he Role of Syntax in Inductive Inference: A Property-Based Study

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Logic is (only) concerned with LOGICAL STRUCTURE

- Does this also hold for non-monotonic (conditional) logic?
- And what does this even mean?

Birds and Flying: Syntax Splitting

Let $\Delta = \Delta_{\text{birds}} \cup \Delta_{\text{geography}}$ with:

$$\begin{aligned} \Delta_{\text{birds}}: &\quad (\text{birds}|\text{penguins}), (\text{fly}|\text{birds}), (\neg\text{fly}|\text{penguins}) \\ \Delta_{\text{geography}}: &\quad (\text{polar}|\text{antarctic}), (\text{africa}|\text{westernCape}) \end{aligned}$$

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$$\mathrm{penguins} \wedge \mathrm{westernCape} \quad \mid\!\sim_{\Delta} \quad \neg\mathrm{fly}$$

just as well as

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Birds and Mamals: Language Independence

$$\Delta_{\mathsf{birds}}: \quad (\text{birds}|\text{penguins}), (\text{fly}|\text{birds}), (\neg\text{fly}|\text{penguins})$$

$$\Delta_{\mathsf{bats}}: \quad (\text{mamals}|\text{bats}), (\neg\text{fly}|\text{mamals}), (\neg\neg\text{fly}|\text{bats})$$

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Structure

Preliminaries

Conditionals

Background on Propositional Logic and Conditionals

$\mathcal{L}$ constructed on the basis of Σ and $\wedge$, $\vee$, $\neg$ and $\rightarrow$.

Possible worlds $\omega \in \Omega(\Sigma)$. $\mathrm{Mod}(A)$ consists of the models of ϕ.

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Conditionals are constructed on the basis of $\mathcal{L}$ as follows
$(\mathcal{L}|\mathcal{L}) = \{(B|A) \mid A, B \in \mathcal{L}\}$.

$$((B|A))(\omega) = \begin{cases} 1 & \omega \models A \wedge B \\ 0 & \omega \models A \wedge \neg B \\ u & \omega \models \neg A \end{cases}$$

Inductive Inference Operators

Definition ([KIBB20])
An inductive inference operator (from conditional belief bases) is a mapping $\mathbf{C} : 2^{(\mathcal{L}|\mathcal{L})} \mapsto 2^{\mathcal{L}^2}$ (or, more readable: $\Delta \to {\mid\!\sim}_{\Delta}$) that satisfies:

DI $(B|A) \in \Delta$ implies $A \mid\!\sim_{\Delta} B$.

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Examples of inductive inference operators are system P, system Z (aka rational closure) and lexicographic closure.

Total Preorders [KLM90]

Given a total preorder (in short, TPO) $\preceq$ on possible worlds:

$A \preceq B$ iff $\omega \preceq \omega'$ for an $\omega \in \min_{\preceq}(\mathrm{Mod}(A))$ and an $\omega' \in \min_{\preceq}(\mathrm{Mod}(B))$.

$$A \mid\!\sim_{\preceq} B \text{ iff } (A \wedge B) \prec (A \wedge \neg B).$$

Example

$$\overline{p}bf, \quad \overline{p}\overline{b}f, \quad \overline{p}\overline{b}\,\overline{f} \quad \prec \quad pb\overline{f}, \quad \overline{p}b\overline{f} \quad \prec \ldots$$

$$\begin{array}{ccc} \top & \mid\!\sim_{\preceq} & \neg p \\ p & \mid\!\sim_{\preceq} & b \end{array}$$

System Z

Z-ranking of conditionals [GP96]

A conditional $(B|A)$ is *tolerated* by a finite set of conditionals Δ if there is a possible world ω with:

1. $(B|A)(\omega) = 1$, and
2. $(B'|A')(\omega) \neq 0$ for all $(B'|A') \in \Delta$.

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The *Z-partitioning* $(\Delta_0, \ldots, \Delta_n)$ of Δ is defined as:

- $\Delta_0 = \{\delta \in \Delta \mid \Delta \text{ tolerates } \delta\}$;
- $\Delta_1, \ldots, \Delta_n$ is the Z-partitioning of $\Delta \setminus \Delta_0$.

