Measurement of electron charge $q_{\rm e}$ and Boltzmann's constant $k_{\rm B}$ by a cheap do-it-yourself undergraduate experiment

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A simple circuit consisting of 4 low noise operational amplifiers with voltage noise lower than $1\ \mathrm{nV}/\sqrt{\mathrm{Hz}}$ and one four-quadrant multiplier with full scale accuracy 0.5% gives the possibility to determine the fundamental constants $q_{\rm e}$ and $k_{\rm B}$ with an accuracy better than 5%. The Boltzmann constant is determined by measurement of thermally averaged square of the voltage of a capacitor and also by the spectral density of thermal noise of resistors. The electron charge is obtained from properties of shot noise of a photodiode illuminated by a lamp. For both the measurements we are using one and the same set-up. The cost of the 3 integrated circuits gives the possibility to realize the experiment in every high-school or university in the world. The analytic calculation for the pass-bandwidth avoids the necessity to use standard instrumentation as oscilloscope and sine-wave generator. We use only a LCR-meter, and a voltmeter, that is why the budget is less than 137 $.

I. IDEA BY EINSTEIN

In 1905 Albert Einstein[1] suggested to determine the Boltzmann constant $k_{\rm B}$ ("elementary quanta") by thermally agitated voltage of a capacitor. It was a lecture delivered by Einstein on statistical mechanics which stimulated a PhD student of Max Planck, Walter Schottky to consider the problem of tube noise and to develop his theory on the shot noise effect which opened the possibility of an alternative method for measurement of electron charge q_e;[2,3] for additional historical remarks see also Ref. 4 and 5. The purpose of the present work is to describe how those ideas can be realized in a day using contemporary low noise operational amplifiers within the budget of 137 $. The experimental set-up is so simple that can be given to students as an exercise for do-it-yourself task for fundamental constants measurement and can be made by every high school physics teacher.

Let us describe the idea. Having a capacitor C switched in parallel with a resistor R, the thermally averaged common voltage is given by the equipartition theorem

$$\frac{1}{2}C\left\langle U^2\right\rangle = \frac{1}{2}k_{\rm B}T'. \qquad (1)$$

Thermal voltages $U(t)$ are too small to be measured directly but using low noise amplifier with almost constant amplification $K \approx 10^6$ or $K^2 = 120$ dB, can be determined by $U(t) = U_a/K$, where U_a is the amplified voltage measured with the multimeter. First attempt was made by Habicht brothers[6] but their electrostatic multiplicator suffers from significant floating of the zero. In the present work we suggest a simple electronic scheme using contemporary integral circuits. Square of the amplified voltage U_a can be determined analogously using multipliers $U_{\rm sq}(t) = U_a^2(t)/U'_m$; the constant U'_m is of order of 1 V. Moreover, time averaging can be performed by a low pass filter with very large time constant $\tau_{\rm av} = R_{\rm av}C_{\rm av}$ of order of 1 minute. This final averaged voltage $U_{\rm dc} = \langle U_{\rm sq}(t)\rangle$ can be measured with a multimeter. Schematically this chain of sequentially transformed voltages is depicted in Fig. ... And finally, the voltage on the capacitor in Eq. (1) can be determined by experimentally measurable quantities

$$\left\langle U^2\right\rangle = \frac{(U_{\rm dc}-U_{\rm off})U_0}{K^2 Z} = \frac{k_{\rm B}T'}{C}, \qquad (2)$$

and non ideality factor which is close to one $Z = 1-\varepsilon$is a matter of routine calculation related to the electronic scheme, typically $\varepsilon < 5\%$. The offset voltage $U_{\rm off}$ is created mainly by internal noise of the amplifier and can be excluded if we measure the averaged voltage $U_{\rm dc}$ for different capacitors and make a linear regression $U_{\rm dc}$ versus $1/C$. The slope of the linear regression gives the temperature in energy units $T = k_{\rm B}T'$, and for known temperature in conditional units we measure the Boltzmann constant $k_{\rm B}$. The offset voltage $U_{\rm off}$ is proportional to the input electric noise $e_{_N} = 0.9\ \mathrm{nV}/\sqrt{\mathrm{Hz}}$ for the used ADA-4898[7] operational amplifier and the pass-bandwidth of the amplifier.

Tracing of the history of determination of Boltzmann constant $k_{\rm B}$ and electron charge q_e by electric noise from the fundamental physics[8,9] to the students laboratory,[10–21] methodological articles and textbooks,[22–26] and contemporary metrology[27–32] gives the milestones of development of physics.

The purpose of the present work is methodical to suggest a do-it-yourself set-up and in appendices to give reference to the state of the art theoretical derivation of the used formulae. From many years the measurement of $q_{\rm e}$ and $k_{\rm B}$ is included in the undergraduate physics laboratories. In all these labworks it is suggested to use standard commercial electronic devices: amplifiers, filters, ohmmeters, oscilloscope, charging batteries, RMS-meter, key attenuator, functional generator, etc.[17–20,31] Continuing this tradition we suggest a simple electronic scheme,which consists of three standard integrated circuits (two double operational amplifiers and one analog multiplier), photodiodes, capacitors, and resistors and can be built for every undergraduate and even high school laboratory. The building of that cheap set-up is possible due to appearance of operational amplifiers whose input voltage noise does not exceed $1\ \mathrm{nV}/\sqrt{\mathrm{Hz}}$, such as the products of Analog Devices,[33] Texas Instruments, etc. In addition, we have four-quadrant analog multipliers (with buried Zener) with nonlinearity better than 1%.[34–36] For many years a made from scratch set-up for measurement of a fundamental constant by statistical methods has been an alternative for students in University of Sofia to pass the exam in statistical physics.

The work is organized as follows:

- In the next Sec. II we will analyze measurement the $k_{\rm B}$ by thermal noise.
- The shot noise as a tool for determination of q_e will be considered in Sec. III.
- The experimental set-up will be described in Sec. IV.
- In order to alleviate the text, state of the art calculation of the bandwidth of the amplifier is given in Appendix A.
- The background theoretical physics is given in other appendixes; the appendixes are to be published in arXiv and are addressed to colleagues involved in construction of similar set-ups for educational physics experiment.

II. SOLUTION BY J. B. JOHNSON

Almost 20 years after Einstein suggestion, Johnson[37] realized the idea to determine Boltzmann constant by measurements of electric noise. The square of the electric voltage can be presented by frequency integration of the voltage spectral density $\left(U^2\right)_f$

$$\left\langle U^2\right\rangle = \lim_{t\to\infty}\frac{1}{t}\int_0^t U^2(t_1)\mathrm{d}t_1 = \int_0^\infty \left(U^2\right)_f \mathrm{d}f,\quad f=\frac{\omega}{2\pi}. \quad (3)$$

Analyzing one dimensional black-body radiation in a transmission line as a *Gedankenexperiment*, Nyquist[38] obtained the general expression for the spectral density of the voltage noise expressed by the real part of the impedance

$$\left(U^2\right)_f = 4\,\Re(Z(\omega))\,\varepsilon_{\rm osc}, \quad (4)$$
$$\varepsilon_{\rm osc} = -\frac{\partial}{\partial\beta}\ln\sum_{n=0}^{\infty}\exp\left[-\beta\hbar\omega\left(n+\frac{1}{2}\right)\right]$$
$$= \frac{\hbar\omega/2}{\tanh\left(\hbar\omega/2T\right)} \approx T, \quad \text{for } \hbar\omega \ll T \equiv k_{\rm B}T'.$$

For example, having a capacitor C parallely connected with a resistor R

$$\frac{1}{Z(\omega)} = \frac{1}{R} + \mathrm{j}\omega C,\ \ \mathrm{j}=-\mathrm{i},\ \ \Re(Z) = \frac{R}{1+(\omega RC)^2}. \quad (5)$$

The substitution of this frequency dependence of the resistance $\Re(Z)$ in the above spectral density Eq. (4) and frequency integration Eq. (3) gives the equipartition formula Eq. (1).

In the experiment for investigation of the noise of resistors a resistor $\Re(Z) = R$ is directly connected to the input of the amplifier. For illustration, we suggest to connect a low pass filter (LPF) after the amplifier, a voltage divider with sequentially connected resistor $R_{\rm LPF}$ and capacitor $C_{\rm LPF}$, with transmission coefficient

$$\Upsilon_{\rm LPF}(\omega) = \frac{1/\mathrm{j}\omega C_{\rm LPF}}{R_{\rm LPF} + 1/\mathrm{j}\omega C_{\rm LPF}}, \quad (6)$$
$$|\Upsilon_{\rm LPF}|^2 = \frac{1}{1+(\omega\tau_{\rm LPF})^2},\quad \tau_{\rm LPF} = R_{\rm LPF}C_{\rm LPF},$$
$$\int_0^\infty |\Upsilon_{\rm LPF}|^2\,\frac{\mathrm{d}\omega}{2\pi} = \frac{1}{4\tau_{\rm LPF}}.$$

Then for the time averaged square of the amplified voltage we obtain

$$\left\langle U_a^2\right\rangle = U_m' U_{\rm dc} = \left[4RT + \left(e_N^2 + 4R_{||}T\right) + \tilde{R}^2 i_N^2\right]B, \quad (7)$$
$$B \approx K^2\frac{1}{4\tau_{\rm LPF}},\quad R_{||}\equiv\frac{rR_f}{r+R_f},\quad \frac{1}{\tau_{\rm LPF}} = \frac{1}{R_{\rm LPF}C_{\rm LPF}}$$

depicted in Fig. 1 for the set-up described in the section Set-upIV. Here the $R_{||}\approx 50\,\Omega$ term describes the thermal noise of the resistors of the amplifiers described in the section Set-up; there will be calculated the pass-bandwidth B. For the

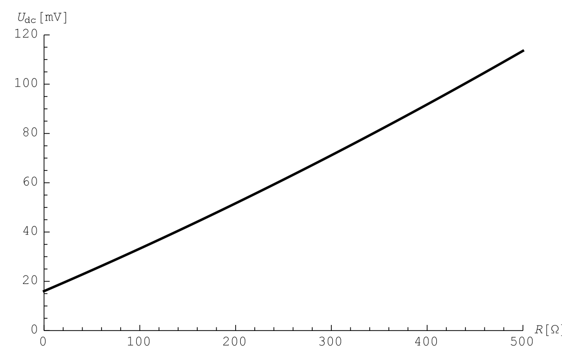

FIG. 1. Thermal noise presented by averaged square of the amplified voltage $U_{\rm dc}$ versus resistivity R according Eq. (7)

special case of $R_{\rm LPF} = R$ and $C = C_{\rm LPF}$ we return to Eq. (2). However, now we will change not the capacitor of the filter but the Johnson resistor R at the input of the amplifier. Amplified voltage has to be squared and time averaged. Measuring the DC voltage $U_{\rm dc}(R)$ for different resistors by a multimeter, we can perform parabolic regression of the data. The curvature is determined by the current noise of the operational amplifier $i_N = 2.4\,\mathrm{pA}/\sqrt{\mathrm{Hz}}$ for ADA 4898.[7] The Boltzmann constant is given by the linear coefficient $\mathrm{d}U_{\rm dc}/\mathrm{d}R$ of the polynomial regression

$$U_{\rm dc} = U_{\rm off} + \frac{\mathrm{d}U_{\rm dc}}{\mathrm{d}R}R + \frac{1}{2}\frac{\mathrm{d}^2U_{\rm dc}}{\mathrm{d}R^2}R^2 \quad (8)$$

describing the slope of the curve $U_{\rm dc}(R)$ at small resistances

$$k_{\rm B} = \frac{U_0}{4T'B}\frac{\mathrm{d}U_{\rm dc}}{\mathrm{d}R}. \quad (9)$$

In the next section we will describe how this set-up can be used for determination of the electron charge q_e. Instead introducing notations for the coefficient of the parabolic fit, we use the notations for derivatives because in some sense polynomial fit is a tool for differentiation of a function tabulated with noise.

III. SCHOTTKY SHOT NOISE AND DETERMINATION OF THE ELECTRON CHARGE

As far as we know the Schottky[39] idea was realized for first time by in 1925 by Hull and Williams;[40] see also the papers of N. H. Williams and W. S. Huxford[9] and Johnson[8].

If a mean current $\langle I\rangle$ is created by incoherent quantum transitions, this current is a sum of independent charge impulses of different electron transitions

$$I(t)=\sum_a q_e\delta(t-t_a), \tag{10}$$

$$\langle I\rangle=\lim_{t\to\infty}\frac{1}{t}\int_0^t I(t_1)\mathrm{d}t_1=\nu q_e. \tag{11}$$

Here we suppose that random electron impulses are homogeneously distributed in time with averaged rate of ν electrons per unit time. Typical examples are electron in vacuum diodes or as in our casephotodiode illuminated by an electrical lamp; definitely the incoherent black body radiation, not a light by light emitting diode (LED), which emits correlated photons and the mean charge of the impulse can exceed many electron charges.

The spectral density of the current of these electron shots is proportional to the electron charge

$$(I^2)_f=2q_e\langle I\rangle. \tag{12}$$

The spectral density is the integrand of averaging of squares

$$\left\langle I^2\right\rangle=\lim_{t\to\infty}\frac{1}{t}\int_0^t I^2(t_1)\mathrm{d}t_1=\int_0^\infty \left(I^2\right)_f\mathrm{d}f. \tag{13}$$

If the current passes through a resistor, for the averaged voltage we have

$$\langle U\rangle=R\langle I\rangle \tag{14}$$

and analogously for the voltage spectral density

$$(U^2)_f=R^2(I^2)_f=2q_eR\langle U\rangle. \tag{15}$$

This shot noise has to be added to the thermal noise and Eq. (7) now reads

$$U_0U_{\mathrm{dc}}=\left[4\tilde{R}T+e_N^2+\tilde{R}^2i_N^2+2q_eR\langle U\rangle\right]B. \tag{16}$$

Quantum shots of the electron current can be created by a photodiode and by changing the intensity of light we can change the averaged voltage $\langle U\rangle$ and the Schottky noise. In such a way we can investigate the dependence $U_{\mathrm{dc}}(\langle U\rangle)$ and perform the linear regression. The slope of the line $\mathrm{d}U_{\mathrm{dc}}/\mathrm{d}\langle U\rangle$ can be graphically calculated as a ratio of the differences, and for the electron charge we finally derive

$$q_e=\frac{U_0}{2RB}\frac{\mathrm{d}U_{\mathrm{dc}}}{\mathrm{d}\langle U\rangle}. \tag{17}$$

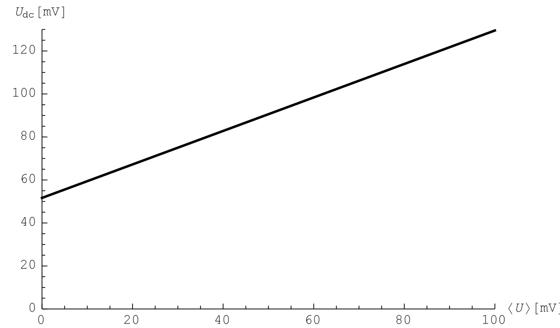

FIG. 2. Thermal shot noise presented by averaged square of the amplified voltage U_{dc} versus resistivity R according Eq. (7)

For applicability of this result, we suppose that electron shots do not overlap significantly $\nu\tau_{\mathrm{LPF}}\ll 1$.

As we mentioned, the purpose of the present work is purely methodological suggesting a simple do-it-yourself set-up which is described in the next section.

IV. SET-UP

The electric circuit of the proposed set-up is depicted at Fig. 3, read the self-explained figure caption.

V. CONCLUSIONS

In conclusion we hope that if all the parameters of the experimental set-up are carefully taken into account, an accuracy better than a percent in extracting the electron charge q_{e}, and Boltzmann constant k_{B} can be reached. In high schools we can use approximation $Z\approx 1$ and within several percent accuracy we need no integration. Additionally, we can use higher frequency Op Amp and many NIA with moderate amplification, and this will increase $f_{-3\mathrm{dB}}$ for the whole amplification sequence. As the budget is less than 137 \$ ($\hbar c/e^2$), the present experimental set-up can be easily reproduced in every university and high-school in the world. The current task which we set in the agenda for the physics education labs is determination of the Boltzmann constant by the method originally suggested by Einstein, not by dissipation, but by measurement of the averaged energy of a capacitor $\langle E_c\rangle=\frac{1}{2}\left\langle U^2\right\rangle=\frac{1}{2}k_{\mathrm{B}}T'$, just as an illustration of the John James Waterston equipartition theorem from the beginning of statistical physics.

