Exponential Sums Related to Maass Cusp Forms

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Acknowledgements

List of Included Articles

This dissertation consists of an introductory part and four research articles. During the introduction we refer to these articles by letters [A]-[D].

[A] Anne-Maria Ernvall-Hytönen, Jesse Jääsaari, and Esa V. Vesalainen: Resonances and Ω-Results for Exponential Sums Related to Maass Forms for $\mathrm{SL}(n, \mathbb{Z})$, Journal of Number Theory, Vol. 153 (2015), $135 - 157$.

[B] Jesse Jääsaari, and Esa V. Vesalainen: On Sums Involving Fourier Coefficients of Maass Forms for $\mathrm{SL}(3, \mathbb{Z})$. Functiones et Approximatio Commentarii Mathematici, Vol. 57, No. 2 (2017), $255 - 275$.

[C] Jesse Jääsaari, and Esa V. Vesalainen: Exponential Sums Related to Maass Forms, to appear in Acta Arithmetica. Preprint available at arXiv:1409.7235.

[D] Jesse Jääsaari: On Short Sums Involving Fourier Coefficients of Maass Forms. Preprint available at arXiv:1804.09702.

The author of this thesis had an equal role in the research and writing of the joint articles.

Contents

1 Overview

Fourier coefficients of cusp forms are interesting objects due to their arithmetic significance but we know very little about them in general. For instance, it is an interesting question, for a fixed form, to ask how these coefficients are distributed or how large their order of magnitude can be. Such coefficients are hard to study individually and therefore it is necessary to have some other ways to study them. A classical theme in analytic number theory is to understand highly oscillatory objects, such as Fourier coefficients of cusp forms, by studying their sums and their correlations against other oscillating objects over certain intervals. These are also the main underlying themes of the present thesis. The dissertation consists of an introductory part and four research articles referred to as [A], [B], [C] and [D] where different aspects of exponential sums weighted by Fourier coefficients of Maass cusp forms for $\mathrm{SL}(n, \mathbb{Z})$ in both the classical case $n = 2$ and for larger n are studied.

Knowledge of sizes of above mentioned correlation sums can improve understanding of the nature of Fourier coefficients we are interested in, denoted by $A(m, 1, ..., 1)$. For example, if such a sum is large, this means that the Fourier coefficients and the test sequence oscillate similarly. This naturally leads to the exponential sums considered in this work. We study weighted sums of these coefficients against various oscillatory exponential phases. More specifically, we consider linear exponential phases $e(m\alpha)$ for fixed $\alpha \in \mathbb{R}$, and varying $m \in \mathbb{N}$. The problems studied in this thesis deal with sums of consecutive terms in the sequence $\{A(m, 1, ..., 1)e(m\alpha)\}_{m \in \mathbb{N}}$ over both long $1 \leq m \leq M$ and short $M \leq m \leq M + \Delta$, $\Delta = o(M)$, intervals.

Such exponential sums have been studied extensively for the Fourier coefficients of holomorphic cusp forms, denoted by $a(m)$. The first estimate for long linear exponential sums involving holomorphic cusp form coefficients was proved by Wilton [118] in the course of proving an analogue of Voronoi's summation formula for these coefficients, having the application that the L-function attached to the Ramanujan τ-function has infinitely many zeroes on the critical line in mind. Wilton's estimate states that the long linear sum is $\ll M^{1/2} \log M$ uniformly in the twist α. The logarithm was later removed by Jutila [62] giving the best possible result

$$\sum_{m \leq M} a(m)e(m\alpha) \ll M^{1/2}$$

one could hope for in light of the Rankin-Selberg asymptotics (see (10) below). This estimate signifies enormous amount of cancellation in the sum meaning that the Fourier coefficients of holomorphic cusp forms are quite far from being aligned with values of any fixed linear additive character.

After long sums, short sums are the next natural focus of investigation. Intuitively it makes sense to study them as one can suspect that short intervals might capture the erratic nature of the Fourier coefficients better than longer ones. Pointwise bounds for short sums have been obtained first by Jutila [62] and later by Ernvall-Hytönen and Karppinen [26] in the GL(2)-setting for holomorphic cusp forms. Furthermore, when Δ is small compared to M, the resulting short sums provide a natural analogue of the classical problems of analytic number theory studying various number theoretic error terms in short intervals.

Also, good estimates for short sums can be used to reduce smoothing error, thereby possibly leading to sharper results in various problems involving automorphic forms[1] such as the subconvexity problem for automorphic L-functions.

In the study of these linear exponential sums, the case in which the twist is near a fraction with a small denominator is often different from the one in which the twist is not close to such a fraction. The behaviour near such rational values is often strongly linked to the behaviour at such rational points, and hence it is also natural to study sums with a rational twist. Besides, rationally twisted linear exponential sums are more closely related to the classical problems of understanding the error terms in the Dirichlet divisor problem and the Gauss circle problem. Indeed, these concepts are closely related, largely due to the fact that both these problems have modular origins. Namely, the divisor function $d(n)$ appears as the nth Fourier coefficient of the modular form $\partial/\partial s E(s, z)|_{s=1/2}$, where $E(s, z)$ is the Eisenstein series for $\mathrm{SL}(2, \mathbb{Z})$. However, this is not a cusp form.

Analytic number theory of automorphic forms has seen many advances in the classical setting over the years but results are sporadic for automorphic forms of higher rank. There are many ways to try to generalise the classical theory to a higher rank setting. The underlying group in the classical theory is $\mathrm{SL}(2, \mathbb{R})$. One possible way is to note that $\mathrm{SL}(2, \mathbb{R})$ is the same as $\mathrm{Sp}(2, \mathbb{R})$ and then pass to $\mathrm{Sp}(2n, \mathbb{R})$. While this yields a rich theory, in this thesis we consider a more natural analogue of automorphic forms for the group $\mathrm{SL}(n, \mathbb{R})$. In particular, the higher rank automorphic forms we consider are Maass forms for $\mathrm{SL}(n, \mathbb{Z})$ for general $n \geq 3$ and in the special case $n = 3$. It turns out that these are natural analogues of classical Maass waveforms of $\mathrm{SL}(2, \mathbb{Z})$ in this higher rank setting. There is no analogous theory of holomorphic forms for $\mathrm{SL}(n, \mathbb{Z})$, with $n \geq 3$, due to the fact that the group $\mathrm{SL}(n, \mathbb{R})$ does not admit a discrete series representation for such n, or because the generalised upper half-plane $\mathbb{H}^n$, defined below, does not have a complex structure for $n \geq 3$.

It has been understood for a long time that periodic functions are central objects throughout science as they describe various natural phenomena that exhibit periodicity. Holomorphic modular forms and Maass forms can be viewed as certain analogues of periodic functions in the hyperbolic plane $\mathbb{H}$ in the following way. A 1-periodic function $f : \mathbb{R} \longrightarrow \mathbb{R}$ can be thought as a function invariant under the natural action of $\mathbb{Z}$ on $\mathbb{R}$. Similarly, classical Maass forms are functions $f : \mathbb{H} \longrightarrow \mathbb{C}$ invariant under the action of $\mathrm{SL}(2, \mathbb{Z})$ by the linear fractional transformations on $\mathbb{H}$ (for holomorphic modular forms one needs a more general transformation rule) satisfying some additional conditions. Higher rank Maass forms are defined similarly with respect to the action of the group $\mathrm{SL}(n, \mathbb{Z})$ in the generalised upper half-plane $\mathbb{H}^n$. We will focus on special types of Maass forms called Maass cusp forms; these are Maass forms that vanish at the cusps of the action of $\mathrm{SL}(n, \mathbb{Z})$ on $\mathbb{H}^n$. These forms have a Fourier-Whittaker expansion involving Fourier coefficients (not in the literal sense) $A(m_1, ..., m_{n-1})$ parametrised by $(n-1)$-tuples of integers. We will focus our attention to the coefficients $A(m, 1, ..., 1)$ as these appear in the standard L-function attached to the underlying Maass cusp form.

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[1]In this thesis, the term automorphic form refers mostly to either integral weight holomorphic modular forms or Maass forms but we also mention half-integral forms in Section 2.

Fourier coefficients are connected to arithmetic due to the fact that there is a large family of symmetries, so-called Hecke operators, acting on the space of cusp forms. It turns out that Hecke eigenvalues, which encode arithmetic data, can be expressed as polynomials of the Fourier coefficients. Furthermore, for each $m \in \mathbb{N}$ there is a particular Hecke operator T_m such that if a Maass cusp form f is an eigenfunction of all Hecke operators and normalised so that $A(1, ..., 1) = 1$, then $T_m f = A(m, 1, ..., 1) f$ for any $m \in \mathbb{N}$. This gives another reason to concentrate on these particular coefficients. The questions of interest are both the size and the distribution of these coefficients. We shall consider both upper and lower bounds for sums involving these coefficients with general exponential twists and also obtain sharper results in the case of an additive rational twist. The average behaviour of these sums is also investigated.

A valuable tool for analysing these sums are the so-called Voronoi summation formulas. These roughly dualise the sums into other sums that are easier to analyse. Voronoi summation formulas are usually established for smoothed sums but in some applications, especially those concerning moments of these sums, it is beneficial to have truncated Voronoi summation formulas with a sharp cut-off. Morally this is the same as replacing the smooth weight function by a characteristic function of an interval but this leads to analytic complications such as delicate issues with convergence.

The outline for this introductory part is as follows. In Section 2 we introduce key concepts and definitions and in particular explain how definitions for general n naturally generalise the classical situation. Section 3 deals with exponential sums weighted by the Fourier coefficients of higher rank Maass cusp forms in more detail. In Section 4 we discuss Voronoi summation formulas which are key tools in proofs of many results contained in this thesis. This section also includes discussion about some results included in Articles [A] and [B]. Sections 5, 6, 7, and 8 contain a summary of the results and techniques from Articles [A], [B], [C], and [D] in the dissertation, respectively. As many proofs are quite involved, we will only give sketches of them in this introductory part to illustrate the main ideas which can get lost beneath the detailed computations.

2 Automorphic forms

Classical modular forms made their first appearance in the late 19$^{\text{th}}$ and early 20$^{\text{th}}$ century in complex analysis. Originally, they arose from the theory of elliptic functions and have since been connected to various other branches of mathematics, e.g. number theory, combinatorics, representation theory and mathematical physics. There are several good introductory texts on the basic theory of modular forms, e.g. [13, 49, 104, 106]. We will not work on the most general level and shall only consider the case of the full (i.e. level 1) modular group $\mathrm{SL}(n, \mathbb{Z})$. We give definitions with respect to this group, but the reader can imagine that there is a similar story for other discrete subgroups of $\mathrm{SL}(n, \mathbb{R})$.

2.1 The upper half-plane

Before we consider automorphic forms, let us say something about the space they live in. The basic facts about hyperbolic geometry can be found, for instance,

in [50, 65]. The upper half-plane of the complex plane is

$$\mathbb{H} := \{ z = x + iy \in \mathbb{C} : y > 0 \}.$$

Equipping $\mathbb{H}$ with a Riemannian metric

$$\mathrm{d}s^2 := \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}$$

makes it a model for the hyperbolic plane which is a two-dimensional Riemannian manifold of constant negative curvature -1. The geometry of $\mathbb{H}$ differs from the standard Euclidean one. For example, geodesics are vertical straight half-lines and half circles perpendicular to the real axis. In small scale the geometry of $\mathbb{H}$, however, resembles the Euclidean one. For instance, the area of a hyperbolic circle of radius r is $\sim \pi r^2$ when $r \longrightarrow 0$. In large scale the situation changes drastically: the area of a circle of radius r is $\sim \pi e^r$ as $r \longrightarrow \infty$.

The group

$$\mathrm{SL}(2, \mathbb{R}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_{2 \times 2}(\mathbb{R}) \ : \ ad - bc = 1 \right\}$$

acts on $\mathbb{H}$ by fractional linear transformations:

$$\gamma.z := \frac{az + b}{cz + d} \quad \text{for all } z \in \mathbb{H} \text{ and for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}).$$

It turns out that $\mathrm{SL}(2, \mathbb{R})$ is the group of isometries of the hyperbolic upper half-plane. As usual, it makes sense to study the discontinuous action of a discrete subgroup $\mathrm{SL}(2, \mathbb{Z})$ of $\mathrm{SL}(2, \mathbb{R})$. The fundamental domain of this action in $\mathbb{H}$ is given by

$$\mathrm{SL}(2, \mathbb{Z}) \backslash \mathbb{H} = \left\{ z \in \mathbb{C} : |z| \geq 1, \quad -\frac{\sqrt{3}}{2} \leq \Re(z) \leq \frac{\sqrt{3}}{2} \right\}.$$

There are several reasons to study this quotient, perhaps the most natural one being the fact that it parametrises elliptic curves over $\mathbb{C}$, or equivalently complex tori.

In order to discuss square-integrable functions on the above quotient, we need an appropriate measure. It turns out that the right measure is

$$\mathrm{d}\mu(z) := \frac{\mathrm{d}x \mathrm{d}y}{y^2},$$

which is indeed invariant under the action of the group $\mathrm{GL}(2, \mathbb{R})$ on $\mathbb{H}$. Now we may define the L^2-space $L^2(\mathrm{SL}(2, \mathbb{Z}) \backslash \mathbb{H})$ in the usual way as the space of square-integrable functions (wrt. measure $\mathrm{d}\mu$) such that $f(\gamma.z) = f(z)$ for every $z \in \mathbb{H}$ and $\gamma \in \mathrm{SL}(2, \mathbb{Z})$. There is a natural inner product in this space, named after Petersson,

$$\langle f, g \rangle := \int_{\mathrm{SL}(2,\mathbb{Z}) \backslash \mathbb{H}} f(z) \overline{g}(z) \, \mathrm{d}\mu(z),$$

for $f, g \in L^2(\mathrm{SL}(2, \mathbb{Z}) \backslash \mathbb{H})$, which makes it a Hilbert space.

Two of the $SL(2, \mathbb{Z})$-orbits in $\mathbb{H}$ have nontrivial stabilisers in $SL(2, \mathbb{Z})$. These are i and $e^{2\pi i/3}$, whose stabilisers in $SL(2, \mathbb{Z})$ have orders 2 and 3, respectively. This gives the quotient $SL(2, \mathbb{Z}) \backslash \mathbb{H}$ the structure of an orbifold.