$Z_\Delta(\delta) = i$ iff $\delta \in \Delta_i$.

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Example ($\Delta = \{(f|b), (b|p), (\neg f|p)\}$)
$\Delta_0 = \{(f|b)\}$ (in view of $\overline{p}bf$),

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Example ($\Delta = \{(f|b), (b|p), (\neg f|p)\}$)
$\Delta_0 = \{(f|b)\}$ (in view of $\overline{p}bf$), and $\Delta_1 = \{(b|p), (\neg f|p)\}$

System Z [GP96]

- $\kappa^Z_\Delta(\omega) = \max\{Z(\delta) \mid \delta(\omega) = 0, \delta \in \Delta\} + 1$, with $\max \emptyset = -1$.
- $A \mid\!\sim^Z_\Delta B$ iff $A \mid\!\sim_{\kappa^Z_\Delta} B$.

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Example

Recall: $\Delta_0 = \{(f|b)\}$ and $\Delta_1 = \{(b|p), (\neg f|p)\}$.

ω	κ^Z_Δ	ω	κ^Z_Δ	ω	κ^Z_Δ	ω	κ^Z_Δ
pbf	2	$pb\overline{f}$	1	$p\overline{b}f$	2	$p\overline{b}\,\overline{f}$	2
$\overline{p}bf$	0	$\overline{p}b\overline{f}$	1	$\overline{p}\overline{b}f$	0	$\overline{p}\overline{b}\,\overline{f}$	0

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$$\overline{p}bf,\quad \overline{p}\overline{b}f,\quad \overline{p}\overline{b}\,\overline{f} \quad\prec\quad pb\overline{f},\quad \overline{p}b\overline{f} \quad\prec\quad pbf,\quad p\overline{b}\,\overline{f},\quad p\overline{b}f$$

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$$\top \mid\!\sim^Z_\Delta \neg p.$$
$$p \wedge f \not\mid\!\sim^Z_\Delta b.$$

Lexicographic Inference

Lexicographic Inference [Leh95]

- Basic idea: compare worlds by the number of falsified conditionals in each Z-partition.
- Given $\omega \in \Omega$ and $\Delta' \subseteq \Delta$, $V(\omega, \Delta') = |(\{(B|A) \in \Delta' \mid (B|A)(\omega) = 0\}|$.
- The lexicographic vector for ω is: $\mathrm{lex}(\omega) = (V(\omega, \Delta_0), \ldots, V(\omega, \Delta_n))$.
- Given two vectors $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$, $(x_1, \ldots, x_n) \preceq^{\mathrm{lex}} (y_1, \ldots, y_n)$ iff there is some $j \leq n$ s.t. $x_k = y_k$ for every $k > j$ and $x_j \leq y_j$.
- $\omega \preceq^{\mathrm{lex}}_{\Delta} \omega'$ iff $\mathrm{lex}(\omega) \preceq^{\mathrm{lex}} \mathrm{lex}(\omega')$.

Lexicographic Inference [Leh95]

Example ($\Delta = \{(f|b), (b|p), (\neg f|p)\}$)

ω	lex(ω)	ω	lex(ω)	ω	lex(ω)	ω	lex(ω)
pbf	(0,1)	$pb\overline{f}$	(1,0)	$p\overline{b}f$	(0,2)	$p\overline{b}\,\overline{f}$	(0,1)
$\overline{p}bf$	(0,0)	$\overline{p}b\overline{f}$	(1,0)	$\overline{p}\overline{b}f$	(0,0)	$\overline{p}\overline{b}\,\overline{f}$	(0,0)

The lex-vectors are ordered as follows:

$$(0,0) \prec^{\mathrm{lex}} (1,0) \prec^{\mathrm{lex}} (0,1) \prec^{\mathrm{lex}} (0,2).$$

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pbf	(0,1)	$pb\overline{f}$	(1,0)	$p\overline{b}f$	(0,2)	$p\overline{b}\,\overline{f}$	(0,1)
$\overline{p}bf$	(0,0)	$\overline{p}b\overline{f}$	(1,0)	$\overline{p}\overline{b}f$	(0,0)	$\overline{p}\overline{b}\,\overline{f}$	(0,0)

