Operator-based semantics for logic programs: recent advances

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November 26, 2024

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Why Operator-Based Semantics?

- Powerful framework for defining and studying semantics rule-based languages

  If you have an idea of how to eavluate rule bodies in a three-valued setting, the framework does the rest of you.
- Single unifying framework that allows to derive most common logic programming semantics (and the semantics of default logic, autoepistemic logic, abstract argumentation, ...)
- Gives a natural justification of the well-founded and stable model semantics.
- Applied in grounders such as gringo.
- Useful for explanations.

Goal of this talk

- Give an overview of the operator-based view on semantics of logic programming [VGRS91, Fit06, DMT00].
- Given an overview of some recent developments (by Bart, Ofer and myself) in this area:
    - Non-deterministic operators,
    - Disjunctive logic programs,
    - Choice logic programs.

Goals and Structure

Syntax of Logic Programs

Semantics of Positive Programs

Semantics of Normal Logic Programs

Non-Deterministic Operators

Stable Semantics

- Stable Semantics for Deterministic Operators
- Stable Non-Deterministic Operators

Approximation Fixpoint Theory

Round up

Syntax of Logic Programs

Syntax of Logic Programs

Set of atoms $\mathcal{A} = \{a, b, c, p, q, r, a_1, a_2, \ldots\}$

$$a \leftarrow b_1, \ldots, b_n, \neg c_1, \ldots, \neg c_m$$

- Program is a set of rules.
- Rule is positive if $m = 0$.
- Program is positive if all the rules are positive.

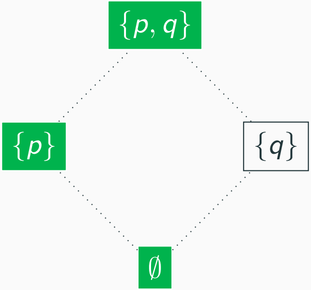
Semantics of Positive Programs

What are the semantics of logic programs?

$$p \leftarrow q.$$

Classical models? $\emptyset, \{p\}, \{p, q\}$.

Notice: a formula follows from every classical model if it follows from the minimal model $\emptyset$.



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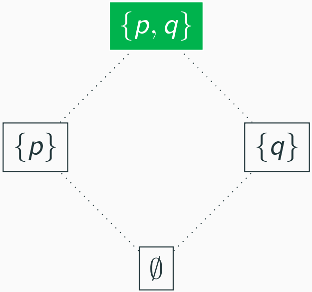


$T_{\mathcal{P}}$-operator

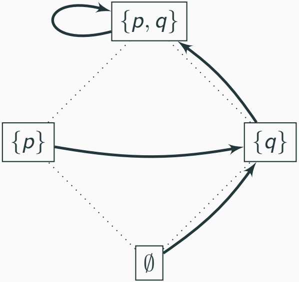
$$T_{\mathcal{P}} : \wp(\mathcal{A}_{\mathcal{P}}) \mapsto \wp(\mathcal{A}_{\mathcal{P}})$$

$$T_{\mathcal{P}}(x) = \{a \mid a \leftarrow b_1, \ldots, b_n \in \mathcal{P} \text{ and } b_1, \ldots, b_n \in x\}$$

Example

$$\mathcal{P} = \{p \leftarrow q., \quad q \leftarrow .\}$$

x	$\emptyset$	$\{p\}$	$\{q\}$	$\{p, q\}$
$T_{\mathcal{P}}(x)$	$\{q\}$	$\{q\}$	$\{p, q\}$	$\{p, q\}$



Models and Fixpoints

Definition
x is a pre-fixpoint of $T_{\mathcal{P}}$ if $T_{\mathcal{P}}(x) \subseteq x$.

Intuition: everything I can derive from x using $\mathcal{P}$ is in x.

Models of $\mathcal{P}$.

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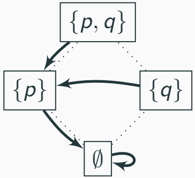

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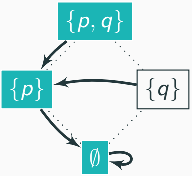

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Definition
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Supported models of $\mathcal{P}$.

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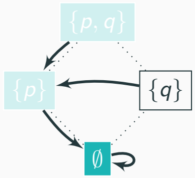

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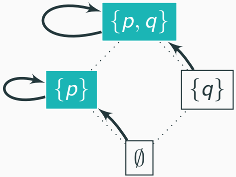

Models and Fixpoints

- For positive programs, $T_{\mathcal{P}}$ has a unique least fixpoint x.
- It is also the least pre-fixpoint.
- We can compute it by iterating $T_{\mathcal{P}}$ starting from $\emptyset$:
  $T_{\mathcal{P}}(\dots T_{\mathcal{P}}(\emptyset)\dots) = \bigcup_{i \geq 0} T^i_{\mathcal{P}}(\emptyset)$
  And this is possible in polynomial time.

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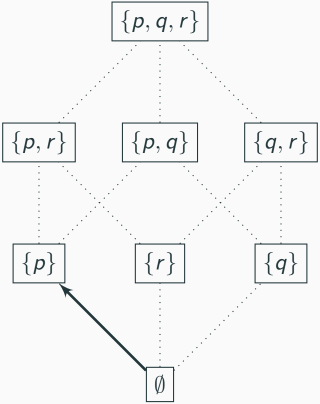
  And this is possible in polynomial time.
  Underlying result: A $\subseteq$-monotonic operator over a complete lattice admits a least fixpoint.

Least fixpoint computation

$$\mathcal{P} = \{p \leftarrow . \quad q \leftarrow p. \quad r \leftarrow p, q.\}$$



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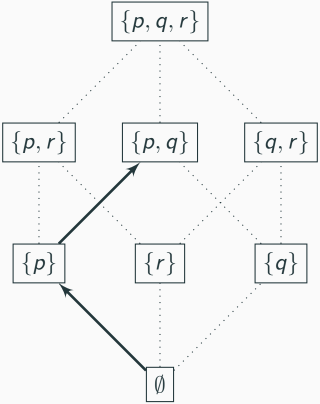

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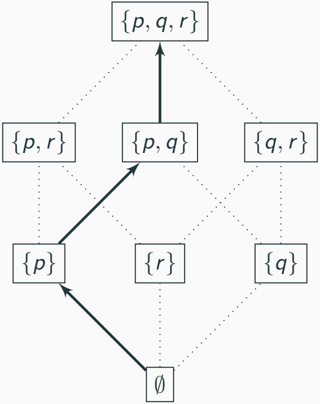

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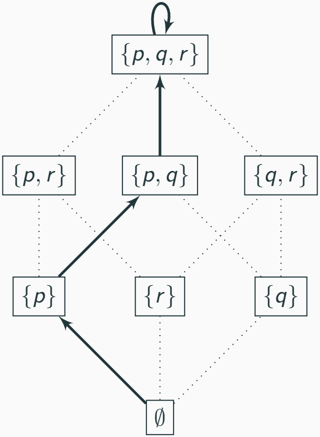
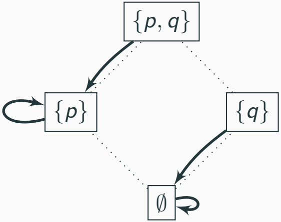


Least fixpoint $\neq$ unique fixpoint

Example ($\mathcal{P} = \{p \leftarrow p.\}$)



Semantics of Normal Logic Programs

Enters Negation

$$p \leftarrow \neg q$$

How to extend our operator?

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Easy:

$$\begin{aligned} T_{\mathcal{P}}(x) = \{a \mid \quad & a \leftarrow b_1, \ldots, b_n, \neg c_1, \ldots, \neg c_m \in \mathcal{P}, \text{ and} \\ & b_1, \ldots, b_n \in x, \text{ and } c_1, \ldots, c_m \notin x\} \end{aligned}$$

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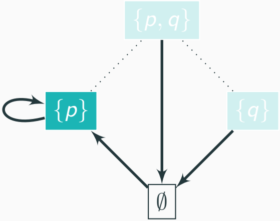

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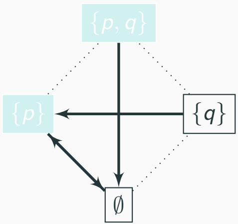
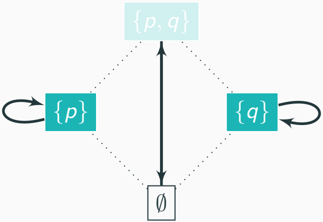
$\mathcal{P} = \{p \leftarrow \neg p\}$



No unique fixpoint

$\mathcal{P} = \{p \leftarrow \neg q; q \leftarrow \neg p\}$



Problems with negation

- There might not be a fixpoint.
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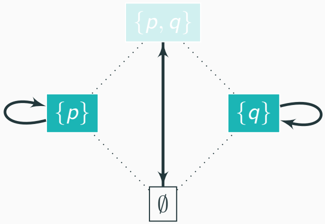

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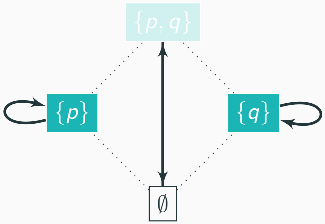

Solution: approximations of operators that are $\leq_i$-monotonic.

