Foundations of logic programming semantics: an operator-based perspective

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Logic Programming

- Specific, powerful family of languages for knowledge representation (problems up to second level of polynomial hierarchy).
- Efficient, user-friendly solvers (clingo[1], DLV) and tools.[2]
- Hallmark of the declarative programming approach: describe a problem (without having to describe how to find solutions).

```
node(1..6).
edge(1,2;1,3;1,4;2,4;2,5;2,6;3,1;3,4;3,5;4,1).
col(r). col(g). col(b).

{ color(X,C) : col(C) } =1 :- node(X).
:- edge(X,Y), color(X,C), color(Y,C).
```

[1] https://potassco.org/clingo/run/

[2] https://potassco.org/related/ and their weekly seminar.

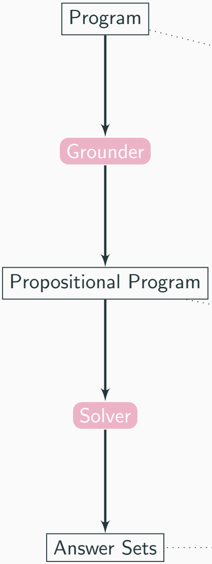
Example Applications: Student Projects

- Puzzles and games:
    - Rush hour
    - Rubics
    - Futoshiki
    - Kakurasu
    - IQ Puzzler Pro
- Generating healthy diets.
- Procedural content generation.
- Parsing grammatical structure of Latin.
- Minimum Sum Partition Problem.

Example Applications: Knowledge Representation

- Solvers or reasoners for:
    - Formal argumentation [DRWW20]
    - AGM Belief Revision [AP17, BNBPW04]
    - Boolean networks [KB19]
    - Ordered disjunction [BNS02]
    - Description Logics [Swi04]
- Inconsistency Measures [KT21]
- Linear temporal logic [KCG23]
- Logic programming (!?) [KRSW23]
- Axiom pinpointing in ontologies [HMP+23]
- ...

The ASP-workflow



penguin(tweety).
bird(zazu).
bird(X)← penguin(X).
flies(X)← bird(X), ¬ ab(X).
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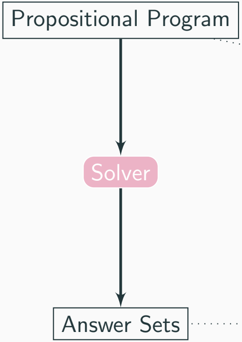
$$\left\{\begin{array}{ccc} \mathit{penguin}(\mathit{tweety}) & \mathit{bird}(\mathit{tweety}) & \mathit{ab}(\mathit{tweety}) \\ \mathit{bird}(\mathit{zazu}) & \mathit{bird}(\mathit{zazu}) & \mathit{flies}(\mathit{zazu}) \end{array}\right\}$$

The ASP-workflow

Focus of today:



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What are answer sets and what is so special about them?

Goals and Structure

- Provide a gentle introduction to the semantics of logic programming:
  - Supported models
  - Kripke-Kleene models
  - Stable models
  - Well-founded model
- Illustrate the operator-based approach to KR with a paradigmatic example.
  - Basic constructions of approximation fixpoint theory (for logic programs).
  - From logic programming to operators.
- Tutorial and invitation to AFT.

Almost nothing of this is my work

- Operator-based approach has driven logic programming since its inception [VGRS91, Fit06].
- Studied algebraically by Denecker, Marek and Truszczyński [DMT00].
- I extended and worked in this algebraic framework with Ofer Arieli and Bart Bogaerts, .

Goals and Structure

Syntax of Logic Programs

Semantics of Positive Programs

Semantics of Normal Logic Programs

Stable Semantics

Approximation Fixpoint Theory

- Operator-Based Studies: Modularity

Round up

Syntax of Logic Programs

Syntax of Logic Programs

Set of atoms $\mathcal{A} = \{a, b, c, p, q, r, a_1, a_2, \ldots\}$

$$a \leftarrow b_1, \ldots, b_n, \neg c_1, \ldots, \neg c_m$$

- Program is a set of rules.
- Rule is positive if $m = 0$.
- Program is positive if all the rules are positive.

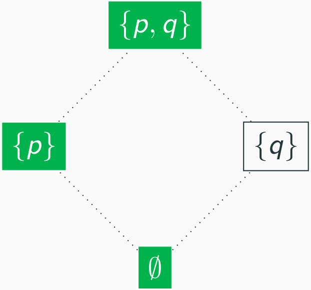
Semantics of Positive Programs

What are the semantics of logic programs?

$$p \leftarrow q.$$

Classical models? $\emptyset, \{p\}, \{p, q\}$.

Notice: a formula follows from every classical model if it follows from the minimal model $\emptyset$.



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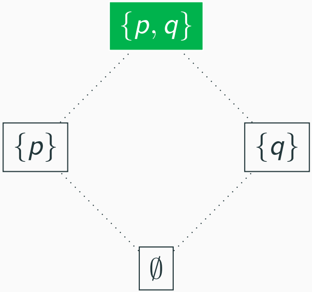


$T_{\mathcal{P}}$-operator

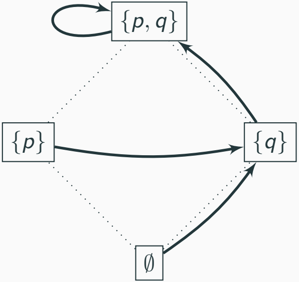
$$T_{\mathcal{P}} : \wp(\mathcal{A}_{\mathcal{P}}) \mapsto \wp(\mathcal{A}_{\mathcal{P}})$$

$$T_{\mathcal{P}}(x) = \{a \mid a \leftarrow b_1, \ldots, b_n \in \mathcal{P} \text{ and } b_1, \ldots, b_n \in x\}$$

Example

$$\mathcal{P} = \{p \leftarrow q., \quad q \leftarrow .\}$$

x	$\emptyset$	$\{p\}$	$\{q\}$	$\{p, q\}$
$T_{\mathcal{P}}(x)$	$\{q\}$	$\{q\}$	$\{p, q\}$	$\{p, q\}$



Models and Fixpoints

Definition
x is a post-fixpoint of $T_{\mathcal{P}}$ if $T_{\mathcal{P}}(x) \subseteq x$.

Intuition: everything I can derive from x using $\mathcal{P}$ is in x.

Models of $\mathcal{P}$.

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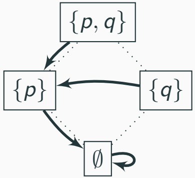

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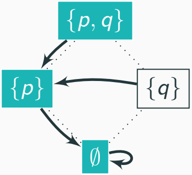

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x is a fixpoint of $T_{\mathcal{P}}$ if $T_{\mathcal{P}}(x) = x$.

Intuition: same, plus everything in x is derivable.

Supported models of $\mathcal{P}$.

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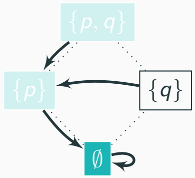

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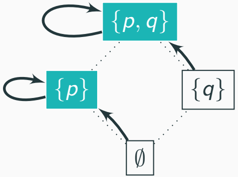

Models and Fixpoints

- For positive programs, $T_{\mathcal{P}}$ has a unique least fixpoint x.
- It is also the least pre-fixpoint.
- We can compute it by iterating $T_{\mathcal{P}}$ starting from $\emptyset$:
  $T_{\mathcal{P}}(\ldots T_{\mathcal{P}}(\emptyset)\ldots) = \bigcup_{i \geq 0} T^{i}_{\mathcal{P}}(\emptyset)$
  And this is possible in polynomial time.

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- We can compute it by iterating $T_{\mathcal{P}}$ starting from $\emptyset$:
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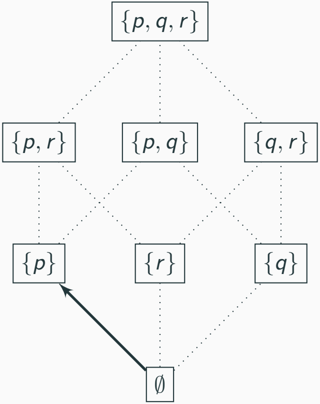
  And this is possible in polynomial time.
  Underlying result: A $\subseteq$-monotonic operator over a complete lattice admits a least fixpoint.

