Explicit construction of Lie algebra representations via characteristic identities

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Chapter 1

Introduction

The origins of representation theory date back to the late 19th century, when F. G. Frobenius invented the notion of a group representation to solve a certain problem in finite group theory[1]. The representation theory of finite groups was later generalised to the case of Lie groups and their corresponding Lie algebras[2,3]. By the early 20th century, the irreducible representations of simple Lie algebras were completely classified, due to the work of Killing, Cartan, Weyl and others. In chapter 2 we will give a very brief summary of some of the main results in the representation theory of Lie algebras, including this classification theorem.

Though the classification thorem for irreducible representations of simple Lie algebras was a major achievement, the problem of explicitly constructing representations in general remained open. By an *explicit construction*, we mean a construction of an explicit basis for the representation, as well as formulae for the matrix elements of the Lie algebra generators in this basis. This problem was eventually solved by Gelfand and Tsetlin in 1950[4,5]. In two short papers, they provided a method for constructing a basis for the Lie algebras $\mathfrak{gl}_n$ and $\mathfrak{o}_n$, parametrised by combinatorial patterns now referred to as *Gelfand-Tsetlin patterns*, as well as explicit formulae for the matrix elements of the generators in this basis. The construction of the basis was a relatively straightforward matter: it relied on the *branching rules* for irreducible representations of $\mathfrak{gl}_n$ and $\mathfrak{o}_n$. If one considers the canonical subalgebra chains

$$\mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \dots \mathfrak{gl}_n$$

and

$$\mathfrak{o}_2 \subset \mathfrak{o}_3 \subset \dots \mathfrak{o}_n$$

then by considering the restriction of a representation to each subalgebra in the chain, one arrives at a decomposition of the representation into distinct one-dimensional representations. Hence one obtains a basis in terms of the weight vectors in each of the subrepresentations occurring in the decomposition. (We will explain this idea in more detail in chapter 2). What was more mysterious was the fact that the (rather complicated) formulae for the matrix elements in Gelfand and Tsetlin's papers were written down without any explanation as to how they were derived.

In an effort to understand Gelfand and Tsetlin's results, several authors developed their own proofs. The first rederivation of Gelfand and Tsetlin's results was due to Baird and Biedenharn[6,7,8]. In a series of four papers, Baird and Biedenharn solved a number of problems related to the explicit construction of representations of the special unitary group SU_n. In particular, they determined a complete set of invariant operators (that is a complete set of operators commuting with the group). Furthermore, they solved the so-called "state-labelling" problem for irreducible SU_n representations[7]. The state-labelling problem consists of determining a set of invariants which is sufficient to uniquely identify (label) the states of a representation. The problem had already been solved in the case of SU_2 and SU_3, the latter case being due to Elliott[9]; Baird and Biedenharn generalised the results to SU_n for arbitrary n.

Moreover, Baird and Biedenharn developed explicit formulae for the matrix elements of SU_n in an irreducible representation. Initially, they showed how one could obtain such formulae using purely algebraic techniques (that is, techniques only involving the generators and commutation relations) for the case of SU_3 and indicated how this could be generalised for SU_n. However, they explained that this technique would actually be "intolerably tedious" to carry out in practice. Therefore they introduced a separate *combinatorial* ("integral") technique to solve the problem for arbitrary n, based on the so-called Schwinger-Bargmann boson calculus. In addition to correcting some mistakes in the original work of Gelfand and Tsetlin, Baird and Biedenharn also revealed the structure of the matrix elements.

In the 1960s an alternative algebraic approach to the problem, using only the generators and their commutation relations, was developed by Nagel and Moshinsky[1]0 and independently by Hou Pei-yu[1]2, the latter of which contains a different proof of the Gelfand-Tsetlin formulas from that of Baird and Biedenharn. In our thesis, we will explain the so-called method of characteristic identities for determining explicit formulae for matrix elements, which is inherently algebraic. These characteristic identities are certain polynomial identities satisfied by the generators of a Lie algebra on an irreducible representation. They were first proved by Bracken and Green in 1970[15,18] (although certain special cases were noticed by Dirac as early as 1936[14]). In the case of $\mathfrak{gl}_n$, if one arranges the generators $\{a^i_j \mid 1 \le i, j \le n\}$ into a matrix A, then on an irreducible representation of $\mathfrak{gl}_n$ the matrix A satisfies an n-degree polynomial identity (analogous to the characteristic polynomial satisfied by an $n \times n$ numerical matrix). Some low-dimensional examples are relatively obvious. Consider a fixed finite-dimensional irreducible representation $V(\lambda)$ of $\mathfrak{gl}_n$ with highest weight

$\lambda = (\lambda_1, \ldots, \lambda_n)$. If one sets

$$A = \begin{pmatrix} a^1_1 & a^1_2 & \ldots & a^1_n \\ a^2_1 & a^2_2 & \ldots & a^2_n \\ \vdots & \vdots & \ddots & \vdots \\ a^n_1 & a^n_2 & \ldots & a^n_n \end{pmatrix}$$

and

$$\sigma_1 = \sum_{i=1}^{n} a^i_i$$
$$\sigma_2 = \sum_{i,j=i}^{n} a^i_j a^j_i,$$

then it is easy to verify that on this representation we have

$$(A - \sigma_1) = 0 \text{ for } n = 1$$
$$A^2 - (\sigma_1 + 1)A + \frac{1}{2}(\sigma_1^2 + \sigma_1 - \sigma_2) = 0 \text{ for } n = 2$$

Bracken and Green used the characteristic identities to develop formulae for invariants of $\mathfrak{gl}_n$. However, the use of the characteristic identities for determining matrix elements was only carried out later by Gould across a series of paper[13,16,17,20].

It is Gould's method for the case of $\mathfrak{gl}_n$ and $\mathfrak{o}_n$ that we will outline in the subsequent chapters of this thesis. If we denote by V^* the contragredient vector representation of $\mathfrak{gl}_n$ (which has highest weight $(0, \ldots, 0, -1)$), then the starting observation is that the matrix A defined earlier can be considered an operator on the tensor product space $V^* \otimes V(\lambda)$ in a natural way. Then, by means of the characteristic identity, one may construct projection operators which project onto one of the components in the direct sum decomposition of $V^* \otimes V(\lambda)$. By means of these projection operators, one may resolve a vector operator on $\mathfrak{gl}_n$ into a sum of "shift components" each of which alters one of the representation labels of $\mathfrak{gl}_n$. Using purely algebraic techniques (and in particular making repeated use of the characteristic identity), one may express the matrix elements of such vector operators in terms of invariants of the algebra.

Chapter 2

Lie algebras and representation theory

In this section we give a brief overview of the basic theory of (semisimple) Lie algebras and their representations. For our purposes the most important results will be the classification of irreducible representations of semisimple Lie algebras and the branching rules for restricted representations. Most of the material in this chapter is from Humphreys[3].

2.1 Basic notions

Recall that an (abstract) **Lie algebra** is a vector space L over a field F endowed with a bilinear map $[\ ,\ ] : L \times L \to L$ (called the **bracket**) satisfying the following two conditions:

$$[x, x] = 0 \ \ \forall x \in L \tag{2.1}$$

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \ \ \forall x, y, z \in L \tag{2.2}$$

Condition (2.2) is often referred to as the **Jacobi identity**. Note that condition (2.1) immediately implies (replacing x by $x - y$ in (2.1) and using the bilinearity of $[\ ,\ ]$)

$$[x, y] = -[y, x] \ \forall x, y \in L$$

In this section we always consider L to be finite-dimensional as a vector space over F. Indeed, we only ever consider finite-dimensional Lie algebras in this thesis, although we briefly mention an application to infinite-dimensional Lie algebras in section 5.3. An important example of a Lie algebra is the **general linear Lie algebra** $\mathfrak{gl}(V)$, which is the set of all linear maps on an n-dimensional F-vector space V endowed with the commutator

$$[x, y] := xy - yx$$

where the "product" on the right hand side is composition of linear maps. It is easy to check that conditions (2.1) and (2.2) are satisfied; indeed, any associative algebra A becomes a Lie algebra when endowed with the commutator. Choosing a basis for V we may identify $\mathfrak{gl}(V)$ with $\mathfrak{gl}_n(F)$, the Lie algebra of $n \times n$ matrices with entries in F.

Definiton *A Lie algebra homomorphism between two Lie algebras L, L' over F is a vector space homomorphism $\phi : L \to L'$ which preserves the bracket, i.e.*

$$\phi([x,y]_L) = [\phi(x), \phi(y)]_{L'}$$

A Lie algebra isomorphism is then a vector space isomorphism $\phi : L \to L'$ which preserves the bracket.

Definiton *A Lie subalgebra of a Lie algebra L is a vector subspace $L' \subset L$ which is closed with respect to the bracket, i.e. $[x,y] \in L'$ $\forall x, y \in L'$.*

We call a Lie subalgebra of $\mathfrak{gl}(V)$ a **linear Lie algebra** for obvious reasons. We have the following striking result (which we will not prove):

Theorem 1 (Ado) *Every (finite-dimensional) abstract Lie algebra is isomorphic to a linear Lie algebra (over a finite-dimensional vector space V).*
So to understand the structure of abstract Lie algebras it suffices to understand the structure of linear Lie algebras. We now define representations of Lie algebras, which are the main subject of this section

Definition *A representation of a Lie algebra L is a Lie algebra homomorphism $\phi : L \to \mathfrak{gl}(V)$.*

Intuitively a representation is a map which "represents" the elements of the Lie algebra as linear transformations on a vector space, while maintaining the structure of the original Lie algebra. Perhaps the most important representation of any Lie algebra is the **adjoint representation** $\mathrm{ad} : L \to \mathfrak{gl}(L)$ defined by $(\mathrm{ad}(x))(y) = [x,y]$. It is clear that $\mathrm{ad}(x)$ is a linear map for each x and that ad is a linear map from L to $\mathfrak{gl}(L)$; the fact that ad is a Lie algebra homomorphism follows from the Jacobi identity. We note in passing that if L is simple then $Z(L) = 0 = \ker \mathrm{ad}$ so that L is isomorphic to a linear Lie algebra. (This proves Ado's theorem in the case that L is simple.)

Now consider the sequence of ideals

$$L^{(0)} = L, L^{(1)} = [L,L], \ldots, L^{(i)} = [L^{(i-1)}, L^{(i-1)}], \ldots$$

called the **derived series** of L. We call L **solvable** if $L^{(n)} = \{0\}$ for some n. This definition is analogous to that of a solvable group. Clearly all abelian Lie algebras are solvable (as $[L,L] = \{0\}$ already), and any simple Lie algebra will not be solvable (as $0 \neq [L,L] = L$, so $L^{(n)} \neq 0$ for all n). (Compare with groups: all abelian groups are solvable and any (non-abelian) simple group is not solvable). We have the following basic properties of solvability

Proposition If a Lie algebra L is solvable, then so are all its subalgebras and homomorphic images. If I and J are solvable ideals of L, then so is $I + J$.

The last of these properties allows us to define the important concept of semisimplicity. Suppose S is a solvable ideal of L which is *maximal* in the sense that it is not contained in a larger solvable ideal, and let I be any other solvable ideal. Then $S + I$ is also solvable by proposition, and by the maximality of S we must have $S + I = S$, so $I \subset S$, which implies that S is unique. So every Lie algebra L has a unique maximal solvable ideal; call this the **radical** of L and denote it by Rad L. Then we have the

Definition *A Lie algebra L is called **semisimple** if Rad $L = \{0\}$.*

We have the following important theorem about the representations of semisimple Lie algebras due to Weyl

Theorem (Weyl) *Every finite-dimensional representation of a semisimple Lie algebras is a direct sum of irreducible represnetations*[3]

Therefore, to classify the representations of semisimple Lie algebras it suffices to classify the irreducible representations. In our thesis, we will only consider semisimple Lie algebras (or reductive Lie algebras, whose representation theory is closely related).

2.2 Classification of irreducible representations

We now briefly outline the classification theorem for irreducible representations of semisimple Lie algebras. Note that because of Weyl's complete reducibility theorem, it is sufficient to classify the irreducible representations (because an arbitrary representation will just be a direct sum of these). The following general method is essentially a generalisation of the case of $\mathfrak{sl}_2$. Let L be a semisimple Lie algebra and let V be an irreducible representation. Suppose we can find a nilpotent subalgebra H of L that is equal to its own normaliser, i.e.

$$H = N_L(H) = \{x \in L | [x, y] \in H \text{ for all } y \in H\}$$

Such a subalegebra is called a Cartan subalgebra, and in the case of finite-dimensional Lie algebras over $\mathbb{C}$ this subalgebra always exists (though is not necessarily unique). Then it can be shown that H acts on L diagonally (via the adjoint action); therefore we can decompose L into a direct sum of H and eigenspaces spaces of H. That is, we have

$$L = H \bigoplus_{\alpha \in H^*} L_\alpha,$$

where H^* is the dual (vector) space of H, and L_α is the eigenspace of L corresponding to α, i.e.

$$L_\alpha = \{x \in L \mid [x, h] = \alpha(h)x \text{ for all } h \in H\}$$

The elements α are called the roots of L and the corresponding spaces L_α are called root spaces. Similarly, given an irreducible representation V there is a decomposition of V into emphweight spaces. The basic result is the following:

Theorem *The finite-dimensional irreducible representations of semisimple Lie algebras contain a unique (up to scaling) vector v of highest weight*[3].

In this sense, such representations are uniquely classified by the highest weight.

2.3 Branching rules and Gelfand-Tsetlin patterns

Given an irreducible representation of a Lie algebra L, it is natural to consider how the representation decomposes upon restriction to a subalgebra K. In general, of course, the restriction to a subalgebra will not be irreducible, and hence will decompose as a direct sum of irreducible representations for the subalgebra. This decomposition is known as the branching rule for this restriction. The general construction of a basis for irreducible representations that is *symmetry adapted* to a certain subalgebra chain is part of the state labelling problem[4]. For the general linear Lie algebra $\mathfrak{gl}_n$ and the orthogonal Lie algebra $\mathfrak{o}_n$, there is a particularly straightforward branching rule for restriction to the subalgebras $\mathfrak{gl}_{n-1}$ and $\mathfrak{o}_{n-1}$. If $V(\lambda)$ denotes an irreducible representation of $\mathfrak{gl}_n$ with highest weight $\lambda = (\lambda_1, \ldots, \lambda_n)$ then the restriction of this representation to $\mathfrak{gl}_{n-1}$ decomposes according to

$$V(\lambda) \downarrow gl_{n-1} \cong \bigoplus_{\mu} V'(\mu),$$

where the direct sum is taken over irreducible $\mathfrak{gl}_{n-1}$ modules with highest weights μ satisfying the betweenness conditions

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \cdots \geq \mu_{n-1} \geq \lambda_n$$

There is a similar branching rule for $\mathfrak{o}_n$ although it depends on the parity of the algebra[2]. In both cases, however, the branching rule is multiplicity free, meaning that each irreducible representation occurring in the decomposition occurs only once. This allows a parametrisation of the basis of the representation in terms of combinatorial patterns associated with the betweenness conditions, because eventually (upon restricting to $\mathfrak{gl}_1$), one obtains a direct sum intro distinct one-dimensional modules. This basis is known as the Gelfand-Tsetlin basis, as it was introduced in their 1950 construction.

Chapter 3

Matrix elements of the $\mathfrak{gl}_n$ generators

3.1 Introduction

We begin by considering the general linear Lie algebra $(gl)_n$. Recall that this is the n^2-dimensional Lie algebra of $n \times n$ complex matrices:

$$\mathfrak{gl}_n = \{x \in M_{n\times n}(\mathbb{C})\}$$

The n^2 generators a^i_j $(i, j = 1, \ldots, n)$ of $\mathfrak{gl}_n$ satisfy the commutation relations

$$[a^i_j, a^k_l] = \delta^k_j a^i_l - \delta^i_l a^k_j \tag{3.1}$$

The starting point[13,20] is the observation that if one arranges these generators in a matrix, then this matrix satisfies a certain degree n polynomial identity on an irreducible representation.

3.2 Characteristic identities and projection operators

We begin by proving the characteristic identity for a finite-dimensional irreducible representation of $U(n)$. The term "characteristic identity" is used in reference to the characteristic polynomial of a square matrix $M \in \mathrm{Mat}_{n\times n}(\mathbb{F})$ with entries from a field $\mathbb{F}$. Recall that this is the degree n polynomial $p_M(t)$ defined by

$$p_M(t) = \det(M - tI)$$

where I is the $n \times n$ identity matrix. The roots of this polynomial are precisely the eigenvalues of the matrix M. The Cayley-Hamilton theorem states that any matrix M satisfies its own characteristic polynomial; that is, $p_M(M) = 0$.

