Monte Carlo Simulation
An Exploration of Variance Reduction Techniques for Pricing European and Asian Options

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1 Introduction

In today's dynamic financial landscape, options play a pivotal role in shaping investment strategies and managing risk. These unique and versatile financial instruments provide investors with the flexibility to hedge against adverse price movements, speculate on market trends, and optimize portfolio performance. Asian options, in particular, stand out for their unique characteristic of deriving their payoff from the average price of the underlying asset over a specified period, rather than relying on the asset's price at a single point in time, like European options. However, the pricing of options in financial markets is very difficult and computationally complex, owing to the intricate interplay of various factors. Market volatility, time decay, interest rates, and the inherent stochastic nature of asset prices present formidable challenges to financial analysts and traders alike. In this context, computational methods such as Monte Carlo simulations have emerged as indispensable tools for option pricing, offering a flexible and robust framework for estimating option values. Yet, the quest for efficient and accurate option pricing remains a perpetual endeavor, necessitating the exploration of innovative techniques to enhance computational efficiency and mitigate estimation errors.

This paper embarks on such a journey, delving into the realm of efficient Monte Carlo pricing methods for Asian options and European options, with a particular focus on variance reduction techniques. We will introduce and implement some of the most common variance reduction techniques – namely Antithetic Variates and an array of Control Variates such as Delta-based, Gamma-based, and a combined antithetic-control variate algorithm which combines the three. Through a comprehensive analysis and empirical investigation, we seek to unveil novel strategies to address the complexities inherent in option pricing.

2 Financial Prerequisites

2.1 Stock Dynamics

Stocks prices are a stochastic process which are assumed to follow Geometric Brownian Motion that satisfies the following stochastic differential equation [5]

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where W_t is a Wiener process and μ, σ is the percentage drift and percentage volatility, respectively.
Under Ito's interpretation, for a initial value S_0 the above SDE has the analytic solution [5]

$$S_t = S_0 e^{(\mu - \sigma^2/2)t + \sigma d W_t}$$

We will be using this formula for simulating our stock prices when we implement the Monte Carlo algorithms discussed previously.

The below figure is 100 simulations of a stock S with the dynamics found in **Table 4.1**. These same parameters will be used throughout this paper.

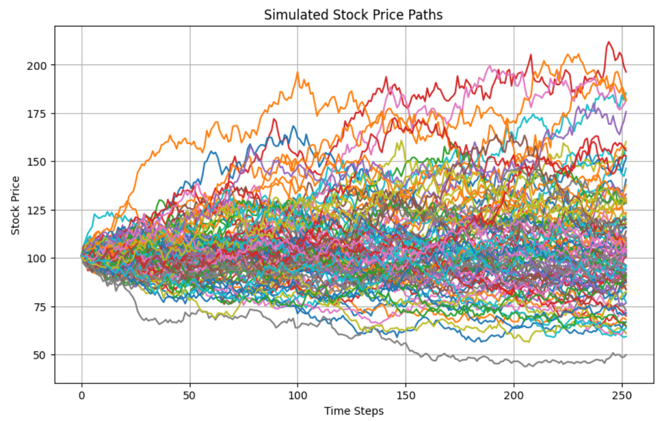

Figure 1 100 simulations of a stock S with dynamics from **Table 4.1**

2.2 European Options

A European option is a financial contract that at time-0 the contract is agreed upon between the buyer and the writer of the option. For the buyer, the call is a non-binding agreement, hence the name 'option', in the sense that they have the right, but not the obligation to buy the underlying asset on the expiration date at time T. The European call option with strike price K has a payoff at time T of

$$C(T) = max(S(T) - K, 0)$$

[4]

2.3 Asian Options

As mentioned previously, Asian option's payoff is determined by the average underlying price over some pre-set period of time. There are a few different types of Asian options, namely, 'Average Price Option' or 'Fixed strike' with payoff:

$$C(T) = max(A(0,T) - K, 0)$$

where the strike K is predetermined and fixed over the lifetime of the option and the averaging price of the underlying is used for payoff calculation.[4] Another type is an 'Average Strike Option' or 'Floating strike' where the averaging price of the underlying over the duration becomes the strike price. The 'Average strike Option' call option has a payoff at time T of

$$C(T) = max(S(T) - kA(0,T), 0)$$

[4]