By the general theory of orbifolds [109], there is a differential operator which is invariant under the action of the group $SL(2, \mathbb{Z})$ on $\mathbb{H}$ given by $-\operatorname{div} \circ \operatorname{grad}$. More explicitly, it turns out that in our case this so-called Laplace-Beltrami operator is given by

$$\Delta := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

It can be proved that $GL(2, \mathbb{R})$-invariant differential operators on $\mathbb{H}$ are polynomials in the operator Δ. Now, the invariant differential operator Δ can be used to decompose the space $L^2(SL(2, \mathbb{Z}) \backslash \mathbb{H})$. The fact that the quotient $SL(2, \mathbb{Z}) \backslash \mathbb{H}$ is both non-compact and has a finite area is highlighted in the spectral theory in the sense that Δ admits both discrete and continuous spectrum.

2.2 Automorphic forms for $SL(2, \mathbb{Z})$

It is still possible to give the quotient $SL(2, \mathbb{Z}) \backslash \mathbb{H}$ the structure of a complex manifold in a natural way, while a little care has to be taken when defining charts around two orbifold points (see [13]). In light of this, it is natural to consider holomorphic functions on the said quotient. This leads to the notion of a holomorphic modular form. These are holomorphic functions on the upper half-plane which are essentially invariant under the action of some discrete subgroup of $SL(2, \mathbb{R})$. As mentioned above, for our purposes it is enough to consider the case of full modular group $SL(2, \mathbb{Z})$. By standard arguments (see e.g. [68]), a holomorphic function satisfying $f(\gamma.z) = f(z)$ for all $\gamma \in SL(2, \mathbb{Z})$, $z \in \mathbb{H}$, is a constant function. But if one relaxes the invariance property slightly, there turns out to be an interesting theory. Namely, we consider invariance up to a certain cocycle $j(z, \gamma)$, that is,

$$f\left(\frac{az+b}{cz+d} \right) = j(z, \gamma) f(z) \quad \text{for every } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}), \ z \in \mathbb{H}. \quad (1)$$

It turns out that $j(z, \gamma) = (cz+d)^\kappa$ for some positive integer κ is a natural choice in light of the theory of elliptic functions by keeping track on the dependence of the underlying lattice, see [49]. A growth condition when approaching the cusp at infinity is assumed for technical reasons.

Definition 1. *A function $f : \mathbb{H} \longrightarrow \mathbb{C}$ is a holomorphic modular form of weight $\kappa \in \mathbb{Z}_+$ for $SL(2, \mathbb{Z})$ if it is a holomorphic function satisfying*

$$f\left(\frac{az+b}{cz+d} \right) = (cz+d)^\kappa f(z) \quad \text{for any} \quad z \in \mathbb{H}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$$

and has a moderate growth $|f(z)| \ll (\Im z)^N$ for some $N \in \mathbb{Z}_+$ as $z \longrightarrow i\infty$.

Examples of modular forms include the holomorphic Eisenstein series

$$E_\kappa(z) := \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{1}{(mz+n)^\kappa}$$

which is of weight κ, for even $\kappa \geq 4$, and the modular discriminant

$$\Delta(z) := e(z) \prod_{n=1}^{\infty} (1 - e(nz))^{24},$$

which is the unique cusp form (defined below) of weight 12 for the group $\mathrm{SL}(2,\mathbb{Z})$. Here and in the rest of this thesis, the notation $e(z)$ means the same as $e^{2\pi i z}$.

Remark 2. *Similarly, one can define modular forms with respect to other discrete subgroups of $\mathrm{SL}(2,\mathbb{R})$ besides $\mathrm{SL}(2,\mathbb{Z})$. The definition is similar but instead of requiring that the transformation property (1) holds for all $\gamma \in \mathrm{SL}(2,\mathbb{Z})$, we require that it holds for all $\gamma \in \Gamma$, where Γ is some discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$.*

Noticing that

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}),$$

the transformation property of modular forms with respect to this matrix takes the form $f(z) = f(z+1)$. The consequence of this and the holomorphicity is that the holomorphic modular form f of weight $\kappa \in \mathbb{Z}_+$ has a representation as a Fourier series

$$f(z) = \sum_{n=0}^{\infty} a(n) n^{\frac{\kappa-1}{2}} e(nz),$$

for some coefficients $a(n) \in \mathbb{C}$. These are called the *Fourier coefficients* of f. Here we have a normalisation by $n^{(\kappa-1)/2}$ so that the absolute values of the Fourier coefficients are bounded by the divisor function (see Section 2.3.2.). If $a(0) = 0$, we say that f is a *cusp form*. Equivalently, the condition of being a cusp form is the same as the condition

$$\int_0^1 f(z)\,\mathrm{d}x = 0.$$

Fourier coefficients of modular forms often have arithmetic significance. They may contain for instance data about the number of ways integers can be represented by a certain quadratic form. Let us give a classical example. Consider the theta-series

$$\vartheta(z) := \sum_{n=-\infty}^{\infty} e^{\pi i n^2 z}.$$

This is a (half-integral) modular form for the group

$$\Gamma_\vartheta := \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle \leq \mathrm{SL}(2,\mathbb{Z}).$$

Let $r_k(n) := \# \left\{ (x_1, ..., x_k) \in \mathbb{Z}^k : x_1^2 + \cdots + x_k^2 = n \right\}$ be the number of ways an integer n can be written as a sum of k squares. Now one easily observes that for every $k \in \mathbb{Z}^+$ we have

$$\vartheta^k(z) = \sum_{n=0}^{\infty} r_k(n) e^{\pi i n^2 z}.$$

Let us specialise to the case $k = 4$. Then one can show that $\vartheta^4(z)$ is a holomorphic modular form of weight 2 for the group Γ_ϑ. On the other hand, $\vartheta^4(z)$ can be realised, by using the so-called valence formula, as a linear combination of Eisenstein series. The upshot is that the Fourier expansion of Eisenstein series can be calculated explicitly. Comparing Fourier coefficients of $\vartheta^4(z)$ obtained in this way from the Fourier coefficients of Eisenstein series with the Fourier coefficients $r_k(n)$ leads to the beautiful formula

$$r_4(n) = 8 \sum_{\substack{d|n \\ 4\nmid d}} d$$

and, by using a similar method, in the case $k = 8$ we have

$$r_8(n) = 16 \sum_{d|n} (-1)^{n-d} d^3.$$

We give two more beautiful examples. Consider the Diophantine equation $y^2 + y = x^3 - x^2$. This is hard to solve by hand and so one needs other ways to tackle it. A natural way to obtain information about the existence of a possible solution is to consider the equation modulo various prime numbers. Let the number of solutions for such congruence modulo prime p be N_p. Then this quantity is related to the Fourier coefficients of the holomorphic cusp form (for the congruence group $\Gamma_0(11)$)

$$g(z) := q \prod_{n=1}^{\infty} (1 - q^n)^2 (1 - q^{11n})^2, \quad q = e(z),$$

in the sense that for every prime p the pth Fourier coefficient of g equals $p - N_p$. The underlying fact here is that the Galois representation attached to the elliptic curve $y^2 + y = x^3 - x^2$ is isomorphic to the Galois representation attached to the cusp form g. This is a special case of the so-called modularity theorem [117, 3].

Another example is the so-called Linnik's theorem. It is a classical result of Legendre that if $n \neq 4^a(8b + 7)$, for $a, b \in \mathbb{Z}_{\geq 0}$, then n can be written as a sum of three squares. For such n we project the solutions to the unit sphere and consider the set

$$\Omega_n := \left\{ \frac{x}{|x|} : x \in \mathbb{Z}^3, \, |x|^2 = n \right\}.$$

Duke [14] showed that such points are equidistributed on the unit sphere S^2 in the sense that for every continuous function $f : S^2 \longrightarrow \mathbb{C}$ we have

$$\frac{1}{|\Omega_n|} \sum_{x \in \Omega_n} f(x) \longrightarrow \frac{1}{4\pi} \int_{S^2} f(x) \, \mathrm{d}x$$

when $n \longrightarrow \infty$ along the set of those n which are square-free and $n \not\equiv 7 \pmod 8$. By using the Weyl criteria for uniform distribution, the proof of this result reduces to bounding Fourier coefficients of half-integral modular forms.

Besides holomorphic modular forms, there is another class of automorphic functions of integral weight first introduced and systematically studied by Maass

[82], now called Maass forms. Originally Maass called them waveforms due to an analogue with a vibrating string. Unlike modular forms these are not holomorphic, but real analytic. This defect is compensated by the fact that they are eigenfunctions of the hyperbolic Laplace-Beltrami operator. It is natural to consider eigenfunctions of Δ for the reason that it is invariant under the action of the group $\mathrm{SL}(2,\mathbb{Z})$. Invariant differential operators can be used to decompose functions invariant under the action of the said group by using their eigenfunctions and they are therefore worth investigating. The primary example of this phenomenon is the classical Fourier theory, which is related to the $\mathbb{Z}$-invariant differential operator $\mathrm{d}^2/\mathrm{d}x^2$.

Now, unlike in the holomorphic case, there are non-trivial eigenfunctions of Δ which are properly invariant under the action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{H}$. It is possible to define Maass forms for general weights (and general characters) but we will only consider weight zero forms (with a trivial character). It is natural to focus on these particular forms as Maass forms of other weights do not appear in the spectral decomposition of $L^2(\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H})$. In fact, our interest lies in special types of Maass forms called Maass cusp forms. Exact definitions vary in the literature; here we shall use the definition in Goldfeld's book [30].

Definition 3. *Let $\nu \in \mathbb{C}$. A function $f : \mathbb{H} \longrightarrow \mathbb{C}$ is a Maass cusp form of type $\nu \in \mathbb{C}$ for $\mathrm{SL}(2,\mathbb{Z})$ if*

(1) $\quad f\left(\dfrac{az+b}{cz+d}\right) = f(z) \quad \textit{for any} \quad z = x + iy \in \mathbb{H}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$

(2) $\quad \Delta f = \nu(1-\nu)f$

(3) $\quad \displaystyle\int_0^1 f(z)\,\mathrm{d}x = 0.$

Remark 4. *Condition (3) is analogous to the statement that the zeroth Fourier coefficient of a holomorphic modular form vanishes. Therefore we call the forms defined above cusp forms.*

Remark 5. *General Maass forms are defined similarly but the third condition is replaced by a more general growth condition.*

It is easy to check that a Maass cusp form of type 0 or 1 must be a constant function [30, Proposition 3.3.], and henceforth we suppose that $\nu(1-\nu) \neq 0$. Deriving the Fourier expansion for these real-analytic forms is more involved than in the holomorphic case due to the lack of holomorphicity. As before, a Maass cusp form f is 1-periodic in the x-variable and therefore it has an expansion

$$f(z) = \sum_{m\in\mathbb{Z}} A_m(y)e(mx). \tag{2}$$

By denoting $W_m(z) := A_m(y)e(mx)$ it follows from the definition of a Maass cusp form that the relations

$$\Delta W_m(z) = \nu(1-\nu)W_m(z), \tag{3}$$

$$W_m\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}.z\right) = W_m(z)e(mu) \tag{4}$$

hold for all $z \in \mathbb{H}$ and $u \in \mathbb{R}$. Functions satisfying these conditions are called Whittaker functions of type ν for the character $u \mapsto e(mu)$.

Hence, $W_m(\cdot)$ is an example of a Whittaker function. Let us now give another example of a function satisfying conditions (3) and (4). A straightforward computation shows that the function $I_\nu(z) := (\Im(z))^\nu$ satisfies the first equation. By using this, one can then show that actually the function

$$W_m^{(\nu)}(z) := \int_{-\infty}^{\infty} I_\nu \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} . z \right) e(-mu) \mathrm{d}u$$

in place of $W_m(\cdot)$ above satisfies both conditions. For more about Whittaker functions, see [53, 105, 96, 69].

It turns out that this integral can be evaluated explicitly and its value equals

$$\sqrt{2} \frac{(\pi|m|)^{\nu - \frac{1}{2}}}{\Gamma(\nu)} \sqrt{2\pi y} K_{\nu - \frac{1}{2}} (2\pi|m|y) e(mx),$$

for $z = x + iy$ with $x, y \in \mathbb{R}$, where $K_.(\cdot)$ is the K-Bessel function. Now the last relevant observation is the multiplicity one theorem stating that every function $W_m(\cdot)$ satisfying the equations (3) and (4) must be a constant multiple of $W_m^{(\nu)}(\cdot)$. Whittaker originally defined his functions as solutions of the confluent hypergeometric differential equation. Indeed, the fact that $W_m(\cdot)$ is an eigenfunction of Δ implies that $A_m(y)$ satisfies the differential equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \frac{1}{y^2} \nu(1 - \nu) - 4\pi^2 m^2 \right) A_m(y) = 0.$$

If $\nu(1 - \nu) \neq 0$ and $m \neq 0$, this differential equation has exactly two solutions $\sqrt{2\pi|m|y} K_{\nu - 1/2}(2\pi|m|y)$ and $\sqrt{2\pi|m|y} I_{\nu - 1/2}(2\pi|m|y)$ (see [93, 116]). Here $K_\nu(\cdot)$ and $I_\nu(\cdot)$ are classical Bessel functions (for more information, see [74]). The cuspidality condition of the underlying Maass cusp form excludes the second option. This gives that a Maass cusp form f of type ν for $\mathrm{SL}(2, \mathbb{Z})$ has a Fourier-Whittaker expansion

$$f(z) = \sqrt{y} \sum_{n \neq 0} t(n) K_{i\sqrt{\lambda - \frac{1}{4}}} (2\pi|n|y) e(nx), \tag{5}$$

where $\lambda = \nu(1 - \nu)$. Here the numbers $t(n) \in \mathbb{C}$ are called the Fourier coefficients of f.

Next, we shall explain how Maass forms originally arose, following [1]. To each Hecke character χ of a quadratic field one can associate a degree two L-function $L(s, \chi)$. In 1927 Hecke [42] constructed a holomorphic modular form f_ψ such that the completed L-function attached to a Hecke character ψ of an imaginary quadratic field K can be obtained from the Mellin transform of f_ψ in the y-direction (this is an analogue of the construction in Riemann's proof of the functional equation of $\zeta(s)$). Roughly speaking, Hecke's construction was as follows. For a Hecke character ψ of an imaginary quadratic field K, define

$$f_\psi(z) := \sum_{\mathfrak{a} \subset \mathcal{O}_K} \psi(\mathfrak{a}) N(\mathfrak{a})^{\frac{k-1}{2}} e(N(\mathfrak{a})z),$$

where $\mathcal{O}_K$ is the ring of integers of K, $N(\mathfrak{a})$ is the norm of the ideal $\mathfrak{a}$, and k is an integer related to the infinity component of ψ. More precisely, it is of the

form $\psi_\infty(z) = (z/|z|)^{k-1}$. The fact that $f_\psi(z)$ is a holomorphic modular form follows by applying the Poisson summation formula.

