# ON THE REAL ZEROES OF HALF-INTEGRAL WEIGHT HECKE CUSP FORMS, II

JESSE JÄÄSAARI

Abstract. We show that for $\gg K^2$ of the half-integral weight Hecke cusp forms in the Kohnen plus subspaces with weight bounded by a large parameter $K$, the number of "real" zeroes grows at the expected rate. A key technical step in the proof is to obtain sharp bounds for the mollified first and second moments of quadratic twists of modular $L$-functions.

## 1. Introduction and the main result

One of the striking consequences of the holomorphic Quantum Unique Ergodicity (QUE) conjecture of Rudnick and Sarnak [27, 22] (which is now a theorem thanks to the work of Holowinsky and Soundararajan [8]) is that the zeroes of holomorphic Hecke cusp forms equidistribute inside the fundamental domain $\mathcal{D} := \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$, where $\mathbb{H}$ is the upper half of the complex plane, as the weight of the form tends to infinity. This was proved by Rudnick [26] building upon the work of Shiffman and Zelditch [28] in the case of compact manifolds. Subsequent investigations concerning the distribution of zeroes in small scales were initiated by Ghosh and Sarnak [7], and then continued by Matomäki [23] and Lester–Matomäki–Radziwiłł [20]. In a recent work [13] the author obtained results related to the latter work in the setting of half-integral weight Hecke cusp forms. We refer to the introduction of [13] for the differences between the half-integral weight case and the integral weight case, but for now we only briefly mention that an important reason why the methods of [7, 23, 20] do not adapt to the half-integral weight setting in a straightforward way is that the Fourier coefficients of half-integral weight Hecke cusp forms are not multiplicative.

In this paper we are interested in the real[1] zeroes of half-integral weight Hecke cusp forms, that is zeroes on the two geodesic segments

$$\delta_1 := \left\{ s \in \mathbb{C} : \mathrm{Re}(s) = -\frac{1}{2} \right\} \qquad \text{and} \qquad \delta_2 := \{ s \in \mathbb{C} : \mathrm{Re}(s) = 0 \}.$$

For motivation and related results for integral weight Hecke cusp forms, we refer to the introduction of [13] as well as the papers [7, 23, 20].

Let us now summarise the main results of [13]. Before that we need to introduce some notation. Throughout the article, let $k$ be a positive integer. We write $S_{k+\frac{1}{2}}(4)$ for the space of cusp forms of weight $k+\frac{1}{2}$ and level 4. Also we denote by $S^+_{k+\frac{1}{2}}(4) \subset S_{k+\frac{1}{2}}(4)$ the Kohnen plus subspace and let $B^+_{k+\frac{1}{2}}$ denote a fixed Hecke eigenbasis for $S^+_{k+\frac{1}{2}}(4)$. Note that for half-integral weight cusp forms we cannot normalise the coefficient $c_g(1)$ to be equal to one without losing the algebraicity of the Fourier coefficients. This means that, unlike in the integral weight case, there is no canonical choice for $B^+_{k+\frac{1}{2}}$, but this causes no problems for us. We write $\mathcal{Z}(g) := \{z \in \mathcal{F} : g(z) = 0\}$ for the set of zeroes of $g \in S^+_{k+\frac{1}{2}}(4)$ and let $\mathcal{F}_Y := \{z \in \mathcal{F} : \mathrm{Im}(z) \geq Y\}$ be a Siegel set, where $\mathcal{F} := \Gamma_0(4)\backslash\mathbb{H}$. Finally, set[2]

$$\mathcal{S}_K := \bigcup_{k \sim K} B^+_{k+\frac{1}{2}}$$

and note that the cardinality of this set is $\sim K^2/4$.

---

2010 *Mathematics Subject Classification.* Primary 11F37; Secondary 11M06.

[1] This notion comes from the fact that half-integral weight form is real-value on the lines $\delta_1$ and $\delta_2$ when it is normalised to have real Fourier coefficients.

[2] Here, and throughout the paper, the notation $\ell \sim L$ means that $L \leq \ell \leq 2L$.

With these notations the main result of [13] may be stated as follows.

**Theorem 1.1.** *([13, Theorems 1.1. and 1.3]) Let $K$ be a large parameter, $\varepsilon > 0$ be an arbitrarily small fixed number, and $j \in \{1, 2\}$.*

*(1) For $\gg_\varepsilon K^2/(\log K)^{3/2+\varepsilon}$ of the forms $g \in \mathcal{S}_K$ we have*

$$\#\{\mathcal{Z}(g) \cap \delta_j \cap \mathcal{F}_Y\} \gg \frac{K}{Y}(\log K)^{-23/2}$$

*for $\sqrt{K \log K} \leq Y \leq K^{1-\delta}$ with any small fixed constant $\delta > 0$.*

*(2) For at least $(1/2 - \varepsilon)\#\mathcal{S}_K$ of the forms $g \in \mathcal{S}_K$ we have*

$$\#\{\mathcal{Z}(g) \cap \delta_j \cap \mathcal{F}_Y\} \gg \sqrt{\frac{K}{Y}}$$

*for $\sqrt{K \log K} \leq Y \leq K^{1-\delta}$ with any small fixed constant $\delta > 0$.*

The first part is proved by studying sign changes of Fourier coefficients $c_g(|d|)$ of a half-integral weight cusp form $g$ at fundamental discriminants $d$. The loss of powers of logarithm in part (i) is ultimately due to the fluctuation in the size of $|c_g(|d|)|$. Indeed, the study of sign changes relies on estimating the second and fourth moments of the Fourier coefficients and a consequence of the variation in the size of $|c_g(|d|)|$ is that these moments are not of the same order of magnitude as the fourth moment amplifies the large values of $|c_g(|d|)|$. The proof of the second part exploited the multiplicativity of $c_g(n)$ at squares following arguments of [20].

In the same article it was alluded that it might be possible to obtain a result that holds for a positive proportion of forms in $\mathcal{S}_K$ with $\gg K/Y$ real zeroes in $\mathcal{F}_Y$ by introducing a suitable mollifier to the argument (we shall review the approach of [13] in the next section). The goal of the present work is to verify this prediction. In the following section we introduce a mollifier that is suitable for our aim. The mollifier $M_g(d)$ is chosen so that typically $|c_g(|d|)M_g(d)| \approx 1$. Our main result is the following, which unifies and improves the results in Theorem 1.1.

**Theorem 1.2.** *Let $K$ be a large parameter and $j \in \{1, 2\}$. Then for $\gg K^2$ of the forms $g \in \mathcal{S}_K$ we have*

$$\#\{\mathcal{Z}(g) \cap \delta_j \cap \mathcal{F}_Y\} \gg \frac{K}{Y}$$

*for $\sqrt{K \log K} \leq Y \leq K^{1-\vartheta}$ with any small fixed constant $\vartheta > 0$.*

**Remark 1.3.** (1) It might be possible to extract an explicit proportion of forms for which Theorem 1.2 holds from our argument, but this seems difficult.

(2) In the range $\sqrt{k \log k} \leq Y < \frac{1}{100}k$ it is expected that

$$\#\{\mathcal{Z}(g) \cap \mathcal{F}_Y\} \asymp \frac{k}{Y}$$

for $g \in S^+_{k+\frac{1}{2}}(4)$.

Theorem 1.2 should be compared to a result of Lester, Matomäki, and Radziwiłł [20] for integral weight forms. They showed that for any fixed $\varepsilon > 0$ there exists a subset[3] $S_k \subset B_k$, containing more than $(1-\varepsilon)\#B_k$ elements, such that for every $f \in S_k$ we have

$$\#\{z \in Z(f) \cap \delta_j \cap \mathcal{D}_Y\} \geq c(\varepsilon)\#\{Z(f) \cap \mathcal{D}_Y\}$$

for each $j \in \{1, 2\}$ provided that $\sqrt{k \log k} < Y < \delta(\varepsilon)k$ and $k \longrightarrow \infty$ for some positive constants $c(\varepsilon)$ and $\delta(\varepsilon)$ depending only on $\varepsilon$. Here of course $\mathcal{D}_Y := \{z \in \mathcal{D} : \operatorname{Im}(z) \geq Y\}$. Furthermore, under the Generalised Lindelöf Hypothesis they showed that

$$\#\{z \in Z(f) \cap \delta_j \cap \mathcal{D}_Y\} \gg \left(\frac{k}{Y}\right)^{1-\varepsilon}$$

[3]Throughout the paper we write $H_k$ for the set of holomorphic cusp forms of integral weight $k$ and full level. We also write $B_k$ for the Hecke eigenbasis of $H_{2k}$.

in the same range of $Y$.

The proof of their result relies on the breakthrough of Matomäki and Radziwiłł [24] on multiplicative functions in short intervals. Concerning the first result mentioned, note that for individual forms $f$ we cannot do as well, even on the assumption of the Generalised Lindelöf or Generalised Riemann Hypothesis (GRH). The reason is that in order to produce real zeroes of $f$ we look at sign changes of the Hecke eigenvalues $\lambda_f(n)$. In order to obtain a positive proportion of the zeroes on the line we need a positive proportion of sign changes between the coefficients of $\lambda_f(n)$ in appropriate ranges of $n$. However, we cannot e.g. rule out the scenario where for all $p \le (\log k)^{2-\varepsilon}$ we have $\lambda_f(p) = 0$.

Broadly the method used to prove Theorem 1.2 is the same as in our previous work [13], which significantly differs from the arguments used in [7, 23, 20]. However, computing the mollified moments of $c_g(|d|)$ is significantly more challenging compared to estimating the unmollified moments in [13]. In particular, the main novelty of the present work is the sharp estimation of the mollified fourth moment of the Fourier coefficients that requires relating the moment in question to a suitable random model. Finally, observe that our result requires considering forms in a larger family $\cup_{k\sim K} B^+_{k+1/2}$ instead of just $B^+_{k+1/2}$. The reason for this is that similarly as in [13] the averages over the forms $g \in B^+_{k+1/2}$ and the weights $k$ are needed in order to evaluate the mollified moments of the Fourier coefficients sharply, see the discussion in [13, Section 2].

## 2. The Strategy

In this section we explain the ideas behind Theorem 1.2. But before that we very briefly outline the structure of the approach in [13]. There the main idea was to study sign changes of the Fourier coefficients $c_g(|d|)$ of a half-integral weight Hecke cusp form $g$ along the (odd) fundamental discriminants $d$. It is convenient to normalise the Fourier coefficients $c_g(|d|)$ by the factor $\sqrt{\alpha_g}$, where

$$\alpha_g := \frac{\Gamma\left(k-\frac{1}{2}\right)}{2(4\pi)^{k-\frac{1}{2}}\|g\|_2^2} > 0. \tag{2.1}$$

Throughout this section we set $X = K/Y$. Very briefly, it follows from a steepest descend argument that in order to obtain real zeroes of $g$ one needs to produce fundamental discriminants $d_\pm$ in short intervals $x \le (-1)^k d \le x+H$ with $H$ as small as possible in terms of $X \sim x$ so that $\sqrt{\alpha_g}c_g(|d_-|) < -k^{-\delta} < k^{-\delta} < \sqrt{\alpha_g}c_g(|d_+|)$ for some sufficiently small $\delta > 1/2$. For the purpose of exposition we pretend here that we are only looking for numbers $d_\pm$ for which $c_g(|d_-|) < 0 < c_g(|d_+|)$. To detect such an event one considers the inequality

$$\left| \sideset{}{^\flat}\sum_{x\le(-1)^k d\le x+H} \sqrt{\alpha_g}c_g(|d|)\right| \le \sideset{}{^\flat}\sum_{x\le(-1)^k d\le x+H} \sqrt{\alpha_g}|c_g(|d|)$$

and notes that the inequality is strict if and only if $c_g(|d|)$ has a sign change in the interval $[x, x+H]$. Here $\sum^\flat$ signifies that we are summing over odd fundamental discriminants. In [13] it was shown that one is able to detect sign changes for many forms if $H$ is chosen to be a power of $\log K$. Indeed, choosing $H = (\log X)^9$ it was shown that for $\gg_\varepsilon K^2/(\log K)^{3/2+\varepsilon}$ of the forms $g \in S_K$ the inequality

$$\left| \sideset{}{^\flat}\sum_{x\le(-1)^k d\le x+H} \sqrt{\alpha_g}c_g(|d|)\right| < \sideset{}{^\flat}\sum_{x\le(-1)^k d\le x+H} \sqrt{\alpha_g}|c_g(|d|)$$

holds for $\gg X/(\log X)^{5/2}$ of $x \sim X$ from which one immediately deduces part (1) of Theorem 1.1. The two auxiliary results needed for this are contained in the following lemma. Here $\omega_f$ (sometimes we write $\omega_g$ if $g \in S^+_{k+1/2}(4)$ corresponds to $f \in H_{2k}$ under the Shimura correspondence. Similarly for the mollifier we shall use $M_g(d)$ and $M_f(d)$ interchangeably) are the standard harmonic weights that appear in the Petersson trace formula.

**Lemma 2.1.** *(*[13, Lemmas 2.7. and 2.8]*) Let $h$ and $\phi$ be compactly supported smooth weight functions on $\mathbb{R}_+$. Then for $X \ll \sqrt{K}$ we have that*

*(1)*

$$\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{g\in B^+_{k+1/2}} \alpha_g \sum_d{}^{\flat} |c_g(|d|)|^2 \phi\left(\frac{|d|}{X}\right) = \frac{2XK}{3\pi^2}\widehat{h}(0)\widehat{\phi}(0) + O_\varepsilon\left(KX^{1/2+\varepsilon}\right).$$

*(2)*

$$\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{g\in B^+_{k+\frac{1}{2}}} \alpha_g^2\omega_g^{-1} \sum_d{}^{\flat} |c_g(|d|)|^4 \phi\left(\frac{|d|}{X}\right) \ll XK\log(XK).$$

The loss of powers of the logarithm in Theorem 1.1 stems from the fact that in the previous lemma the sizes of the second and fourth moment of the Fourier coefficients are not of the same order of magnitude. As mentioned above, the reason for this is that the size of $|c_g(|d|)|$ fluctuates as $d$ traverses over the fundamental discriminants.

To remedy this one can introduce a positive quantity $M_g(d)$ with the property that $|c_g(|d|)M_g(d)| \approx 1$ and evaluate the moments of $c_g(|d|)M_g(d)$. This then yields information about the sign changes of $c_g(|d)$ as $M_g(d) > 0$. Below we shall choose the mollifier $M_g(d)$ so that, assuming $X \ll \sqrt{K}$,

$$\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{g\in B^+_{k+1/2}} \alpha_g^2\omega_g^{-1} \sum_d{}^{\flat} |c_g(|d|)|^4 M_g(d)^4 \phi\left(\frac{|d|}{X}\right)$$
$$\asymp \sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{g\in B^+_{k+1/2}} \alpha_g \sum_d{}^{\flat} |c_g(|d|)|^2 M_g(d)^2 \phi\left(\frac{|d|}{X}\right) \asymp XK$$

for compactly supported smooth weight functions $h$ and $\phi$, which then implies Theorem 1.2 by repeating the arguments of [13] verbatim except with $c_g(|d|)$ replaced by $c_g(|d|)M_g(|d|)$.

We now compare the strategies to compute the mollified moments to the methods used to establish Lemma 2.1 in [13]. Concerning the second moment, in [13] one executes the $g$-sum using the half-integral weight variant of the Petersson trace formula after which the $d$- and $k$-sums are evaluated by the Poisson summation. For the mollified second moment we first instead use Waldspurger's formula to express $|c_g(|d|)|^2$ as a central $L$-value on which an approximate functional equation is then applied. After that the $d$- and $k$-sums are evaluated by the Poisson summation. The final step is to relate the resulting main term to an Euler product that is estimated by Mertens' theorem.

The unmollified fourth moment was treated by starting with Waldspurger's formula and an approximate functional equation, after which a large sieve inequality of Deshouillers and Iwaniec was used to estimate the double $(g,k)$-sum. The final step was to average over the fundamental discriminants trivially. With a mollifier this strategy does not work and we need to proceed differently. However, the first steps are identical: by Waldspurger's formula we need to estimate the sum

$$\sum_d{}^{\flat} \phi\left(\frac{|d|}{X}\right) \sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{f\in B_k} \omega_f L\left(\frac{1}{2}, f\otimes\chi_d\right)^2 M_f(d)^4.$$

Using an approximate functional equation we express the central $L$-value by essentially a finite Dirichlet series. After expanding the definition of $M_f(d)$ we execute the sum over $B_k$ using the Petersson formula. This splits the sum into two parts, diagonal and off-diagonal, in a natural way. The diagonal term can be written as an expectation value

$$\sum_d{}^{\flat} \phi\left(\frac{|d|}{X}\right) \sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \mathbb{E}(L(X;d,k)M_2(X;d)), \tag{2.2}$$

where $L(X;d,k)$ is a random $L$-function build from random variables $X$, which model the behaviour of Hecke eigenvalues $\lambda_f(n)$. Similarly, $M_2(X;d)$ is random mollifier modelling the behaviour of $M_f(d)^4$. Now, using

Mellin inversion we may express (2.2) as a double line integral

$$\sum_{d}^{\flat}\phi\left(\frac{|d|}{X}\right)\sum_{k\in\mathbb{Z}}h\left(\frac{2k-1}{K}\right)\mathbb{E}\left(\frac{1}{(2\pi i)^2}\int\limits_{(1)}\int\limits_{(1)}\left(\frac{2\pi}{d}\right)^{-s_1-s_2}\frac{\Gamma(s_1+k)}{\Gamma(k)}\cdot\frac{\Gamma(s_2+k)}{\Gamma(k)}e^{s_1^2+s_2^2}F(s_1,s_2;X,d)\,\frac{\mathrm{d}s_1\,\mathrm{d}s_2}{s_1s_2}\right)$$

for certain function $F(s_1,s_2;X,d)$ for which $\mathbb{E}(F(s_1,s_2;X,d))$ has poles at $s_2=0$ and $s_2=-s_1$. It will turn out that the contribution of the latter pole cancels the off-diagonal from the initial application of the Petersson formula. This gives a new instance of a similar phenomenon appearing in some prior works [3, 4, 16, 30]. The computations here have similarities with the arguments of Khan [16] and we are able to use some technical results he proved. The pole at $s_2=0$ gives the dominant contribution, which turns out to be $\ll XK$ after some calculations with Euler products. Here the presence of random variables greatly simplifies the computations, especially allowing us to control the effect of "small" primes in the mollifier using some ideas from [6].

2.1. **Organisation of the article.** This paper is organised as follows. In Section 4 we describe the mollifier $M_g(d)$ used in this work. In the following section we gather basic facts about half-integral weight modular forms and other auxiliary results we need. In Section 6 we introduce a random model for the Hecke eigenvalues and construct a random mollifier from these random variables. In Section 7 we evaluate a character sum that arises in the computation of the fourth moment of the Fourier coefficients. The second mollified moment is estimated in Section 8. The estimation of the fourth moment takes the following three sections. Finally the main result, Theorem 1.2, is proved in Section 12.

2.2. **Acknowledgements.** This work was supported by the Finnish Cultural Foundation. The author is grateful to Kaisa Matomäki for interesting discussions, and to Steve Lester for encouragement and suggesting that the methods of [6] might be useful when computing the mollified fourth moment of the Fourier coefficients.

## 3. Notations

We use standard asymptotic notation. If $f$ and $g$ are complex-valued functions defined on some set, say $\mathcal{G}$, then we write $f\ll g$ to signify that $|f(x)|\leqslant C|g(x)|$ for all $x\in\mathcal{G}$ for some implicit constant $C\in\mathbb{R}_+$. The notation $O(g)$ denotes a quantity that is $\ll g$, and $f\asymp g$ means that both $f\ll g$ and $g\ll f$. We write $f=o(g)$ if $g$ never vanishes in $\mathcal{G}$ and $f(x)/g(x)\longrightarrow 0$ as $x\longrightarrow\infty$. Moreover, we write $f\sim g$ if $f(x)/g(x)\longrightarrow 1$ as $x\longrightarrow\infty$. The letter $\varepsilon$ denotes a positive real number, whose value can be fixed to be arbitrarily small, and whose value can be different in different instances in a proof. All implicit constants are allowed to depend on $\varepsilon$, on the implicit constants appearing in the assumptions of theorem statements, and on anything that has been fixed. When necessary, we will use subscripts $\ll_{\alpha,\beta,\ldots}, O_{\alpha,\beta,\ldots}$, etc. to indicate when implicit constants are allowed to depend on quantities $\alpha,\beta,\ldots$

We define $\chi_d(\cdot):=\left(\frac{d}{\cdot}\right)$, the Kronecker symbol, for all non-zero odd integers $d$. For such $d$ we set

$$\varepsilon_d:=\begin{cases}1 & \text{if } d\equiv 1\,(4)\\ i & \text{if } d\equiv 3\,(4)\end{cases}$$

Let us also write $1_{m=n}$ for the characteristic function of the event $m=n$. Furthermore, $\mathrm{Re}(s)$ and $\mathrm{Im}(s)$ are the real- and imaginary parts of $s\in\mathbb{C}$, respectively, and occasionally we write $\sigma$ for $\mathrm{Re}(s)$. We write $e(x):=e^{2\pi ix}$. For a compactly supported smooth function $\phi$, we define its Fourier transform $\widehat{\phi}(y)$ by

$$\widehat{\phi}(y):=\int\limits_{\mathbb{R}}\phi(x)e(-xy)\,\mathrm{d}x.$$

The sum $\sum_{a\,(c)}^{*}$ means that the summation is over residue classes coprime to the modulus. We also write $\tau(\chi)$ for the quadratic Gauss sum associated to a character $\chi$. Given coprime integers $a$ and $c$, we write $\overline{a}\,(\mathrm{mod}\ c)$ for the multiplicative inverse of $a$ modulo $c$. As usual, $\Gamma$ denotes the Gamma function and $\mu$ denotes the Möbius function. In addition, $d(n)$ is the usual divisor function and $\varphi(n)$ is Euler's totient function. The notation $n=\square$ means that a natural number $n$ is a perfect square. We also write $[a,b]$ for the least common multiple of two natural numbers $a$ and $b$, this should not be confused with a notation $[a,b]$ used for a real interval. The notation $p^\alpha \| n$ means that $p^\alpha | n$, but $p^{\alpha+1} \nmid n$. Of course $\zeta$ denotes the Riemann zeta function and write its local factors as $\zeta_p(s) := (1-p^{-s})^{-1}$ so that $\zeta(s) = \prod_p \zeta_p(s)$ for $\mathrm{Re}(s) > 1$. We also write $\mathrm{Tr}(A)$ for the trace of a matrix $A$, $\mathrm{rad}(n)$ for the radical of $n \in \mathbb{N}$, and $\mathbb{E}(X)$ for the expectation value of a random variable $X$. Finally, $\sum^\flat$ means that we are summing over all odd fundamental discriminants.