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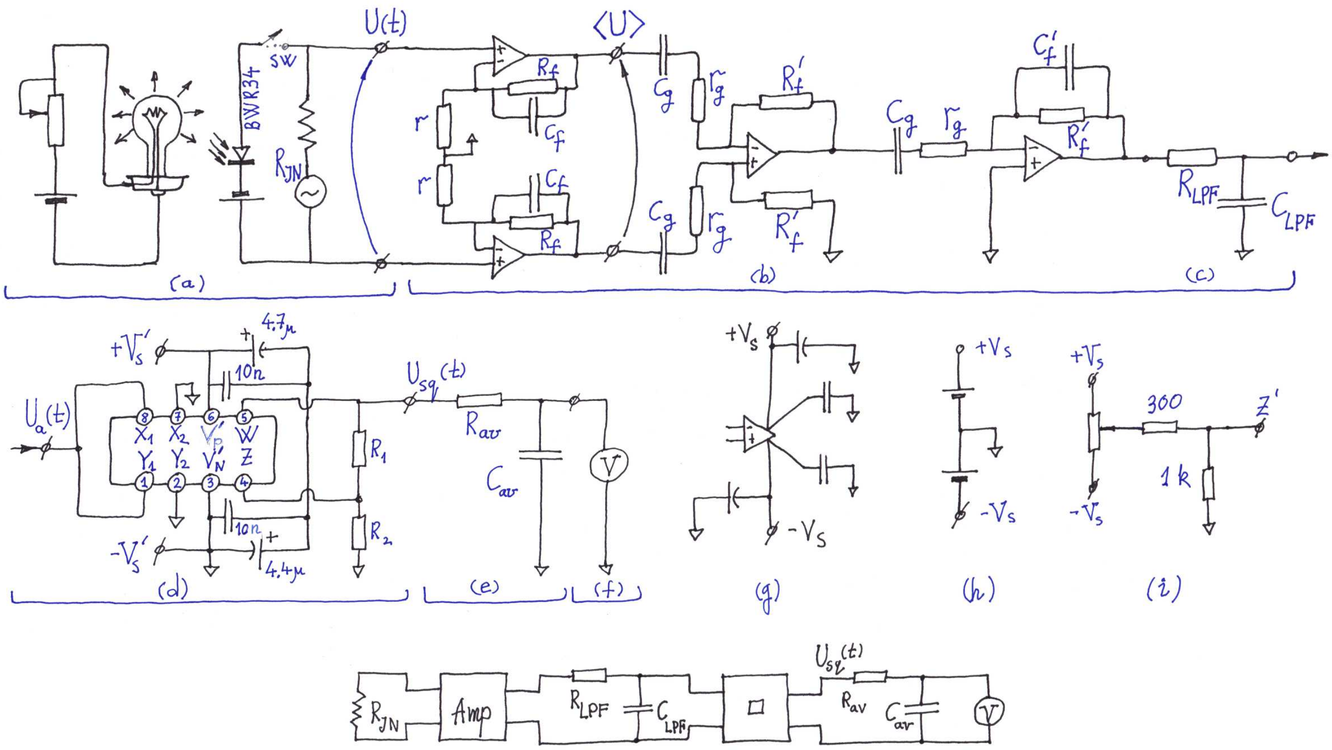

FIG. 3. Circuit for measurement of k_B and q_e. 1st row – linear part, 2 row – squaring and averaging, 3rd row – block scheme. a) Shot and thermal noise sources: lamp, battery and potentiometer; photodiode BWR34 giving Schottky noise, battery and switch; Johnson-Nyquist resistor; noise voltage $U(t)$. b) Linear amplifier: instrumentation amplifier followed by non-inverting amplifier, total amplification $\Upsilon \approx -(1 + R_f/r)(R_f'/r_g)^2 \approx 10^6$. c) Low pass filter with partial bandwidth $B_\mathrm{LPF} = 1/4\tau_{_\mathrm{LPF}}$. d) Squaring part $U_\mathrm{sq} = U_a^2/U_m'$, $U_m' = U_m R_1/(R_1+R_2)$, $U_m = 1\,\mathrm{V}$. e) Averaging filter $\tau_\mathrm{av} \gg \tau_{_\mathrm{LPF}}$. f) Multimeter switched in voltmeter mode. g) Voltage supply for ADA4898-2/ h) Batteries for bipolar supply. i) Zeroing of the multiplier. 3rd row: Voltage noise, amplifier, LPF, squaring by multiplier, averaging, measuring. Parameters: $r = 10\,\Omega$, $R_f = 1\,\mathrm{k}\Omega$, $C_f = 10\,\mathrm{pF}$, $R_f/r = 100$, $C_g = 10\,\mu\mathrm{F}$, $r_g = 100\,\Omega$, $R_f' = 10\,\mathrm{k}\Omega$, $R_f'/r_g = 100$, $R_1 = 100\,\Omega$, $R_2 = 1\,\mathrm{k}\Omega$, $R_\mathrm{av} = 100\,\mathrm{k}\Omega$, and $C_\mathrm{av} = 150\,\mu\mathrm{F}$. The mean photovoltage $\langle U\rangle$ is measured before the large gain capacitors C_g stopping further amplification of the offset voltages.

in the development of our set-up. University of Sofia St. Clement of Ohrid received many free of charge OpAmps from Analog Devices, which is actually an important help to our university research and education. As we are going to collect and cite all works on the problem and we thank in advance to colleagues sending us references on determination ofq_e and k_B by measurements of fluctuations. Especially we are interested in contemporary students lab equipments.

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Appendix A: Analog electronics in a nutshell

This appendix is addressed to colleagues involved with construction of similar devices and modifying the scheme. This recall of the standard electronics notion is not addressed to the students.

1. Operational amplifier master equation

In the beginning was the approximate master equation of operational amplifiers

$$\hat{G}^{-1}U_0(t) = \left(\frac{1}{G_0} + \tau\frac{\mathrm{d}}{\mathrm{d}t}\right)U_0 = U_+ - U_-, \qquad \text{(A1)}$$

giving the relation between the output voltage $U_0(t)$ and difference of (+) and (-) inputs of operational amplifier, cf. Ref. 41. For harmonic signals, introducing j-imaginary unit

$$U \propto \mathrm{e}^{\mathrm{j}\omega t} = \mathrm{e}^{st}, \quad \mathrm{j} = -\mathrm{i}, \quad s \equiv \mathrm{j}\omega, \qquad \text{(A2)}$$

we have

$$G^{-1}(\omega)U_0 = \left(\frac{1}{G_0} + \mathrm{j}\omega\tau\right)U_0 = U_+ - U_-, \qquad \text{(A3)}$$

$$G^{-1} = \left(\frac{1}{G_0} + s\tau\right)\frac{1 + a_1 s + a_2 s^2 + a_3 s^3 + \dots}{1 + b_1 s + b_2 s^2 + b_3 s^3 + \dots}, \qquad \text{(A4)}$$

where in the second row we present a Padé approximant for the frequency dependent open loop gain. For the used by us operational amplifier ADA4898[7] the static open loop gain is approximately 100 dB

$$G_0 = 10^5, \qquad f_{\mathrm{crossover}} \equiv \frac{1}{2\pi\tau} = 100\ \mathrm{MHz} \qquad \text{(A5)}$$

and time constant is parameterized by the crossover frequency $f_{\mathrm{crossover}}$.

In the next subsections we will recall how the frequency dependent open loop gain determines the frequency dependent transmission function of different amplifiers. Later we will take into account the influence of the internal resistance of the operational amplifiers in differential mode, and finally we will analyze scaling voltage of the true RMS meter by an analog multiplier AD633[34].

2. Differential Amplifier

Let us trace, see Fig. 6, the voltage drop from the negative (n) input of the differentialamplifier and the output (0) voltage U_0. The input current is given by the Ohm law and the voltage at the (-) input of the operational amplifier (OpAmp) is given by the proportion of the voltage divider

$$U_- = \left(I_n = \frac{U_n - U_0}{z + R_f}\right)R_f + U_0 = \frac{R_f}{z + R_f}U_n + \frac{z}{z + R_f}U_0. \qquad \text{(A6)}$$

Analogous voltage divider gives the voltage of (+) input of the OpAmp expressed by the voltage drop between positive (p) input of the ground

$$U_+ = \left(I_n = \frac{U_p - 0}{z + R_f}\right)R_f + 0 = \frac{R_f}{z + R_f}U_n. \qquad \text{(A7)}$$

Then we calculate the voltage difference at the inputs of the OpAmp

$$U_+ - U_- = \frac{\Delta U \equiv U_p - U_n}{z + R_f}R_f - \frac{z}{z + R_f}U_0 \qquad \text{(A8)}$$

which together with Eq. (A3) gives

$$\begin{aligned}
\Upsilon_\Delta(\omega) &\equiv \frac{U_0}{U_p - U_n} \qquad \text{(A9)}\\
&= \frac{1}{\Lambda(\omega) + [\Lambda(\omega) + 2\epsilon_a(\omega)]\,G^{-1}(\omega) + G^{-1}(\omega)},\\
\Lambda(\omega) &= \frac{z_g(\omega)}{Z_f(\omega)}, \quad z_g = r + \frac{1}{\mathrm{j}\omega C_g}, \quad Z_f = R_f\\
\varepsilon_a(\omega) &= z_g(\omega)\sigma_a(\omega), \quad \sigma_a(\omega) \equiv \frac{1}{R_a} + \mathrm{j}\omega C_a,
\end{aligned}$$

cf. Eq. (3.14) of Ref. 42. Here we added the correction ε_a of internal conductivity σ_a between inputs of OpAmp in differential mode. A detailed derivation will be given later in subsection A 7.

3. Inverting Amplifier

If for the differential amplifier, see Fig. 4, the positive (p) input is grounded $U_p = 0$, this leads to $U_+ = 0$ and $\Delta U = -U_n$ hence for the transmission function we get $\Upsilon_{\mathrm{IA}}(\omega) =$

$-\Upsilon_\Delta(\omega)$or finally

$$\Upsilon_{\mathrm{IA}}(\omega) \equiv \frac{U_0}{U_n} = -\frac{1}{\Lambda(\omega) + [\Lambda(\omega) + \epsilon_a(\omega)]\, G^{-1}(\omega) + G^{-1}(\omega)}, \tag{A10}$$

$$\Lambda(\omega) = \frac{z_g(\omega)}{Z_f(\omega)}, \quad z_g = r + \frac{1}{\mathrm{j}\omega C_g}, \quad Z_f = R_f$$

$$\varepsilon_a(\omega) = z_g(\omega)\sigma_a(\omega), \quad \sigma_a(\omega) \equiv \frac{1}{R_a} + \mathrm{j}\omega C_a,$$

The subtle difference with Eq. (A9) is only the coefficient in front of ϵ_a, cf. Eq. (3.10) of Ref. 42. For negligible the internal conductivity σ_a

$$\Upsilon_{\mathrm{IA}}(\omega) = -\frac{1}{\Lambda(\omega) + [1 + \Lambda(\omega)]\, G^{-1}(\omega)}. \tag{A11}$$

A large capacitance of the gain impedance stops the offset voltages coming from former amplification blocks. Additionally, for high enough working frequencies $\omega\, r\, C_g \gg 1$ and negligible static-inverse-open-loop gain $G_0^{-1} \ll 1$ we obtain the well-known formulae Eq. (7) and Eq. (8) of Ref. 43, see also Ref. 42. For frequencies much lower than the crossover frequency $f_{\mathrm{crossover}}$ of the OpAmp the amplification is in a wide band frequency independent

$$\Upsilon_{\mathrm{IA}}(\omega) \approx -\frac{2\pi f_{\mathrm{crossover}} R_f}{(R_f + r)s + 2\pi f_{\mathrm{crossover}}\, r}, \tag{A12}$$

$$\Upsilon_{\mathrm{IA}} = -\frac{R_f}{r} \qquad \text{for} \quad f = \frac{\omega}{2\pi} \ll f_{\mathrm{crossover}}. \tag{A13}$$

For calculation of the pass-bandwidth we will need to know the square of the modulus of the complex amplification

$$|\Upsilon_{\mathrm{IA}}(\omega)|^2 = |\Upsilon_\Delta(\omega)|^2 = \frac{(\omega\tau_g)^2}{\left[1 + G_0^{-1} - M\omega^2\tau\tau_g\right]^2 + (\omega\tau_g)^2 \left[\dfrac{\tau}{\tau_g} + \Lambda_0 + G_0^{-1} M\right]^2}, \tag{A14}$$

$$\Lambda_0 \equiv \frac{r}{R_f}, \qquad M \equiv 1 + \Lambda_0, \qquad \tau_g \equiv C_g R_f.$$

For large enough amplification $r \ll R_f$ from Eq. (A11) we obtain

$$\frac{R_f}{r} \gg 1, \quad \omega r C_g \gg 1, \quad \omega\tau \ll 1,$$

$$\Upsilon_{\mathrm{IA}}(\omega) \approx -\frac{R_f}{r + \frac{1}{\mathrm{j}\omega C_g} + \mathrm{j} L_{\mathrm{eff}}\omega}, \tag{A15}$$

$$L_{\mathrm{eff}} \equiv \tau R_f, \quad Q = \frac{\sqrt{L_{\mathrm{eff}}/C_g}}{r} = \frac{\sqrt{\tau R_f / C_g}}{r} \ll 1,$$

and in this approximation the frequency dependence of the amplification reminds the frequency response of an over-damped oscillator with low quality factor Q.

4. Non-inverting amplifier

For the non-inverting amplifier, depicted at Fig. 5, the input voltage U_i is applied directly to the (+) input of the OpAmp $U_+ = U_i$. If we trace the voltage drop from the output voltage U_0 through the feedback Z_f and gain z impedances to the ground, we obtain the potential of (-) input U_- of the OpAmp as a voltage divider

$$U_- = \left(I = \frac{U_0}{Z_f + z}\right) z. \tag{A16}$$

The master equation of OpAmps Eq. (A3) $U_+ - U_- = G^{-1}(\omega) U_0$ gives after an elementary substitution the frequency dependent amplification

$$\Upsilon_{\mathrm{NIA}}(\omega) \equiv \frac{U_0}{U_+} = \frac{1}{Y^{-1}(\omega) + G^{-1}(\omega)[1 + \epsilon_a(\omega)]}, \tag{A17}$$

$$Y(\omega) = \frac{Z_f(\omega)}{z_g(\omega)} + 1, \quad z_g = r, \quad \frac{1}{Z_f} = \frac{1}{R_f} + \mathrm{j}\omega C_f,$$

$$\epsilon_a(\omega) = Y^{-1} r \sigma_a, \quad \sigma_a \equiv \frac{1}{R_a} + \mathrm{j}\omega C_a. \tag{A18}$$

If the influence of the small feedback capacitor C_f parallelly switched to the feedback resistor is negligible, i.e.for $\omega\, R_f\, C_f \ll 1$ and $G_0^{-1} \ll 1$, we obtain the well-known formulae Eq. (4) and Eq. (5) of Ref. 43, see also Eq. (3.12) of Ref. 42

$$\Upsilon_{\mathrm{NIA}}(\omega) = \frac{2\pi f_{\mathrm{crossover}}(R_f + r)}{(R_f + r)s + 2\pi f_{\mathrm{crossover}}\, r}, \tag{A19}$$

$$\Upsilon_{\mathrm{NIA}} = Y_0 \equiv \frac{R_f}{r} + 1, \ \text{for}\ f = \frac{\omega}{2\pi} \ll f_{\mathrm{crossover}}. \tag{A20}$$

For measurement of $f_{\mathrm{crossover}}$ we have to perform linear regression for $|\Upsilon_{\mathrm{NIA}}|^{-2}(\omega^2)$ using the experimentally determined ratio of input to output voltages

$$\frac{|U_i|^2}{|U_0|^2} = \frac{1}{(1 + R_f/r)^2} + \frac{f^2}{f^2_{\mathrm{crossover}}}, \qquad f = \frac{\omega}{2\pi}. \tag{A21}$$

Here the first term is negligible if reciprocal amplification coefficient $1/(1 + R_f/r)$ is smaller than accuracy of measurement, say 1%.