The characteristic identity for $\mathfrak{gl}_n$ (and, more generally, for any semisimple Lie algebra; see section 5.1) is a kind of analogue of this theorem. In this case, we fix a finite-dimensional irreducible representation π_λ of $\mathfrak{gl}_n$. Recall that the finite-dimensional irreducible representations of $\mathfrak{gl}_n$ are in one-to-one correspondence with n-tuples of complex numbers

$$\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$$

where the components λ_i satisfy

$$\lambda_i - \lambda_{i+1} \in \mathbb{Z}_+ \text{ for } 1 \leq i \leq n-1 \tag{3.2}$$

λ is the highest weight of the corresponding $\mathfrak{gl}_n$-module V_λ. Given such a representation, we form the $n \times n$ matrix A whose (i, j) entry is the element $\pi_\lambda(a^i_j)$. That is,

$$A = \begin{pmatrix} \pi_\lambda(a^1_1) & \pi_\lambda(a^1_2) & \ldots & \pi_\lambda(a^1_n) \\ \pi_\lambda(a^2_1) & \pi_\lambda(a^2_2) & \ldots & \pi_\lambda(a^2_n) \\ \vdots & \vdots & \ddots & \vdots \\ \pi_\lambda(a^n_1) & \pi_\lambda(a^n_2) & \ldots & \pi_\lambda(a^n_n) \end{pmatrix} \tag{3.3}$$

From now on, we drop the labels π_λ and write a^i_j in place of $\pi_\lambda(a^i_j)$. Note that A may be regarded as an element of the tensor product space

$$\mathrm{End}(V^* \otimes V_\lambda)$$

where V^* denotes the contragredient vector representation, $V^* \cong V_{(0,\ldots,1)}$. To see this, note that A may be written as

$$\begin{aligned} A &= \sum_{i,j=1}^{n} E^i_j \otimes a^i_j \\ &= -\sum_{i,j=1}^{n} \pi^*(a^j_i) \otimes a^i_j, \end{aligned}$$

where $E^i_j \in \mathrm{End}(V^*)$ is the elementary matrix with 1 in the (i, j)th position and 0's elsewhere, and $\pi^*(a^i_j) := -E^j_i$ by definition. Now recall from section 2 that given two irreducible representations V_μ, V_λ of $U(n)$ we have the decomposition

$$V_\mu \otimes V_\lambda \cong \bigoplus_\nu m(\nu) V_\nu$$

where the direct sum is taken over certain irreducible representations V_ν, and $m(\nu)$ denotes the multiplicity of V_ν in the direct sum. When $V_\mu = V^*$ we have the particularly simple decomposition

$$V^* \otimes V_\lambda \cong V_{\lambda-\Delta_1} \oplus \cdots \oplus V_{\lambda-\Delta_n}, \tag{3.4}$$

where Δ_r is the weight with 1 in the rth position and 0's elsehwere. So A may also be thought of as an operator on the direct sum $V_{\lambda-\Delta_1} \oplus \cdots \oplus V_{\lambda-\Delta_n}$.

In order to prove the characteristic identity we wish to write A in an *invariant* form. That is, we want to express A in terms of invariants of $\mathfrak{gl}_n$. The reason for this is that by Schur's lemma, any $\mathfrak{gl}_n$ invariant reduces to a scalar multiple of the identity on an irreducible $\mathfrak{gl}_n$ representation. Hence, if we can express A in terms of invariants, it will reduce to a scalar multiple of the identity on each of the summands in the decomposition (3.4). To this end, we introduce the operator

$$A(z) = -\frac{1}{2}(\pi^* \otimes \pi_\lambda(z) - \pi^*(z) \otimes I - I \otimes \pi_\lambda(z)),$$

where z is an element of the centre of the universal enveloping algebra $U(\mathfrak{gl}_n)$ and I is the identity matrix (the size of which depends on context). In particular, if we take $z = \sigma_2 = \sum_{i,j=1}^n a^i_j a^j_i$ (the Casimir element) then we can write the matrix A as

$$A = A(\sigma_2) = -\frac{1}{2}(\pi^* \otimes \pi_\lambda(\sigma_2) - \pi^*(\sigma_2) \otimes I - I \otimes \pi_\lambda(\sigma_2)) \qquad (3.5)$$

To see this, note that by definition of tensor product of representations, we have

$$\begin{aligned} \pi^* \otimes \pi_\lambda(a^i_j) &= \pi^*(a^i_j) \otimes I + I \otimes \pi_\lambda(a^i_j) \\ &= -E^j_i \otimes I + I \otimes \pi_\lambda(a^i_j) \end{aligned}$$

Therefore,

$$\begin{aligned} \pi^* \otimes \pi_\lambda(\sigma_2) &= (\pi^* \otimes \pi_\lambda)\left(\sum_{i,j=1}^n a^i_j a^j_i\right) \\ &= \sum_{i,j=1}^n (\pi^* \otimes \pi_\lambda(a^i_j))(\pi^* \otimes \pi_\lambda(a^j_i)) \\ &= \sum_{i,j=1}^n (-E^j_i \otimes I + I \otimes \pi_\lambda(a^i_j))(-E^i_j \otimes I + I \otimes \pi_\lambda(a^j_i)) \\ &= \sum_{i,j=1}^n (E^j_i E^i_j \otimes I - E^j_i \otimes \pi_\lambda(a^j_i) - E^i_j \otimes \pi_\lambda(a^i_j) + I \otimes \pi_\lambda + I \otimes \pi_\lambda(a^i_j)\pi_\lambda(a^j_i)) \\ &= (\pi^*(\sigma_2) \otimes I + I \otimes \pi_\lambda(\sigma_2)) - \sum_{i,j=1}^n (E^j_i \otimes \pi_\lambda(a^j_i)) - \sum_{i,j=1}^n (E^i_j \otimes \pi_\lambda(a^i_j)) \\ &= (\pi^*(\sigma_2) \otimes I + I \otimes \pi_\lambda(\sigma_2)) - 2A \end{aligned}$$

Rearranging this expression, we find that A indeed has the form (3.5). Therefore, on each space $V(\lambda - \Delta_r)$ in the tensor product decomposition, A takes the constant value

$$\begin{aligned} A(V(\lambda-\Delta_r)) &= -\frac{1}{2}(\pi_{\lambda-\Delta_r(\sigma_2)} - \pi^*(\sigma_2) - \pi_\lambda(\sigma_2)(V(\lambda-\Delta_r)) \\ &= (n-r-\lambda_r)(V(\lambda-\Delta_r)) \end{aligned}$$

This implies the characteristic identity

$$\prod_{r=1}^{n}(A-\lambda_r-n+r)=0 \tag{3.6}$$

For convenience we set

$$\alpha_r=\lambda_r+n-r$$

Then we may write (3.6) as

$$\prod_{r=1}^{n}(A-\alpha_r)=0 \tag{3.7}$$

Similarly, we derive the identity

$$\prod_{r=1}^{n}(\overline{A}-\overline{\alpha}_r)=0, \tag{3.8}$$

where $\overline{\alpha}_r=r-\lambda_r-1$. In light of (3.7) and (3.8) we define the n **projection operators**

$$P[r]=\prod_{l\neq r}\frac{(A-\alpha_l)}{(\alpha_r-\alpha_l)} \tag{3.9}$$

$$\overline{P}[r]=\prod_{k=1,k\neq r}^{n}\left(\frac{\overline{A}-\overline{\alpha}_k}{\overline{\alpha}_r-\overline{\alpha}_k}\right) \tag{3.10}$$

Note that the numerators of $P[r]$ and $\overline{P}[r]$ are the same as the left-hand side of (3.7) and (3.8) respectively but with the rth term omitted. In particular, since these operators are polynomials in the matrix A they belong to the same space that A does, namely $\mathrm{End}(V^*\otimes V_\lambda)$. In light of the decomposition (3.4) the reason for the term "projection operator" is clear: $P[r]$ projects the tensor product $V^*\otimes V_\lambda$ onto the rth summand $V_{\lambda-\Delta_r}$. To see this, note that

$$A(V(\lambda-\Delta_k))=\alpha_k V(\lambda-\Delta_k),$$

and if $k\neq r$ then

$$\begin{aligned} P[r]V(\nu_k) &= \prod_{l\neq r}\frac{(A-\alpha_l)}{(\alpha_r-\alpha_l)}V(\nu_k) \\ &= \prod_{l\neq r}\frac{(\alpha_k-\alpha_l)}{(\alpha_r-\alpha_l)}V(\nu_k) \\ &= 0.V(\nu_k), \end{aligned}$$

while if $k = r$ then

$$P[r]V(\nu_r) = \prod_{l \neq r} \frac{(\alpha_r - \alpha_l)}{(\alpha_r - \alpha_l)} V(\nu_r)$$
$$= I.V(\nu_r)$$

Hence

$$\begin{aligned}
P[r](V^* \otimes V(\lambda)) &\cong P[r](V_{\lambda-\Delta_1} \oplus \cdots \oplus V_{\lambda-\Delta_n}) \\
&= P[r](V_{\lambda-\Delta_1}) \oplus \cdots \oplus P[r](V_{\lambda-\Delta_r}) \oplus \cdots \oplus P[r](V_{\lambda-\Delta_n} \\
&= 0.(V_{\lambda-\Delta_1}) \oplus \cdots \oplus I.(V_{\lambda-\Delta_r}) \oplus \cdots \oplus 0.(V_{\lambda-\Delta_1}) \\
&= V_{\lambda-\Delta_r}
\end{aligned}$$

We note here a few basic properties of the projection operators which will be useful later.

Proposition 1

1. The projection operators sum to the identity operator, i.e.

$$\sum_{r=1}^{n} P[r] = I$$

2. The projection operators are mutually orthonormal in the sense that

$$P[k]P[r] = \delta_r^k P[r]$$

3. For any polynomial $p(x)$ with coefficients in $\mathbb{C}$ we have

$$p(A) = \sum_{r=1}^{n} p(\alpha_r) P[r]$$

Proof. 1. This follows from the fact that each $P[r]$ reduces to the identity on exactly one of the n components in the tensor product decomposition, and hence the sum of all of them is just the identity on the tensor product space.

2. If $r \neq k$ then we have that

$$P[r]P[k] = \prod_{l=1, l\neq r} \left(\frac{A - \alpha_l}{\alpha_r - \alpha_l} \right) \prod_{m=1, m\neq r} \left(\frac{A - \alpha_m}{\alpha_r - \alpha_m} \right) = 0$$

from the characteristic identity. On the other hand, if $k = r$ then

$$P[r]^2 = \prod_{l=1, l\neq r} \left(\frac{A - \alpha_l}{\alpha_r - \alpha_l} \right) \prod_{l=1, l\neq r} \left(\frac{A - \alpha_l}{\alpha_r - \alpha_l} \right) = \prod_{l=1, l\neq r} \left(\frac{A - \alpha_l}{\alpha_r - \alpha_l} \right) \left(\frac{A - \alpha_l}{\alpha_r - \alpha_l} \right)$$

and then using the fact that

$$A \prod_{m=1, m \neq r} (A - \alpha_m) = \alpha_r \prod_{m=1, m \neq r} (A - \alpha_m)$$

this is just equal to $P[r]$

3. Again from the characteristic identity we have that

$$A \left(\prod_{m=1, m \neq r} (A - \alpha_m) \right) = \alpha_r \left(\prod_{m=1, m \neq r} (A - \alpha_m) \right)$$

which implies that

$$A = \sum_{r=1}^{n} \alpha_r P[r]$$

Then by induction this is true for any power of A, and hence for any polynomial by linearity

$\square$

We note in passing that the matrix elements of the projection operators can be shown to determine squares of Wigner coefficients.

3.3 Vector operators

Eventually we will derive a formula for expressing the matrix elements of vector operators in terms of the matrix elements of the projection operators. It will turn out that this formula will be essentially sufficient for determining the matrix elements of the $\mathfrak{gl}_n$ generators. To begin with, however, we need to develop some notation and basic properties. We define a $\mathfrak{gl}_n$ vector operator as a vector

$$\psi = \begin{pmatrix} \psi^1 \\ \vdots \\ \psi^n \end{pmatrix}$$

whose components ψ^k transform with respect to the $\mathfrak{gl}_n$ generators a^i_j as

$$[a^i_j, \psi^k] = \delta^k_j \psi^i$$

Similarly we define a $\mathfrak{gl}_n$ contragredient vector operator as a vector

$$\psi^\dagger = \begin{pmatrix} \psi^\dagger_1 \\ \vdots \\ \psi^\dagger_n \end{pmatrix}$$

whose components transform as

$$[a^i_j, \psi^\dagger_k] = -\delta^i_k \psi^\dagger_j$$

We make the following

Definition The rth shift component $(1 \leq r \leq n)$ of the $\mathfrak{gl}_n$ vector operator ψ is given by

$$\psi[r] = P[r]\psi = ((\psi)^T \overline{P}[r])^T$$

Similarly the rth shift component of the $\mathfrak{gl}_n$ contragredient vector operator $\psi^\dagger$ is given by

$$\psi^\dagger[r] = \overline{P}[r]\psi^\dagger = ((\psi^\dagger)^T P[r])^T$$

The reason for the term shift component is clear in light of the following:

Proposition 3 We have that

1. $\psi = \sum_{r=1}^n \psi[r]$ and $\psi^\dagger = \sum_{r=1}^n \psi^\dagger[r]$
2. The (non-zero) shift components $\psi[r]$ and $\psi^\dagger[r]$ form linearly independent sets
3. The shift components have the following effect on the representation labels of $\mathfrak{gl}_n$:

$$\begin{aligned} \lambda_k \psi[r] &= \psi[r](\lambda_k + \delta^r_k) \\ \lambda_k \psi^\dagger[r] &= \psi^\dagger[r](\lambda_k - \delta^r_k) \end{aligned} \tag{3.11}$$

Proof. 1. This is a direct consequence of the fact that the projection operators sum to the identity. In particular,

$$\sum_{r=1}^n \psi[r] = \sum_{r=1}^n (P[r])\psi = I\psi = \psi,$$

and

$$\sum_{r=1}^n \psi^\dagger[r] = \sum_{r=1}^n (\overline{P}[r])\psi^\dagger = I\psi^\dagger = \psi^\dagger$$

2. Suppose there existed some $\mathfrak{gl}_n$ invariants γ_r such that

$$\sum_{r=1}^n \gamma_r \psi[r] = 0$$

Then multiplying both sides by $P[k]$ and using the orthogonality of the projections, we have

$$\begin{aligned}0 &= \sum_{r=1}^{n} \gamma_r P[k]P[r]\psi \\ &= \sum_{r=1}^{n} \gamma_r \delta_r^k P[r]\psi \\ &= \sum_{r=1}^{n} \gamma_r \delta_r^k \psi[r] \\ &= \gamma_k \psi[k],\end{aligned}$$

and so assuming the shift operators are non-zero we must have that $\gamma_k = 0$ for all k, so $\{\psi[r]\}_{r=1}^{n}$ forms a linearly independent set. The proof that the $\psi^\dagger[r]$ are linearly independent is analogous.

3.

□

Note that because of the form of the characteristic roots α_k, equation (3.11) also implies

$$\begin{aligned}\alpha_k \psi[r] &= \psi[r](\alpha_k + \delta_k^r) \\ \alpha_k \psi^\dagger[r] &= \psi^\dagger[r](\alpha_k - \delta_k^r)\end{aligned} \tag{3.12}$$

3.4 Matrix elements: generators of the form a_n^{n+1} and a_{n+1}^{n}

Now consider the subalgebra embedding $\mathfrak{gl}_n \subset \mathfrak{gl}_{n+1}$, and consider a fixed $\mathfrak{gl}_{n+1}$ irreducible representation $V(\mu)$ with highest weight $\mu = (\mu_1, \ldots, \mu_{n+1})$. The $(n+1)^2$ generators a_j^i $(i, j = 1, \ldots, n+1)$ of $\mathfrak{gl}_{n+1}$ may also be arranged in a matrix, say B:

$$B = \begin{pmatrix} a_1^1 & a_2^1 & \ldots & a_{n+1}^1 \\ a_1^2 & a_2^2 & \ldots & a_{n+1}^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{n+1} & a_2^{n+1} & \ldots & a_{n+1}^{n+1} \end{pmatrix},$$

B is now an operator on the tensor product space $V^* \otimes V_\mu$ where V^* is the $(n+1)^2$-dimensional contragredient vector representation of $\mathfrak{gl}_{n+1}$. Then B satisfies the analogous identity to (3.7), i.e.

$$\prod_{r=1}^{n+1} (B - \beta_r) = 0 \tag{3.13}$$

where β_r are the $\mathfrak{gl}_{n+1}$ characteristic roots, given by $\beta_r = \mu_r + n + 1 - r$. Similarly we have the adjoint identity

$$\prod_{r=1}^{n+1}(\overline{B} - \overline{\beta}_r) = 0 \tag{3.14}$$

where $\overline{B} = -B^T$ and $\overline{\beta_r} =$. We similarly define the $\mathfrak{gl}_{n+1}$ projection operators

$$Q[r] = \prod_{k=1,k\neq r}^{n+1}\left(\frac{B - \beta_k}{\beta_r - \beta_k}\right)$$

$$\overline{Q}[r] = \prod_{k=1,k\neq r}^{n+1}\left(\frac{\overline{B} - \overline{\beta}_k}{\overline{\beta}_r - \overline{\beta}_k}\right)$$

We now aim to establish a relationship between the $\mathfrak{gl}_{n+1}$ projection operators and the "induced" projection operators of the $\mathfrak{gl}_n$ subalgebra (later we will generalise this relationship to the full subalgebra chain $\mathfrak{gl}_1 \subset \cdots \subset \mathfrak{gl}_{n+1}$). To this end, consider the first n rows and n columns of the matrix B. Denote this matrix by A, so

$$A = \begin{pmatrix} a^1_1 & a^1_2 & \dots & a^1_n \\ a^2_1 & a^2_2 & \dots & a^2_n \\ \vdots & \vdots & \ddots & \vdots \\ a^n_1 & a^n_2 & \dots & a^n_n \end{pmatrix},$$

where each entry a^i_j is now a $\dim V(\mu) \times \dim V(\mu)$ matrix. Now if we order the basis of V_μ in accordance with the branching rule, then each of the matrix elements of the generators in the matrix A will have a block diagonal structure. Explicitly, if the branching rule is

$$V(\mu)\downarrow_{\mathfrak{gl}_n} \cong V'(\nu[1]) \oplus \cdots \oplus V'(\nu[l])$$

then each of the matrix elements of a^i_j (for $1 \leq i, j \leq n$) will have the form

$$\pi_\mu(a^i_j) = \left(\begin{array}{c:c:c} \pi_{\nu[1]}(a^i_j) & 0 & 0 \\ \hdashline 0 & \ddots & 0 \\ \hdashline 0 & 0 & \pi_{\nu[l]}(a^i_j) \end{array}\right)$$

Then multiplying any two of these generators has the following effect:

$$\pi_\mu(a^i_j)\pi_\mu(a^k_l) = \left(\begin{array}{c:c:c} \pi_{\nu[1]}(a^i_j) \times \pi_{\nu[1]}(a^k_l) & 0 & 0 \\ \hdashline 0 & \ddots & 0 \\ \hdashline 0 & 0 & \pi_{\nu[l]}(a^i_j) \times \pi_{\nu[l]}(a^k_l) \end{array}\right)$$