## 4. Choosing the mollifier

4.1. **Heuristics.** Let $d$ denote a fundamental discriminant, and $\chi_d(\cdot) = \left(\frac{d}{\cdot}\right)$ denote the primitive quadratic character of conductor $|d|$. Let $f \otimes \chi_d$ denote the twist of $f \in B_k$ by the character $\chi_d$, and let $L(s, f \otimes \chi_d)$ denote the twisted $L$-function

$$L(s, f \otimes \chi_d) := \sum_{m=1}^{\infty} \frac{\lambda_f(m)\chi_d(m)}{m^s} = \prod_p L_p(s, f \otimes \chi_d)$$

for $\mathrm{Re}(s) > 1$. Here

$$L_p(s, f \otimes \chi_d) := \left(1 - \frac{\lambda_f(p)\chi_d(p)}{p^s} + \frac{\chi_d(p)^2}{p^{2s}}\right)^{-1}.$$

By Waldspurger's formula the absolute square of $c_g(|d|)$ is proportional to the central $L$-value $L(1/2, f \otimes \chi_d)$, where $f \in H_{2k}$ is a classical holomorphic cusp form attached to $g \in S^+_{k+\frac{1}{2}}(4)$ via the Shimura correspondence. This motivates us to choose the mollifier $M_g(d)$ so that it behaves like $L(1/2, f \otimes \chi_d)^{-1/2}$, at least for typical $g$ and $d$.

For $|d| \gg k^\varepsilon$ we expect that $\log L(1/2, f \otimes \chi_d)$ to be approximated by

$$\sum_{p^\ell < |d|^\varepsilon} \frac{\left(\alpha_p^\ell + \beta_p^\ell\right)\chi_d(p)^\ell}{\ell p^{\ell/2}},$$

where $\alpha_p, \beta_p$ are the Satake parameters of $f$ at $p$. Recalling that $\alpha_p + \beta_p = \lambda_f(p)$ and $\alpha_p^2 + \beta_p^2 = \lambda_f(p^2) - 1$ it is expected (and can be shown under GRH) that the sum above is

$$\sum_{p < |d|^\varepsilon} \frac{\lambda_f(p)\chi_d(p)}{\sqrt{p}} - \frac{1}{2}\log\log|d|\,(1 + o(1)).$$

Thus heuristically we expect (unconditionally) that

$$L\left(\frac{1}{2}, f \otimes \chi_d\right) \approx (\log|d|)^{-1/2} \exp\left(\sum_{p<|d|^\varepsilon} \frac{\lambda_f(p)\chi_d(p)}{\sqrt{p}}\right). \tag{4.1}$$

Here the approximation sign $\approx$ should not be taken too literally. Recall that we wish to choose the mollifier so that it approximates $L(1/2, f \otimes \chi_d)^{-1/2}$. By the discussion above a natural choice for the mollifier would be, for suitable $x$ depending on $X$ when $d \sim X$,

$$M_g(d) = (\log x)^{1/4} \exp(P(f,d)),$$

where

$$P(f,d) := -\frac{1}{2}\sum_{p \le x} \frac{\lambda_f(p)\chi_d(p)}{\sqrt{p}}.$$

In order for $M_g(d)$ to be a good approximation for $L(1/2, f \otimes \chi_d)^{-1/2}$ one wishes to choose $x \sim X^\varepsilon$ for some fixed $\varepsilon > 0$. However, with this choice one faces a technical problem that expanding the exponential factor into a Taylor series leads to a long Dirichlet polynomial, which in turn makes it impossible to facilitate certain necessary computations. On the other hand, for rather small $x$, say $x \asymp \log X$, the resulting Dirichlet polynomial has a short length but in this case we do not expect $M_g(d)$ to be a good approximation for $L(1/2, f \otimes \chi_d)^{-1/2}$.

The idea that allows us to take $x$ to be a small power of $X$ is to use an iterative scheme of Radziwiłł and Soundararajan [25], which is partly motivated by the Brun–Hooley sieve. Heuristically we expect that $P(f,d)$ has a Gaussian limiting distribution over $d \sim x$ with mean zero and variance $\sim \log\log x$. We consider the disjoint sets of primes in the intervals $I_0, I_1, ..., I_J$ with $I_j = [K^{\theta_{j-1}}, K^{\theta_j}[$ for $j = 1, ..., J$ and set $I_0 :=]c, K^{\theta_0}]$, where $c > 2$ is fixed and chosen to be sufficiently large. Due to the expected Gaussian distribution, choosing $\theta_j$'s appropriately, we anticipate that each of the terms

$$\exp\left(-\frac{1}{2}\sum_{p\in I_j}\frac{\lambda_f(p)\chi_d(p)}{\sqrt{p}}\right)$$

should typically be approximated by a short Dirichlet polynomial depending on the size of $\theta_j$ as the sum inside the argument is expected to be small for most $d$. We take $J$ to be so large that $\theta_J \geq \eta_2$ for some fixed $\eta_2 > 0$, meaning that $J \asymp \log\log\log K$. This leads one to expect that for typical $d \sim X$ we should have that

$$\exp\left(-\frac{1}{2}\sum_{p\leq X^{\varepsilon}}\frac{\lambda_f(p)\chi_d(p)}{\sqrt{p}}\right) \approx \prod_{j=0}^{J}\exp\left(-\frac{1}{2}\sum_{p\in I_j}\frac{\lambda_f(p)\chi_d(p)}{\sqrt{p}}\right)$$

can be approximated by a short Dirichlet polynomial. Due to technical reasons it is better to replace $\lambda_f(m)$ with the completely multiplicative function $a_f(m)$ defined by $a_f(p) = \lambda_f(p)$ at the primes. The key observation is that using the Taylor expansion[4] and binomial theorem we have

$$\begin{aligned}\exp\left(-\frac{1}{2}\sum_{p\in I_j}\frac{a_f(p)\chi_d(p)}{\sqrt{p}}\right) &\approx \sum_{0\leq \ell\leq 10\ell_j}\frac{1}{\ell!}\left(-\frac{1}{2}\sum_{p\in I_j}\frac{a_f(p)\chi_d(p)}{\sqrt{p}}\right)^{\ell}\\ &= \sum_{\substack{p|n\Rightarrow p\in I_j\\ \Omega(n)\leq 10\ell_j}}\frac{\lambda(n)a_f(n)\chi_d(n)\nu(n)}{2^{\Omega(n)}\sqrt{n}}\end{aligned}$$

if

$$\left|\sum_{p\in I_j}\frac{a_f(p)\chi_d(p)}{\sqrt{p}}\right| \leq \ell_j.$$

Here $\Omega(n)$ is the number of prime divisor of a positive integer $n$ (counted with multiplicity), $\nu$ is a multiplicative function such that $\nu(p^{\alpha}) = 1/\alpha!$, and $\lambda(n) = (-1)^{\Omega(n)}$ is the Liouville function.

A concrete choice of parameters that suffices for us is the same one as in [21] (see also [6] for a similar choice in a little different context), namely

$$\theta_j := \eta_1\frac{e^j}{(\log\log K)^5} \qquad \text{and} \qquad \ell_j := 2\lfloor \theta_j^{-3/4}\rfloor,$$

where $\eta_1 > 0$ is sufficiently large absolute constant. We choose $J$ so that $\eta_2 \leq \theta_J \leq e\eta_2$ for sufficiently small constant $\eta_2 > 0$.

The discussion above motivates us to choose our mollifier as

$$M_g(d) := (\log K)^{1/4}\prod_{j=0}^{J} M_g(d;j), \tag{4.2}$$

where

$$M_g(d;j) := \sum_{\substack{p|n\Rightarrow p\in I_j\\ \Omega(n)\leq \ell_j}}\frac{a_f(n)\lambda(n)\nu(n)\chi_d(n)}{2^{\Omega(n)}\sqrt{n}}.$$

[4]More precisely, in the form saying that $e^z = (1 + O(e^{-9V}))\sum_{0\leq j\leq 10V} z^j/j!$ for $|z| \leq V$.

This mollifier is designed so that it behaves like $L(1/2, f\otimes\chi_d)^{-1/2}$ for typical $f\in B_k$ and fundamental discriminant $d$. Note that as $c>2$ it follows that $M_g(d;j)$ is supported only on odd $n$'s. Throughout the paper we set

$$\delta_0 := \sum_{j=0}^{J} \ell_j\theta_j$$

and note that $M_g(d) \ll K^{\delta_0}(\log K)^{1/4}$. By choosing $\eta_2$ to be small we can guarantee that $\delta_0$ is small enough. This will be important in Section 12.

The method used to prove Theorem 1.2 relies crucially on the estimation of certain mollified moments of quadratic twists of modular $L$-functions. The relevant results are the content of the following two propositions, which replace Lemmas 2.7. and 2.8. in [13].

**Proposition 4.1.** *Let $\phi$ and $h$ be smooth compactly supported functions on $\mathbb{R}_+$. Suppose that $X \ll \sqrt{K}$. Then we have*

$$\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{g\in B^+_{k+\frac12}} \alpha_g {\sum_d}^\flat |c_g(|d|)|^2 M_g(d)^2 \phi\left(\frac{|d|}{X}\right) \gg XK.$$

**Proposition 4.2.** *Let $\phi$ and $h$ be smooth compactly supported functions on $\mathbb{R}_+$. Suppose that $X \ll \sqrt{K}$. Then we have*

$$\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{g\in B^+_{k+\frac12}} \alpha_g^2\omega_g^{-1} {\sum_d}^\flat |c_g(|d|)|^4 M_g(d)^4 \phi\left(\frac{|d|}{X}\right) \ll XK. \tag{4.3}$$

In the latter proposition and throughout the paper we write as an abuse of notation $\omega_g$ for $\omega_f$ when $f$ is the Shimura lift of $g$.

Given these the proof of Theorem 1.2 can be completed similarly as in [13]. This deduction is achieved in Section 12.

4.2. **Evaluating the fourth moment.** The proof of Proposition 4.2 warrants some comments. But before that, one easily observes using the complete multiplicativity that for any $\ell\in\mathbb{N}$,

$$M_g(d)^{2\ell} = (\log K)^{\ell/2} \sum_{n\le K^{2\ell\delta_0}} \frac{h_\ell(n)a_f(n)\lambda(n)\chi_d(n)}{2^{\Omega(n)}\sqrt{n}}. \tag{4.4}$$

Here

$$h_\ell(n) := \sum_{\substack{n_0\cdots n_J=n\\ p|n_j\Rightarrow p\in I_j\,\forall\, 0\le j\le J\\ \Omega(n_j)\le 2\ell\ell_j\,\forall\, 0\le j\le J}} \nu_{2\ell}(n_0;\ell_0)\cdots\nu_{2\ell}(n_J;\ell_J)$$

and

$$\nu_r(n;\ell) := \sum_{\substack{n_1\cdots n_r=n\\ \Omega(n_j)\le\ell\,\forall\, 1\le j\le r}} \nu(n_1)\cdots\nu(n_r).$$

Let $\nu_j(n) := (\nu*\cdots*\nu)(n)$ denote the $j$-fold convolution of $\nu$. Note that $\nu_r(n;\ell)\le\nu_r(n)$ and that if $\Omega(n)\le\ell$, then $\nu_r(n;\ell)=\nu_r(n)$.

After opening $M_g(d)^4$ using (4.4) and using Waldspurger's formula to express $|c_g(|d|)|^2$ as a central value of an $L$-function, in order to prove Proposition 4.2 we would require an asymptotic evaluation of the multiple average

$$\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{f\in B_k} \omega_f {\sum_d}^\flat L\left(\frac12, f\otimes\chi_d\right)^2 a_f(r)\chi_d(r)\phi\left(\frac{|d|}{X}\right).$$

This can be done, but applying the resulting formula to the left-hand side of (4.3) results with an unwieldy expression for the main term, which is hard to evaluate. Instead we introduce a random $L$-function in which the Hecke eigenvalues $\lambda_f(m)$ are modelled by random variables $X(m)$ (defined shortly), and then match our expression with the random analogue. Comparison with a random model allows us to sidestep a number of technical points that would otherwise require involved effort to resolve. In particular, using the independence of the random variables $\{X(p)\}_p$ at the primes reduces many of the computations to "local" ones evaluated at each prime.

## 5. Preliminaries

5.1. **Half-integral weight forms.** Let $S_{k+\frac{1}{2}}(4)$ denote the space of holomorphic cusp forms of weight $k+\frac{1}{2}$ for the Hecke congruence group $\Gamma_0(4)$. Any such form $g$ has a Fourier expansion of the form

$$g(z) = \sum_{n=1}^{\infty} c_g(n) n^{\frac{k}{2}-\frac{1}{4}} e(nz), \tag{5.1}$$

where $c_g(n)$ are the Fourier coefficients of $g$.

For $g, h \in S_{k+\frac{1}{2}}(4)$, we define the Petersson inner product $\langle g, h\rangle$ to be

$$\langle g, h\rangle := \int_{\Gamma_0(4)\backslash\mathbb{H}} g(z)\overline{h}(z) y^{k+\frac{1}{2}} \frac{\mathrm{d}x\,\mathrm{d}y}{y^2}. \tag{5.2}$$

For any odd prime $p$ there exists a Hecke operator $T(p^2)$ acting on the space of half-integral weight modular forms. We call a half-integral weight cusp form a Hecke cusp form if $T(p^2)g = \gamma_g(p)g$ for all $p > 2$ for some $\gamma_g(p) \in \mathbb{C}$.

The Kohnen plus subspace $S^+_{k+\frac{1}{2}}(4) \subset S_{k+\frac{1}{2}}(4)$ consists of all weight $k+\frac{1}{2}$ Hecke cusp forms whose $n^{\text{th}}$ Fourier coefficient vanishes whenever $(-1)^k n \equiv 2, 3 \pmod 4$. This space has a basis consisting of simultaneous eigenfunctions of $T(p^2)$ for odd $p$. It is well-known that, as $k \longrightarrow \infty$, asymptotically one third of half-integral weight cusp forms lie in the Kohnen plus space.

Kohnen proved [18] that there exists a Hecke algebra isomorphism between $S^+_{k+\frac{1}{2}}(4)$ and the space of level 1 cusp forms of weight $2k$. That is, $S^+_{k+\frac{1}{2}}(4) \simeq H_{2k}$ as Hecke modules. Also recall that every Hecke cusp form $g \in S^+_{k+\frac{1}{2}}(4)$ can be normalised so that it has real Fourier coefficients and throughout the article we assume that $g$ has been normalised in this way.

The proof of our main result uses the explicit form of Waldspurger's formula due to Kohnen and Zagier [17].

**Lemma 5.1.** *For a Hecke cusp form $g \in S^+_{k+\frac{1}{2}}(4)$ we have*

$$|c_g(|d|)|^2 = L\left(\frac{1}{2}, f \otimes \chi_d\right) \cdot \frac{(k-1)!}{\pi^k} \cdot \frac{\langle g, g\rangle}{\langle f, f\rangle} \tag{5.3}$$

*for each fundamental discriminant $d$ with $(-1)^k d > 0$, where $f$ is a holomorphic modular form attached to $g$ via the Shimura correspondence, normalised so that $\lambda_f(1) = 1$.*

**Remark 5.2.** Using the normalisation (2.1) the above formula can be written in the form

$$\alpha_g |c_g(|d|)|^2 = \omega_f L\left(\frac{1}{2}, f \otimes \chi_d\right).$$

We also remark that $L(1/2, f \otimes \chi_d)$ vanishes when $(-1)^k d < 0$ due to the sign in the functional equation. It follows directly from (5.3) that $L(1/2, f \otimes \chi_d) \geq 0$ otherwise. Here for $f_1, f_2 \in H_{2k}$ the Petersson inner product, which is still denoted by $\langle f_1, f_2\rangle$, is defined to be

$$\langle f_1, f_2\rangle := \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} f_1(z)\overline{f_2}(z) y^{2k} \frac{\mathrm{d}x\,\mathrm{d}y}{y^2}.$$

We write $\|f\|_2^2 := \langle f, f\rangle$.

Using the Hecke relations we have for any $\alpha \in \mathbb{N}$ and prime $p$ that

$$\lambda_f(p)^\alpha = \sum_{c=0}^{\alpha} h_\alpha(c)\lambda_f(p^c),$$

where $h_\alpha(c)$ are non-negative integers given by

$$h_\alpha(c) := \frac{2}{\pi}\int_0^\pi (2\cos\theta)^\alpha \sin((c+1)\theta)\sin\theta \,\mathrm{d}\theta.$$

We call these Chebyshev coefficients. Recall that $a_f(n) = \prod_{p^\alpha || n} \lambda_f(p)^\alpha$. By the repeated use of Hecke relations it is easy to see that

$$a_f(n) = \sum_{u|n} c_n(u)\lambda_f(u), \tag{5.4}$$

where

$$c_n(u) := \prod_{\substack{p^c||u \\ p^\alpha||n}} h_\alpha(c).$$

Later in the paper we require some values of $c_n(u)$ with $n$ (and $u$) powers of a fixed prime. These are easily computed and listed here:

| | | | | |
|---|---|---|---|---|
| $c_1(1) = h_0(0) = 0$ | $c_p(1) = h_1(0) = 0$ | $c_p(p) = h_1(1) = 1$ | $c_{p^2}(1) = h_2(0) = 1$ | $c_{p^2}(p) = h_2(1) = 0$ |
| $c_{p^2}(p^2) = h_2(2) = 1$ | $c_{p^3}(1) = h_3(0) = 0$ | $c_{p^3}(p) = h_3(1) = 2$ | $c_{p^3}(p^2) = h_3(2) = 0$ | $c_{p^3}(p^3) = h_3(3) = 1$ |

Table 1. Certain values of $c_n(u)$.

We also note the easy bounds $0 \leq h_\alpha(c) \leq 2^{\alpha+1}$ from which we immediately deduce the trivial bound $c_n(u) \leq 2^{\Omega(n)}$.

An approximate functional equation also plays a key role in our work. The following is an easy modification of [25, Lemma 5].

**Lemma 5.3.** *Let $f$ be a Hecke cusp form of weight $2k$ for the full modular group $\mathrm{SL}_2(\mathbb{Z})$. Then for any fundamental discriminant $d$ for which $(-1)^k d > 0$ we have that*

$$L\left(\frac{1}{2}, f\otimes\chi_d\right) = 2\sum_{m=1}^{\infty} \frac{\lambda_f(m)\chi_d(m)}{\sqrt{m}} V_k\left(\frac{m}{|d|}\right),$$

*where, for any $\sigma > 1/2$,*

$$V_k(x) := \frac{1}{2\pi i}\int_{(\sigma)} \eta(s)x^{-s}e^{s^2}\frac{\mathrm{d}s}{s} \quad \textit{with} \quad \eta(s) := (2\pi)^{-s}\frac{\Gamma(s+k)}{\Gamma(k)}. \tag{5.5}$$

*Furthermore, we have that*

$$V_k(\xi) = 1 + O\left(\left(\frac{\xi}{k}\right)^{1/2-o(1)}\right) \tag{5.6}$$

*as $\xi \longrightarrow 0$.*

*We also have the estimates*

$$V_k(\xi) \ll_A \left(\frac{k}{\xi}\right)^A,$$

$$V_k^{(B)}(\xi) \ll_{A,B} \xi^{-B}\left(\frac{k}{\xi}\right)^A$$

*for any $A \geq 1$ and integer $B \geq 0$.*

*In addition, using Stirling's formula there exists a holomorphic function $R(s,k)$ so that for* $\mathrm{Re}(s) \geq -k/2$ *we have $R(s,k) \ll |s|^2/k$ and*

$$\frac{\Gamma(s+k)}{\Gamma(k)} = k^s e^{R(s,k)} \left(1 + O\left(k^{-1}\right)\right). \tag{5.7}$$

*In particular, for* $|\mathrm{Re}(s)| \leq \sqrt{k}$ *we have*

$$\frac{\Gamma(s+k)}{\Gamma(k)} = k^s \left(1 + O\left(\frac{|s|^2}{k}\right)\right).$$

Note also that $V_k = V_{Ku+1}$ for $u = (k-1)/K$.

To see that (5.7) holds, note that for $|\arg(s)| \leq \pi - \delta$, Stirling's formula gives

$$\Gamma(s) = \sqrt{\frac{2\pi}{s}} \left(\frac{s}{e}\right)^s \left(1 + O\left(\frac{1}{|s|}\right)\right),$$

where the implied constant depends at most on $\delta$. Hence for $\mathrm{Re}(s) \geq -k/2$ we have that

$$\begin{aligned}\frac{\Gamma(k+s)}{\Gamma(k)} &= \left(1 + O\left(\frac{1}{k}\right)\right) \sqrt{\frac{k}{k+s}} \left(\frac{k}{e}\right)^{-k} \left(\frac{k+s}{e}\right)^{k+s} \\ &= \left(1 + O\left(\frac{1}{k}\right)\right) k^s e^{-s} \left(1 + \frac{s}{k}\right)^{k+s-\frac{1}{2}}.\end{aligned}$$

For $\mathrm{Re}(z) > -1/2$, we have that $\log(1+z) = z + O(|z|^2)$. Hence, for $\mathrm{Re}(s) > -k/2$ we have that

$$\begin{aligned}\left(1 + \frac{s}{k}\right)^{k+s-\frac{1}{2}} &= \exp\left((k + s - \tfrac{1}{2})\left(\frac{s}{k} + O\left(\frac{|s|^2}{k^2}\right)\right)\right) \\ &= \exp\left(s + O\left(\frac{|s|^2}{k^2}\right)\right).\end{aligned}$$

Thus we conclude that for $\mathrm{Re}(s) \geq -k/2$ we have that

$$\frac{\Gamma(k+s)}{\Gamma(k)} = k^s \exp\left(O\left(\frac{|s|^2}{k^2}\right)\right)\left(1 + O\left(\frac{1}{k}\right)\right),$$

as desired.