The calculation of the pass-bandwidth requires the square of themodulus of the complex amplification, which after a straight forward calculation from A17

$$|\Upsilon_{\mathrm{NIA}}(\omega)|^2 = \frac{\mathcal{N}^2(\omega)}{\left[G_0^{-1}\mathcal{N} + (Y_0^{-1} + \omega^2\tau_s^2)\right]^2 + (\omega\tau_s)^2 \left[\dfrac{\tau}{\tau_s}\mathcal{N} + I_0\right]^2}, \tag{A22}$$

where

$$\mathcal{N}(\omega) = 1 + \omega^2\tau_s^2, \quad I_0 \equiv 1 - Y_0^{-1}, \quad \tau_s \equiv \frac{C_f R_f}{Y_0}. \tag{A23}$$

5. Instrumentation amplifier

The instrumentation amplifier from our set-up consists of 2 symmetrical non-inverting amplifiers followed by a differential amplifier and for the amplification of instrumentation amplifier we have

$$\Upsilon_{\mathrm{Instr}}(\omega) = \Upsilon_{\mathrm{NIA}}(\omega)\Upsilon_\Delta(\omega). \tag{A24}$$

6. Amplifier and pass-bandwidth

The instrumentation amplifier is followed by an inverting amplifier and for the whole amplifier we have

$$\Upsilon_{\mathrm{Amp}}(\omega) = \Upsilon_{\mathrm{Instr}}(\omega)\Upsilon_{\mathrm{IA}}(\omega). \tag{A25}$$

a. Low-pass filter

After the amplification the signal passes trough a low pass filter which is also a voltage divider

$$U_C = \left(I = \frac{U}{Z_R + Z_C}\right) Z_C, \tag{A26}$$

$$Z_C = \frac{1}{\mathrm{j}\omega C_{\mathrm{LPF}}}, \quad Z_R = R_{\mathrm{LPF}}$$

with transmission function Υ_{LPF} and bandwidth B_{LPF}

$$\Upsilon_{\mathrm{LPF}}(\omega) = \frac{U_C}{U} = \frac{1}{1+\mathrm{j}\omega\tau_{\mathrm{LPF}}}, \tag{A27}$$

$$\tau_{\mathrm{LPF}} = R_{\mathrm{LPF}}C_{\mathrm{LPF}}, \quad |\Upsilon_{\mathrm{LPF}}|^2 = \frac{1}{1+(\omega\tau_{\mathrm{LPF}})^2},$$

$$B_{\mathrm{LPF}} = \int_0^\infty |\Upsilon_{\mathrm{LPF}}|^2 \frac{\mathrm{d}\omega}{2\pi} = \frac{1}{4\tau_{\mathrm{LPF}}} = \frac{1}{4R_{\mathrm{LPF}}C_{\mathrm{LPF}}}.$$

b. Pass-bandwidth

The linear part of our set-up consist of non-inverting amplifiers and differential amplifier (included in the instrumentation amplifier block) followed by non-inverting amplifier and a low pass filter. In such a way for the total frequency dependent amplification we have

$$\Upsilon(\omega) = \Upsilon_{\mathrm{NIA}}\Upsilon_{\Delta}\Upsilon_{\mathrm{IA}}\Upsilon_{\mathrm{LPF}}, \tag{A28}$$

and for the pass-bandwidth

$$B = \int_0^\infty |\Upsilon(\omega)|^2 \frac{\mathrm{d}\omega}{2\pi}. \tag{A29}$$

In a broad frequency intervals the amplification of different amplifiers is almost constant

$$K_{\mathrm{NIA}} = \frac{R_f}{r}+1, \quad K_{\Delta} = \frac{R_f}{r}, \quad K_{\mathrm{IA}} = -\frac{R_f}{r} \tag{A30}$$

and the frequency dependence is concentrated mainly in the low pass filter. Approximately for the bandwidth we have

$$\begin{aligned} B_0 &= (K_{\mathrm{NIA}}K_{\Delta}K_{\mathrm{IA}})^2 B_{\mathrm{LPF}} \\ &= \left(\frac{R_f}{r}+1\right)^2 \left(\frac{R_f}{r}\right)^4 \frac{1}{4R_{\mathrm{LPF}}C_{\mathrm{LPF}}}. \end{aligned} \tag{A31}$$

The exact result can be parameterized by a renormalizing factor Z or by a small correction to the simple approximative formula

$$B = ZB_0, \quad Z = 1-\varepsilon \equiv \frac{B}{B_0}. \tag{A32}$$

In short, for the small correction ε we arrive at a programmable expression

$$\begin{aligned} Z = 1-\varepsilon = {} & 4R_{\mathrm{LPF}}C_{\mathrm{LPF}}\left(\frac{r}{R_f}\right)^4\left(\frac{r}{r+R_f}\right)^2\int_0^\infty\frac{\mathrm{d}\omega}{2\pi} \\ & \times\frac{\mathcal{N}^2(\omega)}{\left[G_0^{-1}\mathcal{N}+(Y_0^{-1}+\omega^2\tau_s^2)\right]^2+(\omega\tau_s)^2\left[\dfrac{\tau}{\tau_s}\mathcal{N}+I_0\right]^2} \\ & \times\left[\frac{(\omega\tau_g)^2}{\left[1+G_0^{-1}-M\omega^2\tau\tau_g\right]^2+(\omega\tau_g)^2\left[\dfrac{\tau}{\tau_g}+\Lambda_0+G_0^{-1}M\right]^2}\right]^2 \\ & \times\frac{1}{1+(\omega\tau_{\mathrm{LPF}})^2}, \end{aligned} \tag{A33}$$

where let us repeat the introduced notations

$$\mathcal{N}(\omega) = 1+\omega^2\tau_s^2, \quad I_0 \equiv 1-Y_0^{-1}, \quad \tau_s \equiv \frac{C_fR_f}{Y_0}.$$

$$Y_0 \equiv \frac{R_f}{r}+1, \quad \tau_{\mathrm{LPF}} = R_{\mathrm{LPF}}C_{\mathrm{LPF}},$$

$$\Lambda_0 \equiv \frac{r}{R_f}, \qquad M \equiv 1+\Lambda_0, \qquad \tau_g \equiv C_gR_f, \tag{A34}$$

which can be slightly elaborated including the small input conductivity of the OpAmp. Substitution in this formula gives $\varepsilon < 5\%$, and for educational purposes can be neglected. In the next subsection however, we will take into account the small internal conductivity of the OpAmp in order to have state of the art calculation of the bandwidth in which we have considered all accessories of electronics.

7. Influence of finite common mode input conductivity of operational amplifiers

In this appendix we will investigate the influence of a small conductivity between the inputs of an operational amplifier in differential mode. For the simplicity of the notations we will write only the big differential mode input resistance R_a and later on we will include the differential mode input capacitance. In the next subsection we will start with the inverting amplifier.

a. Inverting amplifier

Standard scheme of the inverting amplifier is presented at Fig. 4. Let us write the Kirchhoff equations. The input current I_i is branching to the current I_f passing trough the feedback resistor R_f and the current I_a flowing between (-) and (+) inputs of the OpAmp

$$I_i = I_a + I_f. \tag{A35}$$

Let trace the voltage drop between the input voltage by gain resistor r_g and input resistor R_a to the ground

$$U_i = R_aI_a + r_gI_i. \tag{A36}$$

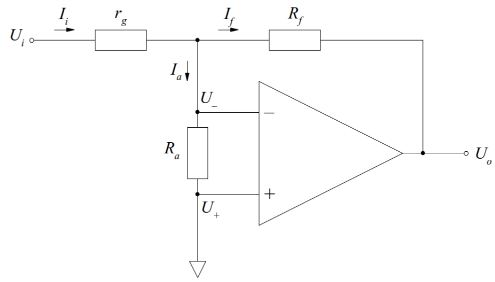

FIG. 4. Inverting amplifier with a finite common mode input resistance R_a (non-zero input conductivity) of the operational amplifier.

The Ohm law of the internal resistor of the OpAmp is

$$-R_a I_a = U_+ - U_- = G^{-1} U_o, \qquad \text{(A37)}$$

where the potential difference between inputs voltages is expressed by the output voltage U_o and reciprocal open loop gain G^{-1}. Finally starting from the output voltage U_o and passing trough the feedback resistor R_f and gain resistor r_g we reach the input voltage U_i.

$$U_i = U_o + R_f I_f + r_g I_i. \qquad \text{(A38)}$$

From Eq. (A37) we express $I_a = -U_o/R_a G$ and substitute in Eq. (A36) which now reads

$$U_i = -G^{-1} U_o + r_g I_i, \qquad \text{(A39)}$$

and gives

$$I_i = \left(U_i + G^{-1} U_o\right)/r_g. \qquad \text{(A40)}$$

The substitution of I_i in Eq. (A35) gives

$$I_f = \frac{U_i}{r_g} + \left(\frac{1}{r_g} + \frac{1}{R_a}\right)\frac{U_o}{G}. \qquad \text{(A41)}$$

Substitution of derived in such a way currents in Eq. (A38) gives for the output voltage U_o the well-known result Eq. (3.10) of Ref. 42

$$\frac{U_o}{U_i} = -\frac{R_f/r_g}{1 + G^{-1}\left(1 + \dfrac{R_f}{r_g} + \dfrac{R_f}{R_a}\right)}. \qquad \text{(A42)}$$

In the next subsection we will perform analogous consideration for the non-inverting amplifier.

b. Non-inverting amplifier

The non-inverting amplifier circuit is depicted in Fig. 5. The current from the ground I_n is branching to the current I_f passing through the feedback resistor R_f and the current I_a flowing through the OpAmp (-) and (+) inputs. The charge conservation in the branching point is

$$I_n = I_f + I_a. \qquad \text{(A43)}$$

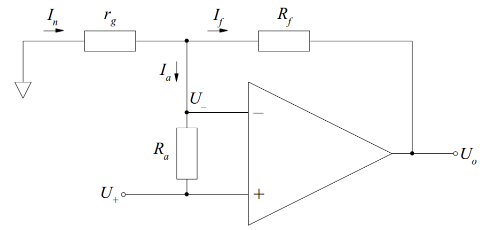

FIG. 5. Non-inverting amplifier with a finite input resistance R_a (non-zero input conductivity) of the operational amplifier.

Tracing the voltage drop from ground through the gain resistor r_g and the OpAmp internal resistance R_a to the input voltage U_i we have

$$U_+ = -r_g I_n - R_a I_a. \qquad \text{(A44)}$$

Where we take into account that the input voltage is directly applied to the positive input of the operational amplifier where $U_i \equiv U_+$. The Ohm law of the internal resistor of the OpAmp is

$$-R_a I_a = U_+ - U_- = G^{-1} U_o, \qquad \text{(A45)}$$

where the potential difference between the input voltage is expressed by the output voltage U_o and reciprocal open loop gain G^{-1}. The last circuit to consider is the voltage drop from the output voltage U_o through R_f and r_g to ground

$$U_o = -I_f R_f - I_n r_g. \qquad \text{(A46)}$$

Expressing the currents I_f from Eq. (A43), I_n from Eq. (A44), I_a from Eq. (A45) and substituting them in Eq. (A46), for the ratio of the output voltage U_o to the input voltage U_i we obtain

$$\frac{U_o}{U_i} = \frac{R_f/r_g + 1}{1 + G^{-1}\left(1 + \dfrac{R_f}{r_g} + \dfrac{R_f}{R_a}\right)}. \qquad \text{(A47)}$$

The differential amplifier analyzed in the next subsection is slightly more complicated.

c. Differential amplifier

The differential amplifier circuit is shown in Fig. 6. The feedback current I_f going through the feedback resistor R_f is a sum of the input current I_n from the input voltage U_n and the current I_a flowing through the OpAmp (+) and (-) inputs

$$I_f = I_n + I_a. \qquad \text{(A48)}$$

The other input current I_p from the input voltage U_p flowing through the gain resistor r_g is branching to the current I_a and the current I_0 flowing through the other feedback resistor R_f to ground

$$I_p = I_0 + I_a. \qquad \text{(A49)}$$

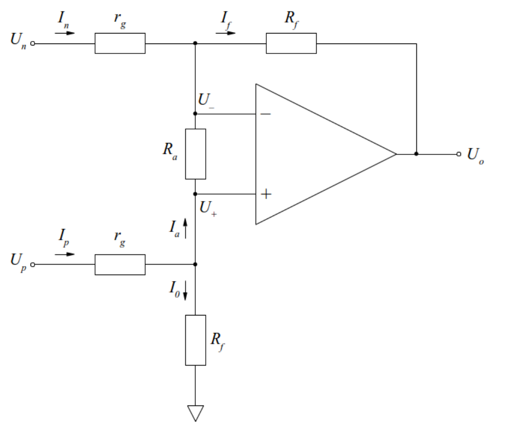

FIG. 6. Differential amplifier with a finite input resistance R_a (non-zero input conductivity) of the operational amplifier.

Now let trace the circuit between U_n and the output voltage U_o passing through r_g and the feedback resistor R_f

$$U_n - U_o = r_g I_n + R_f I_f. \tag{A50}$$

Another circuit to consider is from U_p through r_g and R_f to ground, for which the Kirchhoff's voltage law reads

$$U_p = r_g I_p + R_f I_0. \tag{A51}$$

Substituting I_f from Eq. (A48) into Eq. (A50), for I_n we derive

$$I_n = \frac{U_n - U_o}{r_g + R_f} - \frac{R_f}{r_g + R_f} I_a. \tag{A52}$$

Expressing I_0 from Eq. (A49) and substituting it into Eq. (A51), for I_p we obtain

$$I_p = \frac{U_p}{r_g + R_f} + \frac{R_f}{r_g + R_f} I_a. \tag{A53}$$

The difference between positive U_+ and negative U_- input of the operational amplifier can be expressed simultaneously by the output voltage U_o and the Ohm law for the internal resistance of the OpAmp

$$R_a I_a = U_+ - U_- = G^{-1} U_o. \tag{A54}$$

The current $I_a = U_o/GR_a$ is easily expressed by the open loop gain G, output voltage U_o and internal resistance of the OpAmp in differential mode R_a. Finally we write down the Kirchhoff's voltage law for the circuit from U_p to U_n through the resistors with values r_g, R_a and r_g

$$U_p - U_n \equiv \Delta U = r_g I_p + R_a I_a - r_g I_n, \tag{A55}$$

where $\Delta U = U_p - U_n$ denotes the input voltage difference between positive U_p and negative U_n input voltages. Substitution of the already derived expressions for the input currents I_p, I_n and the internal OpAmp current I_a into Eq. (A55) yields for the transmission coefficient, i.e. ratio of the output voltage U_o and input voltage difference ΔU,

$$\frac{U_o}{\Delta U} = \frac{R_f/r_g}{1 + G^{-1}\left(1 + \dfrac{R_f}{r_g} + 2\dfrac{R_f}{R_a}\right)}. \tag{A56}$$

For high frequencies we have to take into account also the input capacitance of the OpAmp in differential mode $R_a \to Z_a = R_a/(1 + \mathrm{j}\omega R_a C_a)$. For the used ADA4898[7] $R_a = 5\,\mathrm{k\Omega}$ and $C_a = 2.5\,\mathrm{pF}$. In all formulae we suppose frequency dependent open loop gain $G^{-1}(\omega) = G_0^{-1} + \mathrm{j}\omega\tau$ given by the master equation of the OpAmps Eq. (A1).

The linear circuits are terminated by a LPF and after that the filtered voltage is squared and averaged by the true RMS meter described in the next subsection.

8. True RMS by multiplier

For true RMS meter we use analog multiplier AD633[34] and let us recall the basic equation of the multiplier output voltage U_W. If X2 and Y2 inputs of the multiplier are connected to the ground, at X1 input is applied voltage U_X, and voltage U_Y is applied to Y1 input, for the voltage U_W according to the equation in Fig. 17 of Ref. 34 and Eq. (4) and Fig. 20 of Ref. 35 we have

$$U_W = \frac{U_X U_Y}{U'_m}, \quad U'_m \equiv (1-k) U_m, \quad k = \frac{R_1}{R_1 + R_2}. \tag{A57}$$

Here R_1 is the resistor between the output W and Z summing input of the multiplier. The summing input of the multiplier Z is grounded through a resistor R_2. Both resistors are part of a trimmer potentiometer with resistivity $R_1 + R_2$. The potentiometer can adjust the scaling voltage U'_m of the true RMS meter. For AD835 the scaling voltage of the multiplier is nominally $U_m^{(\mathrm{AD835})} = 1.05\,\mathrm{V}$, while for the used in our set-up AD633 $U_m^{(\mathrm{AD633})} = 10\,\mathrm{V}$.