A question from the reader that may arise is since there are thousands of daily price ticks from the underlying stock, what quotes do we consider when computing the average? Most commonly, Asian options are discretely monitored – monitoring at times $0 = t_0, t_1, t_2, ..., t_n = T$ and $t_i = i \cdot \frac{T}{n}$. The discrete average is then given by:

$$A(0,T) = \frac{1}{n}\sum_{i=1}^{n} S(t_i)$$

2.4 Classical Monte Carlo

As a review to the reader, it is worth quickly reviewing the Classical Monte Carlo algorithm for pricing an option. The pseudocode below is an outline for pricing an Asian Call Option. [2]

Algorithm 2.1 Asian Option Price using Classical Monte Carlo

Require: Sample size N, number of control points d, Option Data: $N, d, S_0, K, r, \sigma, T$

1: $\Delta t \leftarrow \frac{T}{d}$ $n = 1$ to N
2: Generate independent standard normal variates, $\xi_j \sim N(0,1)$, $j = 1, ..., d$. $j = 1$ to d
3: $S(t_j) \leftarrow S(t_{j-1}) \times \exp\Big((r - \frac{\sigma^2}{2}) \times \Delta t + \sigma \times \sqrt{\Delta t} \times \xi_j\Big)$
4: $Y_i \leftarrow \exp(-rT) \times \max\left(\frac{\sum_{j=1}^{d} S(t_j)}{d} - K, 0\right)$
5: Compute the sample mean $\bar{Y}$ and the sample standard deviation s of Y_i's.
6: **return** $\widehat{C}_{MC} = \bar{Y}$ and the standard error $\frac{\sigma_Y}{\sqrt{N}}$

Note that the algorithm can be adjusted for pricing a European call option by simply replacing the term $\max\left(\frac{\sum_{j=1}^{d} S(t_j)}{d} - K, 0\right)$ with $\max(S_T - K, 0)$

3 Variance Reduction Techniques

Option prices and other financial derivatives are quoted on a very frequent basis. It is imperative for market-making trading firms and other institutions to have efficient and accurate pricing to carry out their day to day and second by second trades. Numerical methods are needed for pricing options, namely Asian options, because these types of options do not have a closed form solution. These types of options are 'path dependent' – they depend on the average price of the underlying asset over a certain period of time.

Different numerical methods yield a range of varied results. Higher variance over a large number of simulations yield large confidence intervals. Thus leading to experiments requiring more sample size to consistently observe a statistically significant result on the same effect size. Reducing variance can lead to shorter experiment run times due to the lower sample required.

3.1 Antithetic Variates

The motivation to implement Antithetic Variates is to exploit negative correlation between samples to cancel out some of the variance inherent in Monte Carlo estimations. The key idea is if X_1 and X_2 are independent random variables with mean μ, then

$$Var\left(\frac{X_1 + X_2}{2}\right) = \frac{1}{4}(Var(X_1) + Var(X_2) + 2Cov(X_1, X_2))$$

where the variance is reduced if X_1 and X_2 have $Cov(X_1, X_2) \leq 0$.
For many simulations M large, a μ estimator is $X_1 = h(U_1, ..., U_n)$ for some h; we

consider the antithetic estimator $X_2 = h(1-U_1, ..., 1-U_n)$, yielding a combined estimator of $(X_1 + X_2)/2$ [2].
The variance of an option price using Antithetic Variates using M simulations is:

$$Var(\widehat{C}_M^{AV}) = Var(\frac{1}{M}\sum_m Y_m^{AV}) = \frac{1}{M^2} M Var(Y^{AV})$$

$$= \frac{1}{M} Var(\frac{Y+\tilde{Y}}{2}) = \frac{1}{M}\frac{Var(Y) + 2Cov(Y,\tilde{Y}) + Var(\tilde{Y})}{4}$$

$$= \frac{1}{M}\Big(\frac{1}{2}Var(Y) + \frac{1}{2}Cov(Y,\tilde{Y})\Big)$$

To see that this is less than Classical Monte Carlo, we show:

$$Var(\widehat{C}^{MC}) = \frac{1}{M} Var(Y)$$

and $Cov(Y,\tilde{Y}) \leq \sqrt{Var(Y)Var(\tilde{Y})} = Var(Y)$ so $Var(\widehat{C}_M^{AV}) \leq Var(\widehat{C}^{MC})$