Maass [82] observed that by replacing the exponential phases in Hecke's construction by classical Whittaker functions he could produce automorphic functions g_ψ whose Mellin transform in the y-direction gives the completed L-function attached to a Hecke character ψ of a real quadratic field. Maass proceeded as follows. The infinite component of a Hecke character ψ of a real quadratic field K is of the form $\psi_\infty(z) = \mathrm{sgn}(x)^a \mathrm{sgn}(y)^b |x/y|^{ir}$, where $z = x+iy$, $a, b \in \{0,1\}$ and $r \in (\pi/\log\varepsilon)\mathbb{Z}$, ε is a fundamental unit in K. Then he set

$$g_\psi(z) := \sum_{\mathfrak{a} \subset \mathcal{O}_K} \psi(\mathfrak{a}) W_{\frac{k}{2}, ir}(N(\mathfrak{a})z),$$

where k is 0 if $a+b$ is even and 1 otherwise. Here $W_{\cdot,\cdot}(\cdot)$ is a classical Whittaker function (see [116]). These functions Maass constructed translate nicely under the action of certain discrete subgroups of $\mathrm{SL}(2,\mathbb{R})$, are eigenfunctions of the hyperbolic Laplace-Beltrami operator, and satisfy suitable growth condition, but are not holomorphic [8, Section 1.9]. This provides a natural definition for a Maass form. Let us also remark that not all Maass forms arise in this way (from Hecke characters), but those which do, are called dihedral. The primary example of a Maass form is the non-holomorphic Eisenstein series. We are, however, interested in Maass cusp forms defined above.

Maass cusp forms are mysterious objects. While explicit examples of such objects exist for congruence subgroups of $\mathrm{SL}(2,\mathbb{Z})$, it was not until Selberg's work [103] that even their existence for the full modular group was known. It was Selberg's principal motivation for developing his celebrated trace formula to prove a Weyl law for the asymptotic count of such objects. Even today we do not know any concrete examples of Maass forms for the full modular group $\mathrm{SL}(2,\mathbb{Z})$ and it is widely believed that they are unconstructable.

2.3 Automorphic forms for higher rank groups

We start by reviewing the classical GL(2)-theory and then generalise to GL(n). As some results obtained in this thesis are specifically for the GL(3)-forms, and because GL(3) is the simplest special case of the higher rank situation, we specialise to this case from time to time for the sake of concreteness. Good references for this section are the books of Goldfeld [30] for general n and Bump [7] in the special case $n = 3$. For more about harmonic analysis on symmetric spaces, see [58].

We consider the following three subgroups of GL(2, $\mathbb{R}$):

$$Z(2,\mathbb{R}) := \left\{ \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} : d \neq 0 \right\}, \quad H(2,\mathbb{R}) := \left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R}, y > 0 \right\},$$

$$\text{and} \quad \mathrm{O}(2,\mathbb{R}) := \left\{ \begin{pmatrix} \pm\cos\theta & -\sin\theta \\ \pm\sin\theta & \cos\theta \end{pmatrix} : \theta \in [0, 2\pi] \right\}.$$

The Iwasawa decomposition says that $\mathrm{GL}(2,\mathbb{R}) = Z(2,\mathbb{R})H(2,\mathbb{R})\mathrm{O}(2,\mathbb{R})$, that is, every matrix $g \in \mathrm{GL}(2,\mathbb{R})$ can be written as

$$g = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \pm\cos\theta & -\sin\theta \\ \pm\sin\theta & \cos\theta \end{pmatrix},$$

where $x, y \in \mathbb{R}$, $y > 0$, $\theta \in [0, 2\pi[$, and $d \neq 0$. Here the middle matrix is uniquely determined and the other two are uniquely determined up to multiplication by $\pm I$. Notice also that there is a decomposition

$$H(2, \mathbb{R}) = N(2, \mathbb{R})A(2, \mathbb{R}),$$

where

$$N(2, \mathbb{R}) := \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}, \quad \text{and} \quad A(2, \mathbb{R}) := \left\{ \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} : y > 0 \right\}.$$

By identifying the upper half-plane $\mathbb{H}$ with $H(2, \mathbb{R})$ via an obvious isomorphism

$$\mathbb{H} \longrightarrow H(2, \mathbb{R})$$
$$x + iy \mapsto \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix},$$

the Iwasawa decomposition gives an isomorphism

$$\mathbb{H} \simeq \mathrm{GL}(2, \mathbb{R})/\langle O(2, \mathbb{R}), Z(2, \mathbb{R}) \rangle.$$

This provides a natural generalisation for the upper half-plane. For $n \in \mathbb{Z}_+$, define

$$Z(n, \mathbb{R}) := \left\{ \begin{pmatrix} d & & \\ & \ddots & \\ & & d \end{pmatrix} : d \neq 0 \right\}$$

and $O(n, \mathbb{R})$ to be the set of real $n \times n$-orthogonal matrices. We simply replace the integer 2 by an integer n and define the generalised upper half-plane $\mathbb{H}^n$ to be $\mathrm{GL}(n, \mathbb{R})/\langle O(n, \mathbb{R}), Z(n, \mathbb{R}) \rangle$. One can show that this quotient can be identified with a product $N(n, \mathbb{R})A(n, \mathbb{R})$ of the groups

$$N(n, \mathbb{R}) := \left\{ \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & x_{1,n} \\ 0 & 1 & x_{2,3} & \cdots & x_{2,n} \\ 0 & 0 & 1 & \cdots & x_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} : x_{i,j} \in \mathbb{R} \right\}$$

and

$$A(n, \mathbb{R}) := \left\{ \begin{pmatrix} y_1 y_2 \cdots y_{n-1} & 0 & 0 & 0 & 0 \\ 0 & y_1 y_2 \cdots y_{n-2} & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & 0 & 0 & y_1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} : y_i > 0 \right\}.$$

Therefore it is natural to define the generalised upper half-plane as follows.

Definition 6. *The generalised upper half-plane $\mathbb{H}^n$ is a set consisting of $n \times n$-matrices of the form $z = x \cdot y$, where*

$$x = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & x_{1,n} \\ 0 & 1 & x_{2,3} & \cdots & x_{2,n} \\ 0 & 0 & 1 & \cdots & x_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

and

$$y = \begin{pmatrix} y_1 y_2 \cdots y_{n-1} & 0 & 0 & 0 & 0 \\ 0 & y_1 y_2 \cdots y_{n-2} & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & 0 & 0 & y_1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

with $x_{i,j} \in \mathbb{R}$ and $y_i > 0$ for every $1 \le i < j \le n$.

In the case $n = 3$, this reads $z = x \cdot y$, where

$$x = \begin{pmatrix} 1 & x_2 & x_3 \\ & 1 & x_1 \\ & & 1 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} y_1 y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix}. \tag{6}$$

The definition above makes $\mathbb{H}^n$ a symmetric space. Associated to the hyperbolic structure of $\mathbb{H}^n$, there is a natural $\mathrm{GL}(n,\mathbb{R})$ left-invariant Haar measure defined by $\mathrm{d}z := \mathrm{d}x\mathrm{d}y$, where

$$\mathrm{d}x := \prod_{1 \le i < j \le n} \mathrm{d}x_{i,j}, \quad \text{and} \quad \mathrm{d}y := \prod_{k=1}^{n-1} \frac{\mathrm{d}y_k}{y_k^{k(n-k)+1}}.$$

For $n = 3$ this measure is

$$\mathrm{d}z = \frac{\mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}y_1 \mathrm{d}y_2}{y_1^3 y_2^3}.$$

One easily establishes that the group $\mathrm{GL}(n,\mathbb{Z})$ acts discretely on the generalised upper half-plane $\mathbb{H}^n$ by left matrix multiplication. Again, we can consider the space $L^2(\mathrm{SL}(n,\mathbb{Z})\backslash\mathbb{H}^n)$. This space carries a natural Petersson inner product

$$\langle f, g \rangle := \int_{\mathrm{SL}(n,\mathbb{Z})\backslash\mathbb{H}^n} f(z)\overline{g}(z)\,\mathrm{d}z$$

for $f, g \in L^2(\mathrm{SL}(n,\mathbb{Z})\backslash\mathbb{H}^n)$, which makes it a Hilbert space.

Next, we generalise the notion of being an eigenfunction of the hyperbolic Laplace-Beltrami operator. Our discussion follows [30] but a more comprehensive treatment on invariant differential operators in symmetric spaces is given in [43]. A natural source for $\mathrm{GL}(n,\mathbb{R})$-invariant differential operators is its associated Lie algebra $\mathfrak{gl}(n,\mathbb{R})$ which is just the additive vector space of $n \times n$-matrices with coefficients on $\mathbb{R}$ and Lie bracket operator $[\alpha, \beta] := \alpha \cdot \beta - \beta \cdot \alpha$. The key point is that one can realise the universal enveloping algebra $\mathcal{U}(\mathfrak{gl}(n,\mathbb{R}))$ of $\mathfrak{gl}(n,\mathbb{R})$ as an algebra D^n generated by differential operators D_α, one for each $\alpha \in \mathfrak{gl}(n,\mathbb{R})$, acting on smooth functions $f : \mathrm{GL}(n,\mathbb{R}) \longrightarrow \mathbb{C}$ via

$$D_\alpha f(g) := \frac{\partial}{\partial t} f(g \exp(t\alpha)) \Big|_{t=0}.$$

Let $\mathfrak{D}^n$ be the center of D^n. Then it can be shown [30, Proposition 2.3.1.] that every differential operator in $\mathfrak{D}^n$ is well-defined on the space of smooth functions $f : \mathrm{GL}(n,\mathbb{Z})\backslash\mathrm{GL}(n,\mathbb{R})/\langle O(n,\mathbb{R}), Z(n,\mathbb{R})\rangle \longrightarrow \mathbb{C}$.

When $n = 2$, the $\mathrm{GL}(2,\mathbb{R})$-invariant differential operators on $\mathbb{H}$ are polynomials in the hyperbolic Laplace-Beltrami operator Δ. Thus, a natural generalisation of the condition being an eigenfunction of the hyperbolic Laplace-Beltrami operator is to be an eigenfunction of every differential operator belonging to the center $\mathfrak{D}^n$ of the algebra D^n. It is possible to show that the center is an $(n-1)$-dimensional algebra over $\mathbb{R}$: $\mathfrak{D}^n = \mathbb{R}[\Delta_1, ..., \Delta_{n-1}]$, [30, Proposition 2.3.5.]. The differential operators Δ_i are called Casimir operators and they can be given explicitly. To be more precise, the are given by

$$\Delta_{m-1} := \sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n} D_{i_1,i_2} \circ D_{i_2,i_3} \circ \cdots \circ D_{i_m,i_1}$$

for every $2 \leq m \leq n$, where $D_{i,j} = D_{E_{i,j}}$ and recall that $E_{i,j}$ is the matrix with a 1 at the $(i,j)^{\text{th}}$ entry and zeroes elsewhere.

For $n = 3$ the center $\mathfrak{D}^3$ is given by $\mathbb{R}[\Delta_1, \Delta_2]$, where

$$\Delta_1 = y_1^2 \frac{\partial^2}{\partial y_1^2} + y_2^2 \frac{\partial^2}{\partial y_2^2} - y_1 y_2 \frac{\partial^2}{\partial y_1 \partial y_2} + y_1^2 (x_2^2 + y_2^2) \frac{\partial^2}{\partial x_3^2} + y_1^2 \frac{\partial^2}{\partial x_1^2}$$
$$+ y_2^2 \frac{\partial^2}{\partial x_2^2} + 2y_1^2 x_2 \frac{\partial^2}{\partial x_1 \partial x_3}$$

and

$$\Delta_2 = -y_1^2 y_2 \frac{\partial^3}{\partial y_1^2 \partial y_2} + y_1 y_2^2 \frac{\partial^3}{\partial y_1 \partial y_2^2} - y_1^3 y_2^2 \frac{\partial^3}{\partial x_3^2 \partial y_1} + y_1 y_2^2 \frac{\partial^3}{\partial x_2^2 \partial y_1}$$
$$- 2y_1^2 y_2 x_2 \frac{\partial^3}{\partial x_1 \partial x_3 \partial y_2} + (-x_2^2 + y_2^2) y_1^2 y_2 \frac{\partial^3}{\partial x_3^2 \partial y_2} - y_1^2 y_2 \frac{\partial^3}{\partial x_1^2 \partial y_2}$$
$$+ 2y_1^2 y_2^2 \frac{\partial^3}{\partial x_1 \partial x_2 \partial x_3} + 2y_1^2 y_2^2 x_2 \frac{\partial^3}{\partial x_2 \partial x_3^2} + y_1^2 \frac{\partial^2}{\partial y_1^2} - y_2^2 \frac{\partial^2}{\partial y_2^2}$$
$$+ 2y_1^2 x_2 \frac{\partial^2}{\partial x_1 \partial x_3} + (x_2^2 + y_2^2) y_1^2 \frac{\partial^2}{\partial x_3^2} + y_1^2 \frac{\partial^2}{\partial x_1^2} - y_2^2 \frac{\partial^2}{\partial x_2^2}$$

in coordinates (6).

Next, we will explain how to attach a spectral parameter to each Maass cusp form. As indicated above, Maass cusp forms will be defined to be eigenfunctions of every differential operator in $\mathfrak{D}^n$. Every Maass cusp form f for the group $\mathrm{SL}(n,\mathbb{Z})$ generates a homomorphism from $\mathfrak{D}^n$ into $\mathbb{C}$ in the following way. Let $D_1, D_2 \in \mathfrak{D}^n$ be differential operators and let the eigenvalues of f under these operators be $\lambda_f(D_1)$ and $\lambda_f(D_2)$. Then, as $(D_1 \circ D_2)f = \lambda_f(D_1)\lambda_f(D_2)f$, it follows that the map $D \mapsto \lambda_f(D)$ is a homomorphism, called the eigencharacter of f.