That is, each of the respective blocks are multiplied and everything else remains 0. Now recall that we only defined the characteristic identity on an irreducible representation, and the representation $V(\mu)$ is not in general irreducible for $\mathfrak{gl}_n$. However, because of the block diagonal structure of the matrix elements, and the multiplication rule above, it is clear that the matrix A will satisfy the analogous identity

$$\prod_{k=1}^{n}(A - \alpha_k) = 0$$

where we now interpret α_k as an operator on the direct sum $V'(\nu[1]) \oplus \cdots \oplus V'(\nu[l])$ which has the form

$$\alpha_k = \left(\begin{array}{c:c:c} \alpha(\nu[1])_k & 0 & 0 \\ \hdashline 0 & \ddots & 0 \\ \hdashline 0 & 0 & \alpha(\nu[l])_k \end{array}\right)$$

Here $\alpha(\nu[i])_k$ denotes the kth characteristic root of the irreducible $\mathfrak{gl}_n$ representation $V'(\nu[i])$ (which we interpret as either a number or a multiple of the $\dim(V'(\nu[i])) \times \dim(V'(\nu[i]))$ identity matrix, depending on context). Finally, note that when we refer to the $\mathfrak{gl}_n$ representation labels ν_k, we mean the operator with the form

$$\nu_k = \left(\begin{array}{c:c:c} \nu[1]_k & 0 & 0 \\ \hdashline 0 & \ddots & 0 \\ \hdashline 0 & 0 & \nu[l]_k \end{array}\right)$$

Because of "blockwise" characteristic identity satisfied by A, we may similarly define the block-diagonal $\mathfrak{gl}_n$ projection operators according to

$$P[r] = \prod_{i=1, i\neq r}^{n} \left(\frac{A - \alpha_i}{\alpha_r - \alpha_i} \right)$$

where this equation is to be understood "blockwise". Explicitly, the entries of the matrix $\prod_{i=1,i\neq r}^{n}(A - \alpha_i)$ will have the block structure described earlier, and we divide each of these blocks by the corresponding product of characteristic roots (i.e., we divide the jth block by the product $\prod_{i=1,i\neq r}^{n}(\alpha(\nu_j)_r - \alpha(\nu_j)_i)$. Then these block diagonal projection operators satisfy the properties of proposition 1 (because each of the blocks does). We then define the adjoint matrix $\overline{A} = -A^T$ and the adjoint $\mathfrak{gl}_n$ projection operators

$$\overline{P}[r] = \prod_{i=1, i\neq r}^{n} \left(\frac{\overline{A} - \overline{\alpha_i}}{\overline{\alpha_r} - \overline{\alpha_i}} \right),$$

With these conventions established, we now aim to evaluate the matrix elements of the generators a_n^{n+1} and a_{n+1}^n. To this end, we set $\psi^i := a_{n+1}^i$ and $\psi_i^\dagger := a_i^{n+1}$, for $1 \leq i \leq n$. Then ψ and $\psi^\dagger$ so defined transform as vector operators and contragredient vector operators respectively with respect to $\mathfrak{gl}_n$. To see this, note that

$$[a^i_j, \psi^k] = [a^i_j, a^k_{n+1}] = \delta^k_j a^i_{n+1} = \delta^k_j \psi^i$$

and

$$[a^i_j, \psi^\dagger_k] = [a^i_j, a^{n+1}_k] = -\delta^i_k a^{n+1}_j = -\delta^i_k \psi^\dagger_j$$

Hence the properties about vector operators established in proposition 3 apply to these operators. In particular, we have the decompositions

$$\begin{aligned} a^i_{n+1} &= \psi^i = \sum_{r=1}^{n} \psi[r]^i \\ a^{n+1}_i &= \psi^\dagger_i = \sum_{r=1}^{n} \psi^\dagger[r]_i, \end{aligned}$$

where

$$\begin{aligned} \psi[r] &= P[r]\psi \\ \psi^\dagger[r] &= \overline{P}[r]\psi \end{aligned}$$

Let us denote the $(n+1, n+1)$th entry of $Q[r]$ by C_r. Note that C_r is a $\mathfrak{gl}_n$ invariant; indeed, if $p(x)$ is any abstract polynomial over $\mathbb{C}$ then the $(n+1, n+1)$th entry of $p(B)$ will be a $\mathfrak{gl}_n$ invariant. The following lemma allows us to evaluate (the matrix element of) C_r in terms of the $\mathfrak{gl}_{n+1}$ and $\mathfrak{gl}_n$ characteristic roots:

Lemma 4: We have that

$$\begin{aligned} Q[r]^i_{n+1} &= \sum_{k=1}^{n} \psi[k]^i (\beta_r - \alpha_k - 1)^{-1} C_r \\ Q[r]^{n+1}_i &= \sum_{k=1}^{n} C_r (\beta_r - \alpha_k - 1)^{-1} \psi^\dagger[k]_i \end{aligned} \tag{3.15}$$

Proof. Note that from the characteristic identity we have that

$$\begin{aligned} (B - \beta_r)Q[r] &= (B - \beta_r) \prod_{k=1, k\neq r}^{n+1} \left(\frac{B - \beta_k}{\beta_r - \beta_k} \right) \\ &= \prod_{k=1}^{n+1} \left(\frac{B - \beta_k}{\beta_r - \beta_k} \right) \\ &= 0 \end{aligned}$$

So

$$BQ[r] = \beta_k Q[r]$$

Now taking the $(i, n+1)$th entry of this equation (where $1 \leq i \leq n$ we have that

$$\sum_{l=1}^{n+1} a_l^i Q[r]_{n+1}^l = \beta_r Q[r]_{n+1}^i$$

(recall that $B_l^i = a_l^i$ and $\beta_l^i = 0$ unless $i = l$). Rearranging this we find

$$\begin{aligned} a_{n+1}^i Q[r]_{n+1}^{n+1} &= \beta_r Q[r]_{n+1}^i - \sum_{l=1}^{n} a_l^i Q[r]_{n+1}^l \\ &= \sum_{l=1}^{n} (\beta_r)_l^i Q[r]_{n+1}^l - \sum_{l=1}^{n} a_l^i Q[r]_{n+1}^l \\ &= \sum_{l=1}^{n} ((\beta_r)_l^i - a_l^i) Q[r]_{n+1}^l \\ &= \sum_{l=1}^{n} (\beta_r - A)_l^i Q[r]_{n+1}^l \end{aligned}$$

We now multiply both sides of this equation by $((\beta_r - A)^{-1})_i^j$ for some $1 \leq j \leq n$, and sum from $i = 1$ to n. Doing so gives:

$$\begin{aligned} \sum_{i=1}^{n} ((\beta_r - A)^{-1})_i^j a_{n+1}^i Q[r]_{n+1}^{n+1} &= \sum_{i=1}^{n} ((\beta_r - A)^{-1})_i^j \sum_{l=1}^{n} (\beta_r - A)_l^i Q[r]_{n+1}^l \\ &= \sum_{i,l=1}^{n} ((\beta_r - A)^{-1})_i^j (\beta_r - A)_l^i Q[r]_{n+1}^l \\ &= \sum_{l=1}^{n} \delta_l^j Q[r]_{n+1}^l \\ &= Q[r]_{n+1}^j \end{aligned} \tag{3.16}$$

Now, since

$$(\beta_r - A) = \sum_{k=1}^{n} (\beta_r - \alpha_k) P[k]$$

and

$$I = \sum_{k=1}^{n} P[k]$$

by proposition 3, it follows that

$$\begin{aligned} I &= (\beta_r - A)^{-1}(\beta_r - A) \\ &= (\beta_r - A)^{-1} \left(\sum_{k=1}^{n} (\beta_r - \alpha_k) P[k] \right) \\ &= \sum_{k=1}^{n} P[k], \end{aligned}$$

and, since $P[k]^2 = P[k]$, we must have

$$(\beta_r - A)^{-1} = \sum_{k=1}^{n} (\beta_r - \alpha_k)^{-1} P[k].$$

So relabelling (3.16) (replacing i with j) and using the above, we find

$$\begin{aligned}
Q[r]^i_{n+1} &= \sum_{j=1}^{n} ((\beta_r - A)^{-1})^i_j a^j_{n+1} C_r \\
&= \sum_{k=1}^{n} \sum_{j=1}^{n} (\beta_r - \alpha_k)^{-1} P[k]^i_j a^j_{n+1} C_r \\
&= \sum_{k=1}^{n} (\beta_r - \alpha_k)^{-1} \psi[k]^i C_r \\
&= \sum_{k=1}^{n} \psi[k]^i (\beta_r - \alpha_k - 1)^{-1} C_r,
\end{aligned}$$

which is the first equation in (3.15). Note that we here we have used the fact that

$$(\beta_r - \alpha_k)^{-1} \psi[k]^i = \psi[k]^i (\beta_r - \alpha_k - 1)^{-1}$$

On the other hand, if we start with

$$Q[r]B = Q[r]\beta_r$$

and take the $(n+1, i)$th entry of this equation, then by a completely analogous procedure we find that

$$Q[r]^{n+1}_i = \sum_{k=1}^{n} C_r (\beta_r - \alpha_k - 1)^{-1} \psi^\dagger[k]_i$$

□

As a result of this, we have the following:

Corollary 5

$$C_r = \prod_{k=1}^{n+1} (\beta_r - \beta_k) \prod_{l=1}^{n} (\beta_r - \alpha_l - 1) \tag{3.17}$$

Proof. First recall that

$$\sum_{r=1}^{n+1} Q[r]^i_{n+1} = \delta^i_{n+1} = 0 \text{ for } 1 \le i \le n$$

So from (3.15) we have that

$$\begin{aligned}\sum_{r=1}^{n+1} Q[r]^i_{n+1} &= \sum_{r=1}^{n+1}\sum_{k=1}^{n} \psi[k]^i(\beta_r - \alpha_k - 1)^{-1}C_r \\ &= \sum_{k=1}^{n} \psi[k]^i \sum_{r=1}^{n+1}(\beta_r - \alpha_k - 1)^{-1}C_r \\ &= 0,\end{aligned}$$

and since the $\psi[k]$ are linearly independent we must have

$$\sum_{r=1}^{n+1}(\beta_r - \alpha_k - 1)^{-1}C_r = 0 \text{ for } k = 1, \ldots, n$$

Furthermore we know that

$$\sum_{r=1}^{n+1} C_r = I$$

where here I is the $\dim(V(\mu)) \times \dim(V(\mu))$ identity matrix. This gives us $n+1$ linear equations for the $n+1$ unknowns C_r, and so we can solve these uniquely for C_r. In particular, we can write these equations compactly in the form

$$M\mathbf{x} = \mathbf{b}$$

where

$$\mathbf{x} = \begin{pmatrix} C_1 \\ \vdots \\ C_r \end{pmatrix},$$

$$\mathbf{b} = \begin{pmatrix} 0 \\ \vdots \\ I \end{pmatrix}, \text{ and}$$

$$M = \begin{pmatrix} (\beta_1 - \alpha_1 - 1)^{-1} & (\beta_2 - \alpha_1 - 1)^{-1} & \ldots & (\beta_n - \alpha_1 - 1)^{-1} \\ (\beta_1 - \alpha_2 - 1)^{-1} & (\beta_2 - \alpha_2 - 1)^{-1} & \ldots & (\beta_n - \alpha_2 - 1)^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ I & I & \ldots & I \end{pmatrix},$$

We can use Cramer's rule to solve this system of equations. The solutions are found to be

$$C_r = \prod_{l=1, l\neq r}^{n+1}(\beta_r - \beta_l)^{-1}\prod_{k=1}^{n}(\beta_r - \alpha_k - 1)$$

$\square$

We note in passing that if $p(x)$ is any polynomial over $\mathbb{C}$, then by means of the above formula we may evaluate any element of the form $p(B)^{n+1}_{n+1}$, since $p(B) = \sum_{r=1}^{n+1} \beta_r Q[r]$. Also, if we start with the adjoint identity, then by following an analogous procedure to that above, we find that

$$\overline{C}_r = \prod_{l=1, l\neq r}^{n+1} (\beta_r - \beta_l)^{-1} \prod_{k=1}^{n} (\beta_r - \alpha_k) \tag{3.18}$$

We may similarly express the matrix elements of $P[r]^n_n$ and $\overline{P}[r]^n_n$ in terms of the $\mathfrak{gl}_n$ and $\mathfrak{gl}_{n-1}$ characteristic roots. This allows us to determine the fundamental Wigner coefficients, by proposition 2.

We need one more lemma:

Lemma 6

$$\begin{aligned} Q[k]^i_{n+1}(C_k)^{-1} Q[k]^{n+1}_j &= Q[k]^i_j \\ \overline{Q}[k]^{n+1}_i(\overline{C}_k)^{-1} \overline{Q}[k]^j_{n+1} &= \overline{Q}[k]^j_i \end{aligned} \tag{3.19}$$

Proof. This essentially follows from the commutation relations. See Gould[16] □

The following formulas follow from lemmas 4 and 6

Corollary 7

$$\begin{aligned} \psi[r]^i \psi^\dagger[r]_j &= \overline{M}_r P[r]^i_j \\ \psi^\dagger[r]_j \psi[r]^i &= M_r \overline{P}[r]^i_j \end{aligned} \tag{3.20}$$

where

$$\overline{M}_r = (-1)^n \prod_{k=1}^{n+1} (\beta_k - \alpha_r) \prod_{l=1, l\neq r}^{n} (\alpha_r - \alpha_l - 1)^{-1}, \text{ and}$$

$$M_r = (-1)^n \prod_{k=1}^{n+1} (\beta_k - \alpha_r - 1) \prod_{l=1, l\neq r}^{n} (\alpha_r - \alpha_l + 1)^{-1}$$

Proof. In order to show this, we would like to invert (3.15) to isolate the terms $\psi[k]^i$. To this end, consider, for each $l = 1, \ldots, n$, the solutions to the $(n+1)$ equations

$$\sum_{r=1}^{n+1} \gamma_{lr} C_r (\beta_r - \alpha_k - 1)^{-1} = \delta^k_l$$

$$\sum_{r=1}^{n+1} \gamma_{lr} C_r = 0$$

Again, since for each l we have $n+1$ equations in the $n+1$ unknowns $\gamma_{l1}, \ldots, \gamma_{l,n+1}$ we may solve these equations uniquely by Cramer's rule. This gives the $n(n+1)$ invariants

$$\gamma_{lr} = \gamma_l (\beta_r - \alpha_l - 1)^{-1}, \tag{3.21}$$

where

$$\gamma_l = (-1)^n \prod_{m=1}^{n+1} (\beta_m - \alpha_l - 1) \prod_{p=1, pr \neq l}^{n} (\alpha_l - \alpha_p)^{-1}$$

Now, multiplying each side of (3.15) by γ_{lr} and summing from $r = 1$ to $r = n+1$, we get

$$\begin{aligned}
\sum_{r=1}^{n+1} Q[r]^i_{n+1} \gamma_{lr} &= \sum_{k=1}^{n} \psi[k]^i \sum_{r=1}^{n+1} \gamma_{lr} C_r (\beta_r - \alpha_k - 1)^{-1} \\
&= \sum_{k=1}^{n} \psi[k]^i \delta^k_l \\
&= \psi[l]^i
\end{aligned}$$

For convenience we relabel this equation:

$$\psi_[r]^i = \sum_{k=1}^{n+1} Q[k]^i_{n+1} \gamma_{rk}$$

Now, recall that $\psi^\dagger[r] = \psi^\dagger P[r]$. Using this and the fact that $P[r]^2 = P[r]$ we find that

$$\begin{aligned}
\psi^\dagger[r]_i \psi_[r]^i &= \psi^\dagger_i \psi_[r]^i \\
&= a_i^{n+1} \psi_[r]^i \\
&= a_i^{n+1} \sum_{k=1}^{n+1} Q[k]^i_{n+1} \gamma_{rk}
\end{aligned}$$

Summing from $i = 1$ to $i = n$, and recalling that $BQ[k] = \beta_k Q[k]$, we have:

$$\begin{aligned}
\sum_{i=1}^{n} \psi^\dagger[r]_i \psi_[r]^i &= \sum_{k=1}^{n+1} \sum_{i=1}^{n} a_i^{n+1} Q[k]^i_{n+1} \gamma_{rk} \\
&= \sum_{k=1}^{n+1} \sum_{i=1}^{n} (a_i^{n+1} Q[k]^i (\beta_k - \alpha_r - 1)^{-1}) \gamma_r \\
&= \sum_{k=1}^{n+1} C_k \gamma_r \\
&= \gamma_r
\end{aligned}$$

Similarly if start with the formula from lemma 6 and use the definition of C_k we arrive at (3.20). □

By means of these formulae we may immediately write down the matrix elements of the generators a_n^{n+1} and a_{n+1}^n. Explicitly, let

$$\left| \begin{array}{cccc} \mu_1 & \mu_2 & \dots & \mu_n \\ \vdots & & \iddots & \\ \nu & & & \end{array} \right\rangle$$

be one of the basis states of the module $V(\mu)$. (Note that we identify the basis states with the corresponding Gelfand-Tsetlin pattern). For convenience let us denote this state by $|\mu_{j,k}\rangle$. Then acting on this state a^n_{n+1} has the effect

$$\begin{aligned} a^n_{n+1} |\mu_{j,k}\rangle &= \sum_{r=1}^{n} \psi[r]^n |\mu_{j,k}\rangle \\ &= \sum_{r=1}^{n} M^n_r |\mu_{j,k} + \Delta_{n,r}\rangle , \end{aligned}$$

where $|\mu_{j,k} + \Delta_{n,r}\rangle$ denotes the state obtained from $|\mu_{j,k}\rangle$ by increasing the label $\mu_{n,r}$ of the subalgebra $\mathfrak{gl}_n$ by one. Note that because of the betweenness conditions, many of the terms on the right hand-side may be zero; if all of them are, then the corresponding column of a^n_{n+1} is zero. In any event, taking the inner product of this equation with the state $|\mu_{j,k} + \Delta_{n,l}\rangle$ (or equivalently, applying the bra $\langle\mu_{j,k} + \Delta_{n,l}|$), we get

$$\langle\mu_{j,k} + \Delta_{n,l}| a^n_{n+1} |\mu_{j,k}\rangle = M^n_l$$

The elements M^n_l are given by

$$M^n_l = \langle\mu_{j,k}| M_r \overline{P}[r]^n_n |\mu_{j,k}\rangle$$

To see this, note that

$$\begin{aligned} \langle\mu_{j,k}| M_r \overline{P}[r]^n_n |\mu_{j,k}\rangle &= \langle\mu_{j,k}| \psi^\dagger[r]_n \psi[r]^n |\mu_{j,k}\rangle \\ &= \langle\mu_{j,k}| \psi^\dagger[r]_n |\mu_{j,k} + \Delta_{n,r}\rangle \langle\mu_{j,k} + \Delta_{n,r}| \psi[r]^n |\mu_{j,k}\rangle \\ &= (\langle\mu_{j,k} + \Delta_{n,r}| \psi[r]^n |\mu_{j,k}\rangle)^2, \end{aligned}$$

since $\langle\mu_{j,k} + \Delta_{n,r}| \psi[r]^n |\mu_{j,k}\rangle = \langle\mu_{j,k}| \psi^\dagger[r]_n |\mu_{j,k} + \Delta_{n,r}\rangle$

Now we establish a relationship between the $\mathfrak{gl}_{n+1}$ and $\mathfrak{gl}_n$ projection operators which will allow us to determine the matrix elements of other generators.