### 5.2. **Basic tools.**

5.2.1. *Summation formulas.* One of the most important tools is the Petersson trace formula.

**Lemma 5.4.** *Let $m$ and $n$ be natural numbers, and $k$ be a positive integer. Then*

$$\sum_{f \in \mathcal{B}_k} \omega_f \lambda_f(m) \lambda_f(n) = 1_{m=n} + 2\pi i^{2k} \sum_{c=1}^{\infty} \frac{S(m,n;c)}{c} J_{2k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right),$$

*where $S(m,n;c)$ is the usual Kloosterman sum and $J_\nu$ is the $J$-Bessel function.*

Let us now define two integral transforms. For a smooth compactly supported function $h$, we set

$$\hbar(y) := \int_0^\infty \frac{h(\sqrt{u})}{\sqrt{2\pi u}} e^{iyu} \,\mathrm{d}u$$

and

$$W_K^{(2)}(m,n,v) := \int_0^\infty \frac{V_{\sqrt{u}K+1}(m) V_{\sqrt{u}K+1}(n) h(\sqrt{u})}{\sqrt{2\pi u}} e^{iuv} \,\mathrm{d}u.$$

Main properties of these integral transforms have been worked out by Khan[5]. He has shown [16, (2.17)] that

$$W_K^{(2)}(n,m,v) \ll_{A_1,A_2,B} \left(\frac{K}{n}\right)^{A_1}\left(\frac{K}{m}\right)^{A_2} v^{-B} \tag{5.8}$$

for any $A_1, A_2 > 0$ and $B \geq 0$. Thus $W_K^{(2)}(n,m,v)$ is essentially supported on $n \leq K^{1+\varepsilon}$, $m \leq K^{1+\varepsilon}$ and $v \leq K^{\varepsilon}$. Moreover, using integration by parts we have estimates for the derivatives;

$$\frac{\partial^j}{\partial \xi^j} W_K^{(2)}\left(\frac{\xi}{|d|}, \frac{z}{|d|}, \frac{K^2 c}{8\pi\sqrt{\xi z}}\right) \ll_{j,A_1,A_2,B} \left(\frac{|d|K}{\xi}\right)^{A_1} \xi^{-j} \left(\frac{|d|K}{z}\right)^{A_2} \left(\frac{\sqrt{\xi z}}{K^2 c}\right)^{B} K^{\varepsilon} \tag{5.9}$$

for any $j \geq 0$, $A_1, A_2, B > 0$.

We also have the identity [16, (2.24)]

$$\begin{aligned} &2\sum_{k\equiv 0\,(2)} i^k h\left(\frac{k-1}{K}\right) V_k(n)V_k(m)J_{k-1}(t) \\ &= -\frac{K}{\sqrt{t}}\mathrm{Im}\left(e^{-2\pi i/8}e^{it}W_K^{(2)}\left(n,m,\frac{K^2}{2t}\right)\right) + O\left(\frac{t}{K^4}\int_{\mathbb{R}} v^4 \left|\int_0^\infty V_{uK+1}(n)V_{uK+1}(m)h(u)e^{iuv}\,\mathrm{d}u\right| \mathrm{d}v\right). \end{aligned} \tag{5.10}$$

A simple application of the Poisson summation gives

$$\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) = \frac{K}{2}\widehat{h}(0) + O_B(K^{-B})$$

for every $B > 0$.

For real $\xi_1 > 0$ and $\xi_2 > 0$ we define

$$W(\xi_1,\xi_2,v) := \frac{1}{(2\pi i)^2}\int_{(A_1)}\int_{(A_2)} (2\pi)^{-x-y}e^{x^2+y^2}\xi_1^{-x}\xi_2^{-y}\hbar_{x+y}(v)\,\frac{\mathrm{d}x\,\mathrm{d}y}{xy},$$

where

$$\hbar_z(v) := \int_0^\infty \frac{h(\sqrt{u})}{\sqrt{2\pi u}} u^{z/2} e^{iuv}\,\mathrm{d}u.$$

Then

$$W_K^{(2)}(m,n,v) = W\left(\frac{m}{K}, \frac{n}{K}, v\right) + O_\varepsilon\left(K^{-1+\varepsilon}\right). \tag{5.11}$$

Integrating by parts shows that $\hbar_z(v) \ll_{\mathrm{Re}(z),B} (1+|z|)^B v^{-B}$ and consequently

$$W(\xi_1,\xi_2,v) \ll_{B,A_1,A_2} \xi_1^{-A_1}\xi_2^{-A_2}v^{-B}$$

for $A_1, A_2 > 0$ and $B \geq 0$.

The treatment of the off-diagonal in the fourth moment computation requires the following auxiliary result [16, Lemma 3.3.] concerning the properties of the Mellin transform of the function $\hbar_z$.

**Lemma 5.5.** *For $0 < \mathrm{Re}(s) < 1$, we have*

$$\widetilde{\hbar}_z(s) = \int_0^\infty \frac{h(\sqrt{u})}{\sqrt{2\pi u}} u^{z/2-s}\Gamma(s)\left(\cos\left(\frac{\pi s}{2}\right) + i\sin\left(\frac{\pi s}{2}\right)\right)\mathrm{d}u \tag{5.12}$$

*and the bound $\widetilde{\hbar}_z(s) \ll_{\mathrm{Re}(z)} (1+|z|)^3|s|^{-2}$.*

[5] In [16] the definition of $V_k(\xi)$ differs slightly from ours, but identical arguments to those in [16] lead to same bounds for expressions involving $V_k(\xi)$ in our situation.

*For $0 < c < 1$ we have*

$$\hbar_z(v) = \frac{1}{2\pi i} \int\limits_{(c)} v^{-s} \widetilde{\hbar}_z(s)\, \mathrm{d}s. \tag{5.13}$$

Another key tool is the Poisson summation formula.

**Lemma 5.6.** *Let $f$ be a Schwartz function and $a$ be a residue class modulo $c$. Then*

$$\sum_{n \equiv a\,(c)} f(n) = \frac{1}{c} \sum_{n} \widehat{f}\left(\frac{n}{c}\right) e\left(\frac{an}{c}\right),$$

*where $\widehat{f}$ denotes the Fourier transform of $f$. Note that this reduces to the classical Poisson summation formula when $c = 1$.*

We shall also need a different variant of the Poisson summation formula. For this, let us define, for any $j \in \mathbb{Z}$, a Gauss-type sum

$$\tau_j(n) := \sum_{b\,(n)} \left(\frac{n}{b}\right) e\left(\frac{jb}{n}\right) = \left(\frac{(1+i)}{2} + \left(\frac{-1}{n}\right)\frac{(1-i)}{2}\right) G_j(n), \tag{5.14}$$

where

$$G_j(n) := \left(\frac{(1-i)}{2} + \left(\frac{-1}{n}\right)\frac{(1+i)}{2}\right) \sum_{a\,(n)} \left(\frac{n}{a}\right) e\left(\frac{aj}{n}\right).$$

It can be shown that for odd coprime $m, n$ one has $G_j(mn) = G_j(m)G_j(n)$ and if $p^\alpha \| j$ for a prime $p$, then the exact formula for $G_j\left(p^\beta\right)$ is given in [29, Lemma 2.3]. The variant of Poisson summation formula needed for our purposes is contained in the following lemma.

**Lemma 5.7.** *([25, Lemma 7.]) Let $n$ be an odd integer and $q$ positive integer so that $(n, q) = 1$. Suppose that $F$ is a smooth and compactly supported function on $\mathbb{R}$. Finally, let $\eta$ be a reduced residue class modulo $q$. Then*

$$\sum_{d \equiv \eta\,(q)} \left(\frac{d}{n}\right) F(d) = \frac{1}{qn} \left(\frac{q}{n}\right) \sum_{\ell \in \mathbb{Z}} \widehat{F}\left(\frac{\ell}{nq}\right) e\left(\frac{\ell \eta \overline{n}}{q}\right) \tau_\ell(n),$$

*where $\widehat{F}$ is the usual Fourier transform.*

The Gauss-type sum in the previous lemma may be evaluated explicitly.

**Lemma 5.8.** *([29, Lemma 2.3.]) If $m$ and $n$ are coprime odd integers, then $\tau_\ell(mn) = \tau_\ell(m)\tau_\ell(n)$. Furthermore, if $p^\alpha$ is the highest power of $p$ that divides $\ell$ (setting $\alpha = \infty$ if $\ell = 0$), then*

$$\tau_\ell\left(p^\beta\right) = \begin{cases} 0 & \text{if } \beta \le \alpha \text{ is odd} \\ \varphi(p^\beta) & \text{if } \beta \le \alpha \text{ is even} \\ -p^\alpha & \text{if } \beta = \alpha + 1 \text{ is even} \\ \left(\frac{\ell p^{-\alpha}}{p}\right) p^\alpha \sqrt{p} & \text{if } \beta = \alpha + 1 \text{ is odd} \\ 0 & \text{if } \beta \ge \alpha + 2 \end{cases} \tag{5.15}$$

5.3. **Other tools.** We also record the following well-known uniform estimate for the $J$-Bessel function [11, (2.11")]. For $\nu \ge 0$ and $x > 0$, the $J_\nu$-Bessel function satisfies the bound

$$J_\nu(x) \ll \frac{x}{\sqrt{\nu + 1}} \left(\frac{ex}{2\nu + 1}\right)^\nu. \tag{5.16}$$

Another crucial auxiliary result is the stationary phase method for estimating oscillatory exponential integrals. We quote the following special case of a result [5, Proposition 8.2.] by Blomer, Khan, and Young that is uniform with respect to multiple parameters (see also [30, Lemma 5.6.]).

**Lemma 5.9.** *Let $X, Y, V, V_1, Q > 0$ and $Z := Q + X + Y + V_1 + 1$, and assume that*

$$Y \geq Z^{3/20}, \qquad V_1 \geq V \geq \frac{QZ^{1/40}}{Y^{1/2}}.$$

*Suppose that $h$ is a smooth function on $\mathbb{R}$ with support on an interval $J$ of length $V_1$, satisfying*

$$h^{(j)}(t) \ll_j XV^{-j}$$

*for all $j \in \mathbb{N} \cup \{0\}$. Suppose that $f$ is a smooth function on $J$ such that there exists a unique point $t_0 \in J$ such that $f'(t_0) = 0$, and furthermore*

$$f''(t) \gg YQ^{-2}, \quad f^{(j)}(t) \ll_j YQ^{-j} \quad \text{for all } j \geq 1 \text{ and } t \in J.$$

*Then*

$$\int_{\mathbb{R}} h(t)e(f(t))\,\mathrm{d}t = e^{sgn(f''(t_0))\pi i/4} \frac{e(f(t_0))}{\sqrt{|f''(t_0)|}} h(t_0) + O\left(\frac{Q^{3/2}X}{Y^{3/2}} \cdot \left(V^{-2} + (Y^{2/3}/Q^2)\right)\right).$$

## 6. Random mollifier

6.1. **Random model.** We need a probabilistic random model for the Hecke eigenvalues $\lambda_f(m)$, which we now introduce following [19]. To motivate the model, let $G^\sharp$ denote the set of conjugacy classes of $G = \mathrm{SU}(2)$ endowed with the Sato–Tate measure. Note that if $m > 1$ has the prime factorisation $m = p_1^{\alpha_1} \cdots p_\ell^{\alpha_\ell}$, then we have

$$\lambda_f(m) = \prod_{j=1}^{\ell} \lambda_f\left(p_j^{\alpha_j}\right) = \prod_{j=1}^{\ell} \mathrm{Tr}\left(\mathrm{Sym}^{\alpha_j}(g_f(p_j))\right),$$

where

$$g_f(p) := \begin{pmatrix} e^{i\alpha_p} & \\ & e^{i\beta_p} \end{pmatrix}$$

with $\alpha_p, \beta_p$ being the Satake parameters of $f$ at a prime $p$. We write $g_f^\sharp(p)$ for the conjugacy class of the matrix $g_f(p)$. Then it can be shown that for a fixed set of primes $\{p_1, ..., p_\ell\}$, the $\ell$-tuple $\{g_f^\sharp(p_1), ..., g_f^\sharp(p_\ell)\}_{f \in B_k}$ of conjugacy classes equidistributes inside $(G^\sharp)^\ell$ as $k \longrightarrow \infty$.

This suggests the following random model for $\lambda_f(m)$. Let $\{g_p^\sharp\}_p$ be a sequence of independent random variables with values in $G^\sharp$ that are distributed according to the Sato–Tate measure. Then we define $X(1) = 1$ and

$$X(m) := \prod_{j=1}^{\ell} \mathrm{Tr}\left(\mathrm{Sym}^{\alpha_j}(g_{p_j}^\sharp)\right)$$

for $m = p_1^{\alpha_1} \cdots p_\ell^{\alpha_\ell}$.

We note that

$$X(m)X(n) = \sum_{d|(m,n)} X\left(\frac{mn}{d^2}\right) \tag{6.1}$$

and $\mathbb{E}(X(m)) = 1_{m=1}$. By linearity of the expectation these give

$$\mathbb{E}(X(m)X(n)) = \sum_{\substack{d|(m,n) \\ mn=d^2}} 1 = 1_{m=n}. \tag{6.2}$$

6.2. **Construction of a random mollifier.** Let $X$ be the random variable as above. Then we define the random mollifier

$$M_1(X;d) := (\log K)^{1/2} \prod_{0\le j\le J} M_{1,j}(X;d),$$

where

$$M_{1,j}(X;d) := \sum_{\substack{p|n\Rightarrow p\in I_j\\ \Omega(n)\le 2\ell_j}} \frac{\lambda(n)\chi_d(n)\nu_2(n;\ell_j)}{2^{\Omega(n)}\sqrt{n}} \sum_{u|n} c_n(u)X(u).$$

Note that we may also write

$$M_1(X;d) = (\log K)^{1/2} \sum_{n\le K^{2\delta_0}} \frac{\lambda(n)\chi_d(n)h_1(n)}{2^{\Omega(n)}\sqrt{n}} \sum_{u|n} c_n(u)X(u).$$

This is a random counterpart for $M_g(d)^2$ as can be seen by comparing this to (4.4) for $\ell = 1$.

## 7. Character sum

In the proof of Proposition 4.2 we require the evaluation of a certain character sum. In this section we achieve this task. In order to do this the following result is needed.

**Lemma 7.1.** ([2, Theorem 2.1.2.]) *Let $p$ be an odd prime and $a, b, c$ be integers with $p \nmid a$. Then*

$$\sum_{x\,(p)} \left(\frac{ax^2+bx+c}{p}\right) = \begin{cases} -\left(\frac{a}{p}\right) & \text{if } b^2-4ac \not\equiv 0\,(p) \\ (p-1)\left(\frac{a}{p}\right) & \text{if } b^2-4ac \equiv 0\,(p) \end{cases}$$

The main result of this section is the following.

**Proposition 7.2.** *Let $d$ be an odd squarefree integer and let $c, u, v,$ and $\eta$ be natural numbers so that $v\eta = u[c,d]^2/c^2$. Then the sum*

$$\sum_{x\,([c,d])} \sum_{w\,([c,d])} \chi_d(x)\chi_d(w) S\left(xw, u; c\right) e\left(\frac{xv+w\eta}{[c,d]}\right)$$

*vanishes unless*[6] *$d|c$, in which case it equals*

$$\frac{c\cdot\varphi(c)}{\varphi(d)} \chi_d(u)(-1)^{\#\{p|d\}} \prod_{\substack{p|d\\ p|\frac{c}{d}}} (1-p),$$

*where $\#\{p|d\}$ is the number of primes dividing $d$.*

*Proof.* Suppose first that $d \nmid c$. Then there exists a prime $p|d$ for which $p \nmid c$. From the assumption $v\eta = u[c,d]^2/c^2$ it follows that $p|v\eta$. By symmetry we may assume that $p|v$.

After opening the Kloosterman sum and rearranging our task is to evaluate

$$\sideset{}{^\star}\sum_{\gamma\,(c)} \sum_{w\,([c,d])} \chi_d(w) e\left(\frac{u\overline{\gamma}}{c} + \frac{w\eta}{[c,d]}\right) \sum_{x\,([c,d])} \chi_d(x) e\left(\frac{xw\gamma}{c} + \frac{xv}{[c,d]}\right). \tag{7.1}$$

Let us focus on the inner sum. We compute

$$\begin{aligned}\sum_{x\,([c,d])} \chi_d(x) e\left(\frac{xw\gamma}{c} + \frac{xv}{[c,d]}\right) &= \sum_{h=0}^{[c,d]/d-1} \sum_{\ell=0}^{d-1} \chi_d(hd+\ell) e\left((hd+\ell)\left(\frac{w\gamma}{c} + \frac{v}{[c,d]}\right)\right) \\ &= \sum_{\ell=0}^{d-1} \chi_d(\ell) e\left(\ell\left(\frac{w\gamma}{c} + \frac{v}{[c,d]}\right)\right) \sum_{h=0}^{[c,d]/d-1} e\left(hd\left(\frac{w\gamma}{c} + \frac{v}{[c,d]}\right)\right).\end{aligned}$$

[6]Note also that the condition $d|c$ forces $v\eta = u$.

Clearly the inner $h$-sum is given by

$$\begin{cases} \frac{[c,d]}{d} & \text{if } w\gamma \cdot \frac{d}{(c,d)} + v \equiv 0 \left(\frac{[c,d]}{d}\right) \\ 0 & \text{otherwise} \end{cases}$$

The $\ell$-sum is an usual Gauss sum and it evaluates to

$$\varepsilon_d \sqrt{d} \chi_d \left( \frac{w\gamma \cdot \frac{d}{(c,d)} + v}{[c,d]/d} \right)$$

using the formula $(c,d) \cdot [c,d] = cd$.

Let us now write

$$w\gamma \cdot \frac{d}{(c,d)} + v = h \cdot \frac{[c,d]}{d}$$

for some integer $h$. By the assumption made in the beginning of the proof, both summands on the left-hand side are divisible by $p$. On the other hand, by construction $p$ does not divide the number $[c,d]/d$ as $d$ is squarefree. Thus $p|h$. But in this case

$$\chi_d \left( \frac{w\gamma \cdot \frac{d}{(c,d)} + v}{[c,d]/d} \right) = \chi_d(h) = 0$$

as $p|d$, and so the whole sum vanishes.

Thus we are reduced to the case $d|c$. Note that in this case $[c,d] = c$ and $(c,d) = d$. All the previous computations are still valid so taking into account the evaluation of the $h$- and $\ell$-sums it follows that (7.1) is given by

$$\varepsilon_d \frac{c}{\sqrt{d}} \sum_{\gamma\,(c)}^{\star} \sum_{w\,(c)} \chi_d(w) e\left( \frac{v\eta\overline{\gamma} + w\eta}{c} \right) \chi_d \left( \frac{d(w\gamma + v)}{c} \right) \cdot 1_{w\gamma + v \equiv 0\,(c/d)}.$$

Note that writing $w\gamma + v = h \cdot \frac{c}{d}$, $w$-sum can be written as

$$\sum_{h\,(d)} \chi_d \left( \overline{\gamma} \left( h \cdot \frac{c}{d} - v \right) \right) \chi_d(h) e\left( \frac{\eta h \overline{\gamma}}{d} \right),$$

and so at this point (7.1) is given by

$$\varepsilon_d \cdot \frac{c}{\sqrt{d}} \sum_{\gamma\,(c)}^{\star} \chi_d(\overline{\gamma}) \sum_{h\,(d)} \chi_d \left( h \left( h \cdot \frac{c}{d} - v \right) \right) e\left( \frac{\eta h \overline{\gamma}}{d} \right).$$

To treat the subtraction inside a multiplicative character we write, using [10, (3.12)],

$$\chi_d \left( h \cdot \frac{c}{d} - v \right) = \frac{1}{\tau(\chi_d)} \sum_{b\,(d)} \chi_d(b) e\left( \frac{b\left(h \cdot \frac{c}{d} - v\right)}{d} \right).$$

Thus

$$\sum_{h\,(d)} \chi_d \left( h \left( h \cdot \frac{c}{d} - v \right) \right) e\left( \frac{\eta h \overline{\gamma}}{d} \right) = \sum_{b\,(d)} \chi_d(b) e\left( -\frac{bv}{d} \right) \frac{1}{\tau(\chi_d)} \sum_{h\,(d)} \chi_d(h) e\left( \frac{h\left(b \cdot \frac{c}{d} + \eta\overline{\gamma}\right)}{d} \right).$$

The inner Gauss sum equals

$$\varepsilon_d \sqrt{d} \chi_d \left( b \cdot \frac{c}{d} + \eta\overline{\gamma} \right)$$

and we also have $\tau(\chi_d)=\varepsilon_d\sqrt{d}$. At this point our sum is given by

$$\begin{aligned}&\varepsilon_d\frac{c}{\sqrt{d}}\sideset{}{^\star}\sum_{\gamma\,(c)}\chi_d(\overline{\gamma})\sum_{b\,(d)}\chi_d\left(b^2\cdot\frac{c}{d}+b\eta\overline{\gamma}\right)e\left(-\frac{bv}{d}\right)\\&=\varepsilon_d\frac{c}{\sqrt{d}}\sum_{b\,(d)}e\left(-\frac{bv}{d}\right)\sideset{}{^\star}\sum_{\gamma\,(c)}\chi_d\left(b\eta\gamma^2+b^2\cdot\frac{c}{d}\gamma\right).\end{aligned}$$

Now we use the complete multiplicativity of $\chi_d$ and the Chinese remainder theorem as well as and apply Lemma 7.1 to the inner sum to see that

$$\begin{aligned}\sideset{}{^\star}\sum_{\gamma\,(c)}\chi_d\left(b\eta\gamma^2+b^2\cdot\frac{c}{d}\gamma\right)&=\frac{\varphi(c)}{\varphi(d)}\sideset{}{^\star}\sum_{\gamma\,(d)}\chi_d\left(b\eta\gamma^2+b^2\cdot\frac{c}{d}\gamma\right)\\&=\frac{\varphi(c)}{\varphi(d)}\chi_d(b\eta)(-1)^{\#\{p|d\}}\prod_{\substack{p|d\\p|\frac{c}{d}}}(1-p).\end{aligned}$$

Thus we have shown that the sum we started with can be written as

$$\varepsilon_d\frac{c\cdot\varphi(c)}{\varphi(d)\cdot\sqrt{d}}\chi_d(\eta)(-1)^{\#\{p|d\}}\prod_{\substack{p|d\\p|\frac{c}{d}}}(1-p)\sum_{b\,(d)}\chi_d(b)e\left(-\frac{bv}{d}\right)$$

when $d|c$.