Approximations

- Pairs of sets of atoms (x, y).
  - x contains all atoms that are definitely true.
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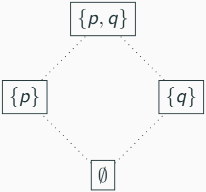
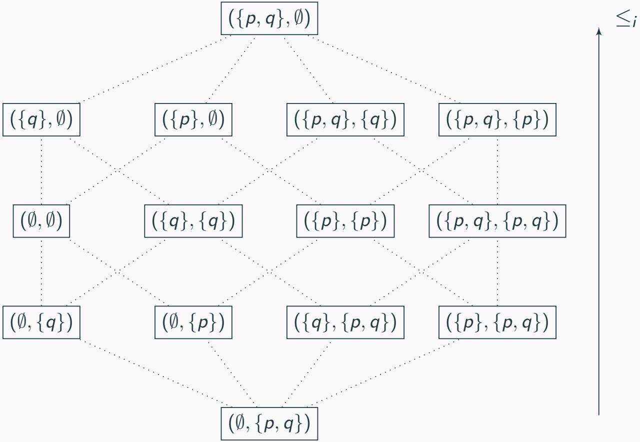
Example (Given $\mathcal{A} = \{p, q, r\}$)
$(\{p\}, \{p, q\})$: *p is true and q can be true.*

$(\{p\}, \{p\}) \leq_t (\{p\}, \{p, q\})$
$(\{p\}, \{p, q\}) \leq_i (\{p\}, \{p\})$

Graphical Depiction





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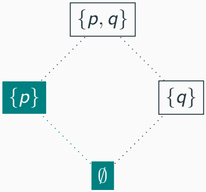

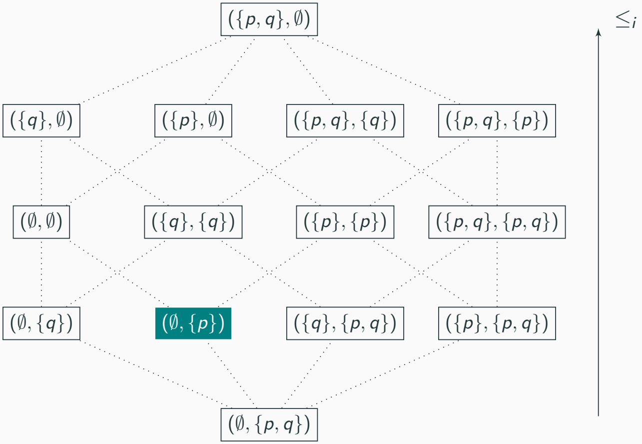

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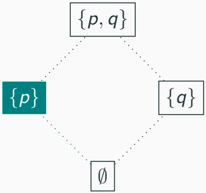

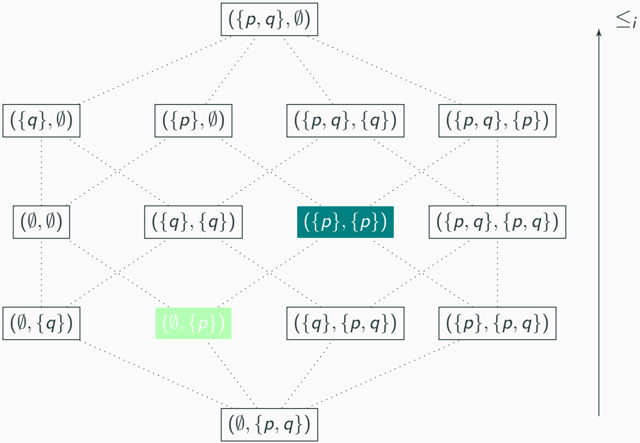

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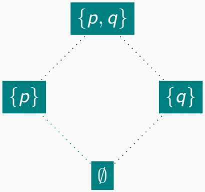
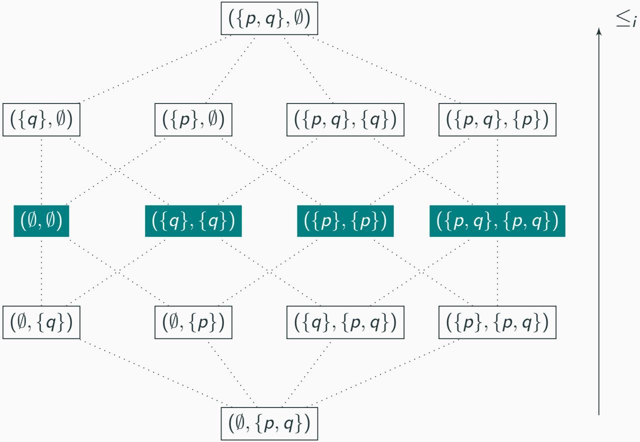
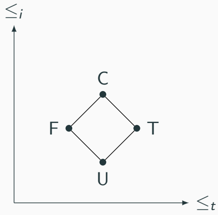




Approximations as Four-Valued Interpretations



- $-\mathsf{F} = \mathsf{T}$, $-\mathsf{T} = \mathsf{F}$, $-\mathsf{U} = \mathsf{U}$ and $-\mathsf{C} = \mathsf{C}$
- $(x, y)(p) = \begin{cases} \mathsf{T} & \text{if } p \in x \text{ and } p \in y, \\ \mathsf{U} & \text{if } p \notin x \text{ and } p \in y, \\ \mathsf{F} & \text{if } p \notin x \text{ and } p \notin y, \\ \mathsf{C} & \text{if } p \in x \text{ and } p \notin y. \end{cases}$
- $(x, y)(\neg\phi) = -(x, y)(\phi)$,
- $(x, y)(\psi \wedge \phi) = lub_{\leq_t}\{(x, y)(\phi), (x, y)(\psi)\}$,
- $(x, y)(\psi \vee \phi) = glb_{\leq_t}\{(x, y)(\phi), (x, y)(\psi)\}$.

Example

$(\{p\}, \{p, q\})(p) = \mathsf{T} \quad (\{p\}, \{p, q\})(q) = \mathsf{U} \quad (\{p\}, \{p, q\})(r) = \mathsf{F}$.

$(\{p\}, \{p, q\})(\neg p) = \mathsf{F} \quad (\{p\}, \{p, q\})(\neg q) = \mathsf{U}$

$(\{p\}, \{p, q\})(p \wedge q) = \mathsf{U} \quad (\{p\}, \{p, q\})(q \vee r) = \mathsf{U}$

Approximating $T_{\mathcal{P}}$ (from below)

$$\mathcal{IC}_{\mathcal{P}} : \mathcal{A} \times \mathcal{A} \mapsto \mathcal{A} \times \mathcal{A}$$

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We input an approximation and output an approximation.

$$\mathcal{IC}^{l}_{\mathcal{P}}(x, y) = \{a \in \mathcal{A} \mid a \leftarrow b_1, \ldots, b_n, \neg c_1, \ldots, \neg c_m \in \mathcal{P}, \\ b_1, \ldots, b_n \in x \text{ and } c_1, \ldots, c_m \notin y\}$$

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Example ($\{p \leftarrow p, \neg q\}$)
$\mathcal{IC}^{l}_{\mathcal{P}}(\{p\}, \{p, q\}) = \emptyset$
$\mathcal{IC}^{l}_{\mathcal{P}}(\{p\}, \{p\}) = \{p\}$

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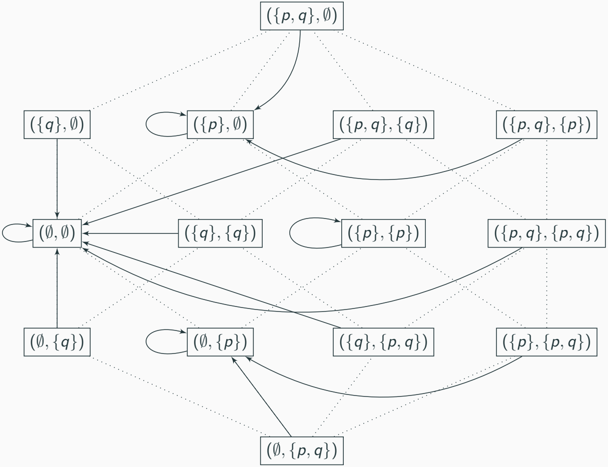
The Approximation Operator $\mathcal{IC}_\mathcal{P}$ (for $\mathcal{P} = \{p \leftarrow p, \neg q\}$)

$$\mathcal{IC}_\mathcal{P}(x, y) = (\mathcal{IC}^l_\mathcal{P}(x, y), \mathcal{IC}^u_\mathcal{P}(x, y))$$



Properties of $\mathcal{IC}_\mathcal{P}$

- $\mathcal{IC}_\mathcal{P}$ approximates $T_\mathcal{P}$:

$\mathcal{IC}_\mathcal{P}(x, x) = (T_\mathcal{P}(x), T_\mathcal{P}(x))$ for any $x \subseteq \mathcal{A}$.

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- $\mathcal{IC}_\mathcal{P}$ approximates $T_\mathcal{P}$:

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We say $\mathcal{IC}_\mathcal{P}$ is an approximation operator. It is also symmetric, in the sense that $\mathcal{IC}_\mathcal{P}(x, y) = (\mathcal{IC}^l_\mathcal{P}(x, y), \mathcal{IC}^l_\mathcal{P}(y, x))$.

Kripke-Kleene Fixpoint

The $\leq_i$-monotonicity is our *indulgentia* back into Tarski's heaven:

Proposition
$\mathcal{IC}_\mathcal{P}$ *has a least fixpoint, obtainable as* $\bigsqcup_{i \geq 0} \mathcal{IC}^i_\mathcal{P}(\emptyset, \mathcal{A})$[1].

[1] $(x_1, y_1) \sqcup (x_2, y_2) = (x_1 \cup x_2, y_1 \cap y_2)$.