Let

$$\mathfrak{a} := \left\{ (\mu_1, ..., \mu_n) \in \mathbb{R}^n : \sum_{j=1}^{n} \mu_j = 0 \right\} \simeq \mathbb{R}^{n-1}$$

denote the Lie algebra of $A(n,\mathbb{R})$ and let its complexification be $\mathfrak{a}_\mathbb{C} := \mathfrak{a} \otimes \mathbb{C}$. Then it is a theorem due to Harish-Chandra [41] (see also [67, 43]) that there is an isomorphism

$$\psi : \mathfrak{D}^n \xrightarrow{\sim} \mathrm{sym}\left(\mathfrak{a}_\mathbb{C}\right)^W,$$

where $\mathrm{sym}(\mathfrak{a}_\mathbb{C})$ is the symmetric algebra of $\mathfrak{a}_\mathbb{C}$, and W is the Weyl group. For a given $\nu \in \mathfrak{a}_\mathbb{C}^*$ we can extend it to an algebra homomorphism $\nu : \mathrm{sym}(\mathfrak{a}_\mathbb{C}) \longrightarrow \mathbb{C}$ [67, Proposition 3.1.]. It can be shown that any homomorphism from the set of Weyl invariants $\mathrm{sym}(\mathfrak{a}_\mathbb{C})^W$ into $\mathbb{C}$ is an evaluation map at some $\nu \in \mathfrak{a}_\mathbb{C}^*$, unique up to an action of the Weyl group W [43, Lemma 3.11 of Section III.3.4]. When composed with the Harish-Chandra isomorphism, this implies that any homomorphism from $\mathfrak{D}^n$ into $\mathbb{C}$ is a map sending a differential operator D to $\nu(\psi(D))$ for some fixed $\nu \in \mathfrak{a}_\mathbb{C}^*$. In particular, the eigencharacter of f is of this form for some linear functional $\nu_f \in \mathfrak{a}_\mathbb{C}^*/W$. This ν_f is called the spectral parameter of f.

Let us then explain how such elements of $\mathfrak{a}_\mathbb{C}^*$ can be identified with elements of $\mathbb{C}^{n-1}$. The basis of $\mathfrak{a}_\mathbb{C}$ is given by the matrices $H_i := E_{i,i} - E_{i+1,i+1}$ for $1 \le i \le n-1$, where $E_{i,i}$ is a matrix with 1 at $(i,i)^{\mathrm{th}}$ entry and zeroes everywhere else. Let $\nu_f(H_i) = n\nu_{i,f} - 1 \in \mathbb{C}$, where $\nu_{i,f} \in \mathbb{C}$. Then we identify $\nu_f \in \mathfrak{a}_\mathbb{C}^*$ with $(\nu_{1,f}, ..., \nu_{n-1,f}) \in \mathbb{C}^{n-1}$. Notice that here we use the same normalisation as in [30]. We also call the elements of such an $(n-1)$-tuple the spectral parameters of f or say that f is of type $(\nu_{1,f}, ..., \nu_{n-1,f}) \in \mathbb{C}^{n-1}$. It follows from the above discussion that the eigenvalue of a Maass cusp form f of type $(\nu_{1,f}, ..., \nu_{n-1,f}) \in \mathbb{C}^{n-1}$ under the given differential operator $D \in \mathfrak{D}^n$ is given by a polynomial (depending only on D) in the spectral parameters of f.

We now give a more concrete description of the spectral parameters. For $\nu = (\nu_1, ..., \nu_{n-1}) \in \mathbb{C}^{n-1}$ we define the power function as

$$I_\nu(z) := \prod_{i=1}^{n-1} \prod_{j=1}^{n-1} y_i^{b_{i,j}\nu_j},$$

where

$$b_{i,j} := \begin{cases} ij, & \text{if } i+j \le n \\ (n-i)(n-j) & \text{if } i+j \ge n \end{cases}$$

This is a character of the group of upper-triangular matrices and an eigenfunction of every differential operator in $\mathfrak{D}^n$. Let the corresponding eigenvalue under $D \in \mathfrak{D}^n$ be $\lambda_\nu(D)$.

Remark 7. *The power function $I_\nu(z)$ is the natural generalisation of the function $z \mapsto (\Im(z))^\nu$.*

For example in the case $n = 3$ the power function for $(\nu_1, \nu_2) \in \mathbb{C}^2$ is given by

$$I_{\nu_1,\nu_2}(z) = y_1^{\nu_1+2\nu_2} y_2^{2\nu_1+\nu_2}.$$

It can be shown that the eigenvalues of $I_\nu(\cdot)$ under the elements of $\mathfrak{D}^n$ give the eigenvalues of a Maass cusp form of type ν under the elements of $\mathfrak{D}^n$. Recall that every differential operator which lies in $\mathfrak{D}^n$ can be expressed as a polynomial (with coefficients in $\mathbb{R}$) in the Casimir operators. Furthermore,

$$D_{i,j}^k I_\nu(z) = \begin{cases} \nu_{n-i}^k \cdot I_\nu(z) & \text{if } i = j \\ 0, & \text{if } i \ne j \end{cases}$$

Therefore, eigenvalues of $I_\nu(\cdot)$ under the differential operators in $\mathfrak{D}^n$ are polynomials in the entries of $\nu \in \mathbb{C}^{n-1}$.

Actually, by noting that $(b_{n-i,j})_{1 \le i,j \le n-1}$ is the inverse of the Cartan matrix $(c_{ij})_{1 \le i,j \le n-1}$, where

$$c_{ij} := \begin{cases} \frac{2}{n} & \text{if } i = j \\ -\frac{1}{n} & \text{if } |i - j| = 1 \\ 0 & \text{otherwise,} \end{cases}$$

it is not hard to calculate that $\lambda_\nu(D)$, equals the the eigenvalue $\lambda_f(D)$ for any Maass cusp form f of type $\nu \in \mathbb{C}^{n-1}$. Hence, the eigenvalues of the power function are sufficient to describe the eigenvalues of Maass cusp forms. Therefore an equivalent formulation for the spectral parameter is the following: a Maass form f for $\mathrm{SL}(n,\mathbb{Z})$ is of type $\nu = (\nu_1, ..., \nu_{n-1}) \in \mathbb{C}^{n-1}$ (or has spectral parameter ν) if it has an eigenvalue $\lambda_\nu(D)$ under every differential operator $D \in \mathfrak{D}^n$.

Next, we give a representation-theoretic parametrisation for Maass cusp forms by so-called Langlands parameters and shortly explain how they are related to the spectral parameters. The spectral parameter $\nu_f \in \mathfrak{a}_\mathbb{C}^*$ of a Maass cusp form f generates an unramified representation of the group $\mathrm{GL}(n,\mathbb{R})$ with a trivial central character, denoted by π_{ν_f} [6, Section 6.2.]. This representation can be realised as an induced representation $\mathrm{Ind}_{B_n}^{\mathrm{GL}(n,\mathbb{R})} \chi(b)$ from the Borel subgroup

$$B_n := \left\{ b = \begin{pmatrix} b_1 & & & \\ & b_2 & & * \\ & & \ddots & \\ & & & b_n \end{pmatrix} : b_i \in \mathbb{R} \right\}$$

of $\mathrm{GL}(n,\mathbb{R})$, where

$$\chi(b) := \prod_{i=1}^n |b_i|^{\mu_{i,f}} \tag{7}$$

for some uniquely determined complex numbers $\mu_{i,f}$ with $\mu_{1,f} + \cdots + \mu_{n,f} = 0$ [6, Section 9.2.]. Entries of this n-tuple $(\mu_{1,f}, ..., \mu_{n,f}) \in \mathbb{C}^n$ are the Langlands parameters of f.

The relation between Langlands parameters and spectral parameters can be described as follows (here we follow [5, 6]). Let $\mathfrak{t}$ be the Lie algebra of the diagonal torus T_n of $\mathrm{GL}(n,\mathbb{R})$. The character χ above (7) can be written as $\chi(b) = e^{\mu_f(H(b))}$ for a uniquely determined $\mu_f \in \mathfrak{t}_\mathbb{C}^*$. Here $H : B_n \longrightarrow \mathfrak{t}_\mathbb{C}$ is the so-called logarithm map given by

$$H\left(\begin{pmatrix} b_1 & & & \\ & b_2 & & * \\ & & \ddots & \\ & & & b_n \end{pmatrix} \right) := \begin{pmatrix} \log|b_1| & & & \\ & \log|b_2| & & \\ & & \ddots & \\ & & & \log|b_n| \end{pmatrix}.$$

Actually, it is easy to see that

$$\mu_f = \sum_{i=1}^n \mu_{i,f} e_i,$$

where $e_i(\operatorname{diag}(a_1, ..., a_n)) := a_i$.

Since the representation π_{ν_f} has trivial central character, μ_f factors through a functional on $\mathfrak{t}/\mathfrak{z}$, where $\mathfrak{z}$ is the Lie algebra of the center of $\operatorname{GL}(n, \mathbb{R})$. By identifying this quotient with $\mathfrak{a}$, we get that μ_f agrees with ν_f on $\mathfrak{a}$. By evaluating μ_f at the matrices H_i we conclude that $\mu_{i,f} - \mu_{i+1,f} = n\nu_{i,f} - 1$ for all $1 \le i \le n-1$. As the sum of $\mu_{j,f}$'s is zero, we can use the above relations to solve $\mu_{j,f}$'s in terms of $\nu_{j,f}$'s. For instance, in $\operatorname{GL}(3)$ we have

$$\mu_{1,f} = 2\nu_{1,f} + \nu_{2,f} - 1$$
$$\mu_{2,f} = -\nu_{1,f} + \nu_{2,f}$$
$$\mu_{3,f} = -\nu_{1,f} - 2\nu_{2,f} + 1.$$

As alluded above, the eigenvalue under a given differential operator is obtained by evaluating the associated polynomial at the Langlands parameters (and hence also spectral parameters). For example, the eigenvalue of an $\operatorname{SL}(3, \mathbb{Z})$ Maass cusp form of type (ν_1, ν_2) (or Langlands parameter (μ_1, μ_2, μ_3)) under the Laplace-Beltrami operator is given by

$$1 - \frac{1}{2}\left(\mu_1^2 + \mu_2^2 + \mu_3^2\right) = -3\nu_1^2 - 3\nu_2^2 + 3\nu_1 + 3\nu_2 + 3\nu_1\nu_2.$$

More generally, the eigenvalue of an $\operatorname{SL}(n, \mathbb{Z})$ Maass cusp form with Langlands parameters $(\mu_1, ..., \mu_n)$ under the Laplace-Beltrami operator is given by

$$\frac{n^3 - n}{24} - \frac{1}{2}\sum_{j=1}^{n} \mu_j^2.$$

Finally, we generalise the cuspidality condition. The key observation is that clearly

$$\int_0^1 f(z)\, \mathrm{d}x = \int_{(\operatorname{SL}(2,\mathbb{Z}) \cap M(2,\mathbb{R})) \backslash M(2,\mathbb{R})} f(uz)\, \mathrm{d}u,$$

where

$$M(2, \mathbb{R}) := \left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} : u \in \mathbb{R} \right\}.$$

As $M(2, \mathbb{R})$ is the group of unipotent upper triangular matrices, it turns out that the natural generalisation for the cuspidality condition is obtained by replacing the integer 2 by an integer n and requiring that

$$\int_{(\operatorname{SL}(n,\mathbb{Z}) \cap M(n,\mathbb{R})) \backslash M(n,\mathbb{R})} f(uz)\, \mathrm{d}u = 0,$$

where $M(n, \mathbb{R})$ is the group of $n \times n$ real unipotent block upper triangular matrices with identity matrices on the diagonal.

For example, a Maass cusp form of type $(\nu_1, \nu_2) \in \mathbb{C}^2$ is a smooth function f on $L^2(\operatorname{SL}(3,\mathbb{Z}) \backslash \mathbb{H}^3)$ such that f is an eigenfunction of every $D \in \mathfrak{D}^3$ with eigenvalue $\lambda_\nu(D)$ and

$$\int_{(\operatorname{SL}(3,\mathbb{Z}) \cap M) \backslash M} f(uz)\, \mathrm{d}u = 0$$

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for $M = M(1), M(2), M(3)$ where

$$M(1) := \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}, \quad M(2) := \left\{ \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\}, \quad \text{and} \quad M(3) := \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Now we have all the ingredients to define Maass forms for $\mathrm{SL}(n, \mathbb{Z})$.

Definition 8. *Let $\nu = (\nu_1, ..., \nu_{n-1}) \in \mathbb{C}^{n-1}$. A Maass form of type ν for $\mathrm{SL}(n, \mathbb{Z})$ is a smooth function $f : \mathbb{H}^n \longrightarrow \mathbb{C}$ satisfying*

(1) $f(\gamma \cdot z) = f(z) \quad$ *for all* $\gamma \in \mathrm{SL}(n, \mathbb{Z}), \quad z \in \mathbb{H}^n,$

(2) $Df = \lambda_\nu(D)f \quad$ *for all* $D \in \mathfrak{D}^n$, *where* $\lambda_\nu(D)$ *is the eigenvalue of* $I_\nu(\cdot)$ *under* D,

(3) $\displaystyle\int_{\mathrm{SL}(n,\mathbb{Z}) \backslash \mathbb{H}^n} |f(z)|^2 \, \mathrm{d}z < \infty.$

Furthermore, we say that f is a Maass cusp form if

$$\int_{(\mathrm{SL}(n,\mathbb{Z}) \cap M) \backslash M} f(uz) \, \mathrm{d}u = 0$$

for all upper-triangular matrix groups

$$M = \left\{ \begin{pmatrix} I_{n_1} & & * \\ & \ddots & \\ & & I_{n_r} \end{pmatrix} \right\} \quad \text{with} \quad n = \sum_{i=1}^r n_i,$$

where I_n is the $n \times n$-identity matrix.

Remark 9. *In Goldfeld's book [30] Maass forms are Maass cusp forms in our language.*

Remark 10. *Here we only consider weight zero Maass forms but analogous to the lower rank case one could define Maass forms with arbitrary weight.*

Maass cusp forms appear naturally in the decomposition of L^2-functions. Generalising Selberg's classical result, Langlands [73] (see also [89]) proved the following general spectral decomposition.

Theorem 11. *We have*

$$L^2(\mathrm{SL}(n,\mathbb{Z}) \backslash \mathbb{H}^n) = \mathbb{C} \oplus L^2_{cusp} \oplus L^2_{residual} \oplus L^2_{Eisenstein},$$

where L^2_{cusp} is spanned by (weight zero) Maass cusp forms, $L^2_{Eisenstein}$ is spanned by Eisenstein series, and $L^2_{residual}$ is spanned by residues of the Eisenstein series at points in the complex plane, all of which are eigenfunctions of all of $\mathfrak{D}^n$.

Now the ring of differential operators $\mathfrak{D}^n$ acts on L^2_{cusp} and this subspace decomposes discretely. Therefore we find a basis of L^2_{cusp} consisting of simultaneous eigenfunctions of elements of $\mathfrak{D}^n$. These eigenfunctions are precisely the Maass cusp forms described above.