The final remaining sum is again a Gauss sum that equals $\varepsilon_d\sqrt{d}\chi_d(v)$. This completes the proof recalling that $v\eta=u$ (for $d|c$) and the fact that $\varepsilon_d^2=1$. □

## 8. Proof of Proposition 4.1

We start with the following result concerning averages of quadratic characters. The proof is standard and similar calculations can be found e.g. in [25, 21, 14, 15].

**Lemma 8.1.** *Suppose that $n$ and $u$ are odd positive integers with $nu\le X^{1-\varepsilon}$. Let $\Phi$ denote a smooth and compactly supported function in $\mathbb{R}^+$. Then*

$$\sideset{}{^\flat}\sum_{d}\chi_d(nu)\Phi\left(\frac{|d|}{X}\right)V_k\left(\frac{u}{|d|}\right)=\frac{X\pi^2}{9}\left(\int_0^\infty\Phi(\xi)\,\mathrm{d}\xi\right)\prod_{p|nu}\left(1+\frac{1}{p}\right)^{-1}\cdot 1_{nu=\square}+O_\varepsilon\left(X^{3/4}k^{-1/4+\varepsilon}+X^{1/2+\varepsilon}\sqrt{nu}\right).$$

*Proof.* We pick out the property that $d$ is squarefree by the identity

$$\sum_{\substack{\alpha=1\\(\alpha,2)=1\\\alpha^2|d}}^\infty\mu(\alpha)=\begin{cases}1 & \text{if } d \text{ is squarefree}\\0 & \text{otherwise}\end{cases}\tag{8.1}$$

Note that the above identity holds without the condition $(\alpha,2)=1$, but this can be added as by construction $(d,2)=1$. Inserting this to the above expression gives that the $d$-sum is given by

$$\sum_{\substack{\alpha=1\\(\alpha,2nu)=1}}^\infty\mu(\alpha)\left(\frac{\alpha^2}{nu}\right)\sum_{r\equiv\overline{\alpha^2}\,(\mathrm{mod}\ 4)}\left(\frac{r}{nu}\right)V_k\left(\frac{u}{r\alpha^2}\right)\Phi\left(\frac{r\alpha^2}{X}\right).\tag{8.2}$$

Let us set $Y:=X^{1/2-\varepsilon}/\sqrt{nu}$. We split the $\alpha$-sum in (8.2) into two parts corresponding to $\alpha\le Y$ and $\alpha>Y$. For the latter terms we estimate the $r$-sum trivially and get the upper bound (recall that $nu\le X^{1-\varepsilon}$ and $r\alpha^2\asymp X$ so $V_k(u/r\alpha^2)\ll 1$ uniformly in $k$ in this range)

$$\ll\sum_{\alpha>Y}\frac{X}{\alpha^2}\ll\frac{X^{1+\varepsilon}}{Y}\ll X^{1/2+\varepsilon}\sqrt{nu}.$$

For the terms with $\alpha \le Y$ we will evaluate the $r$-sum by applying Lemma 5.7. The terms where $nu$ is a square will contribute the main term in the zero frequency term on the dual side and the rest will give the error term.

Using Lemma 5.8 the zero frequency contribution is given by

$$\frac{X}{2}\cdot\frac{1}{2\pi i}\int_{(c)}\left(\int_0^\infty \Phi(\xi)\xi^s \,\mathrm{d}\xi\right)\frac{\Gamma(s+k)}{\Gamma(k)}\left(\frac{X}{2\pi}\right)^s e^{s^2}\sum_{\substack{\alpha=1\\(\alpha,2u)=1}}^{\infty}\frac{\mu(\alpha)}{\alpha^2}u^{-s}\frac{\varphi(nu)}{nu}\cdot 1_{nu=\square}\frac{\mathrm{d}s}{s}, \tag{8.3}$$

where we have added back in the terms with $\alpha > Y$ at the cost of an error term of size $\ll_\varepsilon X^{1/2+\varepsilon}\sqrt{nu}$.

A simple computation shows that, recalling $nu$ is odd,

$$\sum_{\substack{\alpha=1\\(\alpha,2nu)=1}}^{\infty}\frac{\mu(\alpha)}{\alpha^2}\cdot\frac{\varphi(nu)}{nu} = \prod_{p\neq 2}\left(1-\frac{1}{p^2}\right)\prod_{p\mid nu}\left(1+\frac{1}{p}\right)^{-1}, \tag{8.4}$$

leading to

$$\frac{X}{2}\prod_{p\neq 2}\left(1-\frac{1}{p^2}\right)\prod_{p\mid nu}\left(1+\frac{1}{p}\right)^{-1}\cdot 1_{nu=\square}\cdot\frac{1}{2\pi i}\int_{(c)}\left(\int_0^\infty \Phi(\xi)\xi^s\,\mathrm{d}\xi\right)\frac{\Gamma(s+k)}{\Gamma(k)}\left(\frac{X}{2\pi}\right)^s e^{s^2}u^{-s}\frac{\mathrm{d}s}{s}. \tag{8.5}$$

The $s$-integrand in (8.5) extends to an analytic function in the above domain (apart from a simple pole at $s=0$). Thus moving the line of integration in (8.5) to the line $\mathrm{Re}(s) = -1/4+\varepsilon$ shows that the expression equals

$$\frac{X\pi^2}{9}\left(\int_0^\infty \Phi(\xi)\,\mathrm{d}\xi\right)\prod_{p\mid nu}\left(1+\frac{1}{p}\right)^{-1}\cdot 1_{nu=\square} + O_\varepsilon\left(X^{3/4}k^{-1/4+\varepsilon}\right),$$

where the main terms comes from the residue at $s=0$ and the error term from the contour shift.

It remains to bound the contribution coming from the other terms (i.e. $\ell\neq 0$) after applying Lemma 5.7. Let

$$F(y) := \Phi\left(\frac{y\alpha^2}{X}\right)V_k\left(\frac{u}{y\alpha^2}\right).$$

The sum we have to estimate after an application of Poisson summation is

$$\frac{1}{nu}\sum_{\substack{\alpha\le Y\\(\alpha,2nu)=1}}^{\infty}\mu(\alpha)\sum_{j\neq 0}\widehat{F}\left(\frac{Xj}{4nu\alpha^2}\right)e\left(\frac{j\overline{\alpha^2 nu}j}{4}\right)\tau_j(nu).$$

But in this case the $j$-sum is, say, $\ll X^{-1}$ due to the rapid decay of $\widehat{F}$ and using the trivial estimate $|\tau_\ell(nu)|\le nu$. Thus the whole sum is bounded by $\ll Y/X \ll X^{1/2+\varepsilon}\sqrt{nu}$, concluding the proof. $\square$

We now begin the computation of the second moment of $|c_g(|d|)|$. By Waldspurger's formula and (4.4) we have to estimate the average

$$\begin{aligned}&\sum_{k\in\mathbb{Z}}h\left(\frac{2k-1}{K}\right)\sum_{f\in B_k}\omega_f\sideset{}{^\flat}\sum_{d}L\left(\frac{1}{2},f\otimes\chi_d\right)\phi\left(\frac{|d|}{X}\right)M_f(d)^2\\ &= (\log K)^{1/2}\sum_{k\in\mathbb{Z}}h\left(\frac{2k-1}{K}\right)\sum_{f\in B_k}\omega_f\sideset{}{^\flat}\sum_{d}L\left(\frac{1}{2},f\otimes\chi_d\right)\phi\left(\frac{|d|}{X}\right)\sum_{n\le K^{2\delta_0}}\frac{h_1(n)a_f(n)\lambda(n)\chi_d(n)}{2^{\Omega(n)}\sqrt{n}}.\end{aligned}$$

Let us first use the approximate functional equation and then execute the $f$-sum. For the latter, note that using the definition of $a_f(n)$ and applying the Petersson formula we have

$$\begin{aligned}\sum_{f\in B_k}\lambda_f(m)a_f(n) &= \sum_{f\in B_k}\lambda_f(m)\sum_{u|n}c_n(u)\lambda_f(u)\\ &= \sum_{u|n}c_n(u)\cdot 1_{m=u}+\text{error},\end{aligned}$$

where the error term contributes

$$\begin{aligned}&(\log K)^{1/2}\sum_{k\in\mathbb{Z}}h\left(\frac{2k-1}{K}\right)i^{2k}\sum_d{}^{\flat}\phi\left(\frac{|d|}{X}\right)\sum_{m=1}^{\infty}\frac{\chi_d(m)}{\sqrt{m}}V_k\left(\frac{m}{|d|}\right)\\ &\times\sum_{n\le K^{2\delta_0}}\frac{h_1(n)\lambda(n)\chi_d(n)}{2^{\Omega(n)}\sqrt{n}}\sum_{u|n}c_n(u)\sum_{c=1}^{\infty}\frac{S(m,u;c)}{c}J_{2k-1}\left(\frac{4\pi\sqrt{mu}}{c}\right)\end{aligned}$$

to the original sum. This gives a negligible contribution by estimating trivially using Weil's bound for Kloosterman sums and the bound (5.16) as $mu\le mn\ll K^{1+2\theta_J+\varepsilon}X$ by recalling that $J$ was chosen so that $\eta_2\le\theta_J\le e\eta_2$ for some very small $\eta_2>0$ as well as $X\ll\sqrt{K}$.

The main term contributes

$$(\log K)^{1/2}\sum_{k\in\mathbb{Z}}h\left(\frac{2k-1}{K}\right)\sum_d{}^{\flat}\phi\left(\frac{|d|}{X}\right)\sum_{n\le K^{2\delta_0}}\frac{h_1(n)\lambda(n)\chi_d(n)}{2^{\Omega(n)}\sqrt{n}}\sum_{u|n}c_n(u)\frac{\chi_d(u)}{\sqrt{u}}V_k\left(\frac{u}{|d|}\right)$$

to the original sum.

Let us define a multiplicative function

$$\iota(n):=\sum_{\substack{u|n\\ nu=\square}}\frac{c_n(u)}{\sqrt{u}}.$$

At this point we evaluate the $d$-sum using Lemma 8.1.

Using this and the properties of $\nu_r(n;\ell)$ listed after (4.4) we see that the main term equals

$$\asymp XK(\log K)^{1/2}\prod_{0\le j\le J}\sum_{\substack{p|n\Rightarrow p\in I_j\\ \Omega(n)\le 2\ell_j}}\frac{\lambda(n)\iota(n)\nu_2(n)}{2^{\Omega(n)}\sqrt{n}}\prod_{p|\mathrm{rad}(n)}\left(1+\frac{1}{p}\right)^{-1}\tag{8.6}$$

after executing the $k$-sum using Poisson summation as the error term from Lemma 8.1 contributes $\ll_\varepsilon K^{3/4+\varepsilon}X^{3/4}+K^{1-\varepsilon}X^{1/2}=o(XK)$.

Next we estimate the individual terms in the product. By Rankin's trick we have

$$\begin{aligned}&\sum_{\substack{p|n\Rightarrow p\in I_j\\ \Omega(n)\le 2\ell_j}}\frac{\lambda(n)\iota(n)\nu_2(n)}{2^{\Omega(n)}\sqrt{n}}\prod_{p|\mathrm{rad}(n)}\left(1+\frac{1}{p}\right)^{-1}\\ &=\sum_{p|n\Rightarrow p\in I_j}\frac{\lambda(n)\iota(n)\nu_2(n)}{2^{\Omega(n)}\sqrt{n}}\prod_{p|\mathrm{rad}(n)}\left(1+\frac{1}{p}\right)^{-1}+O\left(\frac{1}{4^{\ell_j}}\sum_{p|n\Rightarrow p\in I_j}\frac{\lambda(n)\iota(n)\nu_2(n)}{\sqrt{n}}\prod_{p|\mathrm{rad}(n)}\left(1+\frac{1}{p}\right)^{-1}\right).\end{aligned}$$

Note that the error term is

$$\ll\frac{1}{4^{\ell_j}}\prod_{p\in I_j}\left(1+O\left(\frac{1}{p}\right)\right)\ll\frac{1_{j=0}\cdot(\log K)^{O(1)}+1}{4^{\ell_j}}.$$

Using multiplicativity we also have

$$\sum_{p|n \Rightarrow p \in I_j} \frac{\lambda(n)\iota(n)\nu_2(n)}{2^{\Omega(n)}\sqrt{n}} \prod_{p|\mathrm{rad}(n)} \left(1+\frac{1}{p}\right)^{-1}$$
$$= \prod_{p \in I_j} \left(1 - \frac{\iota(p)\nu_2(p)}{2\sqrt{p}}\left(1+\frac{1}{p}\right)^{-1} + \frac{\iota(p^2)\nu_2(p^2)}{4p}\left(1+\frac{1}{p}\right)^{-1} + O\left(\frac{1}{p^{3/2}}\right)\right).$$

Combining the previous estimates with the fact that (which follows from $\iota(p) = 1/\sqrt{p}$)

$$\prod_{p \in I_j} \left(1 - \frac{\iota(p)\nu_2(p)}{2\sqrt{p}}\left(1+\frac{1}{p}\right)^{-1} + \frac{\iota(p^2)\nu_2(p^2)}{4p}\left(1+\frac{1}{p}\right)^{-1} + O\left(\frac{1}{p^2}\right)\right)^{-1} \ll 1_{j=0} \cdot (\log K)^{O(1)} + 1$$

it follows that the main term equals

$$XK(\log K)^{1/2} \prod_{c<p\le K^{\theta_J}} \left(1 - \frac{\iota(p)\nu_2(p)}{2\sqrt{p}}\left(1+\frac{1}{p}\right)^{-1} + \frac{\iota(p^2)\nu_2(p^2)}{4p}\left(1+\frac{1}{p}\right)^{-1} + O\left(\frac{1}{p^2}\right)\right)$$
$$\times \prod_{0\le j\le J}\left(1+O\left(\frac{1_{j=0}\cdot(\log K)^{O(1)}+1}{4^{\ell_j}}\right)\right).$$

To estimate the latter product from below note that

$$\prod_{0\le j\le J}\left(1+O\left(\frac{1_{j=0}\cdot(\log K)^{O(1)}+1}{4^{\ell_j}}\right)\right) = 1 + O\left(\frac{1}{4^{\ell_J}}\right) \ge \frac{1}{2}$$

as $\eta_2$ was chosen to be sufficiently small.

Moreover, by Mertens' theorem the first product is bounded from below by

$$\prod_{c<p<K^{\theta_J}}\left(1-\frac{1}{2p}+O\left(\frac{1}{p^2}\right)\right) \gg (\log K)^{-1/2}$$

as $\iota(p) = 1/\sqrt{p}$, $\iota(p^2) = 1 + 1/p$, and $\nu_2(p) = \nu_2(p^2) = 2$.

Hence, using this we get that (8.6) is

$$\gg XK(\log K)^{1/2}(\log K)^{-1/2} \gg XK.$$

This finishes the proof of Proposition 4.1. □

## 9. Relating the Fourth moment to a random model

Here our argument combines ingredients from [6] and [16]. Indeed, in order to compute the mollified fourth moment many technical calculations simplify by relating the original sum to a random model. We begin with a lemma, which plays a key role in the proof of Proposition 4.2. Let us define a random Dirichlet series

$$L(X;d,k) := 4\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\frac{X(m_1)X(m_2)\chi_d(m_1m_2)}{\sqrt{m_1m_2}}V_k\left(\frac{m_1}{|d|}\right)V_k\left(\frac{m_2}{|d|}\right).$$

**Lemma 9.1.** *Let $n \le K^{4\delta_0}$ be an odd positive integer. Then we have*

$$\begin{aligned}
&\sum_d{}^{\flat}\chi_d(n)\phi\left(\frac{d}{X}\right)\sum_{k\in\mathbb{Z}}h\left(\frac{2k-1}{K}\right)\sum_{f\in B_k}\omega_f L\left(\frac{1}{2},f\otimes\chi_d\right)^2 a_f(n)\\
&=\sum_d{}^{\flat}\phi\left(\frac{|d|}{X}\right)\chi_d(n)\sum_{k\in\mathbb{Z}}h\left(\frac{2k-1}{K}\right)\mathbb{E}\left(L(X;d,k)\sum_{u|n}c_n(u)X(u)\right)\\
&\quad+2K\widehat{h}(0)\sum_d{}^{\flat}\frac{\varphi(d)}{d}\phi\left(\frac{|d|}{X}\right)\sum_{u|n}\frac{c_n(u)}{\sqrt{u}}\chi_d(nu)\cdot\frac{1}{2\pi i}\int_{(\frac{1}{2}+\varepsilon)}\frac{e^{2x^2}}{x^2}\cdot u^{-x}\sum_{v|u}v^{2x}\,\mathrm{d}x+O\left(X^{3/4}K^{1+\varepsilon}\right).
\end{aligned}$$

*Proof.* Using the approximate functional equation we have that the sum on the left-hand side is

$$\begin{aligned}
&4\sum_d{}^{\flat}\phi\left(\frac{|d|}{X}\right)\sum_{k\in\mathbb{Z}}h\left(\frac{2k-1}{K}\right)\\
&\qquad\times\sum_{f\in B_k}\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\frac{\lambda_f(m_1)\lambda_f(m_2)\chi_d(m_1m_2n)}{\sqrt{m_1m_2}}V_k\left(\frac{m_1}{|d|}\right)V_k\left(\frac{m_2}{|d|}\right)\sum_{u|n}c_n(u)\lambda_f(u).
\end{aligned}$$

Rearranging and using the Hecke relations this can be written as

$$\begin{aligned}
&4\sum_d{}^{\flat}\phi\left(\frac{|d|}{X}\right)\sum_{k\in\mathbb{Z}}h\left(\frac{2k-1}{K}\right)\\
&\qquad\times\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\frac{\chi_d(m_1m_2n)}{\sqrt{m_1m_2}}V_k\left(\frac{m_1}{|d|}\right)V_k\left(\frac{m_2}{|d|}\right)\sum_{u|n}c_n(u)\sum_{j|(m_1,m_2)}\sum_{f\in B_k}\lambda_f\left(\frac{m_1m_2}{j^2}\right)\lambda_f(u).
\end{aligned}$$

Now an application of the Petersson formula shows that this is

$$\begin{aligned}
&4\sum_d{}^{\flat}\phi\left(\frac{|d|}{X}\right)\sum_{k\in\mathbb{Z}}h\left(\frac{2k-1}{K}\right)\\
&\qquad\times\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\frac{\chi_d(m_1m_2n)}{\sqrt{m_1m_2}}V_k\left(\frac{m_1}{|d|}\right)V_k\left(\frac{m_2}{|d|}\right)\sum_{u|n}c_n(u)\sum_{j|(m_1,m_2)}1_{m_1m_2/j^2=u}+\text{error term},
\end{aligned}$$

where the error term is given by

$$\begin{aligned}
&8\pi\sum_d{}^{\flat}\phi\left(\frac{|d|}{X}\right)\sum_{k\in\mathbb{Z}}h\left(\frac{2k-1}{K}\right)\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\frac{\chi_d(m_1m_2n)}{\sqrt{m_1m_2}}V_k\left(\frac{m_1}{|d|}\right)V_k\left(\frac{m_2}{|d|}\right)\\
&\qquad\times\sum_{u|n}c_n(u)\sum_{j|(m_1,m_2)}i^{2k}\sum_{c=1}^{\infty}\frac{S(m_1m_2/j^2,u;c)}{c}J_{2k-1}\left(\frac{4\pi\sqrt{m_1m_2u}}{jc}\right).
\end{aligned}$$

For the main term we use the relations (6.1) and (6.2) to write it as

$$4\sum_d^\flat \phi\left(\frac{|d|}{X}\right)\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right)$$
$$\times\sum_{m_1=1}^\infty\sum_{m_2=1}^\infty \frac{\chi_d(m_1m_2n)}{\sqrt{m_1m_2}}V_k\left(\frac{m_1}{|d|}\right)V_k\left(\frac{m_2}{|d|}\right)\sum_{u|n}c_n(u)\sum_{j|(m_1,m_2)}\mathbb{E}\left(X\left(\frac{m_1m_2}{j^2}\right)X(u)\right)$$
$$=4\sum_d^\flat \phi\left(\frac{|d|}{X}\right)\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right)\sum_{m_1=1}^\infty\sum_{m_2=1}^\infty \frac{\chi_d(m_1m_2n)}{\sqrt{m_1m_2}}V_k\left(\frac{m_1}{|d|}\right)V_k\left(\frac{m_2}{|d|}\right)\sum_{u|n}c_n(u)\mathbb{E}\left(X(m_1)X(m_2)X(u)\right)$$
$$=\sum_d^\flat \phi\left(\frac{|d|}{X}\right)\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right)\mathbb{E}\left(L(X;d,k)\chi_d(n)\sum_{u|n}c_n(u)X(u)\right).$$

Next we will analyse the error term. We start by executing the $k$-sum using (5.10) to see that the off-diagonal equals

$$-2\sqrt{\pi}K\sum_d^\flat \phi\left(\frac{|d|}{X}\right)\chi_d(n)\sum_{m_1=1}^\infty\sum_{m_2=1}^\infty\frac{\chi_d(m_1)\chi_d(m_2)}{m_1^{3/4}m_2^{3/4}}\sum_{u|n}c_n(u)u^{-1/4}$$
$$\times\operatorname{Im}\left(e^{-2\pi i/8}\sum_{j|(m_1,m_2)}\sum_{c=1}^\infty\frac{S(m_1m_2/j^2,u;c)}{\sqrt{c}}\cdot e\left(\frac{2\sqrt{m_1m_2u}}{jc}\right)W_K^{(2)}\left(\frac{m_1}{|d|},\frac{m_2}{|d|},\frac{K^2cj}{8\pi\sqrt{m_1m_2u}}\right)\right)$$
$$+O\left(X^{3+\varepsilon}/K^2\right)$$
$$=-2\sqrt{\pi}K\sum_d^\flat \phi\left(\frac{|d|}{X}\right)\chi_d(n)\sum_{m_1=1}^\infty\sum_{m_2=1}^\infty\frac{\chi_d(m_1)\chi_d(m_2)}{m_1^{3/4}m_2^{3/4}}\sum_{u|n}c_n(u)u^{-1/4}$$
$$\times\operatorname{Im}\left(e^{-2\pi i/8}\sum_{(j,d)=1}\frac{1}{j}\sum_{c=1}^\infty\frac{S(m_1m_2,u;c)}{\sqrt{c}}\cdot e\left(\frac{2\sqrt{m_1m_2u}}{c}\right)W_K^{(2)}\left(\frac{m_1j}{|d|},\frac{m_2j}{|d|},\frac{K^2c}{8\pi\sqrt{m_1m_2u}}\right)\right)$$
$$+O\left(X^{3+\varepsilon}/K^2\right), \tag{9.1}$$

where the error comes from estimating the error term in (5.10) trivially.