The pseudocode for implementing Antithetic Variates to price an Asian call option is below [8]

Algorithm 3.1 Asian Option Price using Antithetic Variates

Require: Sample size N, number of control points d, Option Data: $N, d, S_0, K, r, \sigma, T$

1: $\Delta t \leftarrow \frac{T}{d}$
2: **for** $n = 1$ to N **do**
3: Generate independent standard normal variate and antithetic variate $Z_1 = \xi_j$ and $Z_2 = 1 - \xi_j$ where $\xi_j \sim N(0,1)$, $j = 1, ..., d$.
4: **for** $j = 1$ to d **do**
5: $S_1(t_j) \leftarrow S_1(t_{j-1}) \times \exp\Big((r - \frac{\sigma^2}{2}) \times \Delta t + \sigma \times \sqrt{\Delta t} \times Z_{1j}\Big)$
6: $S_2(t_j) \leftarrow S_2(t_{j-1}) \times \exp\Big((r - \frac{\sigma^2}{2}) \times \Delta t + \sigma \times \sqrt{\Delta t} \times Z_{2j}\Big)$
7: **end for**
8: Compute $\bar{S}_{1n}(t) = \frac{\sum_{i=1}^T S_{1i}}{d}$; $\bar{S}_{2n}(t) = \frac{\sum_{i=1}^T S_{2i}}{d}$
9: Compute $Y_{1n} \leftarrow \exp(-rT) \times \max\Big(\bar{S}_{1n}(t) - K, 0\Big)$; $Y_{2n} \leftarrow \exp(-rT) \times \max\Big(\bar{S}_{2n}(t) - K, 0\Big)$
10: Compute $Y_n^{AV} = \frac{Y_{1n}+Y_{2n}}{2}$
11: **end for**
12: Compute the sample mean $\bar{Y}^{AV}$ and the sample standard deviation s of Y_n's.
13: **return** $\widehat{C}_{AV} = \bar{Y}$ and the standard error $\frac{\sigma_Y}{\sqrt{N}}$

Note that the algorithm can be adjusted for pricing a European call option by simply skipping step 5 and using $S_{1n}(t), S_{2n}(t)$ in the payoff instead of the $\bar{S}_{1n}(t), \bar{S}_{2n}(t)$

3.2 Control Variates

Recall that the control variate is a random variable with a known expected value, which is correlated with the variable we are trying to estimate – the option.

3.2.1 Delta-based Control Variates

Delta, denoted as $\frac{\partial C}{\partial S}$, is simply the amount the option price changes for a change of \$1 in the underlying asset. Let us consider the effect of delta-hedging the call option. The hedging procedure consists of selling the option, putting the premium in a bank account, and rebalancing the holding in the asset at discrete intervals with resultant cash flows into and out of the bank account. It turns out that the hedge at maturity T, consisting of the cash account plus the underlying asset, closely replicates the pay-off of the option. [6]

$$C_{t_0} e^{r(T-t_0)} - \sum_{i=0}^{N} \left[\frac{\partial C_{t_i}}{\partial s} - \frac{\partial C_{t_{i-1}}}{\partial S} \right] S_{t_i} e^{r(T-t_i)} = C_T + \eta$$

where $\partial C_{t-1}/\partial S = 0$. The expression in the square brackets is the delta hedge. By expanding the summation term in the square brackets gives

$$\begin{aligned} &\frac{\partial C_{t_0}}{\partial S} S_{t_0} e^{r(T-t_0)} + \frac{\partial C_{t_1}}{\partial S} S_{t_1} e^{r(T-t_1)} + \cdots + \frac{\partial C_{t_{N-1}}}{\partial S} S_{t_{N-1}} e^{r(T-t_{N-1})} + \frac{\partial C(t_N)}{\partial S} S_{t_N} \\ &\quad - \left(\frac{\partial C_{t_0}}{\partial S} S_{t_1} e^{r(T-t_1)} + \frac{\partial C_{t_1}}{\partial S} S_{t_2} e^{r(T-t_2)} + \cdots + \frac{\partial C_{t_{N-1}}}{\partial S} S_{t_N} \right) \end{aligned} \tag{3.1}$$