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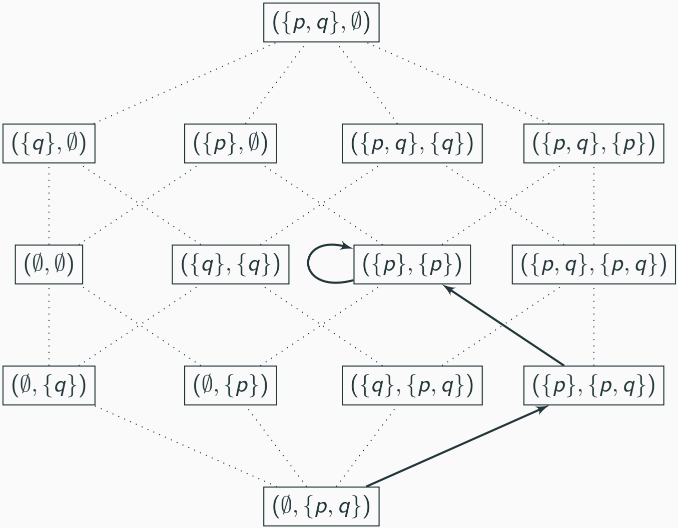
If $(x, y) = \mathcal{IC}_{\mathcal{P}}(x, y)$ then we call it a partial supported model.

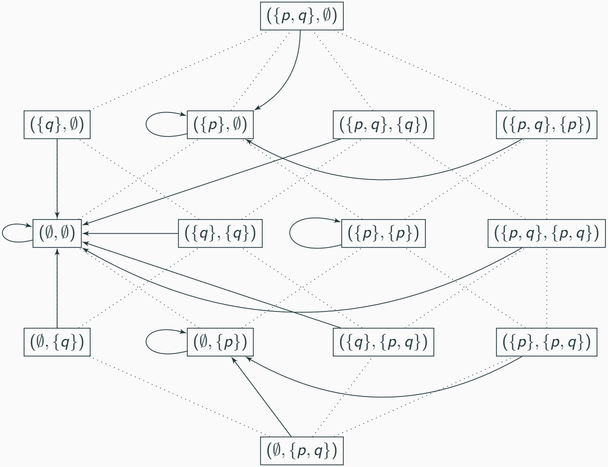
[1] $(x_1, y_1) \sqcup (x_2, y_2) = (x_1 \cup x_2, y_1 \cap y_2)$.

Example: $\mathcal{P} = \{p \leftarrow; q \leftarrow \neg p\}$



The Approximation Operator $\mathcal{IC}_\mathcal{P}$ (for $\mathcal{P} = \{p \leftarrow p, \neg q\}$)

$$\mathcal{IC}_\mathcal{P}(x, y) = (\mathcal{IC}^l_\mathcal{P}(x, y), \mathcal{IC}^u_\mathcal{P}(x, y))$$



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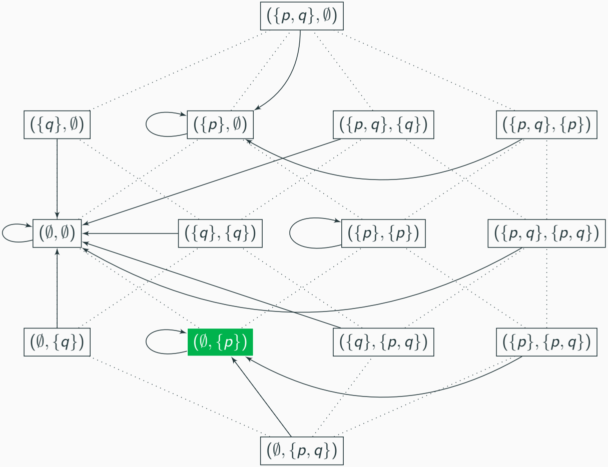

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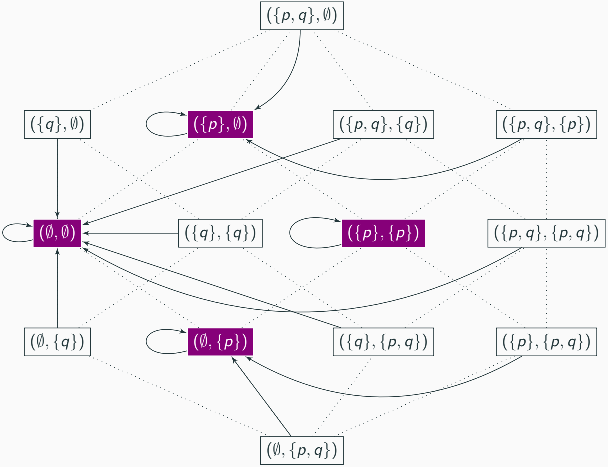
Partial Supported models



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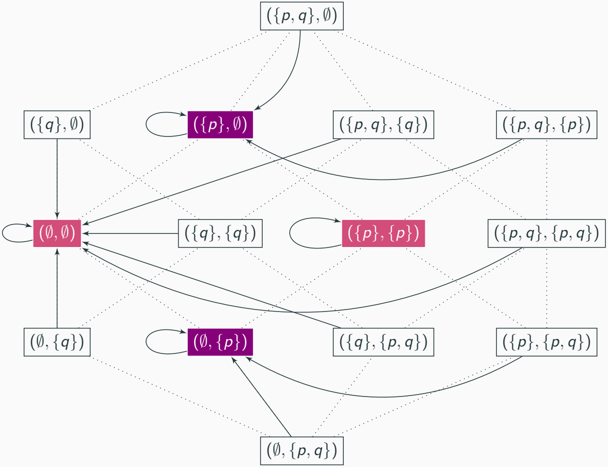
Supported models



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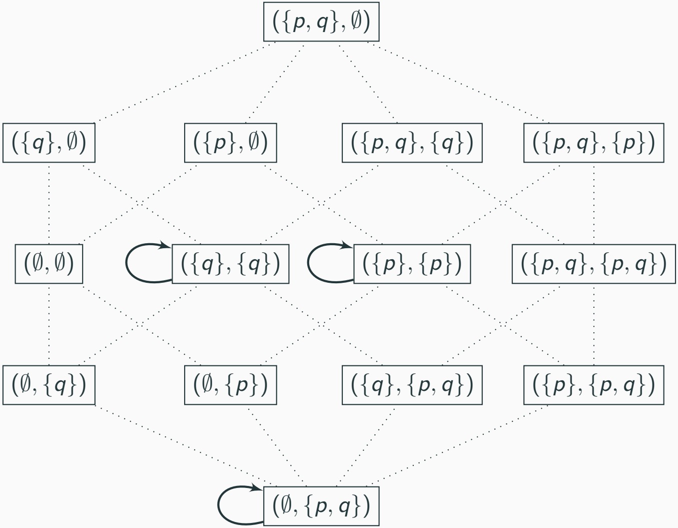

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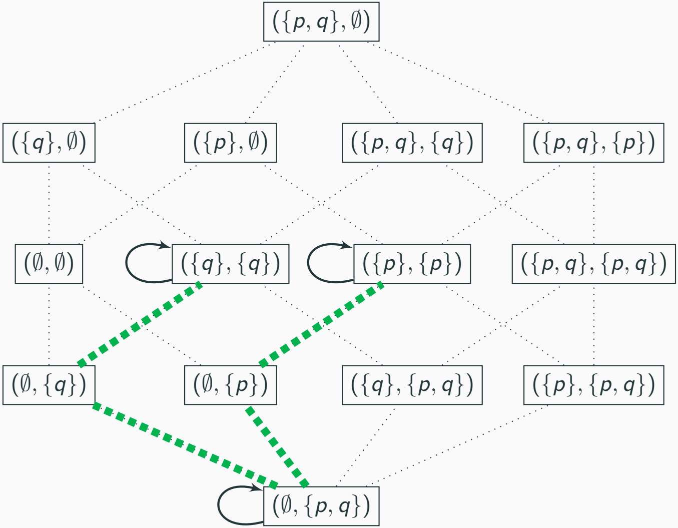

Non-Deterministic Operators

Motivation

```
{processed(M, T, 1..A): availability(M,T,A)} C :-
 capacity(T,C).
processed(M, T, N-1) :- processed(M, T, N), N>1.
result(P,0,I) :- inventory(P,0,I), I>=0.
result(P,0,-I) :- backlog(P,0,I), I>0.
result(P,T,R1) :- demand(P,T,D), result(P,T-1,R), T>0,
S=#sum{Y,M,X : processed(M,T,X), yield(M,P,Y)}, R1=S-D+R.
```

Choice Atoms

A *choice atom* is an expression $C = (\mathsf{dom}, \mathsf{sat})$ where $\mathsf{dom} \subseteq \mathcal{A}$ and $\mathsf{sat} \subseteq \wp(\mathsf{dom})$.

Example

$1\{p, q, r\}2$ *corresponds to the choice atom* $C_1 = (\{p, q, r\}, \{\{p\}, \{q\}, \{r\}, \{p, q\}, \{p, r\}, \{q, r\}\})$.

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A set of atoms x satisfies $(\mathsf{dom}, \mathsf{sat})$ if $x \cap \mathsf{dom} \in \mathsf{sat}$.

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$1\{p, q, r\}2$ *corresponds to the choice atom*
$C_1 = (\{p, q, r\}, \{\{p\}, \{q\}, \{r\}, \{p, q\}, \{p, r\}, \{q, r\}\})$.

- $\{p, q, s\}$ *satisfies* C_1 *as* $\{p, q, s\} \cap \{p, q, r\} = \{p, q\} \in \{\{p\}, \{q\}, \{r\}, \{p, q\}, \{p, r\}, \{q, r\}\}$.
- $\{p, q, r\}$ *does not satisfy* C_1 *as* $\{p, q, r\} \cap \{p, q, r\} = \{p, q, r\} \notin \{\{p\}, \{q\}, \{r\}, \{p, q\}, \{p, r\}, \{q, r\}\}$.