Splitting the $m_1$- and $m_2$-sums into congruence classes modulo $[c,d]$, these sums may be written as

$$\sum_{x\ ([c,d])}\sum_{w\ ([c,d])}S(xw,u;c)\chi_d(x)\chi_d(w)$$
$$\times\sum_{m_1\equiv x\ ([c,d])}\sum_{m_2\equiv w\ ([c,d])}m_1^{-3/4}m_2^{-3/4}e\left(\frac{2\sqrt{m_1m_2u}}{c}\right)W_K^{(2)}\left(\frac{m_1j}{|d|},\frac{m_2j}{|d|},\frac{K^2c}{8\pi\sqrt{m_1m_2u}}\right). \tag{9.2}$$

Applying Poisson summation to the $m_1$-sum gives that it equals

$$\frac{1}{[c,d]}\sum_{v\in\mathbb{Z}}e\left(\frac{xv}{[c,d]}\right)\int_{\mathbb{R}}y^{-3/4}e\left(\frac{2\sqrt{ym_2u}}{c}-\frac{yv}{[c,d]}\right)W_K^{(2)}\left(\frac{yj}{|d|},\frac{m_2j}{|d|},\frac{K^2c}{8\pi\sqrt{ym_2u}}\right)\mathrm{d}y.$$

Similarly applying Poisson summation to the $m_2$-sum gives, after some computations, that our double sum equals

$$\sum_{x\ ([c,d])}\sum_{w\ ([c,d])}S\left(xw,u;c\right)\chi_d(x)\chi_d(w)\frac{1}{[c,d]^2}\sum_{v\in\mathbb{Z}}\sum_{\eta\in\mathbb{Z}}e\left(\frac{xv+w\eta}{[c,d]}\right)\cdot\mathcal{I},$$

where

$$\mathcal{I} := \int_{\mathbb{R}} \int_{\mathbb{R}} y^{-3/4} z^{-3/4} e\left(\frac{2\sqrt{yzu}}{c} - \frac{yv}{[c,d]} - \frac{z\eta}{[c,d]}\right) W_K^{(2)}\left(\frac{yj}{|d|}, \frac{zj}{|d|}, \frac{K^2 c}{8\pi\sqrt{yzu}}\right) \mathrm{d}y\, \mathrm{d}z.$$

We first focus on the $y$-integral. Let us set

$$f(y) := \frac{2\sqrt{yzu}}{c} - \frac{yv}{[c,d]} - \frac{z\eta}{[c,d]}.$$

An easy computation shows that the integral has a saddle-point at

$$y_0 := \frac{[c,d]^2 zu}{c^2 v^2}.$$

Similarly a straightforward computation gives

$$f(y_0) = z\left(\frac{u[c,d]}{c^2 v} - \frac{\eta}{[c,d]}\right) \qquad \text{and} \qquad f''(y_0) = -\frac{1}{2} \cdot \frac{c^2 v^3}{zu[c,d]^3}.$$

Note that by (5.8) the function $W_K^{(2)}$ decays rapidly unless

$$|y| \ll |d|K^{1+\varepsilon}/j, \qquad |z| \ll |d|K^{1+\varepsilon}/j, \quad \text{and} \quad |yz| \gg K^{4-\varepsilon}c^2/u. \tag{9.3}$$

We truncate the $y$-integral smoothly to $|y| \leq K^{1+\varepsilon}|d|/j$ with a negligible error and then apply Lemma 5.9 with the choices $Q = V = V_1 = K|d|/j$, $X = K^{\varepsilon}$, and $Y = \sqrt{zuK|d|}/c\sqrt{j}$. It is straightforward to check that the conditions of the lemma are met under (9.3). We conclude that

$$\mathcal{I} \sim \sqrt{2} e^{-\pi i/4} \sqrt{c} u^{-1/4} \int_{\mathbb{R}} z^{-1} e\left(z\left(\frac{u[c,d]}{c^2 v} - \frac{\eta}{[c,d]}\right)\right) W_K^{(2)}\left(\frac{[c,d]^2 zuj}{c^2 v^2 d}, \frac{zj}{d}, \frac{K^2 c^2 v}{8\pi zu[c,d]}\right) \mathrm{d}z$$

and so it follows that the expression (9.1) is, after some simplification, equal to

$$\begin{aligned} &\frac{-2\sqrt{2\pi}}{\sqrt{u}} K \sum_d^{\flat} \phi\left(\frac{|d|}{X}\right) \chi_d(n) \sum_{(j,d)=1} \frac{1}{j} \mathrm{Im}\Bigg(e^{-\pi i/2} \sum_{c=1}^{\infty} \frac{1}{[c,d]^2} \sum_{v\in\mathbb{Z}} \sum_{\eta\in\mathbb{Z}} \sum_{x\ ([c,d])} \sum_{w\ ([c,d])} \chi_d(x)\chi_d(w) \\ &\times S(xw,u;c) e\left(\frac{xv+w\eta}{[c,d]}\right) \int_{\mathbb{R}} z^{-1} e\left(z\left(\frac{u[c,d]}{c^2 v} - \frac{\eta}{[c,d]}\right)\right) W_K^{(2)}\left(\frac{zu[c,d]^2 j}{v^2 c^2 |d|}, \frac{zj}{|d|}, \frac{K^2 c^2 v}{8\pi zu[c,d]}\right) \mathrm{d}z\Bigg) \\ &\quad + O\left(X^{3/4} K^{1+\varepsilon}\right). \end{aligned} \tag{9.4}$$

Note that the exponential phase in the last integral vanishes when $\eta v = u[c,d]^2/c^2$. If this not the case, we may bound the integral using the first derivative test [9, Section 1.5.]. Indeed, note that in this case the absolute value of the derivative of the phase function is $\geq 1/c^2 v[c,d]$. Also by (5.9) the weight function $W^{(2)}$ is negligible unless

$$v \ll_{\varepsilon} \frac{[c,d]|d|uK^{\varepsilon}}{c^2 Kj}$$

and the same holds for $\eta$. Note that as $|d| \ll X \ll \sqrt{K}$ and $u \leq K^{4\delta_0} \leq K^{\varepsilon}$ this shows that the integral is negligible unless $v, \eta \leq K^{2\varepsilon}/c$, which effectively truncates also the $c$-sum at $K^{2\varepsilon}$. Note also that we may truncate the $\eta$-sum at $X^2$ as otherwise the derivative of the phase function is $\gg [c,d]^2/c^2 v$. Now using the first derivative test and estimating everything else trivially using the Weil bound for Kloosterman sums shows that (9.4) is $\ll X^{3/4} K^{1+\varepsilon}$.

When $\eta v = u[c,d]^2/c^2$ we use Proposition 7.2 to deduce that the main part of the off-diagonal is

$$
\begin{aligned}
&-2\sqrt{2\pi}K\sum_d{}^\flat \phi\left(\frac{|d|}{X}\right)(-1)^{\#\{p|d\}}\sum_{u|n} c_n(u)\frac{\chi_d(nu)}{\sqrt{u}}\\
&\times\sum_{v|u}\mathrm{Im}\left(e^{-\pi i/2}\sum_{(j,d)=1}\frac{1}{j}\sum_{\substack{c=1\\ d|c}}^{\infty}\frac{\varphi(c)}{c\cdot\varphi(d)}\prod_{\substack{p|d\\ p|\frac{c}{d}}}(1-p)\int_{\mathbb{R}} z^{-1}W_K^{(2)}\left(\frac{zuj}{|d|v^2},\frac{zj}{|d|},\frac{K^2c|d|v}{8\pi zu}\right)\mathrm{d}z\right).
\end{aligned}
\tag{9.5}
$$

One easily computes

$$
\sum_{\substack{c=1\\ d|c}}^{\infty}\frac{\varphi(c)}{c\cdot\varphi(d)}\prod_{\substack{p|d\\ p|\frac{c}{d}}}(1-p)=\sum_{c=1}^{\infty}\frac{\varphi(c)}{c}\cdot\frac{1}{d}\prod_{\substack{p|d\\ p|c}}(-p),
$$

where the summand on the right-hand side is a multiplicative function of $c$. We also have by a simple calculation that, for $\mathrm{Re}(x+y)>1$,

$$
\begin{aligned}
\sum_{c=1}^{\infty}\frac{\varphi(c)}{c^{1+x+y}}\cdot\frac{1}{d}\prod_{\substack{p|d\\ p|c}}(-p)&=\frac{1}{d}\prod_p\left(1+\sum_{j=1}^{\infty}\frac{\varphi(p^j)}{p^{j(1+x+y)}}\prod_{\substack{q|d\\ q|p^j}}(-q)\right)\\
&=\frac{1}{d}\prod_{p|d}\left(1+\sum_{j=1}^{\infty}\frac{\varphi(p^j)}{p^{j(1+x+y)}}(-p)\right)\prod_{p\nmid d}\left(1+\sum_{j=1}^{\infty}\frac{\varphi(p^j)}{p^{j(1+x+y)}}\right)\\
&=\frac{1}{d}\prod_{p|d}\frac{1+\sum_{j=1}^{\infty}\frac{\varphi(p^j)}{p^{j(1+x+y)}}(-p)}{1+\sum_{j=1}^{\infty}\frac{\varphi(p^j)}{p^{j(1+x+y)}}}\prod_p\left(1+\sum_{j=1}^{\infty}\frac{\varphi(p^j)}{p^{j(1+x+y)}}\right)\\
&=\frac{\zeta(x+y)}{\zeta(1+x+y)}\cdot\prod_{p|d}\left(\frac{p^{x+y}-p}{p(p^{x+y}-1)+p-1}\right),
\end{aligned}
$$

where in the last step we have used the fact that $d$ is squarefree.

Making the change of variables $z\mapsto cz$, using the approximation (5.11) and Poisson summation we see that the integral in (9.5) equals

$$
\int_{\mathbb{R}} z^{-1}W\left(\frac{zuj}{K|d|v^2},\frac{zj}{K|d|},\frac{K^2|d|v}{8\pi zu}\right)\mathrm{d}z=\sum_{h=1}^{\infty}h^{-1}W\left(\frac{hcuj}{K|d|v^2},\frac{hcj}{K|d|},\frac{K^2|d|v}{8\pi nu}\right)
$$

up to a negligible error.

By Mellin inversion we have, for any $\varepsilon>0$,

$$
\begin{aligned}
&W\left(\frac{hcuj}{K|d|v^2},\frac{hcj}{K|d|},\frac{K^2|d|v}{8\pi nu}\right)\\
&=\frac{1}{(2\pi i)^2}\int_{(\frac12+\varepsilon)}\int_{(\frac12+\varepsilon)}(2\pi)^{-x-y}e^{x^2+y^2}\left(\frac{hcuj}{K|d|v^2}\right)^{-x}\left(\frac{hcj}{K|d|}\right)^{-y}\hbar_{x+y}\left(\frac{K^2|d|v}{8\pi hu}\right)\frac{\mathrm{d}x\,\mathrm{d}y}{xy}\\
&=\frac{1}{(2\pi i)^3}\int_{(\frac12+\varepsilon)}\int_{(\frac12+\varepsilon)}\int_{(1-\varepsilon)}(2\pi)^{-x-y}e^{x^2+y^2}\left(\frac{hcuj}{K|d|v^2}\right)^{-x}\left(\frac{hcj}{K|d|}\right)^{-y}\left(\frac{8\pi hu}{K^2|d|v}\right)^{z}\widetilde{\hbar}_{x+y}(z)\,\mathrm{d}z\,\frac{\mathrm{d}x\,\mathrm{d}y}{xy},
\end{aligned}
$$

where we have used (5.13) in the final step.

Substituting these into (9.5) leads us to consider the multiple integral

$$
\begin{aligned}
&2\sqrt{2\pi}K\sum_d{}^{\flat}\phi\left(\frac{|d|}{X}\right)(-1)^{\#\{p|d\}}\sum_{u|n}\frac{c_n(u)}{\sqrt{u}}\chi_d(nu)\\
&\quad\times\sum_{v|u}\frac{1}{(2\pi i)^3}\int\limits_{(\frac12+\varepsilon)}\int\limits_{(\frac12+\varepsilon)}\int\limits_{(1-\varepsilon)}(2\pi)^{-x-y}e^{x^2+y^2}u^{z-x}(8\pi)^zK^{x+y-2z}d^{x+y-z}v^{2x-z}\widetilde{h}_{x+y}(z)\\
&\qquad\times\left(\sum_{(j,d)=1}\frac{1}{j^{1+x+y}}\right)\left(\sum_{h=1}^{\infty}\frac{1}{h^{1+x+y-z}}\right)\left(\sum_{c=1}^{\infty}\frac{\varphi(c)}{c^{1+x+y}}\cdot\frac1d\prod_{\substack{p|d\\p|c}}(-p)\right)\mathrm{d}z\frac{\mathrm{d}x\,\mathrm{d}y}{xy}\\
&=2\sqrt{2\pi}K\sum_d{}^{\flat}\phi\left(\frac{|d|}{X}\right)(-1)^{\#\{p|d\}}\sum_{u|n}\frac{c_n(u)}{\sqrt{u}}\chi_d(nu)\sum_{v|u}\\
&\quad\times\frac{1}{(2\pi i)^3}\int\limits_{(\frac12+\varepsilon)}\int\limits_{(\frac12+\varepsilon)}\int\limits_{(1-\varepsilon)}(2\pi)^{-x-y}e^{x^2+y^2}u^{z-x}(8\pi)^zK^{x+y-2z}d^{x+y-z}v^{2x-z}\widetilde{h}_{x+y}(z)
\end{aligned}
$$

(9.6)

$$
\times\,\zeta(1+x+y)\prod_{p|d}\left(1-p^{-1-x-y}\right)\cdot\frac{\zeta(x+y)}{\zeta(1+x+y)}\prod_{p|d}\left(\frac{p^{x+y}-p}{p(p^{x+y}-1)+p-1}\right)\zeta(1+x+y-z)\,\mathrm{d}z\,\frac{\mathrm{d}x\,\mathrm{d}y}{xy}.
$$

Using (5.12) this equals

$$
\begin{aligned}
&2K\sum_d{}^{\flat}\phi\left(\frac{|d|}{X}\right)(-1)^{\#\{p|d\}}\sum_{u|n}\frac{c_n(u)}{\sqrt{u}}\chi_d(nu)\sum_{v|u}\\
&\quad\times\frac{1}{(2\pi i)^3}\int\limits_{(\frac12+\varepsilon)}\int\limits_{(\frac12+\varepsilon)}\int\limits_{(1-\varepsilon)}\int_0^{\infty}(2\pi)^{-x-y}e^{x^2+y^2}u^{z-x}(8\pi)^zK^{x+y-2z}d^{x+y-z}v^{2x-z}\frac{h(\sqrt{\xi})}{\sqrt{\xi}}\xi^{(x+y)/2-z}\\
&\quad\times\Gamma(z)\left(\cos\left(\frac{\pi z}{2}\right)+i\sin\left(\frac{\pi z}{2}\right)\right)\zeta(1+x+y)\prod_{p|d}\left(1-p^{-1-x-y}\right)\\
&\quad\times\frac{\zeta(x+y)}{\zeta(1+x+y)}\prod_{p|d}\left(\frac{p^{x+y}-p}{p(p^{x+y}-1)+p-1}\right)\zeta(1+x+y-z)\,\mathrm{d}\xi\,\mathrm{d}z\,\frac{\mathrm{d}x\,\mathrm{d}y}{xy}.
\end{aligned}
$$

Shifting the $y$-integral to the line $\sigma=-\varepsilon$ we cross simple poles at $y=z-x$ with residue 1. Thus (9.6) takes the form

$$
\begin{aligned}
&2K\sum_d{}^{\flat}\phi\left(\frac{|d|}{X}\right)(-1)^{\#\{p|d\}}\sum_{u|n}\frac{c_n(u)}{\sqrt{u}}\chi_d(nu)\sum_{v|u}\frac{1}{(2\pi i)^2}\int\limits_{(\frac12+\varepsilon)}\int\limits_{(1-\varepsilon)}\int_0^{\infty}(2\pi K)^{-z}u^{z-x}(8\pi)^ze^{x^2+(z-x)^2}v^{2x-z}\frac{h(\sqrt{\xi})}{\sqrt{\xi}}\xi^{-z/2}\\
&\quad\times\Gamma(z)\left(\cos\left(\frac{\pi z}{2}\right)+i\sin\left(\frac{\pi z}{2}\right)\right)\zeta(z)\prod_{p|d}\left(\left(1-p^{-1-z}\right)\cdot\frac{p^z-p}{p(p^z-1)+p-1}\right)\mathrm{d}\xi\frac{\mathrm{d}z\,\mathrm{d}x}{x(z-x)}.
\end{aligned}
$$

Shifting the $z$-integral to the line $\sigma=-\varepsilon$ we encounter a simple pole at $z=0$ coming from the Gamma function with a contribution

(9.7)

$$
-2\zeta(0)K\sum_d{}^{\flat}\phi\left(\frac{|d|}{X}\right)(-1)^{\#\{p|d\}}\sum_{u|n}\frac{c_n(u)}{\sqrt{u}}\chi_d(nu)\sum_{v|u}\frac{1}{2\pi i}\int\limits_{(\frac12+\varepsilon)}\int_0^{\infty}e^{2x^2}u^{-x}v^{2x}\frac{h(\sqrt{\xi})}{\sqrt{\xi}}\prod_{p|d}\left(1-p^{-1}\right)\left(\frac{1-p}{p-1}\right)\mathrm{d}\xi\frac{\mathrm{d}x}{x^2}.
$$

Simplifying this by recalling that $\zeta(0) = -1/2$ and making a change of variables in the $\xi$-integral we see that (9.5) equals, up to a negligible error,

$$2\widehat{h}(0)K \sum_d{}^{\flat} \phi\left(\frac{|d|}{X}\right) \frac{\varphi(d)}{d} \sum_{u|n} \frac{c_n(u)}{\sqrt{u}} \chi_d(nu) \sum_{v|u} \frac{1}{2\pi i} \int_{(\frac{1}{2}+\varepsilon)} e^{2x^2} u^{-x} v^{2x} \frac{\mathrm{d}x}{x^2}.$$

This finishes the proof. □

Let us define the random mollifier by

$$\begin{aligned} M_2(X;d) &:= (\log K) \sum_{n \le K^{4\delta_0}} \frac{h_2(n)\lambda(n)\chi_d(n)}{2^{\Omega(n)}\sqrt{n}} \sum_{u|n} c_n(u) X(u) \\ &= (\log K) \prod_{j=0}^{J} \underbrace{\sum_{\substack{p|n \Rightarrow p \in I_j \\ \Omega(n) \le 4\ell_j}} \frac{\lambda(n)\chi_d(n)\nu_4(n;\ell_j)}{2^{\Omega(n)}\sqrt{n}} \sum_{u|n} c_n(u) X(u)}_{=: M_{2,j}(X;d)} . \end{aligned}$$

This is a random counterpart for $M_g(d)^4$ (compare to (4.4) with $\ell = 2$).

An immediate consequence of the previous lemma is the evaluation of the mollified second moment.

**Corollary 9.2.** *We have*

$$\begin{aligned} &\sum_d{}^{\flat} \phi\left(\frac{|d|}{X}\right) \sum_{k \in \mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{f \in B_k} \omega_f L\left(\frac{1}{2}, f \otimes \chi_d\right)^2 M_f(d)^4 \\ &= \sum_d{}^{\flat} \phi\left(\frac{|d|}{X}\right) \sum_{k \in \mathbb{Z}} h\left(\frac{2k-1}{K}\right) \mathbb{E}\left(L(X;d,k) M_2(X;d)\right) \\ &\quad + 2K(\log K)\widehat{h}(0) \sum_{n \le K^{4\delta_0}} \frac{h_2(n)\lambda(n)}{2^{\Omega(n)}\sqrt{n}} \sum_d{}^{\flat} \frac{\varphi(d)}{d} \phi\left(\frac{|d|}{X}\right) \sum_{u|n} \frac{c_n(u)}{\sqrt{u}} \chi_d(nu) \frac{1}{2\pi i} \int_{(\frac{1}{2}+\varepsilon)} \frac{e^{2x^2}}{x^2} \cdot u^{-x} \sum_{v|u} v^{2x} \, \mathrm{d}x \qquad (9.8) \\ &\quad + O\left(X^{3/4} K^{1+\varepsilon}\right) \end{aligned}$$

## 10. A RANDOM COMPUTATION

Let us define for $\mathrm{Re}(s_1) > 1$ and $\mathrm{Re}(s_2) > 1$,

$$L(s_1, s_2, X; d) = \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \frac{X(m_1)X(m_2)\chi_d(m_1 m_2)}{m_1^{s_1} m_2^{s_2}} = \prod_p \underbrace{\sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \frac{X(p^{j_1})X(p^{j_2})\chi_d(p^{j_1+j_2})}{p^{j_1 s_1 + j_2 s_2}}}_{=: L_p(s_1, s_2, X; d)} .$$

Our next goal is to estimate the contribution of primes of different sizes.

