Covering Edge-Coloured Graphs by Monochromatic Paths and Cycles

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A Dissertation submitted to the Department of Mathematics of the London School of Economics and Political Science for the degree of Master of Science

September 9, 2016

Summary

A problem in graph theory asks for the minimum number of vertex-disjoint monochromatic paths or cycles needed to cover the vertices of any r-edge-coloured complete graph. This problem is closely related to calculating generalised Ramsey numbers and has recently received a lot of attention.

Our aim is to make the proofs of these results more accessible to those less familiar with the field, as well as to correct any errors. We also offer a new elementary proof of the $r = 2$ case of a conjecture by Pokrovskiy, which says that at most $2r - 1$ vertex-disjoint monochromatic paths are needed to cover the vertices of any r-edge-coloured complete balanced bipartite graph.

A conjecture due to Gyárfás asks whether the minimum number of vertex-disjoint monochromatic paths needed to cover the vertices of any r-edge-coloured complete graph is at most r. We will see an elegant proof by Gyárfás of the $r = 2$ case of this conjecture. The $r = 3$ case was more recently verified by Pokrovskiy. Gyárfás' conjecture remains open for all $r \geq 4$.

A stronger conjecture, posed by Erdös, Gyárfás and Pyber, says that the same result holds if we replace "paths" by "cycles". Pokrovskiy recently established that this is not in fact true for any number of colours greater than two. An approximate version of the two-colour case was shown to be true by Gyárfás using the fact that any two-edge-coloured complete graph contains a Hamilton path with at most one colour change. A large part of this dissertation is devoted to a proof by Bessy and Thomassé of Lehel's conjecture, which says that the vertices of any two-edge-coloured complete graph can be covered by two vertex-disjoint cycles of different colours.

Contents

Chapter 1

Introduction

Ramsey theory is a branch of combinatorics concerned with finding organised substructures in what seem to be highly disorganised mathematical objects. Problems in Ramsey theory seek to determine the exact conditions under which these substructures appear. A simple and well-known example is the Pigeonhole Principle, which says that if you place $n + 1$ pigeons into n boxes then at least one box will contain more than one pigeon.

There exist many interesting graph-theoretic examples of Ramsey theory (for a comprehensive introduction to graph theory, we recommend [4]). For instance, a special case of Ramsey's famous theorem says that for all positive integers s and t, there exists a number $R(K_s, K_t)$ such that any complete graph of order at least $R(K_s, K_t)$, whose edges have been coloured in red and blue, contains either a complete red subgraph of order s, or a complete blue subgraph of order t. The numbers $R(K_s, K_t)$ are known as *Ramsey numbers*. In what follows, we will call a graph whose edges have been arbitrarily assigned the colours $1, \ldots, r$ an *r-edge-coloured* graph.

In this dissertation we investigate a number of graph theoretic results associated with Ramsey theory. In particular, we consider the problem of finding the minimum number of vertex-disjoint monochromatic paths or cycles needed to cover the vertices of an r-edge-coloured graph. Hereafter, for the sake of brevity, we will use "disjoint" to mean "vertex-disjoint". For the sake of simplicity, we consider only graphs having a finite number of vertices, single edges and no loops.

In what follows we will use Diestel's [4] terminology and notation unless otherwise specified. For instance, unlike Diestel, we will regard the length of a path or cycle Q as the number of vertices on Q, and we will denote by K_n the complete graph of order n. Also, we will use "log" to denote the natural logarithm and we will consider the empty set, single vertices and single edges as monochromatic paths and cycles. Removing a vertex x from a path or cycle Q will be denoted by $Q - x$. Similarly removing an edge xy from Q will be denoted by $Q - xy$. Adding an edge xy to a path Q which already contains the vertices x and y will be denoted by $Q + xy$.

The following conjecture due to Gyárfás [8] was inspired by a result of Rado [22]:

Conjecture 1. *At most r disjoint monochromatic paths are needed to cover the vertices of any r-edge-coloured K_n.*

In [15], Lang and Stein showed that Conjecture 1 is best possible by constructing, for each $r \geq 2$, an r-edge-coloured complete graph whose vertices cannot be covered by fewer than r disjoint monochromatic paths. The $r = 2$ case of Conjecture 1 follows from a theorem of Gyárfás and Gerencsér, first proved in a footnote of [6]. Their theorem states that every two-edge-coloured K_n contains a Hamilton path with at most one colour change. We give an exposition of Gyárfás' algorithmic proof [7] of this theorem in Chapter 2. The $r = 3$ case of Conjecture 1 was proved by Pokrovskiy in [20]. The $r \geq 4$ case remains open.

In [5], Erdös, Gyárfás and Pyber showed by an example that the minimum number of disjoint monochromatic cycles needed to cover the vertices of any r-edge-coloured K_n is at least r. Consequently, in the same paper, they proposed the following stronger version of Conjecture 1:

Conjecture 2. *At most r disjoint monochromatic cycles are needed to cover the vertices of any r-edge-coloured K_n.*

Notice that the $r = 1$ case of Conjectures 1 and 2 is trivially true. Also, if Conjecture 2 holds then Conjecture 1 also holds, but the reverse is not true. The $r = 2$ case of Conjecture 2 is a weaker version of the following, earlier conjecture by Lehel:

Conjecture 3 (Lehel)**.** *The vertices of any two-edge-coloured K_n can be covered by two disjoint monochromatic cycles of different colours.*

Lehel's conjecture was first proved only for graphs with a very large number of vertices by Łuczak, Rödl and Szemerédi [17]. In 2008 Allen [1] proved this result for graphs with a smaller, but still very large number of vertices. In Chapters 3 and 4 we give an exposition of a more recent proof by Bessy and Thomassé [3], which holds for complete graphs on any number of vertices. Bessy and Thomassé use as a starting point the following weaker version of Lehel's conjecture by Gyárfás [7]:

Theorem 4. *The vertices of any two-edge-coloured K_n can be covered by two monochromatic cycles of different colours which share at most one vertex.*

We give an exposition of Gyárfás' algorithmic proof [7] of Theorem 4 in Chapter 2. Bessy and Thomassé [3] consider the case where the cycles in Theorem 4 do share a vertex. They show, by considering certain properties of these structures, that one can always find two *disjoint* monochromatic cycles of different colours covering the vertices of any two-edge-coloured K_n.

In [12] Gyárfás, et al. showed that all but $o(n)$ vertices of any three-edge-coloured K_n can be covered by three disjoint monochromatic cycles. In the same paper, they used this result to show that at most 17 cycles are needed to cover *all* the vertices. Lang, Schaudt and Stein [14] claim that this number can be reduced to 10.

In [5], Erdös, Gyárfás and Pyber showed that, for general r, at most $cr^2 \log r$ cycles are needed, where c is a constant. Gyárfás, et al. [11] have since shown that, for large enough graphs, only $100r \log r$ cycles are needed. However, in Chapter 5 we describe counterexamples by Pokrovskiy [20] to Conjecture 2 for all $r \geq 3$.

There has also been some progress in covering the vertices of edge-coloured graphs other than the complete graph. We call a bipartite graph *balanced* if the two parts of its bipartition contain the same number of vertices. We will denote the complete balanced bipartite graph by $K_{n,n}$, where n is the number of vertices in each part of the bipartition.

With the hope of using their results to solve related problems on complete graphs, in [5] Erdös, Gyárfás and Pyber investigated the problem of covering the vertices of an edge-coloured $K_{n,n}$ by disjoint monochromatic cycles. They asked whether the number of disjoint monochromatic cycles needed to cover the vertices of any r-edge-coloured $K_{n,n}$ is a function or r alone. Haxell [13] confirmed that this is true. In [20] Pokrovskiy proposed the following conjecture:

Conjecture 5. *At most $2r-1$ disjoint monochromatic paths are needed to cover the vertices of any r-edge-coloured $K_{n,n}$.*

Pokrovskiy [20] showed that Conjecture 5 is optimal by constructing, for each r, an r-edge-coloured $K_{n,n}$ whose vertices he claims cannot be covered by $2r-2$ disjoint monochromatic paths. In the same paper, Pokrovskiy shows that if certain colourings of a two-edge-coloured $K_{n,n}$ are avoided, then its vertices can be covered by two disjoint monochromatic paths. He then shows that if we allow for these problematic colourings, then at most three paths are needed. This verifies the $r = 2$ case of Conjecture 5. So we have the following theorem:

Theorem 6. *At most three disjoint monochromatic paths are needed to cover the vertices of any two-edge-coloured $K_{n,n}$.*

In Chapter 6 we give a self-contained proof of Theorem 6. We also offer a second, simpler proof of this result in Chapter 7.

In [13] Haxell proved that at most 1695 disjoint monochromatic cycles are needed to cover the vertices of any three-edge-coloured $K_{n,n}$. Lang, Schaudt and Stein [14] have since shown that at most 18 disjoint monochromatic cycles are needed for large enough graphs. They also showed [14] that at most 5 disjoint monochromatic cycles are needed to cover all but $o(n)$ vertices of any three-edge-coloured $K_{n,n}$. Conjecture 5 remains open for all $r \geq 3$.

For general r, Peng, Rödl and Ruciński [18] claim that at most $O(r^2 \log r)$ disjoint monochromatic cycles are needed to cover the vertices of any r-edge-coloured $K_{n,n}$, improving the earlier bound of $O((r \log r)^2)$ cycles by Haxell [13].

Chapter 2

Results by Gyárfás

2.1 Motivation

We define a simple path [7] to be a two-edge-coloured path with at most one colour change, so the path is either monochromatic or the union of a red path and a blue path. In this chapter we give an exposition of Gyárfás' algorithmic proofs [7] of the following two theorems:

Theorem 7 (Gerencsér, Gyárfás [6])**.** *Every two-edge-coloured K_n contains a simple Hamilton path.*

Theorem 4 (Gyárfás [7])**.** *The vertices of any two-edge-coloured K_n can be covered by two monochromatic cycles of different colours which share at most one vertex.*

Observe that Theorem 7 verifies the $r = 2$ case of Conjecture 1. To prove Theorem 7 we construct simple paths in K_n of increasing length, starting with a path on any vertex in K_n. The path of length n is the simple Hamilton path we are after.

The algorithmic proof of Theorem 4 uses as a subroutine the algorithmic proof of Theorem 7. Its output, a red cycle and a blue cycle sharing at most one vertex, is the starting point of Bessy and Thomassé's proof [3] of Lehel's conjecture discussed in Chapters 3 and 4. Recall that the empty set, singletons and single edges are considered monochromatic paths and cycles.

2.2 Existence of a Simple Hamilton Path

In this section we state and prove Theorem 7.

Proof. Suppose that the edges of some K_n have been coloured in red and blue. We claim that Algorithm 1 constructs a simple Hamilton path in K_n. An outline of this algorithm is as follows. If $n \leq 2$ then K_n is a simple path so we are done. If $n > 2$ then we first construct the simple paths P_1 and P_2 on any one and two vertices of K_n respectively. Then, for $3 \leq i \leq n$, we construct the simple path P_i by joining an unused vertex in K_n to the path P_{i-1}. The path P_n is the simple Hamilton path we are after.

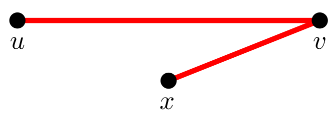
This algorithm takes as input a two-edge-coloured K_n and returns a simple Hamilton path in K_n. Note that in lines 3 and 4 we make an arbitrary choice about which endvertex of P_{i-1} is the "first" vertex of the path and which is the "last" vertex; this is not intended to imply that P_{i-1} is a directed path. We will then use xP_{i-1} to denote joining x to the "first" vertex of P_{i-1}, and $P_{i-1}x$ to denote joining x to the "last" vertex of P_{i-1}.

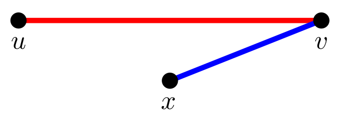
Algorithm 1: SimplePath(K_n)

```
if n ≤ 2 then
 |  return K_n;
end
Choose vertices a, b ∈ K_n;
1 Set P_1 := a;
2 Set P_2 := ab;
foreach 3 ≤ i ≤ n do
 |  Choose a vertex x in the set K_n − P_{i−1};
3|  Set u to be the first vertex on P_{i−1};
4|  Set v to be the last vertex on P_{i−1};
 |  if P_{i−1} is monochromatic or P_{i−1}x is simple then
 |   |  Set P_i = P_{i−1}x;
 |  else if xP_{i−1} is simple then
 |   |  Set P_i = xP_{i−1};
 |  else if xv(P_{i−1} − v) is simple then
 |   |  Set P_i = xv(P_{i−1} − v);
 |  else
 |   |  Set P_i = (P_{i−1} − u)ux;
end
return P_n;
```

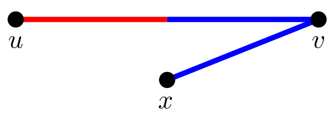
Observe that Algorithm 1 terminates and that all lines are possible. We will now prove that the algorithm is correct. If $n \leq 2$ then K_n is a simple path and is returned by the algorithm so we are done. For $n \geq 3$ we use induction to show that for all $1 \leq k \leq n$ the algorithm finds a simple path P_k of length k in K_n. For $k = 1$ or $k = 2$ (base cases) P_k is defined in lines 1 and 2 and is trivially simple. So suppose that $k \geq 3$ and assume that P_{k-1} was constructed correctly. We suppose arbitrarily that the first edge of P_{k-1} is red. If the first if condition within the for-loop is satisfied then one of the following cases must hold:

- The path P_{k-1} is only red, so joining x to the last vertex of P_{k-1} forms a simple path of length k.





- The path P_{k-1} consist of a red path followed by a blue path and vx is blue so $P_{k-1}x$ is a simple path of length k.

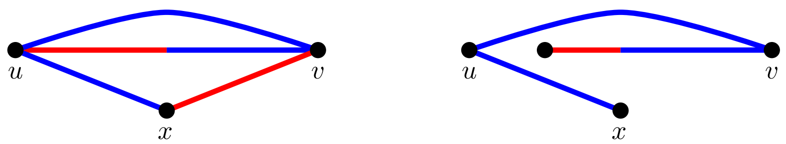


Thus, if the first if condition is *not* satisfied, then P_{k-1} consists of a red path followed by a blue path and vx is red (as in Figure 2.1). Then if the second if condition is satisfied ux must be red so that xP_{k-1} is a simple path.

Figure 2.1

If the second if condition is not satisfied ux must be blue. Then if the third if condition is satisfied uv must be red so that $xv(P_{k-1} - v)$ is a simple path.

Finally, if the third if condition is not satisfied then uv must be blue, but then $(P_{k-1} - u)ux$ is a simple path so we are done.



□

2.3 Two Almost Disjoint Cycles of Different Colours

Here we prove Theorem 4.

Proof. Suppose that the edges of some K_n have been coloured in red and blue. We claim that Algorithm 2 finds a red cycle and a blue cycle which cover the vertices of K_n and which share at most one vertex. The outline of the algorithm follows.

If $n = 1$ or $n = 2$ then K_n is a single vertex or a single edge respectively. In the former case K_n is both a red cycle and a blue cycle, so K_n is returned by the algorithm "twice". In the latter case K_n is a monochromatic cycle of one colour, and any vertex on K_n is a monochromatic cycle of the other colour, so K_n and this vertex are returned by the algorithm.

If $n \geq 3$ then the algorithm uses Algorithm 1 to obtain a simple Hamilton cycle D in K_n. If D is monochromatic then D and any vertex on D form our two cycles and are returned by the algorithm. Otherwise D is the union of a red path and a blue path. Joining the two vertices u and v on D where the colour changes produces a monochromatic cycle A and a monochromatic path B of different colours. If $|A| \geq n - 2$ then removing the endvertices from B produces a monochromatic cycle which is disjoint from A. This cycle and A are then returned by the algorithm. If A contains fewer than $n - 2$ vertices then the algorithm iteratively increases the length of A until either it finds a second cycle of the other colour which shares at most one vertex with A, or the condition $|A| \geq n - 2$ is satisfied.

This algorithm takes as input a two-edge-coloured K_n and returns two monochromatic cycles of different colours sharing at most one vertex and which cover the vertices of K_n.

Algorithm 2: TwoCycles(K_n)

if $n \leq 2$ **then**
    Choose a vertex $x \in K_n$;
    return (K_n, x);
Set $P := SimplePath(K_n)$;
Join the endvertices of P to form the simple cycle D;
if *D is monochromatic* **then**
    Choose a vertex $x \in D$;
    return (D, x);
Set u and v to be the vertices on D where the colour changes;
Set A to be the path on D of the same colour as uv and set B to be the path of the other colour;
Set $A = A + uv$;
Set u' and v' to be the neighbours of u and v respectively on B;
while $|A| < n - 2$ **do**
    if *uv' is of the same colour as B* **then**
        return $(A, (B - v) + uv')$;
    else if *$u'v$ is of the same colour as B* **then**
        return $(A, (B - u) + u'v)$;
    else if *$u'v'$ is of the same colour as B* **then**
        return $(A, (B - \{u, v\}) + u'v')$;
    else
        Set $A = (A - uv) + \{uv', vu', u'v'\}$;
        Set $B = B - \{u, v\}$;
        Set $u = u'$ and $v = v'$;
        Set u' and v' to be the neighbours of u and v respectively on B;
    end
end
1 **return** $(A, B - \{u, v\})$;

Observe that Algorithm 2 terminates and that all lines are possible. So it remains to prove that the algorithm is correct. For $n \leq 2$ and the case where D is monochromatic it is easy to check that the algorithm returns the two cycles we are after. So consider the case where $n \geq 3$ and D is the union of a red and a blue path. Suppose arbitrarily that the vertices u and v where the two paths meet are joined in red. Then, as shown in Figure 2.2, A is a red cycle and B is a blue path whose endvertices are shared by A.

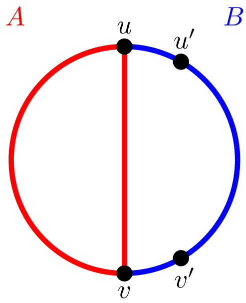
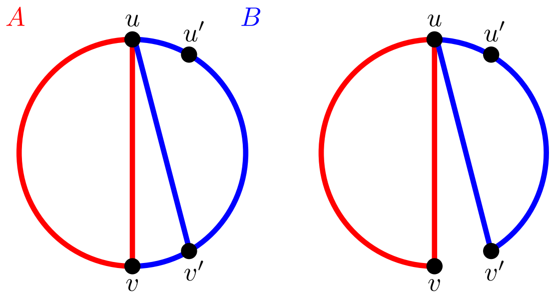


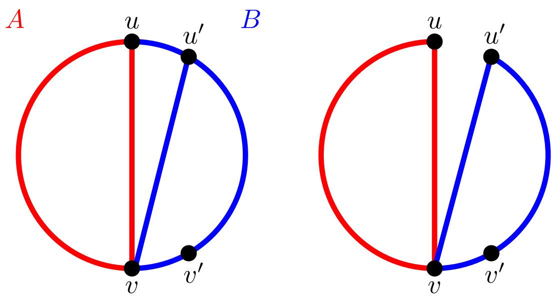
Figure 2.2

If $|A| \geq n-2$ then the while loop does not run and the algorithm returns the red cycle A and the blue cycle $B-\{u,v\}$ such that $B-\{u,v\}$ contains either a single vertex or a single edge (line 1). So in this case the algorithm is correct. If $|A| < n-2$ then we enter the while loop. We now show that after any iteration of the while loop, either the algorithm has returned a red cycle and a blue cycle sharing at most one vertex, or the length of A has increased and B is still a blue path sharing only its endvertices with A.

If the first if condition is satisfied then uv' must be blue and the algorithm correctly returns the red cycle A and the blue cycle $(B-v)+uv'$ sharing the single vertex u.



Otherwise uv' must be red. Then if the second if condition is satisfied $u'v$ is blue and the algorithm correctly returns the red cycle A and the blue cycle $(B-u)+u'v$ sharing the single vertex v.



Otherwise $u'v$ must be red (as in Figure 2.3). Then if the third if condition is satisfied $u'v'$ must be blue and the algorithm returns the disjoint monochromatic cycles A and $(B-\{u,v\})+u'v'$.

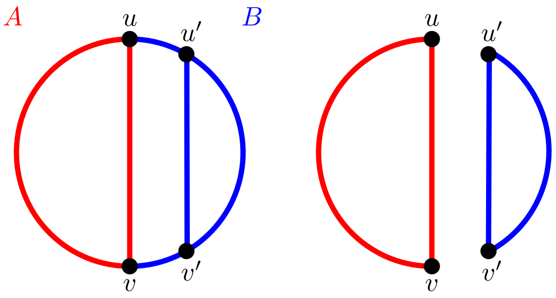
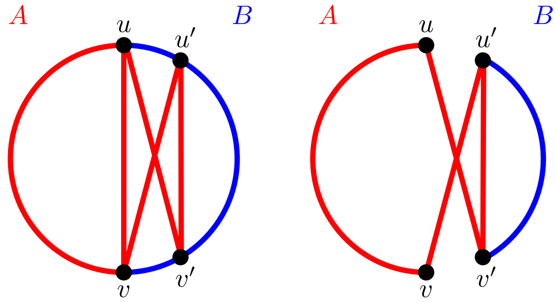


Figure 2.3

Finally, if the third if condition is not satisfied then $u'v'$ must be red. Since $u'v$ and uv' are also red $(A-uv)+\{uv',vu',u'v'\}$ is a red cycle longer than A, and $B-\{u,v\}$ is a blue path whose endvertices are shared by $(A-uv)+\{uv',vu',u'v'\}$, so we are done.



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Chapter 3

Some Preparatory Lemmas

3.1 Motivation

Lehel's conjecture asks whether the vertices of any two-edge-coloured K_n can be covered by two disjoint monochromatic cycles of different colours. In this and the next chapter we give an exposition of Bessy and Thomassé's proof [3] of this conjecture.

The additional explanation and diagrams included in our exposition have resulted in a substantially longer proof than that given by Bessy and Thomassé. For this reason, the proof given here is split into several lemmas, stated and proven in this chapter, and a short proof of the conjecture which makes use of these lemmas given in the next chapter.

Recall Theorem 7 from Chapter 2, which says that if we colour the edges of any K_n using only red and blue then we can cover its vertices by a red cycle and a blue cycle sharing at most one vertex. In the case where the cycles are disjoint we have the two cycles described by Lehel. Bessy and Thomassé [3] consider the case where the cycles *do* share a vertex. If one of the cycles has fewer than four vertices then removing the shared vertex from this cycle produces disjoint cycles of different colours. If not, then removing the shared vertex from the red cycle produces a blue cycle C and a disjoint red path P such that $|P| \geq 3$ and $|C| \geq 4$. We assume that C is maximal. The structures C and P are the focus of this and the next chapter. In this chapter we show via Lemmas 8 to 35 that if C and P have certain properties then we can cover the vertices of K_n by one of the following pairs of structures:

- A red path and a disjoint blue cycle longer than C.
- A red cycle and a disjoint blue cycle.

Since we have assumed that C is maximal, the former case gives us a contradiction. The latter case gives us the two cycles we are after. The order of the lemmas matters because the lemma statements made earlier in the chapter are used in the proofs of those made later.

In Chapter 4 we consider the case where C and P do not have any of these properties. We show that it is still possible to cover the vertices of K_n by a red path and a disjoint blue cycle longer than C, a contradiction, or two disjoint cycles of different colours, thus completing the proof. Recall that the empty set, singletons and single edges are all considered monochromatic cycles and paths. We define a *proper cycle* to be a cycle with at least three vertices.

3.2 Lemmas

Lemma 8. *Suppose that the vertices of a two-edge-coloured K_n are covered by a maximal blue cycle C and a disjoint red path P such that $|P| \geq 3$ and $|C| \geq 4$. If the endvertices of P are joined in red then the vertices of K_n can be covered by disjoint monochromatic cycles of different colours.*

Proof. Denote the endvertices of P by x and y. If xy is red then $P + xy$ is a red cycle, so we are done. □

Definition 9. Suppose that the vertices of a two-edge-coloured K_n are covered by a maximal blue cycle C and a disjoint red path P such that $|P| \geq 2$. We call a vertex on C **red** if it is joined to both endvertices of P in red and we call it **blue** otherwise.

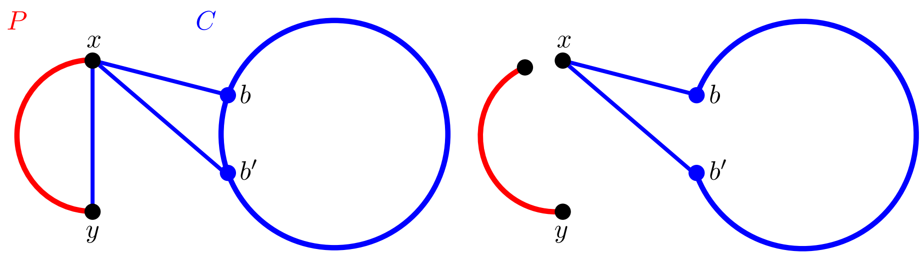
Note that in every diagram in this and the next chapter vertices and edges whose colour is unknown will be coloured black.