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$$p \wedge f \mathrel{|\!\!\sim}^{\mathrm{lex}}_{\Delta} b.$$

Postulates

(Conditional) Syntax Splitting

Splitting Conditional Belief Bases [KIBB20]

We assume a conditional belief base Δ that can be split into subbases Δ_1, Δ_2 s.t. $\Delta_i \subset (\mathcal{L}_i | \mathcal{L}_i)$ with $\mathcal{L}_i = \mathcal{L}(\Sigma_i)$ for $i = 1, 2$ s.t. $\Sigma_1 \cap \Sigma_2 = \emptyset$ and $\Sigma_1 \cup \Sigma_2 = \Sigma$, writing:

$$\Delta = \Delta^1 \bigcup_{\Sigma_1, \Sigma_2} \Delta^2.$$

Example

$$\{(a|\top), (b|\top)\} = \{(a|\top)\} \bigcup_{\{a\},\{b\}} \{(b|\top)\}$$

Syntax Splitting= Independence + Relevance [KIBB20]

Definition (Independence (Ind))
An inductive inference operator $\mathbf{C}$ satisfies **(Ind)** if for any $\Delta = \Delta_1 \bigcup_{\Sigma_1,\Sigma_2} \Delta_2$ and for any $A, B \in \mathcal{L}_i$, $C \in \mathcal{L}_j$ $(i, j \in \{1, 2\}$, $j \neq i)$,

$$A \mid\sim_{\Delta} B \text{ iff } AC \mid\sim_{\Delta} B$$

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Inferences about Σ_1 are independent form information about Σ_2.

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Definition (Relevance (Rel))
An inductive inference operator **C** satisfies **(Rel)** if for any $\Delta = \Delta_1 \bigcup_{\Sigma_1,\Sigma_2} \Delta_2$ and for any $A, B \in \mathcal{L}_i$ $(i \in \{1,2\})$,

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Derivations about Σ_1 only depend on conditionals in Δ about Σ_1.

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Derivations about Σ_1 only depend on conditionals in Δ about Σ_1.

An inductive inference operator **C** satisfies **(SynSplit)** if it satisfies (**Ind**) and (**Rel**).

Proposition

C^{lex} and C^{Z} satisfy **Rel**.

Proposition

C^{lex} satisfies **Ind**.

Proposition
C^{Z} does not satisfy **Ind**.

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C^{lex} and C^{Z} satisfy **Rel**.

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Example

Let $\Delta = \{(a|\top), (b|\top)\}$. Then:

$$\begin{array}{lllll} ab & \prec^{Z}_{\Delta} & a\overline{b}, \overline{a}b, \overline{a}\overline{b} & & \\ ab & \prec^{\text{lex}}_{\Delta} & a\overline{b}, \overline{a}b & \prec^{\text{lex}}_{\Delta} & \overline{a}\overline{b} \end{array}$$

$$\top \mathrel{|\!\sim}^{Z}_{\Delta} a \qquad \neg b \mathrel{|\!\not\sim}^{Z}_{\Delta} a$$

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Respect for Equivalence

Equivalence of Conditional KBs

$(B_1|A_1) \equiv (B_2|A_2)$ iff $\omega(B_1|A_1) = \omega(B_2|A_2)$ for every $\omega \in \Omega$. This is equivalent to $A_1 \equiv A_2$ and $B_1 \wedge A_1 \equiv B_2 \wedge A_2$.

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Two conditional knowledge bases Δ_1 and Δ_2 are:

- bijective pairwise equivalent if there is a bijection $f : \Delta_1 \to \Delta_2$ s.t. $\delta \equiv f(\delta)$ for every $\delta \in \Delta_1$
- pairwise equivalent if for every $\delta_1 \in \Delta_1$ there is some $\delta_2 \in \Delta_2$ s.t. $\delta_1 \equiv \delta_2$, and vice versa.
- globally equivalent if for every tpo $\preceq$, Δ_1 is valid w.r.t. $\preceq$ iff Δ_2 is valid w.r.t. $\preceq$.

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Not bijective pairwise equivalent, but pairwise equivalent, and globally equivalent.