Least fixpoint computation

$$\mathcal{P} = \{p \leftarrow . \quad q \leftarrow p. \quad r \leftarrow p, q.\}$$



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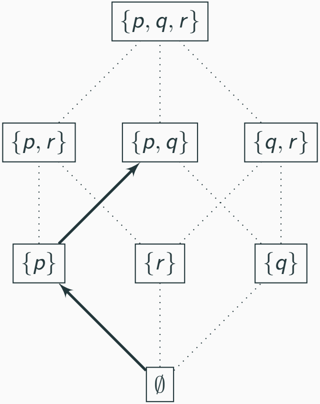

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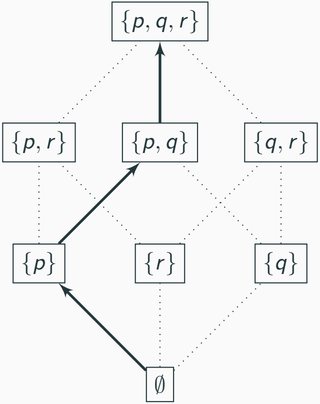

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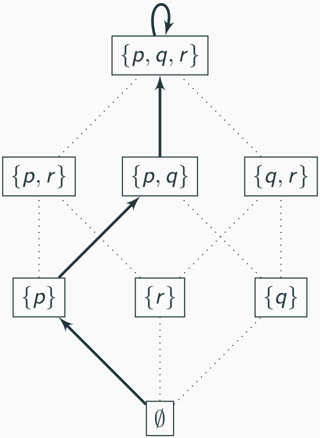
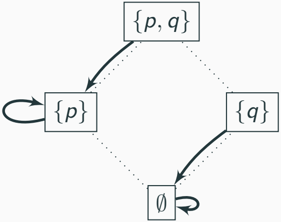


Least fixpoint $\neq$ unique fixpoint

Example ($\mathcal{P} = \{p \leftarrow p.\}$)



Semantics of Normal Logic Programs

Enters Negation

$$p \leftarrow \neg q$$

How to extend our operator?

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Easy:

$$\begin{aligned} T_{\mathcal{P}}(x) = \{a \mid \quad & a \leftarrow b_1, \ldots, b_n, \neg c_1, \ldots, \neg c_m \in \mathcal{P}, \text{ and} \\ & b_1, \ldots, b_n \in x, \text{ and } c_1, \ldots, c_m \notin x\} \end{aligned}$$

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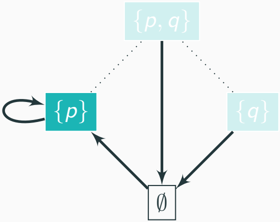

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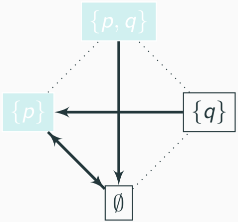
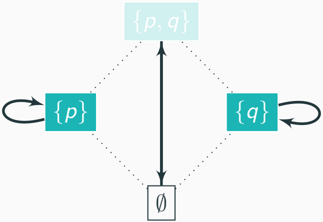
$\mathcal{P} = \{p \leftarrow \neg p\}$



No unique fixpoint

$\mathcal{P} = \{p \leftarrow \neg q; q \leftarrow \neg p\}$



Problems with negation

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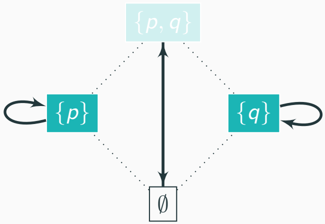

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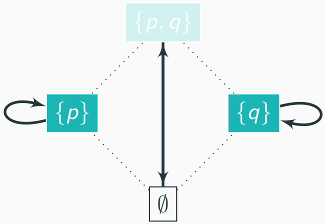

Solution: approximations of operators that are $\leq_i$-monotonic.

Commercial break

November 2-4, 2024, Hanoi, Vietnam
Co-located with KR 2024

- https://nmr.krportal.org/2024
- Submissions can be:
  - New material
  - Previously published papers
- Paper submission deadline: July 12th, 2024

Approximations

- Pairs of sets of atoms (x, y).
  - x contains all atoms that are definitely true.
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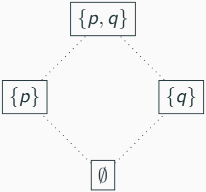
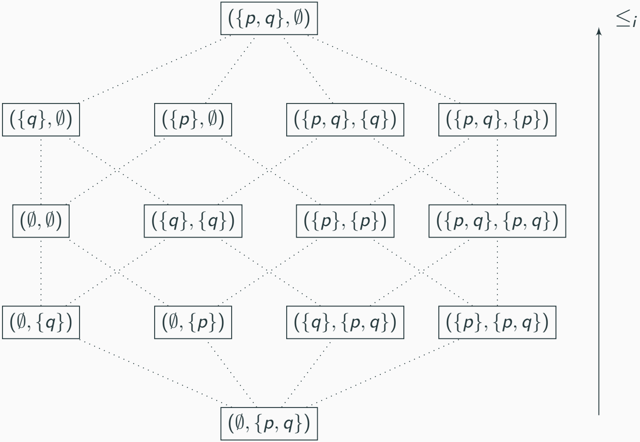
Example

$(\{p\}, \{p, q\})$: *p is true and q can be true.*

$(\{p\}, \{p\}) \leq_t (\{p\}, \{p, q\})$
$(\{p\}, \{p, q\}) \leq_i (\{p\}, \{p\})$

Graphical Depiction





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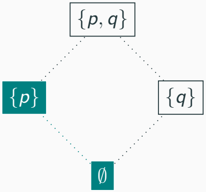

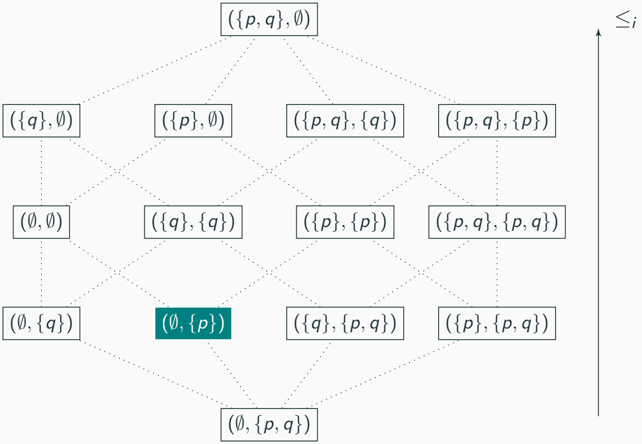

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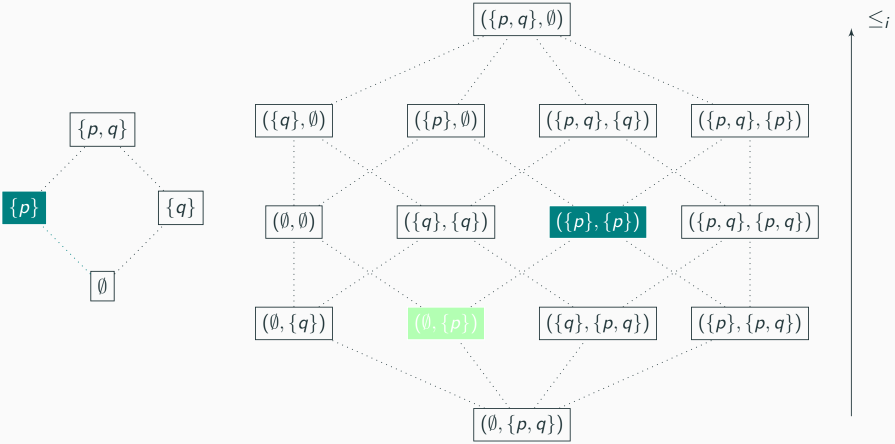