[6]

Rewriting the equation above, grouping terms with $\partial C_{t_i}/\partial S$ at the same time step

$$\begin{aligned} &- \frac{\partial C_{t_0}}{\partial S} (S_{t_1} - S_{t_0} e^{r\Delta t}) e^{r(T-t_1)} - \frac{\partial C_{t_1}}{\partial S} (S_{t_2} - S_{t_1} e^{r\Delta t}) e^{r(T-t_2)} \ \cdots \\ &- \frac{\partial C_{t_{N-1}}}{\partial S} (S_{t_N} - S_{t_{N-1}} e^{r\Delta t}) + \frac{\partial C_{t_N}}{\partial S} S_{t_N} \end{aligned} \tag{3.2}$$

If we assume the final term in the above equation is zero, meaning that we do not buy the final delta amount of the asset, but rather we liquidate the holding from the previous rebalancing date into cash, then the hedged portfolio becomes

$$C_{t_0} e^{r(T-t_0)} + \left[\sum_{i=0}^{N-1} \frac{\partial C_{t_i}}{\partial S} (S_{t_{i+1}} - S_{t_i} e^{r\Delta t}) e^{r(T-t_{i+1})} \right] = C_T + \eta$$

[6]

This gives us the formula for the Delta-based martingale Control Variate (cv_Δ) below, introduced by Mark Broadie and Paul Glasserman in their paper titled "Estimating Security Price Derivatives Using Simulation" in 1996 [1].

$$cv_\Delta = \sum_{i=0}^{N-1} \frac{\partial C_{t_i}}{\partial S} (S_{t_{i+1}} - \mathbb{E}[S_{t_i}]) \exp\left[r(T - t_{i+1})\right]$$

where the martingale $\mathbb{E}[S_{t_i}] = S_{t_{i-1}} \exp(r \cdot dt)$. The forward payoff, therefore, is

$$C_{t_0} \exp rT = C_T - cv_\Delta + \eta \qquad \text{where } \eta \text{ is an error term}$$

In other words, the above is the forward payoff at time T is the option payoff at time T minus the discrete rebalancing plus an error term.

The control variate technique for options simulates the difference $\widehat{C} - \beta \cdot cv$, where β_1 is a constant. Ideally, the control variate should by highly correlated with C but cheaper to compute. The scaling factor β, should be chosen such that the variance of the difference is minimized.

We want to use delta as a control variate because delta is perfectly correlated (1 to 1) to the option price, hence why we will choose $\beta_1 = -1$ as the coefficient of the delta-based control variate. [8]

Remark 3.1. The probability distribution of the payoff of an option after delta hedging has smaller standard deviation compared to its portfolio's unhedged counter part.

To quickly see this, denote option payoff without delta hedging as V_{RAW} and option payoff using delta hedging as V_Δ. Without delta hedging, V_{RAW} follows a certain probability distribution, $f_{V_{RAW}}(S_t)$, with mean $\mu_{V_{RAW}}$ and standard deviation $\sigma_{V_{RAW}}$, and V_Δ similarly. After delta hedging, the portfolio's pay-off is a combination of the option pay-off and the pay-off from the delta-hedged position in the underlying asset. The pay-off from the delta-hedged position in the underlying asset is designed to offset the changes in the option's value due to small movements in the underlying's price. Therefore, it introduces a certain level of certainty into the pay-off. Claiming $\sigma_{V_{RAW}} < \sigma_{V_\Delta}$, by delta hedging, we introduce a hedge that is designed to offset the fluctuations in the option's value due to small changes in the underlying asset's price. This hedge effectively reduces the overall risk (standard deviation) of the portfolio.[7]

The below pseudocode is for implementing the Delta-based Control Variates algorithm for a European Call option.