Remark 12. *It turns out that the part in the above decomposition related to the theory of Eisenstein series, $L^2_{residual} \oplus L^2_{Eisenstein}$, is well-understood if we understand the cuspidal part $L^2_{cusp}(\mathrm{SL}(n,\mathbb{Z})\backslash\mathbb{H}^n)$ for lower rank groups. Hence, it is important to concentrate on the cuspidal part in the spectral decomposition. This gives one more motivation for studying Maass cusp forms.*

As indicated above, the existence of Maass cusp forms is not clear at all. However, it turns out that there exists infinitely many of them. Moreover, there is a Weyl law

$$\sum_{\substack{f \text{ Maass cusp form of type } \nu_f \\ |\nu_f| \leq T}} 1 \sim c \cdot T^d$$

for some non-zero constant c and $d = \dim \mathbb{H}^n$. Here $|\cdot|$ is the euclidean norm on $\mathbb{C}^{n-1}$. For $n = 2$ this is work of Selberg [103], for $n = 3$ of Miller [84] and for general n due to Müller [91, 92] and independently Lindenstrauss and Venkatesh [78].

Next, we sketch how the Fourier-Whittaker expansion for such Maass cusp form f of type $\nu \in \mathbb{C}^{n-1}$ for the group $\mathrm{SL}(n,\mathbb{Z})$ is derived. For more details, see [30, Chapter 5]. Analogous to the GL(2)-case, the translations correspond to the maximal unipotent subgroup

$$U(n,\mathbb{R}) := \left\{ \begin{pmatrix} 1 & u_{1,2} & u_{1,3} & \cdots & u_{1,n} \\ 0 & 1 & u_{2,3} & \cdots & u_{2,n} \\ 0 & 0 & 1 & \cdots & u_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} : u_{i,j} \in \mathbb{R} \right\}.$$

However, as opposed to the case $n = 2$, this is not an abelian group for $n \geq 3$. Still, there is a natural way to generalise the theory to higher rank following the ideas of Piatetski-Shapiro [95, 96] and Shalika [105] in the adelic setting. Namely, the maximal unipotent subgroup itself has an abelian subgroup

$$V(n,\mathbb{R}) := \left\{ \begin{pmatrix} 1 & v_1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & v_2 & 0 & \cdots & 0 \\ 0 & 0 & 1 & v_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & v_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} : v_i \in \mathbb{R} \right\}.$$

Note that for $n = 2$ this group coincides with $U(2,\mathbb{R})$. To do Fourier analysis we need a character of this group. For each $m = (m_1, ..., m_{n-1}) \in \mathbb{Z}^{n-1}$, we have a character of $V(n,\mathbb{R})$ given by

$$\psi_m(v) := e(m_1 v_1 + m_2 v_2 + \cdots + m_{n-1} v_{n-1})$$

for $v = (v_1, ..., v_{n-1}) \in V(n,\mathbb{R})$. Notice that $\psi_m(\cdot)$ extends trivially to a character of the group $U(n,\mathbb{R})$ as $\psi_m(u) = e(m_1 u_{1,2} + m_2 u_{2,3} + \cdots + m_{n-1} u_{n-1,n})$. By the classical Fourier theory of abelian groups, we have

$$f(z) = \sum_{m \in \mathbb{Z}^{n-1}} \widetilde{f}_m(z), \tag{8}$$

where

$$\widetilde{f}_m(z) := \int_0^1 \cdots \int_0^1 f(v \cdot z)\overline{\psi_m}(v) \prod_{i=1}^{n-1} \mathrm{d}v_i.$$

Here the integration is over the superdiagonal, not over the whole group $U(n, \mathbb{R})$. Hence this does not constitute as a Fourier expansion. However, by using group-theoretic arguments [30] it is possible to show that the above decomposition (8) can be written as

$$f(z) = \sum_{\gamma \in U(n-1, \mathbb{Z}) \backslash \mathrm{SL}(n-1, \mathbb{Z})} \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \cdots \sum_{m_{n-1} \neq 0} \widehat{f}_m\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} \cdot z \right),$$

where

$$\widehat{f}_m(z) := \int_0^1 \cdots \int_0^1 f(u \cdot z)\overline{\psi_m}(u) \prod_{1 \le i < j \le n} \mathrm{d}u_{i,j}.$$

This roughly corresponds to (2) on page 8. To get the actual Fourier coefficients to appear, we need to study the function $\widehat{f}_m$, for fixed $m = (m_1, ..., m_{n-1}) \in \mathbb{Z}^{n-1}$, more closely. It is simple to check that $\widehat{f}_m(\cdot)$ satisfies the following three properties:

(1) $\widehat{f}_m(u \cdot z) = \overline{\psi_m}(u)\widehat{f}_m(z)$ for every $u \in U(n, \mathbb{R})$

(2) $D\widehat{f}_m(z) = \lambda_\nu(D)\widehat{f}_m(z)$ for every $D \in \mathfrak{D}^n$, where $\lambda_\nu(D) \in \mathbb{C}$ is as in p. 14

(3) $\displaystyle\int_{\Sigma_{\frac{\sqrt{3}}{2}, \frac{1}{2}}} |\widehat{f}_m(z)|^2 \mathrm{d}z < \infty,$

where $\Sigma_{\frac{\sqrt{3}}{2}, \frac{1}{2}}$ is the usual Siegel set (which is a good approximation for the fundamental domain of the action of $\mathrm{SL}(n, \mathbb{Z})$ on $\mathbb{H}^n$, see [30, Definition 1.3.1]). Inspired by this and the classical GL(2)-case we make the following definition.

Definition 13. *Fix a character ψ_m of the group $U(n, \mathbb{R})$. A Whittaker function of type $\nu = (\nu_1, ..., \nu_{n-1}) \in \mathbb{C}^{n-1}$ (with respect to the character ψ_m) for $\mathrm{SL}(n, \mathbb{Z})$ is a smooth function $W_\nu : \mathbb{H}^n \longrightarrow \mathbb{C}$ satisfying the conditions*

(1) $W_\nu(u \cdot z) = \overline{\psi_m}(u)W_\nu(z)$ *for every* $u \in U(n, \mathbb{R})$

(2) $DW_\nu(z) = \lambda_\nu(D)W_\nu(z)$ *for every* $D \in \mathfrak{D}^n$

(3) $\displaystyle\int_{\Sigma_{\frac{\sqrt{3}}{2}, \frac{1}{2}}} |W_\nu(z)|^2 \mathrm{d}z < \infty.$

Remark 14. *This is not the most general definition of a Whittaker function.*

Thus $\widehat{f}_m(z)$ is an example of a Whittaker function. Jacquet introduced a way to construct another example of such a function. The idea is to simply define

$$W_{\mathrm{Jacquet}}(z, \nu; \psi_m) := \int_{U(n, \mathbb{R})} I_\nu(w \cdot u \cdot z)\overline{\psi_m}(u) \prod_{1 \le i < j \le n} \mathrm{d}u_{i,j},$$

where

$$w = \begin{pmatrix} & & & (-1)^{\lfloor n/2 \rfloor} \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{pmatrix}$$

is the long Weyl element. This is called the Jacquet-Whittaker function of type ν for the character ψ_m.

The integral on the right-hand side converges absolutely and uniformly on the compact subsets of $\mathbb{H}^n$ assuming $\Re(\nu_i) > 1/n$ for $i = 1, ..., n-1$. Furthermore, it has a meromorphic continuation to all $\nu \in \mathbb{C}^{n-1}$. More importantly, the Jacquet-Whittaker function for ψ_m satisfies the identity

$$W_{\text{Jacquet}}(z; \nu, \psi_m) = c_{\nu,m} \psi_m(x) W_{\text{Jacquet}}\left(My; \nu, \psi_{(1,...,1)}\right),$$

where $c_{\nu,m} \neq 0$ is a constant depending only on ν and m, and

$$M = \begin{pmatrix} |m_1 m_2 ... m_{n-1}| & & & & \\ & \ddots & & & \\ & & |m_1 m_2| & & \\ & & & |m_1| & \\ & & & & 1 \end{pmatrix}.$$

For the proofs of these facts, see [30, Chapter 5.5]. Now, just like in the GL(2)-situation, the fundamental multiplicity one theorem of Shalika [105] can be used to show that any Whittaker function is a constant multiple of the Jacquet-Whittaker function W_{Jacquet}:

$$W_{\nu}(z) = c \cdot W_{\text{Jacquet}}(z; \nu, \psi_m)$$

for some constant $c \neq 0$. Let us quickly explain how this follows from Shalika's results in the case $n = 3$ following [7]. The type of the underlying Whittaker function, $\nu = (\nu_1, \nu_2) \in \mathbb{C}^2$, parametrises an induced representation of GL(3, $\mathbb{R}$) as follows. Consider the character ψ of the Borel subgroup $B_3 \subset$ GL(3, $\mathbb{R}$) given by

$$\psi\left(\begin{pmatrix} y_1 y_2 & y_1 x_2 & x_3 \\ & y_1 & x_1 \\ & & 1 \end{pmatrix}\right) = y_1^{\nu_1 + 2\nu_2} y_2^{2\nu_1 + \nu_2}$$

in coordinates (6).

Then we study the representation of GL(3, $\mathbb{R}$) induced from the character ψ. By the multiplicity one theorem of Shalika, this representation has a unique Whittaker model (for the basic theory of Whittaker models, see [8, Chapter 3.5]) corresponding to the character $e(x_1 + x_2)$ of the unipotent subgroup of B_3. The Jacquet-Whittaker function $W_{\nu}(z)$ lies in the space of the Whittaker model. Furthermore, the space of the Whittaker model has a unique one-dimensional subspace of SO(3)-stable vectors. As any Whittaker function is right-invariant under the action of SO(3), it follows that any Whittaker function $W_{\nu}(z)$ must be a constant multiple of a Jacquet-Whittaker function of the same type.

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These observations together yield a Fourier-Whittaker expansion

$$f(z) = \sum_{\gamma \in U(n-1,\mathbb{Z}) \backslash \mathrm{SL}(n-1,\mathbb{Z})} \sum_{m_1=1}^{\infty} \cdots \sum_{m_{n-2}=1}^{\infty} \sum_{m_{n-1} \neq 0} C(m_1, ..., m_{n-1})$$
$$\cdot W_{\mathrm{Jacquet}} \left(M \cdot \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} \cdot z, \nu, \psi_{\left(1,...,1,\frac{m_{n-1}}{|m_{n-1}|}\right)} \right),$$

where $C(m_1, ..., m_{n-1})$ depends only on $m_1, ..., m_{n-1}$, and ν. We write

$$C(m_1, ..., m_{n-1}) = \frac{A(m_1, ..., m_{n-1})}{\prod_{k=1}^{n-1} |m_k|^{k(n-k)/2}}$$

and call the coefficients $A(m_1, ..., m_{n-1}) \in \mathbb{C}$ the Fourier coefficients of the Maass cusp form f. We remark that the above normalisation simplifies several formulas appearing in the literature. Hence we have the following theorem:

Theorem 15. *Let f be a Maass form of type $\nu \in \mathbb{C}^{n-1}$ for $\mathrm{SL}(n,\mathbb{Z})$. Then it has a Fourier-Whittaker expansion*

$$f(z) = \sum_{\gamma \in U(n-1,\mathbb{Z}) \backslash \mathrm{SL}(n-1,\mathbb{Z})} \sum_{m_1=1}^{\infty} \cdots \sum_{m_{n-2}=1}^{\infty} \sum_{m_{n-1} \neq 0} \frac{A(m_1, ..., m_{n-1})}{\prod_{k=1}^{n-1} |m_k|^{k(n-k)/2}}$$
$$\cdot W_{Jacquet} \left(M \cdot \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} \cdot z, \nu, \psi_{\left(1,...,1,\frac{m_{n-1}}{|m_{n-1}|}\right)} \right).$$

Next, we define an important notion of a dual Maass cusp form. Let f be a Maass cusp form of type $(\nu_1, ..., \nu_{n-1}) \in \mathbb{C}^{n-1}$ for $\mathrm{SL}(n,\mathbb{Z})$. Then

$$\widetilde{f}(z) := f(w \cdot {}^t(z^{-1}) \cdot w), \quad \text{where} \quad w = \begin{pmatrix} & & & (-1)^{\lfloor n/2 \rfloor} \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{pmatrix},$$

is a Maass cusp form of type $(\nu_{n-1}, ..., \nu_1) \in \mathbb{C}^{n-1}$ for $\mathrm{SL}(n,\mathbb{Z})$. We say that $\widetilde{f}$ is a dual Maass cusp form of f. It turns out that the $(m_1, ..., m_{n-1})^{\mathrm{th}}$ Fourier coefficient of f equals the $(m_{n-1}, ..., m_1)^{\mathrm{th}}$ Fourier coefficient of $\widetilde{f}$.

2.3.1 Automorphic L-functions

In order to discuss Voronoi type summation formulas below, we need to briefly say a few words about L-functions attached to Maass cusp forms. We will not discuss theory of Hecke operators here, instead we refer the reader to [106, Chapter 3] for the classical case $n = 2$ and to [30, Chapter 9.3] for the higher rank case. The main results we need state that such an L-function has a meromorphic continuation to the complex plane and satisfies a functional equation essentially equating values of the L-function at s and $1 - s$. Let f be a Maass cusp form of type $\nu \in \mathbb{C}^{n-1}$ for $\mathrm{SL}(n,\mathbb{Z})$ with Fourier coefficients $A(m_1, ..., m_{n-1})$ which is an eigenfunction of every Hecke operator and normalised so that $A(1, ..., 1) = 1$. Then we define a Dirichlet series attached to f by

$$L(s,f) := \sum_{m=1}^{\infty} \frac{A(m, 1, ..., 1)}{m^s},$$

which converges for $\Re(s) > 1$. This has an analytic continuation to the whole complex plane via the functional equation

$$\pi^{-ns/2} \prod_{i=1}^{n} \Gamma\left(\frac{s - \lambda_i(\nu)}{2}\right) L(s, f) = \pi^{-n(1-s)/2} \prod_{i=1}^{n} \Gamma\left(\frac{1 - s - \widetilde{\lambda}_i(\nu)}{2}\right) L(1 - s, \widetilde{f}),$$

where $\widetilde{f}$ is the dual Maass cusp form of f and $\lambda_i(\nu)$ and $\widetilde{\lambda}_i(\nu)$ are the Langlands parameters of f and $\widetilde{f}$, respectively. The resulting function is called the Godement-Jacquet L-function of the Maass cusp form f and it is still denoted by $L(s, f)$. Such an L-function can be related to the sum of consecutive Fourier coefficients via Perron's formula.