Lemma 10. *Suppose that the vertices of a two-edge-coloured K_n are covered by a maximal blue cycle C and a disjoint red path P such that $|P| \geq 3$ and $|C| \geq 4$. If C contains a pair of consecutive blue vertices then the vertices of K_n can be covered by either:*

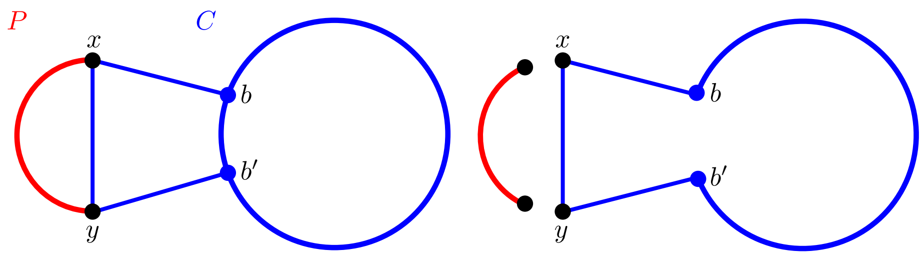
- *A red path and a disjoint blue cycle longer than C, or*
- *A red cycle and a disjoint blue cycle.*

Proof. Let b and b' be a pair of distinct consecutive blue vertices on C. By Lemma 8 if the endvertices of P are joined in red then we are done. So denote the endvertices of P by x and y and suppose that xy is blue. At least one of bx and by must be blue since b is a blue vertex. Suppose arbitrarily that bx is blue. Similarly, at least one of $b'x$ and $b'y$ is blue. We consider each case in turn.

- If $b'x$ is blue then $(C - bb')x$ is a blue cycle longer than C, and $P - x$ is a disjoint red path covering the remaining vertices of K_n, so we are done.



- If $b'y$ is blue then $(C - bb')xy$ is a blue cycle longer than C and $P - \{x, y\}$ is a disjoint red path covering the remaining vertices of K_n.



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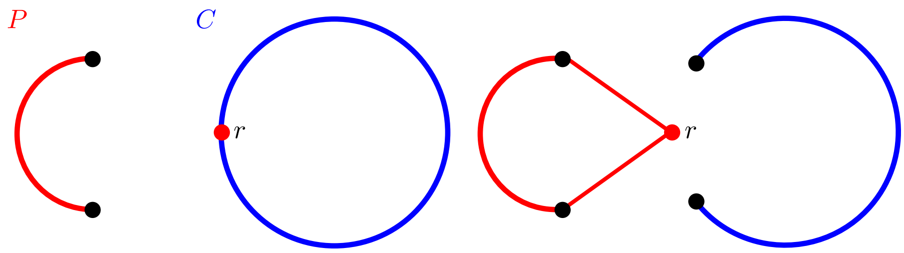
Lemma 11. *Suppose that the vertices of a two-edge-coloured K_n are covered by a maximal blue cycle C and a disjoint red path P such that $|P| \geq 3$ and $|C| \geq 4$. If $|C| \leq |P|$ then the vertices of K_n can be covered by either:*

- *A red path and a disjoint blue cycle longer than C, or*
- *A red cycle and a disjoint blue cycle.*

Proof. Suppose that $|C| \leq |P|$. If C contains a pair of consecutive blue vertices then by Lemma 10 we are done, so suppose that C contains at least one red vertex r. But then Pr is a red cycle longer than C and $C - r$ is a disjoint blue path covering the remaining vertices of K_n. Changing the colour of every edge in K_n produces a red path and a disjoint blue cycle longer than C.



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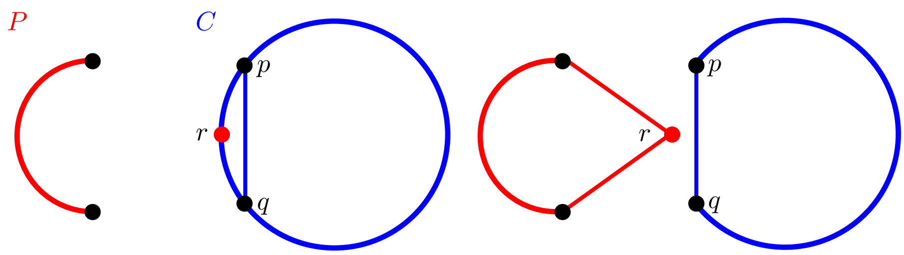
Lemma 12. *Suppose that the vertices of a two-edge-coloured K_n are covered by a maximal blue cycle C and a disjoint red path P such that $|P| \geq 3$ and $|C| \geq 4$. If distinct neighbours on C of a red vertex are joined by a blue edge then the vertices of K_n can be covered by either:*

- *A red path and a disjoint blue cycle longer than C, or*
- *A red cycle and a disjoint blue cycle.*

Proof. Denote by r a red vertex on C and suppose that the neighbours p and q of r are joined by a blue edge. Then Pr is a red cycle and $(C - r) + pq$ is a disjoint blue cycle covering the remaining vertices of K_n.



□

Lemma 13. *Suppose that the vertices of a two-edge-coloured K_n are covered by a maximal blue cycle C and a disjoint red path P such that $|P| \geq 3$ and $|C| \geq 4$. If C contains a set of consecutive blue, red, blue, red, blue vertices such that the first and last vertex in the sequence may be the same, then the vertices of K_n can be covered by either:*

- *A red path and a disjoint blue cycle longer than C, or*

- *A red cycle and a disjoint blue cycle.*

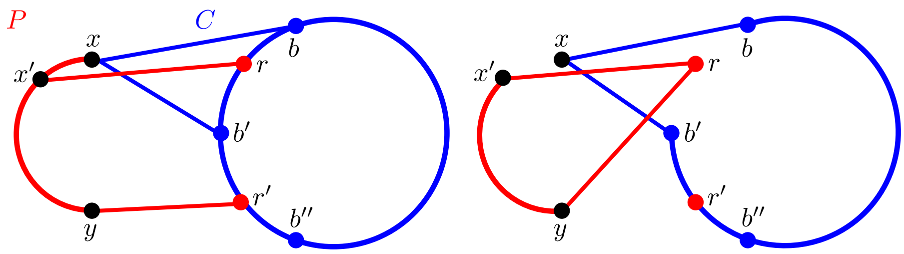
Proof. Suppose that b, r, b', r', b'' is a sequence of blue, red, blue, red, blue vertices on C, with possibly $b = b''$. By Lemma 8, if the endvertices of P are joined in red then we are done. So denote the endvertices of P by x and y and suppose that xy is blue. Denote the neighbour of x on P by x'. We now show that if x' is joined in red to r or r' then we are done. If, on the other hand, $x'r$ and $x'r'$ are both blue, then we will see that we can cover the vertices of K_n by a red path and a disjoint blue cycle longer than C.

At least one of $b'x$ and $b'y$ is blue, so suppose arbitrarily that $b'x$ is blue.

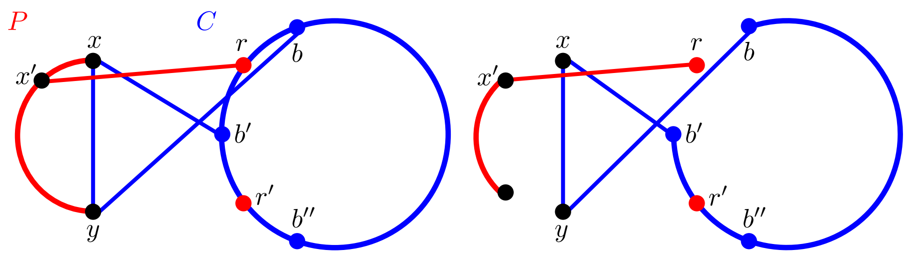
Claim 14. *If $x'r$ is a red edge then we are done.*

Proof. Suppose that $x'r$ is red. At least one of bx and by is blue. We consider each case in turn.

- If bx is blue then $(C - r)x$ is a blue cycle and $(P - x)r$ is a disjoint red cycle covering the remaining vertices of K_n.



- If by is blue then $(C - r)xy$ is a blue cycle longer than C and $(P - \{x, y\})r$ is a disjoint red path.



□

So suppose that $x'r$ is blue. The following claim was stated but not proven in [3].

Claim 15. *If $x'r'$ is a red edge we are done.*

Proof. Suppose that $x'r'$ is red and recall that $b'x$ is blue. At least one of $b''x$ and $b''y$ is blue. We consider each case in turn.

- If $b''x$ is blue (Figure 3.1) then $(C - r')x$ is a blue cycle and $(P - x)r'$ is a disjoint red cycle spanning the remaining vertices of K_n.

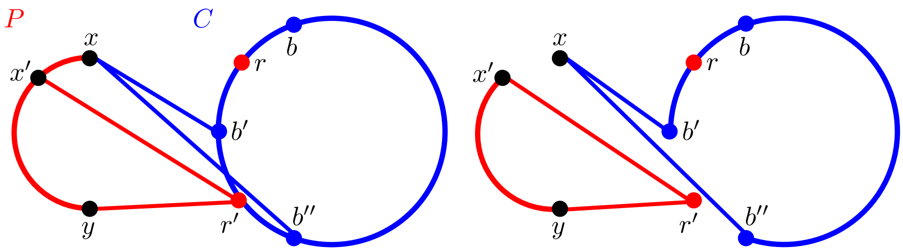
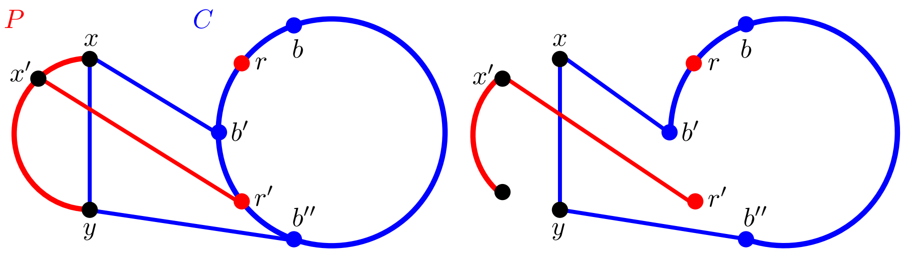


Figure 3.1

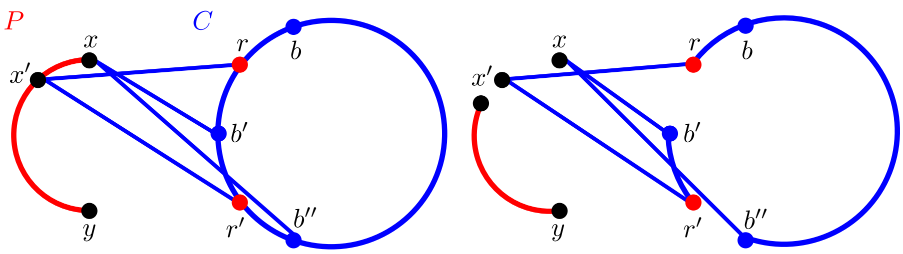
- If $b''y$ is blue then $(C - r')xy$ is a blue cycle longer than C and $(P - \{x, y\})r'$ is a disjoint red path so we are done.



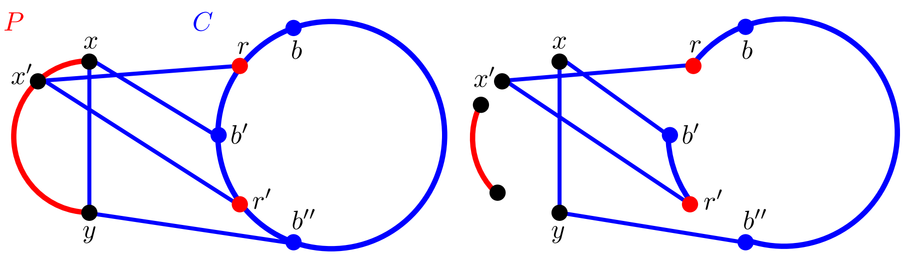
□

So suppose that $x'r'$ is blue. We complete the proof by showing that we can cover the vertices of K_n by a red path and a disjoint blue cycle longer than C. At least one of xb'' and yb'' is blue. We consider each case in turn.

- If xb'' is blue then replacing the path $brb'r'b''$ in C by $brx'r'b'xb''$ increases the length of C and $P - \{x, x'\}$ is a disjoint red path, so we are done.



- If yb'' is blue then replacing the path $brb'r'b''$ in C by $brx'r'b'xyb''$ increases the length of C and $P - \{x, x', y\}$ is a disjoint red path, so we are done.



□

Definition 16. Suppose that the vertices of a two-edge-coloured K_n are covered by a maximal blue cycle C and a disjoint red path P such that $|P| \geq 3$ and $|C| \geq 4$. If acb is a sequence of distinct consecutive vertices on C such that c is red, then we call ab a **special edge**. Denote by G_s the graph on the same vertex set as C whose edges are the special edges.

A well-known result in graph theory says that a graph with maximum degree two is a disjoint union of paths and cycles. Observe that the maximum degree of G_s is two, so G_s is a disjoint union of paths and cycles. The following corollary follows from Lemma 12.

Corollary 17. *Suppose that the vertices of a two-edge-coloured K_n are covered by a maximal blue cycle C and a disjoint red path P such that $|P| \geq 3$ and $|C| \geq 4$. If any special edge is blue then the vertices of K_n can be covered by either:*

- *A red path and a disjoint blue cycle longer than C, or*
- *A red cycle and a disjoint blue cycle.*

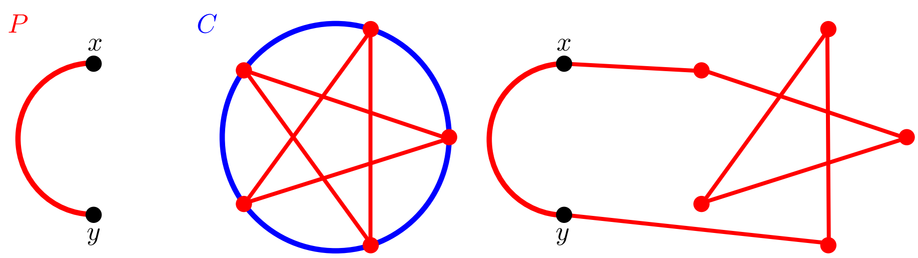
Lemma 18. *Suppose that the vertices of a two-edge-coloured K_n are covered by a maximal blue cycle C and a disjoint red path P such that $|P| \geq 3$ and $|C| \geq 4$. If C contains only red vertices then the vertices of K_n can be covered by either:*

- *A red path and a disjoint blue cycle longer than C, or*
- *A red cycle and a disjoint blue cycle.*

Proof. Suppose that every vertex on C is red. The outline of the proof is as follows. We first show that if certain properties of C and P hold then by previous lemmas we are done. We then consider the case where C has an odd number of vertices and the case where C has an even number of vertices in turn. In the former case we show that there exists a red Hamilton cycle in K_n, so we are done. In the latter case we will see that G_s forms the union of two disjoint red cycles A and B. We then show that for each of the cases $|P| = 3$, $|P| = 4$ $|P| = 5$ and $|P| \geq 6$ the vertices of K_n can be covered as described.

By Lemma 8 if the endvertices of P are joined in red then we are done, so suppose that the endvertices x and y of P are joined in blue. If $|C| \leq |P|$ then by Lemma 11 we are done, so suppose not. Note that G_s contains every vertex of C. By Corollary 17 if any special edge is blue then we are done, so suppose that all special edges are red.

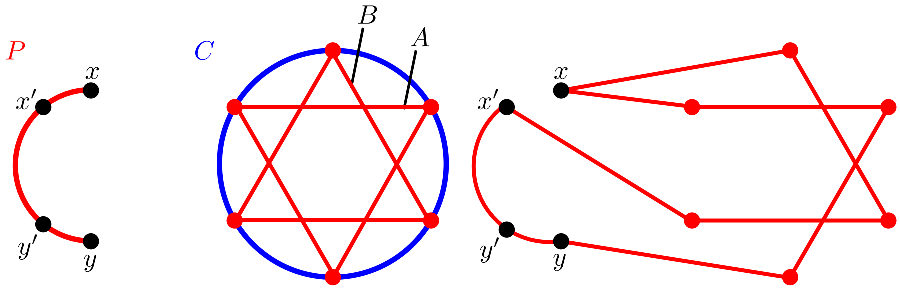
First consider the case where C has an odd number of vertices, so G_s is a red Hamilton cycle in C. Then G_sP is a red Hamilton cycle in K_n, so we are done.



Now consider the case where C has an even number of vertices, so G_s forms the union of two disjoint red nonempty cycles A and B. Since A and B alternate along C they must contain the same number of vertices. Denote by x' and y' the respective neighbours of x and y on P, where $x' = y'$ if $|P| = 3$.

Claim 19. *If there is a red edge from x' to A then we are done.*

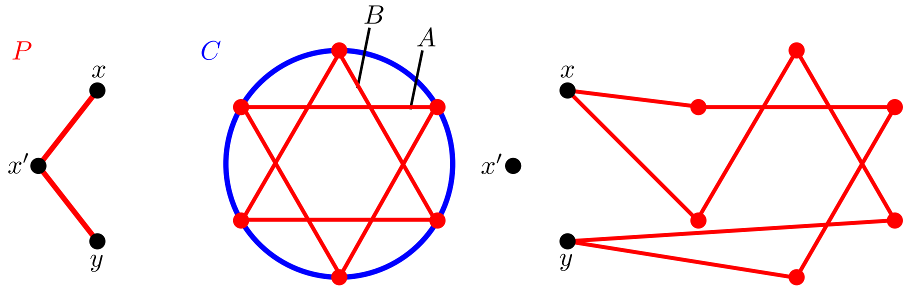
Proof. Suppose there is such an edge. Then we can form the red Hamilton cycle $(P-x)AxB$ in K_n.



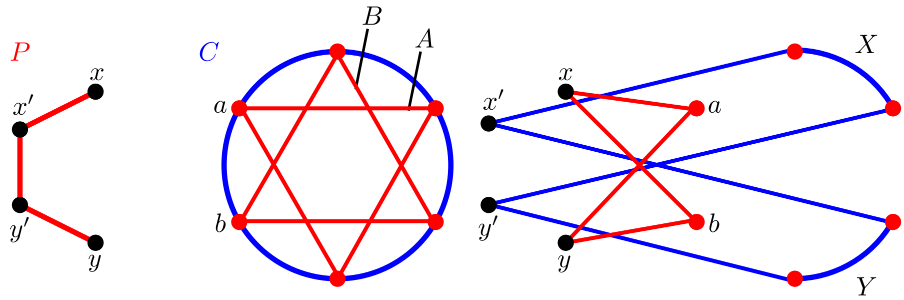
□

By symmetry if there is a red edge from x' to B or from y' to vertices in the set $A \cup B$ then we are done, so suppose that all edges from x' and y' to vertices in the set $A \cup B$ are blue. To complete the proof we consider each of the cases $|P| = 3, |P| = 4, |P| = 5$ and $|P| \geq 6$ in turn.

- If $|P| = 3$ then $AxBy$ is a red cycle and x' forms a blue cycle so we are done.



- If $|P| = 4$ then choose a vertex a on A and a vertex b on B, consecutive along C, and form the red cycle $axby$. Since $|C| > 4$, we can then partition what remains of C into two nonempty subpaths X and Y and form the blue cycle $x'Xy'Y$, so we are done.



Now suppose that P has at least five vertices. Denote by x'' and y'' the second neighbours of x' and y' respectively on P, where $x'' = y''$ if $|P| = 5$. We will need the following three claims.

Claim 20. *If there is a red edge from x'' to A then we are done.*

Proof. Suppose there is such an edge (as in Figure 3.2). Then $(P - \{x', x\})AxB$ forms a red cycle and x' is a disjoint blue cycle, so we are done.

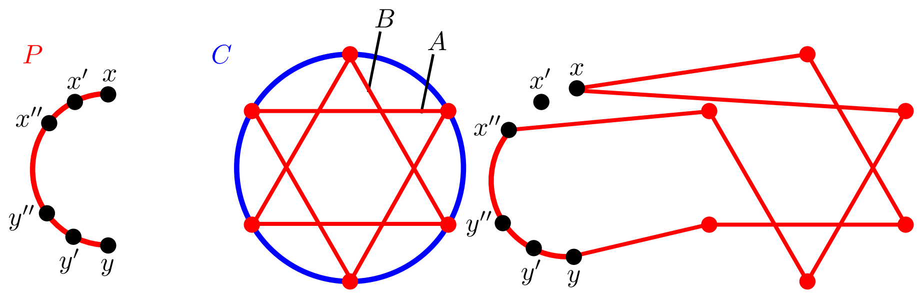


Figure 3.2

□

By symmetry if there is a red edge from x'' to B or from y'' to any vertices in the set $A \cup B$ then we are done. So suppose that all edges from x'' and y'' to vertices in the set $A \cup B$ are blue.

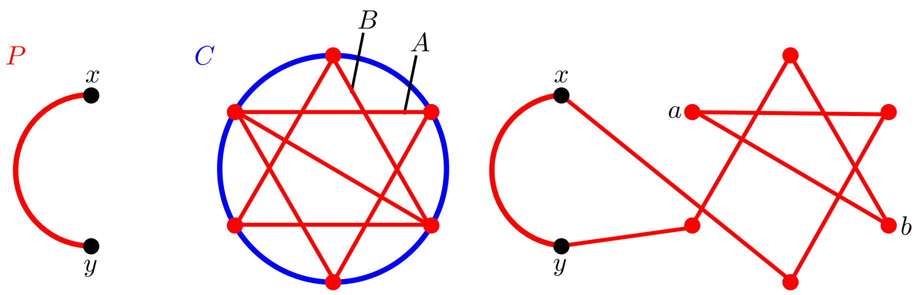
Claim 21. *If xx'' is red then we are done.*

Proof. If xx'' is a red edge then $P - x'$ forms a red path. Then, since we have assumed that all edges between x' and vertices in the set $A \cup B$ are blue, we can insert x' between two adjacent vertices on C to form a disjoint blue cycle longer than C. □

By symmetry if yy'' is red then we are done. So suppose that xx'' and yy'' are blue.

Claim 22. *If there is a red edge between A and B then we are done.*

Proof. Suppose that there is such an edge. Then PAB is a red Hamilton cycle in K_n, so we are done.



□

So suppose that there are only blue edges between A and B. We now consider the case where $|P| = 5$ and the case where $|P| \geq 6$ in turn.

- Suppose that $|P| = 5$. Then, since we have assumed that $|C| > |P|$, we know that C has at least six vertices. Pick a vertex a on A and a vertex b on B and form the red cycle $axby$ (as in Figure 3.3). We will now show that there exists a blue cycle covering the remaining vertices of K_n. Since we have assumed that every edge between A and B is blue, then every edge between $A - a$ and $B - b$ must also be blue. Thus, since $|A| = |B|$, we can form a blue cycle which spans the set $(A - a) \cup (B - b)$ by alternating between vertices in $A - a$ and vertices in $B - b$. This cycle has at least four vertices since it contains exactly two fewer vertices than C, so we can insert into it the vertices x', y' and x'', so we are done.

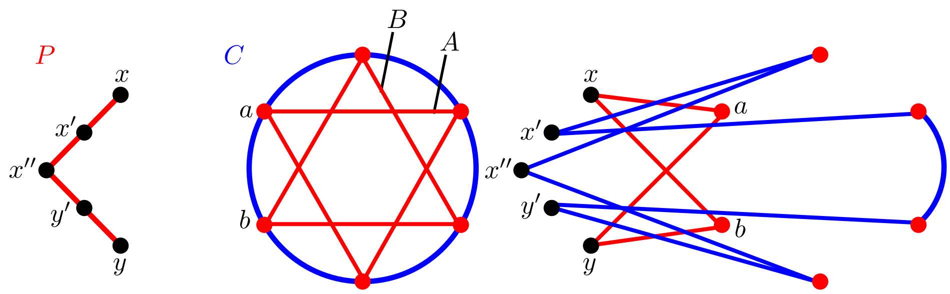
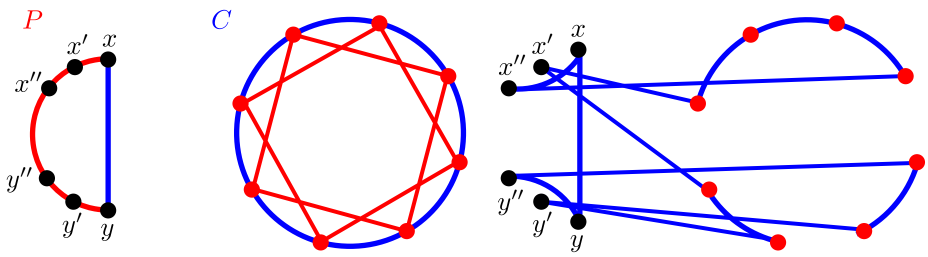


Figure 3.3

- Finally, suppose that $|P| \geq 6$. Then, since $|C| > 6$, we can insert into C the three blue paths x', $x''xyy''$ and y' to obtain a blue cycle longer than C. The red path $P - \{x, y, x', y', x'', y''\}$ covers the remaining vertices of K_n, so we are done.



□

Note that the following definition is not the definition of left and right vertices given by Bessy and Thomassé in [3]. In particular, our definition is more precise and does not allow the set of left vertices and the set of right vertices to intersect.

Definition 23. Suppose that the vertices of a two-edge-coloured K_n are covered by a maximal blue cycle C and a disjoint red path P. Fix an orientation of C. We define the set L of **left vertices** to be the set of vertices which are *left* neighbours on C of a blue vertex. We define the set R of **right vertices** to be the vertices which are *right* neighbours on C of a blue vertex and left neighbours of a *red* vertex. Observe that L and R are disjoint.

The following corollary follows directly from Lemma 10.