For true RMS meter the filtered signal is applied simultaneously to X1 and Y1 inputs $U_X = U_Y = U_a(t)$ and finally the the output voltage $U_W(t) = U_a^2/U'_m$ is averaged by another low pass filter with very large time constant

$$\tau_{\mathrm{dc}} = R_{\mathrm{dc}} C_{\mathrm{dc}} = 150\,\mu\mathrm{F} \times 100\,\mathrm{k\Omega} = 15\,\mathrm{s}. \tag{A58}$$

In such a way for constant spectral density of the signal, we can measure after 2 minutes ($8\tau_{\mathrm{dc}}$) the averaged dc voltage

$$U_{\mathrm{dc}} \approx \int_0^\infty U_W(-t) \exp(-t/\tau_{\mathrm{dc}}) \frac{\mathrm{d}t}{\tau_{\mathrm{dc}}} = \frac{\langle U_a^2 \rangle}{U'_m}. \tag{A59}$$

For our set-up $U'_m = 1.00\,\mathrm{V}$.

Before the multiplier the filtered by LPF signal is correlated in time intervals of order of τ_{LPF}. Averaging time τ_{dc} is much longer and we have $N = \tau_{\mathrm{dc}}/\tau_{\mathrm{LPF}}$ independent measurements and uncertainty of the final voltage averaging is of order of $1/\sqrt{N} < 10^{-3}$. Such fluctuations give the accuracy limit of the set-up for determination of the electron charge and the Boltzmann constant. Finally, the accuracy can be slightly improved if we record the averaging signal and perform further digital averaging.

Appendix B: Thermal noise

In this appendix we will consider the Langevin approach for the classical thermodynamic fluctuations in Sec. B 1 then we will recall the Nyquist quantum mechanical approach to electric fluctuations in Sec. B 2 and finally in Sec. B 3 we will rederive the Callen-Welton fluctuation dissipative theorem as consequence of the Nyquist theorem.

1. Langevin stochastic differential equation approach

The Langevin method of stochastic differential equations is a very convenient tool which unfortunately is not given in regular courses on calculus or statistical physics. The purpose of this subsection is to give a detailed pedagogical derivation.

Let us analyze a simple circuit with sequentially connected ohmic resistance R and inductance L as it is depicted in Fig. 7. The thermal fluctuations in the resistor create a random

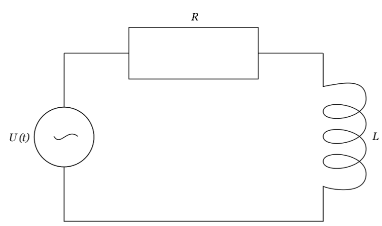

FIG. 7. An equivalent RL circuit for describing thermal noise in the resistor R.

(stochastic) voltage $U(t)$ as we have sequentially connected ideal resistor and random voltage generator. Kirchhoff's law applied to this RL circuit gives the time derivative of the current

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} = -\frac{R}{L}I(t) + \frac{U(t)}{L}. \tag{B1}$$

This stochastic differential equation has the same form as Langevin's equation for Brownian motion.[44] Following Langevin we assume that values of the voltage in different moments t_1 and t_2 are uncorrelated, i.e. we have a δ-function like voltage-voltage correlator

$$\langle U(t_1)U(t_2)\rangle = \Gamma\delta(t_1 - t_2), \tag{B2}$$

where Γ is a constant determined by the temperature T and resistance R. In order to determine this coefficient Γ we have to analyze the condition for thermal equilibrium and the equipartition theorem.

The inhomogeneous linear differential equation for the current Eq. (B1) has a solution which for further derivation we intentionally write two times

$$\begin{aligned} I(t) &= \mathrm{e}^{-t/\tau}\left(I_0 + \frac{1}{L}\int_0^t U(t_1)\mathrm{e}^{t_1/\tau}\mathrm{d}t_1\right), \\ I(t) &= \mathrm{e}^{-t/\tau}\left(I_0 + \frac{1}{L}\int_0^t U(t_2)\mathrm{e}^{t_2/\tau}\mathrm{d}t_2\right). \end{aligned} \tag{B3}$$

Here $\tau \equiv L/R$ and I_0 is the initial current at $t = 0$ which gives negligible influence in the solution for $t \gg \tau$.

Now we have to calculate the noise averaged square of the current applying the voltage-voltage correlator Eq. (B2) in the solution[26] of the stochastic differential equation Eq. (B3)

$$\begin{aligned} \left\langle I^2(t)\right\rangle &= \frac{\mathrm{e}^{-2t/\tau}}{L^2}\int_0^t\int_0^t \langle U(t_1)U(t_2)\rangle\, \mathrm{e}^{(t_1+t_2)/\tau}\mathrm{d}t_1\mathrm{d}t_2 \\ &= \frac{\Gamma}{L^2}\mathrm{e}^{-2t/\tau}\int_0^t \mathrm{e}^{2t_1/\tau}\mathrm{d}t_1 = \frac{\Gamma\tau}{2L^2}\left(1 - \mathrm{e}^{-2t/\tau}\right) \\ &\approx \frac{\Gamma\tau}{2L^2} = \frac{\Gamma}{2LR}, \qquad \text{for } t \gg \tau. \end{aligned} \tag{B4}$$

Now we can apply equipartition theorem for the energy of the inductance for $t \gg \tau$

$$\langle E_L\rangle \equiv \frac{1}{2}L\left\langle I^2\right\rangle = \frac{1}{2}k_\mathrm{B}T' = \frac{1}{2}T, \tag{B5}$$

where T' is the temperature in Kelvin, T is the temperature in energetic units and k_B is the Boltzmann constant. The comparison gives $\Gamma = 2RT$ and for the correlator we have

$$\langle U(t_1)U(t_2)\rangle = 2RT\delta(t_1 - t_2). \tag{B6}$$

The inductance in this consideration was an auxiliary detail, with the same success we can use not RL but a RC-circuit

$$R\frac{\mathrm{d}Q(t)}{\mathrm{d}t} + \frac{1}{C}Q(t) = U(t) \tag{B7}$$

and averaged energy of a capacitor $\langle E_C\rangle = \left\langle U^2\right\rangle/2C = \frac{1}{2}k_\mathrm{B}T'$.

The next step is to calculate the spectral density of the random voltage noise. We have to work with frequency dependent variables Fourier transform of the voltage and the filtered signal $\widetilde{U}(\omega)$

$$U(\omega) = \int_{-\infty}^{\infty} U(t)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t, \quad \widetilde{U}(\omega) = \Phi(\omega)U(\omega). \tag{B8}$$

Let us analyze an ideal band filter with filter function $\Phi = 1$ in a narrow frequency range $(\omega_a,\ \omega_b)$

$$\Phi(\omega) = \theta(\omega_a < |\omega| < \omega_b) = \begin{cases} 1, & \text{for } |\omega| \in (\omega_a, \omega_b) \\ 0, & \text{for } |\omega| \ni (\omega_a, \omega_b) \end{cases}. \tag{B9}$$

Following Eq. (B3), we write twice the time dependent filtrated signal at the moment t expressed by the inverse Fourier transformation

$$\begin{aligned} \widetilde{U}(t) &= \int_{-\infty}^{\infty}\frac{\mathrm{d}\omega_1}{2\pi}\mathrm{e}^{\mathrm{i}\omega_1 t}\Phi(\omega_1)\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\omega_1 t_1}U(t_1)\mathrm{d}t_1, \\ \widetilde{U}(t) &= \int_{-\infty}^{\infty}\frac{\mathrm{d}\omega_2}{2\pi}\mathrm{e}^{\mathrm{i}\omega_2 t}\Phi(\omega_2)\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\omega_2 t_2}U(t_2)\mathrm{d}t_2. \end{aligned} \tag{B10}$$

Now we can calculate the mean square of the filtered voltage sequentially substituting the correlator Eq. (B6) and the even filter function Eq. (B9)

$$\begin{aligned}
\left\langle \widetilde{U}^2(t)\right\rangle &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega_1}{2\pi}\frac{\mathrm{d}\omega_2}{2\pi}\mathrm{e}^{\mathrm{i}(\omega_1+\omega_2)t}\Phi(\omega_1)\Phi(\omega_2)\\
&\quad\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}t_1\mathrm{d}t_2\mathrm{e}^{-\mathrm{i}(\omega_1t_1+\omega_2t_2)}\left\langle U(t_1)U(t_2)\right\rangle\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega_1}{2\pi}\frac{\mathrm{d}\omega_2}{2\pi}\mathrm{e}^{\mathrm{i}(\omega_1+\omega_2)t}\Phi(\omega_1)\Phi(\omega_2)\\
&\quad\times\int_{-\infty}^{\infty}\mathrm{d}t_1\Gamma\mathrm{e}^{-\mathrm{i}(\omega_1+\omega_2)t_1}\\
&= 2RT\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega_1}{2\pi}\Phi(\omega_1)\\
&\quad\times\int_{-\infty}^{\infty}\mathrm{d}\omega_2\mathrm{e}^{\mathrm{i}(\omega_1+\omega_2)t}\Phi(\omega_2)\delta(\omega_1+\omega_2)\\
&= 4RT\int_{0}^{\infty}\frac{\mathrm{d}\omega_1}{2\pi}\Phi(\omega_1)\Phi(-\omega_1)\\
&= 4RT\left(\int_{0}^{\infty}\frac{\mathrm{d}\omega_1}{2\pi}\Phi^2(\omega_1)=\int_0^{\infty}\Phi^2(f)\,\mathrm{d}f\right)\\
&= 4RTB,
\end{aligned}\tag{B11}$$

where $B=(\omega_b-\omega_a)/2\pi\ll\omega_a,\omega_b$ is the bandwidth of the ideal filter; we are writing this chain of derivation in order derivation to be traced by a student. In such a way we have derived the Nyquist[38] theorem

$$\mathrm{e}_\mathrm{n}^2B\equiv\left\langle\widetilde{U}^2(t)\right\rangle=4RTB \tag{B12}$$

for the spectral density e_n^2 of the noise of a resistor. In order to apply this classical Langevin approach, we supposed that temperature is high enough $T\gg\hbar\omega$.[45] If we prefer frequency integration to be in the interval $(-\infty,\ \infty)$, it is better to use twice smaller spectral density,[46] which is an even function $(\mathcal{E}^2)_\omega=\frac{1}{2}\mathrm{e}_\mathrm{n}^2$.

2. Nyquist one-dimensional black body radiation model

Now black body radiation is described in every textbook on statistical physics, see for example the Landau-Lifshitz encyclopedia[47] and no doubts one dimensional (1D) case is simpler than three dimensional (3D). Nyquist considered a transmission line and one dimensional black body radiation. Technically it is simpler to use periodic boundary conditions; imagine, for example, a ring of coaxial cable with length L[24] and mode summation to be substituted by momentum or wave-vector integration

$$\sum_p=\frac{L}{2\pi\hbar}\int_{-\infty}^{+\infty}. \tag{B13}$$

The analog of the averaged Pointing vector is 1D energy flux for wave propagating in right $p>0$ direction

$$\tilde{S}_\rightarrow=\frac{L}{2\pi\hbar}\int_0^\infty\frac{\hbar\omega}{L}\,\tilde{c}\,\frac{\theta(\omega_a<\omega<\omega_b)}{\mathrm{e}^{\hbar\omega/T}-1}\mathrm{d}p, \tag{B14}$$

where $\tilde{c}$ is the velocity of the waves $\omega=\tilde{c}k$. This expression has a simple interpretation $\hbar\omega$ is the energy of one quantum and $\hbar\omega/L$ is the one dimensional energy density created by this quantum. The thermal averaged number of quanta according the Bose-Einstein statistics is $\langle n_p(\omega)\rangle=1/(\exp(\hbar\omega_p/T)-1)$. The energy density of these quanta $(\hbar\omega/L)\langle n_p\rangle$ moves with velocity $v_p=\partial\hbar\omega/\partial p=c$, as we have for the waves in the waveguide $\omega=c(p/\hbar)$.We have to perform summation for all right propagating $(p>0)$ wave modes $\sum_p\approx L\int\frac{\mathrm{d}p}{2\pi\hbar}$. Additionally an ideal bandpass filter Eq. (B9) cuts an infinitesimal frequency range $\Delta\omega\ll\omega_a,\,\omega_b$, as is introduced in the above formula Eq. (B14). Taking into account the relation $p=\hbar\omega/c$, and using the integral

$$\int_0^\infty\frac{x\mathrm{d}x}{\mathrm{e}^x}=\frac{\pi^2}{6} \tag{B15}$$

we obtain for the 1D black-body radiation

$$S_\rightarrow=\frac{\hbar}{2\pi}\int_0^\infty\frac{\omega\mathrm{d}\omega}{\mathrm{e}^{\hbar\omega/T}-1}=\sigma_{1\mathrm{D}}T^2,\quad\sigma_{1\mathrm{D}}\equiv\frac{\pi}{12\hbar}. \tag{B16}$$

The 1D Stefan-Boltzmann constant $\sigma_{1\mathrm{D}}$ does not depend on the velocity $\tilde{c}$, i.e. in 1D case sound and electromagnetic waves transport the same power.

For the filtered signal write

$$\tilde{S}_\rightarrow=\frac{\hbar}{2\pi}\int_{\omega_a}^{\omega_b}\frac{\omega\mathrm{d}\omega}{\mathrm{e}^{\hbar\omega/T}-1} \tag{B17}$$

and for a narrow frequency range $2\pi B=\omega_b-\omega_a\ll\omega_a,\,\omega_b$ we have

$$\tilde{S}_\rightarrow=\frac{\hbar\omega}{\mathrm{e}^{\hbar\omega/T}-1}B. \tag{B18}$$

In thermal equilibrium the leftward flux is equal to the rightward flux $\tilde{S}_\rightarrow=\tilde{S}_\leftarrow$ and we can replace the periodic boundary condition by a long transmitting line terminated by consistent resistors. This is a *gedanken* experiment with thermodynamic style. Now we consider a simple circuit: a resistor with resistance R connected with a long waveguide (coaxial cable) with the same characteristic impedance R, which is terminated with another resistor with resistance R in order wave reflection coefficient to be negligible – this is a 1D analog of a black body with zero albedo. The total dissipated power in the circuit is $\tilde{W}=2\tilde{S}_\rightarrow$ and this power in the equilibrium is equal to the Ohmic heating $\tilde{W}=\langle\tilde{U}_{\mathrm{JN}}^2\rangle/(2R)$ in sequential resistors with total resistance $2R$.

The condition for dynamic thermal equilibrium gives the equivalence of the power of emitted waves and Ohmic dissipation

$$\frac{2\hbar\omega}{\mathrm{e}^{\hbar\omega/T}-1}B=\tilde{W}=\frac{\left\langle\tilde{U}_{\mathrm{JN}}^2\right\rangle}{2R},\quad B=\frac{\omega_b-\omega_a}{2\pi}. \tag{B19}$$

In such a way, we took into account the contribution of the wave propagation to the mean square of the filtered voltage. The introduction of the ground state energy of the wave oscillators gives for filtered voltage square the final expression known as Nyquist theorem for the Johnson-Nyquist noise

$$\left\langle \tilde{U}^2_{\scriptscriptstyle\mathrm{JN}} \right\rangle = 4R\left(\frac{\hbar\omega}{\mathrm{e}^{\hbar\omega/T}-1}+\frac{\hbar\omega}{2}\right)B \tag{B20}$$

and the transmission line can be omitted in the further analysis. Using the identity

$$\frac{1}{\mathrm{e}^x-1}+\frac{1}{2}=\frac{1}{2}\frac{\mathrm{ch}(x/2)}{\mathrm{sh}(x/2)}=\frac{1}{2}\mathrm{cth}\frac{x}{2} \tag{B21}$$

we can rewrite the time-averaged square of the voltage as

$$\left\langle \tilde{U}^2_{\scriptscriptstyle\mathrm{JN}} \right\rangle = 4R(\omega)\frac{\hbar\omega}{2}\coth\left(\frac{\hbar\omega}{2T}\right)B, \quad R(\omega)=\mathrm{Re}\,Z(\omega). \tag{B22}$$

Here the coefficient

$$e_n^2=2(\mathcal{E}^2)_\omega\equiv 2R(\omega)\,\hbar\omega\coth\left(\hbar\omega/2T\right) \tag{B23}$$

in front of B is known as the spectral density of the voltage noise.[46] This general formula includes quantum effects.