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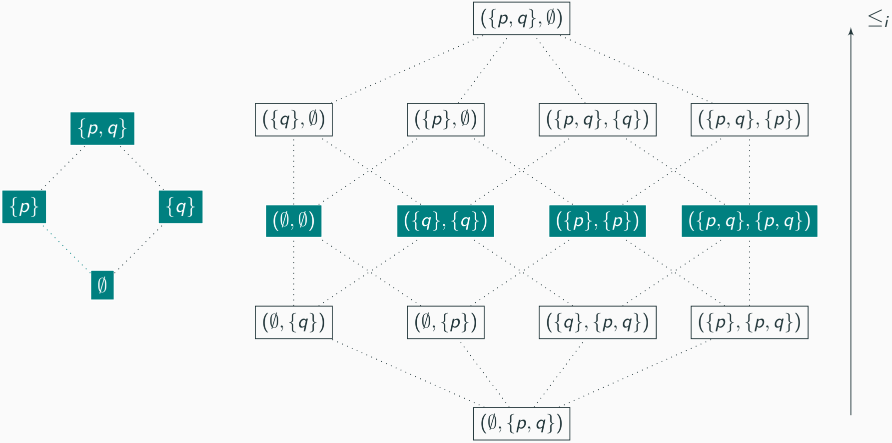
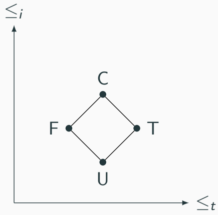


Approximations as Four-Valued Interpretations



- $-\mathsf{F} = \mathsf{T}$, $-\mathsf{T} = \mathsf{F}$, $-\mathsf{U} = \mathsf{U}$ and $-\mathsf{C} = \mathsf{C}$
- $(x, y)(p) = \begin{cases} \mathsf{T} & \text{if } p \in x \text{ and } p \in y, \\ \mathsf{U} & \text{if } p \notin x \text{ and } p \in y, \\ \mathsf{F} & \text{if } p \notin x \text{ and } p \notin y, \\ \mathsf{C} & \text{if } p \in x \text{ and } p \notin y. \end{cases}$
- $(x, y)(\neg\phi) = -(x, y)(\phi)$,
- $(x, y)(\psi \wedge \phi) = lub_{\leq_t}\{(x, y)(\phi), (x, y)(\psi)\}$,
- $(x, y)(\psi \vee \phi) = glb_{\leq_t}\{(x, y)(\phi), (x, y)(\psi)\}$.

Example

$(\{p\}, \{p, q\})(p) = \mathsf{T} \quad (\{p\}, \{p, q\})(q) = \mathsf{U} \quad (\{p\}, \{p, q\})(r) = \mathsf{F}$.

$(\{p\}, \{p, q\})(\neg p) = \mathsf{F} \quad (\{p\}, \{p, q\})(\neg q) = \mathsf{U}$

$(\{p\}, \{p, q\})(p \wedge q) = \mathsf{U} \quad (\{p\}, \{p, q\})(q \vee r) = \mathsf{U}$

Approximating $T_{\mathcal{P}}$ (from below)

$$\mathcal{IC}_{\mathcal{P}} : \mathcal{A} \times \mathcal{A} \mapsto \mathcal{A} \times \mathcal{A}$$

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We input an approximation and output an approximation.

$$\mathcal{IC}^{l}_{\mathcal{P}}(x, y) = \{a \in \mathcal{A} \mid a \leftarrow b_1, \ldots, b_n, \neg c_1, \ldots, \neg c_m \in \mathcal{P},$$
$$b_1, \ldots, b_n \in x \text{ and } c_1, \ldots, c_m \notin y\}$$

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Example ($\{p \leftarrow p, \neg q\}$)
$\mathcal{IC}^{l}_{\mathcal{P}}(\{p\}, \{p, q\}) = \emptyset$
$\mathcal{IC}^{l}_{\mathcal{P}}(\{p\}, \{p\}) = \{p\}$

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$$\mathcal{IC}^{u}_{\mathcal{P}}(x, y) = \{a \in \mathcal{A} \mid a \leftarrow b_1, \ldots, b_n, \neg c_1, \ldots, \neg c_m \in \mathcal{P}, \\ b_1, \ldots, b_n \in y \text{ and } c_1, \ldots, c_m \notin x\}$$

Approximating $T_{\mathcal{P}}$ (from above)

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Example ($\{p \leftarrow p, \neg q\}$)
$\mathcal{IC}^u_{\mathcal{P}}(\{p\}, \{p, q\}) = \{p\}$
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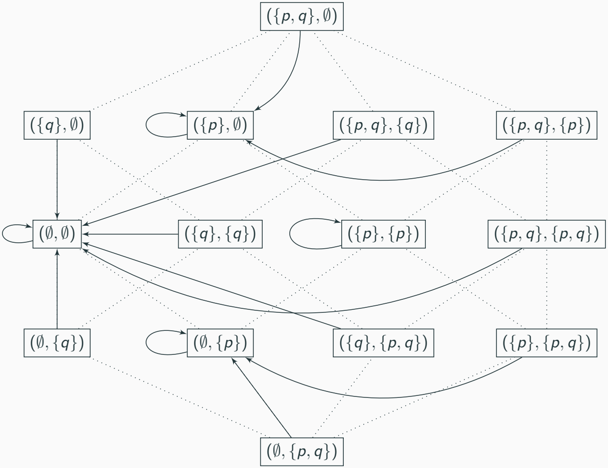

Properties of $\mathcal{IC}_{\mathcal{P}}$

- $\mathcal{IC}_{\mathcal{P}}$ approximates $T_{\mathcal{P}}$:

$\mathcal{IC}_{\mathcal{P}}(x, x) = (T_{\mathcal{P}}(x), T_{\mathcal{P}}(x))$ for any $x \subseteq \mathcal{A}$.

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We say $\mathcal{IC}_{\mathcal{P}}$ is an approximation operator. It is also symmetric, in the sense that $\mathcal{IC}_{\mathcal{P}}(x,y) = (\mathcal{IC}^l_{\mathcal{P}}(x,y), \mathcal{IC}^l_{\mathcal{P}}(y,x))$.

Kripke-Kleene Fixpoint

The $\leq_i$-monotonicity is our *indulgentia* back into Tarski's heaven:

Proposition
$\mathcal{IC}_\mathcal{P}$ *has a least fixpoint, obtainable as* $\bigsqcup_{i\geq 0} \mathcal{IC}^i_\mathcal{P}(\emptyset, \mathcal{A})$[3].

[3] $(x_1, y_1) \sqcup (x_2, y_2) = (x_1 \cup x_2, y_1 \cap y_2)$.

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If $(x, y) = \mathcal{IC}_{\mathcal{P}}(x, y)$ then we call it a partial supported model.

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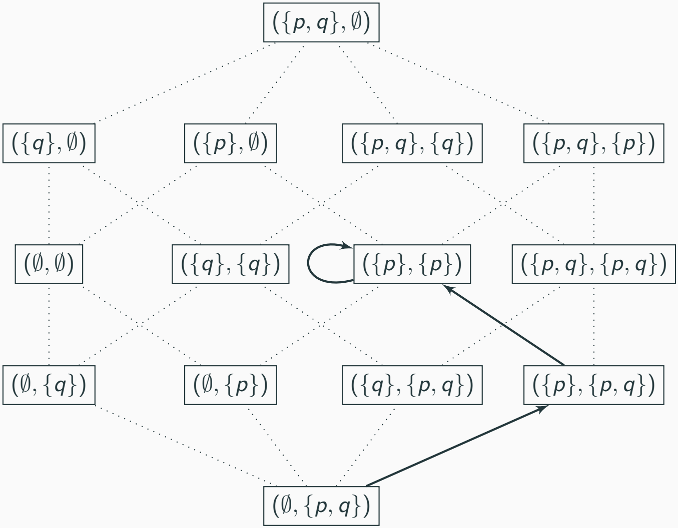
Example: $\mathcal{P} = \{p \leftarrow; q \leftarrow \neg p\}$



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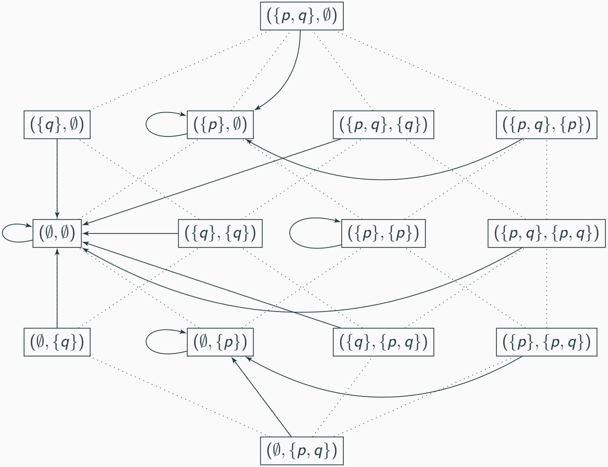