Corollary 24. *Suppose that the vertices of a two-edge-coloured K_n are covered by a maximal blue cycle C and a disjoint red path P such that $|P| \geq 3$ and $|C| \geq 4$. If there is a blue vertex in the set $L \cup R$ then the vertices of K_n can be covered by either:*

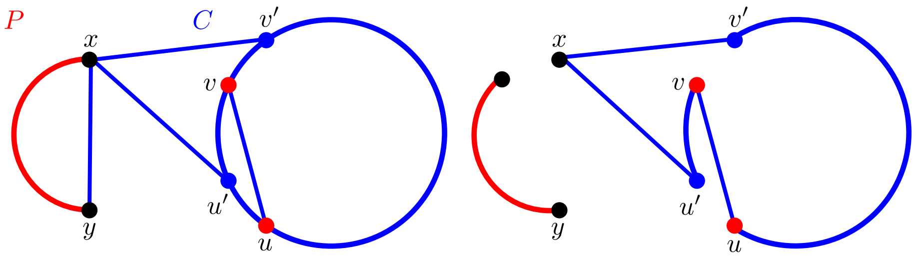
- *A red path and a disjoint blue cycle longer than C, or*
- *A red cycle and a disjoint blue cycle.*

Lemma 25. *Suppose that the vertices of a two-edge-coloured K_n are covered by a maximal blue cycle C and a disjoint red path P such that $|P| \geq 3$ and $|C| \geq 4$. If there is a blue edge between vertices in the set L or between vertices in the set R then the vertices of K_n can be covered by either:*

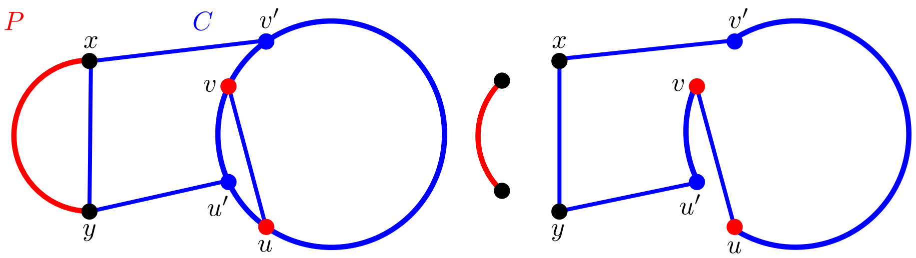
- *A red path and a disjoint blue cycle longer than C, or*
- *A red cycle and a disjoint blue cycle.*

Proof. Let u and v be distinct vertices in L and suppose that uv is blue. Denote the right neighbours of u and v by u' and v' respectively. By Lemma 8 if the endvertices of P are joined in red then we are done. So denote the endvertices of P by x and y and suppose that xy is blue. Suppose arbitrarily that C is oriented clockwise. Since v' is blue at least one of $v'x$ and $v'y$ is blue, so suppose arbitrarily that $v'x$ is blue. Similarly at least one of $u'x$ and $u'y$ is blue. We consider each case in turn.

- If $u'x$ is blue then $(C - \{u', v\})xu'v$ is a blue cycle longer than C, and $P - x$ covers the remaining vertices in K_n.



- If $u'y$ is blue then $(C - \{u', v\})xyu'v$ is a blue cycle longer than C, and $P - \{x, y\}$ is a red path covering the remaining vertices of K_n.



Since in this proof we have considered only the right neighbours of u and v, the case where u and v are right vertices holds by symmetry. $\square$

Recall that a proper cycle is a cycle with at least three vertices.

Lemma 26. *Suppose that the vertices of a two-edge-coloured K_n are covered by a maximal blue cycle C and a disjoint red path P such that C contains at least one blue vertex. If the graph G_s of special edges in C contains a proper cycle Z then the following hold:*

(26.1) *$|C|$ is even.*

(26.2) *Z is unique.*

(26.3) *Z contains all the blue vertices on C.*

Note that Z may also contain red vertices on C.

Proof. Suppose that G_s contains a proper cycle Z. Note that this implies that $|C| \geq 6$. We prove each of the above statements in turn.

(26.1) If $|C|$ is odd then Z visits every vertex on C. Since Z contains only special edges, this implies that every vertex on C is red, which we have assumed is not true.

(26.2) Suppose that G_s contains a proper cycle T which is distinct from Z. By (26.1) we know that $|C|$ must be even, so Z visits every other vertex on C. Since Z contains only special edges, the vertices on C which are not on Z are red. But, following the same reasoning, the cycle T also visits every other vertex on C and the vertices on C which are not on T are red. The cycles T and Z cannot share vertices since then Z and T would not be distinct, so T visits exactly those vertices on C which are not on Z. This implies that every vertex on C is red, which we have assumed is not true.

(26.3) Since all vertices on C which are not on Z are red there can be no blue vertices on C which are not on Z.

□

Definition 27. Suppose that the vertices of a two-edge-coloured K_n are covered by a maximal blue cycle C and a disjoint red path P such that C contains at least one blue vertex. Denote by S the set of vertices on C which have two red neighbours on C and which are not on a proper cycle in G_s (by Lemma 26 part (26.2) there can is at most one such cycle). That is, $S = C - (L \cup R \cup Z)$ where Z is a possible proper cycle in G_s.

Lemma 28. *Suppose that the vertices of a two-edge-coloured K_n are covered by a maximal blue cycle C and a disjoint red path P such that $|P| \geq 3$ and $|C| \geq 4$. Suppose further that C contains at least one blue vertex. Then at least one of the following holds.*

(a) The vertices of K_n can be covered by a red path and a disjoint blue cycle longer than C.

(b) The vertices of K_n can be covered by a red cycle and a disjoint blue cycle.

(c) The set $S \cup L \cup R$ is spanned by a red path with endvertices in $L \cup R$. More precisely:

- *If $|R|$ is even then for all distinct vertices of R (resp. L) u and v there is a red path of special edges from u to v which spans the set $S \cup L \cup R$.*
- *If $|R|$ is odd, then for all $u \in R$ and $v \in L$ such that u and v are not endvertices of the same maximal path of G_s there is a red path from u to v which spans the set $S \cup L \cup R$. Only if $|R| = 1$ do we allow u and v to the be the endvertices of the single maximal path of G_s.*

Proof. To begin we show that either the vertices in K_n can be covered as described in part (a) or part (b) of the lemma statement, or else the set S is covered by the set of $|R|$ disjoint maximal paths in G_s. We then show that in the latter case we can join these paths to form a single red path spanning $S \cup L \cup R$ with endvertices u and v.

If any neighbour of a blue vertex on C is also blue then by Lemma 10 we are done, so suppose not. If C contains a sequence of blue, red, blue, red, blue vertices, where possibly the first and last vertex in the sequence are the same, then by Lemma 13 we are done, so suppose not. Then C contains at least one vertex with one red neighbour and one blue neighbour on C, so $|R| \geq 1$. Also, every vertex in R is adjacent to a single special edge, so is an endvertex of a single path in G_s.

By following a maximal path in G_s with an endvertex in R we find that its other endvertex must be in L. In fact, any maximal path in G_s must have one endvertex in R and one endvertex in L. So there are exactly $|R|$ disjoint maximal paths in G_s, each with one endvertex in R the other endvertex in L. We make the following claim.

Claim 29. *The $|R|$ maximal paths in G_s cover the set S.*

Proof. By Definition 27 all vertices in S have two red neighbours, so are adjacent to exactly two special edges. These vertices are also not on any proper cycle of G_s, so they must all be nonendvertices of paths in G_s, so we are done. $\square$

Now, by Lemma 25 if there are any blue edges between vertices in the set L or between vertices in the set R then we can cover the vertices of K_n as described in part (a) or part (b) of the lemma statement, so suppose not. Also, by Corollary 17, if any special edge is blue then we are done, so suppose that all special edges are red. We complete the proof by constructing a red path from u to v which spans the set $S \cup L \cup R$.

If $|R| = 1$ then $|L| = 1$, otherwise we have a sequence of blue, red, blue, red, blue vertices on C. Then by Claim 29 the graph G_s contains a single maximal red path which covers the set $S \cup L \cup R$ and we are done. So suppose that $|R| \geq 2$. If u is in L and has two blue neighbours, so is not an endvertex of a maximal path in G_s, then begin by following a red path in L from u to another vertex in L which *is* an endvertex of a maximal path in G_s. If, on the other hand, u has a red neighbour then u is an endvertex of a maximal path in G_s. In either case we then follow this maximal path until its end. Since L and R are red cliques we can then go to an endvertex of an unvisited maximal path in G_s and follow it. We continue until we have covered all but one maximal path in G_s. Note that at some stage in this process we visit all vertices in L except u and v which are not endvertices of any maximal path in G_s.

If v is an endvertex of a maximal path P_v in G_s then we ensure that P_v is the last path we cover. Then v will always be the last vertex we visit due to the parity of $|R|$. If v is not an endvertex of any maximal path in G_s then v is in L and we follow the remaining maximal path until its end, which will be in L due to the parity of $|R|$, and then follow a red edge to v.

Since this path covers all $|R|$ maximal paths in G_s and any "leftover" vertices in L, this path indeed covers the set $S \cup L \cup R$, so we are done. Figure 3.4 illustrates an example of this construction for the case where $|R|$ is even.

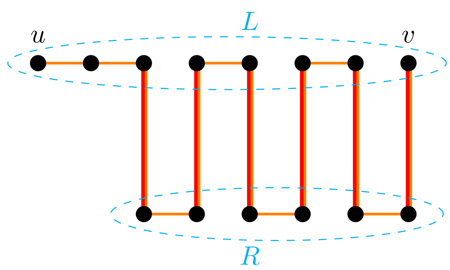


Figure 3.4: Red Path Spanning $S \cup L \cup R$ for even $|R|$

$\square$

Lemma 30. *Suppose that the vertices of a two-edge-coloured K_n are covered by a maximal blue cycle C and a red path P such that $|P| \geq 3$ and $|C| \geq 4$. Suppose further that C contains at least one blue vertex. If the graph G_s does not contain a proper cycle then the vertices of K_n can be covered by either:*

- *A red path and a disjoint blue cycle longer than C, or*
- *A red cycle and a disjoint blue cycle.*

Proof. Suppose that G_s does not contain a proper cycle. By Lemma 28 if the set $S \cup L \cup R$ is not covered by a red path with endvertices in $L \cup R$ then we can cover the vertices of K_n as described. So let Q be such a path. Since S contains all the vertices on C which are not left or right vertices, then Q spans C. If the set $L \cup R$ contains a blue vertex then by Corollary 24 we are done, so suppose not. Then the endvertices of Q are red and we can form the red Hamilton cycle QP in K_n. □

The following corollary follows from Lemma 13.

Corollary 31. *Suppose that the vertices of a two-edge-coloured K_n are covered by a maximal blue cycle C and a disjoint red path P such that $|P| \geq 3$ and $|C| \geq 4$. Suppose further that G_s contains a proper cycle Z. Then if a blue vertex on Z has two blue neighbours on Z the vertices of K_n can be covered by either:*

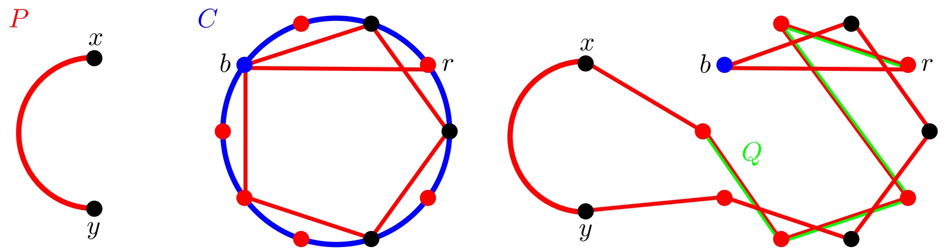
- *A red path and a disjoint blue cycle longer than C, or*
- *A red cycle and a disjoint blue cycle.*

Lemma 32. *Suppose that the vertices of a two-edge-coloured K_n are covered by a maximal blue cycle C and a disjoint red path P such that $|P| \geq 3$ and $|C| \geq 4$. If a blue vertex is joined in red to a vertex in the set $L \cup R$ then the vertices of K_n can be covered by either:*

- *A red path and a disjoint blue cycle longer than C, or*
- *A red cycle and a disjoint blue cycle.*

Proof. Suppose that some blue vertex b is joined in red to a vertex r in R. By Corollary 24 if r is blue then we are done, so suppose that r is red. By Lemma 30 if G_s does not contain a proper cycle then we are done, so suppose that Z is a proper cycle in G_s. By Lemma 26 we know that Z is unique and that b is on Z. By Corollary 17 if any special edge is blue then we are done, so suppose not. Then Z is a red cycle. We will now show that either we are already done or we can find a red Hamilton cycle in K_n.

By Lemma 33 if the set $S \cup L \cup R$ is not spanned by a red path with endvertices in $L \cup R$ then we are done. So suppose that Q is such a path with r as one of its endvertices. By Corollary 31 if b has two blue neighbours on Z then we are done, so suppose not. Then since r is joined to b in red and b has a red neighbour on Z we can form the red Hamilton cycle PQZ in K_n.



□

Lemma 33. *Suppose that the vertices of a two-edge-coloured K_n are covered by a maximal blue cycle C and a disjoint red path P such that $|P| \geq 3$ and $|C| \geq 4$. Suppose further that C contains at least one blue vertex. Then at least one of the following is true.*

(a) There exists a red cycle spanning $S \cup R \cup L$.

(b) The vertices of K_n can be covered either:

- *A red path and a disjoint blue cycle longer than C, or*
- *A red cycle and a disjoint blue cycle.*

Proof. In what follows we show that either we can cover the vertices of K_n as described in part (b) of the lemma statement, or there exists a red path spanning $S \cup L \cup R$ whose endvertices are joined in red, so we are done. We will consider the case where $|R|$ is even and the case where $|R|$ is odd in turn. When $|R|$ is even we will see that it is easy to find a red cycle spanning $S \cup R \cup L$. For the case where $|R|$ is odd we consider the subcase where C has only one blue vertex and the subcase where C has at least two blue vertices in turn.

By Lemma 28 either we can cover the vertices of K_n as described in part (b) of the lemma statement, or there exists a red path spanning the set $S \cup R \cup L$ with endvertices in $L \cup R$. So let Q be such a path and denote its endvertices by u and v. If u or v is blue then by Corollary 24 we are done, so suppose not.

First suppose that $|R|$ is even, so by Lemma 28 the endvertices of Q are either both in L or they are both in R. By Lemma 25 if there is any blue edge between vertices in L or between vertices in R then we are done, so suppose not. Then uv is red and we are done.

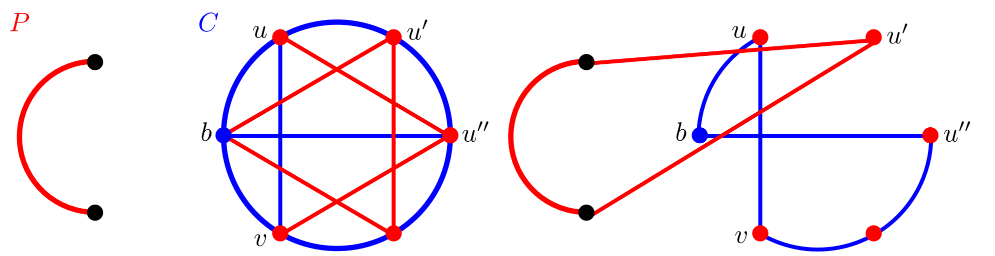
Now suppose that $|R|$ is odd, so by Lemma 28 one endvertex of Q is in L and the other is in R. If G_s doesn't contain a proper cycle then by Lemma 30 we are done, so suppose that Z is a proper cycle in G_s. Then $|C| \geq 6$ and by Lemma 26 we know that $|C|$ is even and Z is unique. By Corollary 17, if any special edge is blue then we are done, so suppose that all special edges are red. So Z consists of only red edges.

We consider first the case where C contains only one blue vertex and then the case where C has two or more blue vertices.

First suppose that there is only one blue vertex b on C, so C contains one left vertex and one right vertex. Then u and v are the neighbours of b on C. If uv is red then $Q + uv$ is the red cycle we are after, so suppose that uv is blue. Denote by u' the second neighbour of u on C and by u'' the second neighbour of u' on C. By Lemma 26 we know that b is on Z, so u' is on Z and u'' is not.

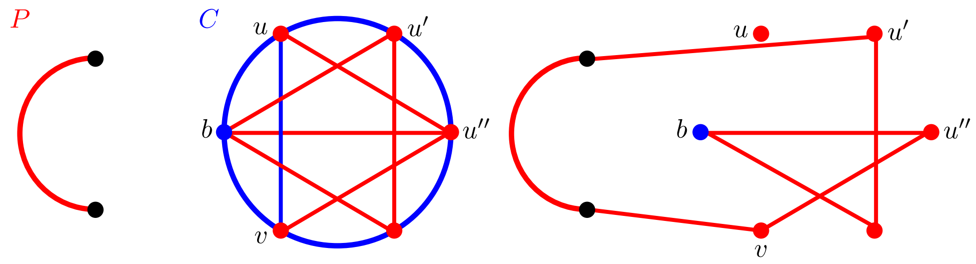
Claim 34. *If bu'' is blue then we are done.*

Proof. If bu'' is blue then we can form a blue cycle which covers the set $C - u'$ by replacing the path $vbuu'u''$ on C by $vubu''$. The red cycle Pu' covers the remaining vertices of K_n, so we are done.



□

So suppose that bu'' is red and form the red cycle $(Z - bu')(Q - u)P$. The singleton u is a blue cycle so we are done.



Now suppose that C has at least two blue vertices b and b'. Since $|R|$ is odd, Lemma 28 implies that if we can prove the existence of a red edge between a vertex in R and a vertex in L which are not endvertices of the same maximal path of G_s then we are done. So let's try.

If b and b' are neighbours on C then by Lemma 10 we are done, so suppose not. By Lemma 13 if there is only one red vertex between b and b' in both directions along C then we are done, so suppose not. Then there exists a blue subpath $br_1 \dots r_k b'$ on C where $r_1 \dots, r_k$ are red vertices and $k > 1$. Suppose arbitrarily that r_1 is a right neighbour of b and r_k is a left neighbour of b', so r_1 is a right vertex and r_k is a left vertex. Then Lemma 28 implies that if we can prove the existence of a red edge between r_1 and a left vertex which is not on the same maximal path of G_s as r_1 we are done. So let's try.

By Lemma 32 if br_k or $b'r_1$ is red then we are done, so suppose they are both blue. Then we can replace the blue path $br_1 \dots r_k b'$ on C by the blue path $br_k r_{k-1} \dots r_1 b'$. Notice that r_1 is now a left vertex, so r_1 is joined in red to all left vertices, except possibly r_k which is now a right vertex. Finally, since C contains at least two blue vertices then C must contain at least two left vertices, so r_1 is joined in red to a left vertex which is not the vertex on the same maximal path in G_s as r_1. So we are done. □

Lemma 35. *Suppose that the vertices of a two-edge-coloured K_n are covered by a maximal blue cycle C and a disjoint red path P such that $|P| \geq 3$, $|C| \geq 4$ and C contains at least one blue vertex. Suppose further that there exists a red cycle W which spans the set $S \cup L \cup R$. Then if any blue vertex is joined in red to a vertex on W, the vertices of K_n can be covered by either:*

- *A red path and a disjoint blue cycle longer than C, or*
- *A red cycle and a disjoint blue cycle.*

Proof. Suppose that bw is a red edge where b is a blue vertex and w is a vertex on W (as in Figure 3.5). By Lemma 30 if G_s does not contain a proper cycle then we are done, so suppose that Z is a proper cycle in G_s. By Lemma 26 we know that Z is unique, b is on Z and all vertices on W are red. If any special edge is blue then by Corollary 17 we are done, so suppose not. Then Z is a red cycle. If b has two blue neighbours on Z then by Lemma 31 we are done, so suppose that b has at least one red neighbour on Z. Then PZW is a red Hamilton cycle in K_n.

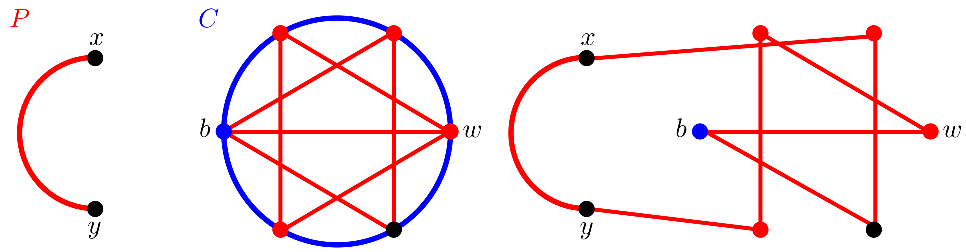


Figure 3.5

□

Chapter 4

Proof of Lehel's Conjecture

4.1 Motivation

In this chapter we complete the proof of Lehel's conjecture. Recall that Lehel's conjecture asks whether the vertices of any two-edge-coloured K_n can be covered by two disjoint monochromatic cycles of different colours.

In Chapter 2 we proved that the vertices of any two-edge-coloured K_n can be covered by monochromatic cycles of different colours with at most one common vertex. If these cycles are disjoint, or if one of them contains fewer than four vertices then we are done. If not, then removing the shared vertex from one cycle produces a monochromatic path P of colour, say, red, and a disjoint monochromatic cycle C of the other colour such that $|P| \geq 3$ and $|C| \geq 4$. We suppose that C is maximal.

In Chapter 3 we showed that if P and C satisfy certain properties then we can cover the vertices of any K_n by either:

- A red path and a disjoint blue cycle longer than C, a contradiction, or
- A red cycle and a disjoint blue cycle.

In this chapter we show that even when these properties aren't satisfied we can still find two disjoint monochromatic cycles which cover the vertices of a two-edge-coloured K_n, thus proving Lehel's conjecture. Recall that the empty set, singletons and single edges are all considered monochromatic cycles and paths.

4.2 Proof Outline

We begin by using Theorem 4 to obtain C and P. The endvertices of P are denoted by x and y, and we denote by y' the neighbour of y on P. We then use several lemmas from Chapter 3 to show that either we are done, or else xy is blue, C contains at least one blue vertex, there exists a unique proper cycle Z of special edges, and there exists a red cycle W spanning the set of vertices on C not covered by Z. We then show that if there is a red edge between W and Z (Claim 37) or between y' and $W \cup Z$ (Claim 38) then we are done.

We complete the proof by considering the case where $|P| = 3$ and the case where $|P| \geq 4$ in turn. In the former case we will construct a red cycle and a disjoint blue cycle covering the vertices of K_n. For the case where $|P| \geq 4$, we denote by y'' the second neighbour of y'

on P. We first show that if the edge yy'' is red (Claim 39), or there is a red edge between y'' and any vertex in the set W (Claim 40), then we are done. If not, then we can construct a red path and a disjoint blue cycle longer than C which cover the vertices of K_n.

4.3 Theorem and Proof

Theorem 36. *The vertices of any two-edge-coloured K_n can be covered by two disjoint monochromatic cycles of different colours.*

Proof. Theorem 4 tells us that the vertices of any two-edge-coloured K_n can be covered by two monochromatic cycles of different colours with at most one common vertex. If these cycles are disjoint then we are done, so suppose not. If either cycle has fewer than four vertices, then removing the shared vertex from this cycle gives us the two cycles we are after. So suppose not. Remove the shared vertex from one of the cycles to obtain a monochromatic cycle C on at least four vertices, and a disjoint monochromatic path P of the other colour on at least three vertices. Assume that C is maximal and suppose arbitrarily that C is blue and P is red. In what follows we show that we can cover the vertices of K_n by either:

- A blue cycle longer than C and a disjoint red path, a contradiction, or
- A blue cycle and a disjoint red cycle.

By Lemma 18 if C contains only red vertices then we are done, so suppose that C contains at least one blue vertex. By Lemma 8 if the endvertices x and y of P are joined in red then we are done, so suppose not. By Lemma 30 if G_s does not contain a proper cycle then we are done. So suppose that Z is a proper cycle in G_s. Then by Lemma 26 we know that Z is unique and $|C|$ is even. By Lemma 33 if the vertices on C which aren't on Z are not spanned by a red cycle then we are done, so suppose that W is a red cycle spanning the set $C - Z$. Since C is even and Z visits every other vertex on C we have that $|Z| = |W|$.

Claim 37. *If there is a red edge between Z and W then we are done.*

Proof. Suppose there is such an edge zw where z is a vertex on Z and w is a vertex on W. If z is blue then by Lemma 35 we are done, so suppose that z is red. Let z' be the first vertex to the right of z on Z which is joined to W by a red edge, so we may have $z' = z$. If z' is blue then by Lemma 35 we are done, so suppose not.

Denote by A the set of vertices on Z to the right of z and between z and z' exclusive, so $|A| < |Z|$ and possibly $A = \emptyset$. Denote by B the set of $|A|$ consecutive vertices to the right of w on W. Note that since $|A| < |Z|$ then B cannot contain w.

We now show that the set $A \cup B$ is spanned by a blue cycle (as in Figure 4.1). If z' is the right neighbour of z then this is trivially true since then A and B are empty, so suppose not. Observe that there are no red edges between A and W. Since $B \subset W$, there are therefore no red edges between A and B. Thus, since $|A| = |B|$, we can form a blue cycle which spans the set $A \cup B$ by alternating between vertices in A and vertices in B, so we are done.

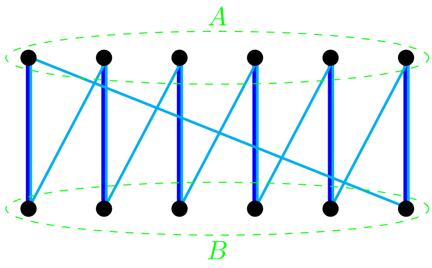


Figure 4.1

If we can now find a red cycle spanning the remaining vertices of K_n then we are done. So let's try.

If any special edge is blue then by Corollary 17 we are done, so suppose that all special edges are red. So Z is a red cycle. Since A is a set of consecutive vertices on Z, the set $Z - A$ is a red path with red endvertices z' and z. Also since $W - B$ is a set of consecutive red vertices on the red cycle W, then $W - B$ is a red path with w as one of its endvertices. Finally, since zw is red we can form the red cycle $(Z - A)(W - B)P$, so we are done. Figure 4.2 illustrates an example of this construction.