Equivalence of Conditional KBs

Proposition
If Δ_1 and Δ_2 are pairwise equivalent, they are globally equivalent.

The other direction does not hold.

$\Delta_1 = \{(q|p), (r|p)\}$ and $\Delta_2 = \{(q \wedge r|p)\}$. Then clearly Δ_1 and Δ_2 are globally equivalent but not pairwise equivalent.

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Definition
Let an inductive inference operator $\mathbf{C}$ be given and $x \in \{$bijective pariwse, pairwise, global$\}$. Then $\mathbf{C}$ respects x equivalence if for any x equivalent knowledge bases Δ_1 and Δ_2, $\mathbf{C}(\Delta_1) = \mathbf{C}(\Delta_1)$.

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Proposition
System Z respects global equivalence.

Equivalence of Conditional KBs

Proposition
Lexicographic inference does not respect pairwise equivalence.

Proof.
Consider the following pairwise equivalent conditional KBs:

$$\Delta_1 = \{(p|q), (r|q)\} \qquad \Delta_2 = \Delta_1 \cup \{(r \wedge q|q)\}$$

The lexicographic vectors for $\overline{p}qr$ and $pq\overline{r}$ are:

$$V(\overline{p}qr, \Delta_1) = 1 \, V(\overline{p}qr, \Delta_2) = 1 \, (\text{as } \overline{p}qr \vdash q \wedge \neg p)$$
$$V(pq\overline{r}, \Delta_1) = 1 \, V(pq\overline{r}, \Delta_2) = 2 \, (\text{as } pq\overline{r} \vdash q \wedge \neg r \wedge \neg(r \wedge q))$$

Which means that

$$\overline{p}qr \approx^{\text{lex}}_{\Delta_1} pq\overline{r} \text{ whereas } \overline{p}qr \prec^{\text{lex}}_{\Delta_2} pq\overline{r}$$

$\square$

Fixing Lexicographic Entailment

$$[(B|A)] = \{(D|C) \in (\mathcal{L}|\mathcal{L}) \mid (B|A) \equiv (C|D)\}.$$

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We count the violations of conditionals in Δ by ω up to equivalence as:

$$V^{\equiv}(\omega, \Delta) := |\,\{\, [(B|A)] \mid ((B|A))(\omega) = 0, (B|A) \in \Delta\}|$$

We can now define, for Δ with Z-ranking $(\Delta_0, \ldots, \Delta_n)$, $\mathtt{lex}^{\equiv}(\omega) = (V^{\equiv}(\omega, \Delta_0), \ldots, V^{\equiv}(\omega, \Delta_n))$.

$\omega_1 \preceq_{\Delta}^{\mathsf{lex},\equiv} \omega_2$ iff $\mathtt{lex}^{\equiv}(\omega_1) \preceq^{\mathsf{lex}} \mathtt{lex}^{\equiv}(\omega_2)$. We denote the corresponding inductive inference relation by $C^{\mathsf{lex},\equiv}$

Fixing Lexicographic Entailment: Example

$$\Delta_1 = \{(p|q), (r|q)\} \qquad \Delta_2 = \Delta_1 \cup \{(r \wedge q|q)\}$$

$[(p|q)] = \{(p|q)\}$ and $[(r|q)] = \{(r|q), (r \wedge q|q)\}$.

$$V^{\equiv}(\overline{p}qr, \Delta_1) = 1 \; V(\overline{p}qr, \Delta_2) = 1 \, (\text{as } \overline{p}qr \vdash q \wedge \neg p)$$

$$V^{\equiv}(pq\overline{r}, \Delta_1) = 1 \; V(pq\overline{r}, \Delta_2) = 1 \, (\text{as } pq\overline{r} \vdash q \wedge \neg r \wedge \neg(r \wedge q)$$

$$\text{and } [(r|q)] \ni (r|q), (r \wedge q|q))$$

We can't have it all

Proposition
There exists no conditional-based inductive inference operation that respects global equivalence and satisfies syntax splitting.

Proof.
Suppose that **C** satisfies syntax splitting.