In the Nyquist theorem Eq. (B22) the resistance $R(\omega)$ can be represented by the impedance $Z(\omega)$ or by the frequency dependent capacitance $C(\omega)=C'+\mathrm{i}C''$

$$\omega R(\omega)=\omega\,\mathrm{Re}\left(Z(\omega)=\frac{\mathrm{i}}{\omega C}\right)=\frac{C''}{|C|^2}=\frac{\mathrm{i}}{2}(C^{-1}-C^{-1*}), \tag{B24}$$

and the spectral density of the thermal noise of the voltage can be rewritten as

$$\begin{aligned}(\mathcal{E}^2)_\omega &= \frac{\hbar C''}{|C|^2}\coth\left(\frac{\hbar\omega}{2T}\right)\\ &= \frac{\mathrm{i}\hbar}{2}\left(C^{-1}-C^{-1*}\right)\coth\left(\frac{\hbar\omega}{2T}\right).\end{aligned} \tag{B25}$$

For the spectral density

$$(Q^2)_\omega=|C(\omega)|^2(\mathcal{E}^2)_\omega \tag{B26}$$

of the the electric charge induced by thermal fluctuations

$$Q(t)=\int_{-\infty}^{+\infty}Q_\omega \mathrm{e}^{-\mathrm{i}\omega t}\frac{\mathrm{d}\omega}{2\pi},\quad Q_\omega=C(\omega)\mathcal{E}_\omega \tag{B27}$$

the Nyquist theorem reads

$$\begin{aligned}(Q^2)_\omega &= \hbar\,\mathrm{Im}\,C(\omega)\coth\left(\hbar\frac{\omega}{2T}\right)\\ &= \frac{\mathrm{i}\hbar}{2}\left(C^*-C\right)\coth\left(\frac{\hbar\omega}{2T}\right).\end{aligned} \tag{B28}$$

and for the thermodynamic mean of the square of the charge we derive

$$\left\langle Q^2\right\rangle=\int_{-\infty}^{+\infty}(Q^2)_\omega\frac{\mathrm{d}\omega}{2\pi}=\frac{\hbar}{\pi}\int_0^\infty C''(\omega)\coth\left(\frac{\hbar\omega}{2T}\right)\mathrm{d}\omega. \tag{B29}$$

Analogously for the spectral density of the current we have

$$\begin{aligned}(I^2)_\omega &= \omega^2(Q^2)_\omega=\frac{(\mathcal{E}^2)_\omega}{|Z(\omega)|^2}\\ &= \frac{\hbar\omega}{|Z(\omega)|^2}R(\omega)\coth\left(\frac{\hbar\omega}{2T}\right).\end{aligned} \tag{B30}$$

If $C''(\omega)$ differs essentially from zero only for $\hbar\omega\ll T$ we have big number of quanta $\langle n(\omega)\rangle\gg 1$ and can use the classical approximation $\coth\left(\hbar\omega/2T\right)\approx 2T/\hbar\omega$. In this classical limit for the thermally averaged square of the charge we have

$$\left\langle Q^2\right\rangle = T\left\{\frac{2}{\pi}\int_0^\infty\frac{C''(\omega)}{\omega}\mathrm{d}\omega=C'(0)\right\}. \tag{B31}$$

The integral here is the Kramers-Kronig relation[47] for the capacity

$$\begin{aligned}&C'(\omega)=\frac{2}{\pi}\mathcal{P}\int_0^\infty\frac{\xi\,C''(\xi)}{\xi^2-\omega^2}\,\mathrm{d}\xi, \qquad (\mathrm{B32})\\ &C''(-\omega)=-C''(\omega),\\ &\mathcal{P}\int_0^\infty\frac{\mathrm{d}\xi}{\xi^2-\omega^2}=\lim_{\rho\to0}\left\{\int_0^{\omega-\rho}\frac{\mathrm{d}\xi}{\xi^2-\omega^2}+\int_{\omega+\rho}^\infty\frac{\mathrm{d}\xi}{\xi^2-\omega^2}\right\}\end{aligned}$$

which is direct consequence of the causality

$$Q(t)=\int_0^\infty C(\tau)\,U(t-\tau)\,\mathrm{d}\tau=\int_{-\infty}^t C(t-t')\,U(t')\,\mathrm{d}t', \tag{B33}$$

where

$$\begin{aligned}&C(t)=\int_{-\infty}^\infty \mathrm{e}^{-\mathrm{i}\omega t}\,C(\omega)\,\frac{\mathrm{d}\omega}{2\pi},\quad C(\omega)=\int_{-\infty}^\infty \mathrm{e}^{\mathrm{i}\omega t}\,C(t)\,\mathrm{d}t,\\ &C(-\omega)=C^*(\omega),\quad C'(-\omega)=C'(\omega), \qquad (\mathrm{B34})\\ &C''(-\omega)=-C''(\omega),\quad C''(0)=0,\quad C'(0)=C(0).\end{aligned}$$

For the detailed explanation see sections 123, 124 and 125 of 5th volume of Landau-Lifshitz course of theoretical physics Ref. 47. In such a way the Nyquist theorem is in agreement with the equipartition theorem and we arrive at

$$\left\langle Q^2\right\rangle = C(0)T. \tag{B35}$$

In the same classical approximation $\hbar\omega\ll T$ the spectra density of the electromotive force Eq. (B23) gives $(\mathcal{E}^2)_\omega=2TR(\omega)=\frac{1}{2}\mathrm{e}_\mathrm{n}^2$ and for a narrow frequency band B we obtain for Johnson–Nyquist (JN) noise

$$\left\langle \tilde{U}^2_{\scriptscriptstyle\mathrm{JN}}(t)\right\rangle = 4RTB=\mathrm{e}_\mathrm{n}^2 B, \tag{B36}$$

in agreement with the former derivation using Langevin approach of stochastic equations Eq. (B12). As the integrand is frequency independent in this formula the narrow bandwidth after the frequency integration is substituted by passbandwidth for which we intentionally use the same notation B.

The formula for the spectral density for the thermal Johnson-Nyquist noise

$$\mathrm{e}^2_{\mathrm{n,JN}}=4Rk_\mathrm{B}T'. \tag{B37}$$

is used often in uncountable number of technical application notes. See for example Refs. 48–50. Additionally, we suppose that for resistors much thinner than the skin depth the resistance is frequency independent. The parameter e_n has dimension $\mathrm{V}/\sqrt{\mathrm{Hz}}$. For example, at room temperature 25°C a resistor with $R = 50\ \Omega$ generate noise with spectral density parameter

$$\mathrm{e}_{\mathrm{n,JN}} = \sqrt{4k_{\mathrm{B}}T'R} \approx 0.9\ \mathrm{nV}/\sqrt{\mathrm{Hz}}. \tag{B38}$$

This is approximately equal to the internal voltage noise $\mathrm{e}_{\mathrm{n,Amp}}$ of OpAmp ADA4898[7] used in our experimental set-up.

Imagine that the voltage $\mathcal{E}$ of the analyzed capacitor C is created by a big capacitor with $C_0 \gg C$ having also a big charge $Q_0 = C_0\mathcal{E} \gg Q$. In this case for the total energy we have

$$\begin{aligned} E &= \frac{Q^2}{2C} + \frac{(Q_0 - Q)^2}{2C_0} \qquad \text{(B39)} \\ &= \frac{Q^2}{2C} - \left\{\mathcal{E} = \frac{Q_0}{C_0}\right\} Q + \frac{Q_0^2}{2C_0}Q + \frac{Q^2}{2C_0} \\ &= \frac{Q^2}{2C} - \mathcal{E}Q + \mathrm{const.} \end{aligned}$$

We can interpret this as a capacitor with energy $Q^2/2C$ perturbed by external voltage and the energy of this perturbation is

$$V = -\mathcal{E}Q. \tag{B40}$$

Nothing specific was assumed for the generalized impedance, whose properties are presented by the frequency dependent capacity $C(\omega)$. In order to emphasize the generality of the Nyquist theorem we can change the notions and notations

$$\begin{aligned} &\alpha(\omega) \equiv C(\omega), \quad \alpha(t) \equiv C(t), f(t) = \mathcal{E}(t), \qquad \text{(B41)} \\ &\hat{x} = Q, \quad \hat{V} = -f(t)\hat{x} \equiv -\mathcal{E}(t)Q, \\ &\langle \hat{x} \rangle = \hat{\alpha} f(t) = \int_0^\infty \alpha(\tau)\, f(t-\tau)\, \mathrm{d}\tau, \\ &\quad \equiv Q(t) = \int_0^\infty C(\tau)\, U(t-\tau)\, \mathrm{d}\tau, \\ &(x^2)_\omega = (Q^2)_\omega, \quad (f^2)_\omega = (\mathcal{E}^2)_\omega, \quad \ldots \end{aligned}$$

Now we can say that we have a quantum system perturbed by a time dependent perturbation $\hat{V}$, where $f(t)$ is the general time dependent force and $\hat{x}$ is the operator describing the quantum system. The quantum averaged value of the general coordinate $\langle \hat{x} \rangle (t)$ described the response of the system at external perturbation and α is the general susceptibility

$$\overline{x}_\omega = \int_{-\infty}^{\infty} \langle x \rangle (t)\, \mathrm{e}^{\mathrm{i}\omega t} \mathrm{d}t, \quad \overline{x}_\omega = \alpha(\omega) f_\omega. \tag{B42}$$

The spectral densities of the general coordinate is $(x^2)_\omega$ and for general force $(f^2)_\omega$. In the new notations Nyquist theorem Eq. (B28) reads

$$\begin{aligned} (x^2)_\omega &= \hbar\alpha'' \coth\left(\frac{\hbar\omega}{2T}\right) \\ &= \frac{\mathrm{i}\hbar}{2}\left(\alpha^* - \alpha\right) \coth\left(\frac{\hbar\omega}{2T}\right), \end{aligned} \tag{B43}$$

and analogously for Eq. (B25) we have

$$\begin{aligned} (f^2)_\omega &= \frac{\hbar\alpha''}{|\alpha(\omega)|^2} \coth\left(\frac{\hbar\omega}{2T}\right) \\ &= \frac{\mathrm{i}\hbar}{2}\left(\alpha^{-1} - \alpha^{-1*}\right) \coth\left(\frac{\hbar\omega}{2T}\right). \end{aligned} \tag{B44}$$

and Eq. (B29) reads

$$\langle x^2 \rangle = \int_{-\infty}^{+\infty} (x^2)_\omega \frac{\mathrm{d}\omega}{2\pi} = \frac{\hbar}{\pi}\int_0^\infty \alpha''(\omega) \coth\left(\frac{\hbar\omega}{2T}\right) \mathrm{d}\omega. \tag{B45}$$

This is exactly Eq. (124.10) of 5th volume of the course on theoretical physics by Landau and Lifshitz. This formula is now called Callen-Welton[51] 1951 Fluctuation Dissipative Theorem (FDP) derived as a consequence of Nyquist[38] 1928 theorem. Callen and Welton proved the Nyquist theorem using the methods of statistical mechanics without any auxiliary constructions and model considerations. That is why sometimes one can read that Nyquist theorem is an interesting application of the FDT, cf. Ref. 46. For methodical derivation and self-education see also the chapter by McCombie in Ref. 52. In the next section we will give a collection of results for generalized susceptibility which marks the development of statistical mechanics in the middle of 20th century. This collection is an extract from the 5th, 9th and 10th volumes of the Landau-Lifshitz course on theoretical physics.

3. Callen and Welton fluctuation-dissipation theorem and generalized susceptibility. A collection of results

Perhaps the first example of connection between fluctuations and dissipation was the Sutherland-Einstein-Smoluchowski relation[53] between diffusion D coefficient and mobility $\mu = v/F$ (the proportion coefficient between velocity v and the external force F), which for a sphere with radius $r_\circ$ in a viscous fluid with viscosity coefficient η reads as

$$D = k_{\mathrm{B}}T' \left(\mu = \frac{v}{F} = \frac{1}{6\pi\eta r_\circ}\right). \tag{B46}$$

For completeness we are listing without derivation and comments some important results related to generalized susceptibility and fluctuations expressed by the matrix elements of the coordinates. This is only a reference collection of formulas whose detailed derivation is given in many monographs in statistical physics. Thermal averaging of perturbation theory gives in a self-explaining notations

$$\begin{aligned}
&\overline{x}(t) = \int_0^\infty \alpha(\tau) f(t-\tau)\,\mathrm{d}\tau = \int_{-\infty}^t \alpha(t-t') f(t')\,\mathrm{d}t' \\
&\alpha(t) = \frac{\mathrm{i}}{\hbar}\left\langle \hat{x}(t)\hat{x}(0) - \hat{x}(0)\hat{x}(t)\right\rangle \theta(t), \\
&\hat{x}(t) = \mathrm{e}^{\mathrm{i}\hat{H}_0/\hbar}\hat{x}(0)\mathrm{e}^{-\mathrm{i}\hat{H}_0/\hbar}, \quad \hat{H} = \hat{H}_0 + \hat{V}, \\
&\alpha(\omega) = \int_0^\infty \alpha(t)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}t \\
&\qquad = -\sum_{m,n} \rho_n \frac{|x_{mn}|^2}{\hbar\omega - \hbar\omega_{mn} + \mathrm{i}0}\left(1 - \mathrm{e}^{-\beta\hbar\omega_{mn}}\right) \\
&\qquad = \frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{e}^{\mathrm{i}\omega t}\left\langle \hat{x}(t)\hat{x}(0) - \hat{x}(0)\hat{x}(t)\right\rangle \mathrm{d}t, \\
&\alpha''(\omega) = \sum_{m,n} \rho_n |x_{mn}|^2 \left(1 - \mathrm{e}^{-\beta\hbar\omega_{mn}}\right)\delta(\hbar\omega - \hbar\omega_{mn}), \\
&\alpha_{\mathrm{M}}(\zeta_s) = \alpha(\mathrm{i}|\zeta_s|) = -\sum_{m,n} \frac{\rho_n |x_{mn}|^2}{\hbar\mathrm{i}\zeta_s - \hbar\omega_{mn}}\left(1 - \mathrm{e}^{-\beta\hbar\omega_{mn}}\right) \\
&\qquad = \int_0^{1/T} \mathrm{e}^{\mathrm{i}\zeta_s\tau}\left\langle \hat{T}_\tau \hat{x}_0^{\mathrm{M}}(\tau)\hat{x}_0^{\mathrm{M}}(0)\right\rangle \mathrm{d}\tau \\
&\qquad = \frac{2}{\pi}\int_0^\infty \frac{\xi\alpha''(\xi)}{\zeta_s^2 + \xi^2}\mathrm{d}\xi, \quad \zeta_s > 0, \\
&\hat{T}_\tau \hat{a}_0^{\mathrm{M}}(\tau_1)\hat{b}_0^{\mathrm{M}}(\tau_2) = \theta(\tau_1 > \tau_2)\hat{a}_0^{\mathrm{M}}(\tau_1)\hat{b}_0^{\mathrm{M}}(\tau_2) \\
&\qquad\qquad + \theta(\tau_2 > \tau_1)\hat{b}_0^{\mathrm{M}}(\tau_2)\hat{a}_0^{\mathrm{M}}(\tau_1), \\
&\hat{x}_0^{\mathrm{M}}(\tau) = \exp(\tau\hat{H}_0)\hat{x}(0)\exp(-\tau\hat{H}_0), \\
&\zeta_s = 2\pi s T, \quad s = 0,\ \pm1,\ \pm2,\ \dots, \quad \beta \equiv \frac{1}{T}, \\
&\rho_n = \mathrm{e}^{\beta(F-E_n)}, \quad \hbar\omega_{mn} = E_m - E_n, \quad \sum_n \rho_n = 1, \\
&\frac{1}{2}\left\langle \hat{x}_\omega \hat{x}_{\omega'} + \hat{x}_{\omega'}\hat{x}_\omega \right\rangle = 2\pi (x^2)_\omega\, \delta(\omega + \omega'), \\
&\frac{1}{2}\left\langle \hat{x}(t)\hat{x}(0) + \hat{x}(0)\hat{x}(t)\right\rangle = \int_{-\infty}^\infty (x^2)_\omega \mathrm{e}^{-\mathrm{i}\omega t}\frac{\mathrm{d}\omega}{2\pi}, \\
&\left\langle \hat{x}^2 \right\rangle = \int_{-\infty}^\infty (x^2)_\omega \frac{\mathrm{d}\omega}{2\pi} \approx T\alpha(0) \quad \text{for classical statistics}, \\
&\mathcal{Q} = \frac{\omega}{2}\alpha''(\omega)|f_0|^2, \quad \text{for } f(t) = \mathrm{Re} f_0 \mathrm{e}^{-\mathrm{i}\omega t}.
\end{aligned} \tag{B47}$$

Here one can recognize the formulas by Kubo[54] (1956) and Matsubara[55] (1955). FDT for several variables by Callen, Barrash, Jackson and Green[56] (1952) reads as

$$\hat{V} = -\hat{x}_i f_i(t), \tag{B48}$$

$$(x_i x_k)_\omega = \frac{\mathrm{i}\hbar}{2}\left(\alpha_{ki}^* - \alpha_{ik}\right)\coth\left(\frac{\hbar\omega}{2T}\right), \tag{B49}$$

$$(f_i f_k)_\omega = \frac{\mathrm{i}\hbar}{2}\left(\alpha_{ik}^{-1} - \alpha_{ki}^{-1*}\right)\coth\left(\frac{\hbar\omega}{2T}\right). \tag{B50}$$

For the dissipation power created by a periodic perturbation

$$f_i(t) = \frac{1}{2}\left(f_{0i}\mathrm{e}^{-\mathrm{i}\omega t} + f_{0i}^*\mathrm{e}^{\mathrm{i}\omega t}\right) \tag{B51}$$

we have

$$\mathcal{Q} = \frac{\mathrm{i}\omega}{4}\left(\alpha_{ik}^* - \alpha_{ki}\right) f_{0i} f_{0k}^*. \tag{B52}$$

After this short review of the properties of the thermal noise, we will analyze in the next appendix the noise related to the discrete nature of the electric charge.