Proposition 8

$$\begin{aligned} \sum_{i,l=1}^{n} P[r]^i_l Q[k]^l_m P[r]^m_j &= C_k \overline{M}_r (\beta_k - \alpha_r - 1)^{-1} (\beta_k - \alpha_r)^{-1} P[r]^i_j \\ \sum_{i,l=1}^{n} \overline{P}[r]^l_i \overline{Q}[k]^m_l \overline{P}[r]^j_m &= \overline{C}_k M_r (\beta_k - \alpha_r - 1)^{-1} (\beta_k - \alpha_r)^{-1} \overline{P}[r]^i_j \end{aligned} \tag{3.22}$$

Proof. Essentially this follows from lemmas 4 and 6. We begin by applying the projection operators to both sides of the first equation in (3.19). □

3.5 Simultaneous shifts

We will now attempt to generalise the formulae obtained in the previous section by considering the full subalgebra chain

$$\mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \dots \mathfrak{gl}_n \subset \mathfrak{gl}_{n+1}$$

This will allow us to determine the matrix elements of all $\mathfrak{gl}_{n+1}$ generators. First we need to establish some notation. Note that most of what follows is just a way of expressing the formulae in section 3.4 in more generality. So, suppose we are working on a fixed finite-dimensional irreducible $\mathfrak{gl}_{n+1}$ representation with highest weight

$$\lambda(n+1) = (\lambda_{n+1,1}, \dots, \lambda_{n+1,n+1})$$

Let us denote by $A(n+1)$ the $(n+1)\times(n+1)$ matrix with the $\mathfrak{gl}_{n+1}$ generator a^i_j in the (i,j)th position. So

$$A(n+1) = \begin{pmatrix} a^1_1 & a^1_2 & \dots & a^1_{n+1} \\ a^2_1 & a^2_2 & \dots & a^2_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a^{n+1}_1 & a^{n+1}_2 & \dots & a^{n+1}_{n+1} \end{pmatrix}$$

Then as usual on the representation $V(\lambda(n+1))$ this matrix satisfies the characteristic identity

$$\prod_{r=1}^{n+1}(A(n+1) - \alpha_{n+1,r}) = 0,$$

where $\alpha_{n+1,r}$ are the $\mathfrak{gl}_{n+1}$ characteristic roots.

In general, for the subalgebra $\mathfrak{gl}_m$ $(1 \le m \le n)$ we denote by $A(m)$ the $m \times m$ matrix with the $\mathfrak{gl}_m$ generator a^i_j in the (i,j)th position. We also denote by

$$\lambda(m) = (\lambda_{m,1}, \dots, \lambda_{m,m})$$

the "highest weight" of the restriction of $V(\lambda(n+1))$ to the subalgebra $\mathfrak{gl}_m$. In general, we have the branching rule

$$V(\lambda(n+1)) \downarrow_{\mathfrak{gl}_m} \cong V'(\lambda(m)[1]) \oplus V'(\lambda(m)[2]) \oplus \dots \oplus V'(\lambda(m)[l])$$

where the direct sum is taken over l irreducible $\mathfrak{gl}_m$ modules with highest weights $\lambda(m)[1], \dots, \lambda(m)[l]$ respectively. So, in accordance with the convention established in the previous section, when we write $\lambda_{m,k}$ we really mean an operator on the direct sum $V'(\lambda(m)[1]) \oplus V'(\lambda(m)[2]) \oplus \dots \oplus V'(\lambda(m)[l])$ which has the block diagonal form

$$\lambda_{m,k} = \left(\begin{array}{c:c:c} \lambda(m)[1]_k & 0 & 0 \\ \hdashline 0 & \ddots & 0 \\ \hdashline 0 & 0 & \lambda(m)[l]_k \end{array}\right)$$

Then if we define the $\mathfrak{gl}_m$ characteristic roots $\alpha_{m,r}$ $(1 \le r \le m)$ according to

$$\alpha_{m,r} = \lambda_{m,r} + m - r$$

(where m and r denote multiples of the identity), the matrix $A(m)$ will satisfy the identity

$$\prod_{r=1}^{m} (A(m) - \alpha_{m,r}) = 0$$

Similarly, we define the adjoint matrix $\overline{A}(m) = -A(m)^T$ which satisfies the adjoint identity

$$\prod_{r=1}^{m} (\overline{A}(m) - \overline{\alpha}_{m,r}) = 0,$$

where $\overline{\alpha}_{m,r} = r - \lambda_{m,r} - 1$. Hence we define the $\mathfrak{gl}_m$ projection operators (following Gould's notation[17]) according to

$$P\binom{m}{r} = \prod_{k=1,k\neq r}^{m} \left(\frac{A(m) - \alpha_{m,k}}{\alpha_{m,r} - \alpha_{m,k}}\right)$$
$$\overline{P}\binom{m}{r} = \prod_{k=1,k\neq r}^{m} \left(\frac{\overline{A}(m) - \overline{\alpha}_{m,k}}{\overline{\alpha}_{m,r} - \overline{\alpha}_{m,k}}\right)$$

Again, note that these formulae are to be interpreted "blockwise", as explained in section 3.4. Denoting the (m, m) entries of these projection operators by $C_{m,r}$ and $\overline{C}_{m,r}$ respectively we have that

$$\begin{aligned} C_{m,r} &= \prod_{k=1}^{m} (\alpha_{m,r} - \alpha_{m,k})^{-1} \prod_{l=1}^{m-1} (\alpha_{m,r} - \alpha_{m-1,l} - 1) \\ \overline{C}_{m,r} &= \prod_{k=1}^{m} (\alpha_{m,r} - \alpha_{m,k})^{-1} \prod_{l=1}^{m-1} (\alpha_{m,r} - \alpha_{m-1,l}) \end{aligned} \tag{3.23}$$

This is the generalisation of (3.17) to the subalgebra $\mathfrak{gl}_m$. The proof is essentially the same (just with the labels altered) so we omit it.

We define $\psi(m)$ to be the $\mathfrak{gl}_m$ vector operator with components a^i_{m+1} for $1 \leq i \leq m$. That is,

$$\psi(m)^i := a^i_{m+1}$$

Similarly, we define the $\mathfrak{gl}_m$ contragredient vector operator $\psi^\dagger(m)$ according to

$$\psi^\dagger(m)_i := a^{m+1}_i$$

as usual we may resolve these vector operators into sums of shift operators $\psi\begin{pmatrix} m \\ r \end{pmatrix}$ and $\psi^\dagger\begin{pmatrix} m \\ r \end{pmatrix}$ which are defined according to

$$\begin{aligned} \psi\begin{pmatrix} m \\ r \end{pmatrix} &= P\begin{pmatrix} m \\ r \end{pmatrix}\psi(m) = \psi(m)\overline{P}\begin{pmatrix} m \\ r \end{pmatrix} \\ \psi^\dagger\begin{pmatrix} m \\ r \end{pmatrix} &= \overline{P}\begin{pmatrix} m \\ r \end{pmatrix}\psi^\dagger(m) = \psi^\dagger(m)P\begin{pmatrix} m \\ r \end{pmatrix} \end{aligned} \tag{3.24}$$

Then we have that

$$\begin{aligned} \psi\begin{pmatrix} m \\ r \end{pmatrix}^i \psi^\dagger\begin{pmatrix} m \\ r \end{pmatrix}_j &= \overline{M}_{m,r}P\begin{pmatrix} m \\ r \end{pmatrix}^i_j \\ \psi^\dagger\begin{pmatrix} m \\ r \end{pmatrix}_i \psi\begin{pmatrix} m \\ r \end{pmatrix}^j &= M_{m,r}\overline{P}\begin{pmatrix} m \\ r \end{pmatrix}^i_j, \end{aligned} \tag{3.25}$$

where

$$M_{m,r} = (-1)^m \prod_{k=1}^{m+1})(\alpha_{m+1,k} - \alpha_{m,r} - 1) \prod_{l=1,l\neq r}^{m} (\alpha_{m,r} - \alpha_{m,l} + 1)^{-1}$$

and

$$\overline{M}_{m,r} = (-1)^m \prod_{k=1}^{m+1})(\alpha_{m+1,k} - \alpha_{m,r}) \prod_{l=1,l\neq r}^{m} (\alpha_{m,r} - \alpha_{m,l} - 1)^{-1}$$

Equation (3.25) is the generalisation of (3.20) to the subalgebra $\mathfrak{gl}_m$. Taking the (m,m) entry of this equation, we may immediately write down the matrix elements of the generators a^m_{m+1} and a^{m+1}_m (cf. the evaluation of the matrix elements of a^n_{n+1} and a^{n+1}_n in section 3.4). In general, we want to consider the generators of the form a^l_{m+1} and a^{m+1}_l (for $l = m, \ldots, 1$). These generators transform as components of vector operators and contragredient vector operators (respectively) with respect to $\mathfrak{gl}_m, \ldots, \mathfrak{gl}_{m-1}$, as can easily be seen from the commutation relations. As above, we know how to evaluate the matrix elements of the generators a^{m+1}_m and a^m_{m+1}, so now we consider generators of the form a^l_{m+1} for $l < m$.

3.5.1 The case $l = m-1$

Let us now consider the generator of the form a^{m-1}_{m+1}. As before, this transforms as a component of a vector operator with respect to $\mathfrak{gl}_m$, so we have the decomposition

$$a^{m-1}_{m+1} = \sum_{r=1}^{m} \psi \begin{pmatrix} m \\ r \end{pmatrix}^{m-1}, \tag{3.26}$$

where $\psi \begin{pmatrix} m \\ r \end{pmatrix}$ is given by (3.24). Moreover, a^{m-1}_{m+1} transforms as a component of a vector operator with respect to $\mathfrak{gl}_{m-1}$, and hence so too does each $\psi \begin{pmatrix} m \\ r \end{pmatrix}^{m-1}$. To see this:

$$[a^i_j, a^{m-1}_{m+1}] = \delta^{m-1}_j a^i_{m+1}$$

and so, substituting (3.26) into the above we find

$$\left[a^i_j, \sum_{r=1}^{m} \psi \begin{pmatrix} m \\ r \end{pmatrix}^{m-1}\right] = \delta^{m-1}_j \sum_{r=1}^{m} \psi \begin{pmatrix} m \\ r \end{pmatrix}^{i},$$

which implies that

$$\sum_{r=1}^{m} \left[\alpha, \psi \begin{pmatrix} m \\ r \end{pmatrix}^{m-1}\right] = \sum_{r=1}^{m} \delta^{m-1}_j \psi \begin{pmatrix} m \\ r \end{pmatrix}^{i},$$

Hence because of the linear independence of the operators $\psi^\dagger \begin{pmatrix} m \\ r \end{pmatrix}$ we must have

$$\left[\alpha^i_j, \psi \begin{pmatrix} m \\ r \end{pmatrix}^{m-1}\right] = \delta^{m-1}_j \psi \begin{pmatrix} m \\ r \end{pmatrix}^{i} \tag{3.27}$$

for each r. Therefore, we can decompose each $\psi \begin{pmatrix} m \\ r \end{pmatrix}$ into a sum of $\mathfrak{gl}_{m-1}$ shift operators:

$$\psi \begin{pmatrix} m \\ r \end{pmatrix}^{m-1} = \sum_{l=1}^{m-1} \psi \begin{pmatrix} m & m-1 \\ r & l \end{pmatrix}^{m-1} \tag{3.28}$$

where each $\psi \begin{pmatrix} m & m-1 \\ r & l \end{pmatrix}$ is now a vector operator with respect to $\mathfrak{o}_m$ *and* $\mathfrak{o}_{m-1}$. These operators are defined, for $1 \le i \le m-1$, by

$$\psi \begin{pmatrix} m & m-1 \\ r & l \end{pmatrix}^{i} = \sum_{j=1}^{m-1} P \begin{pmatrix} m-1 \\ l \end{pmatrix}^i_j \psi \begin{pmatrix} m \\ r \end{pmatrix}^{j},$$

where $P\begin{pmatrix} m-1 \\ l \end{pmatrix}$ is the $(m-1)\times(m-1)$ $\mathfrak{gl}_m$ projection operator, defined as usual by

$$P\begin{pmatrix} m-1 \\ l \end{pmatrix} = \prod_{k=1,k\neq l}^{m-1}\left(\frac{A(m-1)-\alpha_{m-1,k}}{\alpha_{m-1,l}-\alpha_{m-1,k}}\right)$$

This leads us to define $P\begin{pmatrix} m-1 & m \\ l & r \end{pmatrix}$ as the $(m-1)\times m$ matrix with entries

$$P\begin{pmatrix} m-1 & m \\ l & r \end{pmatrix}^i_j = \sum_{k=1}^{m-1}\overline{P}\begin{pmatrix} m-1 \\ l \end{pmatrix}^i_k \overline{P}\begin{pmatrix} m \\ r \end{pmatrix}^k_l \quad (1 \leq i \leq m-1, i \leq j \leq m)$$

Similarly, we define the $m\times(m-1)$ matrix $\overline{P}\begin{pmatrix} m & m-1 \\ r & l \end{pmatrix}$ according to

$$\overline{P}\begin{pmatrix} m & m-1 \\ r & l \end{pmatrix}^i_j = \sum_{k=1}^{m-1}\overline{P}\begin{pmatrix} m \\ r \end{pmatrix}^i_k \overline{P}\begin{pmatrix} m-1 \\ l \end{pmatrix}^k_j \quad (1 \leq i \leq m, i \leq j \leq m-1),$$

so that we have

$$\begin{aligned}\psi\begin{pmatrix} m & m-1 \\ r & l \end{pmatrix} &= P\begin{pmatrix} m-1 & m \\ l & r \end{pmatrix}\psi(m) \\ &= \psi^\dagger(m)\overline{P}\begin{pmatrix} m & m-1 \\ r & l \end{pmatrix}.\end{aligned}$$

Since each $\psi\begin{pmatrix} m & m-1 \\ r & l \end{pmatrix}$ is a vector operator with respect to $\mathfrak{gl}_m$ and $\mathfrak{gl}_{m-1}$, each one alters the representation labels of both $\mathfrak{gl}_{m-1}$ and $\mathfrak{gl}_{m-1}$ according to

$$\begin{aligned}\lambda_{m,i}\psi\begin{pmatrix} m & m-1 \\ r & l \end{pmatrix} &= \psi\begin{pmatrix} m & m-1 \\ r & l \end{pmatrix}(\lambda_{m,i}+\delta_{r,i}) \\ \lambda_{m-1,j}\psi\begin{pmatrix} m & m-1 \\ r & l \end{pmatrix} &= \psi\begin{pmatrix} m & m-1 \\ r & l \end{pmatrix}(\lambda_{m-1,j}+\delta_{l,j})\end{aligned} \tag{3.29}$$

Hence we refer to these operators as "simultaneous" shift operators.

3.5.2 The general case

In general, the generator a^l_{m+1} $(1 \leq l \leq m)$ transforms as a component of a vector operator with respect to $\mathfrak{gl}_m, \ldots, \mathfrak{gl}_l$, so we have the decomposition

$$a^l_{m+1} = \sum_{i_m=1}^{m}\cdots\sum_{i_l=1}^{l}\psi\begin{pmatrix} m & \ldots & l \\ i_m & \ldots & i_l \end{pmatrix}^l, \tag{3.30}$$

where each of the $m!/(l-1)!$ operators $\psi\begin{pmatrix} m & \dots & l \\ i_m & \dots & i_l \end{pmatrix}$ is a vector operator which simultaneously alters the representation labels of $\mathfrak{gl}_m, \dots, \mathfrak{gl}_l$ according to

$$\lambda_{p,j}\psi\begin{pmatrix} m & \dots & l \\ i_m & \dots & i_l \end{pmatrix} = \psi\begin{pmatrix} m & \dots & l \\ i_m & \dots & i_l \end{pmatrix}(\lambda_{p,j} + \delta_{i_p,j} + \delta_{k,p+1-i_p}) \tag{3.31}$$

In (3.31), $\lambda_{p,j}$ denotes the jth component of the $\mathfrak{gl}_p$ highest weight $\lambda(p)$ (where $l \le p \le m$, $1 \le j \le [p/2]$). If we define the $l \times m$ matrix $P\begin{pmatrix} l & \dots & m \\ i_l & \dots & i_m \end{pmatrix}$ according to

$$P\begin{pmatrix} l & \dots & m \\ i_l & \dots & i_m \end{pmatrix} = .. \tag{3.32}$$

and similarly define the $m \times l$ matrix $\overline{P}\begin{pmatrix} m & \dots & l \\ i_m & \dots & i_l \end{pmatrix}$ as

$$\overline{P}\begin{pmatrix} m & \dots & l \\ i_m & \dots & i_l \end{pmatrix}^{i}_{j} = ... \tag{3.33}$$

then the shift components can be defined by

$$\psi\begin{pmatrix} m & \dots & l \\ i_m & \dots & i_l \end{pmatrix} = P\begin{pmatrix} l & \dots & m \\ i_l & \dots & i_m \end{pmatrix}\psi(m) = \psi(m)\overline{P}\begin{pmatrix} m & \dots & l \\ i_m & \dots & i_l \end{pmatrix}$$

Also, note that by taking the Hermitian conjugate of (3.30), we arrive at

$$a_l^{m+1} = \sum_{i_m=1}^{m} \cdots \sum_{i_l=1}^{l} \psi^\dagger\begin{pmatrix} m & \dots & l \\ i_m & \dots & i_l \end{pmatrix}_l,$$

where each $\psi^\dagger\begin{pmatrix} m & \dots & l \\ i_m & \dots & i_l \end{pmatrix}$ is defined by

$$\psi^\dagger\begin{pmatrix} m & \dots & l \\ i_m & \dots & i_l \end{pmatrix} = \overline{P}\begin{pmatrix} l & \dots & m \\ i_l & \dots & i_m \end{pmatrix}\psi^\dagger(m) = \psi^\dagger(m)P\begin{pmatrix} m & \dots & l \\ i_m & \dots & i_l \end{pmatrix},$$

and $\overline{P}\begin{pmatrix} l & \dots & m \\ i_l & \dots & i_m \end{pmatrix}$ and $P\begin{pmatrix} m & \dots & l \\ i_m & \dots & i_l \end{pmatrix}$ are defined in the same way as before but with the order reversed. All of this notation is due to Gould[13,20] (it is perhaps worth noting that the author experimented with different notations, but decided that Gould's was the most elegant).