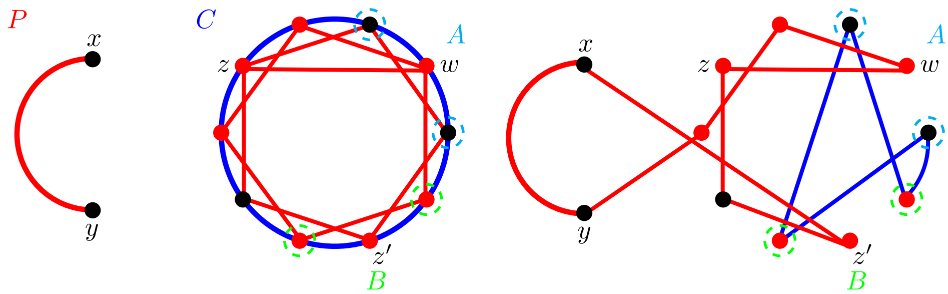


Figure 4.2

□

So suppose that all edges between W and Z are blue. Denote by y' the neighbour of y on P.

Claim 38. *If there is a red edge from y' to $Z \cup W$ then we are done.*

Proof. We consider the case where there is a red edge from y' to Z and the case where there is a red edge from y' to W in turn. First suppose that zy' is a red edge where z is a vertex on Z. Let Q be a minimal path in Z from z to a red vertex z' on Z (inclusive). By Corollary 31 if any blue vertex on Z has two blue neighbours then we are done, so suppose not. Then Q has length one or two (Figure 4.3 illustrates the case where $|Q| = 2$). Denote by Q' a red path on W with the same length as Q. By Lemma 26 part (26.3) all vertices in W are red, so we can form the red cycle $(P - y)QyQ'$.

To complete the proof we show that the remaining vertices on C can be covered by a blue cycle. Since $|Z| = |W|$ and $|Q| = |Q'|$, the set $C - (Q \cup Q')$ contains the same number of vertices on W as on Z. So we can form a blue cycle which covers the set $C-(Q\cup Q')$ by alternating between vertices in $Z-Q$ and vertices in $W-Q'$, so we are done.

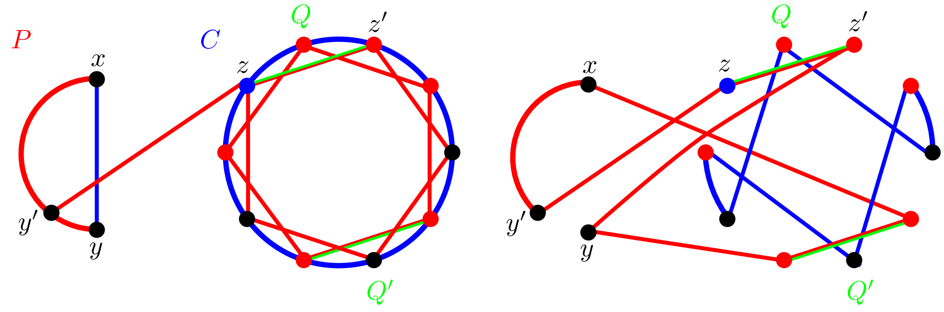
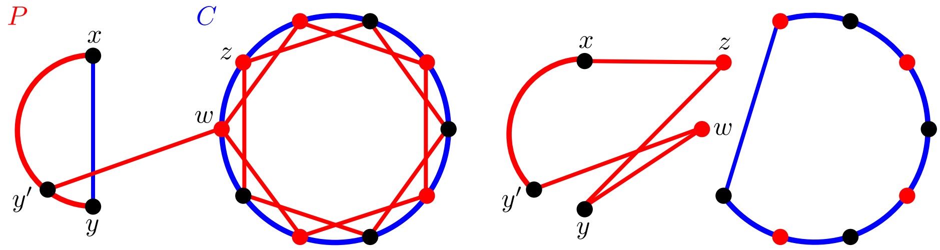


Figure 4.3

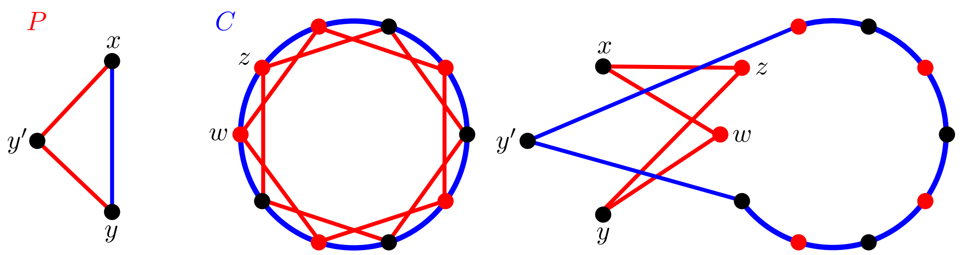
Now suppose that there is some red edge wy', where w is a vertex on W, and denote by z a red vertex on Z (Corollary 31 tells us that there is at least one). To complete the proof we show that the vertices of K_n can be covered by a red cycle and a disjoint blue cycle. Construct the red cycle $(P-y)wyz$. Since $C-\{z,w\}$ contains the same number of vertices in Z as in W, we can form a blue cycle which covers the set $C-\{z,w\}$ by alternating between vertices in W and vertices in Z, so we are done.



□

So suppose that all edges from y' to $Z\cup W$ are blue. We now achieve the proof of the theorem. Denote by w and z a red vertex in W and Z respectively. We consider the case where $P=3$ and that where $P\geq 4$ in turn.

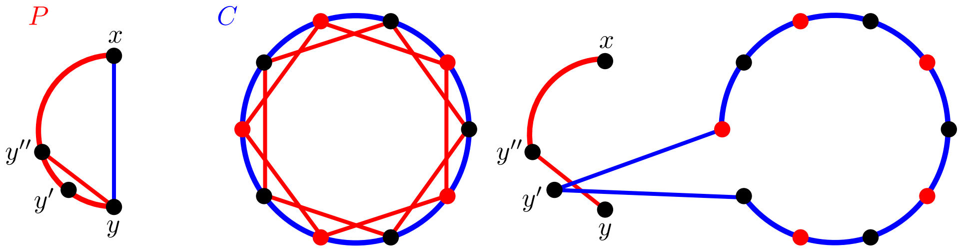
If $|P|=3$ then form the red cycle $xzyw$. We will show that there exists a blue cycle covering the remaining vertices of K_n. Since $C-\{z,w\}$ has the same number of vertices in Z as in W, we can construct a blue path spanning the set $C-\{w,z\}$ by alternating between vertices in $Z-z$ and vertices in $W-w$. Also, since we have assumed that y' is joined to $Z\cup W$ in only blue, we can join both ends of this path to y' to form a blue cycle which covers the set $C-\{w,z\}\cup y'$, so we are done.



If $|P| \geq 4$ then denote by y'' the second neighbour of y' on P. We will need the following two claims.

Claim 39. *If yy'' is red then we are done.*

Proof. If yy'' is red then $P - y'$ is a red path. Since C contains the same number of vertices in W as in Z, we can form a blue path which visits all the vertices on C. Joining both ends of this path to y' forms a blue cycle longer than C so we are done.

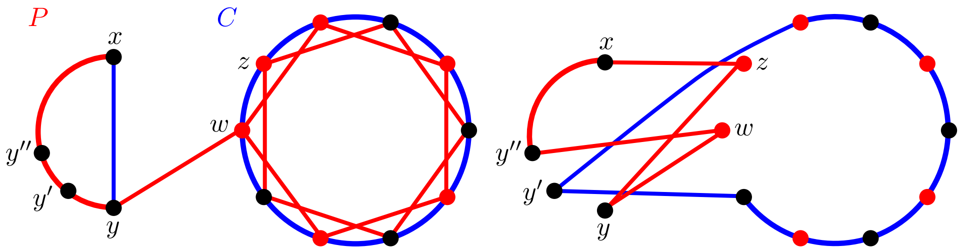


□

So suppose that yy'' is blue.

Claim 40. *If there is a red edge between y'' and W then we are done.*

Proof. Suppose that $y''w$ is red, where w is a vertex w on W and let z be a red vertex on Z. Then we can form the red cycle $(P - \{y, y'\})zyw$ by inserting z between x and y and inserting w between y and y''. We can then form a blue path spanning the set $(C - \{z, w\})$ by alternating between vertices in $Z - z$ and vertices in $W - w$. Joining both ends of this blue path to y' forms a blue cycle spanning the set $(C - \{z, w\}) \cup y'$, so we are done.



□

So suppose that all edges between y'' and W are blue. Since we have assumed that there are no consecutive blue vertices on Z then Z must contain at least two red vertices. Denote two such vertices by z and z', and let w be some vertex on W. Form a blue cycle C' which visits all the vertices of C by alternating between W and Z. Ensure that C' contains the subpath zwz' (as in Figure 4.4). Recall that we have made the following assumptions.

- All edges between y' and $W \cup Z$ are blue.
- The edge yy'' is blue.
- All edges between y'' and W are blue.

So we can replace the path z, w, z' in C' by z, x, y, y'', w, y', z' to obtain a blue cycle longer than C (as in Figure 4.5). Then $P - \{y, y', y'', x\}$ is a red path covering the remaining vertices in K_n and so we complete the proof.

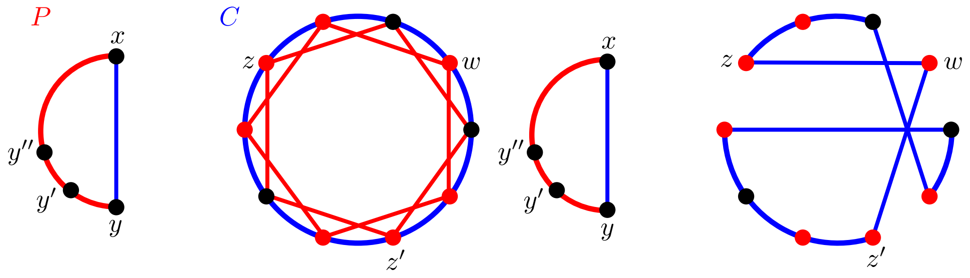


Figure 4.4

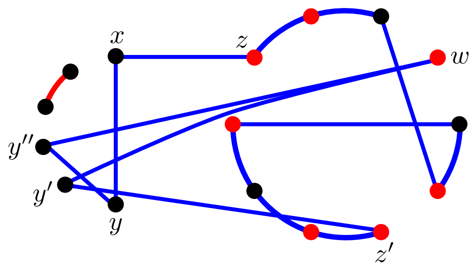


Figure 4.5

□

Chapter 5

Counterexamples to the Conjecture of Erdös, Gyárfás and Pyber

5.1 Motivation

In this chapter we prove the following theorem by Pokrovskiy [20].

Theorem 41. *For all $r \geq 3$ there exist infinitely many r-edge-coloured complete graphs whose vertices cannot be covered by r disjoint monochromatic cycles.*

We prove Theorem 41 by constructing these graphs. We will see, however, that all but one of the vertices of each graph *can* be covered by r disjoint monochromatic cycles, so the counterexamples are quite weak [20]. Indeed, the following theorem by Gyárfás, et al. [12] still holds.

Theorem 42. *At most three disjoint monochromatic cycles are needed to cover all but $o(n)$ of the vertices of any three-edge-coloured K_n.*

In light of this, Pokrovskiy [20] proposed the following weaker versions of Conjecture 2.

Conjecture 43. *The vertices of any r-edge-coloured K_n can be covered by r (not necessarily disjoint) monochromatic cycles.*

Conjecture 44. *At most r disjoint monochromatic cycles are needed to cover all but c_r of the vertices of any r-edge-coloured K_n.*

Conjecture 43 remains open for all $r \geq 3$. Pokrovskiy [21] proved the $r = 3$ case of Conjecture 44 with $c_3 = 43000$ for sufficiently large graphs. It was mentioned in [21] that Letzter [16] proved the same result with the improved constant $c_3 = 60$.

In [19], Pokrovskiy remarked that his counterexamples make Conjecture 1 appear questionable. On the other hand, he believes that the following facts make Conjecture 1 look "very plausible" [19].

1. Conjecture 1 holds for all $r \leq 3$ (see [6] and [20]).

2. Each of Pokrovskiy's counterexamples [20] to Conjecture 2 can be covered by r disjoint monochromatic *paths*.

The second of these facts was mentioned but not proved in [19]. We show that it is true in Section 5.4.

5.2 Proof Outline

To prove Theorem 41 we construct, for each $r \geq 3$ and $m \geq 1$, a complete r-edge-coloured graph J_r^m whose vertices cannot be covered by r disjoint monochromatic cycles. Since there is no upper bound on m this gives, for each r, infinitely many such graphs.

To construct the graph J_r^m we require Lemma 46, which tells us that, for each $r \geq 3$ and $m \geq 1$, there exists a complete graph H_r^m whose vertices cannot be covered by $r-1$ disjoint monochromatic paths, or by r disjoint monochromatic paths of different colours. This lemma is stated and proven before we prove Theorem 41. To construct H_3^m we take a vertex set of size depending on m, partition it into four sets, and then colour the edges between the vertices in H_r^m according to the partition. For all $r \geq 4$, we construct the graph H_r^m by adding to H_{r-1}^{5m} a set of vertices and colouring the extra edges carefully. We then prove by induction on r that H_r^m satisfies Lemma 46 for all $r \geq 3$ and $m \geq 1$. For the base case $r = 3$ we require a simple but technical lemma which states that if G is a graph, P any path in G, and X an independent set in G, then the intersection of P and X cannot be much larger than the intersection of P and $G - X$. This lemma is stated and proven before Lemma 46.

We then construct, for each $m \geq 1$ and $r \geq 3$, the graph J_r^m by adding to a copy of H_r^m a set of r vertices and colouring the extra edges carefully. We then prove Theorem 41 by contradiction. In particular, we suppose that there exists a set of r disjoint monochromatic cycles which cover all the vertices of J_r^m. We will see that every one of these cycles contains edges in H_r^m, and that this implies that one of the colours $1, \ldots, r$ cannot be present in any of the cycles. This, in turn, implies that there is some vertex in J_r^m which cannot be covered by any of the cycles, a contradiction.

5.3 Proof

We begin by proving the following lemma.

Lemma 45. *Let G be a graph, X an independent set in G, and P any path in G. Then we have:*

$$|P \cap X| \leq |P \cap (G - X)| + 1 \tag{5.1}$$

Proof. Let $p_1, \ldots, p_k$ be the vertex sequence of P. Since X is an independent set then, for all $1 \leq i \leq k-1$, if p_i is in X then p_{i+1} must be in $G - X$. So $|P \cap X|$ is of size at most $\lceil \frac{P}{2} \rceil$, implying the result. □

We now use Lemma 45 to prove Lemma 46. We in fact prove a stronger and slightly more complex version of Lemma 46 than was described in the proof outline. Namely, we prove that for each $r \geq 3$ and $m \geq 1$, there exists a graph H_r^m and a set $T \subseteq V(H_r^m)$ with $|T| \leq m$ such that the vertices in $H_r^m - T$ cannot be covered by $r - 1$ disjoint monochromatic paths, or by r disjoint monochromatic paths of different colours. The purpose of this strengthened version is to enable us to prove the $r \geq 4$ case of Lemma 46 given that we have shown the lemma holds when $r = 3$. Notice that if we let $T = \emptyset$ in Lemma 46 then we have the simpler version described in the proof outline.

Lemma 46. *For each $r \geq 3$ and $m \geq 1$ there exists an r-edge-coloured complete graph H_r^m which satisfies the following, where T is any subset of $V(H_r^m)$ with $|T| \leq m$.*

(46.1) *The vertices of $H_r^m - T$ cannot be covered by $r-1$ disjoint monochromatic paths.*

(46.2) *The vertices of $H_r^m - T$ cannot be covered by r disjoint monochromatic paths of different colours.*

Proof. To prove Lemma 46 we first construct H_r^m for each $m \geq 1$ and $r \geq 3$. We then prove by induction on r that for any set T of at most m vertices from $V(H_r^m)$, the graph $H_r^m - T$ satisfies parts (46.1) and (46.2) of Lemma 46. This construction is different for each of $r = 3$ and $r \geq 4$, so we consider each case in turn. Figure 5.1 illustrates the case where $r = 3$, colour 1 is red, colour 2 is blue and colour 3 is green.

Construction 47. For $r = 3$ and each $m \geq 1$ the graph H_3^m is a complete graph of order $43m$ constructed as follows. Partition the vertices into four sets A_1, A_2, A_3 and A_4 such that $|A_1| = 10m$, $|A_2| = 13m$, $|A_3| = 7m$ and $|A_4| = 13m$. The edges between A_1 and A_2 and between A_3 and A_4 are colour 1. The edges between A_1 and A_3 and between A_2 and A_4 are colour 2. The edges between A_1 and A_4 and between A_2 and A_3 are colour 3. Colour the edges within A_1 and A_2 colour 3, and the edges within A_3 and A_4 colour 2.

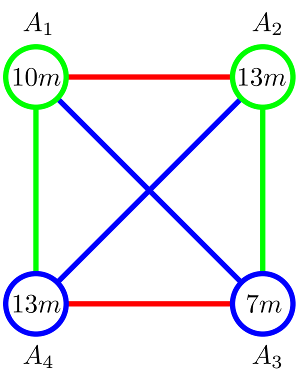


Figure 5.1: Example colouring of H_3^m

We now construct H_r^m for all $r \geq 4$ and $m \geq 1$.

Construction 48. For each $r \geq 4$ and $m \geq 1$ we construct the graph H_r^m as follows. Take a copy H of H_{r-1}^{5m} and add to it a set K of $2m$ vertices. The edges within H are already coloured with colours $1, \ldots, r-1$ and we colour all edges within K, and all edges between K and H using colour r.

We now prove that Lemma 46 holds for $r = 3$. We then use induction on r to show that the lemma also holds for all $r \geq 4$.

Recall from Construction 47 that H_3^m is partitioned into the four sets A_1, A_2, A_3 and A_4. Let $B_i = A_i - T$ for each $1 \leq i \leq 4$. Since $|A_i| \geq 7m$ for each i, and $|T| \leq m$, the sets B_1, B_2, B_3 and B_4 are all nonempty. We will need the following claim.

Claim 49. *The following hold:*

(a) *B_2 cannot be covered by a colour 1 path.*

(b) *B_1 cannot be covered by a colour 2 path.*

(c) *B_4 cannot be covered by a colour 3 path.*

(d) *B_4 cannot be covered by a colour 1 path.*

(e) *$B_1 \cup B_3$ cannot be covered by a colour 1 path contained in $B_1 \cup B_2$ and a disjoint colour 3 path contained in $B_2 \cup B_3$.*

(f) *$B_2 \cup B_3$ cannot be covered by a colour 1 path contained in $B_3 \cup B_4$ and a disjoint colour 2 path contained in $B_2 \cup B_4$.*

Proof. We prove parts *(a)* to *(f)* in turn. The proofs for parts *(a)* to *(d)* follow a similar structure. The proofs for parts *(e)* and *(f)* follow a (different) similar structure.

(a) Let P be any colour 1 path in $H_3^m - T$ which intersects B_2. We must show that P cannot cover all the vertices of B_2. By Construction 47 the path P must be contained in the subgraph $B_1 \cup B_2$ of $H_3^m - T$. Since there are no colour 1 edges between vertices in the set B_2 we can consider B_2 an independent set in the colour 1 subgraph of $B_1 \cup B_2$. So we can apply Lemma 45 to obtain:

$$|P \cap B_2| \leq |P \cap B_1| + 1. \tag{5.2}$$

Since $|A_1| = 10m$ we have that:

$$|P \cap B_1| + 1 = |P \cap (A_1 - T)| + 1 \leq |A_1| + 1 = 10m + 1. \tag{5.3}$$

Finally, since $m \geq 1$, we can now combine (5.2) and (5.3) to get:

$$|P \cap B_2| \leq 10m + 1 < 12m \leq |B_2|.$$

So P cannot cover all of B_2.

(b) Let P be any colour 2 path in $H_3^m - T$ which intersects B_1. This part is proved similarly to (a) using the fact that P must be contained in the colour 2 subgraph $B_1 \cup B_3$. Since there are no colour 2 edges between vertices in the set B_1 we can apply Lemma 45 to obtain:

$$|P \cap B_1| \leq |P \cap B_3| + 1 \leq |A_3| + 1 = 7m + 1 < 9m \leq |B_1|.$$

So P does not cover all of B_1.

(c) Let P be a colour 3 path in $H_3^m - T$ which intersects B_4. This part is proved similarly to (a) using the fact that P must be contained in $B_1 \cup B_4$ and B_4 does not contain any colour 3 edges.

(d) Let P be a colour 1 path in $H_3^m - T$ which intersects B_4. This part is proved similarly to (a) using the fact that P must be contained in $B_3 \cup B_4$ and B_4 does not contain any colour 1 edges.

(e) Let P be a colour 1 path contained in $B_1 \cup B_2$ and let Q be a disjoint colour 3 path contained in $B_2 \cup B_3$. We must show that P and Q do not cover $B_1 \cup B_3$. There are no colour 1 edges in the set B_1, so by Lemma 45 we have that:

$$|P \cap B_1| \leq |P \cap B_2| + 1. \tag{5.4}$$

Similarly, B_3 does not contain any colour 3 edges, so applying Lemma 45 gives:

$$|Q \cap B_3| \leq |Q \cap B_2| + 1. \tag{5.5}$$

Since P does not intersect B_3, and Q does not intersect B_1, then (5.4) and (5.5) imply that:

$$|(P \cup Q) \cap (B_1 \cup B_3)| \leq |(P \cap B_2) \cup (Q \cap B_2)| + 2 = |(P \cup Q) \cap B_2| + 2. \tag{5.6}$$

Finally, since $m \geq 1$ and $|A_2| = 13m$, equation (5.6) implies the following:

$$|(P \cup Q) \cap (B_1 \cup B_3)| \leq |A_2| + 2 = 13m + 2 < 16m \leq |(B_1 \cup B_3)|.$$

So P and Q cannot cover $B_1 \cup B_3$.

(f) This part is proved similarly to (e), using the fact that B_2 does not contain any colour 2 edges and B_3 does not contain any colour 1 edges.

□

We now use Claim 49 to prove Lemma 46 for $r = 3$. We prove parts *(46.1)* and *(46.2)* separately.

(46.1) Suppose for a contradiction that P and Q are two disjoint monochromatic paths which cover the vertices of $H_3^m - T$. Construction 47 tells us that P and Q cannot have different colours since any two monochromatic paths with different colours can cover at most three of B_1, B_2, B_3 and B_4. We now consider each of the possible colours of P and Q.

- If P and Q are both colour 1 then by Construction 47 one of the paths must cover $B_1 \cup B_2$. This contradicts part *(a)* of Claim 49.
- If P and Q are both colour 2 then by Construction 47 one of the paths must cover $B_1 \cup B_3$. This contradicts part *(b)* of Claim 49.
- If P and Q are both colour 3 then by Construction 47 one of the paths covers $B_1 \cup B_4$. This contradicts part *(c)* of Claim 49.

(46.2) Suppose for a contradiction that P_1, P_2 and P_3 are three disjoint monochromatic paths which cover the vertices of $H_3^m - T$ such that P_j is of colour j for $1 \leq j \leq 3$. By Construction 47, each path can be contained in at most two of the sets B_1, B_2, B_3 and B_4, and P_2 must be contained in either $B_1 \cup B_3$ or $B_2 \cup B_4$. We consider each case in turn.

- First suppose that $P_2 \subseteq B_1 \cup B_3$. Since parts *(c)* and *(d)* of Claim 49 tell us that B_4 cannot be covered by a single path of either colour 1 or colour 3, both P_1 and P_3 must intersect B_4. By Construction 47, this implies that $P_1 \subseteq B_3 \cup B_4$ and $P_3 \subseteq B_1 \cup B_4$. This leads to a contradiction since then none of the paths P_1, P_2, and P_3 intersect B_2.
- Now suppose that $P_2 \subseteq B_2 \cup B_4$. By Construction 47 the path P_1 is contained in either $B_1 \cup B_2$ or $B_3 \cup B_4$. In the former case, for B_3 to be covered P_3 must be contained in $B_2 \cup B_3$. This is a contradiction of part *(e)* of Claim 49. In the latter case, for B_1 to be covered P_3 must be contained in $B_1 \cup B_4$. This is a contradiction of part *(f)* of Claim 49, so we are done.

We now prove Lemma 46 for all $r \geq 3$ by induction on r. The initial case $r = 3$ was proved above. Assume that the lemma holds for H^m_{r-1} for all $m \geq 1$. We will now show that the lemma holds for H^m_r.

Recall from Construction 48 that, for all $m \geq 1$ and $r \geq 4$, the graph H^m_r is formed by adding to a copy H of H^{5m}_{r-1} a set K of $2m$ vertices and colouring all edges within K, and between K and H using colour r. Suppose that the vertices of $H^m_r - T$ are covered by r disjoint monochromatic paths $P_1, \dots, P_r$ (with some of these possibly empty). We must show that none of these paths can be empty and that they cannot all be of different colours. Without loss of generality we may assume that each of the paths $P_1, \dots, P_k$ intersects K, and that each of the paths $P_{k+1}, \dots, P_r$ is disjoint from K. Note that since the paths $P_1, \dots, P_k$ are disjoint we must have that $k \leq |K| = 2m$.

Since H does not contain any colour r edges, Lemma 45 implies that for each $1 \leq i \leq k$ we have:

$$|H \cap P_i| \leq |K| + 1 \tag{5.7}$$

If we now let $S = H \cap (P_1 \cup \ldots \cup P_k)$, then, since $k \leq |K| = 2m$, equation (5.7) implies that:

$$|S| \leq |K| + k \leq 4m. \tag{5.8}$$

Since $|T| \leq m$, equation (5.8) implies that $|S \cup T| \leq 5m$. Recall that $H - (S \cup T)$ is covered by the $r - k$ disjoint monochromatic paths $P_{k+1}, \dots, P_r$, and that H is a copy of H^{5m}_{r-1}. Then since $|S \cup T| \leq 5m$, by our induction assumption we can apply Lemma 46 to show that $H - (S \cup T)$ cannot be covered by $r - 2$ disjoint monochromatic paths or by $r - 1$ disjoint monochromatic paths of different colours. Thus $k = 1$ and the paths $P_2, \dots, P_r$ are all nonempty and not all of different colours. This completes the proof since we know that P_1 intersects K so P_1 cannot be empty and hence $P_1, \dots, P_r$ are all nonempty and are not all of different colours. $\square$

We now use Lemma 46 to prove Theorem 41.