Appendix C: Schottky shot noise

When a photo-current is applied in our set-up along with the thermal noise $\left\langle \tilde{U}_{\mathrm{JN}}^2 \right\rangle$, there is an additional shot noise related to discrete photo-current through the photo-diode $\left\langle \tilde{U}_{\mathrm{Schot}}^2 \right\rangle$ shown in Fig. 8.

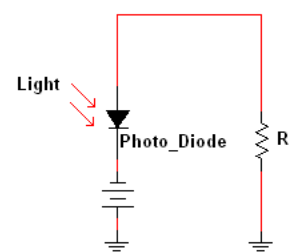

FIG. 8. Schottky shot noise produced by the discrete photo-current through the photo-diode.

In order to alleviate the notations, we will use for a mathematical simplification discrete Fourier series instead of integral ones. Let us suppose that the current through the resistor R_1 is periodic function $I(t+L) = I(t)$ of the time, where the time period L is much longer than the time of the measurement. Thus, we can represent $I(t)$ by discrete Fourier series

$$\begin{aligned}
I(t) &= \sum_m I_m \mathrm{e}^{-\mathrm{i}\omega_m t}, \\
I_m &= \int_0^L \frac{\mathrm{d}t}{L} I(t)\mathrm{e}^{\mathrm{i}\omega_m t},
\end{aligned} \tag{C1}$$

where $\omega_m = m\Delta\omega$ and $\Delta\omega = 2\pi/L$, $m = 0, \pm1, \pm2, \dots$

Introducing the δ-function in physics, an electrical engineer[57] said: “Every electrical engineer knows the term impulse, and this function just represents it in a mathematical form” Ref. 78 of the book by Jemmer.[58] We will consider the quantized contribution from each electron emitted in photoconductive mode of the photodiode to be $I^{(1)}(t) = q_{\mathrm{e}}\delta(t - t_1)$, then Eq. (C1) can be written as

$$\begin{aligned}
I_m^{(1)} &= \int_0^L \frac{\mathrm{d}t}{L} q_{\mathrm{e}}\delta(t - t_1)\mathrm{e}^{\mathrm{i}\omega_m t}, \\
I_m^{(1)} &= \frac{q_{\mathrm{e}}}{L}\mathrm{e}^{\mathrm{i}\omega_m t}, \qquad |I_m^{(1)}|^2 = q_{\mathrm{e}}^2/L^2.
\end{aligned} \tag{C2}$$

For an ideal band filter function $\Phi(\omega_m)$ Parseval's theorem for the filtered current $\tilde{I}^{(1)}(t) = \Phi(\omega_m) I_m^{(1)}$ gives

$$\left\langle \tilde{I}^{(1)}(t)^2 \right\rangle = \sum_m |\Phi(\omega_m)|^2 |I_m^{(1)}|^2. \tag{C3}$$

The even filter function is the same as in Eq. (B9)

$$\Phi(\omega) = \begin{cases} 1, & \text{for } |\omega| \in (\omega_a, \omega_b) \\ 0, & \text{for } |\omega| \ni (\omega_a, \omega_b) \end{cases}. \tag{C4}$$

For big enough time intervals L and we can approximate the sum over ω_m by integral

$$\sum_m f(\omega_m) = \frac{1}{\Delta\omega} \sum_m f(\omega_m) \Delta\omega \approx \frac{1}{\Delta\omega} \int_{-\infty}^{\infty} f(\omega) \mathrm{d}\omega, \tag{C5}$$

and for averaged square of the current we obtain

$$\left\langle \tilde{I}^{(1)}(t)^2 \right\rangle = 2L \int_{\omega_a}^{\omega_b} |I_m^{(1)}|^2 \frac{\mathrm{d}\omega}{2\pi}. \tag{C6}$$

The multiplier 2 in front of the above integral appears because the integrand is an even function. Integrating with the constant integrand Eq. (C2) we obtain

$$\left\langle \tilde{I}^{(1)}(t)^2 \right\rangle = 2L \frac{q_\mathrm{e}^2}{L^2} \frac{B}{2\pi} = \frac{2q_\mathrm{e}^2}{L} B, \tag{C7}$$

where $B = (\omega_b - \omega_a)/2\pi \ll \omega_a, \omega_b$. As electron transitions are statistically independent, if we use black body radiation of a lamp (not a light-diode), the averaged square of the current by all electrons is the contribution by one electron multiplied by the number of electrons passed for time L through the circuit $N = \langle I \rangle L / q_\mathrm{e}$. In such a way we arrive at the well-known result by Schottky[39]

$$i_\mathrm{n}^2 \Delta f \equiv \left\langle \tilde{I}^2 \right\rangle = N \left\langle \tilde{I}^{(1)}(t)^2 \right\rangle = 2q_\mathrm{e} \langle I \rangle B, \tag{C8}$$

where so introduced i_n parameterizes the shot noise current spectral density, analogically as $\mathrm{e_n}$ in Eq. (B12). From Ohm's law applied to a resistor (actually R_1 from Fig. 2) we have for the voltage $\tilde{U} = R\tilde{I}$, and therefore $\left\langle \tilde{U}^2 \right\rangle = R^2 \left\langle \tilde{I}^2 \right\rangle$. The spectral density of the voltage for the Schotky shot noise is

$$e_{\mathrm{n,Sch}}^2 = R^2 i_\mathrm{n}^2 = 2R^2 q_\mathrm{e} \langle I \rangle. \tag{C9}$$

Here we will allow ourselves a lateral speculation on the current noise of OpAmps. For OpAmps the input bias curren I_B is always small and obviously related to incoherent quantum transitions through an insulating barrier. For example, tunneling from the gate to source-drain channel in a field effect transistor through the narrow insulating barrier. It isn't always specified on data sheets, but the spectral density of the input current noise may be calculated in cases like simple BJT or JFETs, where all the bias current flows in the input junction, because in these cases it is simply the Schottky noise of the bias current[59] $i_\mathrm{n}^2 \approx 2q_\mathrm{e} I_\mathrm{B}$. For a check with a logarithmic accuracy we can use even data sheets and to plot a linear regression $\ln i_\mathrm{n}$ versus $\ln I_\mathrm{B}$

$$2 \ln i_\mathrm{n} \approx \ln I_\mathrm{B} + \ln(2q_e). \tag{C10}$$

If we use electro-meter OpAmp AD549[60] with extremely low bias current of $I_B = 200$ fA the whole bias current is related with incoherent processes of quantum tunneling and we can use thermal evaporation of the charge carriers from the gate as a perfect source of shot noise for determination of q_e without photo-current using only integral circuits.

Such plot is given on Fig. 9 for all low input bias current amplifiers taken from the selection table of the Analog Devices Inc. web site.[61]

Integrating spectral density of the voltage noise we obtain for the average-squared-shot-noise voltage

$$\left\langle \tilde{U}_\mathrm{Schot}^2 \right\rangle = 2Rq_\mathrm{e} \langle U \rangle B, \qquad \langle U \rangle \equiv R \langle I \rangle \tag{C11}$$

For constant integrand the integration is trivial the narrow frequency interval is substituted by pass-bandwidth for which we use the same notation B.

Appendix D: Old experimental set-up

Due to extra volume this appendix is only for refereeing in order to demonstrate that idea is working. The idea that a new simple set-up can be made. In this appendix we are giving details for the old experimental set-up which we used to have before double now noise OpAmps (ADA4898-2). This device is realized not by instrumental amplifier buffer but by non-inverting amplifier in the beginning and requires careful screening and BNC connections between the different modules placed in different metallic boxes.

The set-up depicted in Fig. 10 comprises of 3 non-inverting amplifiers (NIA) with ADA4898-1[7] OpAmp, 1 four-quadrant multiplier (with buried Zener) AD633JN[34], 2 high pass filters (HPF) and 2 LPFs. In order to analyze the set-up, in the next subsection D 1 we will give formula for the total noise created by all elements of the circuit in the approximation of negligible conductivity between the input electrodes of the OpAmps. After that, we will continue with the linear part of the circuit which comprises 3 amplifiers with filters and low noise operational amplifiers (OpAmps) depicted in units U1, U2 and U3 in Fig. 10. In subsection D 3 we will describe the squaring of the voltage (unit U4) and its averaging unit U5.

1. Calculation of total noise

The total noise of the resistor R in Fig. 10 is the sum of two components $\left\langle \tilde{U}_\mathrm{Schot}^2 \right\rangle + \left\langle \tilde{U}_\mathrm{JN}^2 \right\rangle$. However, we can not ignore the internal noise of the amplifying circuit $\left\langle \tilde{U}_\mathrm{Amp}^2 \right\rangle$ and we have to add its contribution to the total noise calculation. That is why it is important to use low noise amplifiers

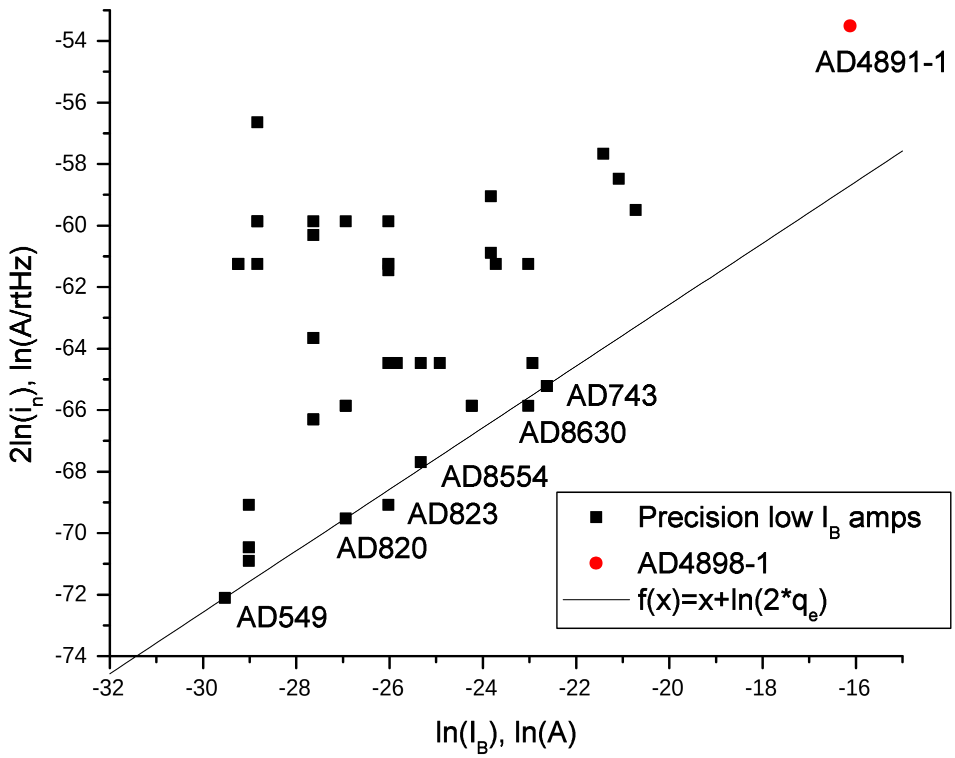

FIG. 9. Logarithmic plot of input noise $i_{\rm n}$ and input bias current $I_{\rm B}$ of all low input bias current amplifiers (< 100 pA) taken from the selection guide table of Analog Devices Inc. In simple BJT and JFET input stages, the current noise is the shot noise of the bias current and may be calculated from the bias current, but for many cases the relation is more complex and dependent upon the input structure of the amplifier. Lowest bias current electrometer AD549 is just on the line.

especially when we are working with small values of the resistor R. According to Eq.(C11), and Eq.(B36), the total voltage noise $\left\langle \tilde U^2_{\rm total}\right\rangle = \left\langle \tilde U^2_{\rm Schot}\right\rangle + \left\langle \tilde U^2_{\rm JN}\right\rangle + \left\langle \tilde U^2_{\rm Amp}\right\rangle$ is

$$\begin{aligned}
&\left\langle \tilde U^2_{\rm total}\right\rangle = \left(2Rq_{\rm e}\left\langle U\right\rangle + 4RT + e^2_{\rm n,circ} + R^2 i^2_{\rm n,Amp+}\right)\Delta f,\\
&e^2_{\rm n,circ} = e^2_{\rm n,Amp} + 4(R_f||r)T + (R_f||r)^2 i^2_{\rm n,Amp-}, \qquad \text{(D1)}\\
&(R_f||r) \equiv \frac{R_f r}{R_f + r},
\end{aligned}$$

where according to Ref. 59, Fig. 1 of Ref 59and Fig. 49 of Ref. 43, in the spectral density of the voltage noise of the circuit MT $e^2_{\rm n,circ}$ we included the contribution of the referred-to-input-voltage-spectral density of the Op Amp $e^2_{\rm n,Amp}$, thermal $4(R_f||r)T$ and current noise $(R_f||r)^2 i^2_{\rm n,Amp-}$ of the minus input of the Op Amp. In this appendix the bandwidth B is denoted also by ΔF. The noise of amplifier is represented by voltage $e^2_{\rm n,Amp}$ and current $i^2_{\rm n,Amp}$ noise spectral densities. This equation for the averaged square of the noise Eq. (D1) can be rewritten as formula for the total voltage spectral density for the total noise voltage

$$\begin{aligned}
\frac{e^2_{\rm n,tot}}{R} &= \frac{\left\langle \tilde U^2_{\rm total}\right\rangle}{R\Delta f} \qquad \text{(D2)}\\
&= 2q_{\rm e}\left\langle U_{\rm Schot}\right\rangle + 4k_{\rm B}T' + \frac{e^2_{\rm n,circ}}{R} + R i^2_{\rm n,Amp},
\end{aligned}$$

where $k_{\rm B}$ is Boltzmann's constant, $T^{'}$ is the temperature measured in Kelvins, and $T = k_{\rm B}T'$; all terms here have dimension energy and can be evaluated in Volts and Kelvins. In order to obtain the mean squared voltage in the general case not only for a model-band-pass filter, it is necessary to integrate with respect to the frequency f. As the spectral density of the noise is frequency independent we have to multiply it by the pass-bandwidth Δf of the experimental set-up Δf, analyzed in detail in Sec. D 2. Formally we have to substitute the bandwidth of the ideal filter with the bandwidth calculated by integration of the square of the modulus of the transfer function of the circuit.

$$\Delta f = \int_0^\infty |\Phi(\omega)|^2 \frac{{\rm d}\omega}{2\pi} \to \Delta f = \int_0^\infty |Y(\omega)|^2 \frac{{\rm d}\omega}{2\pi}. \qquad \text{(D3)}$$