Finally, we obtain a generalisation of the result (3.25) for simultaneous shift operators. We have that

$$\psi^\dagger \begin{pmatrix} m & \dots & l \\ i_m & \dots & i_l \end{pmatrix}_l \psi \begin{pmatrix} m & \dots & l \\ i_m & \dots & i_l \end{pmatrix}^l = \sum_{i,j=1}^{m-1} \overline{P} \begin{pmatrix} l & \dots & m-1 \\ i_l & \dots & i_(m-1) \end{pmatrix}_i^l \psi^\dagger(m)_i \psi(m)^j$$

$$\times \overline{P} \begin{pmatrix} m-1 & \dots & l \\ i_(m-1) & \dots & i_l \end{pmatrix}_l^j$$

$$= \sum_{i,j=1}^{m-1} \overline{P} \begin{pmatrix} l & \dots & m-1 \\ i_l & \dots & i_(m-1) \end{pmatrix}_i^l M_{m,i_m} \overline{P} \begin{pmatrix} m \\ i_m \end{pmatrix}_j^i$$

$$\times \overline{P} \begin{pmatrix} m-1 & \dots & l \\ i_(m-1) & \dots & i_l \end{pmatrix}_l^j$$

and by repeatedly applying equation (3.22) we can evaluate these elements in terms of the characteristic roots

3.6 Matrix elements: general formulae

We will now determine explicit formulae for matrix elements of generators of the form a^l_{m+1} for $l = 1, \dots, m$. In the case $l = m$ the determination is completely analogous to the case α^n_{n+1} already considered (and indeed the two cases coincide when $m = n$). Given a basis state $|\lambda_{j,k}\rangle$ we have that

$$a^m_{m+1} |\lambda_{j,k}\rangle = \sum_{r=1}^{m} \psi \begin{pmatrix} m \\ r \end{pmatrix}^m |\lambda_{j,k}\rangle$$

$$= \sum_{r=1}^{m} M^m_r |\lambda_{j,k} + \Delta_{m,r}\rangle \,,$$

where again $|\lambda_{j,k} + \Delta_{m,r}\rangle$ denotes the basis state obtained from $|\lambda_{j,k}\rangle$ by adding one to the rth label $\lambda_{m,r}$ of the subalgebra $\mathfrak{gl}_m$. As before, the elements M^m_r are simply given by

$$M^m_r = \langle \lambda_{j,k}| M_{m,r} \overline{C}_{m,r} |\lambda_{j,k}\rangle \,,$$

and we have an explicit formula for $M_{m,r}\overline{C}_{m,r}$ by which we may evaluate this matrix element. We obtain the formulae for the matrix elements of a^l_{m+1} $(l < m)$ in essentially the same way, except that now we have a decomposition into simultaneous shifts, so the state vectors will be shifted on multiple levels. We have that

$$a^l_{m+1} |\lambda_{j,k}\rangle = \sum_{i_k} \psi \begin{pmatrix} m & \dots & l \\ i_m & \dots & i_l \end{pmatrix}^l |\lambda_{j,k}\rangle$$

$$= N^{m \cdots l}_{i_m \dots i_l} |\lambda_{j,k} + \Delta_{m,i_m} + \dots + \Delta_{l,i_l}\rangle$$

where here $|\lambda_{j,k} + \Delta_{m,i_m} + \cdots + \Delta_{l,i_l}\rangle$ denotes the state obtained from $|\lambda_{j,k}\rangle$ by increasing the representation labels $\lambda_{m,i_m}, \ldots, \lambda_{l,i_l}$ of the subalgebras $\mathfrak{gl}_m, \ldots, \mathfrak{gl}_l$ by one. The elements $N^{m\cdots l}_{i_m\cdots i_l}$ are then given by

$$N^{m\cdots l}_{i_m\cdots i_l} = \left(\langle\lambda_{j,k}|\, \psi^\dagger \begin{pmatrix} m & \ldots & l \\ i_m & \ldots & i_l \end{pmatrix}_l \psi \begin{pmatrix} m & \ldots & l \\ i_m & \ldots & i_l \end{pmatrix}^l |\lambda_{j,k}\rangle \right)^{1/2},$$

and we may calculate this explicitly using the formula (**??**).

3.7 Explicit calculation: $\mathfrak{gl}_3$

We will now go through an explicit calculation which will hopefully shed light on the preceding general method. In this case, we consider the subalgebra chain

$$\mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \mathfrak{gl}_3,$$

and we work with the irreducible $\mathfrak{gl}_3$ representation with highest weight $(1,1,0)$. This representation is six-dimensional, and the branching rule in this case is

$$V(1,1,0) \downarrow_{\mathfrak{gl}_2} \cong V'(1,1) \oplus V'(1,0)$$

The $\mathfrak{gl}_2$-module $V'(1,1)$ is one-dimensional and the $\mathfrak{gl}_2$-module $V'(1,0)$ is two-dimensional (it decomposes as a direct sum of two irreducible $\mathfrak{gl}_1$-modules). To make the connection with our general notation clear, note that in this case we have $\lambda(3) = (1,1,0) \equiv (\lambda_{3,1}, \lambda_{3,2}, \lambda_{3,3})$ and $\lambda(2)[1] = (1,1)$, $\lambda(2)[2] = (1,0)$. If we let $\lambda(2) = (\lambda_{2,1}, \lambda_{2,2})$ then $\lambda_{2,1}$ is an operator on $V'(1,1) \oplus V'(1,0)$ with the form

$$\lambda_{2,1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and similarly we have

$$\lambda_{2,2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Note that the lower right 2×2 block corresponds to the two-dimensional $\mathfrak{gl}_2$ module $V'(1,0)$ while the upper left 1×1 block corresponds to the one-dimensional $\mathfrak{gl}_2$ module $V'(1,1)$.

We denote the orthonormal basis states of $V(1,1,0)$ by the corresponding Gelfand-Tsetlin patterns. So, let v_1 be the highest weight state of $V(1,1,0)$. Then

$$v_1 \equiv \left| \begin{matrix} 1 & 1 & 0 \\ 1 & 1 & \\ 1 & & \end{matrix} \right\rangle \text{ (assume normalised)}$$

The other orthonormal basis states are then given by

$$v_2 = a^2_1 v_1 \equiv \left| \begin{array}{lll} 1 & 1 & 0 \\ 1 & 0 & \\ 1 & & \end{array} \right\rangle$$

and

$$v_3 = a^3_2 a^2_1 v_1 \equiv \left| \begin{array}{lll} 1 & 1 & 0 \\ 1 & 0 & \\ 0 & & \end{array} \right\rangle$$

Then we may easily verify that $\langle v_i | v_j \rangle = \delta^i_j$. In this case, it is relatively straightforward to work out the matrix elements of the generators in this basis explicitly by considering the action of each generator on the basis elements and using the commutation relations. We find that

$$a^1_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, a^1_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, a^1_3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$a^2_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, a^2_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, a^2_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$a^3_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, a^3_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, a^3_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Because we already have the matrix elements, everything that follows in this section is essentially redundant. The main purpose is simply to demonstrate what all of the objects defined in section 3.4 actually look like. Note that each of the $\mathfrak{gl}_2$ generators $(a^1_1, a^1_2, a^2_1, a^2_2)$ has a block diagonal form corresponding to the branching rule: the top left entry is the matrix of each generator in the one-dimensional $\mathfrak{gl}_2$ representation $V'(1,1)$ and the 2×2 block in the bottom right is the matrix of each generator in the two-dimensional $\mathfrak{gl}_2$ representation $V'(1,0)$. Following our general notation, we denote by $A(3)$ the matrix of $\mathfrak{gl}_3$ generators. So

$$A(3) = \begin{pmatrix} a^1_1 & a^1_2 & a^1_3 \\ a^2_1 & a^2_2 & a^2_3 \\ a^3_1 & a^3_2 & a^3_3 \end{pmatrix}$$

$$= \begin{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{pmatrix}$$

As before, we can think of $\alpha_{2,1}$ and $\alpha_{2,2}$ as operators that represent the characteristic roots of all of the $\mathfrak{gl}_2$ modules in the branching rule at once. So we can write

$$\alpha_{2,1} = \begin{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \end{pmatrix}$$

$$\alpha_{2,2} = \begin{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{pmatrix},$$

and note that acting on the module $V(1,1,0)$ these will just reduce to numbers. Considering the subalgebra $\mathfrak{gl}_2 \subset \mathfrak{gl}_3$ we have the matrix $A(2)$ of $\mathfrak{gl}_2$ generators given by

$$\begin{aligned} A(2) &= \begin{pmatrix} a^1_1 & a^1_2 \\ a^2_1 & a^2_2 \end{pmatrix} \\ &= \begin{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{pmatrix} \end{aligned}$$

Then $A(2)$ satisfies the characteristic identity

$$(A(2) - \alpha_{2,1})(A(2) - \alpha_{2,2}) = 0,$$

as may be verified explicitly. (Note that in this case we extend $\alpha_{2,1}$ and $\alpha_{2,1}$ to 6×6 matrices). We then define the $\mathfrak{gl}_2$ projection operators according to

$$\begin{aligned} P\begin{pmatrix} 2 \\ 1 \end{pmatrix} &= \left(\frac{A(2) - \alpha_{2,2}}{\alpha_{2,1} - \alpha_{2,2}} \right) \\ &= \begin{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ th0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \end{pmatrix}, \end{aligned}$$

and

$$P\binom{2}{2} = \left(\frac{A(2)-\alpha_{2,1}}{\alpha_{2,2}-\alpha_{2,1}}\right)$$

$$= \begin{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \end{pmatrix}$$

Again, note the block diagonal form of each element. Also note that we have the $\mathfrak{gl}_2$ shift operator

$$\psi(2) = \begin{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{pmatrix}$$

By means of such formulae we may explicitly verify the results in the preceding sections.

Chapter 4

Matrix elements of the $\mathfrak{o}_n$ generators

4.1 Introduction

We now turn our attention to the orthogonal Lie algebra $\mathfrak{o}_n$ over $\mathbb{C}$, defined by

$$\mathfrak{o}_n = \{x \in M_{n\times n}(\mathbb{C}) : x^T + x = 0\}$$

This Lie algebra has dimension $\frac{1}{2}n(n-1)$ and is spanned by n^2 generators α^i_j $(i, j = 1, \dots, n)$ satisfying the commutation relations

$$[\alpha^i_j, \alpha^k_l] = \delta^k_j \alpha^i_l - \delta^i_l \alpha^k_j - \delta\alpha + \delta\alpha \tag{4.1}$$

and the conditions

$$\alpha^i_j = -\alpha^j_i,\ (\alpha^i_j)^\dagger = \alpha^i_j$$

Conversely, any Lie algebra with generators satisfying (4.1) will be isomorphic to $\mathfrak{o}_n$. Note that we can write the $\mathfrak{o}_n$ generators in terms of the $\mathfrak{gl}_n$ generators a^i_j via

$$\alpha^i_j = a^i_j - a^j_i$$

4.2 Characteristic identities and projection operators

Following the $\mathfrak{gl}_n$ procedure, we consider a fixed finite-dimensional irreducible representation of $\mathfrak{o}_n$:

$$\pi_\lambda : \mathfrak{o}_n \to \mathfrak{gl}(V)$$

Such a representation is classified by its highest weight $\lambda = (\lambda_1, \dots, \lambda_h)$, where

$$h = \left\lfloor \frac{n}{2} \right\rfloor = \begin{cases} n/2, & \text{if } n \text{ is even} \\ (n-1)/2, & \text{if } n \text{ is odd} \end{cases}$$

and the components λ_i satisfy

$$\lambda_i - \lambda_{i+1} \in \mathbb{Z}_+ (i = 1, \ldots, n-1)$$

and

$$\begin{cases} 2\lambda_h \in \mathbb{Z}_+, & \text{if } n \text{ is odd} \\ \lambda_{h-1} + \lambda_h \in \mathbb{Z}_+ & \text{if } n \text{ is even} \end{cases}$$

We denote the corresponding $\mathfrak{o}_n$-module by $V(\lambda)$. As in the $\mathfrak{gl}_n$ case, we arrange the $\mathfrak{o}_n$ generators (in this representation) into an $n \times n$ matrix A whose (i,j)th entry is the generator α^i_j; that is,

$$A := \begin{pmatrix} \alpha^1_1 & \alpha^1_2 & \ldots & \alpha^1_n \\ \alpha^2_1 & \alpha^2_2 & \ldots & \alpha^2_n \\ \vdots & \vdots & \ddots & \vdots \\ \alpha^n_1 & \alpha^n_2 & \ldots & \alpha^n_n \end{pmatrix} \tag{4.2}$$

Note that for brevity we write α^i_j where we mean $\pi_\lambda(\alpha^i_j)$, as before. Also, because of the skew symmetry of the generators in this case, this matrix reduces to

$$A = \begin{pmatrix} \mathbf{0} & \alpha^1_2 & \ldots & \alpha^1_n \\ -\alpha^1_2 & \mathbf{0} & \ldots & \alpha^2_n \\ \vdots & \vdots & \ddots & \vdots \\ -\alpha^1_n & -\alpha^2_n & \ldots & \mathbf{0} \end{pmatrix},$$

where $\mathbf{0}$ denotes the $\dim V(\lambda) \times \dim V(\lambda)$ zero matrix. As before, the matrix A may be regarded as an operator on the space $V^* \otimes V(\lambda)$, where $V^* \cong V(0, \ldots, 0, -1)$ is the contragredient vector representation of $\mathfrak{o}_n$. To see this note that we can write A as

$$\begin{aligned} A &= \frac{1}{2} \sum_{i,j=1}^{n} (E^i_j - E^j_i) \otimes \alpha^i_j \\ &= -\frac{1}{2} \sum_{i,j=1}^{n} \pi^*(\alpha^j_i) \otimes \alpha^i_j \end{aligned}$$

Since A is an operator on $V^* \otimes V(\lambda)$, we consider the decomposition of this tensor product space into a direct sum of irreducible $\mathfrak{o}_n$ modules. The result is analogous to the $\mathfrak{gl}_n$ case, namely that

$$V^* \otimes V(\lambda) \cong V(\lambda - \Delta_1) \oplus \ldots V(\lambda - \Delta_n) \tag{4.3}$$

Here Δ_r (for $r = 1, \ldots, h$) denotes the weight with 1 in the rth position and zeroes elsewhere, and we define Δ_r for $r > h$ according to

$$\Delta_r = -\Delta_{n+1-r}$$

(Note that when $n = 2h + 1$ this implies $\Delta_{h+1} = 0$). Similarly, we define representation labels λ_r for $r > h$ according to

$$\lambda_r = 1 - \lambda_{n+1-r} \tag{4.4}$$

for $r = n - h + 1, \ldots, n$, with the extra label

$$\lambda_{h+1} = 1$$

if $n = 2h+1$. Note that the decomposition (4.3) has n terms in general because there are exactly n weights that occur in the contragredient vector representation, each of which occurs with multiplicity 1. However, some of the weights $\lambda - \Delta_i$ may not be dominant, in which case the corresponding module does not occur in the decomposition .