Proof of Theorem 41. We first construct, for each $m \geq 1$ and $r \geq 3$, the graph J^m_r whose vertices cannot be covered by r disjoint monochromatic cycles. Figure 5.2 illustrates an example of this construction for the case where $r = 3$, colour 1 is red, colour 2 is blue and colour 3 is green. Note that we will make use of Construction 47 and Construction 48.

Construction 50. For each $m \geq 1$ and $r \geq 3$ we form J^m_r by adding to a copy D of H^m_r a set of r vertices $\{v_1, \dots, v_r\}$. Then for each $i \in \{1, \dots, r\}$, we colour all edges between v_i and D with colour i. The edge v_1v_2 has colour 3. For $j \geq 3$ the edge v_1v_j has colour 2 and the edge v_2v_j has colour 1. For $3 \leq i < j$, the edge v_iv_j has colour 1.

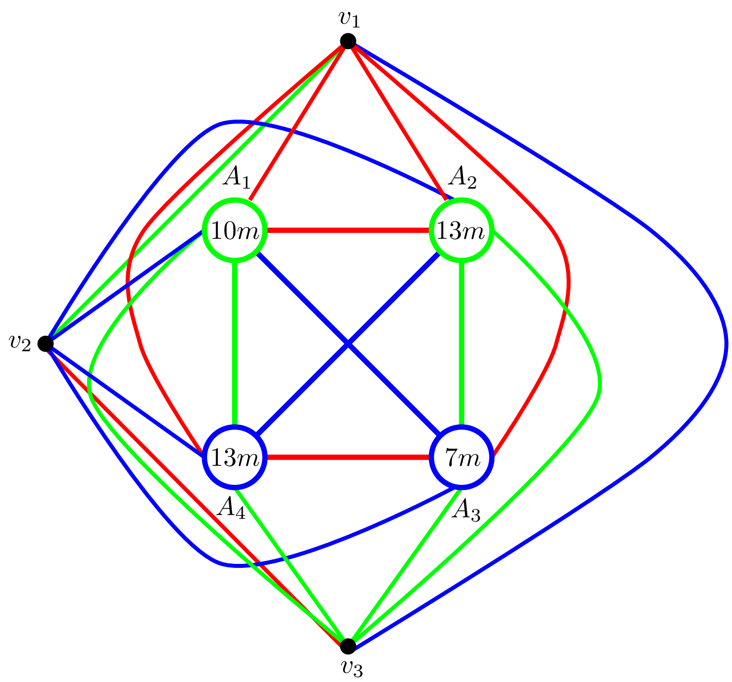


Figure 5.2: Colouring of J_3^m

We will now prove that the vertices of J_r^m cannot be covered by r disjoint monochromatic cycles. Let $C_1, \ldots, C_r$ be r disjoint monochromatic cycles in J_r^m and suppose for a contradiction that $C_1 \cup \ldots \cup C_r = J_r^m$. Note that none of these cycles can be empty since Conjecture 2 is best possible [5]. Also since, for any $1 \leq i, j \leq r$ with $i \neq j$, the edge $v_i v_j$ has a different colour to the edges between v_i and D, a monochromatic cycle in J_r^m cannot pass through both edges in $\{v_1, \ldots, v_r\}$ and vertices in D.

Let $P_i = C_i - \{v_1, \ldots, v_r\}$ for all $1 \leq i \leq r$. Then for each i the path P_i is a (possibly empty) monochromatic path in D and $P_1, \ldots, P_r$ cover D. Since D is a copy of H_r^m, by Lemma 46 (with $T = \emptyset$) the paths $P_1, \ldots, P_r$ are all nonempty and are not all of different colours. So there must be a colour, say colour i, which is not present in any of the cycles $C_1, \ldots, C_r$.

Also, since the monochromatic paths $P_1, \ldots, P_r$ are all nonempty and contained in D, none of the cycles $C_1, \ldots, C_r$ can contain any edges between the vertices $\{v_1, \ldots, v_r\}$. But since none of the cycles has colour i this implies that the vertex v_i is not contained in any of the cycles $C_1, \ldots, C_r$ since v_i is joined to all vertices in D by edges of colour i. This is a contradiction since we assumed that $C_1 \cup \ldots \cup C_r = J_r^m$, so we are done.

Note that if all colours except i are present among the cycles $C_1, \ldots, C_r$ then these cycles may cover every vertex of J_r^m except v_i. □

5.4 Covering The Counterexamples by Monochromatic Paths

We will now show that we can cover the vertices of the graph J_r^m from Construction 50 by r disjoint monochromatic paths for all $r \geq 3$ and $m \geq 1$. Recall from Construction 47 that

J_r^m is formed by adding to a copy of H_r^m a set of r vertices $\{v_1, \ldots, v_r\}$ such that v_i is joined to the copy of H_r^m in colour i for $1 \leq i \leq r$. By Lemma 46, for each $r \geq 3$, at least r disjoint monochromatic paths are needed to cover the vertices of H_r^m, and these paths cannot all be of different colours. We first show that r disjoint monochromatic paths, with at least two of the same colour, are sufficient.

For $r = 3$ it is easy to see from Construction 47 that we cannot cover H_3^m by 3 disjoint paths of the same colour. So one possibility is a colour 2 path covering all the vertices in the set $A_2 \cup A_4$, another colour 2 path covering all the vertices in A_3, and a colour 3 path covering all the vertices in A_1. For $r \geq 4$, Construction 48 tells us that H_r^m is constructed by adding some vertices to a copy of H_{r-1}^m, and that these "extra" vertices are joined to each other using colour r. So by induction, for each $r \geq 4$, we can cover the vertices of H_r^m with a colour r path (covering the "extra" vertices) and a further $r - 1$ disjoint monochromatic paths (covering the copy of H_{r-1}^m) with colours among $1, \ldots, r-1$ and at least two of the same colour. Note that this implies that, for each $r \geq 4$, among the r disjoint monochromatic paths needed to cover H_r^m, there is a path of colour k for all $4 \leq k \leq r$, two paths of colour i for some $1 \leq i \leq 3$ and one path of colour j for some $1 \leq j \leq 3$ with $i \neq j$.

We will now show that, for each $r \geq 3$, we can cover the vertices of J_r^m by r disjoint monochromatic paths. Suppose that, for each $r \geq 3$, among the r paths needed to cover H_r^m, there are two paths of colour j and there is no path of colour k for $j, k \leq 3$. Then for all $1 \leq i \leq r$ with $i \neq j, k$ we can join v_i to the path of colour i, and we can join v_j to an endvertex of each path of colour j to form a single colour j path. Then v_k is our rth path and we are done.

Chapter 6

Covering Complete Balanced Bipartite Graphs by Monochromatic Paths

6.1 Background

Recall that the bipartite graph $K_{n,m}$ is called *balanced* if $n = m$. Here we will call a path in $K_{n,n}$ *alternating* if its endvertices are in different parts of $K_{n,n}$, otherwise we call the path *nonalternating*.

In [20], Pokrovskiy showed that if and only if certain edge-colourings of a two-edge-coloured $K_{n,n}$ are avoided, then its vertices can be covered by two disjoint monochromatic paths. In the same paper Pokrovskiy claimed, but did not prove, that if we allow for these problematic colourings then at most three disjoint monochromatic paths are needed. This can be written as follows:

Theorem 6. *At most three disjoint monochromatic paths are needed to cover the vertices of any two-edge-coloured $K_{n,n}$.*

In this chapter we give a new self-contained proof of Theorem 6. In [20], Pokrovskiy also proved the following lemma.

Lemma 51. *Colour the edges of K_n using only red and blue. Then the vertices of K_n can be covered by a red path and a disjoint blue balanced complete bipartite graph.*

By combining Theorem 6 and Lemma 51, Pokrovskiy [20] proved the following:

Theorem 52. *The vertices of any three-edge-coloured K_n can be covered by four disjoint monochromatic paths.*

To see that Theorem 52 holds [20], colour the edges of K_n using only red, blue and yellow. If we treat yellow and blue as a single colour then Lemma 51 tells us that we can cover the vertices of K_n by a single red path and a blue-yellow $K_{m,m}$, where $m \leq n$. By Theorem 6, now treating yellow and blue as separate colours, we can cover the vertices of $K_{m,m}$ by at most three disjoint monochromatic paths, so we are done.

In [20] Pokrovskiy uses more complex versions of Theorem 6 and Lemma 51 to show that at most three disjoint monochromatic paths are needed to cover a three-edge-coloured K_n, verifying the $r = 3$ case of Conjecture 1.

In what follows we consider single edges and singletons as monochromatic paths. We now give a proof outline for our proof of Theorem 6.

6.2 Proof Outline

Consider some two-edge-coloured $K_{n,n}$. If $n \leq 3$ then we are done; we can cover the vertices by at most three pairwise non-adjacent edges. So suppose that $n \geq 4$. We begin by constructing any three maximal disjoint monochromatic paths P_1, P_2 and P_3 on the vertices of $K_{n,n}$. Note that any path on a single vertex is considered a nonalternating path. We call a vertex in $K_{n,n}$ which is not covered by P_1, P_2 or P_3 *uncovered*. If P_1, P_2 and P_3 cover all the vertices of $K_{n,n}$ then we are done. If not, then we are in one of the following situations:

1. $K_{n,n}$ contains a pair a and b of adjacent uncovered vertices.
2. $K_{n,n}$ contains an uncovered vertex c which is not adjacent to any other uncovered vertex.

If we are in situation 1. then we consider the complete balanced subgraph $K_{m,m}$ of $K_{n,n}$. The graph $K_{m,m}$ contains a, b, all the vertices on P_1, P_2 and P_3, and possibly some other vertices, as well as all edges between these vertices. We then use case analysis to find at most three disjoint monochromatic paths which cover a, b and all the vertices on P_1, P_2 and P_3. In particular, we consider each of the following cases:

(a) All of the paths P_1, P_2 and P_3 are alternating.

(b) Two of the paths P_1, P_2 and P_3 are alternating and one is nonalternating.

(c) Two of the paths P_1, P_2 and P_3 are nonalternating and one is alternating.

(d) All of the paths P_1, P_2 and P_3 are nonalternating.

Clearly cases (a) to (d) cover all possibilities. In each case, we must consider the different possible colours of the three paths. For cases (b) to (d), we must also consider which part of $K_{m,m}$ the endvertices of the nonalternating paths are contained in. We then show, by considering the colours of edges in $K_{m,m}$ other than those on P_1, P_2 and P_3, that we can always find at most three disjoint monochromatic paths which cover a, b and all the vertices on P_1, P_2 and P_3. Case (a) is considered in Lemma 54, case (b) in Lemma 58, case (c) in Lemma 62 and case (d) in Lemma 66. In each case, when P_1, P_2 and P_3 are all the same colour we use Lemma 53 to obtain the result.

If instead we are in situation 2. then we consider the complete balanced subgraph $K_{t,t}$ of $K_{n,n}$. The graph $K_{t,t}$ contains c, all the vertices on P_1, P_2 and P_3, and possibly some other vertices, as well as all edges between these vertices. We then consider each of the following cases:

(A) Two of the paths P_1, P_2 and P_3 are alternating and one is nonalternating.

(B) Two of the paths P_1, P_2 and P_3 are nonalternating and one is alternating.

(C) All of the paths P_1, P_2 and P_3 are nonalternating.

Observe that cases (A) to (C) cover all possibilities since, if P_1, P_2 and P_3 are all alternating, then c must be adjacent to another uncovered vertex, so we are in situation 1.. For each case we must consider the possible colours of the paths P_1, P_2 and P_3, as well as which part of $K_{t,t}$ the endvertices of each nonalternating path are contained in. By considering the colours of edges in $K_{t,t}$ other than those on P_1, P_2 and P_3, we show that we can always find at most three disjoint monochromatic paths which cover c and all the vertices on P_1, P_2 and P_3. For the subcase where P_1, P_2 and P_3 are all the same colour we use Lemma 53 to obtain the result.

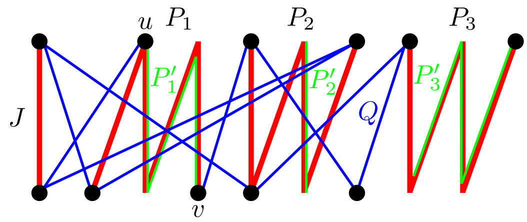
If the "new" disjoint monochromatic paths that we find do not cover all the vertices in $K_{n,n}$, then we can extend each path so that it is maximal and repeat the above process to obtain at most three disjoint monochromatic paths which cover "another" vertex in $K_{n,n}$. Repeating this process until we have covered all vertices in $K_{n,n}$ with at most three disjoint monochromatic paths gives us the result.

6.3 Preparatory Lemmas

We begin by stating and proving Lemmas 53 to 66. We then use these lemmas to prove Theorem 6.

Lemma 53. *Suppose that a two-edge-coloured $K_{m,m}$ contains three disjoint monochromatic paths P_1, P_2 and P_3 of the same colour and a fourth path J which is disjoint from P_1, P_2 and P_3. Suppose further that there exists a monochromatic path Q which covers all the vertices on J and at least one vertex on each of P_1, P_2 and P_3. Denote the endvertices of Q by u and v. Suppose furthermore that $P'_i = P_i - (Q - \{u, v\})$ is a path for each $i \in \{1, 2, 3\}$, and suppose that u and v are the endvertices of P'_1. Then if P'_1, P'_2 and P'_3 are all alternating at most three disjoint monochromatic paths are needed to cover all vertices on J, P_1, P_2 and P_3.*

Proof. If the vertices of any P_i are all covered by Q then we are done, so suppose not. Suppose arbitrarily that u and v are the endvertices of P'_1.



To prove Lemma 53 we show that either $P'_i P'_j$ is a monochromatic path for distinct i and j with $j \in \{1, 2, 3\}$, or we can keep extending Q until it covers all the vertices on one of P_1, P_2 and P_3.

If either u or v is joined to an endvertex of P'_k for some $k > 1$ by an edge of the same colour as P'_k then $P'_1 P'_k$, the remaining path among P'_2 and P'_3, and the path $Q - \{u, v\}$ are disjoint monochromatic paths which cover the vertices of interest. So suppose not. Denote by x and y the endvertices of P'_k (Figure 6.1). Set $Q = xQy$, so x and y are the new endvertices of Q, and set $P'_1 = P'_1 - \{u, v\}$. If P'_1 is now empty then we are done, so suppose not. Observe that P'_1, P'_2 and P'_3 are still disjoint alternating paths. Continuing in this way we eventually find that either there exists a single monochromatic path $P'_i P'_j$ for distinct i and j, or Q covers all the vertices on one of P_1, P_2 and P_3. So we are done.

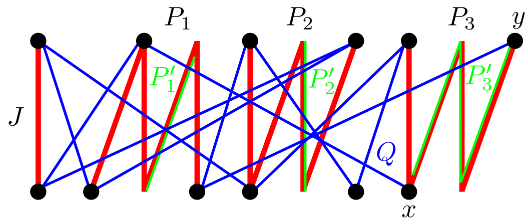


Figure 6.1

□

Lemma 54. *Suppose that a two-edge-coloured $K_{m,m}$ contains three disjoint maximal monochromatic alternating paths P_1, P_2 and P_3 and a pair of adjacent vertices a and b which are not covered by any of these paths. Then at most three disjoint monochromatic paths are needed to cover a, b and all the vertices on P_1, P_2 and P_3.*

Proof. Suppose arbitrarily that ab is red. Denote the bipartition of $K_{m,m}$ by (M_a, M_b). We suppose arbitrarily that a is contained in M_a, so b is contained in M_b. Denote the endvertices of P_i by u_i and v_i for $i \in \{1, 2, 3\}$ where, say, u_i is in M_b and v_i is in M_a. We consider each of the following cases in turn:

(a) The paths P_1, P_2 and P_3 are all the same colour.

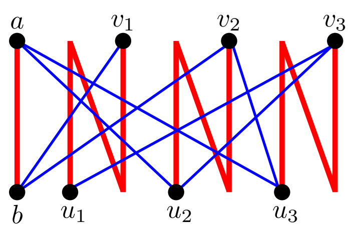
(b) Exactly two of P_1, P_2 and P_3 are red.

(c) Exactly two of P_1, P_2 and P_3 are blue.

Observe that this covers all possibilities. For case (a) we use Lemma 53 to obtain the result. For cases (b) and (c) we obtain the result by considering the colours of edges in $K_{m,m}$ other than those on P_1, P_2 and P_3.

Claim 55. *If P_1, P_2 and P_3 are all the same colour then we are done.*

Proof. Suppose arbitrarily that P_1, P_2 and P_3 are red. If a or b is joined in red to an endvertex of P_i for any i then we have a contradiction of the maximality of P_i. Notice that for all distinct i and j the edge $u_i v_j$ must be blue, otherwise $P_i P_j$ is a single red path, a contradiction. Construct a blue path which covers only a, b and both endvertices of each of P_1, P_2 and P_3. Ensure that the endvertices of this blue path are the endvertices of some P_i. Then by Lemma 53 we are done.



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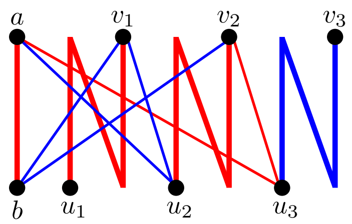
Claim 56. *If two of P_1, P_2 and P_3 are red and one is blue then we are done.*

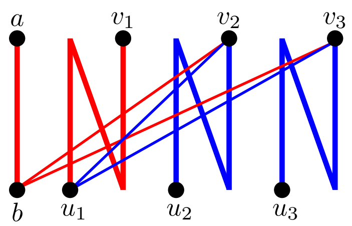
Proof. Suppose arbitrarily that P_3 is the blue path. If a or b is joined in red to an endvertex of P_1 or P_2, or in blue to an endvertex of P_3 then we have a contradiction. Also if u_2v_1 is red then we have a contradiction. If v_2u_3 is blue then $P_3v_2bv_1u_2a$ is a blue path and we are done, so suppose not. Then P_2u_3ab is a red path so we are done.



□

Claim 57. *If two of P_1, P_2 and P_3 are blue and one is red then we are done.*

Proof. Suppose arbitrarily that P_1 is the red path. If b is joined in blue to v_2 or v_3 then we have a contradiction. If u_1v_3 is red then P_1v_3ba is a red path and we are done, so suppose not. Similarly, if u_1v_2 is red then we are done, so suppose not. Then $P_2u_1P_3$ is a single blue path, so we are done.



□

□

Lemma 58. *Suppose that a two-edge-coloured $K_{m,m}$ contains three disjoint maximal monchromatic paths P_1, P_2 and P_3 and two adjacent vertices a and b which are not covered by any of these paths. Suppose further that two of P_1, P_2 and P_3 are alternating and one is nonalternating. Then at most three disjoint monochromatic paths are needed to cover a, b and all the vertices on P_1, P_2 and P_3.*

Proof. Suppose arbitrarily that ab is red. Denote the bipartition of $K_{m,m}$ by (M_a, M_b) where, say, a is in M_a, so b is in M_b. Denote the endvertices of P_i by u_i and v_i for $i \in \{1, 2, 3\}$. Suppose arbitrarily that P_1 is the nonalternating path and that u_1 and v_1 are contained in M_b. Suppose arbitrarily that u_j is contained in M_b and v_j is contained in M_a for all $j \geq 2$.

We denote by x a vertex in M_a which is distinct from a and which is not covered by any path P_i. We consider each of the following cases in turn:

(a) The paths P_1, P_2 and P_3 are all the same colour.

(b) The path P_1 is of a different colour to both P_2 and P_3.

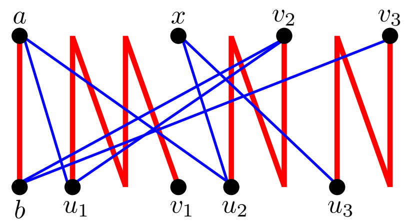
(c) The paths P_2 and P_3 are of different colours.

Within cases (b) and (c) we consider the subcase where P_1 is blue and that where P_1 is red in turn. Observe that this covers all possibilities. For case (a) we make use of Lemma 53.

If a, b or x is joined to an endvertex of P_i by an edge of the same colour as P_i then we have a contradiction. Similarly, if there is an edge of the same colour as P_i between an endvertex of P_i and an endvertex of a different path P_k of the same colour then P_iP_k is a single monochromatic path, a contradiction.

Claim 59. *If P_1, P_2 and P_3 are all the same colour then we are done.*

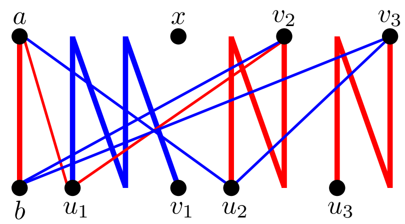
Proof. Suppose arbitrarily that P_1, P_2 and P_3 are all red. Construct a blue path which covers only a, b, x, u_1 and both endvertices of P_2 and P_3. Ensure that the endvertices of this blue path are the endvertices of either P_2 or P_3. Then by Lemma 53 we are done.



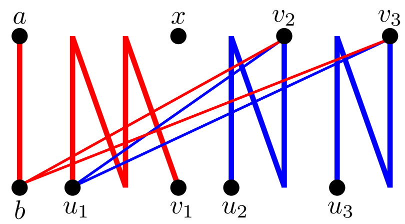
□

Claim 60. *If P_1 is of a different colour to P_2 and P_3 then we are done.*

Proof. We consider the case where P_1 is blue and the case where P_1 is red in turn. First consider the case where P_1 is blue. If u_1v_2 is blue then $P_1v_2bv_3u_2a$ is a blue path, so suppose not. Then P_2u_1ab is a red path so we are done.



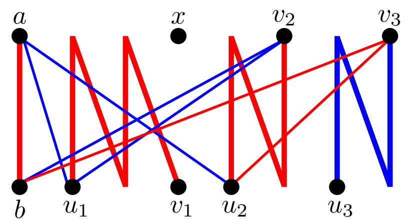
Now consider the case where P_1 is red. If u_1v_2 is red then P_1v_2ba is a red path and we are done, so suppose not. Similarly, if u_1v_3 is red then we are done, so suppose not. Then $P_2u_1P_3$ is a single blue path so we are done.



□

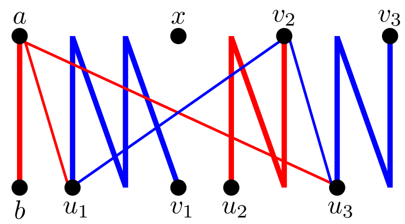
Claim 61. *If P_2 and P_3 are of different colours then we are done.*

Proof. Suppose arbitrarily that P_2 is red, so P_3 is blue. First consider the case where P_1 is red. If u_2v_3 is blue then $P_3u_2au_1v_2b$ is a blue path, so suppose not. Then P_2v_3ba is a red path, so we are done.



□

Now consider the case where P_1 is blue. If u_1v_2 is red then P_2u_1ab is a red path and we are done, so suppose not. Similarly, if u_3v_2 is red then we are done, so suppose not. Then $P_1v_2P_3$ is a blue path so we are done.



□

Lemma 62. *Suppose that a two-edge-coloured $K_{m,m}$ contains three disjoint maximal monchromatic paths P_1, P_2, P_3 and two adjacent vertices a and b which are not covered by any of these paths. Suppose further that two of P_1, P_2 and P_3 are nonalternating and one is alternating. Then at most three disjoint monochromatic paths are needed to cover a, b and all the vertices on P_1, P_2 and P_3.*

Proof. Suppose arbitrarily that ab is red and that P_3 is the alternating path. Denote the bipartition of $K_{m,m}$ by (M_a, M_b) where, say, a is contained in M_a, so b is contained in M_b. Denote the endvertices of P_i by u_i and v_i for $i \in \{1, 2, 3\}$ and suppose arbitrarily that u_1, v_1 and u_3 are in M_b, so v_3 is in M_a. We consider each of the following cases in turn:

(a) The paths P_1, P_2 and P_3 are all the same colour.

(b) Two of P_1, P_2 and P_3 are red and one is blue.

(c) Two of P_1, P_2 and P_3 are blue and one is red.

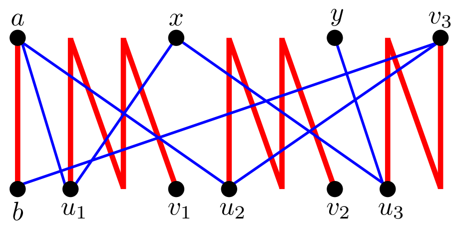
For cases (b) and (c) we must consider the case where P_3 is the path of a different colour to the other two, and that where P_2 is the path of a different colour to the other two. The case where P_1 is the path of a different colour to the other two then holds by symmetry. Within case (a), and within each subcase of cases (b) and (c), we must consider the case where $u_2, v_2 \in M_a$ and that where $u_2, v_2 \in M_b$. Observe that this covers all possibilities. For case (a) we use Lemma 53 to obtain the result.

For the case where $u_2, v_2 \in M_b$ we denote by x and y two vertices in M_a which are both distinct from a and not covered by any path P_i with $i \in \{1, 2, 3\}$.