Algorithm 3.2 European Option Price using Delta-Based Control Variates

Require: Sample size N, number of control points d, β value, Option Data: $N, d, S_0, K, r, \sigma, T$

$\Delta t \leftarrow \frac{T}{d}$
for $n = 1$ to N **do**
  Generate independent standard normal variates, $\xi_j \sim N(0,1)$, $j = 1, ..., d$.
  for $j = 1$ to d **do**
    $S(t_j) \leftarrow S(t_{j-1}) \times \exp\Big((r - \frac{\sigma^2}{2}) \times \Delta t + \sigma \times \sqrt{\Delta t} \times \xi_j\Big)$
    $cv_j \leftarrow cv_{j-1} + \frac{\partial C}{\partial S}[S(t_j) - S(t_{j-1}) \times \exp(r \times \Delta t)]$
  end for
  $Y_n \leftarrow \exp(-rT) \times \max(S(t_n) - K, 0) + \beta cv_d$
end for
Compute the sample mean $\bar{Y}$ and the sample standard deviation s of Y_n's.
return $\widehat{C}_{AV} = \bar{Y}$ and the standard error $\frac{\sigma_Y}{\sqrt{N}}$

The next subsection will introduce a similar control variate, Gamma-based Control Variates, along with discussions of an implementation to price a European Call Option.

3.2.2 Gamma-based Control Variates

One can even go further and classify another control variate cv_2 which is Gamma-based. Introduced by Paul Glasserman and Philip Heidelberger in their 2000 paper titled "Variance Reduction Techniques for Estimating Value-at-Risk"[3], we define a Gamma-based control variate cv_2 as the following [8, 3]

$$cv_2 = \sum_{i=0}^{N-1} \frac{\partial^2 C_{t_i}}{\partial S^2}((\Delta S_{t_{i+1}})^2 - \mathbb{E}[(\Delta S_{t_{i+1}})^2]) \exp[r(T - t_{i+1})]$$

where $\mathbb{E}[(\Delta S_{t_{i+1}})^2]) = S_{t_i}^2(\exp[(2r + \sigma^2)\Delta t_i] - 2\exp(r\Delta t_i) + 1)$

The algorithm to implement Gamma-Based Control Variates is very similar to the Delta-Based approach, except for redefining the control variate cv. Note that the coefficient $\beta_2 = -\frac{1}{2}$ in the payoff estimation formula arises from the quadratic term in the Taylor series expansion of the payoff function.

The following section will show a way to combine both Antithetic Variates and Delta-Based Control Variates.

3.2.3 Combined Antithetic and Delta-based Control Variates

So far we have implemented option pricing algorithms using Antithetic Variates, Delta-based Control Variates, and Gamma-based Control Variates. In this section, we will implement and discuss a variance reduction technique that uses two techniques – both Delta-based Control Variates and Antithetic Variates. We will show that combining these two techniques will yield even further variance reduction for pricing European options. In fact, the final section will implement an algorithm that uses both Delta-based and Gamma-based CV and Antithetic Variates.

In essence, this algorithm computes delta-based control variates using antithetic sampling producing cv_Δ and $\tilde{cv}_\Delta$ – both perfectly negatively correlated underlyings. For the sake of notation, we will define this new control variate by taking the average of the two variates and multiplying by β_1 like we did previously [6, 8, 1]

$$cv_{\Delta,AV} = \beta_1 \frac{(cv_\Delta + \tilde{cv}_\Delta)}{2}$$

The following section will introduce a sampling algorithm that implements Antithetic Variates with the two control-variates – Delta-based and Gamma-based – discussed thus far.