To prove the characteristic identity in the case of $\mathfrak{o}_n$, we would again like to express the matrix A in terms of invariants of $\mathfrak{o}_n$ (so that it will reduce to constant multiples of the identity when acting on an irreducible representation, by Schur's lemma). To this end, we consider the operator

$$A(z) = -\frac{1}{2}((\pi^* \otimes \pi_\lambda)(z) - \pi^*(z) \otimes I - I \otimes \pi_\lambda(z)),$$

where I denotes the identity matrix in the representations π_λ and π^* respectively, and z is an element of the centre of the universal enveloping algebra of $\mathfrak{o}_n$. Then if we take

$$z = \sigma_2 = \frac{1}{2}\sum_{i,j=1}^{n} \alpha^i_j \alpha^j_i,$$

we can write the matrix A as

$$A = -\frac{1}{2}((\pi^* \otimes \pi_\lambda)(\sigma_2) - \pi^*(\sigma_2) \otimes I - I \otimes \pi_\lambda(\sigma_2)) \tag{4.5}$$

To see this, just note that

$$\begin{aligned}
(\pi^* \otimes \pi_\lambda)(\sigma_2) &= (\pi^* \otimes \pi_\lambda)\left(\frac{1}{2}\sum_{i,j=1}^{n} a^i_j a^j_i\right) \\
&= \frac{1}{2}\sum_{i,j=1}^{n}((\pi^* \otimes \pi_\lambda)(\alpha^i_j))((\pi^* \otimes \pi_\lambda)(\alpha^j_i)) \\
&= \frac{1}{2}\sum_{i,j=1}^{n}(\pi^*(\alpha^i_j) \otimes I + I \otimes \pi_\lambda(a^i_j))(\pi^*(\alpha^j_i) \otimes I + I \otimes \pi_\lambda(a^j_i)) \\
&= \frac{1}{2}\sum_{i,j=1}^{n}(\pi^*(\alpha^i_j)\pi^*(\alpha^j_i) \otimes I + I \otimes \pi_\lambda(a^i_j)\pi_\lambda(a^j_i)) + \frac{1}{2}\sum_{i,j=1}^{n}(\pi^*(\alpha^i_j) \otimes \pi_\lambda(a^j_i) + \pi^*(\alpha^j_i) \otimes \pi_\lambda(a^i_j)) \\
&= (\pi^*(\sigma_2) \otimes I + I \otimes \pi_\lambda(\sigma_2)) - 2A,
\end{aligned}$$

and so (upon rearranging) we see that A indeed has the form (4.5). Hence on each irreducible module $V(\lambda-\Delta_r)$ in the decomposition (4.3) the matrix A reduces to

$$\begin{aligned}A(V(\lambda-\Delta_r)) &= -\frac{1}{2}(\pi_{\lambda-\Delta_r}(\sigma_2)-\pi^*(\sigma_2)v-\pi_\lambda)(V(\lambda-\Delta_r))\\ &= (r+1-n-\lambda_r)(V(\lambda-\Delta_r)),\end{aligned}$$

which implies the characteristic identity

$$\prod_{r=1}^{n}(A-\alpha_r)=0 \tag{4.6}$$

Similarly, if we define the adjoint matrix $\overline{A}=-A^T$ then we derive the adjoint characteristic identity

$$\prod_{r=1}^{n}(\overline{A}-\overline{\alpha}_r)=0, \tag{4.7}$$

where $\overline{\alpha}_r=\lambda_{n+1-r}+r-2$. In this case, we note that $\overline{A}$ is an operator on the tensor product space $V\otimes V(\lambda)$ where $V\equiv V(1,\dot{0})$ is the vector representation, and note that we may write $\overline{A}$ as

$$\overline{A}=-\frac{1}{2}((\pi_{(1,\dot{0})}\otimes\pi_\lambda)(\sigma_2)-\pi_{(1,\dot{0})}(\sigma_2)\otimes I-I\otimes\pi_\lambda(\sigma_2))$$

With these identities in mind, we define the $\mathfrak{o}_n$ projection operators according to

$$\begin{aligned}P[r] &= \prod_{l=1,l\neq r}^{n}\left(\frac{A-\alpha_l}{\alpha_r-\alpha_l}\right)\\ \overline{P}[r] &= \prod_{l=1,l\neq r}^{n}\left(\frac{\overline{A}-\overline{\alpha}_l}{\overline{\alpha}_r-\overline{\alpha}_l}\right)\end{aligned} \tag{4.8}$$

These operators project onto the rth component in the decomposition of the tensor products $V^*\otimes V(\lambda)$ and $V\otimes V(\lambda)$ respectively. To see this in the case of $P[r]$, note that

$$\begin{aligned}P[r](V^*\otimes V(\lambda)) &\cong P[r](V(\lambda-\Delta_1)\oplus\cdots\oplus V(\lambda-\Delta_n))\\ &= P[r]V(\lambda-\Delta_1)\oplus\cdots\oplus P[r]V(\lambda-\Delta_r)\oplus\cdots\oplus P[r]V(\lambda-\Delta_n)\\ &= 0.V(\lambda-\Delta_1)\oplus\cdots\oplus I.V(\lambda-\Delta_r)\oplus\cdots\oplus 0.V(\lambda-\Delta_n)\\ &= V(\lambda-\Delta_r),\end{aligned}$$

since A takes the value α_l on each space $V(\lambda-\Delta_l)$. Similarly, $\overline{P}[r]$ projects onto the rth summand in the decomposition of the tensor product $V\otimes V(\lambda)$.

We will make repeated use of the following properties of projection operators:

Proposition 1

1. $\sum_{r=1}^{P}[r] = I$

2. $P[r]P[k] = \delta_k^r P[r]$

The proof is analogous to the $\mathfrak{gl}_n$ case.

4.3 Vector operators

The notion of an $\mathfrak{o}_n$ vector operator plays a fundamental role in all that follows. In accordance with the general definition in section 2.7, we define an $\mathfrak{o}_n$ vector operator as a vector

$$\psi = \begin{pmatrix} \psi^i \\ \vdots \\ \psi^n \end{pmatrix}$$

whose components are linear maps on $V(\lambda)$ which transform with respect to $\mathfrak{o}_n$ according to

$$[\alpha_j^i, \psi^k] = \delta_j^k \psi^i - \delta_i^k \psi^j \tag{4.9}$$

Similarly, we define an $\mathfrak{o}_n$ contragredient vector operator as a vector

$$\psi^\dagger = \begin{pmatrix} \psi_1^\dagger \\ \vdots \\ \psi_n^\dagger \end{pmatrix}$$

whose components $\psi_j^\dagger$ are operators on $V(\lambda)$ which transform according to

$$[\alpha_j^i, \psi_k^\dagger] = \delta_j^k \psi_i^\dagger - \delta_i^k \psi_j^\dagger \tag{4.10}$$

As in the case of $\mathfrak{gl}_n$, we may resolve these vector operators into shift components by means of the projection operators. We define the shift components of ψ and $\psi^\dagger$ respectively according to

$$\begin{aligned} \psi[r] &:= P[r]\psi = \psi\overline{P}[r] \\ \psi^\dagger[r] &:= \overline{P}[r]\psi^\dagger = \psi^\dagger P[r] \end{aligned} \tag{4.11}$$

These shift components satisfy analogous properties to those of the $\mathfrak{gl}_n$ shift operators, which we record in the following proposition.

Proposition 2 *We have that*

1. *The shift components provide a spectral decomposition of the vector operators ψ and $\psi^\dagger$ in the sense that*

$$\psi = \sum_{r=1}^{n} \psi[r]$$

and

$$\psi^\dagger = \sum_{r=1}^{n} \psi^\dagger[r]$$

2. *The (non-zero) shift components $\psi[r]$ and $\psi^\dagger[r]$ form linearly independent sets*
3. *The shift components have the following effect on the representation labels of $\mathfrak{o}_n$:*

$$\lambda_k \psi[r] = \psi[r](\lambda_k + \delta_k^r)$$
$$\lambda_k \psi^\dagger[r] = \psi^\dagger[r](\lambda_k - \delta_k^r)$$

The proof of this proposition is almost identical to the proof of the analogous proposition (3.3) in the $\mathfrak{gl}_n$ case, so we omit it.

4.4 Matrix elements: generators of the form a_n^{n+1} and a_{n+1}^n

Now consider the Lie algebra $\mathfrak{o}_{n+1}$ which has $(n+1)^2$ generators α_j^i, $1 \leq i, j \leq n+1$. The subalgebra generated by $\{\alpha_j^i \mid 1 \leq i, j \leq n\}$ is isomorphic to $\mathfrak{o}_n$; this is called the *canonical* embedding of $\mathfrak{o}_n$ in $\mathfrak{o}_{n+1}$. We consider a fixed finite-dimensional irreducible o_{n+1} representation $V(\mu)$ with highest weight $\mu = (\mu_1, \ldots, \mu_{h'})$ (where $h' = \lfloor (n+1)/2 \rfloor$). Let B denote the $(n+1)\times(n+1)$ matrix whose (i, j) entry is the $\mathfrak{o}_{n+1}$ generator α_j^i, i.e.

$$B = \begin{pmatrix} \alpha_1^1 & \alpha_2^1 & \ldots & \alpha_{n+1}^1 \\ \alpha_1^2 & \alpha_2^2 & \ldots & \alpha_{n+1}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{n+1} & \alpha_2^{n+1} & \ldots & \alpha_{n+1}^{n+1} \end{pmatrix}$$

It is clear that on the representation $V(\mu)$, B will satisfy the analogous characteristic identity to the one satisfied by A; that is, we have the identity

$$\prod_{r=1}^{n+1} (B - \beta_r) = 0$$

where β_r are the $\mathfrak{o}_{n+1}$ characteristic roots given by $\beta_r = \mu_r + n - 1 - r$ (and we define the labels μ_r for $r > h'$ analogously to (4.4)). Similarly, for the "adjoint" matrix $\overline{B}$ defined by $\overline{B} = -B^T$, we have the identity

$$\prod_{r=1}^{n+1} (\overline{B} - \overline{\beta}_r) = 0,$$

where $\overline{\beta}_r = \mu_{n+2-r} + r - 2$. We then define the $\mathfrak{o}_{n+1}$ projection operators according to

$$Q[r] = \prod_{l=1,l\neq r}^{n+1} \left(\frac{B - \beta_l}{\beta_r - \beta_l} \right)$$

$$\overline{Q}[r] = \prod_{l=1,l\neq r}^{n+1} \left(\frac{B - \beta_l}{\beta_r - \beta_l} \right)$$

Now consider the matrix A that is formed by taking the first n rows and n columns of the matrix B, i.e.

$$A = \begin{pmatrix} \alpha_1^1 & \alpha_2^1 & \dots & \alpha_n^1 \\ \alpha_1^2 & \alpha_2^2 & \dots & \alpha_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^n & \alpha_2^n & \dots & \alpha_n^n \end{pmatrix}$$

The entries of A are generators of $\mathfrak{o}_n$ as mentioned earlier, so because of the branching rule

$$V(\mu) \downarrow_{\mathfrak{o}_n} \cong V'(\nu[1]) \oplus \cdots \oplus V'(\nu[l]),$$

each of the entries in A has a block diagonal form (provided we are working in an appropriately ordered basis of $V(\mu)$). Therefore, because each block in the entries of A corresponds to an irreducible representation, A satisfies the "block-diagonal" characteristic identity

$$\prod_{k=1}^{n} (A - \alpha_k) = 0$$

Here as before we interpret α_k as an operator (matrix) which contains the kth characteristic roots of each of the modules $V'(\nu[1]), \dots, V'(\nu[l])$ occurring in the branching rule. Of course, when acting on a particular basis state of $V(\mu)$, the operator α_k just reduces to a scalar. By means of this "induced" $\mathfrak{o}_n$ identity we define the $\mathfrak{o}_n$ projection operators (for $1 \leq r \leq n$):

$$P[r] = \prod_{i=1,i\neq r}^{n} \left(\frac{A - \alpha_i}{\alpha_r - \alpha_i} \right)$$

Similarly, we define the adjoint matrix $\overline{A} = -A^t$ and the adjoint $\mathfrak{o}_n$ projection operators

$$\overline{P}[r] = \prod_{i=1,i\neq r}^{n} \left(\frac{A - \alpha_i}{\alpha_r - \alpha_i} \right)$$

Our aim now is to determine the matrix elements of the generators α_{n+1}^n and α_n^{n+1} (note that since $\alpha_{n+1}^n = -\alpha_n^{n+1}$, once we know the matrix elements of one of these, we know the matrix elements of both). Eventually we will generalise this procedure to determine the matrix elements of all the generators. To begin with we define the vectors

$$\psi := \begin{pmatrix} \alpha^1_{n+1} \\ \vdots \\ \alpha^n_{n+1} \end{pmatrix}$$

and

$$\psi^\dagger := \begin{pmatrix} \alpha^{n+1}_1 \\ \vdots \\ \alpha^{n+1}_n \end{pmatrix}$$

Then ψ an $\psi^\dagger$ transform as (respectively) vector operators and contragredient vector operators with respect to $\mathfrak{o}_n$. To see this note that, for $1 \leq i, j \leq n$,

$$\begin{aligned}
[\alpha^i_j, \psi^k] &= [\alpha^i_j, \alpha^k_{n+1}] \\
&= \delta^k_j \alpha^i_{n+1} - \delta^k_i \alpha^j_{n+1} \\
&= \delta^k_j \psi^i - \delta^k_i \psi^j,
\end{aligned}$$

and similarly

$$\begin{aligned}
[\alpha^i_j, \psi^\dagger_k] &= [\alpha^i_j, \alpha^{n+1}_k] \\
&= \delta^k_j \alpha^{n+1}_i - \delta^k_i \alpha^{n+1}_j \\
&= \delta^k_j \psi^\dagger_i - \delta^k_i \psi^\dagger_j,
\end{aligned}$$

Hence the properties established in proposition 4.2 apply to ψ and $\psi^\dagger$; in particular, we have that

$$\begin{aligned}
\alpha^i_{n+1} = \psi^i &= \sum_{r=1}^{n} \psi[r]^i \text{ and} \\
\alpha^{n+1}_i = \psi^\dagger &= \sum_{r=1}^{n} \psi^\dagger[r]_i,
\end{aligned}$$

where

$$\begin{aligned}
\psi[r] &:= P[r]\psi \\
\psi^\dagger[r] &:= \overline{P}[r]\psi
\end{aligned}$$

Let us denote the $(n+1, n+1)$th entry of the projection operator $Q[r]$ by C_k, and similarly denote the $(n+1, n+1)$th entry of $\overline{Q}[k]$ by $\overline{C}_k$. We will soon establish a formula for C_k and $\overline{C}_k$ in terms of the characteristic roots α_r, β_l, which will then lead to a formula for the elements $\psi[r]^n \psi^\dagger[r]_n$ in terms of these roots. First we need some lemmas.

Lemma 4.3 *We have that*

$$\begin{aligned} Q[k]^i_{n+1} &= \sum_{r=1}^{n} \psi[r]^i(\beta_k - \alpha_r - \eta_r)^{-1} C_k \text{ and} \\ Q[k]^{n+1}_i &= \sum_{r=1}^{n} (\beta_k - \alpha_r - \eta_r)^{-1} C_k \psi^\dagger[r]_i \end{aligned} \tag{4.12}$$

Proof. Note that the proof of this result is almost identical to the proof of the analogous result for $\mathfrak{gl}_n$ (lemma 3.4), so will only give the key details. As in the $\mathfrak{gl}_n$ case, one starts with the fact that

$$(B - \beta_k)Q[k] = 0$$

which follows immediately from the $\mathfrak{gl}_{n+1}$ characteristic identity and the definition of $Q[k]$. Hence

$$BQ[k] = \beta_k Q[k]$$

Taking the $(i, n+1)$th entry of this matrix equation (where $i \in \{1, \ldots, n\}$) we have:

$$\sum_{j=1}^{n+1} B^i_j Q[k]^j_{n+1} = \sum_{j=1}^{n+1} (\beta_k)^i_j Q[k]^j_{n+1}$$

Expanding the left-hand side (and using the definition of B) we have

$$\alpha^i_{n+1} Q[k]^{n+1}_{n+1} + \sum_{j=1}^{n} \alpha^i_j Q[k]^j_{n+1} = \sum_{j=1}^{n} (\beta_k)^i_j Q[k]^j_{n+1}$$

Note that $(\beta_k)^i_{n+1} = 0$ so this term can be omitted from the sum. Rearranging we get

$$\alpha^i_{n+1} C_k = (\beta_k - A)^i_j Q[k]^j_{n+1}$$

Inverting this equation we find

$$Q[k]^i_{n+1} = \sum_{j=1}^{n} [(\beta_k - A)^{-1}]^i_j \alpha^j_{n+1} C_k$$

Then we use the fact that we can express any polynomial in A in terms of the projection operators, and then use the decomposition of the vector operators ψ into shift operators. The argument is analogous to the $\mathfrak{gl}_n$ case.[13] □

Note that in the proof of this statement we needed the fact that

$$(\beta_k - \alpha_r)^{-1}\psi[r] = \psi[r](\beta_k - \alpha_r - \eta_r)^{-1},$$

where

$$\eta_r = \eta_{n+1-r} = \begin{cases} 1 \text{ if } n = 2h \\ 1 - \delta_{r,h+1} \text{ if } n = 2h+1 \end{cases} \tag{4.13}$$

By means of this lemma we may evaluate the $\mathfrak{o}_n$ invariants C_k and $\overline{C}_k$ explicitly. We have the following results:

Corollary 4.4

$$\begin{aligned} C_k &= \prod_{l\neq k}(\beta_k - \beta_l)^{-1} \prod_{r=1}^{n}(\beta_k - \alpha_r - \eta_r) \\ \overline{C}_k &= C_{n+2-k} \end{aligned} \tag{4.14}$$

Lemma 4.5

$$Q[k]^i_{n+1}(C_k)^{-1}Q[k]^{n+1}_j = Q[k]^i_j \tag{4.15}$$

Proposition 4.6

$$\begin{aligned} \psi[r]^i\psi^\dagger[r]_j &= \overline{M}_r P[r]^i_j \\ \psi^\dagger[r]_i\psi[r]^j &= M_r\overline{P}[r]^i_j, \end{aligned} \tag{4.16}$$

where

$$\overline{M}_r = (-1)^n \prod_{k=1}^{n+1}(\bar{\beta}_k - \bar{\alpha}_r)\prod_{l\neq r}(\bar{\alpha}_r - \bar{\alpha}_l - \eta_l - \delta_{l,n+1-r})^{-1}$$

$$M_r = (-1)^n \prod_{k=1}^{n+1}(\beta_k - \alpha_r)\prod_{l\neq r}(\alpha_r - \alpha_l - \eta_l - \delta_{l,n+1-r})^{-1}$$

and η_l is given by (4.13).