Choice Rules

Where C, $C_1, \ldots, C_n$ are choice atoms, a choice rule is of the form:

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- A choice rule is normal if $\mathrm{sat}(C_i) = \{\{a\}\}$ (for some $a \in \mathcal{A}$) or $\mathrm{sat}(C_i) = \{\emptyset\}$ for every $i = 1 \ldots n$,
  - $(\{a\}, \{\{a\}\})$ is denoted by a, and
  - $(\{a\}, \{\emptyset\})$ is denoted by $\neg a$.

Non-Deterministic Immediate Consequence Operator

Given a choice program $\mathcal{P}$ and a set of atoms x:

- A rule $r \in \mathcal{P}$ is x-applicable (in symbols, $r \in \mathcal{P}(x)$) if x satisfies the body of r.
- $IC_{\mathcal{P}}(x) = \{z \subseteq \bigcup_{r \in \mathcal{P}(x)} \mathrm{dom}(\mathrm{hd}(r)) \mid \forall r \in \mathcal{P}(x) : z \models \mathrm{hd}(r)\}$

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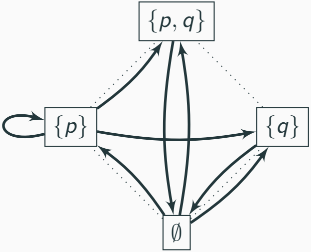

Non-Deterministic Approximators for Normal Choice Programs

$$\mathcal{HD}^l_{\mathcal{P}}(x, y) = \{C \mid C \leftarrow a_1, \ldots, a_n, \neg b_1, \ldots, \neg b_m \in \mathcal{P},$$
$$\{a_1, \ldots, a_n\} \subseteq x, \{b_1, \ldots, b_m\} \cap y = \emptyset\}\}$$
$$\mathcal{IC}^l_{\mathcal{P}}(x, y) = \{z \subseteq \bigcup_{C \in \mathcal{HD}^l_{\mathcal{P}}(x,y)} \mathrm{dom}(C) \mid \forall C \in \mathcal{HD}^l_{\mathcal{P}}(x, y), z \cap \mathrm{dom}(C) \in \mathrm{sat}(C)\}$$
$$\mathcal{IC}^u_{\mathcal{P}}(x, y) = \mathcal{IC}^l_{\mathcal{P}}(y, x)$$
$$\mathcal{IC}_{\mathcal{P}}(x, y) = (\mathcal{IC}^l_{\mathcal{P}}(x, y), \mathcal{IC}^u_{\mathcal{P}}(x, y))$$

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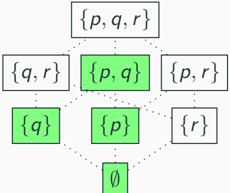
Example: $\mathcal{P} = \{1\{p, q\}2 \leftarrow \neg r.\}$

- $\mathcal{HD}^{c,l}_{\mathcal{P}}(\emptyset, \{r\}) = \emptyset$,
- $HDc^{c,u}_{\mathcal{P}}(\emptyset, \{r\}) = \{1\{p, q\}2\}$,
- $\mathcal{IC}^c_{\mathcal{P}}(\emptyset, \{r\}) = (\{\emptyset\}, \{\{p\}, \{q\}, \{p, q\}\})$.



Approximators for Choice Programs (aka the real fun)

Given a choice program $\mathcal{P}$ and pair of sets of atoms (x, y) let:

$$\mathcal{HD}^{\mathsf{GZ},l}_{\mathcal{P}}(x,y) = \{C \mid \exists C \leftarrow C_1, \ldots, C_i \in \mathcal{P}, \forall i = 1 \ldots n : \\ x \cap \mathsf{dom}(C_i) = y \cap \mathsf{dom}(C_i) \in \mathsf{sat}(C_i)\},$$

$$\mathcal{HD}^{\mathsf{LPST},l}_{\mathcal{P}}(x,y) = \{C \mid \exists C \leftarrow C_1, \ldots, C_n \in \mathcal{P}, \forall i = 1 \ldots n : \\ \forall z \in [x,y] : z(C_i) = \mathsf{T}\},$$

$$\mathcal{HD}^{\mathsf{MR},l}_{\mathcal{P}}(x,y) = \{C \mid \exists C \leftarrow C_1, \ldots, C_n \in \mathcal{P}, \exists z \subseteq x : \\ \forall i = 1 \ldots n : y(C_i) = \mathsf{T} \text{ and } z(C_i) = \mathsf{T}\},$$

For $\mathsf{x} \in \{\mathsf{LPST}, \mathsf{MR}, \mathsf{GZ}\})$ let:

$$\mathcal{IC}^{\mathsf{x},l}(x,y) = \{z \subseteq \bigcup_{C \in \mathcal{HD}^{\mathsf{x},l}_{\mathcal{P}}(x,y)} \mathsf{dom}(C) \mid \forall C \in \mathcal{HD}^{\mathsf{x},l}_{\mathcal{P}}(x,y) : z \cap \mathsf{dom}(C) \in \mathsf{sat}(C)\}$$

For $\dagger \in \{\mathsf{MR}, \mathsf{LPST}, \mathcal{U}\}$ let:

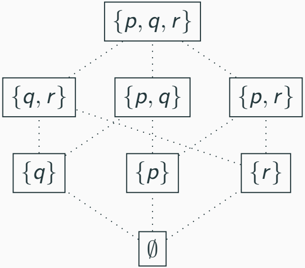
$$\mathcal{IC}^{\mathcal{U},l}_{\mathcal{P}}(x,y) = \mathcal{IC}^{\dagger,u}_{\mathcal{P}}(x,y) \qquad = \bigcup_{x \subseteq z \subseteq y} IC_{\mathcal{P}}(z)$$

whereas $\mathcal{IC}^{\mathsf{GZ},u}_{\mathcal{P}}(x,y) = \mathcal{IC}^{\mathsf{GZ},l}_{\mathcal{P}}(x,y)$. $\mathcal{IC}^{\mathsf{x}}(x,y) = (\mathcal{IC}^{\mathsf{x},l}(x,y), \mathcal{IC}^{\mathsf{x},u}(x,y))$ (for $\mathsf{x} \in \{\mathsf{LPST}, \mathsf{MR}, \mathsf{GZ}, \mathcal{U}\}$).

Intermezzo on Orders

Let $L = \langle \mathcal{L}, \leq \rangle$ be a lattice and $X, Y \in 2^{\mathcal{L}}$.

- $X_1 \preceq_L^S X_2$ if for every $x_2 \in X_2$ there is an $x_1 \in X$ s.t. $x_1 \leq x_2$.
- $Y_1 \preceq_L^H Y_2$ if for every $y_1 \in Y_1$ there is a $y_2 \in Y_2$ s.t. $y_1 \leq y_2$.
- $(X_1,\ Y_1) \preceq_i^A (X_2,\ Y_2)$ iff $X_1 \preceq_L^S X_2$ and $Y_2 \preceq_L^H Y_1$.



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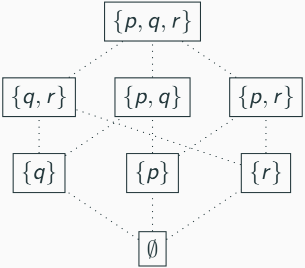

Pairs of sets (X, Y) as convex sets.

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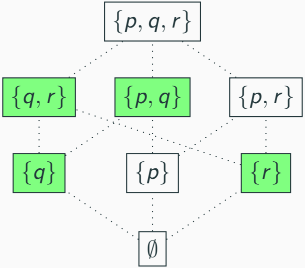

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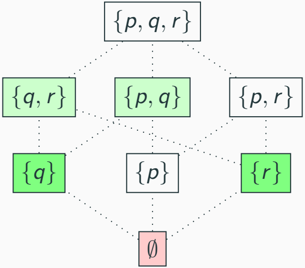

$$\{\emptyset\} \preceq_L^S \{\{q\}, \{r\}\}$$

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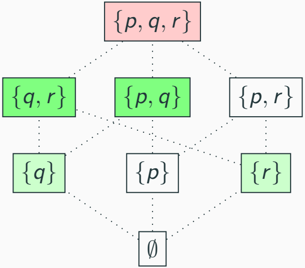

$$\{\{q, r\}, \{p, q\}\} \preceq_L^H \{\{p, q, r\}\}$$

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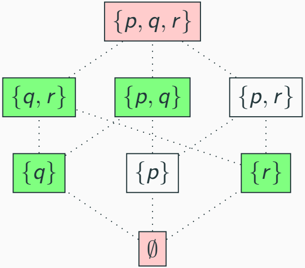

$$(\{\emptyset\},\ \{\{p, q, r\}\}) \preceq_i^A (\{\{q\}, \{r\}\},\ \{\{q, r\}, \{p, q\}\})$$

Non-Deterministic Approximation Operators

O is approximated using a non-deterministic approximation operator $\mathcal{O} : \mathcal{L}^2 \to 2^{\mathcal{L}} \times 2^{\mathcal{L}}$:

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Output of $\mathcal{O}(x, y) = (\{x_1, x_2, \ldots\},\ \{y_1, y_2, \ldots\})$ is a set of lower bounds respectively upper bounds on choices $O(z) = \{z_1, z_2, \ldots\}$ (with $x \leq z \leq y$).