3.2.4 Combined Antithetic, Delta-based and Gamma-based Variates

For this final method, we propose a sampling algorithm that includes all of the variance reduction methods we have discussed thus far. This algorithm uses antithetic sampling for both Delta-based and Gamma-based control variates to produce this combined sampling algorithm. Namely, we have [8]

$$C_T = \frac{1}{2}[max(0, S_{1,t} - K) + max(0, S_{2,t} - K) + \beta_1 cv_1 + \beta_2 cv_2]$$

where cv_1, cv_2 is the delta variate, gamma variate, respectively and $\beta_1 = -1$ and $\beta_2 = -\frac{1}{2}$. When combining this with an antithetic technique, we need to apply the following [8]

$$cv_1 = \frac{1}{2}\beta_1(cv_{11} + cv_{12})$$

where $cv_{11} = \Delta_{S_{1,t}}[S_{1,t_{i+1}} - S_{1,t_i} \exp(r\Delta t_i)]$ and $cv_{12} = \Delta_{S_{2,t}}[S_{2,t_{i+1}} - S_{2,t_i} \exp(r\Delta t_i)]$

$$cv_2 = \frac{1}{2}\beta_2(cv_{21} + cv_{22})$$

where $cv_{21} = \gamma_{S_{1,t}}\Big[(S_{1,t_{i+1}} - S_{1,t_i})^2 - S_{1,t_i}^2(\exp([2r + \sigma^2]t_i) - 2\exp(r\Delta t_i) + 1)]\Big]$
and $cv_{22} = \gamma_{S_{2,t}}\Big[(S_{2,t_{i+1}} - S_{2,t_i})^2 - S_{2,t_i}^2(\exp([2r + \sigma^2]t_i) - 2\exp(r\Delta t_i) + 1)]\Big]$

4 Results of Variance Reduction Methods in Option Pricing

For the results below, we have the following option parameters, $d = 252$ because our option has a 1-year expiration and there are 252 trading days per year.

Option Parameter	Value
Initial Stock Price S_0	101
Strike Price (K)	100
Time to Expiration (T)	1
Volatility (σ)	30%
Risk-free Rate (r)	8%
Number of Steps (d)	252
Number of Simulations (M)	$100,000$

Table 4.1 Parameters for Option Contracts

4.1 European Option Prices

The analysis of the algorithms' performances for the European Call Option are summarized in **Table 4.2**.

Remark 4.1. We see that when compared to Classical Monte Carlo estimation, while the Antithetic Variates appears to have reduced variance over the 100,000 simulations, it is obvious that the methods using Control Variates converges faster and more efficiently. In fact, we see a near 95x decrease in the standard error of the combined Delta-Gamma-Antithetic Variates algorithm compared to Classical Monte Carlo. It is worth noting that we see a spike in the Gamma-based CV standard error because it is a second-order approximation and works best combined with Delta-based CV.