The fact that the underlying Maass cusp form f is a Hecke eigenform guarantees that the associated L-function $L(s, f)$ has a representation as an Euler product:

$$L(s, f) = \prod_p \prod_{i=1}^{n} \left(1 - \alpha_{i,p}(f)p^{-s}\right)^{-1},$$

where $\alpha_{i,p}(f) \in \mathbb{C}$, $i = 1, ..., n$, are the Satake parameters of the form f at a prime p. If g is another Hecke-Maass cusp form for $\mathrm{SL}(n, \mathbb{Z})$ with an L-function

$$L(s, g) = \sum_{m=1}^{\infty} \frac{B(m, 1, ..., 1)}{m^s} = \prod_p \prod_{i=1}^{n} \left(1 - \beta_{i,p}(g)p^{-s}\right)^{-1}$$

we can form a Rankin-Selberg L-function by

$$L(s, f \times g) := \zeta(ns) \sum_{m_1, ..., m_{n-1} \geq 1} \frac{A(m_1, ..., m_{n-1})\overline{B(m_1, ..., m_{n-1})}}{(m_1^{n-1} m_2^{n-2} \cdots m_{n-1})^s}$$

initially defined for $\Re(s)$ large enough, where $\zeta(s)$ is the Riemann zeta-function. Analytic properties of $L(s, f \times \widetilde{f})$ will be useful in Article [D]. The Euler product representation for the Rankin-Selberg L-function is given by

$$L(s, f \times g) = \prod_p \prod_{i=1}^{n} \prod_{k=1}^{n} \left(1 - \alpha_{i,p}(f)\overline{\beta_{k,p}(g)}p^{-s}\right)^{-1}.$$

In Article [C] we need to consider a twisted L-series for $\mathrm{SL}(3, \mathbb{Z})$ Maass cusp forms defined by

$$L_j\left(s + j, \frac{h}{k}\right) := \sum_{m=1}^{\infty} \frac{A(m, 1)}{m^s} \left(e\left(\frac{mh}{k}\right) + (-1)^j e\left(\frac{-mh}{k}\right)\right)$$

for $j \in \{0, 1\}$. This also has an entire analytic continuation to the whole complex plane via a functional equation [32] relating values at s and $1 - s$:

$$L_j\left(s + j, \frac{h}{k}\right) = i^{-j} k^{-3s+1} \pi^{3s-3/2} G_j(s + j) \widetilde{L}_j\left(1 - j - s, \frac{\overline{h}}{k}\right),$$

where

$$G_j(s+j) := \frac{\Gamma\left(\dfrac{1-s+j+\alpha}{2}\right)\Gamma\left(\dfrac{1-s+j+\beta}{2}\right)\Gamma\left(\dfrac{1-s+j+\gamma}{2}\right)}{\Gamma\left(\dfrac{s+j-\alpha}{2}\right)\Gamma\left(\dfrac{s+j-\beta}{2}\right)\Gamma\left(\dfrac{s+j-\gamma}{2}\right)},$$

where α, β and γ are the Langlands parameters

$$\alpha = -\nu_1 - 2\nu_2 + 1, \quad \beta = -\nu_1 + \nu_2, \quad \text{and} \quad \gamma = 2\nu_1 + \nu_2 - 1.$$

Here ν_1 and ν_2 are of course the spectral parameters of the underlying Maass cusp form, and $\widetilde{L}_j(s, \overline{h}/k)$ is the Dirichlet series

$$\widetilde{L}_j\left(s-j, \frac{\overline{h}}{k}\right) := \sum_{d|k} \sum_{m=1}^{\infty} \frac{A(d,m)}{d^{2s-1}\,m^s}\left(S\left(\overline{h}, m; \frac{k}{d}\right) + (-1)^j\, S\left(\overline{h}, -m; \frac{k}{d}\right)\right),$$

where $S(a, b; c)$ is the ordinary Kloosterman sum.

In Article [D] the main result is conditional on the generalised Lindelöf hypothesis. This conjecture predicts that the Godement-Jacquet L-function attached to Hecke-Maass cusp form f for $SL(n, \mathbb{Z})$ satisfies an estimate of the form

$$L\left(\frac{1}{2} + it, f\right) \ll_\varepsilon (1 + |t|)^\varepsilon$$

for every $\varepsilon > 0$ on the critical line. More discussion about this conjecture and other issues related to automorphic L-functions, see [52].

2.3.2 The Ramanujan-Petersson conjecture

Ramanujan's original conjecture was concerned with estimating the Fourier coefficients of the unique weight 12 holomorphic cusp form $\Delta(z)$ for $SL(2, \mathbb{Z})$. This was generalised by Petersson to concern Fourier coefficients of any holomorphic cusp form and also for Maass cusp forms.

Normalised Fourier coefficients of holomorphic cusp forms are known to satisfy the conjecture, i.e. an estimate

$$|a(n)| \leq d(n) \ll_\varepsilon n^\varepsilon$$

for every $\varepsilon > 0$ which is a consequence of Deligne's work on Weil conjectures [12]. Such a result is not known for even Maass cusp forms for $SL(2, \mathbb{Z})$, essentially because Maass forms do not have a direct connection to geometry. There are, however, approximations towards this conjecture.

Let f be a Maass cusp form for $SL(n, \mathbb{Z})$. We seek estimates of the form

$$A(m, 1, ..., 1) \ll_\varepsilon m^{\vartheta_n + \varepsilon}$$

for some $\vartheta_n \geq 0$ and every $\varepsilon > 0$. The generalised Ramanujan-Petersson conjecture predicts that for all $n \geq 2$ the value $\vartheta_n = 0$ is admissible. For general n the current best known result is

$$\vartheta_n \leq \frac{1}{2} - \frac{1}{n^2 + 1}$$

due to Luo, Rudnick, and Sarnak [79]. Better results are known for small values of n. We have $\vartheta_2 \leq 7/64$, $\vartheta_3 \leq 5/14$ and $\vartheta_4 \leq 9/22$ [66].

There is a closely related conjecture, the Selberg eigenvalue conjecture, which concerns the Langlands and spectral parameters. Namely, Selberg's eigenvalue conjecture holds for a Maass cusp form f with spectral parameters $\nu = (\nu_1, ..., \nu_{n-1}) \in \mathbb{C}^{n-1}$ if and only if $\Re(\lambda_j(\nu)) = 0$ for every $j = 1, ..., n$. The bounds for this are analogous to the bounds for the Fourier coefficients. The trivial bound is due to Jacquet and Shalika [56] stating that $\Re(\lambda_j(\nu)) < 1/2$. The best current bound is

$$\Re(\lambda_j(\nu)) \leq \frac{1}{2} - \frac{1}{n^2 + 1}$$

proved in [79]. Yet another reformulation of Selberg's eigenvalue conjecture is that a type $\nu = (\nu_1, ..., \nu_{n-1}) \in \mathbb{C}^{n-1}$ Maass cusp form for $\mathrm{SL}(n, \mathbb{Z})$ satisfies the conjecture if its spectral parameters satisfy $\Re(\nu_j) = 1/n$ for every $j = 1, ..., n-1$.

Although it is not needed in this thesis, we remark that there is a representation-theoretic formulation for the Ramanujan-Petersson conjecture: if π_f is a cuspidal automorphic representation of $\mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}})$ generated by the Maass cusp form f for $\mathrm{SL}(n, \mathbb{Z})$, then every local representation π_p, at finite primes p, of π_f is tempered (for more about this and the relevant terminology, see [54, 8, 31]).

While the bounds for the individual coefficients are quite far from the expected truth, the situation is better on average. Indeed, it follows from the analytic properties of the Rankin-Selberg L-function $L(s, f \times \widetilde{f})$ that

$$\sum_{m_1^{n-1} m_2^{n-2} \cdots m_{n-1} \leq x} |A(m_1, ..., m_{n-1})|^2 \sim c_f \cdot x, \tag{9}$$

for some constant $c_f \neq 0$ depending only on the underlying Maass form [30, Remark 12.1.8]. For $n = 2$ we have more precise estimates

$$\sum_{m \leq x} |a(m)|^2 = c_f' \cdot x + O\left(x^{3/5}\right) \tag{10}$$

in the holomorphic case (see [98, 102]), and

$$\sum_{m \leq x} |t(m)|^2 = \widetilde{c_f} \cdot x + O\left(x^{7/8}\right) \tag{11}$$

in the real-analytic case (see e.g. [50]), for some non-zero constants c_f', $\widetilde{c_f}$ depending on the underlying cusp forms. These results can be interpreted as saying that the Fourier coefficients are of constant size on average.

3 Exponential sums related to automorphic forms

Let f be a Maass cusp form for $\mathrm{SL}(n, \mathbb{Z})$ with Fourier coefficients $A(m_1, ..., m_{n-1})$. In this thesis we are interested in linear exponential sums of the form

$$\sum_m A(m, 1, ..., 1)e(m\alpha).$$

If the summation range is $1 \leq m \leq M$, the sum is called long, and if the summation range is $M \leq m \leq M + \Delta$ with $\Delta = o(M)$, we say that the sum is short. These long and short sums display very different behaviour.

Such sums exhibit somewhat analogous behaviour as the error term in the Dirichlet divisor problem

$$\Delta(x) := \sum_{n \leq x} d(n) - x(\log x + 2\gamma - 1)$$

as well as its twisted variants, short interval analogues, and higher rank generalisations. We will discuss more on these matters in connection to Voronoi summation formulas in Section 4.

For long sums the conjecture is that we expect square-root cancellation:

$$\sum_{m \leq M} A(m, 1, ..., 1)e(m\alpha) \ll M^{1/2}$$

uniformly on $\alpha \in \mathbb{R}$. For holomorphic cusp forms this is a result of Jutila [62] and for GL(2)-Maass cusp forms this is proved in Article [C] contained in the present thesis. For GL(3)-Maass cusp forms the best-known upper bound $\ll_\varepsilon M^{3/4+\varepsilon}$, for every $\varepsilon > 0$, is due to Miller [85]. In the higher rank situation nothing beyond the trivial bound $\ll M^{1+\varepsilon}$ is known in general. In the joint length and spectral parameter aspects uniform bounds in GL(2)-setting have been considered by Godber [29] and in the higher rank setting by Li and Young [77]. It is possible to obtain better bounds for specific values of α (e.g. when α is a rational number) as explained below.

For holomorphic cusp forms, upper bounds for short linear exponential sums have been considered first by Jutila [62] and the best known upper bounds are due to Ernvall-Hytönen and Karppinen [26].

Also, average sizes of such sums can be studied in the case of rationally additive twist. For long additively rationally twisted sums weighted by the Fourier coefficients of holomorphic cusp forms, Jutila [61] has proved that for $1 \leq k \ll M$ and $(h, k) = 1$,

$$\int_1^M \left| \sum_{m \leq x} a(m)e\left(\frac{mh}{k}\right) \right|^2 \mathrm{d}x = CkM^{3/2} + O(k^2 M^{1+\varepsilon}) + O(k^{3/2} M^{5/4+\varepsilon}),$$

where

$$C := \frac{1}{6\pi^2} \sum_{m=1}^\infty \frac{|a(m)|^2}{m^{3/2}}.$$

This means that on average the long linear additively rationally twisted sum is of size $k^{1/2}M^{1/4}$. For short sums, Ernvall-Hytönen [24] has shown that for $k^2 M^{1/2+\delta} \ll \Delta \ll M$, $\delta > 0$, one has

$$\int_M^{M+\Delta} \left| \sum_{x \le m \le x+\sqrt{x}} a(m)e\left(\frac{mh}{k}\right) \right|^2 w(x)\mathrm{d}x \ll \Delta M^{1/2+\varepsilon}$$

for a sufficiently nice weight function w. This largely shows that the expected size of a quarter power for sums of square-root length holds on average. Recently, Vesalainen [112] has studied the mean-square behaviour of additively rationally twisted linear sums of less than the square-root length and he proved that for $1 \le \Delta \ll \sqrt{M}$, $1 \le k \ll \Delta^{1/2}M^{-\varepsilon}$ one has

$$\int_M^{2M} \left| \sum_{x \le m \le x+\Delta} a(m)e\left(\frac{mh}{k}\right) \right|^2 \mathrm{d}x \ll M\Delta,$$

showing that square-root cancellation happens for such short sums on average in this range. Finally, let us mention that Ernvall-Hytönen [25] has studied the average behaviour of rationally additively twisted short sums with length greater than the square-root size.

One may also consider more general sums

$$\sum_m A(m,1,...,1)e(\alpha m^\beta),$$

where linear sums correspond to a special case $\beta = 1$. For $\beta \ne 1$ we call such exponential sums non-linear. The behaviour of these sums for different values of β has been studied recently by Ren and Ye [99, 100, 101].

For short linear sums involving Fourier coefficients of Maass cusp forms for $\mathrm{SL}(2,\mathbb{Z})$, the known upper bounds looks like the following. Figure 1 describes the bounds $\ll M^\beta$ for sums over the short interval $[M, M + M^\gamma]$.
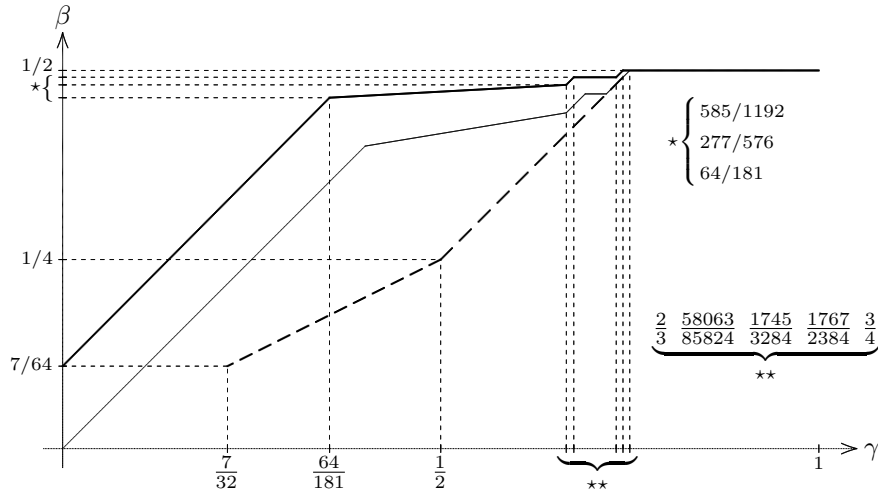

Figure 1. Various results for Maass cusp forms for $\mathrm{SL}(2,\mathbb{Z})$ from Article [C] with approximation $\vartheta = 7/64$ towards the Ramanujan-Petersson conjecture.