Non-Deterministic Approximation Operators

O is approximated using a non-deterministic approximation operator $\mathcal{O} : \mathcal{L}^2 \rightarrow 2^{\mathcal{L}} \times 2^{\mathcal{L}}$:

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Non-Deteterministic AFT: fixpoints

$\mathcal{IC}_{\mathcal{P}}$ allows to generalize supported models to the three-valued case:

Example $\mathcal{P} = \{\{p, q\} = 1 \leftarrow \neg p. \quad p \leftarrow q.\}$

- No (two-valued) supported models

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Example $\mathcal{P} = \{\{p, q\} = 1 \leftarrow \neg p. \quad p \leftarrow q.\}$

- No (two-valued) supported models
- $\mathcal{IC}^c_{\mathcal{P}}(\emptyset, \{p\}) = (\{\emptyset\}, \{\{p\}, \{q\}, \{p, q\}\})$.
- (among others)

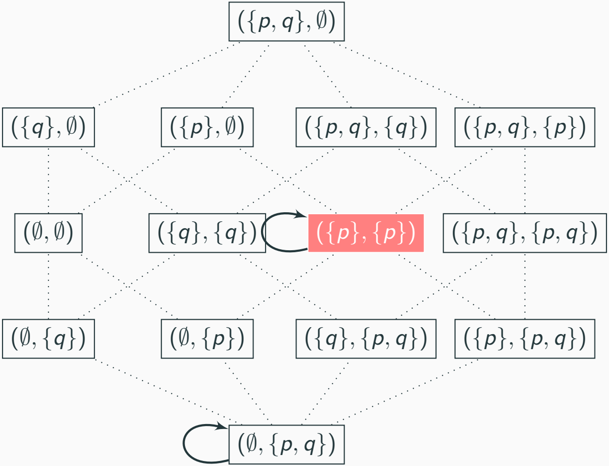
Proposition
Let a normal logic program $\mathcal{P}$ be given. Then $(x, y) \in \mathcal{IC}_{\mathcal{P}}(x, y)$ iff (x, y) is a partial supported model of $\mathcal{P}$.

Proposition
Let a choice program $\mathcal{P}$ and $\mathrm{x} \in \{\mathrm{LPST}, \mathrm{MR}, \mathrm{GZ}, \mathcal{U}\}$ be given. If $(x, y) \in \mathcal{IC}^{\mathrm{x}}_{\mathcal{P}}(x, y)$ then for every $a \in y$, there is a $C \leftarrow C_1, \ldots, C_n$ with $a \in \mathrm{dom}(C)$ s.t.
$\mathcal{IC}^{\mathrm{x}}_{\{p \leftarrow C_1, \ldots, C_n\}}(x, y) = (\{\{p\} \cap x\}, \{\{p\} \cap y\})$.

Stable Semantics

Example: Kripke-Kleene is rather weak

$$p \leftarrow \neg q;\ q \leftarrow q$$



Example: Kripke-Kleene is rather weak

$$\mathcal{P} = \{p \leftarrow \neg q;\ q \leftarrow q\}$$

Construction of the Kripke-Kleene fixpoint:

- $\mathcal{IC}_{\mathcal{P}}(\emptyset, \{p, q\}) = (\emptyset, \{p, q\})$.
- Fixpoint reached.

Can't get rid of the self-supporting atom q in the upper bound.

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As $\mathcal{IC}^u_{\mathcal{P}}(\emptyset, \cdot)$ is a $\subseteq$-monotonic operator, it admits a least fixed point.

Stable Operator

$$S(\mathcal{IC}^{l}_{\mathcal{P}})(y) = lfp(\mathcal{IC}^{l}_{\mathcal{P}}(\cdot, y))$$

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Example ($\{p \leftarrow \neg q;\ q \leftarrow q\}$)

$S(\mathcal{IC}^{l}_{\mathcal{P}})(\{p, q\}) = \emptyset$ *since:*

$\mathcal{IC}^{l}_{\mathcal{P}}(\emptyset, \{p, q\}) = \emptyset$*: fixpoint reached.*

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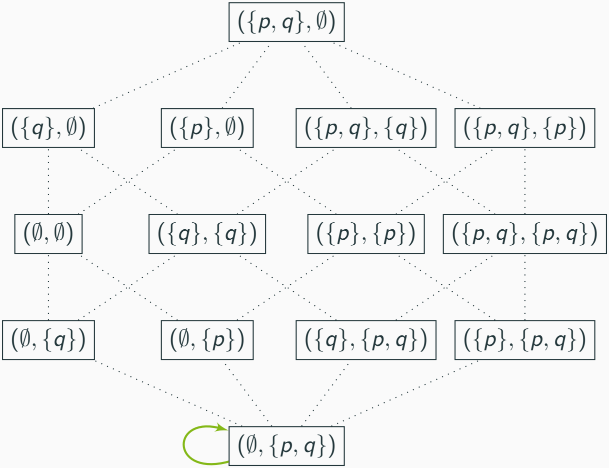
Stable Operator: Example $\{p \leftarrow \neg q; q \leftarrow q\}$

$S(\mathcal{IC}^l_{\mathcal{P}})(\{p, q\}) = lfp(\mathcal{IC}^l_{\mathcal{P}}(\cdot, \{p, q\}))$



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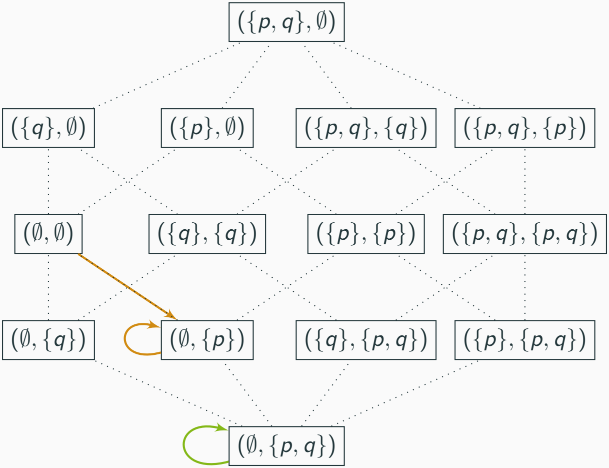
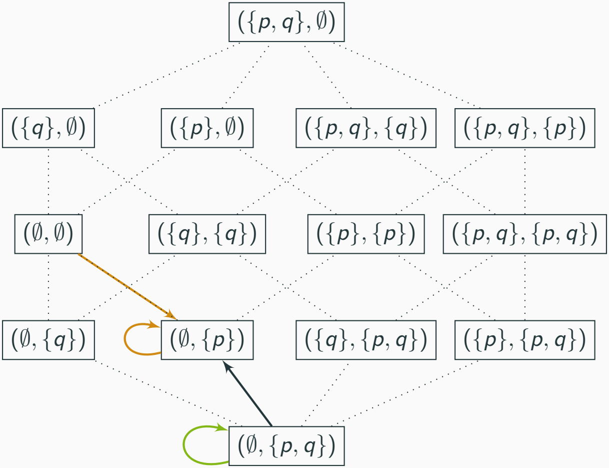


Stable Operator: Example $\{p \leftarrow \neg q; q \leftarrow q\}$

$$S(\mathcal{IC}_{\mathcal{P}})(\emptyset, \{p, q\}) = (S(\mathcal{IC}^{l}_{\mathcal{P}})(\{p, q\}),\ S(\mathcal{IC}^{u}_{\mathcal{P}})((\emptyset))$$



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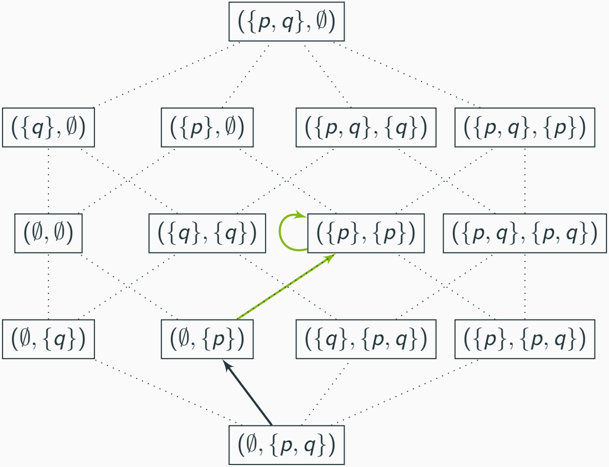

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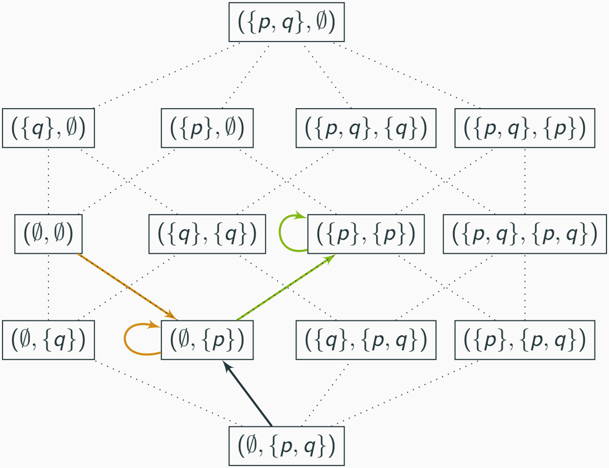

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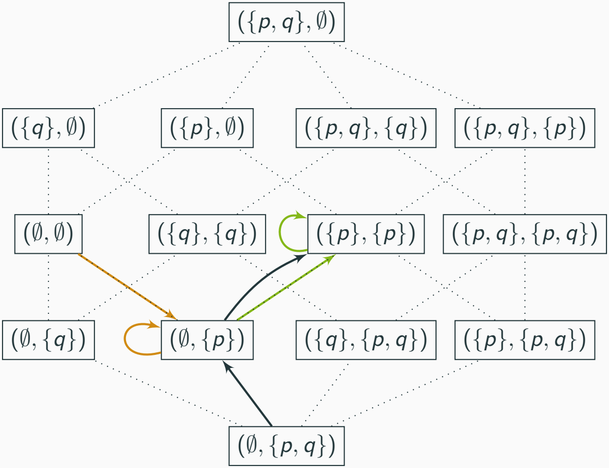