The proofs of these statements are very similar to the proofs of the analogous statements for $\mathfrak{gl}_n$ so we omit them (again, see Gould[13]). Note that the formulas (4.16) and their generalisations to the subalgebra chain $\mathfrak{o}_n \supset \mathfrak{o}_{n-1} \supset \cdots \supset \mathfrak{o}_2$ will be key in determining the matrix elements of the $\mathfrak{o}_n$ generators in the Gelfand-Tsetlin basis. In particular, taking the (n,n) entry of these equations we find

$$\begin{aligned} \psi[r]^n\psi^\dagger[r]_n &= \overline{M}_r P[r]^n_n \text{ and} \\ \psi^\dagger[r]^n\psi[r]_n &= M_r\overline{P}[r]^n_n \end{aligned}$$

Therefore, since we have expressions for $P[r]^n_n$, $\overline{P}[r]^n_n$, $\overline{M}_r$ and M_r in terms of the characteristic roots, we may evaluate the matrix elements of the $\mathfrak{o}_n$ generators α^{n+1}_n in terms of these roots. Again the procedure is analogous to the $\mathfrak{gl}_n$ case. We will carry it out in general in section 4.6. Finally we need the following:

Proposition 4.7

$$\sum_{l,m=1}^{n} P[r]^i_l Q[k]^l_k P[r]^m_j = \alpha\begin{pmatrix} n+1 & n \\ k & r\end{pmatrix} P[r]^i_j,$$

where

$$\begin{pmatrix} n+1 & n \\ k & r \end{pmatrix} = C_k\overline{M}_r(\beta_k-\alpha_r\eta_r)^{-1}(\beta_k-\alpha_r)^{-1}(\beta_k-\alpha_{n+1-r}-\eta_r)^{-1}(\beta_k-\alpha_{n+1-r}-2\eta_r)$$

Proof. Apply the projection operators to the identity

$$Q[k]^i_j = Q[k]^i_{n+1}(C_k)^{-1}Q[k]^{n+1}_j$$

(Again this calculation is analogous to the $\mathfrak{gl}_n$ case; for details see Gould[13]). $\square$

4.5 Simultaneous Shifts

In order to determine matrix elements of the other $\mathfrak{o}_n$ generators in the Gelfand-Tsetlin basis, we attempt to generalise equations (4.16) by proceeding down the canonical subalgebra chain

$$\mathfrak{o}_{n+1} \supset \mathfrak{o}_n \supset \mathfrak{o}_{n-1} \supset \cdots \supset \mathfrak{o}_2$$

First we establish some notation. It is important to note that we consider a fixed irreducible $\mathfrak{o}_{n+1}$ representation $V(\lambda(n+1))$ with highest weight

$$\lambda(n+1) = (\lambda_1, \ldots, \lambda_h)$$

. Everything that follows takes place in this representation. For the subalgebra $\mathfrak{o}_m$, $1 \leq m \leq n$, let $A(m)$ be the $m \times m$ matrix whose (i,j)th entry is the $\mathfrak{o}_m$ generator α^i_j $(i,j = 1,\ldots,m)$. That is,

$$A(m) = \begin{pmatrix} \alpha^1_1 & \alpha^1_2 & \ldots & \alpha^1_m \\ \alpha^2_1 & \alpha^2_2 & \ldots & \alpha^2_m \\ \vdots & \vdots & \ddots & \vdots \\ \alpha^m_1 & \alpha^m_2 & \ldots & \alpha^m_m \end{pmatrix}$$

Furthermore let

$$h_m = \left[\frac{m}{2}\right]$$

so that either $m = 2h$ or $m = 2h+1$. In general, we have the branching rule

$$V(\lambda(n+1))\downarrow_{\mathfrak{gl}_m} \cong V'(\lambda(m)[1]) \oplus V'(\lambda(m)[2]) \oplus \cdots \oplus V'(\lambda(m)[l]) \qquad (4.17)$$

where the direct sum is taken over irreducible $\mathfrak{o}_m$ modules with highest weights $\lambda(m;1),\ldots,\lambda(m;l)$ respectively. We denote by $\lambda(m) = (\lambda_{m,1},\ldots,\lambda_{m,h_m})$ the "generalised" $\mathfrak{o}_m$ highest weight, where each of the components $\lambda_{m,i}$ is an operator representing the ith label of all of the $\mathfrak{o}_m$ irreducible representations occurring in (4.17) at once. That is,

$$\lambda_{m,i} = \left(\begin{array}{c:c:c} \lambda(m;1)_i & 0 & 0 \\ \hdashline 0 & \ddots & 0 \\ \hdashline 0 & 0 & \lambda(m;l)_i \end{array}\right)$$

Note however that $\lambda_{m,i}$ just reduces to a scalar when acting on a basis state of $V(\lambda(n+1))$. The matrix $A(m)$ satisfies the "induced" characteristic identity

$$\prod_{r=1}^{n}(A(m) - \alpha_{m,r}) = 0,$$

where the characteristic roots $\alpha_{m,r}$ are given by

$$\alpha_{m,r} = \bar{\alpha}_{m+1-r,m} = \lambda_{m,r} + m - 1 - r$$

Here m, r and 1 denote multiples of the identity, and we define $\lambda_{m,r}$ for $r > h$ as before. Similarly, we define the adjoint matrix $\overline{A}(m) := -A(m)^t$, which satisfies the adjoint identity

$$\prod_{r=1}^{m}(\overline{A}(m) - \overline{\alpha}_{m,r}) = 0,$$

By virtue of these characteristic identity we may construct the $\mathfrak{o}_m$ projection operators $P\binom{m}{r}$ and $\overline{P}\binom{m}{r}$ (for $1 \leq r \leq m$) via

$$P\binom{m}{r} = \prod_{l \neq r}\left(\frac{A_m - \alpha_{l,m}}{\alpha_{r,m} - \alpha_{l,m}}\right)$$

$$\overline{P}\binom{m}{r} = \prod_{l \neq r}\left(\frac{\overline{A}_m - \overline{\alpha}_{l,m}}{\overline{\alpha}_{r,m} - \overline{\alpha}_{l,m}}\right)$$

We denote the (m, m) entries of these projection operators by $C_{m,r}$ and $\overline{C}_{m,r}$ respectively. As before, these operators determine squares of Wigner coefficients and we may express their eigenvalues in terms of the $\mathfrak{o}_m$ and $\mathfrak{o}_{m-1}$ characteristic roots.
Let $\psi(m)$ be the vector with components

$$\psi(m)^i = \alpha^i_{m+1}$$

for $1 \leq i \leq m$, and let $\psi^\dagger(m)$ be the vector with components

$$\psi^\dagger(m)^i = \alpha_i^{m+1}$$

Then $\psi(m)$ and $\psi^\dagger(m)$ transform as vector operators and contragredient vector operators (respectively) with respect to $\mathfrak{o}_m$. To demonstrate this we use the commutation relations: for $1 \leq i, j \leq m$ and $1 \leq k \leq m$ we have

$$\begin{aligned}[][\alpha_j^i, \psi(m)^k] &= [\alpha_j^i, \alpha_{m+1}^k] \\ &= \delta_j^k \alpha_{m+1}^i - \delta_i^k \alpha_{m+1}^j \\ &= \delta_j^k \psi(m)^i - \delta_i^k \psi(m)^j,\end{aligned}$$

and

$$\begin{aligned}[][\alpha_j^i, \psi^\dagger(m)_k] &= [\alpha_j^i, \alpha_k^{m+1}] \\ &= \delta_j^k \alpha_i^{m+1} - \delta_i^k \alpha_j^{m+1} \\ &= \delta_j^k \psi^\dagger(m)_i - \delta_i^k \psi^\dagger(m)_j\end{aligned}$$

So all of the properties of vector operators established previously apply to these operators. In particular, we may resolve $\psi(m)$ and $\psi^\dagger(m)$ into a sum of m shift components

$$\begin{aligned}\psi(m) &= \sum_{r=1}^{m} \psi \binom{m}{r} \\ \psi^\dagger(m) &= \sum_{r=1}^{m} \psi^\dagger \binom{m}{r}\end{aligned}$$

where $\psi \binom{m}{r}$ and $\psi^\dagger \binom{m}{r}$ are vector operators defined by

$$\begin{aligned}\psi \binom{m}{r} &= P \binom{m}{r} \psi(m) \\ \psi^\dagger \binom{m}{r} &= \overline{P} \binom{m}{r} \psi^\dagger(m)\end{aligned} \tag{4.18}$$

These operators alter the representation labels of $\mathfrak{gl}_m$ according to

$$\lambda_{m,k} \psi \binom{m}{r} = \psi \binom{m}{r} (\lambda_{m,k} + \delta_{r,k}) \tag{4.19}$$

Note that for $m = n$ we recover the operators considered in section 4.4; i.e., $\psi(n) \equiv \psi$, $\psi \binom{n}{r} \equiv \psi[r]$, and so on. Furthermore we have that

$$\begin{aligned}\psi \binom{m}{r}^i \psi^\dagger \binom{m}{r}_j &= \overline{M}_{m,r} P \binom{m}{r} \text{ and} \\ \psi^\dagger \binom{m}{r}_i \psi \binom{m}{r}^j &= M_{m,r} \overline{P} s \binom{m}{r},\end{aligned} \tag{4.20}$$

where

$$\overline{M}_{m,r} = M_{m+1-r,m} = (-1)^m \prod_{k=1}^{n+1} (\alpha_{m+1,k} - \alpha_{m,r}) \prod_{l \neq r} (\alpha_{m,r} - \alpha_{m,l} - \eta_{m,l} - \delta_{m+1-r,l})^{-1}$$

This is the generalisation of equation (4.16) to the subalgebra $\mathfrak{o}_m$, and the proof is analogous (just with the labels changed).

We now make a crucial observation which underlies the rest of the analysis. We have already seen that α_i^{m+1} $(1 \leq i \leq m)$ transforms as a component of a contragredient $\mathfrak{o}_m$ vector operator. Moreover, we have that α_l^{m+1} (for $2 < l \leq m$) transforms as a component of a contragredient vector operator with respect to $\mathfrak{o}_m, \ldots, \mathfrak{o}_l$. Again this can easily be seen from the commutation relations. However, when $l = 2, 1$ we run into a difficulty. The components α_2^{m+1} and α_1^{m+1} do not transform as components of vector operators with respect to $\mathfrak{o}_2$. This is because $\mathfrak{o}_2$ is not a semisimple Lie algebra; in fact it is a one-dimensional abelian Lie algebra, and so all of its irreducible representations are one-dimensional. However, the vector representation of $\mathfrak{o}_2$ is two-dimensional and hence is not irreducible. Hence a special derivation is required in the case $l = 2, 1$, and we deal with this later.

4.5.1 The case $l = m$

So we consider the $\mathfrak{o}_m$ generators of the form α_l^{m+1} for $3 \leq l \leq m$. Our aim is to determine the matrix elements of these generators by generalising equations (4.20). For the case $l = m$ we know that α_m^{m+1} transforms as a component of a contragredient vector operator with respect to $\mathfrak{o}_m$, so we have the decomposition

$$\alpha_m^{m+1} = \sum_{r=1}^{m} \psi^\dagger \begin{pmatrix} m \\ r \end{pmatrix}_m \tag{4.21}$$

where each $\psi^\dagger \begin{pmatrix} m \\ r \end{pmatrix}$ is a shift operator which alters the $\mathfrak{gl}_m$ representation labels $\lambda_{m,i}$ according to (4.19). Note also that α_m^{m+1} does not transform as a component of a contragredient vector operator with respect to $\mathfrak{o}_k$ for $k < m$; this is because α_m^{m+1} commutes with everything in $\mathfrak{o}_{m-1}$. So α_m^{m+1} cannot alter any of the representation labels corresponding to subalgebra $\mathfrak{o}_{m-1}$.

Therefore equation (4.21) cannot be "decomposed" any further. Using (4.21) and (4.20) we can explicitly determine the matrix elements of α_m^{m+1}.

4.5.2 The case $l = m - 1$

Now consider a_{m-1}^{m+1} where $m - 1 > 2$. We know that this transforms as a component of a contragredient vector operator with respect to $\mathfrak{o}_m$ and $\mathfrak{o}_{m-1}$ (as argued previously). Moreover, it does not transform as a component of a contragredient vector operator with respect to $\mathfrak{o}_k$ for $k < m-1$ because it commutes with everything in $\mathfrak{o}_{m-2}$, analogously to the above argument. Now because α^{m+1}_{m-1} is a component of a contragredient vector operator with respect to $\mathfrak{o}_m$, we have as usual the resolution

$$\alpha^{m+1}_{m-1} = \sum_{r=1}^{m} \psi^\dagger \begin{pmatrix} m \\ r \end{pmatrix}_{m-1} \tag{4.22}$$

where the operators $\psi^\dagger \begin{pmatrix} m \\ r \end{pmatrix}$ are given by (4.18). But α^{m+1}_{m-1} also transforms as a component of a contragredient vector operator with respect to $\mathfrak{o}_{m-1}$ and hence so must each of $\psi^\dagger \begin{pmatrix} m \\ r \end{pmatrix}_{m-1}$. To see this, note that (for $1 \leq i, j \leq m-1$)

$$[\alpha^i_j, \alpha^{m+1}_{m-1}] = -\delta^i_{m-1}\alpha^{m+1}_j + \delta^j_{m-1}\alpha^{m+1}_i$$

Then substituting (4.22) into the above, and using the linear independence of the operators $\psi^\dagger \begin{pmatrix} m \\ r \end{pmatrix}$, we find that

$$\left[\alpha^i_j, \psi^\dagger \begin{pmatrix} m \\ r \end{pmatrix}_{m-1}\right] = -\delta^i_{m-1}\psi^\dagger \begin{pmatrix} m \\ r \end{pmatrix}_j + \delta^j_{m-1}\psi^\dagger \begin{pmatrix} m \\ r \end{pmatrix}_i \tag{4.23}$$

(This is analogous to the $\mathfrak{gl}_n$ argument). Therefore, we can decompose each $\psi^\dagger \begin{pmatrix} m \\ r \end{pmatrix}$ into a sum of $\mathfrak{o}_{m-1}$ shift operators:

$$\psi^\dagger \begin{pmatrix} m \\ r \end{pmatrix}_{m-1} = \sum_{l=1}^{m-1} \psi^\dagger \begin{pmatrix} m & m-1 \\ r & l \end{pmatrix}_{m-1} \tag{4.24}$$

where each $\psi^\dagger \begin{pmatrix} m & m-1 \\ r & l \end{pmatrix}$ is now a component of a contragredient vector operator with respect to $\mathfrak{o}_m$ *and* $\mathfrak{o}_{m-1}$. These operators are defined, for $1 \leq i \leq m-1$, by

$$\psi^\dagger \begin{pmatrix} m & m-1 \\ r & l \end{pmatrix}_i = \sum_{j=1}^{m-1} \overline{P} \begin{pmatrix} m-1 \\ l \end{pmatrix}^i_j \psi^\dagger(m;r)_j,$$

where $\overline{P} \begin{pmatrix} m-1 \\ l \end{pmatrix}$ is the $(m-1) \times (m-1)$ $\mathfrak{o}_m$ projection operator, defined as usual by

$$\overline{P} \begin{pmatrix} m-1 \\ l \end{pmatrix} = \prod_{k=1, k\neq l}^{m-1} \left(\frac{\overline{A}(m-1) - \overline{\alpha}_{m-1,k}}{\overline{\alpha}_{m-1,l} - \overline{\alpha} m-1, k} \right)$$

This leads us to define $\overline{P} \begin{pmatrix} m-1 & m \\ l & r \end{pmatrix}$ as the $(m-1) \times m$ matrix with entries

$$\overline{P} \begin{pmatrix} m-1 & m \\ l & r \end{pmatrix}^i_j = \sum_{k=1}^{m-1} \overline{P} \begin{pmatrix} m-1 \\ l \end{pmatrix}^i_k \overline{P} \begin{pmatrix} m \\ r \end{pmatrix}^k_l \quad (1 \leq i \leq m-1, i \leq j \leq m)$$

Similarly, we define the $m \times (m-1)$ matrix $P\begin{pmatrix} m & m-1 \\ r & l \end{pmatrix}$ according to

$$P\begin{pmatrix} m & m-1 \\ r & l \end{pmatrix}^i_j = \sum_{k=1}^{m-1} P\begin{pmatrix} m \\ r \end{pmatrix}^i_k P\begin{pmatrix} m-1 \\ l \end{pmatrix}^k_j,$$

so that we have

$$\begin{aligned} \psi^\dagger \begin{pmatrix} m & m-1 \\ r & l \end{pmatrix} &= \overline{P}\begin{pmatrix} m-1 & m \\ l & r \end{pmatrix} \psi^\dagger(m) \\ &= \psi^\dagger(m) P\begin{pmatrix} m & m-1 \\ r & l \end{pmatrix}. \end{aligned}$$

Since each $\psi^\dagger \begin{pmatrix} m & m-1 \\ r & l \end{pmatrix}$ is a vector operator with respect to $\mathfrak{o}_m$ and $\mathfrak{o}_{m-1}$, each one alters the representation labels of both $\mathfrak{o}_{m-1}$ and $\mathfrak{o}_{m-1}$ according to

$$\begin{aligned} \lambda_{m,i}\psi^\dagger \begin{pmatrix} m & m-1 \\ r & l \end{pmatrix} &= \psi^\dagger \begin{pmatrix} m & m-1 \\ r & l \end{pmatrix} (\lambda_{m,i} - \delta_{r,i} + \delta_{m+1-r,k}) \\ \lambda_{m-1,j}\psi^\dagger \begin{pmatrix} m & m-1 \\ r & l \end{pmatrix} &= \psi^\dagger \begin{pmatrix} m & m-1 \\ r & l \end{pmatrix} (\lambda_{m-1,j} - \delta_{l,j} + \delta_{j,m-1}) \end{aligned} \tag{4.25}$$

Hence we refer to these operators as "simultaneous" shift operators.