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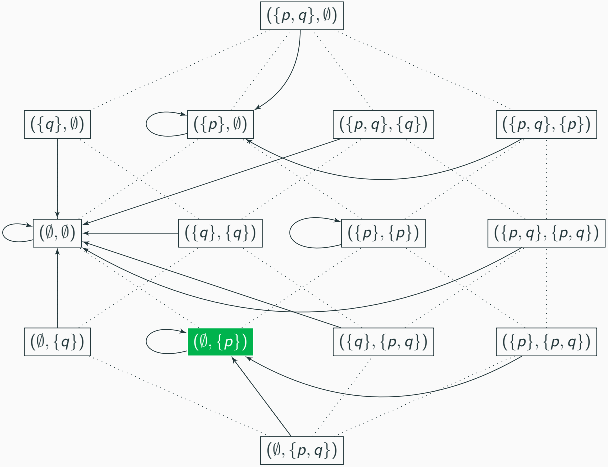

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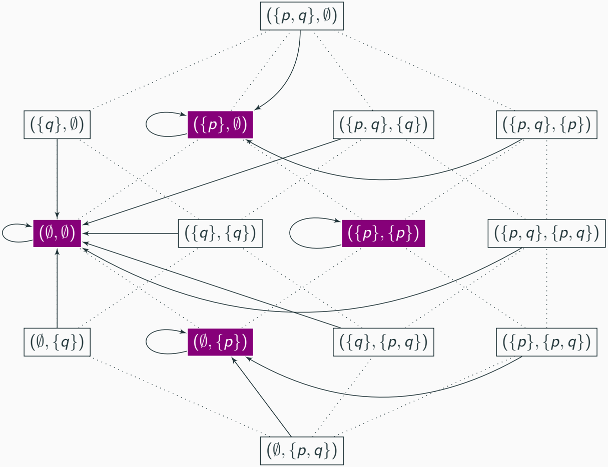
Partial Supported models



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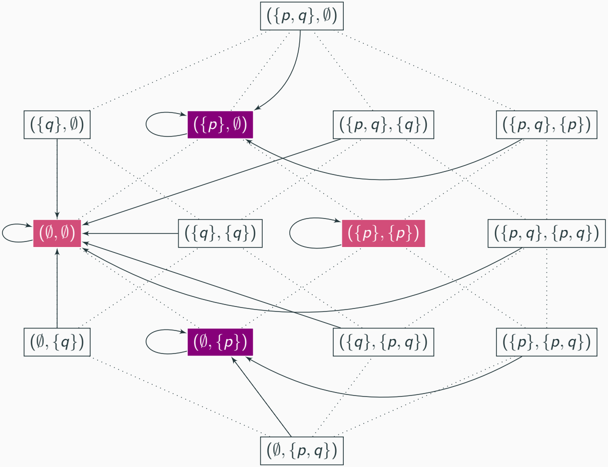
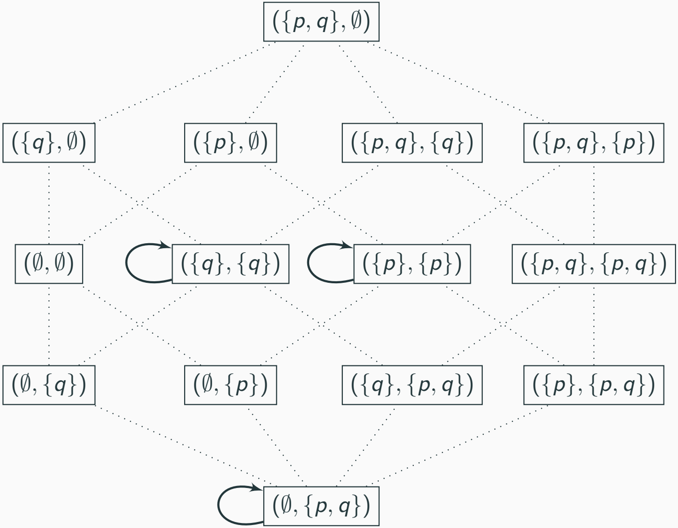


Example: $\mathcal{P} = \{p \leftarrow \neg q;\, q \leftarrow \neg p\}$



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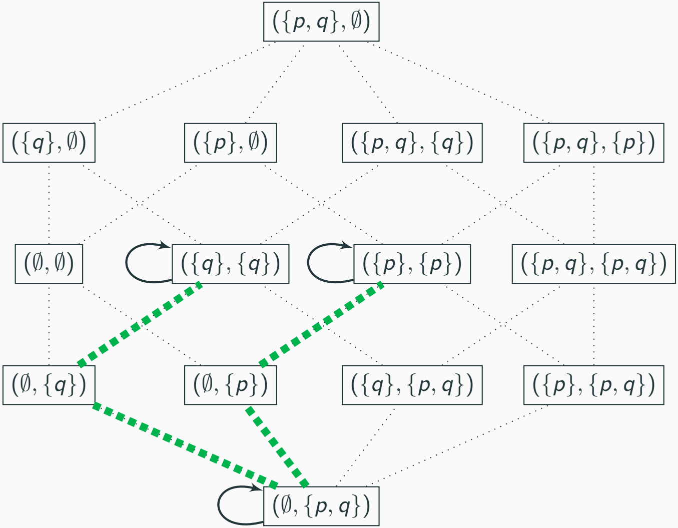
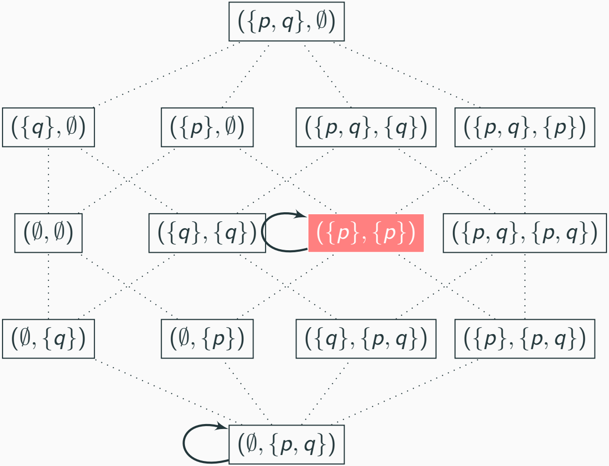


Stable Semantics

Example: Kripke-Kleene is rather weak

$$p \leftarrow \neg q;\ q \leftarrow q$$



Example: Kripke-Kleene is rather weak

$$\mathcal{P} = \{p \leftarrow \neg q;\ q \leftarrow q\}$$

Construction of the Kripke-Kleene fixpoint:

- $\mathcal{IC}_{\mathcal{P}}(\emptyset, \{p, q\}) = (\emptyset, \{p, q\})$.
- Fixpoint reached.

Can't get rid of the self-supporting atom q in the upper bound.

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Assuming that no atom is certainly true, construct the smallest upper bound possible:

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As $\mathcal{IC}^{u}_{\mathcal{P}}(\emptyset, \cdot)$ is a $\subseteq$-monotonic operator, it admits a least fixed point.