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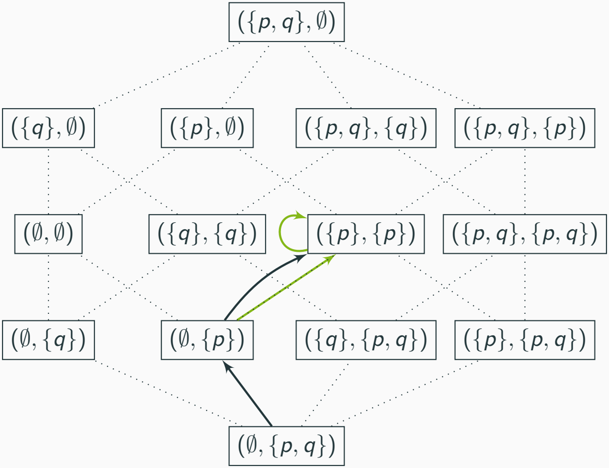
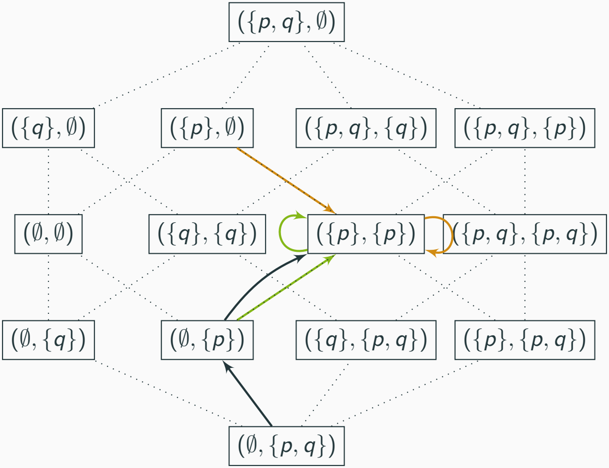


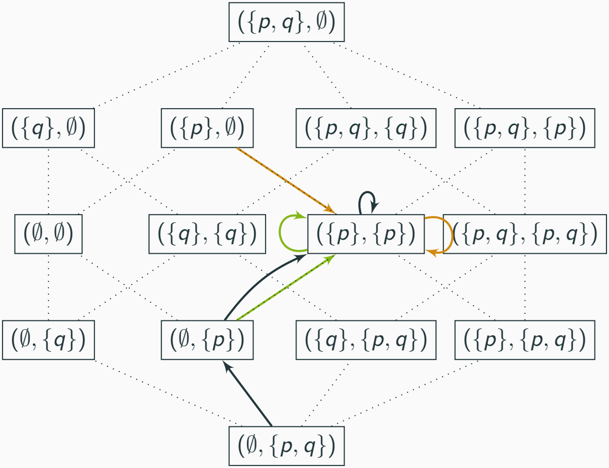
Stable Operator: Example $\{p \leftarrow \neg q; q \leftarrow q\}$

$S(\mathcal{IC}^u_{\mathcal{P}})(x) = lfp(\mathcal{IC}^u_{\mathcal{P}}(\{p\}, \cdot))$



Stable Operator: Example $\{p \leftarrow \neg q; q \leftarrow q\}$

$$S(\mathcal{IC_P})(\{p\}, \{p\}) = (S(\mathcal{IC}^l_\mathcal{P})(\{p\}),\ S(\mathcal{IC}^u_\mathcal{P})(\{p\}))$$



Stable Operator and Well-Founded Model

$$S(\mathcal{IC}^l_{\mathcal{P}})(y) = \mathit{lfp}(\mathcal{IC}^l_{\mathcal{P}}(\cdot, y)) \qquad S(\mathcal{IC}^u_{\mathcal{P}})(x) = \mathit{lfp}(\mathcal{IC}^u_{\mathcal{P}}(x, \cdot))$$

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We call this the well-founded model, denoted $\mathrm{WF}(\mathcal{P})$.

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• Any fixpoint of $S(\mathcal{IC}_{\mathcal{P}})$ is a minimal model of $\mathcal{P}$.
If $(x, y) = S(\mathcal{IC}_{\mathcal{P}})(x, y)$, we call it a (partial) stable model.
If $x = S(\mathcal{IC}_{\mathcal{P}})(x)$, we call it a stable model.

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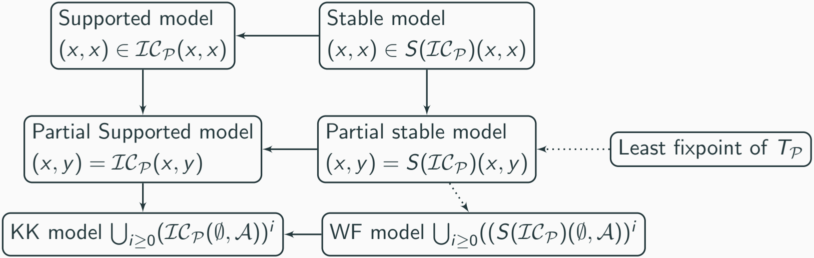
• If $T_{\mathcal{P}}$ has a least fixpoint, it coincides with the well-founded model.

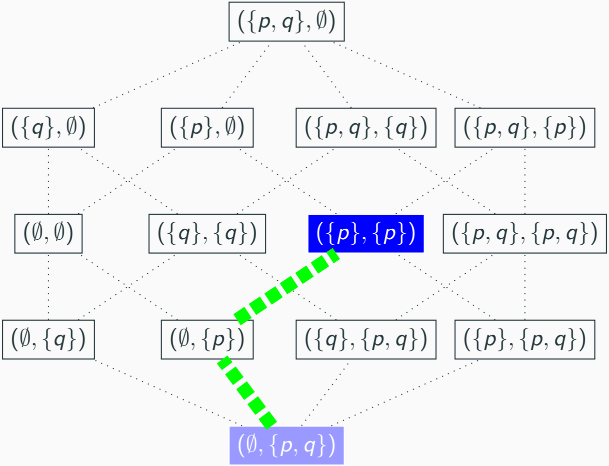
Taking Stock

Semantics =	Stable? +	Operator +	Fixpoint
	S	$(\mathcal{O})(x,y)$	$=(x,y)$
Kripke-Kleene	No	$\mathcal{IC}_{\mathcal{P}}$	least fp
Partial Supported	No	$\mathcal{IC}_{\mathcal{P}}$	any fp
Supported	No	$\mathcal{IC}_{\mathcal{P}}$	total fp $(x=y)$
Well-founded	yes	$\mathcal{IC}_{\mathcal{P}}$	least fp
Partial Stable	yes	$\mathcal{IC}_{\mathcal{P}}$	any fp
Stable	yes	$\mathcal{IC}_{\mathcal{P}}$	total fp $(x=y)$



Stable Operator: Example

$p \leftarrow \neg q; q \leftarrow q$



Stable Operator: Example 2

$$\mathcal{P} = \{p \leftarrow \neg q; \quad q \leftarrow \neg p; \quad r \leftarrow r; \quad s \leftarrow \neg r\}$$

- Kripke-Kleene fixpoint: $(\emptyset, \{p, q, r, s\})$.
- Well-founded model: $(\{s\}, \{p, q, s\})$.
- Stable models: $(\{p, s\}, \{p, s\})$, $(\{q, s\}, \{q, s\})$.

Stable Semantics and Reducts

$$\frac{\mathcal{P}}{x} = \{a \leftarrow b_1, \ldots, b_n \mid a \leftarrow b_1, \ldots, b_n, \neg c_1, \ldots, \neg c_m \in \mathcal{P}$$

$$c_1, \ldots, c_n \notin x\}$$

Definition

x is a stable model of $\mathcal{P}$ if it is a minimal model of $\frac{\mathcal{P}}{x}$.

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Example ($\mathcal{P} = \{p \leftarrow \neg p; q \leftarrow \neg p; p \leftarrow \neg q\}$)
$\frac{\mathcal{P}}{\{q\}} = \{p \leftarrow; q \leftarrow\}$. *$\{q\}$ is not a minimal model of $\mathcal{P}$, thus $\{q\}$ is not a stable model.*

$\frac{\mathcal{P}}{\{p\}} = \{p \leftarrow\}$. *$\{p\}$ is a minimal model of $\mathcal{P}$.$\{q\}$ is not a minimal model of $\mathcal{P}$, thus $\{p\}$ is a stable model.*

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x is a stable model of $\mathcal{P}$ if it is a minimal model of $\frac{\mathcal{P}}{x}$.

Proposition
$S(\mathcal{IC}^l_{\mathcal{P}})(y)$ is the set of minimal models of $\frac{\mathcal{P}}{y}$.

Proposition
$(x, x) = S(\mathcal{IC}_{\mathcal{P}})(x, x)$ if and only if x is a stable model of $\mathcal{P}$ (iff $x = S(\mathcal{IC}^l_{\mathcal{P}})(x)$).

Stable Operator for Deterministic Approximation Operators

Basic idea: look for smallest fixpoint that the upper bound allows us to construct:

$$S(\mathcal{O}_l)(y) = glb\{x \in \mathcal{L} \mid x = \mathcal{O}_l(x, y)\}$$

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Basic idea: look for smallest fixpoint that the upper bound allows us to construct:

$$S(\mathcal{O}_I)(y) = glb\{x \in \mathcal{L} \mid x = \mathcal{O}_I(x, y)\} = \bigcup_{i=1}^{\infty} \mathcal{O}_I^i(\emptyset, y).$$

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Stable Operator: Example

$$\mathcal{P} = \{1\{p, q\}2 \leftarrow .\}$$

x	$\emptyset$	$\{p\}$	$\{q\}$	$\{p, q\}$
$\mathcal{IC}^l_{\mathcal{P}}(x, y)$	$\{p\}, \{q\}, \{p, q\}$	$\{p\}, \{q\}, \{p, q\}$	$\{p\}, \{p\}, \{p, q\}$	$\{p\}, \{q\}, \{p, q\}$

1. glb of fixpoints of $\mathcal{IC}^{c,l}_{\mathcal{P}}(., y)$: $\{p\} \cap \{q\} \cap \{p, q\} = \emptyset$.
2. Minimal fixpoints: $\{p\}$ and $\{q\}$.
3. $\bigcup_{i=1}^{\infty} (\mathcal{IC}^l_{\mathcal{P}})^i(., y) = ?$

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We have two choices now: minimality based or iterative.