Consider first $\Delta_1 = \{(a|\top), (b|\top)\}$. With **DI**, $\top \mathrel{|\!\sim}^{\mathbf{C}}_{\Delta_1} a$ and $\top \mathrel{|\!\sim}^{\mathbf{C}}_{\Delta_1} b$ (which implies $ab \prec \omega$ for any $\omega \in \Omega \setminus \{ab\}$). Then since $\Delta_1 = \{(a|\top)\} \bigcup_{\{a\},\{b\}} \{(b|\top)\}$, $\neg b \mathrel{|\!\sim}^{\mathbf{C}}_{\Delta_1} a$. This means that $a\overline{b} \prec^{\mathbf{C}}_{\Delta_1} \overline{a}\overline{b}$. With symmetry, we establish that $\overline{a}b \prec^{\mathbf{C}}_{\Delta_1} \overline{a}\overline{b}$.

Consider now $\Delta_2 = \{(a \wedge b|\top)\}$. Notice that Δ_1 and Δ_2 are globally equivalent. Thus, by global equivalence, $\prec^{\mathbf{C}}_{\Delta_1} = \prec^{\mathbf{C}}_{\Delta_2}$. As $((a \wedge b|\top))(a\overline{b}) = ((a \wedge b|\top))(\overline{a}b) = ((a \wedge b|\top))(\overline{a}\overline{b}) = 0$ (with conditional-basedness), we see that $a\overline{b} \approx^{\mathbf{C}}_{\Delta_2} \overline{a}b \approx^{\mathbf{C}}_{\Delta_2} \overline{a}\overline{b}$, contradiction. $\square$

Language-Independence

Symbol Translations

$$\Delta_{\mathsf{birds}} : \quad (\mathrm{birds}|\mathrm{penguins}), (\mathrm{fly}|\mathrm{birds}), (\neg\mathrm{fly}|\mathrm{penguins})$$

$$\Delta_{\mathsf{bats}} : \quad (\mathrm{mamals}|\mathrm{bats}), (\neg\mathrm{fly}|\mathrm{mamals}), (\neg\neg\mathrm{fly}|\mathrm{bats})$$

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Definition
A mapping $\sigma : \Sigma \to \mathcal{L}(\Sigma')$ is *belief-amount preserving symbol translation* (in short, a BAP-translation) if there is a bijection $\gamma : \Omega(\Sigma) \to \Omega(\Sigma')$ s.t. for every $\phi \in \mathcal{L}(\Sigma)$ $\mathrm{Mod}(\sigma(\phi)) = \{\gamma(\omega) \mid \omega \in \mathrm{Mod}(\phi)\}$.

Symbol Translations

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$$\begin{aligned} \mathrm{birds} &\mapsto \mathrm{mamals} \\ \mathrm{penguins} &\mapsto \mathrm{bats} \\ \mathrm{fly} &\mapsto \neg\mathrm{fly} \end{aligned} \qquad \begin{aligned} \mathrm{birds}\,\mathrm{penguins}\,\mathrm{fly} &\mapsto \mathrm{mamals}\,\mathrm{bats}\,\overline{\mathrm{fly}} \\ \mathrm{birds}\,\mathrm{penguins}\,\overline{\mathrm{fly}} &\mapsto \mathrm{mamals}\,\mathrm{bats}\,\mathrm{fly} \\ &\dots \end{aligned}$$

Language-Independence

Definition
An inductive inference operator **C** satisfies *language-independence* if for every BAP-translation σ, $\psi \mid\!\sim_{\Delta} \phi$ iff $\sigma(\phi) \mid\!\sim_{\sigma(\Delta)} \sigma(\psi)$.

Language-Independence

Definition

An inductive inference operator **C** satisfies *language-independence* if for every BAP-translation σ, $\psi \mid\!\sim_{\Delta} \phi$ iff $\sigma(\phi) \mid\!\sim_{\sigma(\Delta)} \sigma(\psi)$.

A *vector mass distribution* (VMD) is a function $F : \{1, 0, u\}^n \mapsto \mathbb{N}$ s.t. $\Sigma_{\overrightarrow{\alpha} \in \{1,0,u\}^n} F(\overrightarrow{\alpha}) = 2^{|\Sigma|}$.