10.1. **Contribution of the primes with $K^{\theta_0} < p \le K^{\theta_J}$.** Here we wish to estimate the expectation

$$\mathbb{E}\left( \prod_{K^{\theta_0} < p < K^{\theta_J}} L_p\left(s_1 + \frac{1}{2}, s_2 + \frac{1}{2}; X, d\right) \prod_{j=1}^{J} M_{2,j}(X;d) \right).$$

This is needed for shifting contours in the proof of Lemma 10.3. To do this, we consider for every $1 \le j \le J$ the term

$$\mathbb{E}\left( M_{2,j}(X;d) \prod_{p \in I_j} L_p\left(s_1 + \frac{1}{2}, s_2 + \frac{1}{2}; X, d\right) \right). \qquad (10.1)$$

Let us start by expanding

$$M_{2,j}(X;d)\prod_{p\in I_j} L_p\left(s_1+\frac{1}{2},s_2+\frac{1}{2};X,d\right)$$
$$=\sum_{p|m_1m_2\Rightarrow p\in I_j}\frac{\chi_d(m_1m_2)}{m_1^{s_1+1/2}m_2^{s_2+1/2}}X(m_1)X(m_2)\sum_{\substack{p|n\Rightarrow p\in I_j\\ \Omega(n)\le 4\ell_j}}\frac{\lambda(n)\chi_d(n)\nu_4(n;\ell_j)}{2^{\Omega(n)}\sqrt{n}}\sum_{u|n}c_n(u)X(u).$$

By taking the expectation and using the Hecke relations for $X(m)$ we see that (10.1) equals

$$\sum_{p|m_1m_2\Rightarrow p\in I_j}\frac{\chi_d(m_1m_2)}{m_1^{s_1+1/2}m_2^{s_2+1/2}}\sum_{\substack{p|n\Rightarrow p\in I_j\\ \Omega(n)\le 4\ell_j}}\frac{\lambda(n)\chi_d(n)\nu_4(n;\ell_j)}{2^{\Omega(n)}\sqrt{n}}\sum_{u|n}c_n(u)\sum_{h|(m_1,m_2)}1_{m_1m_2/h^2=u}$$

$$=\sum_{(h,d)=1}\frac{1}{h^{1+s_1+s_2}}\sum_{p|m_1m_2\Rightarrow p\in I_j}\frac{\chi_d(m_1m_2)}{m_1^{s_1+1/2}m_2^{s_2+1/2}}\sum_{\substack{p|n\Rightarrow p\in I_j\\ \Omega(n)\le 4\ell_j\\ m_1m_2|n}}\frac{\lambda(n)\chi_d(n)\nu_4(n;\ell_j)}{2^{\Omega(n)}\sqrt{n}}c_n(m_1m_2). \tag{10.2}$$

For any $r>0$ and $n,\ell\in\mathbb{N}$,

$$1_{\Omega(n)=\ell}=\frac{1}{2\pi i}\int_{|z|=r} z^{\Omega(n)-\ell}\frac{\mathrm{d}z}{z},$$

so that for $r\neq 1$ we have

$$1_{\Omega(n)\le\ell_j}=\frac{1}{2\pi i}\int_{|z|=r} z^{\Omega(n)}\frac{1-z^{-\ell_j-1}}{1-z^{-1}}\frac{\mathrm{d}z}{z}.$$

Therefore, for $1<r\le 2$ the right-hand side of (10.2) is given by

$$\frac{1}{2\pi i}\int_{|z|=r}\frac{1-z^{-4\ell_j-1}}{1-z^{-1}}\Sigma(z)\frac{\mathrm{d}z}{z}, \tag{10.3}$$

where

$$\Sigma(z):=\sum_{(h,d)=1}\frac{1}{h^{1+s_1+s_2}}\sum_{p|m_1m_2\Rightarrow p\in I_j}\frac{\chi_d(m_1m_2)}{m_1^{s_1+1/2}m_2^{s_2+1/2}}\sum_{\substack{p|n\Rightarrow p\in I_j\\ m_1m_2|n}}\frac{z^{\Omega(n)}\lambda(n)\chi_d(n)\nu_4(n;\ell_j)}{2^{\Omega(n)}\sqrt{n}}c_n(m_1m_2).$$

Changing the order of summation, the inner double sum can be written as an Euler product

$$\sum_{p|n\Rightarrow p\in I_j}\frac{z^{\Omega(n)}\lambda(n)\chi_d(n)\nu_4(n)}{2^{\Omega(n)}\sqrt{n}}\sum_{\substack{m_1,m_2\\ m_1m_2|n}}\frac{\chi_d(m_1m_2)}{m_1^{s_1+1/2}m_2^{s_2+1/2}}c_n(m_1m_2)$$
$$=\prod_{\substack{p\in I_j\\ p\nmid d}}\left(1-\frac{z}{\sqrt{p}}\left(\frac{1}{p^{s_1+1/2}}+\frac{1}{p^{s_2+1/2}}\right)+\frac{3z^2}{2p}\left(1+\frac{1}{p^{1+2s_1}}+\frac{1}{p^{1+2s_2}}+\frac{1}{p^{1+s_1+s_2}}\right)+O\left(\frac{1}{p^{3/2+\min\{\mathrm{Re}(s_1),\mathrm{Re}(s_2)\}}}\right)\right)$$

From this it is easily seen that $|\Sigma(z)|\ll 1$ uniformly for $|z|\le 2$ when $\min\{\mathrm{Re}(s_1),\mathrm{Re}(s_2)\}>0$ (recall that $j\neq 0$). Applying this bound in (10.3) we see that

$$\mathbb{E}\left(M_{2,j}(X;d)\prod_{p\in I_j}L_p\left(s_1+\frac{1}{2},s_2+\frac{1}{2};X,d\right)\right)\ll 1.$$

Thus we have that

$$\mathbb{E}\left(\prod_{K^{\theta_0}<p<K^{\theta_J}} L_p\left(s_1+\frac{1}{2},s_2+\frac{1}{2};X,d\right)\prod_{j=1}^{J} M_{2,j}(X;d)\right) \ll (\log\log K)^{O(1)} \tag{10.4}$$

by applying the above bound for each $1 \leq j \leq J$ and recalling that $J \asymp \log\log\log K$.

10.2. **Contribution of the small primes.** Next we need to understand contribution of the primes with $c \leq p \leq K^{\theta_0}$. This involves understanding the interaction between $L(s_1+1/2, s_2+1/2; X, d)$ and $M_{2,0}(X;d)$. A key point is that since $M_{2,0}(X;d)$ consists of relatively small primes we can express it in terms of an Euler product with negligible loss since $\ell_0$ is large. This allows us to simplify our later analysis by reducing the problem to understanding the contribution from each prime $p \in I_0$ individually. Let

$$\widetilde{M}_{2,0}(X;d) := \sum_{p|n \Rightarrow p\in I_0} \frac{\lambda(n)\chi_d(n)\nu_4(n;\ell_0)}{2^{\Omega(n)}\sqrt{n}} \sum_{u|n} c_n(u)X(u). \tag{10.5}$$

We have the following result.

**Lemma 10.1.** *For* $\min\{\mathrm{Re}(s_1),\mathrm{Re}(s_2)\} \geq -(\log\log K)^2/\log K$ *we have that*

$$\begin{aligned}\mathbb{E}&\left(M_{2,0}(X;d)\prod_{p\in I_0} L_p\left(s_1+\frac{1}{2},s_2+\frac{1}{2};X,d\right)\right)\\ &= \mathbb{E}\left(\widetilde{M}_{2,0}(X;d)\prod_{p\in I_0} L_p\left(s_1+\frac{1}{2},s_2+\frac{1}{2};X,d\right)\right) + O\left((\log K)^{-10}\right).\end{aligned}$$

*Proof.* Let us write

$$R(X;d) := \widetilde{M}_{2,0}(X;d) - M_{2,0}(X;d) = \sum_{\substack{p|n\Rightarrow p\in I_0\\ \Omega(n)>4\ell_0}} \frac{\lambda(n)\chi_d(n)\nu_2(n;\ell_j)}{2^{\Omega(n)}\sqrt{n}} \sum_{u|n} c_n(u)X(u)$$

and let us denote $L_0(s_1,s_2;X,d) := \prod_{p\in I_0} L_p(s_1,s_2;X,d)$. Then by the Cauchy–Schwarz inequality we have

$$\mathbb{E}\left(\left|L_0\left(s_1+\frac{1}{2},s_2+\frac{1}{2};X,d\right)R(X;d)\right|\right)^2 \leq \mathbb{E}\left(\left|L_0\left(s_1+\frac{1}{2},s_2+\frac{1}{2};X,d\right)\right|^2\right)\mathbb{E}\left(|R(X;d)|^2\right).$$

Since $L_0(s_1,s_2;X,d)$ is a finite product this function is analytic for $\mathrm{Re}(s_1)$, $\mathrm{Re}(s_2) > 0$. Let us first analyse $\mathbb{E}(|L_0(s_1+1/2,s_2+1/2;X,d)|^2)$. We simply compute

$$\mathbb{E}\left(\left|L_0\left(s_1+\frac{1}{2},s_2+\frac{1}{2};X,d\right)\right|^2\right) = \prod_{p\in I_0}\mathbb{E}\left(\left|\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\frac{X(p^{j_1})X(p^{j_2})\chi_d(p^{j_1+j_2})}{p^{j_1(s_1+1/2)+j_2(s_2+1/2)}}\right|^2\right).$$

Set

$$\rho(X,p^\alpha;s_1,s_2)) := \chi_d(p^\alpha)\sum_{j_1+j_2+j_3+j_4=\alpha}\frac{X(p^{j_1})X(p^{j_2})X(p^{j_3})X(p^{j_4})}{p^{(j_1+j_2)(s_1+1/2)+(j_3+j_4)(s_2+1/2)}}.$$

An easy computation shows that

$$\begin{aligned}\mathbb{E}(\rho(X,p;s_1,s_2))) = 0 \qquad \text{and} \qquad \mathbb{E}(\rho(X,p^2;s_1,s_2)) &= \chi_d(p^2)\left(\frac{1}{p^{1+2s_1}}+\frac{1}{p^{1+2s_2}}+\frac{4}{p^{1+s_1+s_2}}\right)\\ &\ll p^{-1-2\min\{\mathrm{Re}(s_1),\mathrm{Re}(s_2)\}}.\end{aligned}$$

Thus

$$\mathbb{E}\left(\left|L_0\left(s_1+\frac{1}{2},s_2+\frac{1}{2};X,d\right)\right|^2\right) = \prod_{p\in I_0}\left(1+O\left(\frac{1}{p^{1+2\min\{\mathrm{Re}(s_1),\mathrm{Re}(s_2)\}}}\right)\right).$$

Using that $\min\{\mathrm{Re}(s_1), \mathrm{Re}(s_2)\} \geq -(\log\log K)^2/\log K$ we have $\frac{1}{p^{1+2\min\{\mathrm{Re}(s_1),\mathrm{Re}(s_2)\}}} \ll \frac{1}{p}$ for $p \in I_0$ and so the right-hand side of the previous display is $\ll (\log K)^{O(1)}$.

We next estimate $\mathbb{E}(|R(X;d)|^2)$, which equals

$$\sum_{\substack{p|n_1 \Rightarrow p \in I_0 \\ \Omega(n_1) > 4\ell_0}} \sum_{\substack{p|n_2 \Rightarrow p \in I_0 \\ \Omega(n_2) > 4\ell_0}} \frac{\lambda(n_1)\lambda(n_2)\chi_d(n_1 n_2)\nu_4(n_1;\ell_0)\nu_4(n_2;\ell_0)}{2^{\Omega(n_1)+\Omega(n_2)}\sqrt{n_1 n_2}} \sum_{u_1|n_1} \sum_{u_2|n_2} c_{n_1}(u_1) c_{n_2}(u_2) \mathbb{E}(X(u_1)X(u_2)).$$

We have $2^{\Omega(n_1)-4\ell_0} \geq 1$ for $\Omega(n_1) > 4\ell_0$. Thus using the bound $c_n(u) \leq 2^{\Omega(n)}$ and estimating the Liouville function and $\chi_d$ trivially, we see that there exists a constant $C > 0$ so that

$$\begin{aligned}
\mathbb{E}(|R(X;d)|^2) &\leq \frac{1}{2^{8\ell_0}} \sum_{p|r \Rightarrow p \in I_0} \sum_{p|n_1 \Rightarrow p \in I_0} \sum_{p|n_2 \Rightarrow p \in I_0} \frac{\lambda(rn_1)\lambda(rn_2)\chi_d(r^2 n_1 n_2)\nu(rn_1)\nu(rn_2)}{r\sqrt{n_1 n_2}} c_{rn_1}(r) c_{rn_2}(r) \\
&\leq \frac{(\log K)^{O(1)}}{2^{8\ell_0}} \sum_{p|r \Rightarrow p \in I_0} \frac{C^{\Omega(r)}}{r} \\
&\leq \frac{(\log K)^{O(1)}}{2^{8\ell_0}}, \qquad (10.6)
\end{aligned}$$

where we have used the fact that $c$ is sufficiently large so that the sum in the middle converges. Combining the two estimates above and recalling that $\eta_1 > 0$ in the definition of $\ell_0$ is taken to be sufficiently large completes the proof. □

10.3. **Estimates for the large primes.** It remains to study the contribution of the primes $p > K^{\theta_J}$. These do not interact with our mollifier and consequently their contribution is easy to understand. Estimating these terms precisely allows us to meromorphically continue $\mathbb{E}(L(s_1 + 1/2, s_2 + 1/2, X; d)M_2(X;d))$ as $M_2(X;d)$ is a Dirichlet polynomial with coefficients supported on integers with prime factors $\leq K^{\theta_J}$.

**Lemma 10.2.** *For* $\min\{\mathrm{Re}(s_1), \mathrm{Re}(s_2)\} > 1/2$ *we have that*

$$\mathbb{E}\left(L_p\left(s_1 + \frac{1}{2}, s_2 + \frac{1}{2}; X, d\right)\right) = \zeta_p(1 + s_1 + s_2) \cdot 1_{p \nmid d}.$$

*Proof.* We simply compute

$$\begin{aligned}
\mathbb{E}\left(L_p\left(s_1 + \frac{1}{2}, s_2 + \frac{1}{2}; X, d\right)\right) &= \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \frac{\chi_d(p^{k_1+k_2})}{p^{k_1(s_1+1/2)+k_2(s_2+1/2)}} \mathbb{E}\left(X(p^{k_1})X(p^{k_2})\right) \\
&= \sum_{j=1}^{\infty} \frac{\chi_d(p^{2j})}{p^{(1+s_1+s_2)j}} \\
&= 1_{p\nmid d} \cdot \sum_{j=1}^{\infty} \frac{1}{p^{(1+s_1+s_2)j}} \\
&= 1_{p\nmid d} \cdot \frac{1}{1 - p^{-(1+s_1+s_2)}} \\
&= 1_{p\nmid d} \cdot \zeta_p(1 + s_1 + s_2).
\end{aligned}$$

□

Thus using the dominated convergence theorem we have for $\min\{\mathrm{Re}(s_1), \mathrm{Re}(s_2)\} > 1/2$ and any $z \geq 2$ that

$$\begin{aligned}\mathbb{E}\left(\prod_{p>z} L\left(s_1+\frac{1}{2}, s_2+\frac{1}{2}; X, d\right)\right) &= \prod_{p>z} \mathbb{E}\left(L\left(s_1+\frac{1}{2}, s_2+\frac{1}{2}; X, d\right)\right) \\ &= \prod_{\substack{p>z \\ p\nmid d}} \zeta_p(1+s_1+s_2) \\ &= \zeta(1+s_1+s_2) \prod_{p\leq z} \zeta_p(1+s_1+s_2)^{-1} \prod_{\substack{p>z \\ p\mid d}} \zeta_p(1+s_1+s_2)^{-1}.\end{aligned}$$

If $u$ is a natural number with all prime factors $\leq K^{\theta_J}$, this provides a meromorphic continuation of $\mathbb{E}(X(u)L(s_1+1/2, s_2+1/2; X, d)$ to $\mathrm{Re}(s_1+s_2) > -1/2$ with simple poles at $s_1+s_2 = 0$. Indeed, by choosing $z = K^{\theta_J}$, we have

$$\begin{aligned}&\mathbb{E}\left(X(u)L(s_1+1/2, s_2+1/2; X, d)\right) \\ &= \mathbb{E}\left(X(u) \prod_{p\leq K^{\theta_J}} L_p\left(s_1+\frac{1}{2}, s_2+\frac{1}{2}; X, d\right)\right) \mathbb{E}\left(\prod_{p>K^{\theta_J}} L_p\left(s_1+\frac{1}{2}, s_2+\frac{1}{2}; X, d\right)\right) \\ &= \zeta(1+s_1+s_2) \prod_{p\leq K^{\theta_J}} \zeta_p(1+s_1+s_2)^{-1} \prod_{\substack{p>K^{\theta_J} \\ p\mid d}} \zeta_p(1+s_1+s_2)^{-1} \mathbb{E}\left(X(u) \prod_{p\leq K^{\theta_J}} L_p\left(s_1+\frac{1}{2}, s_2+\frac{1}{2}; X, d\right)\right).\end{aligned} \tag{10.7}$$

We further compute using the Hecke relations for $X(n)$ and (6.2) that

$$\begin{aligned}\mathbb{E}\left(X(u) \prod_{p\leq K^{\theta_J}} L_p\left(s_1+\frac{1}{2}, s_2+\frac{1}{2}; X, d\right)\right) &= \mathbb{E}\left(X(u) \sum_{\substack{m_1,m_2 \\ p\mid m_1m_2 \Rightarrow p\leq K^{\theta_J}}} \frac{X(m_1)X(m_2)\chi_d(m_1m_2)}{m_1^{s_1+1/2} m_2^{s_2+1/2}}\right) \\ &= \sum_{\substack{m_1,m_2 \\ p\mid m_1m_2 \Rightarrow p\leq K^{\theta_J}}} \frac{\chi_d(m_1m_2)}{m_1^{s_1+1/2} m_2^{s_2+1/2}} \sum_{h\mid(m_1,m_2)} 1_{u=m_1m_2/h^2} \\ &= \sum_{\substack{(h,d)=1 \\ p\mid h \Rightarrow p\leq K^{\theta_J}}} \frac{1}{h^{1+s_1+s_2}} \sum_{\substack{m_1,m_2 \\ p\mid m_1m_2 \Rightarrow p\leq K^{\theta_J}}} \frac{\chi_d(m_1m_2)}{m_1^{s_1+1/2} m_2^{s_2+1/2}} \cdot 1_{u=m_1m_2} \\ &= \frac{\chi_d(u)}{u^{1/2+s_2}} \prod_{\substack{p\leq K^{\theta_J} \\ p\nmid d}} \left(1-\frac{1}{p^{1+s_1+s_2}}\right)^{-1} \sum_{v\mid u} \frac{1}{v^{s_1-s_2}}.\end{aligned} \tag{10.8}$$

Let us define

$$\widetilde{h}_2(n) := \sum_{\substack{n_0\cdots n_J = n \\ p\mid n_j \Rightarrow p\in I_j\ \forall\, 0\leq j\leq J \\ \Omega(n_j)\leq 4\ell_j\ \forall 1\leq j\leq J}} \nu_4(n_0;\ell_0)\cdots\nu_4(n_r;\ell_r).$$

Our next goal is to prove the following.

**Lemma 10.3.** *We have*

$$\begin{aligned}
&\sum_d^\flat \phi\left(\frac{|d|}{X}\right)\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right)\mathbb{E}\left(L(X;d,k)M_2(X;d)\right)\\
&= 2K(\log K)\widehat{h}(0)\sum_{n\le K^{4\delta_0}}\frac{\widetilde{h}_2(n)\lambda(n)}{2^{\Omega(n)}\sqrt{n}}\sum_d^\flat \phi\left(\frac{|d|}{X}\right)\frac{\varphi(d)}{d}\sum_{u|n}\frac{c_n(u)d(u)}{\sqrt{u}}\chi_d(nu)\left(\log\left(\frac{Kd}{2\pi\sqrt{u}}\right)+O(1)\right)\\
&- 2K(\log K)\widehat{h}(0)\sum_{n\le K^{4\delta_0}}\frac{h_2(n)\lambda(n)}{2^{\Omega(n)}\sqrt{n}}\sum_d^\flat \phi\left(\frac{|d|}{X}\right)\frac{\varphi(d)}{d}\sum_{u|n}\frac{c_n(u)}{\sqrt{u}}\chi_d(nu)\sum_{v|u}\frac{1}{2\pi i}\int_{(\frac12+\varepsilon)} e^{2s_1^2}u^{-s_1}v^{2s_1}\frac{\mathrm{d}s_1}{s_1^2}\\
&\qquad + O\left(KX(\log K)^{-7}\right).
\end{aligned}$$

*Proof.* By Mellin inversion we have

$$\sum_d^\flat \phi\left(\frac{|d|}{X}\right)\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right)\mathbb{E}\left(L(X;d,k)M_2(X;d)\right) \tag{10.9}$$

$$\begin{aligned}
&= 4\sum_d^\flat \phi\left(\frac{|d|}{X}\right)\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right)\\
&\quad\times\mathbb{E}\left(\frac{1}{(2\pi i)^2}\int_{(\frac12+\varepsilon)}\int_{(\frac12+\varepsilon)}\left(\frac{2\pi}{|d|}\right)^{-s_1-s_2}\frac{\Gamma(s_1+k)}{\Gamma(k)}\cdot\frac{\Gamma(s_2+k)}{\Gamma(k)}e^{s_1^2+s_2^2}F(s_1,s_2;X,d)\frac{\mathrm{d}s_1\,\mathrm{d}s_2}{s_1s_2}\right),
\end{aligned}\tag{10.10}$$

where

$$F(s_1,s_2;X,d) := M_2(X;d)\sum_{m_1=1}^\infty\sum_{m_2=1}^\infty\frac{X(m_1)X(m_2)\chi_d(m_1m_2)}{m_1^{s_1+1/2}m_2^{s_2+1/2}} = M_2(X;d)L\left(s_1+\frac12,s_2+\frac12;X,d\right).$$

Recall that $\mathbb{E}(F(s_1,s_2;X,d))$ has a simple pole at $s_1+s_2=0$.