4.5.3 The general case

In general, the generator α_l^{m+1} $(3 \le l \le m)$ transforms as a component of a contragredient vector operator with respect to $\mathfrak{o}_m, \dots, \mathfrak{o}_l$, so we have the decomposition

$$a_l^{m+1} = \sum_{i_m=1}^{m} \cdots \sum_{i_l=1}^{l} \psi^\dagger \begin{pmatrix} m & \dots & l \\ i_m & \dots & i_l \end{pmatrix}_l, \tag{4.26}$$

where each of the $m!/(l-1)!$ operators $\psi^\dagger \begin{pmatrix} m & \dots & l \\ i_m & \dots & i_l \end{pmatrix}$ is a contragredient vector operator which simultaneously alters the representation labels of $\mathfrak{o}_m, \dots, \mathfrak{o}_l$ according to

$$\lambda_{p,j}\psi^\dagger \begin{pmatrix} m & \dots & l \\ i_m & \dots & i_l \end{pmatrix} = \psi^\dagger \begin{pmatrix} m & \dots & l \\ i_m & \dots & i_l \end{pmatrix} (\lambda_{p,j} - \delta_{j,i_p} + \delta_{k,p+1-i_p}) \tag{4.27}$$

In (4.27), $\lambda_{p,j}$ denotes the jth component of the $\mathfrak{o}_p$ highest weight $\lambda(p)$ (where $l \le p \le m$, $1 \le j \le [p/2]$). If we define the $l \times m$ matrix $\overline{P}\begin{pmatrix} l & \dots & m \\ i_l & \dots & i_m \end{pmatrix}$ according to

$$\overline{P}\begin{pmatrix} l & \dots & m \\ i_l & \dots & i_m \end{pmatrix} = \sum_{r=1}^{l} \cdots \sum_{p=1}^{m-1} P\begin{pmatrix} l \\ i_l \end{pmatrix}^i_r \dots P\begin{pmatrix} m \\ i_m \end{pmatrix}^p_j \tag{4.28}$$

and define the $m \times l$ matrix $P\begin{pmatrix} m & \dots & l \\ i_m & \dots & i_l \end{pmatrix}$ similarly, then the shift components can be defined by

$$\psi^\dagger\begin{pmatrix} m & \dots & l \\ i_m & \dots & i_l \end{pmatrix} = \overline{P}\begin{pmatrix} l & \dots & m \\ i_l & \dots & i_m \end{pmatrix}\psi^\dagger(m) = \psi^\dagger(m)P\begin{pmatrix} m & \dots & l \\ i_m & \dots & i_l \end{pmatrix}$$

In analogy with the $\mathfrak{gl}_n$ case, we can generalise equation (4.20) for the simultaneous shift operators. The procedure is similar and so we just quote the result: we have

$$\psi\begin{pmatrix} m & \dots & l \\ i_m & \dots & i_l \end{pmatrix}^l \psi^\dagger\begin{pmatrix} m & \dots & l \\ i_m & \dots & i_l \end{pmatrix}_l$$

4.5.4 The case $l = 2, 1$

Although the general procedure we have outlined so far is very similar to the procedure for the $\mathfrak{gl}_n$ case, we run into a slight difficulty when considering generators of the form α_1^{m+1} and α_2^{m+1} (which in our vector operator notation correspond to $\psi^\dagger(m)_1$ and $\psi^\dagger(m)_2$). These generators do not transform as components of vector operators with respect to $\mathfrak{o}_2$, so we do not get a decomposition analogous to (4.26) for the case $l = 2, 1$. By definition, a tensor operator transforms with respect to the generators of a Lie algebra like an irreducible representation of the algebra. So while it is still true that the components α_1^{m+1} and α_2^{m+1} transform with respect to $\mathfrak{o}_2$ according to

$$[\alpha^i_j, \alpha^{m+1}_k] = \delta^j_k \alpha^{m+1}_i - \delta^i_k \alpha^{m+1}_j,$$

this no longer corresponds to an irreducible representation of $\mathfrak{o}_2$. In fact, this transformation corresponds to the vector representation of $\mathfrak{o}_2$, which is two-dimensional; but $\mathfrak{o}_2$ is an abelian Lie algebra, and so its irreducible representations are all one-dimensional.

To overcome this problem, we introduce the new generators

$$\psi(m)^\pm = \frac{1}{\sqrt{2}}(\alpha^1_{m+1} \pm i\alpha^2_{m+1})$$

$$\psi^\dagger(m)_\pm = \frac{1}{\sqrt{2}}(\alpha^{m+1}_1 \pm i\alpha^{m+1}_2)$$

Then we have

$$[\alpha^1_2, \psi(m)^\pm] = \pm i\psi(m)^\pm$$

and similarly

$$[\alpha^1_2, \psi^\dagger(m)_\pm] = \pm i\psi^\dagger m)_\pm$$

So these new generators transform according to a one-dimensional irreducible representation of $\mathfrak{o}_2$. Hence these new generators transform as components of vector operators with respect to $\mathfrak{o}_2$. So we define the new $\mathfrak{o}_m$ vector $\tilde{\psi}(m)$ with components

$$\begin{aligned}\tilde{\psi}(m)^1 &= \psi(m)_+ \\ \tilde{\psi}(m)^2 &= \psi(m)_- \\ \tilde{\psi}(m)^i &= \psi(m)^i \ \ 3 \leq i \leq m\end{aligned}$$

Clearly this vector can be obtained from the original $\psi(m)$ by applying the change of basis matrix

$$M = M' \oplus I_{m-2}$$

where I_{m-2} is the $(m-2) \times (m-2)$ identity matrix and

$$M' = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$$

We now proceed as before, working with the matrix

$$\tilde{A} = MAM^{-1}$$

The procedure is essentially the same. Also note that the matrix $\tilde{A}$ satisfies the same characteristic equation as A[19]. The result is that we get a decomposition

$$\psi(m)_\pm = \sum_{i_k} \psi \begin{pmatrix} m & \dots & 3 & 2 \\ i_m & \dots & i_3 & \pm \end{pmatrix}$$

and

$$\psi^\dagger(m)_\pm = \psi^\dagger \begin{pmatrix} m & \dots & 3 & 2 \\ i_m & \dots & i_3 & \pm \end{pmatrix}$$

(Here $\sum_{i_k}$ is shorthand notation for the sum over each of the indices i_k for $3 \leq m \leq k$, following Gould[13]). Hence we get the decomposition into shift components

$$\alpha_i^{m+1} = \sum_{i_k} \psi^\dagger \begin{pmatrix} m & \dots & 3 & 2 \\ i_m & \dots & i_3 & i_2 \end{pmatrix}$$

4.6 Matrix elements: general formulae

To calculate the formulae for matrix elements of the generators α_l^{m+1} ($3 \leq l \leq m$) we use the same method as in the case of $\mathfrak{gl}_n$. Acting on the basis state $|\lambda_{j,k}\rangle$, α_l^{m+1} reduces to

$$\begin{aligned}\alpha_l^{m+1}\,|\lambda_{j,k}\rangle &= \sum_{i_k} \psi^\dagger \begin{pmatrix} m & \dots & l \\ i_m & \dots & i_l \end{pmatrix}_l |\lambda_{j,k}\rangle \\ &= \sum_{i_k} N^{m\cdots l}_{i_m \dots i_l}\,|\lambda_{j,k} - \Delta_{m,i_m} - \dots - \Delta_{l,i_l}\rangle\,,\end{aligned}$$

where the elements $N^{m\cdots l}_{i_m\cdots i_l}$ are given by

$$N^{m\cdots l}_{i_m\cdots i_l} = \left(\langle\lambda_{j,k}|\, \psi \begin{pmatrix} m & \dots & l \\ i_m & \dots & i_l \end{pmatrix}^l \psi^\dagger \begin{pmatrix} m & \dots & l \\ i_m & \dots & i_l \end{pmatrix}_l |\lambda_{j,k}\rangle \right)^{1/2},$$

which again we may evaluate explicitly. For the special case α_2^{m+1} we have that

$$\begin{aligned} \alpha_2^{m+1}\,|\lambda_{j,k}\rangle &= \sum_{i_k} \psi^\dagger \begin{pmatrix} m & \dots & 2 \\ i_m & \dots & i_2 \end{pmatrix}_2 |\lambda_{j,k}\rangle \\ &= \sum_{i_k} N^{m\cdots 2}_{i_m\cdots i_2}\,|\lambda_{j,k} - \Delta_{m,i_m} - \dots - \Delta_{2,i_2}\rangle\,, \end{aligned}$$

where

$$N^{m\cdots 2}_{i_m\cdots 1} = \frac{1}{\sqrt{2}} \left(\langle\lambda_{j,k}|\, \psi \begin{pmatrix} m & \dots & 2 \\ i_m & \dots & + \end{pmatrix} \psi^\dagger \begin{pmatrix} m & \dots & 2 \\ i_m & \dots & + \end{pmatrix} |\lambda_{j,k}\rangle \right)^{1/2}$$

and similarly

$$N^{m\cdots 2}_{i_m\cdots 2} = \frac{1}{\sqrt{2}} \left(\langle\lambda_{j,k}|\, \psi \begin{pmatrix} m & \dots & 2 \\ i_m & \dots & - \end{pmatrix} \psi^\dagger \begin{pmatrix} m & \dots & 2 \\ i_m & \dots & - \end{pmatrix} |\lambda_{j,k}\rangle \right)^{1/2}$$

Finally, we may obtain the matrix elements of α_1^{m+1} via the commutation relation

$$\alpha_1^{m+1} = [\alpha_2^{m+1}, \alpha_1^2]$$

Also note that strictly speaking each of these matrix elements is only true up to a phase factor[13].

4.7 Explicit calculation: $\mathfrak{o}_4$

We now consider an explicit example which hopefully will make it clear what the objects defined in the previous sections look like. We consider the Lie algebra $\mathfrak{o}_4$ which has the canonical subalgebra chain

$$\mathfrak{o}_2 \subset \mathfrak{o}_3 \subset \mathfrak{o}_4$$

Let $V(\lambda(4))$ be the irreducible $\mathfrak{o}_4$ representation with highest weight

$$\lambda(4) = (1, 0)$$

Then, following our general notation, we let $A(4)$ be the 4×4 matrix whose (i, j)th entry is the generator α_j^i in this representation. So

$$A(4) = \begin{pmatrix} \alpha_1^1 & \alpha_2^1 & \alpha_3^1 & \alpha_4^1 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 & \alpha_4^2 \\ \alpha_1^3 & \alpha_2^3 & \alpha_3^3 & \alpha_4^3 \\ \alpha_1^4 & \alpha_2^4 & \alpha_3^4 & \alpha_4^4 \end{pmatrix}$$

Now in this case we have the branching rule

$$V(1,0)\downarrow_{\mathfrak{o}_3}\cong V'(1)\oplus V'(0)$$

where $V'(1)$ is the 3-dimensional irreducible $\mathfrak{o}_3$-module with highest weight 1 and $V'(0)$ is the 1-dimensional irreducible $\mathfrak{o}_3$-module with highest weight 0. In the notation of section 4.5 we have that $\lambda(3;1) = (1)$ and $\lambda(3;2) = (0)$. Hence we set $\lambda(3) = (\lambda_{3,1})$ where $\lambda_{3,1}$ is an operator with the form

$$\lambda_{3,1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Note that the upper left 3×3 block corresponds to the $\mathfrak{o}_3$ module $V'(1)$ while the lower right 1×1 block corresponds to the $\mathfrak{o}_3$ module $V'(0)$. Again, this operator reduces to a scalar when acting on a basis state of $V(1,0)$. We denote the basis states of $V(1,0)$ by the corresponding Gelfand-Tsetlin pattern (ordered so as to make the branching rule apparent). We have that

$$v_1 \equiv \left| \begin{array}{ll} 1 & 0 \\ 1 & \\ 1 & \end{array} \right\rangle,$$

$$v_2 \equiv \left| \begin{array}{ll} 1 & 0 \\ 1 & \\ 0 & \end{array} \right\rangle,$$

$$v_3 \equiv \left| \begin{array}{ll} 1 & 0 \\ 1 & \\ -1 & \end{array} \right\rangle,$$

and

$$v_4 \equiv \left| \begin{array}{ll} 1 & 0 \\ 0 & \\ 0 & \end{array} \right\rangle$$

Note that the first three basis vectors correspond to the $\mathfrak{o}_3$ module $V'(1)$ while the fourth basis vector corresponds to the $\mathfrak{o}_3$ module $V'(0)$. Also in this case the $\mathfrak{o}_4$ characteristic roots are $\alpha_{4,1} = 3$, $\alpha_{4,2} = 1$, $\alpha_{4,3} = 1$, $\alpha_{4,1} = -1$. Then using the formulae established in the previous section we may easily evaluate the matrix elements of the generators. It is worth noting that for an example of small dimension such as this the formulae are relatively easy to use by hand; for larger dimensions they become somewhat unwieldy.

Chapter 5

Further results

5.1 Characteristic identities for semisimple Lie algebras

The characteristic identities satisfied by the matrices of $\mathfrak{gl}_n$ and $\mathfrak{o}_n$ generators are by no means the only examples of such identities. In fact, they are merely special cases of a more general phenomenon, as detailed by Gould[16]. In particular, let L be a complex semisimple Lie algebra and let H be a Cartan subalgebra of L, and let U be the universal enveloping algebra of L. Let $V(\lambda)$ be an irreducible representation of L. Consider the coproduct on U,

$$\Delta : U \to U \otimes U$$

defined by $\Delta(x) = x \otimes 1 + 1 \otimes x$ for $x \in L$, and extended to a homomorphism on all of U. Then for z in the centre of U consider the operator

$$A(z) = -\frac{1}{2}((\pi_\lambda \otimes \mathrm{id}) \circ \Delta(z) - \pi_\lambda(z) \otimes 1 - I \otimes z),$$

where 1 denotes the unit element of U and I is the identity matrix in the representation $V(\lambda)$. For a fixed irreducible representation π_μ we have

$$\pi_\mu(A(z)) = -\frac{1}{2}(\pi_\lambda \otimes \pi_\mu(z) - \pi_\lambda(z) \otimes I - I \otimes z)$$

This may be regarded as an operator on the tensor product space $V(\lambda) \otimes V(\mu)$, and its eigenvalues on this space are all of the form[16]

$$\alpha_i = -\frac{1}{2}(\chi_{\mu+\lambda_i}(z) - \chi_\lambda(z) - \chi_\mu(z))$$

for $i = 1, \ldots, \mathrm{dim}L = k$. Here $\{lambda_1, \ldots, \lambda_k\}$ are the distinct weights occurring in the representation $V(\lambda)$. From this it follows that the operator $\pi_\mu(A(z))$ satisfies the identity

$$\prod_{i=1}(\pi_\mu(A(z)) - \alpha_i) = 0$$

In the case that $V(\lambda)$ is the contragredient vector representation, we recover the characteristic identities considered earlier.

5.2 The problem with $\mathfrak{sp}_{2n}$

Recall that the symplectic Lie algebra $\mathfrak{sp}_{2n}$ is the simple Lie algebra defined by

$$\mathfrak{sp}_{2n} = \{X \in \mathfrak{gl}_{2n}(\mathbb{C}) \mid X^t J + JX = 0\},$$

where $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Since $\mathfrak{sp}_{2n}$ is simple (and therefore semisimple), we know from the previous section that the matrix of generators will satisfy a characteristic identity on a (finite-dimensional) irreducible representation $V(\lambda)$. It is natural to ask whether the procedure outlined in the previous two chapters can be adapted to the case of $\mathfrak{sp}_{2n}$ in order to determine the matrix elements of the generators. Unfortunately there is a difficulty in that the canonical Gelfand-Tsetlin basis construction does not work in the case of $\mathfrak{sp}_{2n}$. The reason for this (as alluded to earlier) is that the branching rule in the case of $\mathfrak{sp}_{2n}$ is not multiplicity-free. Explicitly, if we consider the subalgebra embedding $\mathfrak{sp}_{2(n-1)} \subset \mathfrak{sp}_{2n}$, then the representation $V(\lambda)$ decomposes under the action of $\mathfrak{sp}_{2(n-1)}$ according to

$$V(\lambda) \downarrow_{\mathfrak{sp}_{2(n-1)}} \cong \bigoplus_{\mu} m(\mu) V'(\mu),$$

where $V'(\mu)$ denotes an $\mathfrak{sp}_{2(n-1)}$ irreducible representation of highest weight μ, and $m(\mu)$ denotes the multiplicity of the representation $V'(\mu)$. In the case of $\mathfrak{gl}_n$ and $\mathfrak{o}_n$, the multiplicities $m(\mu)$ are always either 0 or 1 (and they are 1 if and only if the components of μ satisfy the "betweenness conditions" discussed earlier), but for $\mathfrak{sp}_{2n}$ we sometimes have $m(\mu) > 1$. Therefore, upon continued restriction to the subalgebra $\mathfrak{sp}_2$, we do *not* necessarily get a direct sum decomposition of $V(\lambda)$ into distinct one-dimensional $\mathfrak{sp}_{2n}$ representations. Hence the canonical Gelfand-Tsetlin basis construction will not work in this case. We note in passing that the problem of constructing a basis for $\mathfrak{sp}_{2n}$ representations has recently been solved by Molev[22] by a method involving "twisted Yangians". This work is an extension of that started by Olshansky and Cherednik[22], who noticed that the standard Gelfand-Tsetlin basis of $\mathfrak{gl}_n$ has an interpretation in terms of Yangians.

5.3 Lie superalgebras and quantum groups

We note that the method of characteristic identities has also been applied to representations of Lie superalgebras, quantum groups, Kac-Moody algebras, and other kinds of algebraic objects[25,27]. Recall that a Lie superalgebra is a $\mathbb{Z}_2$-graded vector space $L = L_0 \oplus L_1$ endowed with a bilinear map $[\ ,\ ] : L \times L \to L$

satisfying

$$[L_a, L_b] \subset L_{a+b} \text{ for } a, b \in \mathbb{Z}_2 \tag{5.1}$$

$$[x, y] = (-1)^{|x||y|}[y, x] \tag{5.2}$$

and

$$(-1)^{|x||z|}[x,[y,z]] + (-1)^{|x||y|}[y,[z,x]] + (-1)^{|y||z|}[z,[x,y]] = 0 \tag{5.3}$$

where $|x|$ denotes the degree of x (either 0 or 1). There is a classification of Lie superalgebras that is similar to that of ordinary Lie algebras, although more complicated (naturally). In particular there are certain infinite families of simple Lie superalgebras that are analogous to the infinite families of simple Lie algebras; these were classified by Victor Kac[23]. There is also a classification of finite-dimensional irreducible representations for simple Lie superalgebras in terms of highest weights. On such representations, Jarvis and Green[25] proved the analogous characteristic identities for the general linear, special linear and orthosymplectic Lie superalgebras, and their work has been extended by Gould[24], Links, Isaac, Werry[27,28], and others.

Chapter 6

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