Stable Operator

$$S(\mathcal{IC}^{l}_{\mathcal{P}})(y) = lfp(\mathcal{IC}^{l}_{\mathcal{P}}(\cdot, y))$$

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Example ($\{p \leftarrow \neg q;\ q \leftarrow q\}$)

$S(\mathcal{IC}^{l}_{\mathcal{P}})(\{p, q\}) = \emptyset$ *since:*

$\mathcal{IC}^{l}_{\mathcal{P}}(\emptyset, \{p, q\}) = \emptyset$: *fixpoint reached.*

Stable Operator

$$S(\mathcal{IC}^l_\mathcal{P})(y) = lfp(\mathcal{IC}^l_\mathcal{P}(\cdot, y))$$

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Stable Operator: Example $\{p \leftarrow \neg q; q \leftarrow q\}$

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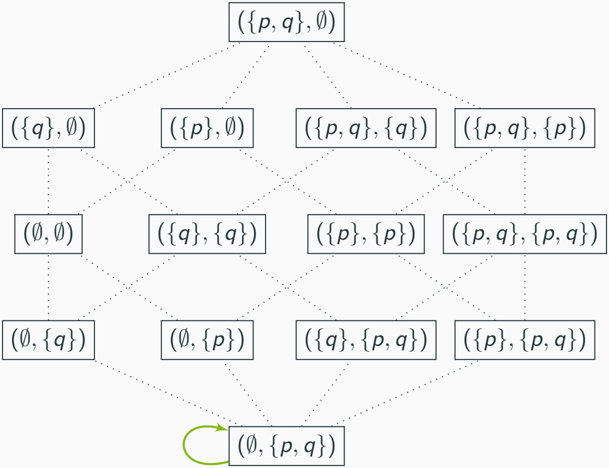

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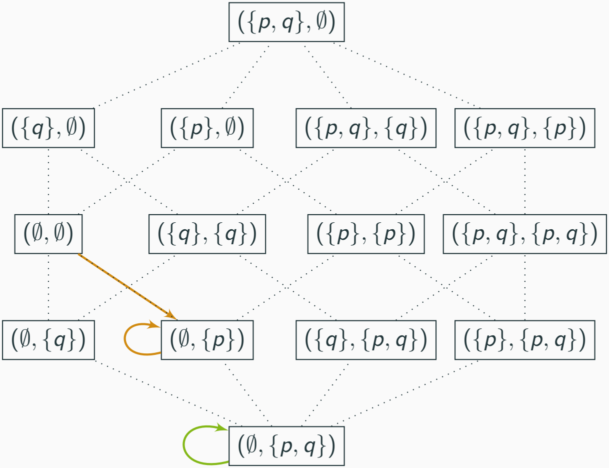

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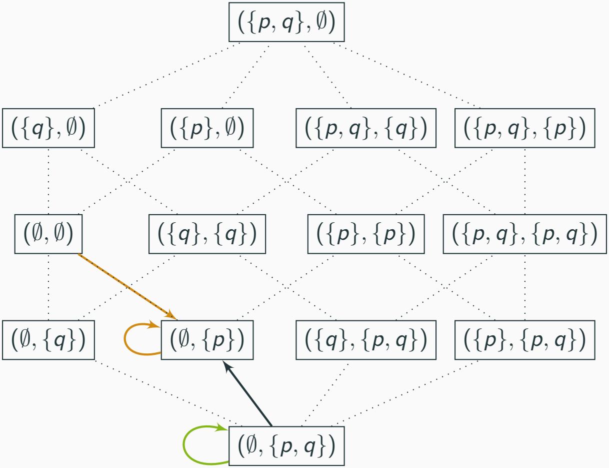

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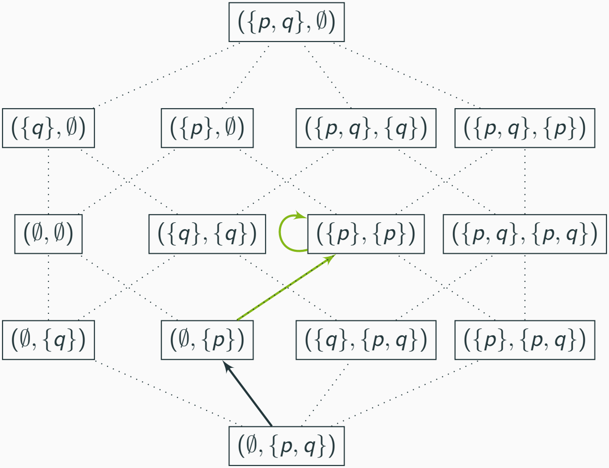

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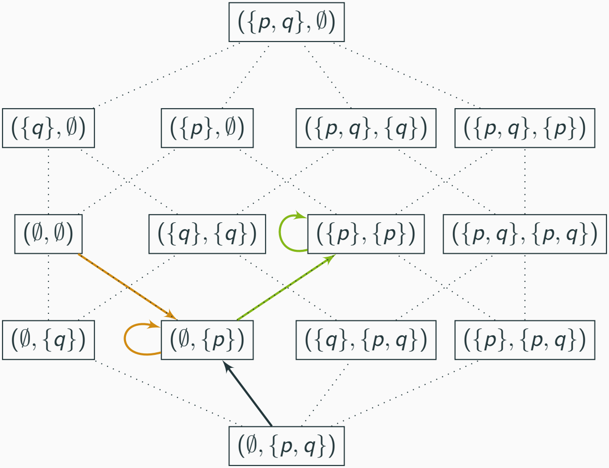
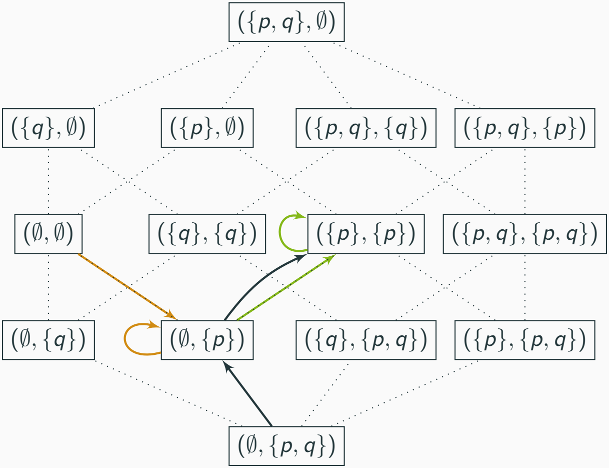


Stable Operator: Example $\{p \leftarrow \neg q; q \leftarrow q\}$

$$S(\mathcal{IC_P})(\emptyset, \{p\}) = (S(\mathcal{IC}^l_\mathcal{P})(\{p\}),\ S(\mathcal{IC}^u_\mathcal{P})(\emptyset))$$



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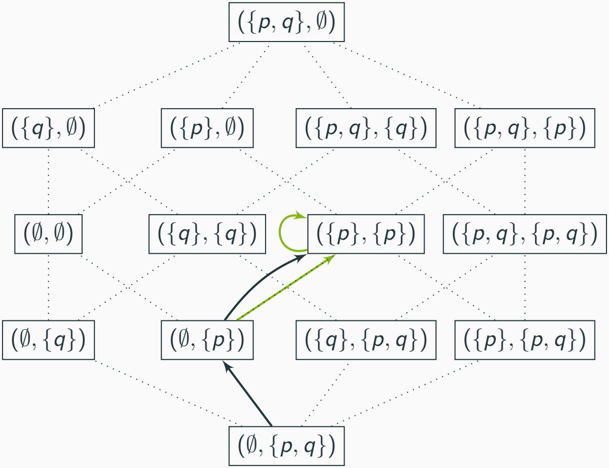
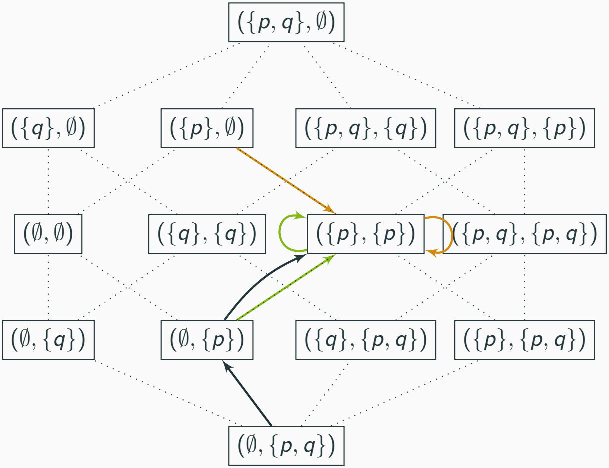