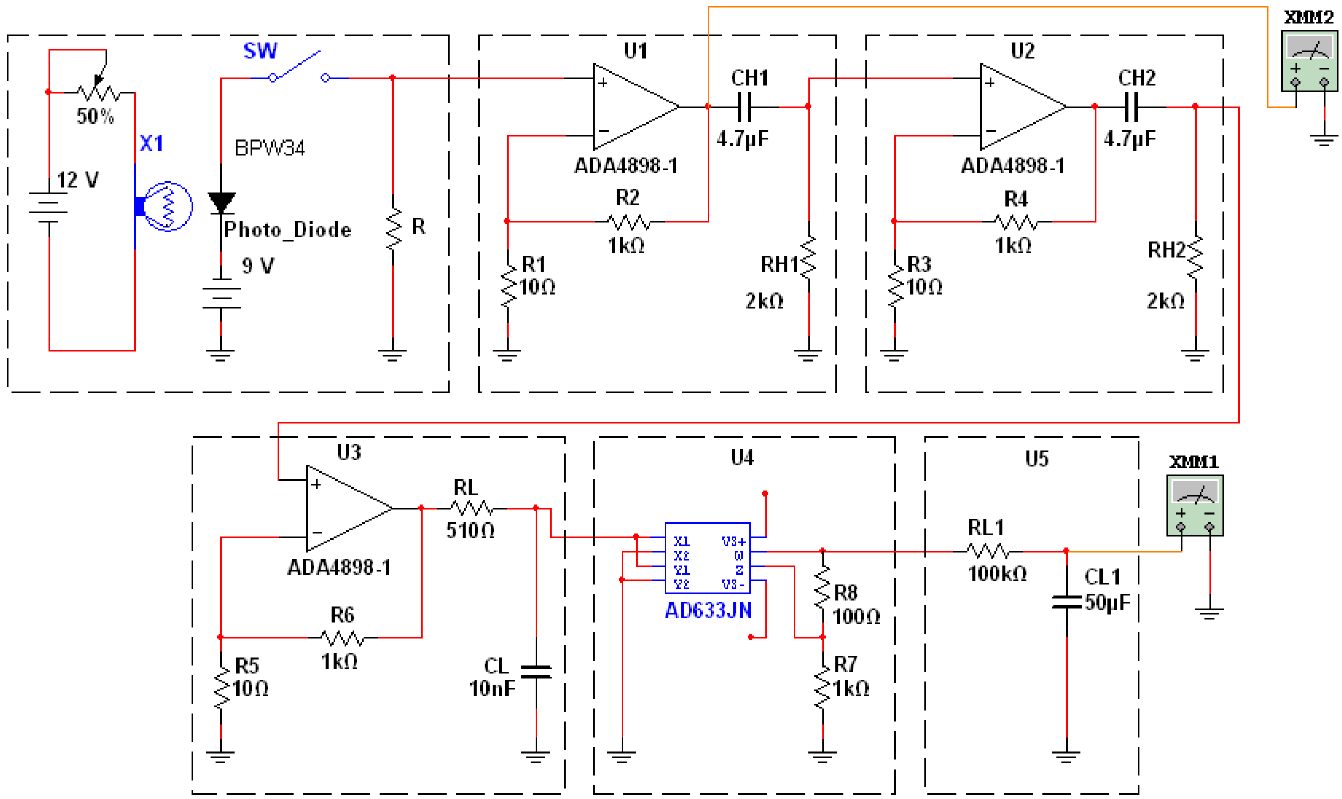

FIG. 10. The experimental set-up for determination of both Boltzmann's constant k_B and electron charge q_e. All operational amplifiers are ADA4898-1. When the switch is open we measure the mean square voltage noise $\langle U^2_\mathrm{JN}\rangle$ from the resistor R_1 with voltmeter 1 to determine the Boltzmann constant k_B, and when the switch is closed voltmeter 1 displays the total mean squared voltage noise $\langle U^2_\mathrm{total}\rangle$ in the circuit and voltmeter 2 measures the average shot noise voltage $\langle U_\mathrm{shot}\rangle$ in the photodiode, determining the electron charge q_e.

Transfer function of the circuit is defined as the ratio of the complex amplitudes of the output and input voltages at some frequency

$$Y(\omega) = \frac{U_\mathrm{out}}{U_\mathrm{in}}. \tag{D4}$$

Some people prefer to factorize transfer function as product of maximal (static in our case) transfer function (amplification $\overline{Y}_0$) and transfer function limited by one

$$Y(\omega) = Y_0 \Upsilon(\omega), \quad |\Upsilon| \le 1. \tag{D5}$$

Introducing bandwidth

$$B \equiv \int_0^\infty |\Upsilon(\omega)|^2 \mathrm{d}f, \quad f \equiv \frac{\omega}{2\pi}, \tag{D6}$$

we have $\Delta f = |\overline{Y}_0|^2 B$. The main idea of the present work is that all details in Eq. (D2) can be measured: the mean square of the voltage

$$\left\langle \tilde{U}^2_\mathrm{total} \right\rangle \tag{D7}$$
$$= \left[2R\left(q_\mathrm{e}\langle U\rangle + 2k_\mathrm{B}T'\right) + e^2_\mathrm{n,circ} + R^2 i^2_\mathrm{n,Amp}\right] |\overline{Y}_0|^2 B,$$

the resistance R, temperature T', mean voltage $\langle U\rangle$, or can be reliably calculated when we know the parameters of the electronic scheme as the pass-bandwidth $\Delta f = |\overline{Y}_0|^2 B$. In such a way experimental data processing using Eq. (D2) gives a method for simultaneous determination of electron charge q_e and Boltzmann constant k_B.

2. Calculation of pass-bandwidth of the amplifiers

The pass-bandwidth is by definition

$$\Delta f = \int_0^\infty |Y(\omega)|^2 \frac{\mathrm{d}\omega}{2\pi}, \tag{D8}$$

where $Y(\omega) = U_\mathrm{out}/U_\mathrm{in}$ stands for the voltage gain.

a. Low pass filter

Let us first calculate the transfer function of the LPF as a voltage divider

$$Y_l(\omega) = \frac{1/(-\mathrm{i}\omega C_l)}{R_l - 1/\mathrm{i}\omega C_l}. \tag{D9}$$

According to technically drawn Fig. 10 : $C_l = \mathrm{CL} = 10.1$ nF and $R_l = \mathrm{RL} = 509\ \Omega$. Taking the modulus of $Y_l(\omega)$ in

Eq. (D9), we arrive at

$$|Y_{\mathrm{l}}(\omega)|^2 = \frac{1}{1+\omega^2 R_{\mathrm{l}}^2 C_{\mathrm{l}}^2} = \frac{1}{1+(\omega\tau_{\mathrm{l}})^2}, \tag{D10}$$

b. High pass filter

On the other hand, the modulus of the transfer function of the HPF is

$$|Y_{\mathrm{h}}(\omega)|^2 = \frac{R_{\mathrm{h}}^2}{R_{\mathrm{h}}^2 + 1/\omega^2 C_{\mathrm{h}}^2} = \frac{(\omega\tau_{\mathrm{h}})^2}{1+(\omega\tau_{\mathrm{h}})^2}. \tag{D11}$$

Here we denote as above $\tau_{\mathrm{h}} \equiv R_{\mathrm{h}} C_{\mathrm{h}} = 8.94$ ms. According Fig. 10: $C_{\mathrm{h}} = \mathrm{CH} \approx \sqrt{\mathrm{CH1}\times\mathrm{CH2}} = 4.47\ \mu\mathrm{F}$ and $R_{\mathrm{h}} = \mathrm{RH} = 2\ \mathrm{k}\Omega$.

c. Non-inverting amplifier

In order to calculate the transfer function of NIA we start with the fundamental equation of an OpAmp

$$\tau_0 \frac{\mathrm{d}U_0}{\mathrm{d}t} = U_+ - U_-, \tag{D12}$$

where U_+ and U_- are the voltages at plus and minus inputs of the OpAmp, U_0 is the voltage of the current output of the OpAmp. τ_0 is a time constant related to the low-frequency behaviour of an Op Amp. Introducing imaginary unit for engineers $\mathrm{j} \equiv -\mathrm{i}$ for harmonic signals $U_+,\ U_-,\ U_0 \propto \mathrm{e}^{-\mathrm{i}\omega t}$ this equation reads as

$$-\mathrm{i}\omega\tau_0 U_0 = U_+ - U_- \text{ or } U_0 = G(U_+ - U_-), \tag{D13}$$
$$G \equiv \frac{1}{\mathrm{j}\omega\tau_0}.$$

For a general review of properties and history of Op Amps see Ref. 62.

According to scheme in Fig. 2 we have a voltage divider $U_- = rU_0/(R_f + r)$, and $U_i = U_+$, hence

$$U_i = \left[\frac{r}{R_f + r} - \mathrm{i}\omega\tau_0\right] U_0. \tag{D14}$$

Now we can compute the transmission function of the NIA

$$\begin{aligned} Y_{\mathrm{a}}(\omega) &= \frac{U_0}{U_i} = \frac{1}{\frac{r}{R_f+r} - \mathrm{i}\omega\tau_0} \\ &= \frac{R_f + r}{r}\frac{1}{1-\mathrm{i}\omega\tau_0\frac{R_f+r}{r}} \\ &= \frac{Y_0}{1-\mathrm{i}\omega\tau_0 Y_0} = \frac{2\pi f_0(R_f+r)}{2\pi f_0 r + (R_f+r)\zeta}, \end{aligned} \tag{D15}$$

where $Y_0 \equiv 1 + R_f/r$ is the amplification coefficient for low frequencies, time constant of the Op Amp is parameterized by the crossover (or cutoff) frequency $f_0 \equiv 1/2\pi\tau_0$, $\zeta \equiv \mathrm{j}\omega$ and $\mathrm{j} \equiv -\mathrm{i}$; cf. Eq. 4 of Ref. 43. For our scheme from Fig. 2: $R_f = \mathrm{R3} = \mathrm{R7} = \mathrm{R9} = 1\ \mathrm{k}\Omega$, $r = \mathrm{R2} = \mathrm{R6} = \mathrm{R5} = 10\ \Omega$, and $Y_0 = 101$ for all 3 NIAs. The transfer function of NIA has a pole at $\omega_{\mathrm{p}} = -\frac{\mathrm{i}}{\tau_0 Y_0}$; all transfer functions are analytical functions in the upper complex semi-plane of the frequency ω.

For the square of the modulus of the transfer function we obtain

$$|Y_a(\omega)|^2 = \frac{Y_0^2}{1+\omega^2\tau_0^2 Y_0^2},$$
$$\frac{1}{|Y_a(\omega)|^2} = \frac{1}{Y_0^2} + \frac{f^2}{(f_0 \equiv 1/2\pi\tau_0)^2}. \tag{D16}$$

A linear fit of a simple experiment with a voltage generator $1/Y_a(\omega)|^2$ versus f^2 determines the cutoff frequency of OpAmp ADA4898-1: $f_0 \approx 80$ MHz or $\tau_0 = 1.99$ ns. Introducing $\tau_a \equiv Y_0\tau_0 = 201$ ns we can rewrite square of the transfer function in similar form as the LPF frequency dependence Eq. (D10)

$$|Y_a(\omega)|^2 = \frac{Y_0^2}{1+(\omega\tau_a)^2} = \frac{Y_0^2}{1+(f/f_{1/2})^2}, \tag{D17}$$

$$f_{1/2} = \frac{f_0}{1+R_f/r}. \tag{D18}$$

For technical applications standard notations are

$$f_{\mathrm{crossover}} \equiv f_0, \quad f_{-3\mathrm{dB}} \equiv f_{1/2}, \quad 10^{-3/10} \approx \frac{1}{2}. \tag{D19}$$

d. All linear part

Now we construct the complete transfer function of our linear circuit consisting of 3 NIAs, 2 HPFs, and 2 LPFs

$$|Y(\omega)|^2 = \left(|Y_{\mathrm{a}}(\omega)|^2\right)^3 \left(|Y_{\mathrm{h}}(\omega)|^2\right)^2 \left(|Y_{\mathrm{l}}(\omega)|^2\right)^1, \tag{D20}$$

and the pass-bandwidth is

$$\Delta f = \int_0^\infty \frac{\mathrm{d}\omega}{2\pi} \left(|Y_{\mathrm{a}}|^2\right)^3 \left(|Y_{\mathrm{h}}|^2\right)^2 \left(|Y_{\mathrm{l}}|^2\right)^1. \tag{D21}$$

In first approximation, we may assume that the HPF is ideal and cuts only $\omega = 0$, hence $|Y_{\mathrm{h}}(\omega)|^2 \approx 1$. Also we consider that amplification of the NIA is frequency independent, so that $|Y_{\mathrm{a}}(\omega)|^2 = Y_0^2$ has its static value for all essential frequencies. In this approximation

$$\tau_a \approx 0.2\ \mu\mathrm{s} \ll \tau_{\mathrm{l}} \approx 5\ \mu\mathrm{s} \ll \tau_{\mathrm{h}} \approx 9000\ \mu\mathrm{s}, \tag{D22}$$

$$b = \frac{\tau_{\mathrm{l}}}{\tau_{\mathrm{h}}} = \frac{R_{\mathrm{l}} C_{\mathrm{l}}}{R_{\mathrm{h}} C_{\mathrm{h}}} \approx 5.75 \times 10^{-4} \ll 1, \tag{D23}$$

$$c = \frac{\tau_{\mathrm{a}}}{\tau_{\mathrm{l}}} = \frac{Y_0\tau_0}{R_{\mathrm{h}} C_{\mathrm{h}}} \approx 3.91 \times 10^{-2} \ll 1. \tag{D24}$$

For the pass-bandwidth now we have

$$\Delta f_0 = \frac{Y_0^6}{2\pi} \int_0^\infty \frac{\mathrm{d}\omega}{1+(\omega\tau_{\mathrm{l}})^2}. \tag{D25}$$

Introducing a new variable $x \equiv \omega\tau_{\mathrm{l}}$, we obtain

$$\Delta f_{0} = \frac{Y_{0}^{6}}{2\pi\tau_{\mathrm{l}}}\int_{0}^{\infty}\frac{\mathrm{d}x}{1+x^{2}} = \frac{Y_{0}^{6}}{4\tau_{\mathrm{l}}} = 51.6\ \mathrm{PHz}. \qquad \text{(D26)}$$

According to Eq. (D21) for the pass-bandwidth we have

$$\Delta f = \frac{1}{2\pi}\int_{0}^{\infty}\left(\frac{Y_{0}^{2}}{1+(\omega\tau_{a})^{2}}\right)^{3}\left(\frac{(\omega\tau_{\mathrm{h}})^{2}}{1+(\omega\tau_{\mathrm{h}})^{2}}\right)^{2} \times\left(\frac{1}{1+(\omega\tau_{\mathrm{l}})^{2}}\right)\mathrm{d}\omega = Z\Delta f_{0}. \qquad \text{(D27)}$$

For the so defined correction factor we have

$$Z = \frac{2}{\pi}\int_{0}^{\infty}\frac{dx}{1+x^{2}}\left[\frac{(x/b)^{2}}{1+(x/b)^{2}}\right]^{2}\left[\frac{1}{1+(cx)^{2}}\right]^{3} \qquad \text{(D28)}$$

and simple numerical calculation for parameters of our set-up gives $Z = 0.932$. Finally for the pass-bandwidth we obtain $\Delta f = Z\Delta f_{0} = 48.1$ PHz

Removing the static amplification $|\overline{Y}_0|^2 = (|Y_0|^2)^3 = 1.06\times10^{12}$ we obtain for the bandwidth $B = \Delta f/|\overline{Y}_0|^2 = 45.3$ kHz.

After this detailed calculation of the bandwidth we can analyze the nonlinear part in the set-up.