In Figure 1, solid thick lines indicate upper bounds and dotted lines indicate Ω-results.

It is commonly believed that optimal upper bounds are

$$\sum_{M \leq m \leq M+M^\gamma} c(m)e(m\alpha) \ll \left\{ \begin{array}{ll} M^{\gamma/2} & \text{for } 0 \leq \gamma \leq 1/2 \\ M^{\gamma-1/4} & \text{for } 1/2 \leq \gamma \leq 3/4 \\ M^{1/2} & \text{for } 3/4 \leq \gamma < 1 \end{array} \right.$$

Here $c(m)$ is a placeholder for $a(m)$ and $t(m)$. Analogously, in the GL(n)-situation, numerical evidence in the GL(2)-case [27] and the shape of the GL(n) Voronoi summation formula suggests that the correct upper bounds should be

$$\sum_{M \leq m \leq M+M^\gamma} A(m,1,...,1)e(m\alpha) \ll \left\{ \begin{array}{ll} M^{\gamma/2} & \text{for } 0 \leq \gamma \leq 1-1/n \\ M^{\gamma+1/2n-1/2} & \text{for } 1-1/n \leq \gamma \leq 1-1/2n \\ M^{1/2} & \text{for } 1-1/2n \leq \gamma < 1 \end{array} \right.$$

It is also interesting to study the special case where α is a rational number, write $\alpha = h/k$ with $(h,k) = 1$. In the GL(2)-situation a folklore conjecture predicts that

$$\sum_{m \leq M} c(m)e\left(\frac{mh}{k}\right) \ll \left\{ \begin{array}{ll} k^{1/2}M^{1/4+\varepsilon} & \text{for } k \ll M^{1/2-\varepsilon} \\ M^{1/2} & \text{for } M^{1/2} \ll k \ll M \end{array} \right.$$

In the higher rank setting we conjecture that the correct upper bounds should be

$$\sum_{m \leq M} A(m,1,...,1)e\left(\frac{mh}{k}\right) \ll \left\{ \begin{array}{ll} k^{1/2}M^{1/2-1/2n+\varepsilon} & \text{for } k \ll M^{1/n-\varepsilon} \\ M^{1/2} & \text{for } M^{1/n} \ll k \ll M \end{array} \right.$$

4 Voronoi summation formulas

Voronoi summation formulas are a cornerstone in proofs of many results included in this thesis. In this section we briefly recall the history and ideas behind summation formulas of this type, following [88] and [110]. Summation formulas have played an important role in analysis and number theory. This is natural as several important questions in number theory are about finite sums of arithmetic quantities. A classical first example of such summation formulas is the Poisson summation formula

$$\sum_{m \in \mathbb{Z}} f(m) = \sum_{m \in \mathbb{Z}} \widehat{f}(m)$$

which relates the sum of values of a function f at integers and its Fourier transform

$$\widehat{f}(m) := \int_{\mathbb{R}} f(x)e(-mx)\,\mathrm{d}x.$$

The Poisson summation formula is valid for functions f with suitable regularity properties such as Schwartz functions. Examples of such interesting problems concerning sums of arithmetic functions are for example the Dirichlet divisor

problem and the Gauss circle problem. Roughly speaking, Dirichlet showed in 1849 that

$$\sum_{m \leq X} d(m) = X \log X + (2\gamma - 1)X + \Delta(X),$$

where γ is the Euler-Mascheroni constant and $\Delta(X) \ll X^{1/2}$. The Dirichlet divisor problem is a deep conjecture that $\Delta(X) \ll_\varepsilon X^{1/4+\varepsilon}$ for every $\varepsilon > 0$. Good surveys on this problem are the one of Tsang [111] and Chapter 13 of Ivić's book [46].

The Gauss circle problem concerns the average size of

$$r_2(m) := \# \left\{ (x, y) \in \mathbb{Z}^2 : x^2 + y^2 = m \right\},$$

towards which Gauss showed that

$$D(X) := \left| \sum_{m \leq X} r_2(m) - \pi X \right| \ll X^{1/2}.$$

Again it is conjectured that the correct upper bound is $\ll_\varepsilon X^{1/4+\varepsilon}$ for every $\varepsilon > 0$.

Both of these problems have geometric interpretations. The quantity $\Delta(X)$ is related to counting lattice points in the region $\{x, y > 0 : xy \leq X\} \subset \mathbb{R}^2$ and $D(X)$ is related to counting lattice points inside the disc $\{x^2 + y^2 \leq X\} \subset \mathbb{R}^2$.

Voronoi [113, 114, 115] developed methods to improve the above bounds of Dirichlet and Gauss. In particular, he generalised the Poisson summation formula for weighted sums by allowing more general integral transforms than the Fourier transform on the right-hand side. In the end, he was able to show that $\Delta(X) \ll X^{1/3} \log X$. The logarithm was removed first by Sierpinski [107] and later independently by Landau [72]. Subsequently this was improved by Cramer [11] and many others. For a long time the best result was due to Huxley [44], $\ll X^{131/416+\varepsilon}$, but this has been recently improved by Bourgain and Watt [2] to $\ll X^{517/1648+\varepsilon}$. In the opposite direction Hardy and Landau [39, 40] proved that $\Delta(X) = \Omega(X^{1/4}(\log X)^{1/4})$. The sharpest Ω-result currently known is due to Soundararajan [108]

$$\Delta(X) = \Omega\left((X \log X)^{1/4} (\log \log X)^b (\log \log \log X)^{-5/8} \right),$$

where $b = 3(2^{4/3} - 1)/4$. Similar, but slightly weaker in some cases, bounds are also true for the quantity $D(X)$.

Many of these results were based on the exact formula

$$\Delta(X) = \frac{1}{4} + \sum_{n=1}^{\infty} d(n) \left(\frac{X}{n} \right)^{1/2} \left(Y_1(4\pi\sqrt{nX}) + K_1(4\pi\sqrt{nX}) \right) \qquad (12)$$

for all $X \notin \mathbb{Z}$, where $Y_1(\cdot)$ and $K_1(\cdot)$ are Bessel functions. This is the first so-called Voronoi summation formula. A smoothed version of the above formula (12) states that

$$\sum_{n=1}^{\infty} d(n)w(n) = \int_0^\infty w(x)(\log x + 2\gamma)\mathrm{d}x + \sum_{n=1}^{\infty} d(n)\breve{w}(n),$$

where for any smooth compactly supported function w on $]0, \infty[$,

$$\breve{w}(y) := \int_0^\infty w(x)(4K_0(4\pi\sqrt{xy}) - 2\pi Y_0(4\pi\sqrt{xy})) \, \mathrm{d}x$$

is the Hankel transform of w.

The modern interpretation of the Voronoi summation is related to the functional equation of the Riemann zeta-function in view of the identity

$$\sum_{m=1}^\infty \frac{d(m)}{m^s} = \zeta(s)^2.$$

Namely, by Perron's formula we have

$$\sum_m d(m)w\left(\frac{m}{X}\right) = \frac{1}{2\pi i} \int_{(3)} \left(\sum_{m=1}^\infty \frac{d(m)}{m^s}\right) X^s \widetilde{w}(s) \, \mathrm{d}s$$

where

$$\widetilde{w}(s) := \int_0^\infty w(x) \, x^{s-1} \, \mathrm{d}x$$

is the Mellin transform of the nicely behaved compactly supported weight function w.

After this observation the standard proof of the Voronoi summation formula proceeds by shifting the line segment of integration to the left, picking the residue of the pole at $s = 1$, and applying the functional equation of the Riemann ζ-function.

More generally, the L-series made of the values of the generalised divisor function $\sigma_k(m) := \sum_{d|m} d^k$ is also related to the Riemann zeta-function:

$$\sum_{m=1}^\infty \frac{\sigma_k(m)}{m^s} = \zeta(s)\zeta(s-k).$$

In this respect it is natural to expect that there is a Voronoi summation formula for sums involving values of the generalised divisor function $d_k(m)$, and this was indeed derived by Oppenheim [94]. Furthermore, values of $d(m)$ are the Fourier coefficients of Eisenstein series on GL(2). This leads to the guess that the sums involving Fourier coefficients of automorphic forms might have Voronoi type summation formulas. As the L-series made of Fourier coefficients of cusp forms have analytic continuations and satisfy suitable functional equations, as explained above, this turns out to be the case.

For holomorphic cusp forms of full level, Duke and Iwaniec [15, 16, 17, 18] derived the Voronoi summation formula for the plain sum of coefficients from the functional equations of the L-functions twisted by Dirichlet characters. For rationally additively twisted exponential sums involving Fourier coefficients of holomorphic cusp forms for the full modular group, the Voronoi summation formula is due to Jutila [61]. Generalisation (in the level aspect) of Jutila's result for cusp forms with trivial character is found in the works of Kowalski, Michel and Vanderkam [70]. This proof combines methods of Jutila as well as the one

of Duke and Iwaniec where the analytic continuation and functional equation of the associated Dirichlet series is established from a direct consideration of the action of the modular group and the Mellin inversion. For classical Maass cusp forms with trivial character, the Voronoi summation formula for rationally additively twisted sums was first established by Meurman [83] following the method of Jutila [61]. Meurman's proof is, however, more complicated, mostly because he considers a wider class of admissible test functions. The general formula for Maass cusp forms with arbitrary character is established by Harcos and Michel [38].

The truncated Voronoi summation formula for holomorphic cusp forms with rational additive twists is due to Jutila [61]. For classical Maass cusp forms analogous results is a work of Meurman [83]. Similar computations have been performed by Friedlander and Iwaniec [28] for sums of Dirichlet series coefficients of a wide class of L-functions.

The Voronoi summation formula for $\mathrm{SL}(n, \mathbb{Z})$ is established by Miller and Schmid who at the same time develop the framework of automorphic distributions [86, 87]. Another proof appears in the works of Goldfeld and Li [32, 33] using the method of Duke and Iwaniec alluded above. A general version in the adelic setting has been given by Ichino and Templier [45] where they derived the Voronoi summation formula from the classical theory of integral representation and both local and global functional equations of L-functions on $\mathrm{GL}(n)$, where they follow the techniques of Jacquet, Piatetski-Shapiro and Shalika [55].

The numerous applications of the Voronoi summation formula include, for example, subconvexity and non-vanishing results of L-values as well as estimates for shifted convolution sums; e.g. the GL(3)-Voronoi summation formula has also been used recently to establish subconvexity for GL(3) L-functions in the t-aspect [90]. Usually standard arguments, such as approximate functional equation and circle methods, reduce interesting problems to estimating sums of the form

$$\sum_m a(m) e\left(\frac{mh}{k}\right) w\left(\frac{m}{X}\right),$$

where $h, k \in \mathbb{Z}$ and w is a compactly supported smooth weight function, the central question being to produce uniform upper-bounds. For that purpose the Voronoi summation formula is widely used: in favourable situations the length of the dual sum is shorter than the original sum and hence can be bounded effectively by using absolute values. To this end, one is typically reduced to understand the asymptotic behaviour of the Bessel transform (a generalisation of the Hankel transform $\breve{w}$ above) of w which is a problem in analysis.

Next, we discuss results of above type obtained in this dissertation. The first result gives an asymptotic expression for certain integral kernel appearing in the twistless GL(n)-Voronoi summation formula. Let us define

$$G(s) := \prod_{\ell=1}^{n} \Gamma\left(\frac{s - \lambda_\ell}{2}\right) \quad \text{and} \quad \widetilde{G}(s) := \prod_{\ell=1}^{n} \Gamma\left(\frac{s - \widetilde{\lambda}_\ell}{2}\right),$$

where $\{\lambda_\ell\}_{\ell=1}^{n}$ are the Langlands parameters of the underlying GL(n) Maass cusp form and $\{\widetilde{\lambda}_\ell\}_{\ell=1}$ are the Langlands parameters of its dual form.

Voronoi summation formulas for $\mathrm{GL}(n)$ are complicated and difficult to deal with not least because of the presence of hyper-Kloosterman sums. Even the twistless $\mathrm{GL}(n)$-Voronoi summation formula [87] states the following:

Theorem 16. *Let $f \in C_{\mathrm{c}}^{\infty}(\mathbb{R}_+)$. Then*

$$\sum_{m=1}^{\infty} A(m, 1, \ldots, 1)\, f(m)$$

$$= \sum_{m=1}^{\infty} \frac{A(1, \ldots, 1, m)}{m} \frac{1}{2\pi i} \int\limits_{(-\sigma_0)} \widetilde{f}(s)\, \pi^{-n/2}\, \frac{\widetilde{G}(1-s)}{G(s)}\, (\pi^n m)^s\, \mathrm{d}s,$$

where σ_0 is a large positive real number, depending on the form.

In Article [A] we derive an asymptotic expansion for the integral

$$\Omega(y) := \frac{1}{2\pi i} \int\limits_{(-\sigma_0)} \int\limits_{0}^{\infty} f(x) x^{s-1} \mathrm{d}x\, \pi^{-n/2}\, \frac{\widetilde{G}(1-s)}{G(s)}\, y^s \mathrm{d}s$$

appearing in this formula. One of the main results in Article [A] is the following.