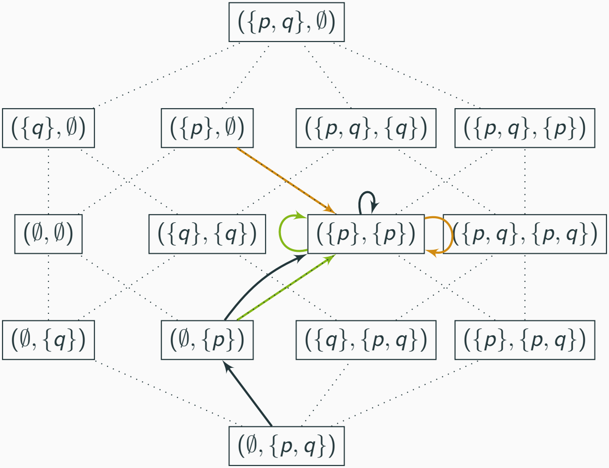
Stable Operator: Example $\{p \leftarrow \neg q; q \leftarrow q\}$

$S(\mathcal{IC}^u_\mathcal{P})(x) = lfp(\mathcal{IC}^u_\mathcal{P}(\{p\}, \cdot))$



Stable Operator: Example $\{p \leftarrow \neg q; q \leftarrow q\}$

$$S(\mathcal{IC}_{\mathcal{P}})(\{p\},\{p\}) = (S(\mathcal{IC}^{l}_{\mathcal{P}})(\{p\}),\ S(\mathcal{IC}^{u}_{\mathcal{P}})(\{p\}))$$



Stable Operator and Well-Founded Model

$$S(\mathcal{IC}^l_{\mathcal{P}})(y) = \mathit{lfp}(\mathcal{IC}^l_{\mathcal{P}}(\cdot, y)) \qquad S(\mathcal{IC}^u_{\mathcal{P}})(x) = \mathit{lfp}(\mathcal{IC}^u_{\mathcal{P}}(x, \cdot))$$

$$S(\mathcal{IC}_{\mathcal{P}})(x, y) = (S(\mathcal{IC}^l_{\mathcal{P}})(y),\ S(\mathcal{IC}^u_{\mathcal{P}})(x))$$

- $S(\mathcal{IC}_{\mathcal{P}})$ is a $\leq_i$-monotonic operator, so it admits a least fixpoint.

We call this the well-founded model, denoted $\mathrm{WF}(\mathcal{P})$.

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• The well-founded model is more precise than the Kripke-Kleene fixpoint: $\mathrm{KK}(\mathcal{P}) \leq_i \mathrm{WF}(\mathcal{P})$.

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• Any fixpoint of $S(\mathcal{IC}_{\mathcal{P}})$ is a minimal model of $\mathcal{P}$.
If $(x, y) = S(\mathcal{IC}_{\mathcal{P}})(x, y)$, we call it a (partial) stable model.
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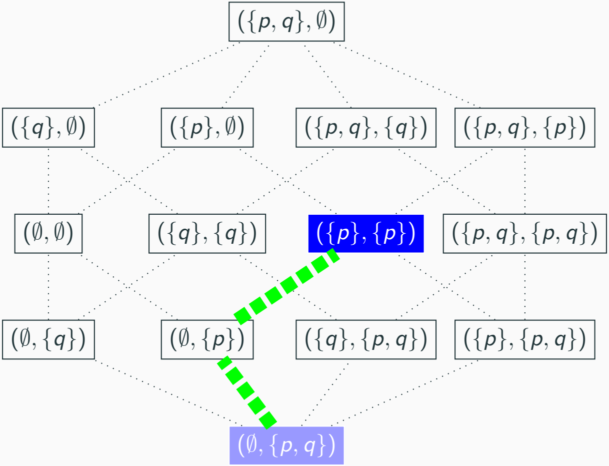
- If $T_{\mathcal{P}}$ has a least fixpoint, it coincides with the well-founded model.

Stable Operator: Example

$p \leftarrow \neg q; q \leftarrow q$



Stable Operator: Example 2

$$\mathcal{P} = \{p \leftarrow \neg q; \quad q \leftarrow \neg p; \quad r \leftarrow r; \quad s \leftarrow \neg r\}$$

- Kripke-Kleene fixpoint: $(\emptyset, \{p, q, r, s\})$.
- Well-founded model: $(\{s\}, \{p, q, s\})$.
- Stable models: $(\{p, s\}, \{p, s\})$, $(\{q, s\}, \{q, s\})$.

Stable Semantics and Reducts

$$\frac{\mathcal{P}}{x} = \{a \leftarrow b_1, \ldots, b_n \mid a \leftarrow b_1, \ldots, b_n, \neg c_1, \ldots, \neg c_m \in \mathcal{P}$$

$$c_1, \ldots, c_n \notin x\}$$

Definition

x is a stable model of $\mathcal{P}$ if it is a minimal model of $\frac{\mathcal{P}}{x}$.

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Definition

x is a stable model of $\mathcal{P}$ if it is a minimal model of $\frac{\mathcal{P}}{x}$.

Example ($\mathcal{P} = \{p \leftarrow \neg p; q \leftarrow \neg p; p \leftarrow \neg q\}$)

$\frac{\mathcal{P}}{\{q\}} = \{p \leftarrow; q \leftarrow\}$. $\{q\}$ is not a minimal model of $\mathcal{P}$, thus $\{q\}$ is not a stable model.

$\frac{\mathcal{P}}{\{p\}} = \{p \leftarrow\}$. $\{p\}$ is a minimal model of $\mathcal{P}$.$\{q\}$ is not a minimal model of $\mathcal{P}$, thus $\{p\}$ is a stable model.

Stable Semantics and Reducts

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$$c_1, \ldots, c_n \notin x\}$$

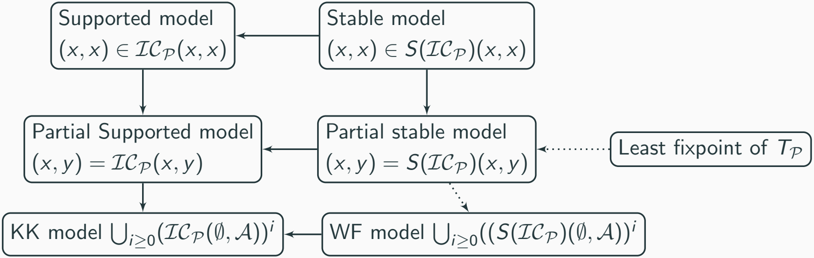
Definition
x is a stable model of $\mathcal{P}$ if it is a minimal model of $\frac{\mathcal{P}}{x}$.

Proposition
$S(\mathcal{IC}^l_{\mathcal{P}})(y)$ is the set of minimal models of $\frac{\mathcal{P}}{y}$.

Proposition
$(x, x) = S(\mathcal{IC}_{\mathcal{P}})(x, x)$ if and only if x is a stable model of $\mathcal{P}$ (iff $x = S(\mathcal{IC}^l_{\mathcal{P}})(x)$).

Approximation Fixpoint Theory

Recap



- Operator-based framework
    - Non-monotonic operator $T_{\mathcal{P}}$,
    - a $\leq_i$-monotonic approximation operator $\mathcal{IC}_{\mathcal{P}}$,
    - and its stable variant $S(\mathcal{IC}_{\mathcal{P}})$.
- Allows us to define semantics as fixpoints of these operators, with attractive properties:
    - KK and WF models exist, can be constructively found, and
    - approximate any fixpoint of $T_{\mathcal{P}}$.
- This story can be told for a great number of formalisms.

Lattices, bilattices, operators

Given a lattice $L = \langle \mathcal{L}, \leq \rangle$.

Interested in operator $O_{\mathcal{L}} : \mathcal{L} \to \mathcal{L}$ and its fixpoints.

- $(x_1, y_1) \leq_i (x_2, y_2)$ iff $x_1 \leq x_2$ and $y_1 \geq y_2$,
- $(x_1, y_1) \leq_t (x_2, y_2)$ iff $x_1 \leq x_2$ and $y_1 \leq y_2$.

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$\langle \mathcal{L}^2, \leq_i, \leq_t \rangle$ is called a bilattice. Approximate $O_{\mathcal{L}}$ with an approximation operator $\mathcal{O} : \mathcal{L}^2 \to \mathcal{L}^2$, which is $\leq_i$-monotonic and for which $\mathcal{O}(x, x) = (O_{\mathcal{L}}(x), O_{\mathcal{L}}(x))$ for any $x \in \mathcal{L}$.

