the listener) overrides the mathematical definition. Therefore, the imperfection of humans and the inconsistency of the human ear and how it perceives music impacts the development of a tuning system and leads us to the idea that no notion of perfection created by humans will ever see the true heavenly perfection with perfect clarity.

This concept is not new, as many have theorized about the idea of a perfect concept existing outside of our perceived reality and how humans can only perceive the imperfect recreations of such concepts. This is touched on by many philosophers and theologians and is one that our discourse adheres to. “Plato theorized that the physical phenomena of everyday objects and human existence were an imperfect realization of an ideal perfect world lying behind them, and that human existence should endeavor to attain to that perfect world” (Parncutt and Hair 2). In concordance with this idea, we would argue that each of the musical tuning systems developed by humans is an imperfect realization of what was created by God in Heaven. This is because we cannot fully see or understand the full splendor of what God has created. This idea is also seen in Dante’s *Paradiso*, when Dante first arrives in Paradise and sees one of his grandfathers. At first Dante is confused and cannot understand what he is saying and can only grasp the emotion that his grandfather conveys. Through this interaction, we see Dante exemplifying how the things in Heaven are beyond human comprehension and cannot be grasped without God’s guidance and provision. It is this train of thought that leads us to believe that it is not until we get to heaven that we will experience music as it is supposed to sound in all of its beauty and consonance, and it will be mathematically perfect as well as conceptually and spiritually perfect. This is because “[His] ways are higher than [our] ways and [His] thoughts higher than [our] thoughts” (Isaiah 55:9 NLT). It is with this idea in mind that we would argue that not only have we yet to create a “perfect” tuning system according to our own criteria, but we will never be able to make or discern a perfect tuning system that meets the true heavenly notions of perfection.

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