We move the line of integration from $\mathrm{Re}(s_2)=1/2+\varepsilon$ to $\mathrm{Re}(s_2)=-1$, crossing simple poles at $s_2=0$ and $s_2=-s_1$. The $s_2$-integral is $O(K^{-1/2+\varepsilon}X^{1/2})$ on the new line. Next we shall study the contribution of the residues at these poles to (10.9) separately.

• Residue at $s_2=0$: Using (10.7) this contribution is given by

$$\begin{aligned}
&4\sum_d^\flat \phi\left(\frac{|d|}{X}\right)\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right)\frac{1}{2\pi i}\int_{(\frac12+\varepsilon)}\left(\frac{2\pi}{|d|}\right)^{-s_1}\frac{\Gamma(s_1+k)}{\Gamma(k)}e^{s_1^2}\zeta(1+s_1)\prod_{p\le K^{\theta_J}}\zeta_p(1+s_1)^{-1}\prod_{\substack{p>K^{\theta_J}\\ p|d}}\zeta_p(1+s_1)^{-1}\\
&\quad\times\mathbb{E}\left(M_{2,0}(X;d)\prod_{p\in I_0}L_p\left(s_1+\frac12,\frac12;X,d\right)\right)\mathbb{E}\left(\prod_{j=1}^J M_{2,j}(X;d)\prod_{p\in I_j}L_p\left(s_1+\frac12,\frac12;X,d\right)\right)\frac{\mathrm{d}s_1}{s_1}.
\end{aligned}\tag{10.11}$$

It is easy to see that we have

$$\mathbb{E}\left(M_2(X;d)L\left(s_1+\frac12,\frac12;X,d\right)\right)\ll(\log K)^{O(1)} \tag{10.12}$$

in the region $\mathrm{Re}(s_1)\ge\frac12+\varepsilon$. Since for fixed $\sigma>0$ Stirling's formula gives $|\Gamma(\sigma+it)|\ll(|t|+1)^{\sigma-\frac12}e^{-\pi|t|}$, by (10.12) we may truncate the $s_1$-integral (10.11) to $|\mathrm{Im}(s_1)|\le B\log K$ at the cost of an error term of size $O(X)$, where $B$ is a sufficiently large absolute constant.

We then shift the line of integration in the (10.11) to $\mathrm{Re}(s_1)=-(\log\log K)^2/\log K$, crossing a double pole at $s_1=0$. The integral on the new line is, say, $O((\log K)^{-10})$. Using Lemma 10.1 and (10.4) we may replace the factor

$$\mathbb{E}\left(M_{2,0}(X;d)\prod_{p\in I_0}L_p\left(s_1+\frac{1}{2},\frac{1}{2};X,d\right)\right) \quad \text{by} \quad \mathbb{E}\left(\widetilde{M}_{2,0}(X;d)\prod_{p\in I_0}L_p\left(s_1+\frac{1}{2},\frac{1}{2};X,d\right)\right)$$

with an error, say, $O(KX(\log K)^{-7})$, where we recall the definition of $\widetilde{M}_{2,0}(X;d)$ from (10.5).

A straightforward computation, using the residue theorem and (10.8), shows that the double pole contributes the amount

$$\begin{aligned}4(\log K)&\sum_{n\le K^{4\delta_0}}\frac{\widetilde{h}_2(n)\lambda(n)}{2^{\Omega(n)}\sqrt{n}}\sum_d{}^{\flat}\phi\left(\frac{|d|}{X}\right)\frac{\varphi(d)}{d}\sum_{u|n}\frac{c_n(u)\chi_d(nu)}{\sqrt{u}}\\&\times\sum_{k\in\mathbb{Z}}h\left(\frac{2k-1}{K}\right)\left(-\log\left(\frac{2\pi}{|d|}\right)\cdot d(u)+\frac{\Gamma'(k)}{\Gamma(k)}\cdot d(u)+\sum_{p|d}\frac{\log p}{p-1}\cdot d(u)-\sum_{v|u}\log v+\gamma\cdot d(u)\right)\end{aligned}$$

to (10.11). Here $\gamma$ is the Euler–Mascheroni constant

The expression above can be simplified by using the easy observation that $\log k=\log(K\cdot(k-1)/K)+O(k^{-1})$, the asymptotic formula

$$\frac{\Gamma'(k)}{\Gamma(k)}=\log(k-1)+O\left(\frac{1}{k}\right),$$

and noting that

$$\sum_{v|u}\log v=\frac{1}{2}d(u)\log u$$

to be

$$\begin{aligned}4(\log K)&\sum_{n\le K^{4\delta_0}}\frac{\widetilde{h}_2(n)\lambda(n)}{2^{\Omega(n)}\sqrt{n}}\sum_d{}^{\flat}\phi\left(\frac{|d|}{X}\right)\frac{\varphi(d)}{d}\sum_{u|n}\frac{c_n(u)d(u)}{\sqrt{u}}\chi_d(nu)\\&\times\sum_{k\in\mathbb{Z}}h\left(\frac{2k-1}{K}\right)\left(\log\left(\frac{k|d|}{2\pi\sqrt{u}}\right)+\sum_{p|d}\frac{\log p}{p-1}+\gamma\right).\end{aligned}$$

Then evaluating the $k$-sum by Poisson summation, the above contribution may be written, up to an error $O(K^{3/4+\varepsilon}X^{3/4})$, as

$$2K(\log K)\widehat{h}(0)\sum_{n\le K^{4\theta_J}}\frac{\widetilde{h}_2(n)\lambda(n)}{2^{\Omega(n)}\sqrt{n}}\sum_d{}^{\flat}\phi\left(\frac{|d|}{X}\right)\frac{\varphi(d)}{d}\sum_{u|n}\frac{c_n(u)d(u)}{\sqrt{u}}\chi_d(nu)\left(\log\left(\frac{K|d|}{2\pi\sqrt{u}}\right)+O(1)\right).$$

• <u>Residue at $s_1=-s_2$</u>: Likewise, using the series representation

$$\zeta(1+x)=\frac{1}{x}+\gamma+\cdots,$$

the term corresponding to this residue equals

$$\begin{aligned}-4(\log K)&\sum_{n\le K^{4\delta_0}}\frac{h_2(n)\lambda(n)}{2^{\Omega(n)}\sqrt{n}}\sum_d{}^{\flat}\phi\left(\frac{|d|}{X}\right)\frac{\varphi(d)}{d}\sum_{u|n}\frac{c_n(u)}{\sqrt{u}}\chi_d(nu)\\&\times\sum_{k\in\mathbb{Z}}h\left(\frac{2k-1}{K}\right)\sum_{v|u}\frac{1}{2\pi i}\int_{(\frac{1}{2}+\varepsilon)}\frac{\Gamma(s_1+k)}{\Gamma(k)}\cdot\frac{\Gamma(-s_1+k)}{\Gamma(k)}e^{2s_1^2}u^{-s_1}v^{2s_1}\frac{\mathrm{d}s_1}{s_1^2}.\end{aligned}$$

Executing the $k$-sum by Poisson summation, it follows that this contribution is given by
(10.13)

$$-2K(\log K)\widehat{h}(0)\sum_{n\le K^{4\delta_0}}\frac{h_2(n)\lambda(n)}{2^{\Omega(n)}\sqrt{n}}\sideset{}{^\flat}\sum_{d}\phi\left(\frac{|d|}{X}\right)\frac{\varphi(d)}{d}\sum_{u|n}\frac{c_n(u)}{\sqrt{u}}\chi_d(nu)\sum_{v|u}\frac{1}{2\pi i}\int\limits_{(\frac{1}{2}+\varepsilon)}e^{2s_1^2}u^{-s_1}v^{2s_1}\frac{\mathrm{d}s_1}{s_1^2}$$
$$+O\left(K^{3/4+\varepsilon}X^{3/4}\right).$$

This completes the proof. □

## 11. Proof of Proposition 4.2

By combining Corollary 9.2 and Lemma 10.3 we immediately get the following result. Here we note that (10.13) precisely cancels (9.8).

**Lemma 11.1.** *We have*

$$\sideset{}{^\flat}\sum_{d}\phi\left(\frac{|d|}{X}\right)\sum_{k\in\mathbb{Z}}h\left(\frac{2k-1}{K}\right)\sum_{f\in B_k}\omega_f L\left(\frac{1}{2},f\otimes\chi_d\right)^2 M_f(d)^4$$
$$=2K(\log K)\widehat{h}(0)\sum_{n\le K^{4\delta_0}}\frac{\widetilde{h}_2(n)\lambda(n)}{2^{\Omega(n)}\sqrt{n}}\sideset{}{^\flat}\sum_{d}\phi\left(\frac{|d|}{X}\right)\frac{\varphi(d)}{d}\sum_{u|n}\frac{c_n(u)d(u)}{\sqrt{u}}\chi_d(nu)\left(\log\left(\frac{K|d|}{2\pi\sqrt{u}}\right)+O(1)\right)$$
$$+O\left(X^{3/4}K^{1+\varepsilon}\right).$$

At this point we need to evaluate the sum

$$\sideset{}{^\flat}\sum_{d}\chi_d(nu)\frac{\varphi(d)}{d}\log\left(\frac{dK}{2\pi\sqrt{u}}\right)\phi\left(\frac{|d|}{X}\right). \tag{11.1}$$

Writing

$$\frac{\varphi(d)}{d}=\sum_{j|d}\frac{\mu(j)}{j},$$

using (8.1) and Lemma 5.7 to see that (11.1) equals

$$\frac{X}{4}\sum_{\substack{\alpha=1\\(\alpha,2nu)=1}}^{\infty}\frac{\mu(\alpha)}{\alpha^2}\sum_{(j,2)=1}\frac{\mu(j)}{j^2}\cdot\frac{\varphi(nu)}{nu}\cdot 1_{nu=\square}\int\limits_0^{\infty}\phi(y)\log\left(\frac{XKy}{2\pi\sqrt{u}}\right)\mathrm{d}y+O\left(X^{1/2+\varepsilon}\sqrt{nu}+K^{-1/4+\varepsilon}X^{3/4}\right).$$

Using the previous estimate to bound the $d$-sum and recalling (8.4) we have that

$$\sideset{}{^\flat}\sum_{d}\phi\left(\frac{|d|}{X}\right)\sum_{k\in\mathbb{Z}}h\left(\frac{2k-1}{K}\right)\sum_{f\in B_k}\omega_f L\left(\frac{1}{2},f\otimes\chi_d\right)^2 M_f(d)^4$$
$$\ll XK(\log K)^2\sum_{n\le K^{4\delta_0}}\frac{\widetilde{h}_2(n)\lambda(n)}{2^{\Omega(n)}\sqrt{n}}\sum_{\substack{u|n\\nu=\square}}\frac{c_n(u)d(u)}{\sqrt{u}}\prod_{p|n}\left(1+\frac{1}{p}\right)^{-1}$$
$$=XK(\log K)^2\left(\sum_{p|n\Rightarrow p\in I_0}\frac{\lambda(n)\nu_4(n)}{2^{\Omega(n)}\sqrt{n}}\sum_{\substack{u|n\\nu=\square}}\frac{c_n(u)d(u)}{\sqrt{u}}\prod_{p|n}\left(1+\frac{1}{p}\right)^{-1}\right)$$
$$\times\prod_{j=1}^{J}\sum_{\substack{p|n\Rightarrow p\in I_j\\\Omega(n)\le 4\ell_j}}\frac{\lambda(n)\nu_4(n)}{2^{\Omega(n)}\sqrt{n}}\sum_{\substack{u|n\\nu=\square}}\frac{c_n(u)d(u)}{\sqrt{u}}\prod_{p|n}\left(1+\frac{1}{p}\right)^{-1}. \tag{11.2}$$

Here we have again used properties of $\nu_r(n;\ell)$ listed after (4.4).

We remove the condition $\Omega(n) \leq 4\ell_j$ by arguing as before: using Rankin's trick and the estimate $c_n(u) \leq 2^{\Omega(n)}$ it is easy to see (as in the proof of (10.6)) that there exists $C > 0$ such that

$$\sum_{\substack{p|n \Rightarrow p\in I_j \\ \Omega(n)\leq 4\ell_j}} \frac{\lambda(n)\nu_4(n)}{2^{\Omega(n)}\sqrt{n}} \sum_{\substack{u|n \\ nu=\square}} \frac{c_n(u)d(u)}{\sqrt{u}} \prod_{p|n}\left(1+\frac{1}{p}\right)^{-1}$$

$$= \sum_{p|n \Rightarrow p\in I_j} \frac{\lambda(n)\nu_4(n)}{2^{\Omega(n)}\sqrt{n}} \sum_{\substack{u|n \\ nu=\square}} \frac{c_n(u)d(u)}{\sqrt{u}} \prod_{p|n}\left(1+\frac{1}{p}\right)^{-1} + O\left(\frac{1}{2^{4\ell_j}} \sum_{p|m\Rightarrow p\in I_j} \frac{C^{\Omega(m)}}{m}\right)$$

The error term is clearly $\ll 2^{-4\ell_j}$. For the main term we use the facts $\nu_4(p) = 4$, $\nu_4(p^2) = 8$, $c_p(p) = 1$ and $c_{p^2}(p) = 0$ to see that it equals

$$\prod_{p\in I_j}\left(1-\frac{2}{p}+O\left(\frac{1}{p^{3/2}}\right)\right).$$

Since the product above is $\asymp 1$, it follows that the product over $1 \leq j \leq J$ is

$$\left(1+O\left(K^{-\theta_0}\right)+O\left(\sum_{j=1}^{J}\frac{1}{2^{4\ell_j}}\right)\right)\prod_{j=1}^{J}\prod_{p\in I_j}\left(1-\frac{2}{p}+O\left(\frac{1}{p^{3/2}}\right)\right).$$

Using that $2^{-\ell_j} \ll 1/\ell_j \asymp \theta_j^{3/4}$, and summing the geometric sum the error term is $\ll \theta_J^{3/4} \ll 1$. Hence, it follows that the right-hand side of (11.2) is

$$\ll XK(\log K)^2 \prod_{c<p\leq K^{\theta_J}}\left(1-\frac{2}{p}+O\left(\frac{1}{p^{3/2}}\right)\right).$$

This is $\ll XK(\log K)^2(\log K)^{-2} \ll XK$ by Mertens' theorem. This completes the proof of Proposition 4.2. $\square$

## 12. Proof of Theorem 1.2

In this section we complete the proof of Theorem 1.2. Towards this the main observation is the following result, which connects finding real zeroes to (essentially) detecting sign changes among the Fourier coefficients.

**Proposition 12.1.** *([13, Proposition 6.1.]) Let $\alpha \in \{-\frac{1}{2}, 0\}$. Then there are positive constants $c_1, c_2$ and $\eta$ such that, for all integers $\ell \in ]c_1, c_2\sqrt{k/\log k}[$ and all Hecke eigenforms $g \in S^+_{k+\frac{1}{2}}(4)$, we have*

$$\sqrt{\alpha_g}\left(\frac{e}{\ell}\right)^{\frac{k}{2}-\frac{1}{4}} g(\alpha+iy_\ell) = \sqrt{\alpha_g}c_g(\ell)e(\alpha\ell) + O(k^{-1/2-\eta}),$$

*where $y_\ell := (k-1/2)/4\pi\ell$ and the implicit constant in the error term is absolute.*

Let $\alpha \in \{-\frac{1}{2}, 0\}$, $\varepsilon > 0$ be any fixed small constant, and let $K$ be large positive parameter. Recall that as $g(\alpha+iy)$ is real-valued for these values of $\alpha$, Proposition 12.1 yields information on the zeroes inside the Siegel sets

$$\mathcal{F}_Y = \{z \in \Gamma_0(4)\backslash\mathbb{H} : \mathrm{Im}(z) \geq Y\}$$

with $c_1'\sqrt{k\log k} \leq Y \leq c_2'k$ for some positive constants $c_1'$ and $c_2'$. Let $\eta$ be as in Proposition 12.1. It follows immediately from that result that if we can find numbers $\ell_1, \ell_2 \in ]c_1, c_2k/Y[$ so that

$$\sqrt{\alpha_g}c_g(\ell_1)e(\alpha\ell_1) < -k^{-\delta} < k^{-\delta} < \sqrt{\alpha_g}c_g(\ell_2)e(\alpha\ell_2) \tag{12.1}$$

for some $\delta < 1/2+\eta$, then $g(z)$ has a zero $\alpha+iy$ with $y$ between $y_{\ell_1}$ and $y_{\ell_2}$. Observe that

$$e(\alpha\ell) = \begin{cases} 1 & \text{if } \alpha = 0 \\ (-1)^\ell & \text{if } \alpha = -1/2 \end{cases}$$

Hence, in order to find real zeroes on the line $\mathrm{Re}(s) = 0$ it suffices (essentially) to detect sign changes among the Fourier coefficients whereas on the line $\mathrm{Re}(s) = -1/2$ one needs to find pairs $(\ell_1, \ell_2)$ with $\ell_i$ odd for which (12.1) holds. As we restrict to odd fundamental discriminants $d$ for which $(-1)^k d > 0$, we automatically obtain real zeroes on both of the individual geodesic segments $\mathrm{Re}(s) = -1/2$ and $\mathrm{Re}(s) = 0$.

Several positive constants appear throughout the proof. We start by summarising their roles. The order in which we fix the constants matters as the constants fixed later may depend on the constants chosen earlier in order to satisfy certain requirements. The constant $C_5$, $C_8$, $C_7$, $C_2$, $C_{11}$, and $C_1$ are fixed in this order.

(1) $C_1$ we are able to choose freely provided that it is large enough compared to $C_7$.
(2) $C_2$ we are able to choose freely provided that it is $< 1$.
(3) $C_3$ is an absolute constant[7] so that

$$\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{g\in B^+_{k+1/2}} \alpha_g^2 \omega_g^{-1} \sideset{}{^\flat}\sum_{d} |c_g(|d|)|^4 \phi\left(\frac{|d|}{X}\right) \leq C_3 XK.$$

(4) $C_4$ is an absolute constant so that

$$\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{g\in B^+_{k+1/2}} \sideset{}{^\flat}\sum_{d} |c_g(|d|)|^2 \phi\left(\frac{|d|}{X}\right) \geq C_4 XK.$$

(5) $C_5$ we are able to choose freely provided that it is sufficiently large.
(6) $C_6 := C_4 - C_3/\sqrt{C_5}$.
(7) $C_7$ we are able to choose freely provided that it is sufficiently small in terms of $C_8$.
(8) $C_8$ we are able to choose freely provided that it is sufficiently large.
(9) $C_9$ is an absolute constant so that

$$\sum_{g\in B^+_{k+1/2}} \omega_g^{1/2} \leq C_9\sqrt{k}.$$

(10) $C_{10} := (C_6 - 2C_7C_8C_9)/C_8\sqrt{C_5}$.
(11) $C_{11}$ we are able to choose freely provided that it is arge enough in terms of $C_2$.
(12) $C_{12} := C_{11}/4$.

We require these to satisfy the following constraints:

(1) $C_6 > 0$, i.e. $C_4 > C_3/\sqrt{C_5}$.
(2) $C_7C_8C_9 < C_6$.
(3) $C_{12} < C_{10}$.
(4) $C_{11} > 1 - C_2$.
(5) $C_{10} - C_{12} > 2C_3/C_2C_1^2$.
(6) $C_1 > C_7$.

Note that the existence of the constants $C_3$ and $C_4$ follows from Propositions 4.1 and 4.2, respectively. Recall that the constants $C_3$, $C_4$ and $C_9$ are absolute. The first condition is satisfied by choosing $C_5$ to be large enough. Similarly, if we have chosen $C_8$ so that it is large enough, the second condition holds by choosing $C_7$ to be small enough. The third condition is true when $C_{11}$ is chosen to be small enough. When $C_{11}$ is chosen, the fourth condition holds when $C_2$ is chosen suitably. The fifth condition is met, note that $C_5$, $C_3$, $C_7$, $C_{11}$ and $C_2$ are already fixed, by choosing $C_1$ to be large enough. This choice of $C_1$ also satisfies the final condition if $C_1$ to be sufficiently large compared to $C_7$.

For the rest of the paper, set $X = K/Y$. Then $K^\vartheta \ll X \ll \sqrt{K/\log K}$ by the assumptions on $Y$. Remember that in order to detect sign changes along the sequence $d \equiv 1 \,(\mathrm{mod}\ 4)$ with $d$ squarefree and

[7]Constants $C_3$ and $C_4$ are allowed to depend on the weight functions $\phi$ and $h$ specified later.