Lattices, bilattices, operators

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Formalism	Lattice Elements	Order
Logic Programming	Possible worlds	$\subseteq$
Default Logic and AEL	Sets of possible worlds	$\supseteq$
Formal Argumentation	Sets of arguments	$\subseteq$
Weighted ADFs	Weighted worlds	Pointwise comparison
SHACL	Interpretations	Truth order

Operator-Based Semantics for Dialects of Logic Programming

- $\checkmark$ Aggregates in the body: $p \leftarrow \#\mathtt{sum}\{2 : p; q : 1; r : 1\} \geq 2$.
- $\checkmark$ Propositional formulas in the body: $p \leftarrow q \wedge (r \vee (s \wedge \neg t))$.
- $\checkmark$ Disjunctions in the head: $p \vee q \leftarrow q \wedge (r \vee (s \wedge \neg t))$.
- $\checkmark$ Choice constructs in the head: $\#\mathtt{count}\{p; q; r\} = 2 \leftarrow \neg r$.
- $\checkmark$ DL-based logic programs: $\mathsf{K}\, C(x) \leftarrow \neg p(X); C \sqsubseteq D$.
- $\checkmark$ Higher-order logic programs: $S(P, Q) \leftarrow; P(X) \leftarrow \neg Q(X)$.
- ? Fuzzy logic programs: $p(X) \leftarrow 0.5 \cdot (q(x) + r(X))$.
- ? Probabilistic logic programs: $0.3 :: p(X)$.
- ? Hex-programs: $tr(S, P, O) \leftarrow \& RDF[uri](S, P, O)$.

Operator-Based Semantics for other KR-formalisms

- autoepistemic logic [DMT03],
- default logic [DMT03],
- abstract argumentation [SW15],
- abstract dialectical frameworks [SW15],
- weighted abstract dialectical frameworks [Bog19],
- SCHACL [BJ21].

Operator-Based Studies

Top-Down approach:

- Instead of studying a concept for a specific framework, define and study it for operators over a lattice (and their approximations).
- We can then apply this concept to all formalisms that are or can be captured in AFT.

Examples:

- $\vee$ Stratification [VGD06]
- $\vee$ Conditional Independence [Hey23]
- $\vee$ Knowledge Compilation [BVdB15]
- $\vee$ Groundedness [BVdB15]
- $\vee$ Strong equivalence [Tru06]
- $\vee$ Argumentative dialogues [HA20]
- ? Belief dynamics
- ? Modular equivalence
- ? Neuro-symbolism

Operator-Based Studies: Modularity

Motivating Example

$$r: \ \mathtt{inf}(\mathtt{X}) \leftarrow \mathtt{inf}(\mathtt{Y}), \mathtt{cnct}(\mathtt{Y}, \mathtt{X}), \mathrm{not}\ \mathtt{vac}(\mathtt{X}).$$

Motivating Example

$$r_1: \ \texttt{inf}(\texttt{b}) \leftarrow \texttt{inf}(\texttt{a}), \texttt{cnct}(\texttt{a}, \texttt{b}), \mathrm{not}\ \texttt{vac}(\texttt{b}).$$

$$r_2: \ \texttt{inf}(\texttt{c}) \leftarrow \texttt{inf}(\texttt{d}), \texttt{cnct}(\texttt{d}, \texttt{c}), \mathrm{not}\ \texttt{vac}(\texttt{c}).$$

$$r_3: \ \texttt{inf}(\texttt{a}).,\ r_4: \ \texttt{cnct}(\texttt{a}, \texttt{b}).,\ r_5: \ \texttt{cnct}(\texttt{d}, \texttt{c}).$$

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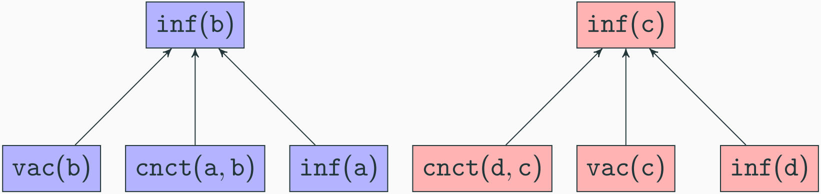

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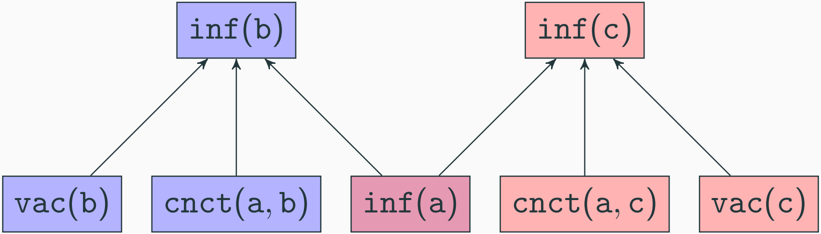

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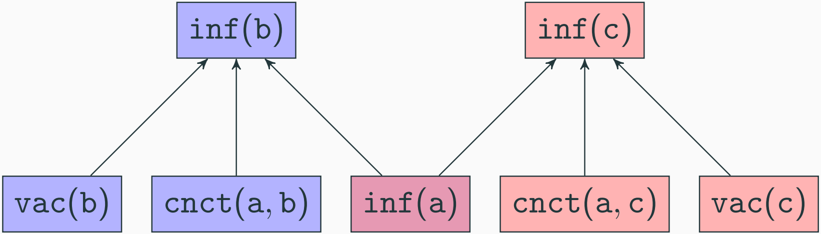

Once we know `inf(a)` (or $\neg$`inf(a)`), we obtain two independent subprograms.

Conditional Independence w.r.t. an operator

Definition
Let $O : L_1 \otimes L_2 \otimes L_3 \to L_1 \otimes L_2 \otimes L_3$ *be given.*
$L_1 \perp\!\!\!\perp_O L_2 \mid L_3$ *if there exist operators*

$$O_{1,3} : L_1 \otimes L_3 \to L_1 \otimes L_3 \text{ and } O_{2,3} : L_2 \otimes L_3 \to L_2 \otimes L_3$$

s.t. for $i, j \in \{1, 2\}$, $i \neq j$, *and for every* $x_i \otimes x_3 \in L_i \otimes L_3$ *and for every* $x_j \in L_j$ *it holds that:*

$$O(x_i \otimes x_j \otimes x_3)_{|i,3} = O_{i,3}(x_i \otimes x_3).$$

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$$O(x_i \otimes x_j \otimes x_3)_{|i,3} = O_{i,3}(x_i \otimes x_3).$$

where $x_i \otimes x_j \otimes x_{3|i,3} = x_i \otimes x_3$

Search for Fixpoints can be Split

Proposition

Let an operator O s.t. $L_1 \perp\!\!\!\perp_O L_2 \mid L_3$ be given.
Then $x_1 \otimes x_2 \otimes x_3 = O(x_1 \otimes x_2 \otimes x_3)$ iff
$x_1 \otimes x_3 = O_{1,3}(x_1 \otimes x_3)$ and $x_2 \otimes x_3 = O_{2,3}(x_2 \otimes x_3)$.

Search for Fixpoints can be Split

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Example

$r_1: \ \mathtt{inf}(\mathtt{b}) \leftarrow \mathtt{inf}(\mathtt{a}), \mathtt{cnct}(\mathtt{a}, \mathtt{b}), \mathrm{not}\ \mathtt{vac}(\mathtt{b}).$

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$r_3: \ \mathtt{inf}(\mathtt{a})., \ r_4: \ \mathtt{cnct}(\mathtt{a}, \mathtt{b})., \ r_5: \ \mathtt{cnct}(\mathtt{a}, \mathtt{c}).$

We can look for supported models of $\mathcal{P}$ by looking for supported models of $\mathcal{P}_1$ and $\mathcal{P}_2$ and combining them afterwards.

