Let X be a non-strict p.u.T. Then let's construct X' which is strict, and such that $X \simeq X'$ as objects of $\mathrm{Fun}(\underline{\mathrm{Mod}}(T), \underline{\mathrm{Set}})$.

$(X, \underline{\Phi})$ ← given this.

transition isos given.

To define X':

(Base case) Start by asking that if M_1, M_2 are not (writable as) ultraproducts,

① $X'(M_1 \xrightarrow{f} M_2) \overset{\mathrm{df}}{=} X(M_1) \xrightarrow{X(f)} X(M_2)$.

(Inductive step)

② If X' has already been defined on the full subcategory $\underline{C} \subseteq \underline{\mathrm{Mod}}(T)$, then extend X' to $\underline{C}'$ the full subcategory of $\underline{\mathrm{Mod}}(T)$ made of anything that is an ultraproduct of things from $\underline{C}$ by setting:

$$\left.\begin{tikzcd} X'\left(\prod_{i\to\mathcal{U}} M_i\right) \arrow[d, "X'(f)"] \\ X'\left(\prod_{j\to\mathcal{V}} N_j\right) \end{tikzcd}\right\} \overset{\mathrm{df}}{\Rightarrow} \begin{tikzcd} \prod_{i\to\mathcal{U}} X'(M_i) \arrow[d, dotted, "!"] & X\left(\prod_{i\to\mathcal{U}} M_i\right) \arrow[l, "\underline{\Phi}'_{(M_i)}", "\sim"'] \arrow[d, "X(f)"] \\ \prod_{j\to\mathcal{V}} X'(N_j) & X\left(\prod_{j\to\mathcal{V}} N_j\right) \arrow[l, "\underline{\Phi}'_{(N_j)}"'] \end{tikzcd}$$

where $\underline{\Phi}'_{(M_i)}$ (resp (N_j)) is defined by

$$\left.\begin{tikzcd} X\left(\prod_{i\to\mathcal{U}} M_i\right) \arrow[r, dotted, "!"] \arrow[d, "\underline{\Phi}_{(M_i)}"'] & \prod_{i\to\mathcal{U}} X'(M_i) \\ \prod_{i\to\mathcal{U}} X(M_i) \arrow[ur, "{[\sigma_i]_{i\to\mathcal{U}}}"'] & \end{tikzcd}\right\}$$

where $\sigma_i \overset{\mathrm{df}}{=} \begin{cases} \mathrm{id}_{X(M_i)} & \text{if } M_i \text{ was in base case,} \\ \boxed{\text{OR}} & \\ \underline{\Phi}'_{(M^i_j)}, & \text{if } M_i = \prod_{j\to W} M^i_j \text{ was defined in an earlier stage.} \end{cases}$

• Functoriality is easy because conjugating by Φ's cancels out. ✓

• (Note if we did base case and are in second stage, all the primes vanish and $\Phi'(M_i) = \Phi(M_i)$.)

• Pre-ultrafunctoriality?

$$\begin{tikzcd}
X'\left(\prod_{i \to \mathcal{U}} M_i\right) \arrow[r, equal] \arrow[d, "?\ X'\left(\prod_{i\to\mathcal{U}} f_i\right)"'] \arrow[dr, phantom, "\circlearrowright"] & \prod_{i \to \mathcal{U}} X'(M_i) \arrow[d, "\prod_{i\to\mathcal{U}} X'(f_i)"] \\
X'\left(\prod_{i \to \mathcal{U}} N_i\right) \arrow[r, equal] & \prod_{i \to \mathcal{U}} X'(N_i)
\end{tikzcd}$$

(So: is $X'\left(\prod_{i\to\mathcal{U}} f_i\right) \overset{?}{=} \prod_{i\to\mathcal{U}} X'(f_i)$?) But now it's an easy calculation!!

$\overset{\text{by df}}{\Longleftrightarrow}$ $\Phi'_{(N_i)} \circ X\left(\prod_{i\to\mathcal{U}} f_i\right) \circ \Phi'^{-1}_{(M_i)} \overset{?}{=} \prod_{i\to\mathcal{U}} X'(f_i)$

$\overset{\text{by df}}{\Longleftrightarrow}$ $\prod_{i\to\mathcal{U}} \sigma_i^N \circ \Phi_{(N_i)} \circ X\left(\prod_{i\to\mathcal{U}} f_i\right) \circ \Phi^{-1}_{(M_i)} \left(\prod_{i\to\mathcal{U}} \sigma_i^M\right)^{-1} \overset{?}{=} \prod_{i\to\mathcal{U}} X'(f_i)$

$\overset{\text{by df}}{\Longleftrightarrow}$ $\prod_{i\to\mathcal{U}} \sigma_i^N \prod_{i\to\mathcal{U}} X f_i \left(\prod_{i\to\mathcal{U}} \sigma_i^M\right)^{-1} \overset{?}{=} \prod_{i\to\mathcal{U}} X'(f_i)$

$\overset{\text{by df}}{\Longleftrightarrow}$ $\prod_{i\to\mathcal{U}} \sigma_i^N X f_i \, {\sigma_i^M}^{-1} \overset{?}{=} \prod_{i\to\mathcal{U}} X'(f_i)$

$\overset{?}{=} \prod_{i\to\mathcal{U}} \Phi'_{N_i} X f_i \, \Phi'^{-1}_{M_i}$ $\overset{\text{by df}}{\Longleftarrow}$

$= \prod_{i\to\mathcal{U}} \sigma_i^N X f_i \, {\sigma_i^M}^{-1}$. □

if N_i or M_i are from base case, Φ'_{N_i} (resp M) is id_{N_i}, so Φ'_{N_i} (resp M) is σ_i^N.

Last thing to check: all of Mod(T) is reached eventually at some stage in this (transfinite) induction.