Minimality-Based Stable Operator

Let an ndao $\mathcal{O} : \mathcal{L}^2 \to 2^{\mathcal{L}} \times 2^{\mathcal{L}}$ and some $x, y \in \mathcal{L}$ be given. Then:

- the m-complete lower stable operator as:

$C^m(\mathcal{O}_l)(y) = \{x \in \mathcal{L} \mid x \in \mathcal{O}_l(x, y) \text{ and } \neg\exists x' < y : x' \in \mathcal{O}_l(x', y)\}$

- the *m-complete upper stable operator* as:

$C^m(\mathcal{O}_u)(x) = \{y \in \mathcal{L} \mid y \in \mathcal{O}_u(x, y) \text{ and } \neg\exists y' < y : z \in \mathcal{O}_u(x, y')\}$

- the m-stable operator as $S^m(\mathcal{O})(x, y) = (C(\mathcal{O}_l)(y), C(\mathcal{O}_u)(x))$.
- a m-stable fixpoint of $\mathcal{O}$ as any $(x, y) \in \mathcal{L}^2$ s.t.

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$(x, y) \in S^m(\mathcal{O})(x, y)$.

Example ($\mathcal{P} = \{1\{p, q\} \leftarrow \neg q\}$)
$C^m(\mathcal{IC}^l_{\mathcal{P}})(\{p\}) \min_{\subseteq}\{x \subseteq \mathcal{A}_{\mathcal{P}} \mid x \in \mathcal{IC}_{\mathcal{P}}(x, \{p\})\} = \{\{p\}, \{q\}\}$.
$(\{p\}, \{p\})$ *is a m-stable fixpoint.*

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$(x, y) \in S^m(\mathcal{O})(x, y)$.

Example ($\mathcal{P} = \{1\{p, q\} \leftarrow \neg q\}$)
$C^m(\mathcal{IC}^l_{\mathcal{P}})(\{p\}) \min_{\subseteq}\{x \subseteq \mathcal{A}_{\mathcal{P}} \mid x \in \mathcal{IC}_{\mathcal{P}}(x, \{p\})\} = \{\{p\}, \{q\}\}$.
$(\{p\}, \{p\})$ *is a m-stable fixpoint.*

Theorem Let a dlp $\mathcal{P}$ and a consistent $x \subseteq y \subseteq \mathcal{A}^2_{\mathcal{P}}$ be given. Then (x, y) is a stable model of $\mathcal{P}$ according to [Prz91] (i.e. the minimal models of the reduct) iff $(x, y) \in S(\mathcal{IC}_{\mathcal{P}})(x, y)$.

Constructive Stable Operator

Given a non-deterministic operator $O : \mathcal{L} \to \wp(\mathcal{L})$, a sequence $x_0, \ldots, x_n \subseteq \mathcal{L}$ is well-founded relative to O if:

- $x_0 = \perp$,
- $x_i \leq x_{i+1}$ and $x_{i+1} \in O(x_i)$ for every successor ordinal $i \geq 0$.
- $x_\lambda = (lub\{x_i\}_{i<\lambda})$ for a limit ordinal λ.

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The complete constructive lower bound operator is defined as:

$$S^c(\mathcal{IC}^l_{\mathcal{P}})(y) = \{x \in \mathcal{O}_l(x, y) \mid \exists x_0, .., x \in \text{wfs}(\mathcal{IC}^l_{\mathcal{P}}(., y))\}$$

The complete constructive upper bound operator is defined analogously, and the constructive stable operator is defined as $S^c(\mathcal{IC}_{\mathcal{P}})(x, y) = (S^c(\mathcal{IC}^l_{\mathcal{P}})(y), S^c(\mathcal{IC}^u_{\mathcal{P}})(x))$.

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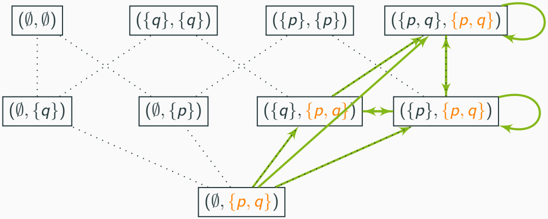
A pair (x, y) is a constructive stable fixpoint iff $(x, y) \in S^c(\mathcal{IC}_{\mathcal{P}})(x, y)$.

Stable Operator: Example 1

$$\mathcal{P} = \{1\{p, q\}2 \leftarrow .,\quad p \leftarrow q\}$$

y	$\emptyset$	$\{p\}$	$\{q\}$	$\{p, q\}$
$\mathcal{IC}^{c,l}_{\mathcal{P}}(., y)$	$\{p\}, \{q\}, \{p, q\}$	$\{p\}, \{q\}, \{p, q\}$	$\{p\}, \{p, q\}$	$\{p\}, \{p, q\}$



$S(\mathcal{IC}^{l}_{\mathcal{P}})(\{p, q\}) =$
$\{\{p\}, \{p, q\}\}$

$(\{p\}, \{p\})$ is a stable fp.

$(\{p, q\}, \{p, q\})$ is a stable fp.

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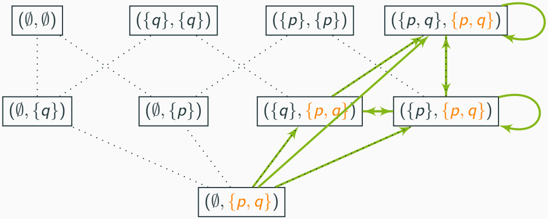

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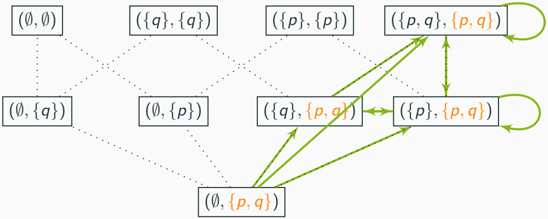
Stable Operator: Example 1

$$\mathcal{P} = \{1\{p,q\}2 \leftarrow ., \quad p \leftarrow q\}$$

y	$\emptyset$	$\{p\}$	$\{q\}$	$\{p,q\}$
$\mathcal{IC}^{c,l}_{\mathcal{P}}(.,y)$	$\{p\},\{q\},\{p,q\}$	$\{p\},\{q\},\{p,q\}$	$\{p\},\{p,q\}$	$\{p\},\{p,q\}$



$S(\mathcal{IC}^{l}_{\mathcal{P}})(\{p,q\}) = \{\{p\},\{p,q\}\}$

$(\{p\},\{p\})$ is a stable fp.

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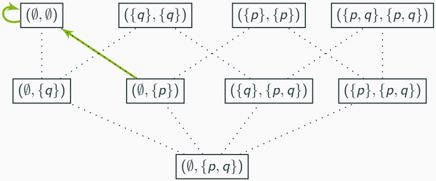
Stable Operator: Example 2

$$\mathcal{P} = \{1\{p,q\} \leftarrow p.; \quad 1\{p,q\} \leftarrow q\}$$



Well-founded sequence of $IC^{l}_{\mathcal{P}}(., y)$ for any $y \subseteq \{p,q\}$: $\emptyset$

$(\emptyset, \emptyset)$ is the unique fixpoint of $S(\mathcal{IC}_{\mathcal{P}})$.

General results and characterisation theorem

Theorem
Let a choice program $\mathcal{P}$ s.t. for every $C_1 \leftarrow C_2, \ldots, C_n \in \mathcal{P}$, $\mathrm{dom}(C_i)$ is finite for $i = 1 \ldots n$, and $\mathrm{x} \in \{\mathrm{MR}, \mathrm{LPST}, \mathcal{U}\}$ and $x \subseteq y \subseteq \mathcal{A}$ be given. Then $S^c(\mathcal{IC}_{\mathcal{P}}^{\mathrm{x}})(x, y) \neq \emptyset$. Furthermore, $C^c(\mathcal{IC}_{\mathcal{P}}^{\mathrm{GZ},l})(y) \neq \emptyset$.

Theorem
Let a choice program $\mathcal{P}$ be given.

1. *x is a stable model according to [LPST10] iff (x, x) is a stable fixpoint of $\mathcal{IC}_{\mathcal{P}}^{\mathrm{LPST}}$.*
2. *x is a stable model according to [MR04] iff (x, x) is a stable fixpoint of $\mathcal{IC}_{\mathcal{P}}^{\mathrm{MR}}$.*
3. *If $\mathcal{P}$ is a aggregate program then x is a stable model according to [GZ14] iff $x \in C^c(\mathcal{IC}_{\mathcal{P}}^{\mathrm{GZ},l})(x)$.*

Disjunctions are Choice Constructs

$\mathtt{D2C}(\mathcal{P}) = \{1\Delta \leftarrow \phi \mid \bigvee \Delta \leftarrow \phi \in \mathcal{P}\}$.

Example $\mathtt{D2C}(\{p \vee q \leftarrow .\}) = \{1\{p \vee q\} \leftarrow .\}$.