An inductive inference operator **C** is *conditional-functional* iff there is a function D that returns, for any VMD F,$(V_F^D, \sqsubseteq_F^D)$ where $V_F^D \subseteq \{1, 0, u\}^n$, and $\sqsubseteq_F^D$ is a TPO on V_F^D such that:

- $\overrightarrow{\alpha} \in V_F^D$ implies $F(\overrightarrow{\alpha}) > 0$, and
- for any permutation σ on $\{1, \ldots, n\}$, $V_{\sigma(F)}^D = \sigma(V_F^D)$ and $\overrightarrow{\alpha} \sqsubseteq_{\sigma(F)}^D \overrightarrow{\beta}$ iff $\sigma^{-1}(\overrightarrow{\alpha}) \sqsubseteq_F^D \sigma^{-1}(\overrightarrow{\beta})$

and such that $\omega_1 \preceq_\Delta \omega_2$ iff $\langle \delta_1(\omega_1), \ldots, \delta_n(\omega_1) \rangle \sqsubseteq_{F_\Delta}^D \langle \delta_1(\omega_2), \ldots, \delta_n(\omega_2) \rangle$, where $\Delta = \{\delta_1, \ldots, \delta_n\}$ and $F_\Delta(\overrightarrow{\alpha}) = |\{\omega \mid \overrightarrow{\alpha} = \delta_1(\omega), \ldots, \delta_n(\omega)\}|$.

Results

Proposition
A TPO-based inductive inference operator **C** is conditional-functional iff it respects bijective pairwise equivalence and satisfies language independence.

Conclusion

Postulates Satisfaction

	System Z	System W	Lex	$\text{Lex}^{\equiv}$	C-rep	System J(LZ)
Independence	$\times$	$\vee$	$\vee$	$\vee$	$\vee$	?
Relevance	$\vee$	$\vee$	$\vee$	$\vee$	$\vee$	?
Global Eq.	$\vee$	$\times$	$\times$	$\times$	$\times$	?
Pairwise Eq.	$\vee$	$\vee$	$\times$	$\vee$	?	?
Cond.-based	$\vee$	$\vee$	$\vee$	$\vee$	?	?
Language-ind.	$\vee$	$\vee$	$\vee$	$\vee$	?	?

This complements the mainly example-driven history predomindant in the literature.

Modularity: wider narrative (Tin foil hat time?)

Formalism	Horizontal	Vertical
Logic Programming	Conditional Independence (AFT)	Splitting
	Treewidth decompositions (?)	Stratification
Argumentation	Non-interference	SCC-recursiviness
Belief revision	Conditional syntax splitting	G. ranking kinematics
Defeasible Conditionals	Conditional syntax splitting	G. ranking kinematics
Probabilistic Reasoning	Conditional Independence	Subset Independence

Conclusion

- "Logicality" can be formalized in different ways:
  - Syntax-Splitting (inference allows modularisation)
  - Respect for Equivalence (inference respects basic semantics)
  - Language-Independence (inference independent of linguistic peculiarities)
- Interesting enough, these notions are not independent:
  - Syntax-splitting and respect for global equivalence are impossible,

Thank you for your attention. Questions?

Moisés Goldszmidt and Judea Pearl.
Qualitative probabilities for default reasoning, belief revision, and causal modeling.
AI, 84(1-2):57–112, 1996.

Gabriele Kern-Isberner, Christoph Beierle, and Gerhard Brewka.
Syntax splitting= relevance+ independence: New postulates for nonmonotonic reasoning from conditional belief bases.
In *Proceedings of the International Conference on Principles of Knowledge Representation and Reasoning*, volume 17, pages 560–571, 2020.

Gabriele Kern-Isberner, Christoph Beierle, and Jesse Heyninck.
Conditional independence for iterated belief revision.
In *Accepted for IJCAI 2022*, 2022.

Sarit Kraus, Daniel Lehmann, and Menachem Magidor.
Nonmonotonic reasoning, preferential models and cumulative logics.
Artificial intelligence, 44(1-2):167–207, 1990.

Daniel Lehmann.
Another perspective on default reasoning.
Annals of mathematics and artificial intelligence, 15(1):61–82, 1995.