Reasoning using Modularity

$r_1: \ \mathtt{inf}(\mathrm{b}) \leftarrow \mathtt{inf}(\mathrm{a}), \mathtt{cnct}(\mathrm{a}, \mathrm{b}), \mathrm{not}\ \mathtt{vac}(\mathrm{b}).$

$r_2: \ \mathtt{inf}(\mathrm{c}) \leftarrow \mathtt{inf}(\mathrm{a}), \mathtt{cnct}(\mathrm{a}, \mathrm{c}), \mathrm{not}\ \mathtt{vac}(\mathrm{c}).$

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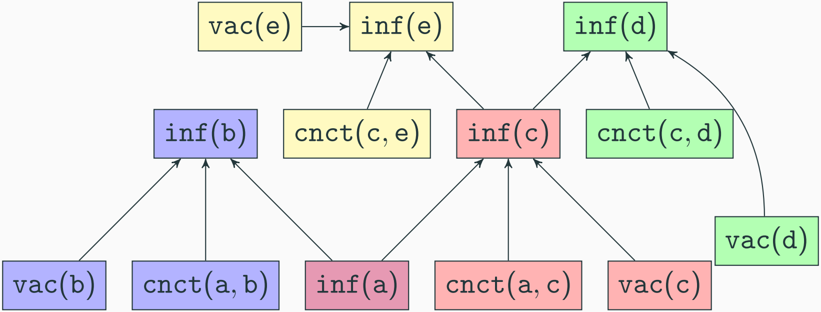
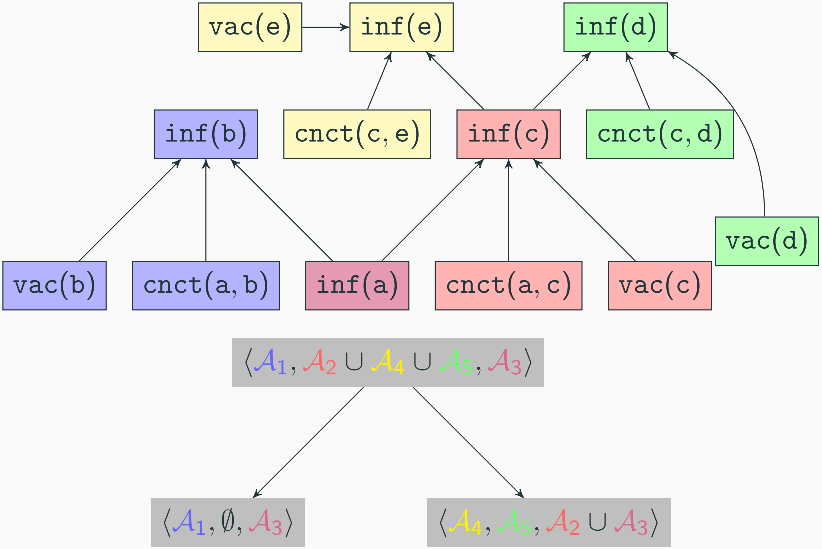
$r_6: \ \mathtt{inf}(\mathrm{d}) \leftarrow \mathtt{inf}(\mathrm{c}), \mathtt{cnct}(\mathrm{c}, \mathrm{d}), \mathrm{not}\ \mathtt{vac}(\mathrm{d}).$

$r_7: \ \mathtt{inf}(\mathrm{e}) \leftarrow \mathtt{inf}(\mathrm{c}), \mathtt{cnct}(\mathrm{c}, \mathrm{e}), \mathrm{not}\ \mathtt{vac}(\mathrm{e}).$

$r_8: \ \mathtt{cnct}(\mathrm{c}, \mathrm{d}).,\ r_9: \ \mathtt{cnct}(\mathrm{c}, \mathrm{e}).$



Reasoning using Modularity



Reasoning using Modularity

$$\begin{tikzcd}[column sep=small]
& \langle \mathcal{A}_1, \mathcal{A}_2 \cup \mathcal{A}_4 \cup \mathcal{A}_5, \mathcal{A}_3 \rangle \arrow[dl] \arrow[dr] & \\
\langle \mathcal{A}_1, \emptyset, \mathcal{A}_3 \rangle & & \langle \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_2 \cup \mathcal{A}_3 \rangle
\end{tikzcd}$$

- $\mathrm{WF}(\mathcal{P}_1 \cup \mathcal{P}_3) = \{\mathtt{inf}(\mathrm{a}), \mathtt{cnct}(\mathrm{a}, \mathrm{b}), \mathtt{inf}(\mathrm{b})\}$
  Search space size: 2^4
- $\mathrm{WF}(\mathcal{P}_4 \cup \mathcal{P}_2 \cup \mathcal{P}_3) = \{\mathtt{inf}(\mathrm{a}), \mathtt{cnct}(\mathrm{a}, \mathrm{c}), \mathtt{inf}(\mathrm{c}), \mathtt{cnct}(\mathrm{c}, \mathrm{e}), \mathtt{inf}(\mathrm{e})\}$
  Search space size: 2^7
- $\mathrm{WF}(\mathcal{P}_5 \cup \mathcal{P}_2 \cup \mathcal{P}_3) = \{\mathtt{inf}(\mathrm{a}), \mathtt{cnct}(\mathrm{a}, \mathrm{c}), \mathtt{inf}(\mathrm{c}), \mathtt{cnct}(\mathrm{c}, \mathrm{d}), \mathtt{inf}(\mathrm{d})\}$
  Search space size: 2^7

Reasoning using Modularity

$$
\begin{tikzcd}[column sep=small]
& \langle \mathcal{A}_1, \mathcal{A}_2 \cup \mathcal{A}_4 \cup \mathcal{A}_5, \mathcal{A}_3 \rangle \arrow[dl] \arrow[dr] & \\
\langle \mathcal{A}_1, \emptyset, \mathcal{A}_3 \rangle & & \langle \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_2 \cup \mathcal{A}_3 \rangle
\end{tikzcd}
$$

- $\mathsf{WF}(\mathcal{P}_1 \cup \mathcal{P}_3) = \{\mathtt{inf}(\mathrm{a}), \mathtt{cnct}(\mathrm{a}, \mathrm{b}), \mathtt{inf}(\mathrm{b})\}$
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  Search space size: 2^7
- $\mathsf{WF}(\mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3 \cup \mathcal{P}_4 \cup \mathcal{P}_5) =$
  $\mathsf{WF}(\mathcal{P}_1 \cup \mathcal{P}_3) \cup \mathsf{WF}(\mathcal{P}_4 \cup \mathcal{P}_2 \cup \mathcal{P}_3) \cup \mathsf{WF}(\mathcal{P}_5 \cup \mathcal{P}_2 \cup \mathcal{P}_3)$.
  Original search space size: 2^{14}

Fixed-Parameter Complexity

Definition

Let an operator O over the power set lattice $\bigotimes_{i\in I}\mathcal{L}_i$ and CIT $T=(V,E,\nu)$ be given s.t. V_l are the leafs of T. The CIT-partition-size of O relative to (V,E,ν) is defined as

$$\max(\{|\bigotimes_{i\in I_j}\mathcal{L}_i \otimes \bigotimes_{i\in I_3}\mathcal{L}_i| \mid v\in V_\mathrm{l}, \nu(v)=\langle I_1,I_2,I_3\rangle, j=1,2\})$$

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$$\max(\{|\bigotimes_{i\in I_j}\mathcal{L}_i \otimes \bigotimes_{i\in I_3}\mathcal{L}_i| \mid v\in V_\text{l}, \nu(v)=\langle I_1, I_2, I_3\rangle, j=1,2\})$$

Proposition
Let a $\leq_\otimes$-monotonic operator O over the product lattice $\bigotimes_{i\in I}\mathcal{L}_i$ and CIT $T=(V,E,\nu)$ with CIT-partition-size s be given. Assume that $O(x,y)$ can be computed to a call to an NP-oracle. *The least fixpoint of O can be computed in time $O(f(s))$.*

Round up

Second Commercial break

Tutorial on Approximation Fixpoint Theory

Jesse Heyninck and Hannes Straß

part of the KR 2024 tutorial program

- https://jesseheyninck.github.io/AFTtutorial/
- Half day workshop
  - Motivation and history: a guided tour through the history of logic programming
  - An abstract view: from logic programming to (general) AFT
  - A second application: (weighted) abstract dialectical frameworks
  - More general insights: complexity, groundedness, modularity

Summary

- Operators as the core for understanding answer set semantics.
- Paved the road towards approximation fixpoint theory.
- Algebraic theory that allows language independent work on KR.
- Requires some buy-in, but in my view a great bargain.
- Interested in cooperating? Questions on AFT? Come talk to me.

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