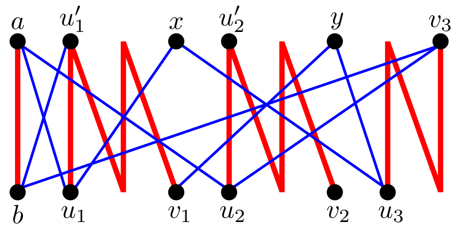
If a or b is joined to an endvertex of a path P_i by an edge of the same colour as P_i then we have a contradiction. Similarly, if there is an edge of the same colour as P_i between an endvertex of P_i and an endvertex of a different path P_j of the same colour, then P_iP_j is a single monochromatic path, a contradiction.

Claim 63. *If P_1, P_2 and P_3 are all the same colour then we are done.*

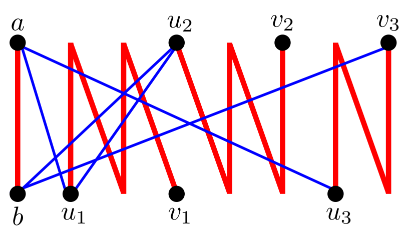
Proof. Suppose arbitrarily that P_1, P_2 and P_3 are red. First consider the case where u_2 and v_2 are contained in M_b. Then if x or y is joined in red to an endvertex of any P_i we have a contradiction. Construct a blue path B covering only a, b, x, y, both endvertices of P_3 and one endvertex of each of P_1 and P_2. Ensure that the endvertices of B are b and y.



Suppose arbitrarily that the vertices on P_1 and P_2 which are covered by B are u_1 and u_2 respectively. If B covers all the vertices of any P_i then we are done, so suppose not. If $u_k = v_k$ for any $k < 3$ then P_kx is a monochromatic path longer than P_k, a contradiction. So denote the neighbour of u_k on P_k by u'_k for each k. If bu'_1 and bu'_2 are both red then $(P_1-u_1)b(P_2-u_2)$ is a red path and u_1au_2 is a blue path, so suppose not. Suppose arbitrarily that bu'_1 is blue and set $B = u'_1Bv_1$, so u'_1 and v_1 are the new endvertices of B. Then by Lemma 53 we are done.



Now consider the case where u_2 and v_2 are contained in M_a. Construct a blue path covering only a, b, both endvertices of P_3 and one endvertex of each of P_1 and P_2. Ensure that the endvertices of this blue path are u_3 and v_3. Then by Lemma 53 we are done.

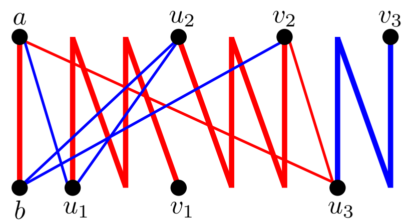


□

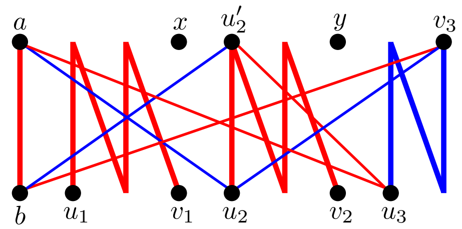
Claim 64. *If two of P_1, P_2 and P_3 are red and one is blue then we are done.*

Proof. First consider the case where P_3 is the blue path. We consider the case where $u_2, v_2 \in M_a$ and the case where $u_2, v_2 \in M_b$ in turn.

Suppose first that $u_2, v_2 \in M_a$. If v_2u_3 is blue then $P_3v_2bu_2u_1a$ is a blue path and we are done, so suppose not. Then P_2u_3ab is a red path and we are done.

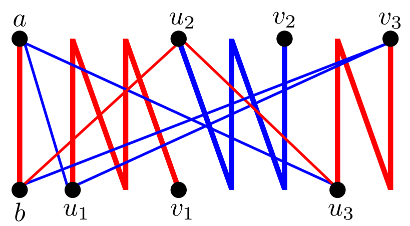


Now consider the case where $u_2, v_2 \in M_b$. If v_3u_1 and v_3u_2 are both red then $P_1v_3P_2$ is a red path and we are done, so suppose not. Suppose arbitrarily that v_3u_2 is blue. If $u_2 = v_2$ then P_2x is a monochromatic path longer than P_2, a contradiction. So denote the neighbour of u_2 on P_2 by u_2'. If bu_2' is red then $(P_2 - u_2)ba$ is a red path and P_3u_2 is a blue path, so suppose not. Then if $u_2'u_3$ is blue $au_2P_3u_2'b$ is a blue path, so suppose not. Then $(P_2 - u_2)u_3ab$ is a red path and $(P_3 - u_3)u_2$ is a blue path so we are done.



Now consider the case where P_2 is the single blue path. We consider the case where $u_2, v_2 \in M_a$ and that where $u_2, v_2 \in M_b$ in turn.

Suppose first that $u_2, v_2 \in M_a$. If u_2u_3 is blue then $P_2u_3au_1v_3b$ is a blue path, so suppose not. Then P_3u_2ba is a red path so we are done.



Now consider the case where $u_2, v_2 \in M_b$ (Figure 6.2). If u_2v_3 is red then P_3u_2ab is a red path, so suppose not. If $u_1 = v_1$ then P_1x is a monochromatic path longer than P_1, a contradiction. Denote the neighbour of u_1 on P_1 by u_1'. Then if bu_1' is red $(P_1 - u_1)ba$ is a red path and $P_2v_3u_1$ is a blue path, so suppose not. Then if $u_1'u_3$ is red $(P_3 - v_3)(P_1 - u_1)$ is a single red path and $P_2v_3u_1$ is a blue path, so suppose not. Then $P_2v_3u_1au_3u_1'b$ is a blue path, so we are done.

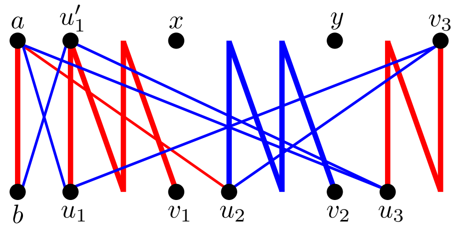
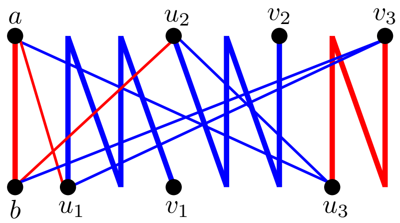


Figure 6.2

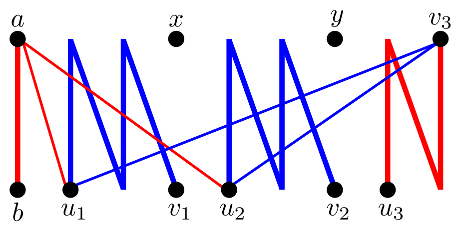
□

Claim 65. *If two of P_1, P_2 and P_3 are blue and one is red then we are done.*

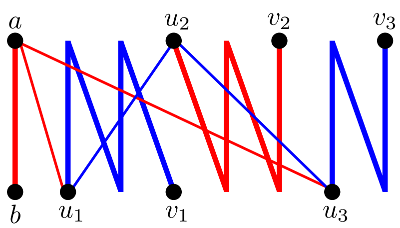
Proof. Suppose first that P_3 is the red path. We consider the case where $u_2, v_2 \in M_a$ and the case where $u_2, v_2 \in M_b$ in turn. Suppose first that $u_2, v_2 \in M_a$. If u_1v_3 is red then P_3u_1ab is a red path, so suppose not. If u_2u_3 is red then P_3u_2ba is a red path, so suppose not. Then P_1v_3b and P_2u_3a are disjoint blue paths, so we are done.



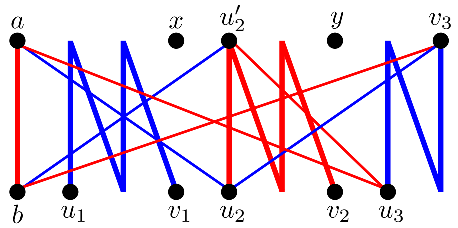
Now consider the case where $u_2, v_2 \in M_b$. If u_1v_3 is red then P_3u_1ab is a red path and we are done, so suppose not. Similarly, if u_2v_3 is red then we are done, so suppose not. Then $P_1v_3P_2$ is a single blue path, so we are done.



Now suppose that P_2 is the single red path. Suppose first that $u_2, v_2 \in M_a$. If u_1u_2 is red then P_2u_1ab is a red path, so suppose not. If u_2u_3 is red then P_2u_3ab is a red path, so suppose not. Then $P_1u_2P_3$ is a single blue path, so we are done.



Now consider the case where $u_2, v_2 \in M_b$. If u_2v_3 is red then P_2v_3ba is a red path, so suppose not. If $u_2 = v_2$ then P_2x is a monochromatic path longer than P_2, a contradiction. Denote the neighbour of u_2 on P_2 by u_2'. If bu_2' is red then $(P_2 - u_2)ba$ is a red path and P_3u_2 is a blue path, so suppose not. Then if $u_2'u_3$ is blue $au_2P_3u_2'b$ is a blue path, so suppose not. Then $(P_2 - u_2)u_3ab$ is a red path and $(P_3 - u_3)u_2$ is a blue path, so we are done.



□

□

Lemma 66. *Suppose that a two-edge-coloured $K_{m,m}$ contains three disjoint maximal non-alternating monchromatic paths P_1, P_2, P_3 and two adjacent vertices a and b which are not covered by any of these paths. Then at most three disjoint monochromatic paths are needed to cover a, b and all the vertices on P_1, P_2 and P_3.*

Proof. Suppose arbitrarily that ab is red. Denote the bipartition of $K_{m,m}$ by (M_a, M_b). Suppose arbitrarily that a is contained in M_a, so b is contained in M_b. Denote the endvertices of P_i by u_i and v_i for all $i \in \{1, 2, 3\}$. Suppose arbitrarily that $u_1, v_1, u_3, v_3 \in M_b$. We consider each of the following cases in turn:

(a) The paths P_1, P_2 and P_3 are all the same colour.

(b) Two of P_1, P_2 and P_3 are red and one is blue.

(c) Two of P_1, P_2 and P_3 are blue and one is red.

For cases (b) and (c) we consider the case where P_3 is the path of a different colour to the other two, and that where P_2 is the path of a different colour to the other two. The case where P_1 is the path of a different colour to the other two then holds by symmetry. Within case (a), and within each subcase of cases (b) and (c), we must consider the case where $u_2, v_2 \in M_a$ and that where $u_2, v_2 \in M_b$. Observe that this covers all possibilities. For case (a) we use Lemma 53 to obtain the result.

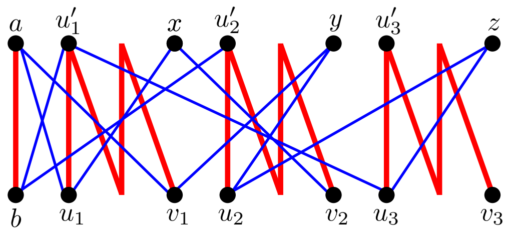
If $u_2, v_2 \in M_b$ we denote by x, y and z three vertices in M_a which are distinct from a and are not covered by any path P_i. If $u_2, v_2 \in M_a$ we denote by w a vertex in M_a which is distinct from a and not covered by any path P_i.

If a or b is joined to an endvertex of any path P_i by an edge of the same colour as P_i then we have a contradiction. Similarly, if there is an edge of the same colour as P_i between an endvertex of P_i and an endvertex of a different path P_j of the same colour, then P_iP_j is a single monochromatic path, a contradiction.

Claim 67. *If P_1, P_2 and P_3 are all the same colour then we are done.*

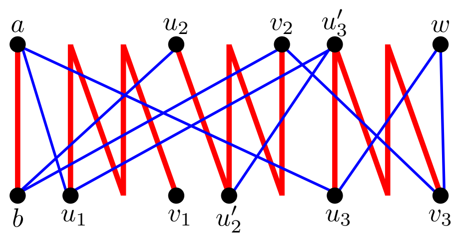
Proof. Suppose arbitrarily that P_1, P_2 and P_3 are red. First consider the case where $u_2, v_2 \in M_b$. If there is a red edge from x, y or z to an endvertex of any P_i then we have a contradiction. If, for any i, we have $u_i = v_i$, then P_ix is a monochromatic path longer than P_i, a contradiction. So denote the neighbour of u_i on P_i by u'_i for all i. If bu'_i and bu'_j are both red for distinct i and j with $j \in \{1,2,3\}$, then $(P_i - u_i)b(P_j - u_j)$ is a single red path and $u_i a u_j$ is a disjoint blue path, so suppose not. We suppose arbitrarily that bu'_1 and bu'_2 are blue.

Now, if both u'_1u_3 and u'_2v_3 are red then $(P_1 - u_1)P_3(P_2 - u_2)$ is a single red path and u_1au_2 is a blue path, so b is our third path and we are done. So suppose that at least one of u'_1u_3 and u'_2v_3 is blue. In either case we can form a blue path which covers only $a, b, x, y, z, u'_1, u'_2$, both endvertices of P_1 and P_2 and a single endvertex of P_3. Ensure that this blue path has u'_k and v_k as endvertices for some $k < 3$. Then by Lemma 53 we are done.



Now consider the case where $u_2, v_2 \in M_a$. If there is a red edge from w to an endvertex of P_i for any i then we have a contradiction. If $u_3 = v_3$ then P_3w is a monochromatic path longer than P_3, a contradiction. If $u_2 = v_2$ then P_2b is a monochromatic path longer than P_2, a contradiction. Denote the neighbour of u_k on P_k by u'_k for each $k > 1$.

Form the blue path $B = u_2bv_2v_3wu_3au_1$. If $u_1u'_3$ is red then $(P_3 - \{u_3, v_3\})P_1$ is a red path and a, b, u_3 and v_3 are covered by $B - u_1$. So suppose that $u_1u'_3$ is blue and set $B = Bu'_3$. Then if $u'_3u'_2$ is red $(P_3 - \{u_3, v_3\})(P_2 - \{u_2, v_2\})$ is red path and $B - u'_3$ is a blue path. So suppose that $u'_3u'_2$ is blue and set $B = Bu'_2$. Then by Lemma 53 we are done.

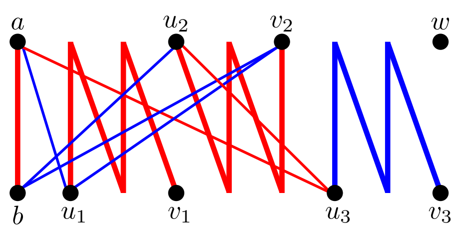


□

Claim 68. *If two of P_1, P_2 and P_3 are red and one is blue then we are done.*

Proof. First suppose that P_3 is the blue path. We consider the case where $u_2, v_2 \in M_a$ and the case where $u_2, v_2 \in M_b$ in turn.

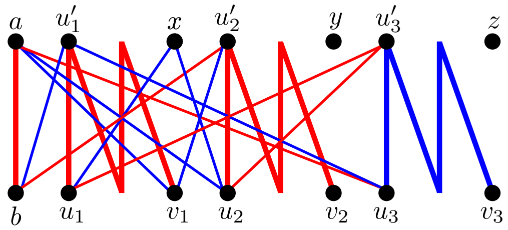
First suppose that $u_2, v_2 \in M_a$. If u_3u_2 is blue then $P_3u_2bv_2u_1a$ is a blue path, so suppose not. Then P_2u_3ab is a red path, so we are done.



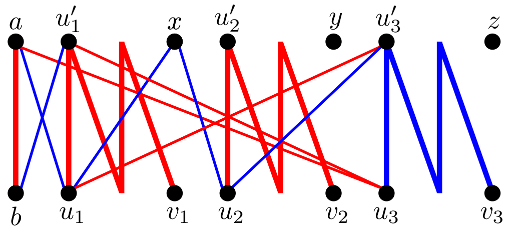
Now consider the case where $u_2, v_2 \in M_b$. If x is joined in blue to an endvertex of P_3, or in red to an endvertex of P_1 or P_2 then we have a contradiction.

If $u_i = v_i$ for any i then $P_i x$ is a monochromatic path longer than P_i, a contradiction. Denote the neighbour of u_i on P_i by u_i. If bu_1' and bu_2' are both red then $(P_1 - u_1)b(P_2 - u_2)$ is a red path and $u_1 a u_2$ is a blue path, so suppose not. Suppose arbitrarily that bu_1' is blue.

First consider the case where $u_1' u_3$ is blue. Then if $v_1 u_2'$ is red $(P_1 - \{u_1, u_1'\})(P_2 - u_2)$ is a red path and $P_3 u_1' b$ and $u_1 a u_2$ are disjoint blue paths. So suppose not. Then if bu_2' is blue $P_3 u_1' b u_2' v_1 a u_1 x u_2$ is a blue path, so suppose not. If $u_1 u_3'$ is blue then $(P_3 - u_3)u_1 x u_2$ is a blue path and $(P_2 - u_2)bau_3$ is a red path, so suppose not. Similarly, if $u_2 u_3'$ is blue then $(P_3 - u_3)u_2$ is a blue path and $(P_2 - u_2)bau_3$ is a red path, so suppose not. Then $P_1 u_3' P_2$ is a single red path and bau_3 is a disjoint red path, so we are done.

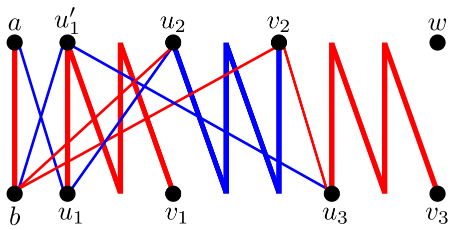


Now suppose that $u_1' u_3$ is red. Then if $u_1 u_3'$ is blue $(P_3 - u_3)u_1$ is a blue path and $(P_1 - u_1)u_3 ab$ is a red path, so suppose not. Then if $u_3' u_2$ is red $P_1 u_3' P_2$ is a single red path and bau_3 is a disjoint red path, so suppose not. Then $(P_3 - u_3)u_2 x u_1$ is a blue path and $(P_1 - u_1)u_3 ab$ is a disjoint red path, so we are done.



Now suppose that P_2 is the blue path. We only need to consider the case where $u_2, v_2 \in M_a$ since the case where $u_2, v_2 \in M_b$ is the same as the case where P_3 is the blue path and $u_2, v_2 \in M_b$.

If $u_1 u_2$ is red then $P_1 u_2 ba$ is a red path, so suppose not. If $u_1 = v_1$ then $P_1 a$ is a monochromatic path longer than P_1, a contradiction. Denote the neighbour of u_1 on P_1 by u_1'. Then if bu_1' is red $(P_1 - u_1)ba$ is a red path and $P_2 u_1$ is a blue path, so suppose not. If $u_1' u_3$ is red then $(P_1 - u_1)P_3$ is a single red path and $P_2 u_1$ is a blue path, so suppose not. Finally, if $v_2 u_3$ is blue then $au_1 P_2 u_3 u_1' b$ is a blue path and we are done, so suppose not. Then $P_3 v_2 ba$ is a red path, so we are done.

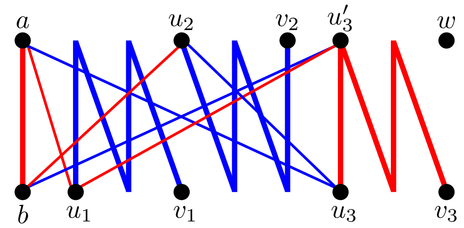


□

Claim 69. *If two of P_1, P_2, P_3 are blue and one is red then we are done.*

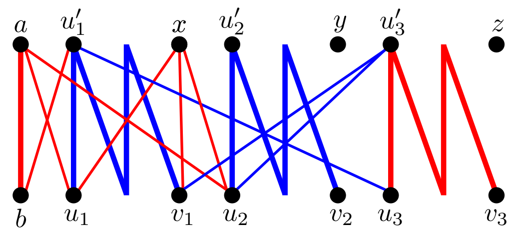
Proof. First consider the case where P_3 is the red path. We consider the case where $u_2, v_2 \in M_a$ and the case where $u_2, v_2 \in M_b$ in turn. First suppose that $u_2, v_2 \in M_a$.

If u_3u_2 is red then P_3u_2ba is a red path, so suppose not. If $u_3 = v_3$ then P_3x is a monochromatic path longer than P_3, a contradiction. Denote the neighbour of u_3 on P_3 by u_3'. If bu_3' is red then $(P_3 - u_3)ba$ is a red path and P_2u_3 is a blue path, so suppose not. Then if $u_3'u_1$ is blue $P_1u_3'b$ is a blue path and P_2u_3a is a disjoint blue path, so suppose not. Then $(P_3 - u_3)u_1ab$ is a red path and P_2u_3 is a blue path, so we are done.



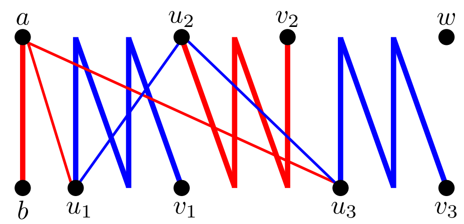
Now consider the case where $u_2, v_2 \in M_b$. If x is joined to an endvertex of P_1 or P_2 in blue, or to an endvertex of P_3 in red then we have a contradiction.

If $u_i = v_i$ for any $i \in \{1, 2, 3\}$ then P_ix is a monochromatic path longer than P_i, a contradiction. So denote the neighbour of u_i on P_i by u_i' for each i. If bu_1' and bu_2' are both blue then $(P_1 - u_1)b(P_2 - u_2)$ is a single blue path and u_1au_2 is a red path, so suppose not. We suppose arbitrarily that bu_1' is red. Then if $u_1'u_3$ is also red $P_3u_1'bau_1$ is a red path, so suppose not. If v_1u_3' is red then $(P_3 - u_3)v_1xu_1ab$ is a red path and $(P_1 - \{u_1, v_1\})u_3$ is a disjoint blue path, so suppose not. If $u_3'u_2$ is red then $(P_3 - u_3)xu_1ab$ is a red path and $(P_1 - u_1)u_3$ is a blue path, so suppose not. Then $P_2u_3'(P_1 - u_1)u_3$ is a single blue path and bau_1 is a red path, so we are done.



Now suppose that P_2 is the red path. We only need to consider the case where $u_2, v_2 \in M_a$ since the case where $u_2, v_2 \in M_b$ is the same as the case where P_3 is the red path and $u_2, v_2 \in M_b$.

If u_2u_1 is red then P_2u_1ab is a red path, so suppose not. Similarly, if u_2u_3 is red then we are done, so suppose not. Then $P_1u_2P_3$ is a single blue path, so we are done.



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6.4 Proof of the Theorem

We are now ready to prove Theorem 6.

Proof of Theorem 6. If $n \leq 3$ then we are done; we can cover the vertices of $K_{n,n}$ by at most three pairwise non-adjacent edges. So suppose that $n \geq 4$. Construct any three maximal disjoint monochromatic paths P_1, P_2 and P_3 on the vertices of $K_{n,n}$. Recall that we call a vertex in $K_{n,n}$ *uncovered* if it is not on any of P_1, P_2 or P_3.

If P_1, P_2 and P_3 cover all the vertices of $K_{n,n}$ then we are done. If not, then we must be in one of the following situations:

1. The graph $K_{n,n}$ contains a pair a and b of adjacent uncovered vertices.
2. The graph $K_{n,n}$ contains an uncovered vertex c which is not adjacent to another uncovered vertex.

First suppose that we are in situation 1. and consider a subgraph $K_{m,m}$ of $K_{n,n}$ containing a, b and the three paths P_1, P_2 and P_3. In the case where P_1, P_2, P_3, a and b do not form a balanced bipartite graph, we include in $K_{m,m}$ just enough uncovered vertices from $K_{n,n}$ to ensure that $K_{m,m}$ is balanced. We also include all edges between the vertices in $K_{m,m}$ from its supergraph. Using Lemmas 53 to 66 we now show that there exist three disjoint monochromatic paths which cover a, b and all the vertices on P_1, P_2 and P_3.

Consider the three paths P_1, P_2 and P_3. If all of these paths are alternating then by Lemma 54 we are done. If only one of P_1, P_2 and P_3 is nonalternating then by Lemma 58 we are done. If exactly two of P_1, P_2 and P_3 are nonalternating then by Lemma 62 we are done. Finally, if all of P_1, P_2 and P_3 are nonalternating then by Lemma 66 we are done.

Now suppose that we are in situation 2. and consider a subgraph $K_{t,t}$ of $K_{n,n}$ containing c and the three paths P_1, P_2 and P_3. In the case where P_1, P_2, P_3 and c do not form a balanced bipartite graph, we include in $K_{t,t}$ just enough uncovered vertices from $K_{n,n}$ to ensure that $K_{t,t}$ is balanced. Observe that these "extra" vertices must be in the same part of $K_{t,t}$ as c, otherwise we would be in situation 1.. We consider each of the following cases in turn:

(a) Two of the paths P_1, P_2 and P_3 are alternating and one is nonalternating.

(b) Two of the paths P_1, P_2 and P_3 are nonalternating and one is alternating.

(c) All of the paths P_1, P_2 and P_3 are nonalternating.

For each case we consider the subcase where the paths P_1, P_2 and P_3 are all of the same colour and that where one of the paths is of a different colour to the other two. Observe that this covers all possibilities.

We now show that in each of cases (a) to (c) we can find three disjoint monochromatic paths which cover c and all the vertices on P_1, P_2 and P_3. Case (a) is considered in Claim 70, case (b) in Claim 71 and case (c) in Claim 72.

Denote the endvertices of P_i by u_i and v_i for all $i \in \{1, 2, 3\}$ and denote the two parts of $K_{t,t}$ by M_b and M_t. Suppose arbitrarily that c is contained in M_b.