3. Squaring and averaging of amplified voltage

The last module U3 of the linear part of the circuit is connected to the input of the nonlinear one U4. The squaring module is implemented with the famous four quadrant multiplier AD633 depicted in Fig. 10. For a general description of multipliers see secs. 2.11 and 4.3 of Ref. 36. On the X1 and Y1 inputs of multiplier is applied one and the same voltage signal $X = Y$. The other inputs X2 and Y2 are connected to the ground. The W output is connected to the ground through the resistors $R_8 = 100\ \Omega$ and $R_7 = 1\ \mathrm{k}\Omega$, which form voltage divider. The Z input is connected to the output of the voltage divider. According to Fig. 8 of the data sheet of AD633[34] for the voltage W we have

$$W = S_{\mathrm{m}}\frac{XY}{U_{\mathrm{m}}}, \quad S_{\mathrm{m}} = \frac{R_7+R_8}{R_7} = 11, \quad U_{\mathrm{m}} = 10\ \mathrm{V}. \qquad \text{(D29)}$$

The scaling factor S_{m} and scaled voltage $U_{\mathrm{m}}/S_{\mathrm{m}}$ can be easily derived from the main equation of the multiplier[34] taking into account the voltage divider in unit U4 in the Fig. 10

$$W = \frac{XY}{U_{\mathrm{m}}} + Z, \qquad Z = \frac{R_7}{R_7+R_8}W. \qquad \text{(D30)}$$

The voltage W is applied to an averaging filter U5 with time constant

$$\tau_{\mathrm{av}} = R_{L1}C_{L1} = 100\ \mathrm{k}\Omega\times 50\ \mu\mathrm{F} = 5\ \mathrm{s}. \qquad \text{(D31)}$$

The filtered signal

$$\overline{W}(t) = \int_{-\infty}^{t}\mathrm{e}^{-(t-t')/\tau_{\mathrm{av}}}\,W(t')\,\mathrm{d}t'/\tau_{\mathrm{av}} \qquad \text{(D32)}$$

is directly measured by a voltmeter V1, i.e. $U_{\mathrm{V1}} = \overline{W}(t)$. We wish to emphasize that voltage $W(t)$ comes through a LPF with time constant τ_{l}, see U3. That is why the voltage values of $W(t)$ are strongly correlated within time intervals τ_{l}. Roughly speaking we have a big number of "independent" samples

$$N = \frac{\tau_{\mathrm{av}}}{\tau_{\mathrm{l}}} \approx 10^{6} \gg 1, \qquad \text{(D33)}$$

and averaging filter gives its statistical averaging $\overline{W}(t) \approx \langle W\rangle$. The relative fluctuation of the voltage U_{V1} can be evaluated as $1/\sqrt{N}\approx 10^{-3}$, i.e. the time averaging $\langle\ \rangle_t$ gives

$$\frac{\sqrt{\langle(\overline{W}(t)-\langle W\rangle)^2\rangle_t}}{\langle W\rangle} \simeq \frac{1}{\sqrt{N}}. \qquad \text{(D34)}$$

This is the essential moment of our statistical approach for measurement of the voltage noise. As $1/\sqrt{N}\approx10^{-3}$ the last digit of the used $3\frac{1}{2}$ digit multimeter is fluctuating. Voltmeter 1 measures the square of averaged signal U_{V1} applied to the multiplier

$$U_{\mathrm{V1}} = S_{\mathrm{m}}\frac{\langle X^2\rangle}{U_{\mathrm{m}}}. \qquad \text{(D35)}$$

On the other hand we apply to the multiplier the amplified and filtered noise signal $X(t) = \tilde{U}(t)$. According to Eq. (D7) voltmeter 1 measures the mean squares of the noise signal

$$U_{\mathrm{V1}} = \left[2R\left(q_{\mathrm{e}}\langle U\rangle + 2k_{\mathrm{B}}T'\right) + e^{2}_{\mathrm{n,circ}} + R^{2}i^{2}_{\mathrm{n,Amp}}\right]F_U. \qquad \text{(D36)}$$

For our set-up $S_{\mathrm{m}}\Delta f = 529$ PHz ≈ 0.5 EHz and for bandwidth divided by scaled voltage constant of the multiplier

$$F_U \equiv \frac{S_{\mathrm{m}}\Delta f}{U_{\mathrm{m}}} = 53\ \mathrm{PHz/V}. \qquad \text{(D37)}$$

The parameter F_U is the most important for our set-up we will see it in many further formulas.

In order to investigate fluctuation phenomena pass-bandwidth Δf times scaling factor of the multiplier has to be in Exa-Herz range.

Appendix E: Determination of constants q_{e} and k_{B} by the old experimental set-up

1. Measurement of the electron charge q_{e}

When the photodiode is enlightened the averaged photo-current $\langle I\rangle$ through the resistor $R_1 = 217.5\ \Omega$ creates an averaged voltage $\langle U\rangle$ which is amplified Y_0 times by the 1st NIA, see Fig. 10. In order to change photo-current we use a 12 V lead battery and a potentiometer. In such a way voltmeter 2 measures the averaged voltage

$$U_{V2} = Y_0\,\langle U\rangle\,. \qquad \text{(E1)}$$

In order to determine the electron charge, we change the light intensity and measure the voltages by the two voltmeters. According to Eqs. (D36) and (E1) we have

$$U_{\rm V1}(U_{\rm V2}) = \frac{2q_{\rm e}R_1F_U}{Y_0}U_{\rm V2} + \mathrm{const} = A_1U_{\rm V2} + A_0. \quad \text{(E2)}$$

The offset voltage A_0 is irrelevant for the result. Our attention is focused on the derivative of the linear approximation

$$\frac{\mathrm{d}U_{\rm V1}}{\mathrm{d}U_{\rm V2}} = A_1. \quad \text{(E3)}$$

The experimental data $U_{\rm V1}$ versus $U_{\rm V2}$ shown in Fig. 11 have very high correlation coefficient. The dimensionless slope from the linear regression $A_1 = 0.0384$ gives the electron charge

$$\begin{aligned} q_e &= \frac{Y_0}{2R_1F_U}\frac{\mathrm{d}U_{\rm V1}}{\mathrm{d}U_{\rm V2}} && \text{(E4)}\\ &= \frac{101\times 0.0384}{2\times 218\times 52.9\times 10^{15}} \\ &= 1.68\times 10^{-19}\ \mathrm{C}. && \text{(E5)}\end{aligned}$$

Our relative error is

$$\frac{(1.68-1.60)\times 10^{-19}\mathrm{C}}{1.60\times 10^{-19}\mathrm{C}} \approx 5\% \quad \text{(E6)}$$

from the well known value of the electron charge $q_e = 1.60217657\times 10^{-19}$ C, which is an acceptable accuracy for a student laboratory with a self-made equipment.

2. Determination of the Boltzmann constant $k_{\rm B}$

One of the possibilities is to measure the thermal noise at fixed resistivity R and different temperatures according to Eq. (D36) we determine $k_{\rm B}$ with simple linear regression for $U_{\rm V1}(T)$

$$U_{\rm V1} = 4Rk_{\rm B}T'F_U + \mathrm{const}. \quad \text{(E7)}$$

An annulation of this constant for $R = 0$ by trimmed resistors connected to Z-input of the multiplier gives the possibility for determination of absolute temperature. However temperature measurements require more experimental time and more sophisticated set-up related to temperature control. Another possibility is to change the resistivity at room the temperature. Let rewrite Eq. (D36) for zero photo-current as

$$\begin{aligned} U_{\rm V1}(R) &= \left(4Rk_{\rm B}T' + e^2_{\rm n,circ} + i^2_{\rm n,Amp}R^2\right)F_U \\ &= a_0 + a_1R + a_2R^2. && \text{(E8)}\end{aligned}$$

Here we inserted the contribution of spectral density of input current noise $i_{\rm N}$ of the first OpAmp. We measure the mean squared of total voltage noise $\left\langle \tilde{U}^2_{\rm total}\right\rangle$, i.e. the voltage U_1 measured by voltmeter V1 for different resistors with R with values between 0 Ω and 218 Ω, as shown in Fig. 12. The continuous line is the parabolic fit U_1 versus R. The constant voltage offset

$$U_{\rm V1}(R=0) = a_0 = e^2_{\rm n,circ}F_U. \quad \text{(E9)}$$

is related to the internal voltage noise of the electronic scheme mainly created by the thermal noise of the transistors and resistors in the first OpAmp. As in our circuit $r \ll R, R_f$ we have as an acceptable approximation $e_{\rm n,circ} \approx e_{\rm n,Amp}$. The constant in front of the quadratic term is the second derivative of the polynomial approximation

$$\frac{1}{2}\left.\frac{\mathrm{d}^2U_{\rm V1}}{\mathrm{d}R^2}\right|_{R=0} = a_2 = i^2_{\rm n,Amp}F_U. \quad \text{(E10)}$$

This parameter is related to the input current noise created by input bias current $I_{\rm B}$ of the first OpAmp. Our experiment is simultaneously a method for determination of the voltage and current noise of OpAmps

$$e_{\rm n,Amp} \approx \sqrt{\frac{a_0}{F_U}}, \quad i_{\rm n,Amp} = \sqrt{\frac{a_2}{F_U}}, \quad i^2_{\rm n,Amp} \approx 2q_eI_{\rm B}. \quad \text{(E11)}$$

The first derivative

$$\left.\frac{\mathrm{d}U_{\rm V1}}{\mathrm{d}R}\right|_{R=0} = a_1 \quad \text{(E12)}$$

of the polynomial approximation at zero resistance has dimension current. The experimental data presented in Fig. 12 give $a_1 = 877\ \mu$A. Taking derivative from Eq. (E8) at $R = 0$ for the Boltzmann constant we obtain

$$\begin{aligned} k_{\rm B} &= \frac{1}{4T'F_U}\left.\frac{\mathrm{d}U_{\rm V1}}{\mathrm{d}R}\right|_{R=0} \\ &= \frac{877\times 10^{-6}}{4\times 300\times 52.9\times 10^{15}} \\ &= 1.3811\times 10^{-23}\ \mathrm{JK}^{-1}. && \text{(E13)}\end{aligned}$$

This value by chance coincides with the well known value

$$\frac{\Delta k_{\rm B}}{k_{\rm B}} = \frac{(1.3811-1.3806)\times 10^{-23}\mathrm{JK}^{-1}}{1.3806\times 10^{-23}\mathrm{JK}^{-1}} \approx 0,04\%\ , \quad \text{(E14)}$$

but typical accuracy is several percent.

Methodically first we measure the Boltzmann constant this is a good test for the set-up. If we use commercial filter and amplifier with unknown pass-bandwidth Δf Boltzmann constant can be used for the calibration of the set-up for the measurement of the electron charge. We wish to remark that the important set-up parameter F_U is canceled in the ratio

$$\frac{q_e}{k_{\rm B}} = 2\left(1+\frac{R_3}{R_2}\right)T'\frac{\left.\frac{\mathrm{d}U_{\rm V1}}{\mathrm{d}U_{\rm V2}}\right|_{U_{\rm V2}=0}}{R\left.\frac{\mathrm{d}U_{\rm V1}}{\mathrm{d}R}\right|_{R=0}}. \quad \text{(E15)}$$

which is an important test for the work of the set-up. Additionally, the resistor R is a platinum wire with resistance of order of 100 Ω, we can test Nyquist theorem in a wide temperature range. And this is an illustration of the thermometry used in some low temperature measurements.

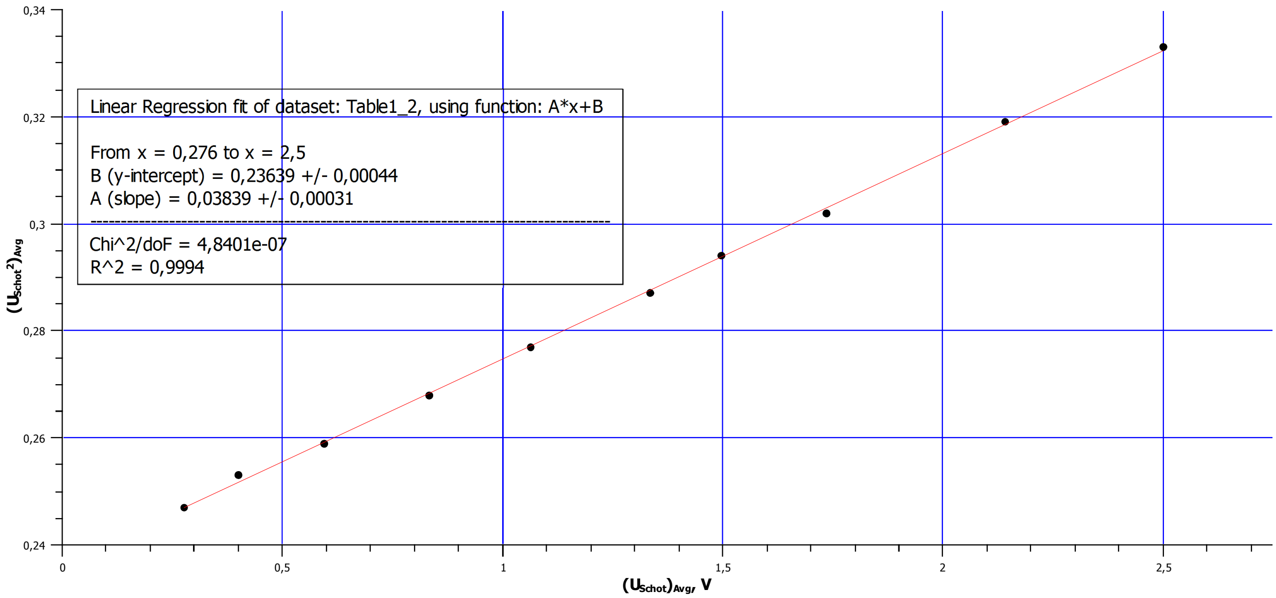

FIG. 11. The mean squared of the Shot noise $U_{V1} \propto \left\langle \tilde{U}^2_{\text{Schot}} \right\rangle$ plotted as a function of the photocurrent $U_{V2} \propto \langle I \rangle$. The experimental points are fitted by linear regression.

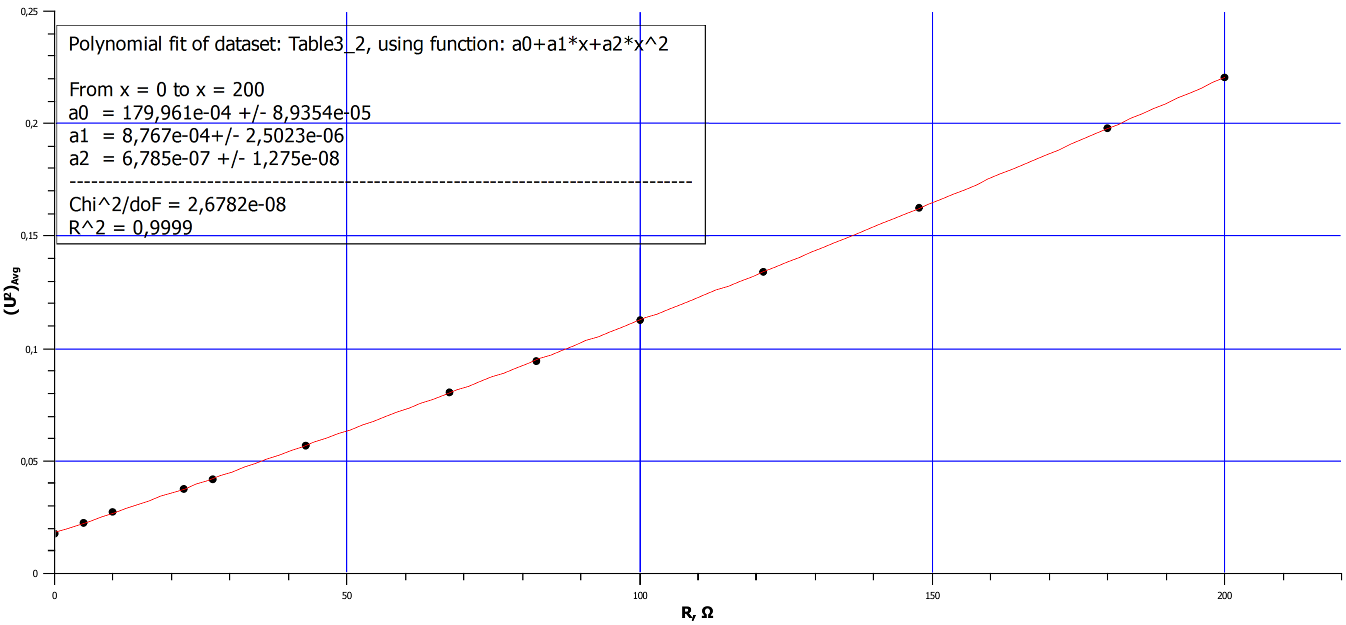

FIG. 12. The the parabolic fit of the experimental data for the mean squared of the thermal noise $U_{V1} \propto \left\langle \tilde{U}^2_{\rm Schot} \right\rangle$ as a function of the resistance of the used resistors R.