$(-1)^k d > 0$ in the short interval $[x, x+H]$, $x \sim X$, it suffices to have

$$\left| \sideset{}{^\flat}\sum_{x \le (-1)^k d \le x+H} \sqrt{\alpha_g} c_g(|d|) M_g(d) \right| < \sideset{}{^\flat}\sum_{x \le (-1)^k d \le x+H} \sqrt{\alpha_g} |c_g(|d|)| M_g(d).$$

Now the proof can be completed following the arguments of [13]. Denote

$$S_{1,g}(x;H) := \left| \sideset{}{^\flat}\sum_{x \le (-1)^k d \le x+H} \sqrt{\alpha_g} c_g(|d|) M_g(d) \right|$$

and

$$\mathcal{T}_{1,g}(X;H) := \# \left\{ x \sim X : S_{1,g}(x;H) \ge C_1 \cdot \sqrt{H} \cdot k^{-1/2} \right\}.$$

Choose $h$ to be a non-negative smooth function that is supported in the interval $[1, 9/2]$ and is identically one in $[3/2, 4]$. With the above notation the quantity we need to bound is by Markov's inequality

$$\begin{aligned}
&\sum_{k \sim K} \sum_{\substack{g \in B^+_{k+\frac{1}{2}} \\ |\mathcal{T}_{1,g}(X;H)| \ge C_2 \cdot X}} 1 \\
&\le \frac{1}{C_2 X} \sum_{k \in \mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{g \in B^+_{k+\frac{1}{2}}} |\mathcal{T}_{1,g}(X;H)| \\
&= \frac{1}{C_2 X} \sum_{k \in \mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{g \in B^+_{k+\frac{1}{2}}} \sum_{\substack{x \sim X \\ |S_{1,g}(x;H)| \ge \sqrt{C}_1 H k^{-1/2}}} 1 \\
&\le \frac{2K}{C_2 C_1^2 H X} \sum_{k \in \mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{g \in B^+_{k+\frac{1}{2}}} \alpha_g \sum_{x \sim X} \left| \sideset{}{^\flat}\sum_{x \le (-1)^k d \le x+H} c_g(|d|) M_g(d) \right|^2 . \qquad (12.2)
\end{aligned}$$

By opening the absolute square the inner sum over $x \sim X$ can be rearranged into

$$\sum_{x \sim X} \sideset{}{^\flat}\sum_{x \le (-1)^k d_1 \le x+H} \sideset{}{^\flat}\sum_{x \le (-1)^k d_2 \le x+H} c_g(|d_1|) c_g(|d_2|) M_g(d_1) M_g(d_2).$$

Let us first focus on the diagonal terms with $d_1 = d_2$. In this case the total contribution to (12.2) is given by

$$\le \frac{2K}{C_2 C_1^2 X} \sum_{k \in \mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{g \in B^+_{k+\frac{1}{2}}} \alpha_g \sideset{}{^\flat}\sum_{(-1)^k d \sim X} |c_g(|d|)|^2 M_g(d)^2.$$

Adding a smooth weight function $\phi$ that localises $d \sim X$ and applying Proposition 4.1 this is bounded by

$$\le \frac{2C_3 K^2}{C_2 C_1^2}.$$

The off-diagonal contribution is negligible identically as in the proof of [13, Proposition 2.5.] using the trivial bound $M_g(d) \ll K^{\delta_0} (\log K)^{1/4}$.

Next we derive a lower bound for the weighted sum of the terms $|c_g(|d|)|$ on average over the forms $g \in \mathcal{S}_K$. Applying Hölder's inequality as in the proof of [13, Proposition 2.6.] we have

$$\sum_{k \in \mathbb{Z}} h\left(\frac{2k-1}{K}\right) \sum_{g \in B^+_{k+\frac{1}{2}}} \sideset{}{^\flat}\sum_{(-1)^k d \sim X} \alpha_g^{1/2} \omega_g^{1/2} |c_g(|d|)| M_g(d) \ge C_4 \cdot XK.$$

On the other hand, by Proposition 4.1 and Waldspurger's formula we also have

$$\begin{aligned}&\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right)\sum_{g\in B^+_{k+\frac12}}\ \sideset{}{^\flat}\sum_{\substack{(-1)^k d\sim X\\ M_g(d)^2L(1/2,f\otimes\chi_d)>C_5}} \alpha_g^{1/2}\omega_g^{1/2}|c_g(|d|)|M_g(d)\\
&\le \frac{1}{\sqrt{C_5}}\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right)\sum_{g\in B^+_{k+\frac12}}\ \sideset{}{^\flat}\sum_{\substack{(-1)^k d\sim X\\ M_g(d)^2L(1/2,f\otimes\chi_d)>C_5}} \alpha_g|c_g(|d|)|^2M_g(d)^2\\
&\le \frac{C_3}{\sqrt{C_5}}\cdot XK.\end{aligned}$$

From this we infer the lower bound

$$\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right)\sum_{g\in B^+_{k+\frac12}}\ \sideset{}{^\flat}\sum_{\substack{(-1)^k d\sim X\\ M_g(d)^2L(1/2,f\otimes\chi_d)\le C_5}} \alpha_g^{1/2}\omega_g^{1/2}|c_g(|d|)|M_g(d)\ge C_6KX.$$

Let us now define the set

$$\mathcal{V}_g := \left\{ x\sim X:\ \sideset{}{^\flat}\sum_{\substack{x\le(-1)^k d\le x+H\\ M_g(d)^2L(1/2,f\otimes\chi_d)\le C_5}} \sqrt{\alpha_g}|c_g(|d|)|M_g(d)\ge \frac{C_7H}{\sqrt{k}}\right\}.$$

From the work above it follows that

$$\begin{aligned}C_6KX &\le \sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right)\sum_{g\in B^+_{k+\frac12}}\ \sideset{}{^\flat}\sum_{\substack{(-1)^k d\sim X\\ M_g(d)^2L(1/2,f\otimes\chi_d)\le C_5}} \alpha_g^{1/2}\omega_g^{1/2}|c_g(|d|)|M_g(d)\\
&\le C_8\cdot\frac{1}{H}\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right)\sum_{g\in B^+_{k+\frac12}}\sum_{x\sim X}\ \sideset{}{^\flat}\sum_{\substack{x\le(-1)^k d\le x+H\\ M_g(d)^2L(1/2,f\otimes\chi_d)\le C_5}} \alpha_g^{1/2}\omega_g^{1/2}|c_g(|d|)|M_g(d)\\
&= \frac{C_8}{H}\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right)\sum_{g\in B^+_{k+\frac12}}\left(\sum_{x\in\mathcal{V}_g}+\sum_{x\notin\mathcal{V}_g}\right)\ \sideset{}{^\flat}\sum_{\substack{x\le(-1)^k d\le x+H\\ M_g(d)^2L(1/2,f\otimes\chi_d)\le C_5}} \alpha_g^{1/2}\omega_g^{1/2}|c_g(|d|)|M_g(d)\\
&\le \frac{C_8}{H}\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right)\sum_{g\in B^+_{k+\frac12}}\left(\sqrt{C_5}\omega_gH\,|\mathcal{V}_g|+\omega_g^{1/2}\frac{C_7XH}{\sqrt{k}}\right),\end{aligned}$$

where we have used the relation $\omega_g^{1/2}\alpha_g^{1/2}|c_g(|d|)| = \omega_g\sqrt{L(1/2,f\otimes\chi_d)}$ in the last estimate. Using an easy estimate $\sum_{g\in B^+_{k+\frac12}}\omega_g^{1/2}\le C_9\sqrt{k}$ (which follows from the Cauchy–Schwarz inequality and the fact that $\sum_{g\in B^+_{k+\frac12}}\omega_g\sim 1$) we conclude that

$$\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right)\sum_{g\in B^+_{k+\frac12}}\omega_g\,|\mathcal{V}_g|\ge C_{10}KX. \tag{12.3}$$

The harmonic weights $\omega_g$ can be removed without affecting the lower bound [1, 12] and so we deduce that

$$\sum_{k\in\mathbb{Z}} h\left(\frac{2k-1}{K}\right)\sum_{g\in B^+_{k+\frac12}}|\mathcal{V}_g|\ge C_{10}K^2X. \tag{12.4}$$

Let us introduce the set

$$\mathcal{U} := \{g \in \mathcal{S}_K : |\mathcal{V}_g| \geq C_{11}X\}.$$

Now from (12.4) we deduce that

$$\begin{aligned} C_{10}XK^2 &\leq \sum_{g\in\mathcal{U}} |\mathcal{V}_g| + \sum_{g\in\mathcal{S}_K\setminus\mathcal{U}} |\mathcal{V}_g| \\ &\leq |\mathcal{U}|X + C_{12}K^2X. \end{aligned}$$

from which we infer the lower bound

$$|\mathcal{U}| \geq (C_{10} - C_{12})K^2$$

by recalling that $C_{12} = C_{11}/4$ and $C_{11}$ is chosen so that $C_{12} < C_{10}$.

Hence we have shown that for $\geq (C_{10} - C_{12})K^2$ of the forms $g \in \mathcal{S}_K$ we have

$$\begin{aligned} &\#\left\{x \sim X : \sideset{}{^\flat}\sum_{x\leq(-1)^k d\leq x+H} \sqrt{\alpha_g}|c_g(|d|)|M_g(d) \geq \frac{C_7H}{\sqrt{k}}\right\} \\ &\geq \#\left\{x \sim X : \sideset{}{^\flat}\sum_{\substack{x\leq(-1)^k d\leq x+H \\ M_g(d)^2L(1/2,f\otimes\chi_d)\leq C_5}} \sqrt{\alpha_g}|c_g(|d|)|M_g(d) \geq \frac{C_7H}{\sqrt{k}}\right\} \\ &\geq C_{11}X. \end{aligned}$$

In conclusion we have shown that for all but $\leq 2C_3 \cdot K^2/C_2C_1^2$ of the forms $g \in S_K$ it holds that

$$\left|\sideset{}{^\flat}\sum_{x\leq(-1)^k d\leq x+H} \sqrt{\alpha_g}c_g(|d|)M_g(d)\right| < \frac{C_1\sqrt{H}}{\sqrt{k}}$$

for a positive proportion of $x \sim X$ with the exceptional set having size $\leq (1 - C_2)X$. Moreover, for at least $(C_{10} - C_{12})K^2$ of the forms $g \in S_K$ it holds that

$$\sideset{}{^\flat}\sum_{x\leq(-1)^k d\leq x+H} \sqrt{\alpha_g}|c_g(|d|)|M_g(d) \geq \frac{C_7H}{\sqrt{k}}$$

for $\geq C_{11}X$ of the numbers $x \sim X$. By recalling that our choices of constants are so that $C_{11} > 1 - C_2$ and $2C_3/C_2C_1^2 < C_{10}$ it follows that for $\geq (C_{10} - C_{12} - 2C_3/C_2C_1^2)K^2$ of the forms $g \in S_K$ the chain of inequalities

$$\left|\sideset{}{^\flat}\sum_{x\leq(-1)^k d\leq x+H} \sqrt{\alpha_g}c_g(|d|)M_g(d)\right| < \frac{C_1 \cdot \sqrt{H}}{\sqrt{k}} < \frac{C_7 \cdot H}{\sqrt{k}} \leq \sideset{}{^\flat}\sum_{x\leq(-1)^k d\leq x+H} \sqrt{\alpha_g}|c_g(|d|)|M_g(d) \tag{12.5}$$

holds for $\geq (C_{11} - 1 + C_2)X$ of the numbers $x \sim X$ if we choose $H > (C_1/C_7)^2$.

Choose $H$ to be large enough ($> (C_1/C_7)^2$), but fixed. With this choice it follows easily from (12.5) that for $\geq (C_{10} - C_{12} - 2C_3/C_2C_1^2)K^2$ of the forms $g \in \mathcal{S}_K$ we have that

$$\left|\sideset{}{^\flat}\sum_{x\leq(-1)^k d\leq x+H} \sqrt{\alpha_g}|c_g(|d|)|M_g(d) \pm \sideset{}{^\flat}\sum_{x\leq(-1)^k d\leq x+H} \sqrt{\alpha_g}c_g(|d|)M_g(d)\right| \geq \frac{(C_7H - C_1\sqrt{H})}{\sqrt{k}} > 0$$

holds for $\geq (C_{11}-1+C_2)X$ of the numbers $x \sim X$. We note that the contribution coming from the summands with $\sqrt{\alpha_g}|c_g(|d|) \leq k^{-\delta}$ is trivially bounded by, say,

$$\leq \sum_{\substack{x\leq(-1)^k d\leq x+H \\ \sqrt{\alpha_g}|c_g(|d|)|\leq k^{-\delta}}} \sqrt{\alpha_g}|c_g(|d|)|M_g(d) \leq 2K^{-\delta} \sum_{\substack{x\leq(-1)^k d\leq x+H \\ \sqrt{\alpha_g}|c_g(|d|)|\leq k^{-\delta}}} M_g(d) \leq 2HK^{-\delta+\delta_0}(\log K)^{1/4}$$

as $\eta_1$ is sufficiently small, and so we conclude, choosing $\delta = 1/2 + \eta/2$ for concreteness so that $\delta > 1/2$, that for the same proportion of $g \in \mathcal{S}_K$ and $x \sim X$ we have

$$\left| \sideset{}{^\flat}\sum_{\substack{x \le (-1)^k d \le x+H \\ \sqrt{\alpha_g}|c_g(|d|)| > k^{-\delta}}} \sqrt{\alpha_g}|c_g(|d|)| M_g(d) \pm \sideset{}{^\flat}\sum_{\substack{x \le (-1)^k d \le x+H \\ \sqrt{\alpha_g}|c_g(|d|)| > k^{-\delta}}} \sqrt{\alpha_g} c_g(|d|) M_g(d) \right| \ge \frac{(C_7 H - C_1\sqrt{H})}{\sqrt{k}} > 0$$

for sufficiently large $k$ provided that $\delta_0 < \eta/2$. Now observe that this implies

$$\begin{aligned} &2 \sideset{}{^\flat}\sum_{\substack{x \le (-1)^k d \le x+H \\ \sqrt{\alpha_g} c_g(|d|) > k^{-\delta}}} \sqrt{\alpha_g}|c_g(|d|)| M_g(d) \\ &= \left| \sideset{}{^\flat}\sum_{\substack{x \le (-1)^k d \le x+H \\ \sqrt{\alpha_g}|c_g(|d|)| > k^{-\delta}}} \sqrt{\alpha_g}|c_g(|d|)| M_g(d) + \sideset{}{^\flat}\sum_{\substack{x \le (-1)^k d \le x+H \\ \sqrt{\alpha_g}|c_g(|d|)| > k^{-\delta}}} \sqrt{\alpha_g} c_g(|d|) M_g(d) \right| \\ &\ge \frac{(C_7 H - C_1\sqrt{H})}{\sqrt{k}} > 0. \end{aligned}$$

Similarly,

$$\begin{aligned} &2 \sideset{}{^\flat}\sum_{\substack{x \le (-1)^k d \le x+H \\ \sqrt{\alpha_g} c_g(|d|) < -k^{-\delta}}} \sqrt{\alpha_g}|c_g(|d|)| M_g(d) \\ &= \left| \sideset{}{^\flat}\sum_{\substack{x \le (-1)^k d \le x+H \\ \sqrt{\alpha_g}|c_g(|d|)| > k^{-\delta}}} \sqrt{\alpha_g}|c_g(|d|)| M_g(d) - \sideset{}{^\flat}\sum_{\substack{x \le (-1)^k d \le x+H \\ \sqrt{\alpha_g}|c_g(|d|)| > k^{-\delta}}} \sqrt{\alpha_g} c_g(|d|) M_g(d) \right| \\ &\ge \frac{(C_7 H - C_1\sqrt{H})}{\sqrt{k}} > 0. \end{aligned}$$

Thus we have shown that for $\ge (C_{10} - C_{12} - 2C_3/C_2C_1^2)K^2$ of the forms $g \in \mathcal{S}_K$ the short interval $[x, x+H]$, $x \sim X$, contains numbers $(-1)^k d_\pm$, with both $d_\pm$ odd fundamental discriminants, for which $\sqrt{\alpha_g} c_g(|d_+|) > k^{-\delta}$ and $\sqrt{\alpha_g} c_g(|d_-|) < -k^{-\delta}$, for $\ge (C_{11} - 1 - C_2)X$ of the numbers $x \sim X$. Thus we deduce a sign change of $c_g(|d|)$ over a positive proportion of intervals of constant size $> (C_1/C_7)^2$ for a positive proportion of forms $g \in S_K$. This leads to

$$\ge \frac{(C_{11} - 1 - C_2)X}{H} \asymp \frac{K}{Y}$$

real zeroes on both of the line segments $\delta_1$ and $\delta_2$, concluding the proof. □

## References


[1] Olga Balkanova and Dmitry Frolenkov. Moments of $L$-functions and the Liouville-Green method. *J. Eur. Math. Soc. (JEMS)*, 23(4):1333–1380, 2021.

[2] Bruce C. Berndt, Ronald J. Evans, and Kenneth S. Williams. *Gauss and Jacobi sums*. Canadian Mathematical Society Series of Monographs and Advanced Texts. John Wiley & Sons, Inc., New York, 1998. A Wiley-Interscience Publication.

[3] Valentin Blomer. Non-vanishing of class group $L$-functions at the central point. *Ann. Inst. Fourier (Grenoble)*, 54(4):831–847, 2004.

[4] Valentin Blomer. On the central value of symmetric square $L$-functions. *Math. Z.*, 260(4):755–777, 2008.

[5] Valentin Blomer, Rizwanur Khan, and Matthew Young. Distribution of mass of holomorphic cusp forms. *Duke Math. J.*, 162(14):2609–2644, 2013.

[6] Hung M. Bui, Natalie Evans, Stephen Lester, and Kyle Pratt. Weighted central limit theorems for central values of $L$-functions. *J. Eur. Math. Soc. (JEMS)*, 27(6):2477–2529, 2025.

[7] Amit Ghosh and Peter Sarnak. Real zeros of holomorphic Hecke cusp forms. *J. Eur. Math. Soc. (JEMS)*, 14(2):465–487, 2012.
[8] Roman Holowinsky and Kannan Soundararajan. Mass equidistribution for Hecke eigenforms. *Ann. of Math. (2)*, 172(2):1517–1528, 2010.
[9] M. N. Huxley. *Area, lattice points, and exponential sums*, volume 13 of *London Mathematical Society Monographs. New Series*. The Clarendon Press, Oxford University Press, New York, 1996. Oxford Science Publications.
[10] Henryk Iwaniec and Emmanuel Kowalski. *Analytic number theory*, volume 53 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2004.
[11] Henryk Iwaniec, Wenzhi Luo, and Peter Sarnak. Low lying zeros of families of $L$-functions. *Inst. Hautes Études Sci. Publ. Math.*, (91):55–131, 2000.
[12] Henryk Iwaniec and Peter Sarnak. The non-vanishing of central values of automorphic $L$-functions and Landau-Siegel zeros. *Israel J. Math.*, 120:155–177, 2000.
[13] Jesse Jääsaari. On the real zeroes of half-integral weight Hecke cusp forms. *To appear in Math. Ann.*, 2025.
[14] Jesse Jääsaari, Stephen Lester, and Abhishek Saha. On fundamental Fourier coefficients of Siegel cusp forms of degree 2. *J. Inst. Math. Jussieu*, 22(4):1819–1869, 2023.
[15] Jesse Jääsaari, Stephen Lester, and Abhishek Saha. Mass equidistribution for Saito-Kurokawa lifts. *Geom. Funct. Anal.*, 34(5):1460–1532, 2024.
[16] Rizwanur Khan. Non-vanishing of the symmetric square $L$-function at the central point. *Proc. Lond. Math. Soc. (3)*, 100(3):736–762, 2010.
[17] W. Kohnen and D. Zagier. Values of $L$-series of modular forms at the center of the critical strip. *Invent. Math.*, 64(2):175–198, 1981.
[18] Winfried Kohnen. Newforms of half-integral weight. *J. Reine Angew. Math.*, 333:32–72, 1982.
[19] Youness Lamzouri. Large sums of Hecke eigenvalues of holomorphic cusp forms. *Forum Math.*, 31(2):403–417, 2019.
[20] Stephen Lester, Kaisa Matomäki, and Maksym Radziwiłł. Small scale distribution of zeros and mass of modular forms. *J. Eur. Math. Soc. (JEMS)*, 20(7):1595–1627, 2018.
[21] Stephen Lester and Maksym Radziwiłł. Signs of Fourier coefficients of half-integral weight modular forms. *Math. Ann.*, 379(3-4):1553–1604, 2021.
[22] Wenzhi Luo and Peter Sarnak. Quantum variance for Hecke eigenforms. *Ann. Sci. École Norm. Sup. (4)*, 37(5):769–799, 2004.
[23] Kaisa Matomäki. Real zeros of holomorphic Hecke cusp forms and sieving short intervals. *J. Eur. Math. Soc. (JEMS)*, 18(1):123–146, 2016.
[24] Kaisa Matomäki and Maksym Radziwiłł. Multiplicative functions in short intervals. *Ann. of Math. (2)*, 183(3):1015–1056, 2016.
[25] Maksym Radziwiłł and K. Soundararajan. Moments and distribution of central $L$-values of quadratic twists of elliptic curves. *Invent. Math.*, 202(3):1029–1068, 2015.
[26] Zeév Rudnick. On the asymptotic distribution of zeros of modular forms. *Int. Math. Res. Not.*, (34):2059–2074, 2005.
[27] Zeév Rudnick and Peter Sarnak. The behaviour of eigenstates of arithmetic hyperbolic manifolds. *Comm. Math. Phys.*, 161(1):195–213, 1994.
[28] Bernard Shiffman and Steve Zelditch. Distribution of zeros of random and quantum chaotic sections of positive line bundles. *Comm. Math. Phys.*, 200(3):661–683, 1999.
[29] K. Soundararajan. Nonvanishing of quadratic Dirichlet $L$-functions at $s = \frac{1}{2}$. *Ann. of Math. (2)*, 152(2):447–488, 2000.
[30] Peter Zenz. Quantum variance for holomorphic hecke cusp forms on the vertical geodesic. *arXiv:2111.04713*, 2021.

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF TURKU, 20014 TURKU, FINLAND
*Email address*: `jesse.jaasaari@utu.fi`