Disjunctions are Choice Constructs

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Example $\mathtt{D2C}(\{p \vee q \leftarrow .\}) = \{1\{p \vee q\} \leftarrow .\}$.

Proposition

For any disjunctive logic program $\mathcal{P}$, $\mathcal{IC}_{\mathcal{P}} = \mathcal{IC}^{c}_{\mathtt{D2C}(\mathcal{P})}$.

Disjunctions are Choice Constructs

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Proposition
For any disjunctive logic program $\mathcal{P}$, $\mathcal{IC}_{\mathcal{P}} = \mathcal{IC}^{c}_{\mathtt{D2C}(\mathcal{P})}$.

Difference between DLP and choice programs: the difference is *not* in the treatment of disjunction and choice atoms (i.e. when they should be made true or false), but rather in how the stable semantics is constructed:

- Disjunctions: minimality-based stable semantics [?]
  $S(\mathcal{O}_u)(x) \;=\; \{y \in \mathcal{L} \mid y \in \mathcal{O}_u(x, y) \text{ and } \neg\exists y' < y : y' \in \mathcal{O}_u(x, y')\}$
  $\{p\}$ and $\{q\}$ are stable
- Choice constructs: constructive stable semantics
  $\{p\}$, $\{p, q\}$ and $\{q\}$ are stable

Taking Stock: Non-Deterministic Operators

Semantics	=	Stable? S	+	Operator $(\mathcal{O})(x,y)$	+	Fixpoint $\ni (x,y)$
Partial Supported		No		$\mathcal{IC}_{\mathcal{P}}$		any fp
Supported		No		$\mathcal{IC}_{\mathcal{P}}$		total fp $(x = y)$
Partial Stable for DLPs		M-stable		$\mathcal{IC}_{\mathcal{P}}$		any fp
Stable for DLPs		M-stable		$\mathcal{IC}_{\mathcal{P}}$		total fp $(x = y)$
Partial Stable for CPs		C-stable		$\mathcal{IC}_{\mathcal{P}}$		any fp
Stable for CPs		C-stable		$\mathcal{IC}_{\mathcal{P}}$		total fp $(x = y)$

Taking Stock: Non-Deterministic Operators

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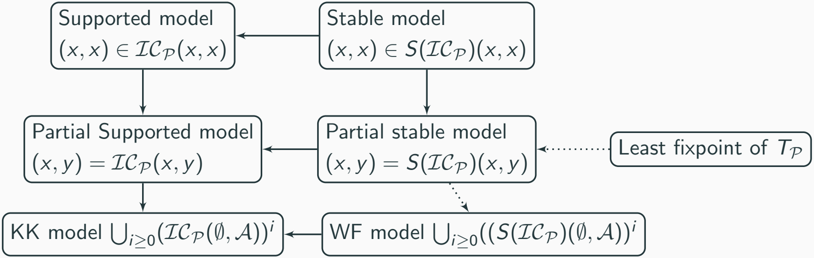
Partial Stable for CPs		C-stable		$\mathcal{IC}_{\mathcal{P}}$		any fp
Stable for CPs		C-stable		$\mathcal{IC}_{\mathcal{P}}$		total fp $(x = y)$
		C-stable		$\mathcal{IC}_{\mathcal{P}}^{\mathsf{LPST}}$		any fp
[LPST10]		C-stable		$\mathcal{IC}_{\mathcal{P}}^{\mathsf{LPST}}$		total fp $(x = y)$
		C-stable		$\mathcal{IC}_{\mathcal{P}}^{\mathsf{MR}}$		any fp
[MR04]		C-stable		$\mathcal{IC}_{\mathcal{P}}^{\mathsf{MR}}$		total fp $(x = y)$
		C-stable		$\mathcal{IC}_{\mathcal{P}}^{\mathsf{GZ}}$		any fp
[GZ14]		C-stable		$\mathcal{IC}_{\mathcal{P}}^{\mathsf{GZ}}$		total fp $(x = y)$

Approximation Fixpoint Theory

Recap



- Operator-based framework
  - Non-monotonic operator $T_{\mathcal{P}}$,
  - a $\leq_i$-monotonic approximation operator $\mathcal{IC}_{\mathcal{P}}$,
  - and its stable variant $S(\mathcal{IC}_{\mathcal{P}})$.
- Allows us to define semantics as fixpoints of these operators, with attractive properties.
- Generalized to the non-deterministic setting, and with other operators.
- This story can be told for a great number of formalisms

Other Semantics

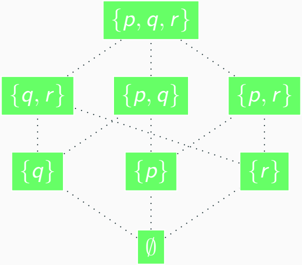
- State semantics for non-deterministic operators:
  - Kripke-Kleene state,
  - Well-founded state (relative to a stable construction),
  - Well-founded state with CWA (relative to a stable construction),
- Semi-equilibrium semantics,
- "Strongly" supported models,
- Regular models,
- M-Stable models,
- L-stable models.

Well-founded State

Basic Idea: generalize operator $\mathcal{O} : \mathcal{L}^2 \to 2^{\mathcal{L}} \times 2^{\mathcal{L}}$ to an operator $\mathcal{O}' : 2^{\mathcal{L}} \times 2^{\mathcal{L}} \to 2^{\mathcal{L}} \times 2^{\mathcal{L}}$ as follows:

$$\mathcal{O}'((X,\ Y)) = \left(\bigcup_{x \in X, y \in Y} \mathcal{O}_l(x,y)\!\uparrow,\ \bigcup_{x \in X, y \in Y} \mathcal{O}_u(x,y)\!\downarrow \right)$$

Example $\mathcal{P} = \{1\{p,q\} \leftarrow .;\ q \leftarrow \neg r.\}$.



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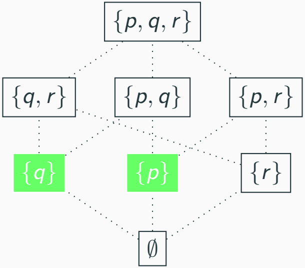

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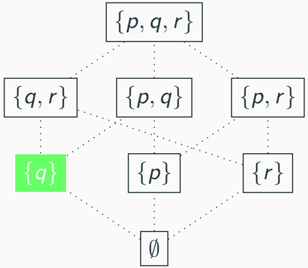

HT-Models, algebraically [Tru06, HB23]

Given an ndao $\mathcal{O}$ approximating a non-determinstic operator O, a pair (x, y) is a HT-*pair* (denoted $(x, y) \in \mathrm{HT}(\mathcal{O})$) if the following three conditions are satisfied:

- $x \leq y$,
- $O(y) \preceq_L^S y$, and
- $\mathcal{O}_l(x, y) \preceq_L^S x$.

PropositionLet some normal disjunctive logic program $\mathcal{P}$ be given. Then: $\mathrm{HT}(\mathcal{P}) = \mathrm{HT}(\mathcal{IC}_{\mathcal{P}})$.

Proposition Given an upwards coherent ndao $\mathcal{O}$, (1) if $(x, x) \in \mathcal{O}(x, x)$ then $(x, x) \in \mathrm{HT}(\mathcal{O})$; and (2) $(x, x) \in \min_{\leq_t}(\mathrm{HT}(\mathcal{O}))$ iff $(x, x) \in S(\mathcal{O})(x, x)$.

Operator-Based Semantics for Dialects of Logic Programming

- $\vee$ Aggregates in the body: $p \leftarrow \#\mathtt{sum}\{2 : p; q : 1; r : 1\} \geq 2$.
- $\vee$ Propositional formulas in the body: $p \leftarrow q \wedge (r \vee (s \wedge \neg t))$.
- $\vee$ Disjunctions in the head: $p \vee q \leftarrow q \wedge (r \vee (s \wedge \neg t))$.
- $\vee$ Choice constructs in the head: $\#\mathtt{count}\{p; q; r\} = 2 \leftarrow \neg r$.
- $\vee$ DL-based logic programs: $\mathbf{K} C(x) \leftarrow \neg p(X); C \sqsubseteq D$.
- $\vee$ Higher-order logic programs: $S(P, Q) \leftarrow; P(X) \leftarrow \neg Q(X)$.
- ... Fuzzy logic programs: $p(X) \leftarrow 0.5 \cdot (q(x) + r(X))$.
- ... Semi-ring constraint programs: $p(X) \leftarrow 0.5 \otimes (q(x) \oplus r(X))$.
- ? Probabilistic logic programs: $0.3 :: p(X)$.
- ? Hex-programs: $tr(S, P, O) \leftarrow \& RDF[uri](S, P, O)$.

Operator-Based Semantics for other KR-formalisms

- autoepistemic logic [DMT03],
- default logic [DMT03],
- abstract argumentation [SW15],
- abstract dialectical frameworks [SW15],
- weighted abstract dialectical frameworks [Bog19],
- SCHACL [BJ21].

Operator-Based Studies

Top-Down approach:

- Instead of studying a concept for a specific framework, define and study it for operators over a lattice (and their approximations).
- We can then apply this concept to all formalisms that are or can be captured in AFT.

Examples:

V Stratification [VGD06]

V Conditional Independence [Hey23]

V Knowledge Compilation [BVdB15]

V Groundedness [BVdB15]

V Strong equivalence [Tru06]

V Argumentative dialogues [HA20]

? Belief dynamics

? Modular equivalence

? Neuro-symbolism

Round up

Summary

- Operators as the core for understanding answer set semantics.
- Paved the road towards approximation fixpoint theory.
- Algebraic theory that allows language independent work on KR.
- Requires some buy-in, but in my view a great bargain.
- Interested in cooperating? Questions on AFT? Come talk to me.

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