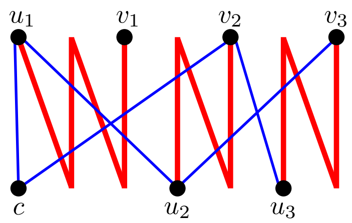
If c is joined to an endvertex of any path P_i by an edge of the same colour as P_i then we have a contradiction of the maximality of P_i. Similarly, if there is an edge of the same colour as P_i between an endvertex of P_i and an endvertex of a different path P_j of the same colour, then P_iP_j is a single monochromatic path, a contradiction.

Claim 70. *If two of the paths P_1, P_2 and P_3 are alternating and one is nonalternating then we are done.*

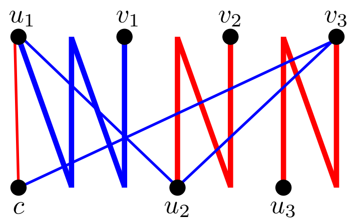
Proof. Suppose arbitrarily that P_1 is the nonalternating path. Observe that u_1 and v_1 must be contained in M_t, otherwise $K_{t,t}$ contains an uncovered vertex which is adjacent to c. Suppose arbitrarily that u_2 and u_3 are contained in M_b, so v_2 and v_3 are contained in M_t.

We first consider the case where P_1, P_2 and P_3 are all of the same colour. We then consider the case where the nonalternating path P_1 is of a different colour to the other two, and finally the case where one of the alternating paths is of a different colour to the other two paths.

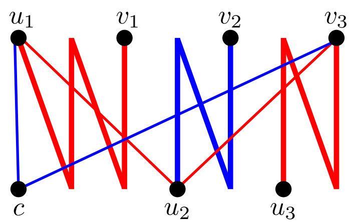
First consider the case where P_1, P_2 and P_3 are all the same colour. Then $v_3u_2u_1cv_2u_3$ is a path of the other colour, so by Lemma 53 we are done.



Now consider the case where P_1 is of a different colour, say blue, to the other two paths. Then if u_1u_2 is red P_2u_1c is a red path and we are done, so suppose not. Then $P_1u_2v_3c$ is a blue path, so we are done.



Now consider the case where P_2 is of a different colour, say blue, to the other two paths blue. The case where P_3 is the single blue path holds by symmetry. If u_2u_1 is blue then P_2u_1c is a blue path and we are done, so suppose not. Similarly, if u_2v_3 is blue then we are done, so suppose not. Then $P_1u_2P_3$ is a single red path, so we are done.



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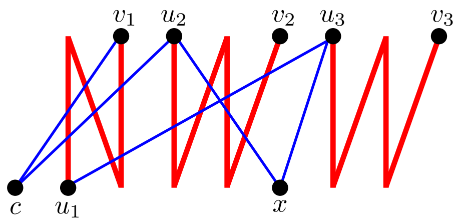
Claim 71. *If two of the paths P_1, P_2 and P_3 are nonalternating and one is alternating then we are done.*

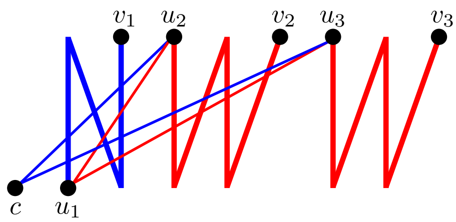
Proof. Suppose arbitrarily that P_1 is the alternating path. Notice that we must have $u_2, v_2, u_3, v_3 \in M_t$, otherwise $K_{t,t}$ contains an uncovered vertex which is adjacent to c. Denote by x an uncovered vertex in M_b which is distinct from c. If x is joined to an endvertex of any P_i by an edge of the same colour as P_i then we have a contradiction.

We first consider the case where P_1, P_2 and P_3 are all of the same colour. We then consider the case where the alternating path is of a different colour to the other two, and finally the case where one of the nonalternating paths is of a different colour to the other two paths.

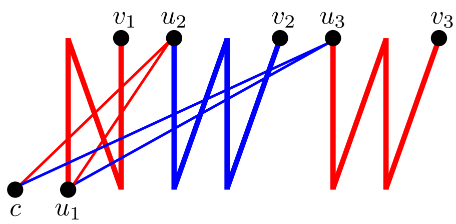
First suppose that P_1, P_2 and P_3 are all the same colour and form the path $v_1cu_2xu_3u_1$ of the other colour. Then by Lemma 53 we are done.



Now consider the case where P_1 is of a different colour, say blue, to the other two paths. If u_1u_2 is blue then P_1u_2c is a blue path and we are done, so suppose not. Similarly, if u_1u_3 is blue then we are done, so suppose not. Then $P_2u_1P_3$ is a single red path, so we are done.



Now consider the case where P_2 is the blue path. The case where P_3 is the blue path holds by symmetry. If u_1u_2 is blue then $P_2u_1u_3c$ is a blue path, so suppose not. Then P_1u_2c is a red path, so we are done.



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Claim 72. *If all of the paths P_1, P_2 and P_3 are nonalternating then we are done.*

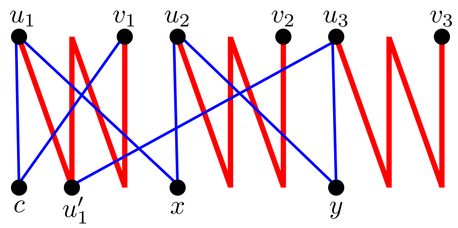
Proof. We consider each of the following cases in turn:

- $u_i, v_i \in M_t$ for all $i \in \{1, 2, 3\}$.
- $u_1, v_1 \in M_b$ and $u_j, v_j \in M_t$ for each $j > 1$.

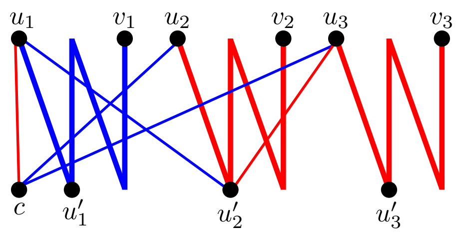
Within each of these cases we must consider the case where P_1, P_2 and P_3 are all of the same colour and that where one of the paths is of a different colour to the other two.

First suppose that $u_i, v_i \in M_t$ for all i. Denote by x and y two uncovered vertices in M_b which are distinct from c. If x or y is joined to an endvertex of any P_i by an edge of the same colour as P_i then we have a contradiction. If $u_i = v_i$ for any i then P_ic is a monochromatic path, a contradiction. So denote the neighbour of u_i on P_i by u_i' for each i.

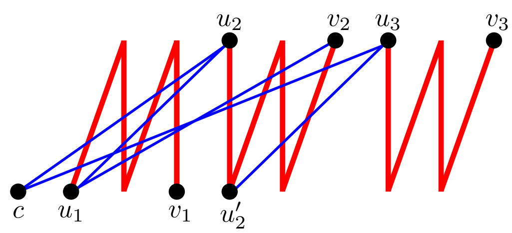
First suppose that P_1, P_2 and P_3 are all the same colour, say red. If u_3u_1' is red then $(P_1 - u_1)P_3$ is a red path and cu_1 is a blue path, so suppose not. Then $v_1cu_1xu_2yu_3u_1'$ is a blue path, so by Lemma 53 we are done.



Now suppose that one of P_1, P_2 and P_3, say P_1, is of a different colour to the other two. Suppose arbitrarily that P_1 is blue. If u_1u_2' and u_1u_3' are both red then $(P_2 - u_2)u_1(P_3 - u_3)$ is a single red path and u_2cu_3 is a blue path, so suppose not. Suppose arbitrarily that u_1u_2' is blue. Then if $u_2'u_3$ is also blue $P_1u_2'u_3cu_2$ is a blue path, so suppose not. Then $P_3(P_2 - u_2)$ is a single red path and cu_2 is a blue path, so we are done.



Now consider the case where u_1 and v_1 are both contained in M_b, while the endvertices of P_2 and P_3 are all contained in M_t. First consider the case where P_1, P_2 and P_3 are all the same colour, say red. If $u_2 = v_2$ then P_2c is a monochromatic path longer than P_2, a contradiction. So denote the neighbour of u_2 on P_2 by u_2'. If $u_2'u_3$ is red then $(P_2 - u_2)P_3$ is a single red path and cu_2 is a blue path, so suppose not. Then $u_2'u_3cu_2u_1v_2$ is a blue path, so by Lemma 53 we are done.



Now consider the case where one of P_1, P_2 and P_3 is blue. First consider the case where P_1 is blue (Figure 6.3). If u_1u_2 is blue then P_1u_2c is a blue path and we are done, so suppose not. Similarly, if u_1u_3 is blue then we are done, so suppose not. Then $P_2u_1P_3$ is a single red path, so we are done.

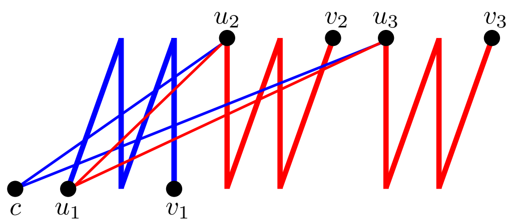
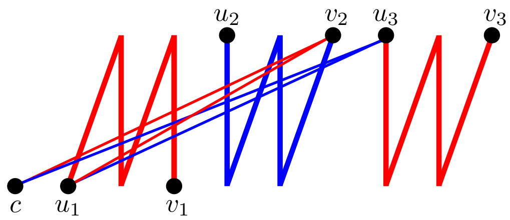


Figure 6.3

Now consider the case where one of P_2 and P_3, say P_2, is the blue path. If u_1v_2 is blue then $P_2u_1u_3c$ is a blue path, so suppose not. Then P_1v_2c is a red path so we are done.



□

We have now shown that we can cover all the vertices on the three maximal paths P_1, P_2 and P_3, and at least one other vertex in $K_{n,n}$ (if it exists) by at most three disjoint monochromatic paths. If these "new" paths do not cover all the vertices in $K_{n,n}$ then we can extend each path until it is maximal and repeat this process. Applying this method until we have covered *all* the vertices in $K_{n,n}$ by at most three disjoint monochromatic paths completes the proof, so we are done. □

Chapter 7

Revised Proof

Here we offer a similar but simpler and shorter proof of Theorem 6 than that given in Chapter 6. Recall that the proof from Chapter 6 involved constructing any three maximal disjoint monochromatic paths P_1, P_2 and P_3 on the vertices of a two-edge-coloured $K_{n,n}$. We called a vertex in $K_{n,n}$ "uncovered" if it was not covered by any of P_1, P_2 and P_3. In the event that $K_{n,n}$ contained an uncovered vertex we were left with the following two cases:

1. The graph $K_{n,n}$ contains two adjacent uncovered vertices.
2. The graph $K_{n,n}$ contains an uncovered vertex which is not adjacent to another uncovered vertex.

We showed that in either case we can find at most three disjoint monochromatic paths which cover P_1, P_2, P_3 and these "extra" vertices. Extending these paths so that they are maximal and repeating this process until we have covered all vertices in $K_{n,n}$ gives us the result.

While considering case 2. it became apparent that this separation into two cases is unnecessary. One can instead consider *any* uncovered vertex in $K_{n,n}$, and find three disjoint monochromatic paths which cover all the vertices on P_1, P_2 and P_3 and also this uncovered vertex. Again, extending these paths so that they are maximal and repeating this process until we have covered all uncovered vertices in $K_{n,n}$ gives us the result.

7.1 Proof Outline

Consider any two-edge-coloured $K_{n,n}$. If $n \leq 3$ then we are done since we can cover the vertices by at most three pairwise non-adjacent edges. For $n \geq 4$ we begin by constructing any three disjoint maximal monochromatic paths P_1, P_2 and P_3 on the vertices of $K_{n,n}$. Recall that any path on a single vertex is considered a nonalternating path.

If P_1, P_2 and P_3 cover all the vertices of $K_{n,n}$ then we are done. If not, then there is an uncovered vertex c in $K_{n,n}$. In this case we consider the complete balanced subgraph $K_{m,m}$ of $K_{n,n}$. The graph $K_{m,m}$ contains c, all the vertices on P_1, P_2 and P_3, as well as all edges between these vertices. If c and all the vertices of P_1, P_2 and P_3 do not form a balanced bipartite graph then we include just enough "extra" vertices in $K_{m,m}$ from its supergraph to ensure that $K_{m,m}$ is balanced. We then use case analysis to find at most three disjoint monochromatic paths which cover c and all the vertices on P_1, P_2 and P_3. In particular, we consider each of the following cases:

(a) All of the paths P_1, P_2 and P_3 are alternating.

(b) Two of the paths P_1, P_2 and P_3 are alternating and one is nonalternating.

(c) Two of the paths P_1, P_2 and P_3 are nonalternating.

(d) All of the paths P_1, P_2 and P_3 are nonalternating.

In each of these cases we must consider the different possible colours of the three paths and, for cases (b) to (d), which part of $K_{m,m}$ the endvertices of each nonalternating path are contained in.

Case (a) is considered in Lemma 73, case (b) in Lemma 74, case (c) in Lemma 75 and case (d) in Lemma 79. In each case, when P_1, P_2 and P_3 are all the same colour we use Lemma 53 from Chapter 6 to find three disjoint monochromatic paths covering the vertices of interest.

If the "new" paths do not cover all the vertices in $K_{n,n}$ then we can extend each path so that it is maximal and repeat the above process to obtain at most three disjoint monochromatic paths which cover another vertex in $K_{n,n}$. Repeating this process until we have covered all vertices in $K_{n,n}$ by at most three disjoint monochromatic paths gives us the result.

7.2 Preparatory Lemmas

We begin by stating and proving Lemmas 73 to 79. We then use these lemmas to prove Theorem 6.

Lemma 73. *Suppose that a two-edge-coloured $K_{m,m}$ contains three maximal disjoint alternating monochromatic alternating paths P_1, P_2 and P_3 and a vertex c which is not covered by any of these paths. Then at most three disjoint monochromatic paths are needed to cover c and all the vertices on P_1, P_2 and P_3.*

Proof. Denote the two parts of $K_{m,m}$ by (M_b, M_t) and suppose arbitrarily that c is contained in M_b. Denote the endvertices of P_i by u_i and v_i for each $i \in \{1, 2, 3\}$. Suppose arbitrarily that for all i the vertex u_i is contained in M_b, so v_i is contained in M_t.

Denote by x a vertex in M_t which is not covered by any path P_i. If c or x is joined to an endvertex of any P_i by an edge of the same colour as P_i then we have a contradiction, so suppose not. Similarly, if there is an edge of the same colour as P_i from an endvertex of P_i to an endvertex of a different path P_j of the same colour, then we have a contradiction.

We first consider the case where P_1, P_2 and P_3 are the same colour. We will see that we can immediately apply Lemma 53 to obtain the desired result. We then consider the case where one of P_1, P_2 and P_3 is of a different colour to the other two.

First suppose that P_1, P_2 and P_3 are all the same colour (as in Figure 7.1). Then $v_1cv_2u_3xu_2v_3u_1$ is a path of the other colour, so by Lemma 53 we are done.

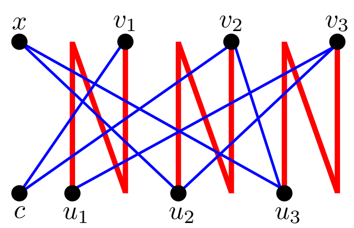
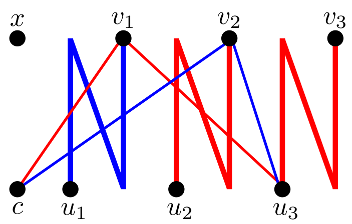


Figure 7.1

Now consider the case where one of the paths, say P_1, is of a different colour to the other two. Suppose arbitrarily that P_1 is blue. If v_1u_3 is blue then $P_1u_3v_2c$ is a blue path and we are done, so suppose not. Then P_3v_1c is a red path, so we are done.



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Lemma 74. *Suppose that a two-edge-coloured $K_{m,m}$ contains three disjoint maximal monchromatic paths P_1, P_2, P_3 and vertex c which is not covered by any of these paths. Suppose further that two of P_1, P_2 and P_3 are alternating and one is nonalternating. Then at most three disjoint monochromatic paths are needed to cover c and all the vertices on P_1, P_2 and P_3.*

Proof. Denote the bipartition of $K_{m,m}$ by (M_b, M_t) where, say, c is contained in M_b. Suppose arbitrarily that P_1 be the nonalternating path. Denote the endvertices of P_i by u_i and v_i for each $i \in \{1, 2, 3\}$ where, say, $u_2, u_3 \in M_b$ and $v_2, v_3 \in M_t$. We consider the case where $u_1, v_1 \in M_b$ and that where $u_1, v_1 \in M_t$ in turn. In both cases we first consider the subcase where all the paths P_1, P_2 and P_3 are the same colour. We will see that we can immediately use Lemma 53 to obtain the result. We then consider the subcase where one of the paths is of a different colour to the other two. Observe that this covers all possibilities.

For the case where $u_1, v_1 \in M_b$ we denote by x and y two vertices in M_t which are not covered by any path P_i.

If c is joined to an endvertex of P_i by an edge of the same colour as P_i for any i then we have a contradiction of the maximality of P_i. Similarly, if there is an edge of the same colour as P_i from an endvertex of P_i to an endvertex of a different path P_j of the same colour, then we have a contradiction.

First consider the case where u_1 and v_1 are both contained in M_b. If x or y is joined to an endvertex of any P_i by an edge of the same colour as P_i then we have a contradiction.

First suppose that P_1, P_2 and P_3 are all the same colour (Figure 7.2). Then $v_3cv_2u_1xu_2yu_3$ is a path of the other colour, so by Lemma 53 we are done.

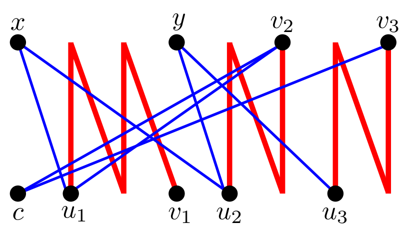
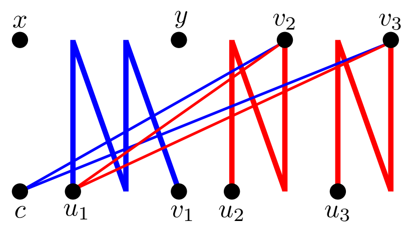
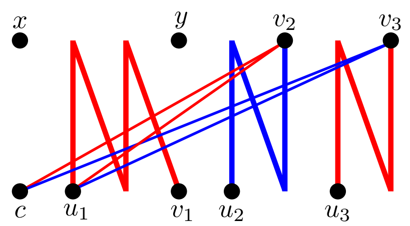


Figure 7.2

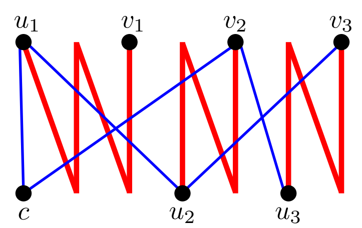
Now suppose that only two of P_1, P_2 and P_3 are of the same colour, say red. First suppose that P_1 is the blue path. If u_1v_2 is blue then P_1v_2c is a blue path and we are done, so suppose not. Similarly, if u_1v_3 is blue then we are done, so suppose not. Then $P_2u_1P_3$ is a single red path, so we are done.



Now suppose that one of P_2 and P_3, say P_2, is the single blue path. If u_1v_2 is blue then $P_2u_1v_3c$ is a blue path, so suppose not. Then P_1v_2c is a red path, so we are done.



Now consider the case where u_1 and v_1 are both contained in M_t. First suppose that P_1, P_2 and P_3 are all the same colour. Then $v_3u_2u_1cv_2u_3$ is a path of the other colour, so by Lemma 53 we are done.



Now suppose that only two of the paths are of the same colour, say red. First suppose that P_1 is the blue path (as in Figure 7.3). Then if u_1u_2 is red P_2u_1c is a red path, so suppose not. Then $P_1u_2v_3c$ is a blue path, so we are done.

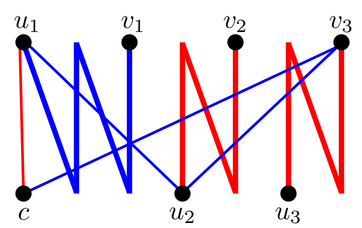
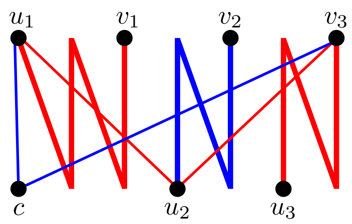


Figure 7.3

Now consider the case where one of P_2 and P_3, say P_2, is the blue path. If u_2v_3 is blue then P_2v_3c is a blue path and we are done, so suppose not. Similarly, if u_2u_1 is blue then we are done, so suppose not. Then $P_1u_2P_3$ is a single red path, so we are done.



□

Lemma 75. *Suppose that a two-edge-coloured $K_{m,m}$ contains three disjoint maximal monchromatic paths P_1, P_2, P_3 and vertex c which is not covered by any of these paths. Suppose further that two of P_1, P_2 and P_3 are nonalternating and one is alternating. Then at most three disjoint monochromatic paths are needed to cover c and all the vertices on P_1, P_2 and P_3.*

Proof. Denote the endvertices of P_i by u_i and v_i for each $i \in \{1, 2, 3\}$. Let P_1 be the single alternating path and suppose arbitrarily that u_1 is contained in M_b, so v_1 is contained in M_t. We consider each of the following cases in turn:

- $u_2, v_2, u_3, v_3 \in M_b$.
- $u_2, v_2, u_3, v_3 \in M_t$.
- $u_2, v_2 \in M_b$ and $u_3, v_3 \in M_t$.

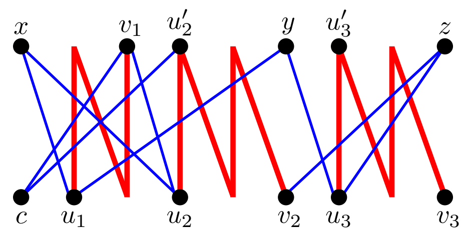
The case where $u_2, v_2 \in M_t$ and $u_3, v_3 \in M_b$ then holds by symmetry. Within each of these cases we first consider the subcase where all the paths P_1, P_2 and P_3 are of the same colour and we use Lemma 53 to obtain the result. We then consider the subcase where one of the paths is of a different colour to the other two. Observe that this covers all possibilities.

If c is joined to an endvertex of any P_i by an edge of the same colour as P_i then we have a contradiction. Similarly, if there is an edge of the same colour as P_i between an endvertex of P_i and an endvertex of a different path P_j of the same colour, then P_iP_j is a single monochromatic path, a contradiction.

Claim 76. *If $u_2, v_2, u_3, v_3 \in M_b$ then we are done.*

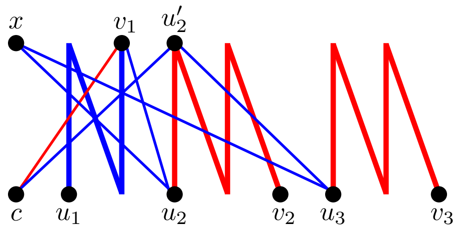
Proof. Denote by x, y and z three vertices in M_t which are not covered by any path P_i. If x, y or z is joined to an endvertex of any P_i by an edge of the same colour as P_i then we have a contradiction.

First suppose that P_1, P_2 and P_3 are all the same colour, say red. If $u_k = v_k$ for any $k > 1$ then P_kx is a monochromatic path longer than P_k, a contradiction. So denote the neighbour of u_k on P_k by u'_k for all $k > 1$. If cu'_2 and cu'_3 are both red then $(P_2 - u_2)c(P_3 - u_3)$ is a single red path and u_2xu_3 is a disjoint blue path, so suppose that at least one of cu'_2 and cu'_3, say cu'_2, is blue. Then $v_2zu_3yu_1xu_2v_1cu'_2$ is a blue path, so by Lemma 53 we are done.

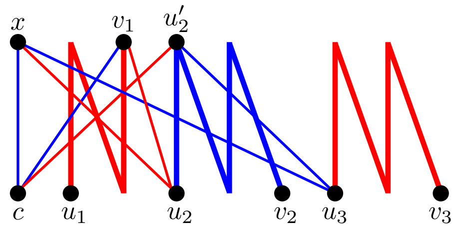


Now suppose that one of P_1, P_2 and P_3 is of a different colour, say blue, to the other two. If $u_2 = v_2$ then P_2x is a monochromatic path longer than P_2, a contradiction. So denote the neighbour of u_2 on P_2 by u_2'.

First consider the case where P_1 is the blue path. If v_1u_2 is red then P_2v_1c is a red path, so suppose not. Then if cu_2' is red $(P_2 - u_2)c$ is a red path and P_1u_2 is a blue path, so suppose not. If u_3u_2' is red then $P_3(P_2 - u_2)$ is a single red path and P_1u_2 is a blue path, so suppose not. Then $P_1u_2xu_3u_2'c$ is a blue path, so we are done.



Now consider the case where one of P_2 and P_3, say P_2, is the single blue path. If v_1u_2 is blue then P_2v_1c is a blue path, so suppose not. Then if cx is red P_1u_2xc is a red path, so suppose that cx is blue. If cu_2' is blue then $(P_2 - u_2)c$ is a blue path and P_1u_2 is a red path, so suppose not. Finally if $u_2'u_3$ is red then $P_3u_2'c$ is a red path and P_1u_2 is a disjoint red path, so suppose not. Then $(P_2 - u_2)u_3xc$ is a blue path and P_1u_2 is a disjoint red path, so we are done.

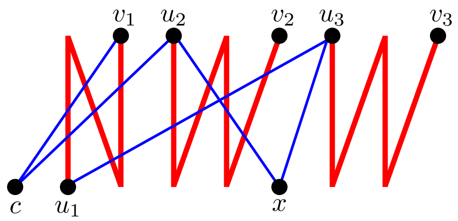


□

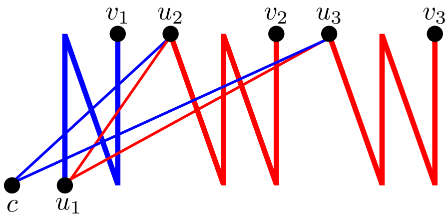
Claim 77. *If $u_2, v_2, u_3, v_3 \in M_t$ then we are done.*

Proof. Denote by x a vertex in M_b which is distinct from c and not covered by any path P_i. If x is joined to an endvertex of any P_i by an edge of the same colour as P_i then we have a contradiction.