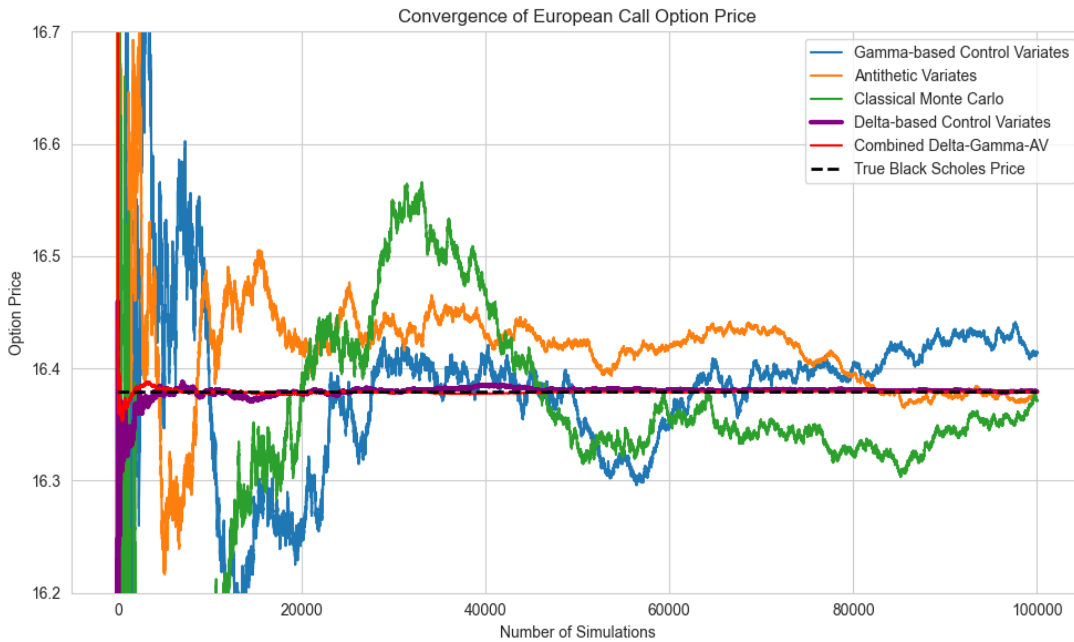

Figure 2 Convergence of European Call Option Price using different variance reduction methods over 100,000 samples.

From **Table 4.2**, although the computation time of the algorithms take longer as the number of control variates increases, we see a major reduction in the standard error. By using these control variate techniques, the standard error reduction could alleviate the large number of simulations necessary to reach convergence. By running less simulations because of faster convergence, computation cost and speed become less and less of a problem.

We see that Antithetic Variates have the same effect versus Classical Monte Carlo for pricing Asian Options in the next section.

4.2 Asian Option Prices

Using the same underlying S dynamics in **Table 4.1**, below is a chart of the convergence of an Asian call price using Antithetic Variates against Classical Monte Carlo over 100,000 simulations. We see that the price of the Asian Call Option converges much quicker using Antithetic Variates versus Classical Monte Carlo.
The standard error reported for the Antithetic Variates was 0.01986 versus Classical Monte Carlo standard error of 0.0401 – more than a 50% decrease.

5 Conclusion and Next Steps

Given the complexity and significance of accurate option pricing in financial markets, this paper has laid a crucial foundation by exploring and implementing efficient Monte Carlo pricing methods for Asian and European options with a focus on variance reduc-

Method	Standard Error	Relative Reduction in SE	Time	Ending Price
Classical Monte Carlo	0.07557	1	4.3	16.372
Antithetic Variates	0.03879	1.68	6.2	16.380
Delta-based CV	0.00401	18.85	6.8	16.380
Gamma-based CV	0.07051	1.07	6.9	16.389
Delta-based and AV	0.002	37.79	7.5	16.381
Delta and Gamma CV, AV	0.0008	94.46	11.6	16.381
True Black Scholes Price	-	-	-	16.38

Table 4.2 Comparison of Different Monte Carlo Methods

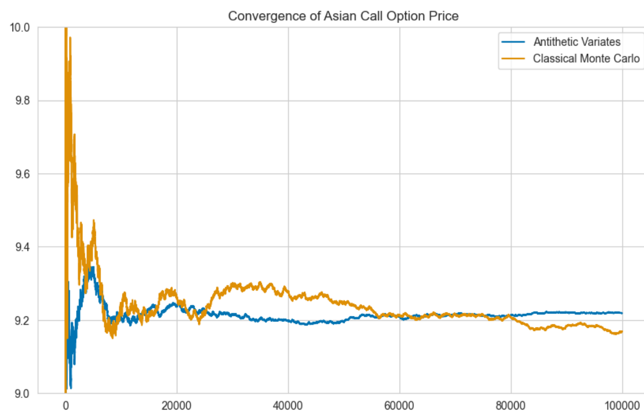

Figure 3 Asian Call Option Convergence using Antithetic Variates and Classical Monte Carlo

tion techniques. Moving forward, several areas merit further exploration to build upon this research.

For one, combining Monte Carlo methods with other numerical techniques, such as finite difference methods or Fourier transform methods, might yield more robust and efficient pricing frameworks. These hybrid models could address the limitations of individual methods and provide a more comprehensive approach to option pricing.

In addition, incorporating the impact of market microstructure elements, such as order flow dynamics and transaction costs, into the pricing models could offer a more realistic representation of market conditions. This would improve the practical applicability of the models for trading and risk management.

In conclusion, this paper discusses strides that have been made in enhancing Monte Carlo methods for option pricing through variance reduction techniques compared to Classical Monte Carlo. Continued research and development in these outlined areas will be instrumental in refining option pricing models, thereby supporting more efficient and informed financial markets.

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