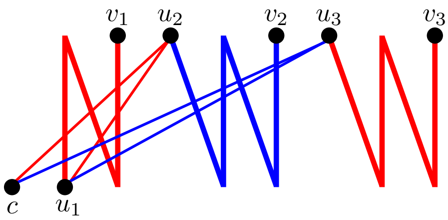
First consider the case where P_1, P_2 and P_3 are all the same colour and construct the path $v_1cu_2xu_3u_1$ of the other colour. Then by Lemma 53 we are done.



Now consider the case where one of P_1, P_2 and P_3 is of a different colour, say blue, to the other two. Suppose first that P_1 is the blue path. Then if u_1u_2 is blue P_1u_2c is a blue path and we are done, so suppose not. Similarly, if u_1u_3 is blue then we are done, so suppose not. Then $P_2u_1P_3$ is a red path, so we are done.



Now consider the case where one of P_2 and P_3, say P_2, is the single blue path. The case where P_3 is the blue path holds by symmetry. If u_1u_2 is blue then $P_2u_1u_3c$ is a blue path, so suppose not. Then P_1u_2c is a red path, so we are done.



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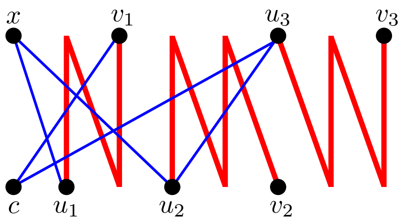
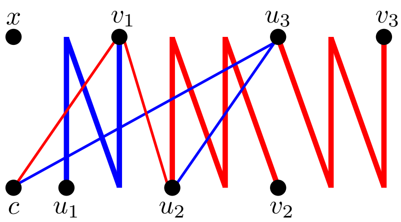
Claim 78. *If $u_2, v_2 \in M_b$ and $u_3, v_3 \in M_t$ then we are done.*

Proof. Denote by x a vertex in M_t which is not covered by any path P_i. If x is joined to an endvertex of any P_i by an edge of the same colour as P_i then we have a contradiction.

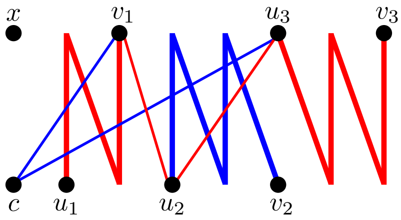
First consider the case where P_1, P_2 and P_3 are all the same colour and form the path $v_1cu_3u_2xu_1$ of the other colour. Then by Lemma 53 we are done.



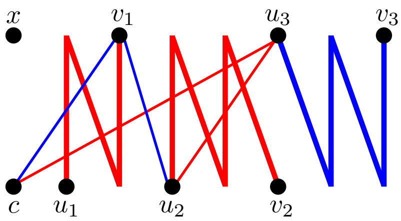
Now suppose that two of P_1, P_2 and P_3 are red and one is blue. First consider the case where P_1 is the blue path. If v_1u_2 is blue then $P_1u_2u_3c$ is a blue path, so suppose not. Then P_2v_1c is a red path, so we are done.



Now consider the case where P_2 is the single blue path. If v_1u_2 is blue then P_2v_1c is a blue path and we are done, so suppose not. Similarly, if u_2u_3 is blue then we are done, so suppose not. Then $P_1u_2P_3$ is a single red path, so we are done.



Finally, consider the case where P_3 is the single blue path. If u_3u_2 is blue then $P_3u_2v_1c$ is a blue path, so suppose not. Then P_2u_3c is a red path, so we are done.



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Lemma 79. *Suppose that a two-edge-coloured $K_{m,m}$ contains three maximal disjoint monochromatic nonalternating paths P_1, P_2, P_3 and a vertex c which is not covered by any of these paths. Then at most three disjoint monochromatic paths are needed to cover c and all the vertices on P_1, P_2 and P_3.*

Proof. We consider each of the following cases in turn:

- $u_i, v_i \in M_t$ for all $i \in \{1, 2, 3\}$.
- $u_i, v_i \in M_b$ for all i.
- $u_1, v_1 \in M_t$ and $u_2, v_2, u_3, v_3 \in M_b$.
- $u_1, v_1 \in M_b$ and $u_2, v_2, u_3, v_3 \in M_t$.

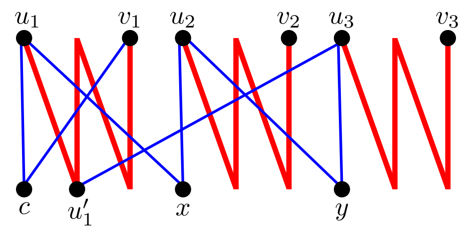
All other cases hold by symmetry. Within each of these cases we first consider the subcase where all the paths P_1, P_2 and P_3 are of the same colour. We will see that we can use Lemma 53 to obtain the result. We then consider the subcase where one of the paths is of a different colour to the other two.

If c is joined to an endvertex of any P_i by an edge of the same colour as P_i then we have a contradiction. Similarly, if there is an edge of the same colour as P_i between an endvertex of P_i and an endvertex of a different path P_j of the same colour, then P_iP_j is a single monochromatic path, a contradiction.

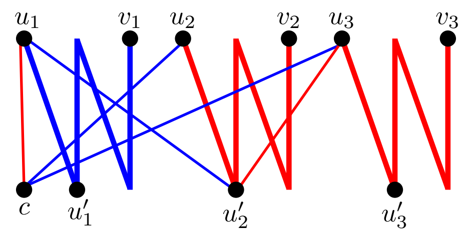
Claim 80. *If $u_i, v_i \in M_t$ for all i then we are done.*

Proof. Denote by x and y two vertices in M_b which are not covered by any path P_i and are distinct from c. If x or y is joined to an endvertex of any P_i by an edge of the same colour as P_i then we have a contradiction. If $u_i = v_i$ for any i then P_ic is a monochromatic path longer than P_i, a contradiction. So denote the neighbour of u_i on P_i by u_i' for all i.

First consider the case where P_1, P_2 and P_3 are all the same colour, say red. If $u_1'u_3$ is red then $(P_1 - u_1)P_3$ is a single red path and cu_1 is a disjoint blue path, so suppose not. Form the blue path $v_1cu_1xu_2yu_3u_1'$, so by Lemma 53 we are done.



Now suppose that one of P_1, P_2 and P_3 is of a different colour, say blue, to the other two. Suppose arbitrarily that P_1 is the blue path. If $u_i = v_i$ for any i then $P_i c$ is a monochromatic path longer than P_i, a contradiction. So denote the neighbour of u_i on P_i by u'_i for all i. If $u_1u'_2$ and $u_1u'_3$ are both red then $(P_2 - u_2)u_1(P_3 - u_3)$ is a single red path and u_2cu_3 is a blue path, so suppose not. Suppose arbitrarily that $u_1u'_2$ is blue. Then if u'_2u_3 is blue $P_1u'_2u_3cu_2$ is a blue path, so suppose not. Then $P_3(P_2 - u_2)$ is a single red path and cu_2 is a disjoint blue path, so we are done.



□

Claim 81. *If $u_i, v_i \in M_b$ for all i then we are done.*

Proof. Denote by w, x, y and z four vertices in M_t which are not covered by any path P_i. If any of these vertices is joined to an endvertex of any P_i by an edge of the same colour as P_i then we have a contradiction. If $u_i = v_i$ for any i then P_ix is a monochromatic path longer than P_i, a contradiction. So denote the neighbour of u_i on P_i by u'_i for all i.

First consider the case where P_1, P_2 and P_3 are all the same colour, say red (Figure 7.4). If cu'_i and cu'_j are both red for distinct i and j then $(P_i - u_i)c(P_j - u_j)$ is a single red path and u_iwu_j is a disjoint blue path. So suppose not. Suppose arbitrarily that cu'_1 and cu'_2 are both blue.

Now, if both u'_1u_3 and u'_2v_3 are red then $(P_1 - u_1)P_3(P_2 - u_2)$ is a single red path and u_1wu_2 is a blue path. So suppose that at least one of u'_1u_3 and u'_2v_3 is blue. In either case we can form a blue path which covers only $c, w, x, y, z, u'_1, u'_2$, both endvertices of P_1 and P_2 and a single endvertex of P_3. Ensure that this blue path has u'_k and v_k as its endvertices for some $k < 3$. Then by Lemma 53 we are done.

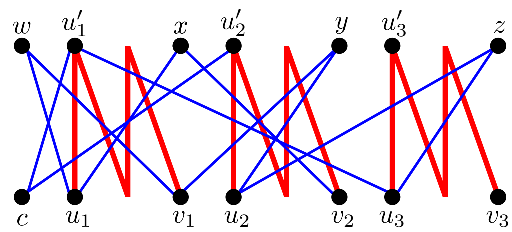
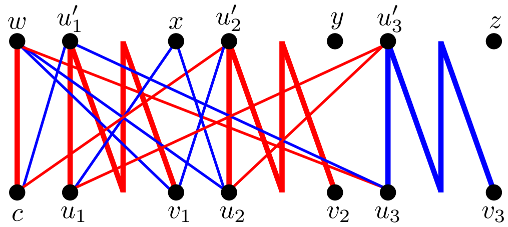


Figure 7.4

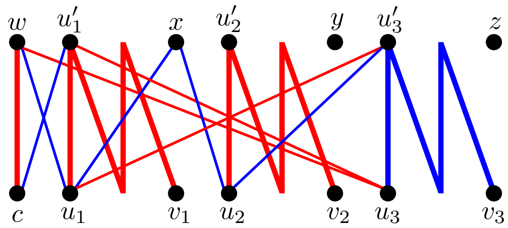
Now consider the case where one of P_1, P_2 and P_3, say P_3, is of a different colour to the other two. Suppose arbitrarily that P_3 is blue.

First consider the case where cw is red. If cu_1' and cu_2' are both red then $(P_1-u_1)c(P_2-u_2)$ is a single red path and u_1wu_2 is a blue path, so suppose not. We suppose arbitrarily that cu_1' is blue.

We consider the case where the edge $u_1'u_3$ is blue and the case where $u_1'u_3$ is red in turn. First suppose that $u_1'u_3$ is blue. Then if v_1u_2' is red $(P_1-\{u_1,u_1'\})(P_2-u_2)$ is a red path, $P_3u_1'c$ a blue path and u_1xu_2 is a third blue path, so suppose not. Then if cu_2' is blue $P_3u_1'cu_2'v_1wu_1xu_2$ is a blue path, so suppose not. If u_1u_3' is blue then $(P_3-u_3)u_1xu_2$ is a blue path and $(P_2-u_2)cwu_3$ is a red path, so suppose not. Similarly, if u_2u_3' is blue then $(P_3-u_3)u_2$ is a blue path and $(P_2-u_2)cwu_3$ is a red path, so suppose not. Then $P_1u_3'P_2$ is a single red path and cwu_3 is a disjoint red path, so we are done.



Now suppose that $u_1'u_3$ is red. Then if u_1u_3' is blue $(P_3-u_3)u_1$ is a blue path and $(P_1-u_1)u_3wc$ is a red path, so suppose not. Then if $u_3'u_2$ is red $P_1u_3'P_2$ is a single red path and cwu_3 is a disjoint red path, so suppose not. Then $(P_3-u_3)u_2xu_1$ is a blue path and $(P_1-u_1)u_3wc$ is a disjoint red path, so we are done.



Now consider the case where cw is blue (as in Figure 7.5). If x is joined to an endvertex of P_1 or P_2 in red, or to an endvertex of P_3 in blue then we have a contradiction. If cu_1' and cu_2' are both red $(P_1-u_1)c(P_2-u_2)$ is a single red path and u_1wu_2 is a blue path, so suppose not. We suppose arbitrarily that cu_1' is blue. Then if $u_1'u_3$ is also blue $P_3u_1'cwu_1$ is a blue path, so suppose not. If v_1u_3' is blue then $(P_3-u_3)v_1xu_1wc$ is a blue path and $(P_1-\{u_1,v_1\})u_3$ is a disjoint red path, so suppose not. Similarly, if $u_3'u_2$ is blue then $(P_3-u_3)u_2xu_1wc$ is a blue path and $(P_1-u_1)u_3$ is a red path, so suppose not. Then $P_2u_3'(P_1-u_1)u_3$ is a single red path and cwu_1 is a blue path so we are done.

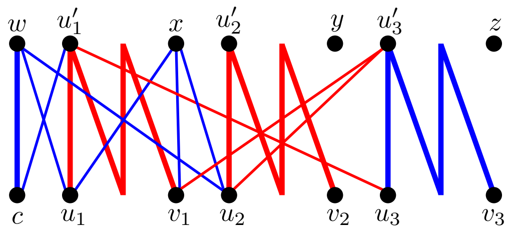


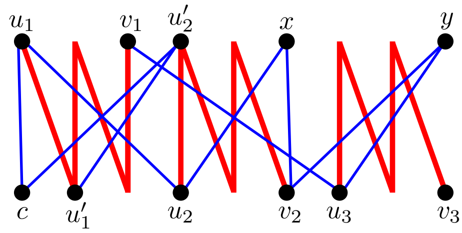
Figure 7.5

□

Claim 82. *If $u_1, v_1 \in M_t$ and $u_2, v_2, u_3, v_3 \in M_b$ then we are done.*

Proof. Denote by x and y two vertices in M_t which are not covered by any path P_i. If x or y is joined to an endvertex of any P_i by an edge of the same colour as P_i then we have a contradiction. If $u_i = v_i$ for any i then we can extend P_i so we have a contradiction. Denote the neighbour of u_i on P_i by u'_i for each i.

First consider the case where P_1, P_2 and P_3 are the same colour, say red. If cu'_2 and cu'_3 are both red then $(P_2 - u_2)c(P_3 - u_3)$ is a red path and u_2xu_3 is a blue path, so suppose not. We suppose arbitrarily that cu'_2 is blue. If $u'_1u'_2$ is red then $(P_1 - u_1)(P_2 - u_2)$ is a single red path and cu_1u_2 is a disjoint blue path, so suppose not. Form the blue path $u'_1u'_2cu_1u_2xv_2yu_3v_1$, so by Lemma 53 we are done.



Now consider the case where one of P_1, P_2 and P_3 is of a different colour, say blue, to the other two. Suppose first that P_1 is the blue path (Figure 7.6). If u_1u_2 is red then P_2u_1c is a red path, so suppose not. Then if cu'_2 is red $(P_2 - u_2)c$ is a red path and P_1u_2 is a blue path, so suppose not. Finally, if u'_2u_3 is blue then $P_1u_2xu_3u'_2c$ is a blue path, so suppose not. Then $(P_2 - u_2)P_3$ is a single red path and P_1u_2 is a blue path so we are done.

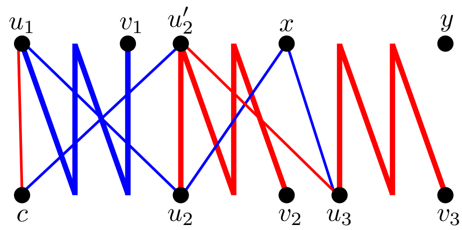
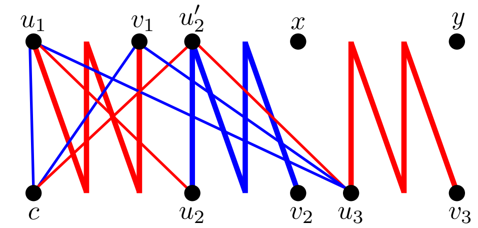


Figure 7.6

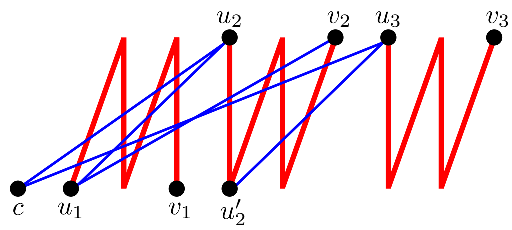
Now suppose that one of P_2 and P_3, say P_2, is blue. If u_1u_2 is blue then P_2u_1c is a blue path, so suppose not. Then if u'_2c is blue $(P_2 - u_2)c$ is a blue path and P_1u_2 is a red path, so suppose not. Finally, if u'_2u_3 is blue then $(P_2 - u_2)u_3v_1c$ is a blue path and $(P_1 - v_1)u_2$ is a red path, so suppose not. Then $P_3u'_2c$ is a red path and P_1u_2 is a disjoint red path so we are done.



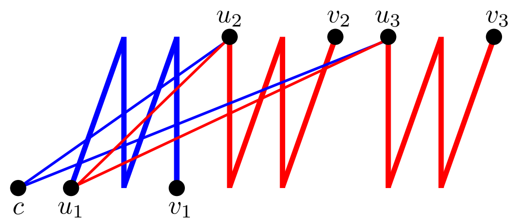
□

Claim 83. *If $u_1, v_1 \in M_b$ and $u_2, v_2, u_3, v_3 \in M_t$ then we are done.*

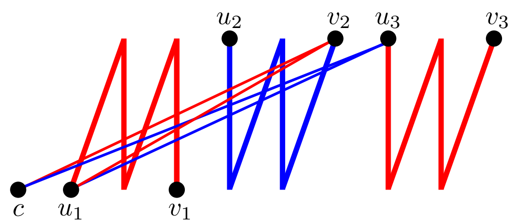
Proof. First consider the case where P_1, P_2 and P_3 are all the same colour, say red. If $u_2 = v_2$ then P_2c is a monochromatic path longer than P_2, a contradiction. Denote the neighbour of u_2 on P_2 by u_2'. If $u_2'u_3$ is red then $(P_2 - u_2)P_3$ is a red path and cu_2 is a blue path, so suppose not. Then $u_2'u_3cu_2u_1v_2$ is a blue path, so by Lemma 53 we are done.



Now consider the case where one of P_1, P_2 and P_3 is of a different colour, say blue, to the other two. First consider the case where P_1 is the blue path. If u_1u_2 is blue then P_1u_2c is a blue path and we are done, so suppose not. Similarly, if u_1u_3 is blue then we are done, so suppose not. Then $P_2u_1P_3$ is a single red path so we are done.



Now consider the case where P_2 is the blue path. The case where P_3 is the blue path holds by symmetry. If u_1v_2 is blue then $P_2u_1u_3c$ is a blue path, so suppose not. Then P_1v_2c is a red path so we are done.



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7.3 Proof of the Theorem

We are now ready to prove Theorem 6.

Proof of Theorem 6. If $n \leq 3$ then we are done; we can cover the vertices of $K_{n,n}$ by at most three pairwise non-adjacent edges. So suppose that $n \geq 4$. Construct any three maximal disjoint monochromatic paths P_1, P_2 and P_3 in $K_{n,n}$. Recall that we call a vertex in $K_{n,n}$ which is not covered by any of P_1, P_2 and P_3 **uncovered**. If $K_{n,n}$ contains no uncovered vertices then we are done. So suppose that there is at least one uncovered vertex c in $K_{n,n}$.

Consider a subgraph $K_{m,m}$ of $K_{n,n}$ which contains c and the three paths P_1, P_2 and P_3. In the case where P_1, P_2, P_3 and c do not form a balanced bipartite graph, we include in $K_{m,m}$ just enough uncovered vertices from $K_{n,n}$ to ensure that $K_{m,m}$ is balanced. We also include all edges between the vertices in $K_{m,m}$ from its supergraph. Using Lemmas 73 to 79 we now show that there exist three disjoint monochromatic paths which cover c and all the vertices on P_1, P_2 and P_3. If P_1, P_2 and P_3 are all alternating then by Lemma 73 we are done. If only one of P_1, P_2 and P_3 is nonalternating then by Lemma 74 we are done. If exactly two of P_1, P_2 and P_3 are nonalternating then by Lemma 75 we are done. Finally, if all three of P_1, P_2 and P_3 are nonalternating then by Lemma 79 we are done.

If the "new" disjoint monochromatic paths that we find do not cover all the vertices in $K_{n,n}$ then we can extend each path so that it is maximal and repeat the above process to obtain at most three disjoint monochromatic paths which cover another vertex in $K_{n,n}$. Repeating this process until we have covered all uncovered vertices in $K_{n,n}$ by at most three disjoint monochromatic paths gives us the result. □

Chapter 8

Conclusion

In this dissertation we have investigated the problem of finding the minimum number of disjoint monochromatic paths or cycles needed to cover the vertices of an edge-coloured graph. In Chapter 2 we gave an exposition of Gyárfás' short and elegant algorithmic proof of the $r = 2$ case of Conjecture 1. We then used this result to verify an approximate version of the $r = 2$ case of Conjecture 2, namely that the vertices of any two-edge-coloured K_n can be covered by monochromatic cycles of *different* colours such that the two cycles share at most one vertex. We contributed to Gyárfás' work by providing a proof outline and a proof of correctness of each algorithm, as well as several illustrative diagrams.

In Chapters 3 and 4 we saw how Bessy and Thomassé used Theorem 4 to prove Lehel's conjecture. We made Bessy and Thomassé's proof more accessible by providing explanation where it was lacking, proving several unproven statements and including many explanatory diagrams. We also made some corrections to the original proof, removed some unnecessary observations and clarified the authors' ambiguous definition of left and right vertices.

Pokrovskiy's counterexamples to Conjecture 2 were presented in Chapter 5. While only a single counterexample to Conjecture 2 is needed for each $r \geq 3$, Pokrovskiy proved the existence of infinitely many. In [19] Pokrovskiy claimed, but did not prove, that each of his counterexamples can be covered by r disjoint monochromatic *paths*. We showed that this is true in Section 5.4 of Chapter 5.

In Chapter 6 we gave a new proof of the $r = 2$ case of Conjecture 5, and in Chapter 7 we offered a simpler version of this proof. A shortcoming of our proof is its considerable length. It seems likely that some of the subcases in the proof, or even some of the cases, can be combined, but it is not clear how. Our method is also unrealistic for solving this problem for any number of colours greater than two, and would have been cumbersome if we did not already know the number of paths needed. An advantage of our approach is that the proof is self-contained and makes use of only simple techniques, so is accessible to those with only a basic knowledge of graph theory. It would not be difficult to develop an algorithm similar to our proof, which finds at most three disjoint monochromatic paths covering the vertices of any two-edge-coloured $K_{n,n}$. It is perhaps also worth investigating whether a modified version of this method can be used to determine the minimum number of disjoint monochromatic *cycles* needed to cover the vertices of any two-edge-coloured $K_{n,n}$.

In almost all of the results that we have seen the proof method has involved working with a similar, existing result to obtain the desired result. Gyárfás proved Theorem 4 by joining the ends of the simple path from Theorem 7 to form a simple cycle, and then joined the vertices on this cycle where the colour changes to form a path and a disjoint cycle of different colours which share two vertices. Gyárfás showed that we can keep increasing the length of the cycle until we obtain two cycles of different colours with at most one common vertex. Bessy and Thomassé then considered the case where these cycles *do* share a vertex. After removing the common vertex from one of the cycles to form a blue cycle and a disjoint red path, Bessy and Thomassé proved, by careful consideration of the properties of these two structures, that the vertices of any two-edge-coloured K_n can be covered by two *disjoint* monochromatic cycles of different colours. Finally, Pokrovskiy's counterexamples [20] were constructed by carefully adding vertices and edges to r-edge-coloured complete graphs whose vertices cannot be covered by r disjoint monochromatic paths of different colours. It was mentioned in [9] that such graphs were first discovered by Heinrich.

In spite of the recent progress in this area, there is still much room for further work, and there remain many interesting open problems. The $r \geq 4$ case of Conjecture 1 and the $r \geq 3$ case of Conjecture 5 are two such examples. It also remains to answer the weaker versions of Conjecture 2 mentioned in Chapter 5. In this dissertation we have considered only vertex coverings of complete and complete bipartite graphs by disjoint paths and cycles. Do we obtain similar results for host graphs other than the complete and complete bipartite graphs? What can we say about vertex coverings by paths and cycles of *different colours*? Or coverings by monochromatic substructures *other* than paths and cycles?

In [5], Gyárfás, Erdös and Pyber suggested that it may be possible to use results on vertex coverings of an edge-coloured $K_{n,n}$ by monochromatic subgraphs to determine similar results on complete graphs. Inspired by Conjecture 5, Schaudt and Stein [23] have asked whether results similar to Conjecture 5 can be found for complete k-partite graphs. As mentioned in Chapter 6, Pokrovskiy [20] used Theorem 6 to solve the $r = 3$ case of Conjecture 1. It may therefore be worth investigating whether results on covering complete multipartite graphs other than the bipartite graph can be used to solve similar problems on complete graphs.

The independence number of a graph is the size of its largest independent set. Some results and open problems on covering the vertices of graphs with specific minimum degree or maximum independence number by monochromatic subgraphs are discussed in [2].

We now briefly consider results and an open problem involving monochromatic paths and cycles of *different colours*. In Chapter 2 we saw that the vertices of any two-edge-coloured K_n can be covered by two disjoint monochromatic paths of different colours. In Chapters 3 and 4 we saw that this is also true if we replace "paths" by "cycles". However, for $T = \emptyset$, Lemma 46 tells us that neither of these statements is true for any number of colours greater than two. On the other hand, it would be interesting to see whether we can cover the vertices of any r-edge-coloured K_n by r (not necessarily disjoint) paths of different colours. Theorem 7 confirms that the answer to this problem is "yes" for $r = 2$, and Pokrovskiy [20] claims that it is also true for $r = 3$.

Many other results and open problems of this nature can be found in a recent survey by Gyárfás [10].

Chapter 9

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