Theorem 17. *(Theorem 10 in [A]) For any $K \in \mathbb{Z}_+$ and $y \gg 1$, we have*

$$\Omega(y) = y^{1/2+1/(2n)} \int\limits_{0}^{\infty} f(x)\, x^{1/(2n)-1/2}\, \mathcal{K}(x, y)\, \mathrm{d}x,$$

where the kernel $\mathcal{K}(x, y)$ has, for $x \gg 1$ and $y \gg 1$, the asymptotics

$$\mathcal{K}(x, y) = \sum_{\ell=0}^{K} (xy)^{-\ell/n} \big(c_\ell^+\, e(\pi^{-1}\, n\, x^{1/n}\, y^{1/n}) + c_\ell^-\, e(-\pi^{-1}\, n\, x^{1/n}\, y^{1/n}) \big)$$

$$+ O\big((xy)^{-(K+1)/n}\big),$$

for some explicit constants $c_\ell^{\pm}$. We emphasise that the implicit constant in the O-term is independent of f. Here the leading coefficients $c_0^{\pm}$ are given by

$$c_0^{\pm} = \pi^{-(n+1)/2}\, \frac{1}{\sqrt{n}}\, e\left(\pm \frac{(n-1)}{8}\right).$$

This generalises the $n = 3$ result of [76], and is an analogue to the result of Ivić [47] concerning the ternary divisor function. For applications of this result, see Section 5. Notice that here the explicit value of constants $c_0^{\pm}$ differs from [A]; see Remark 24 below. The main ideas of the proof are the following. The first step is to replace the quotient of Γ-factors by a quotient of simpler Γ-factors. Indeed, a simple computation shows that

$$\frac{\widetilde{G}(1-s)}{G(s)} = \frac{\Gamma\left(\frac{1-ns}{2}\right)}{\Gamma\left(\frac{ns-(n-1)}{2}\right)} \cdot n^{ns-n/2} \left(1 + \sum_{k=1}^{K} \frac{c_k}{s^k} + O\left(|s|^{-K-1}\right)\right)$$

for some $K \in \mathbb{N}$ and coefficients c_k only depending on the Langlands parameters λ_ℓ and $\widetilde{\lambda}_\ell$. Next, we split the integral into parts:

$$\Omega(y) = \frac{1}{2\pi i} \int_{(-\sigma)} \int_0^\infty f(x) x^{s-1} \,\mathrm{d}x \pi^{-n/2} n^{ns-n/2} \frac{\Gamma\left(\frac{1-ns}{2}\right)}{\Gamma\left(\frac{ns-(n-1)}{2}\right)} y^s \mathrm{d}s$$

$$+ \frac{1}{2\pi i} \int_{(-\sigma)} \int_0^\infty f(x) x^{s-1} \,\mathrm{d}x \pi^{-n/2} n^{ns-n/2} \frac{\Gamma\left(\frac{1-ns}{2}\right)}{\Gamma\left(\frac{ns-(n-1)}{2}\right)} \left(\underbrace{\sum_{k=1}^K \frac{c_k}{s^k} + O\left(|s|^{-K-1}\right)}_{=:H(s)} \right) y^s \mathrm{d}s.$$

By using an additional parameter Λ we write $H(s)$ as a Taylor series at infinity obtaining

$$H(s) = \sum_{k=1}^K \frac{c_k'}{(s+\Lambda)^k} + O\left((s+\Lambda)^{-K-1}\right)$$

for some coefficients c_k' depending on the coefficients c_k. Here the parameter Λ is some sufficiently large parameter needed for an application of Stirling's formula. Each of the K terms in the above sum gives rise to an integral of the form

$$\frac{1}{2\pi i} \int_{(-\sigma)} \int_0^\infty f(x) x^{s-1} \,\mathrm{d}x \pi^{-n/2} n^{ns-n/2} \frac{\Gamma\left(\frac{1-ns}{2}\right)}{\Gamma\left(\frac{ns-(n-1)}{2}\right)} (s+\Lambda)^{-k} y^s \,\mathrm{d}s$$

and an error whose contribution can be estimated to be small. Actually we shall consider slightly more general integrals

$$\Omega_{\nu,k}(y) := \frac{1}{2\pi i} \int_{(-\sigma)} \int_0^\infty f(x) x^{s-1} \,\mathrm{d}x \pi^{-n/2} n^{ns-n/2} \frac{\Gamma\left(\frac{1-ns}{2}\right)}{\Gamma\left(\frac{ns-(n-1)}{2}+\nu\right)} (s+\Lambda)^{-k} y^s \,\mathrm{d}s$$

for $\nu \in \{0,1,2,3,...\}$.

A simple use of residue theorem shows that

$$\Omega_{\nu,0}(y) = 2\pi^{-n/2} y^{1/2+(1-\nu)/n} n^{-\nu} \int_0^\infty f(x) x^{(1-\nu)/n-1/2} J_{\nu-n/2}\left(2n x^{1/n} y^{1/n}\right) \,\mathrm{d}x,$$

where $J_\cdot(\cdot)$ is the usual J-Bessel function.

Now we need two more observations. The first is that for $\nu + k \geq K + 2$ shifting the line of the integration to $\sigma = -K/n + 1/2n - 1/2$ together with Stirling's formula and estimating by absolute values gives

$$\Omega_{\nu,k}(y) \ll y^{1/2+1/2n} \int_0^\infty |f(x)| x^{1/2n-1/2} (xy)^{-(k+1)/n} \,\mathrm{d}x. \tag{13}$$

When these are established, it is possible to prove the required statement by induction on k. The base case $k = 0$ is proved by using residue theorem as indicated above. For the inductive step, the relevant observation is that

$$\Omega_{\nu,k}(y) = \frac{n}{2} \Omega_{\nu+1,k-1}(y) + \left(\frac{1}{2} + \nu - \frac{n}{2} - \frac{n\Lambda}{2}\right) \Omega_{\nu+1,k}(y).$$

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By iterating this $N \in \mathbb{Z}_+$ times we end up with

$$\Omega_{\nu,k}(y) = \alpha_1 \Omega_{\nu+1,k-1}(y) + \alpha_2 \Omega_{\nu+2,k-1}(y) + \alpha_3 \Omega_{\nu+3,k-1}(y)$$
$$+ \cdots + \alpha_N \Omega_{\nu+N,k-1}(y) + \beta_N \Omega_{\nu+N,k}(y)$$

for some real coefficients $\alpha_1, ..., \alpha_N, \beta_N$. Provided that $\nu + k + N \geq K + 2$, the last term can be treated by the estimate (13). All other terms are covered by the induction assumption. Now the theorem follows from the asymptotics of J-Bessel functions.

In Article [B] we need a truncated Voronoi summation formula involving Fourier coefficients of $\mathrm{SL}(3, \mathbb{Z})$ Maass cusp forms for application we have in mind. In the GL(2)-case Jutila [61] has proved the following truncated Voronoi identity:

Theorem 18. *Let $x \geq 1$, $k \leq x$ and $1 \leq N \ll x$. Then we have*

$$\sum_{m \leq x} a(m) e\left(\frac{mh}{k}\right) = \frac{k^{1/2} x^{1/4}}{\pi \sqrt{2}} \sum_{m \leq N} \frac{a(m)}{m^{3/4}} e\left(-\frac{m\bar{h}}{k}\right) \cos\left(\frac{4\pi\sqrt{mx}}{k} - \frac{\pi}{4}\right)$$
$$+ O\left(kN^{-1/2} x^{1/2+\varepsilon}\right),$$

where $\bar{h}$ is the multiplicative inverse of h modulo k.

Remark 19. *Notice that Jutila uses slightly different normalisations than us.*

In Article [B] we prove an analogous truncated Voronoi identity for rationally additively twisted sums in GL(3).

Theorem 20. *(Theorem 1 in [B]) Let $x, N \in [2, \infty[$ with $N \ll x$, and let h and k be coprime integers with $1 \leqslant k \leqslant x$, $k \leqslant N$ and $k \ll (Nx)^{1/3}$, the latter having a sufficiently small implicit constant depending on the underlying Maass form. Then we have*

$$\sum_{m \leqslant x} A(m, 1) e\left(\frac{mh}{k}\right)$$
$$= \frac{x^{1/3}}{\pi \sqrt{3}} \sum_{d|k} \frac{1}{d^{1/3}} \sum_{d^2 m \leqslant N} \frac{A(d, m)}{m^{2/3}} S\left(\bar{h}, m; \frac{k}{d}\right) \cos\left(\frac{6\pi d^{2/3}(mx)^{1/3}}{k}\right)$$
$$+ O(k\, x^{2/3+\vartheta+\varepsilon}\, N^{-1/3}) + O(k\, x^{1/6+\varepsilon}\, N^{1/6+\vartheta}).$$

Here $\vartheta \geq 0$ is an exponent towards the generalised Ramanujan-Petersson conjecture.

Remark 21. *There is a misprint in the formulation of Theorem 1 in Article [B], which is corrected in the formulation of Theorem 20 above. Notice that on the right-hand side the term $1/d^{1/3}$ appears which is wrongly written to be $1/d$ in [B]. However, this does not affect to Theorem 28 and Corollary 29 below.*

Next, we briefly describe the proof of this result. The idea is to prove a summation formula for

$$\sum_{m \leq x} A(m, 1) \left(e\left(\frac{mh}{k}\right) + (-1)^j e\left(-\frac{mh}{k}\right) \right),$$

where $j \in \{0, 1\}$. Up to an error $\ll x^{1+\vartheta+\varepsilon}T^{-1}$ the truncated Perron's formula allows us to express the above sum as an integral

$$\frac{1}{2\pi i} \int_{1+\delta-iT}^{1+\delta+iT} L_j\left(s+j, \frac{h}{k}\right) x^s \frac{\mathrm{d}s}{s}$$

for a fixed $\delta > 0$, where

$$L_j\left(s+j, \frac{h}{k}\right) = \sum_{m=1}^{\infty} \frac{A(m,1)}{m^s} \left(e\left(\frac{mh}{k}\right) + (-1)^j e\left(-\frac{mh}{k}\right)\right).$$

Next, we shift the line of integration from $\sigma = 1 + \delta$ to the line $\sigma = -\delta$. This can be done with sufficiently small error. After applying the functional equation of $L_j(s+j, h/k)$ (see Section 2.3.1.), our integral becomes

$$\frac{1}{2\pi i} \sum_{d|k} \frac{1}{d} \sum_{m=1}^{\infty} \frac{A(d,m)}{m} \left(S\left(\overline{h}, m; \frac{k}{d}\right) + (-1)^j S\left(\overline{h}, -m; \frac{k}{d}\right)\right)$$

$$\cdot \int_{-\delta-iT}^{-\delta+iT} \frac{1}{i} k^{-3s+1} d^{2s} \pi^{3s-\frac{3}{2}} 3^{3s-\frac{3}{2}} \frac{\Gamma\left(\frac{1-3s}{2}\right)}{\Gamma\left(\frac{3s-2}{2}\right)} \left(1 + O\left(|s|^{-1}\right)\right) m^s x^s \frac{\mathrm{d}s}{s}.$$

Then we choose T so that the integral has no saddle points for $d^2 m > N$. These terms are estimated to contribute the error terms by using Stirling's formula and the first derivative test.

The remaining terms contribute the main term which arises in the similar fashion as in the proof of Theorem 17. We also have error terms which are again estimated to be small enough by using standard methods. Now the theorem follows by adding these derived summation formulas for $j = 0$ and 1.

5   Short resonance sums

As mentioned before, one way to study oscillations of the Fourier coefficients $A(m, 1, ..., 1)$ is to test how the correlate with other oscillatory objects. In particular, if we can detect the resonating frequencies of the second system, we gain information on the system of the Fourier coefficients. For a holomorphic cusp form f (of weight $k \in \mathbb{Z}_+$) for $\mathrm{SL}(2,\mathbb{Z})$, Iwaniec, Luo and Sarnak [51] showed that the Fourier coefficients $a(m)$ resonate against exponential phases $e(\alpha m^\beta)$ if and only if $\beta = 1/2$ and α is close to $\pm 2\sqrt{\ell}$ for some positive integer ℓ. To be more precise, they showed that

$$\sum_m a(m) e\left(-2\sqrt{\ell m}\right) w\left(\frac{m}{X}\right)$$

$$= \frac{a(\ell)}{\ell^{1/4}} X^{3/4} i^k \frac{1-i}{2} \int_0^\infty w(x) x^{-1/4} \,\mathrm{d}x + O\left((\ell X)^{1/4+\varepsilon}\right)$$

for a compactly supported smooth weight function w. Therefore the sum is $\gg \ell^{-1/4} X^{3/4+\varepsilon}$ if $a(\ell) \neq 0$ and the weight function w is such that the integral on the right-hand side does not vanish. In Article [A] we consider an analogous problem for short exponential sums attached to a Maass cusp form for $\mathrm{SL}(n, \mathbb{Z})$. In many sums in analytic number theory one expects that the square root heuristic holds:

the sum of oscillating terms of roughly constant size is bounded from above by the square root of the length of summation. In Article [A] we show that for a short sum

$$\sum_{M \le m \le M+\Delta} A(m, 1, ..., 1) e(m\alpha),$$

the square root cancellation philosophy does not hold for every $\alpha \in \mathbb{R}$ when Δ is large enough compared to M. We show that for $d \in \mathbb{Z}_+$ the exponential sum

$$\sum_{M \le m \le M+\Delta} A(m, 1, ..., 1) e\left(\frac{d^{1/n} m}{M^{1-1/n}}\right) \tag{14}$$

is larger than the square root size $\Delta^{1/2}$ under some conditions. Similar resonance sums involving Fourier coefficients of both $\mathrm{GL}(2)$ and $\mathrm{GL}(n)$ forms have been considered before, see [21, 22, 23, 99, 101].

In particular, we show that the following holds.

Theorem 22. *(Corollary 3 in [A]) Let $d \in \mathbb{Z}_+$ be fixed, and assume that $A(1, \ldots, 1, d) \ne 0$. Then, for $M^{1-1/n+\varepsilon} \ll \Delta \ll M^{1-1/(2n)}$, the sum*

$$\sum_{M \le m \le M+\Delta} A(m, 1, \ldots, 1) e\left(\frac{d^{1/n} m}{M^{1-1/n}}\right)$$

is $\Omega(\Delta M^{1/(2n)-1/2})$, and for $M^{1-1/(2n)} \ll \Delta \ll M$, the sum is $\Omega(M^{1/2})$.

Notice that in the range $M^{1-1/n+\varepsilon} \ll \Delta$ we have $\Delta M^{1/(2n)-1/2} \gg \Delta^{1/2+\varepsilon}$ and also for $\Delta \ll M$ we have $M^{1/2} \gg \Delta^{1/2}$. Therefore for $\Delta \gg M^{1-1/n+\varepsilon}$ the sum (14) has a lower bound greater than the square root-cancellation bound.

This result follows from the following resonance result, which shows that the sum we are interested in essentially depends on the dth-Fourier coefficient of the underlying Maass cusp form.

Theorem 23. *(Theorem 1 in [A]) Let $M^{1-1/n+\varepsilon} \ll \Delta \ll M$, and let d be a fixed positive integer. Also, let $w \in C_c^\infty(\mathbb{R}_+)$ be supported in the interval $[M, M+\Delta]$ with $w^{(\nu)}(x) \ll_\nu \Delta^{-\nu}$ for $\nu \in \mathbb{Z}_+ \cup \{0\}$. Then*

$$\sum_{M \le m \le M+\Delta} A(m, 1, ..., 1) \, w(m) \, e\left(\frac{d^{1/n} m}{M^{1-1/n}}\right) = \frac{A(1, ..., 1, d)}{d^{1/2-1/(2n)} \sqrt{n}} e\left(-\frac{(n-1)}{8}\right)$$

$$\cdot \int_M^{M+\Delta} w(x) \, e\left(\frac{d^{1/n} x}{M^{1-1/n}} - n \, x^{1/n} \, d^{1/n}\right) x^{1/(2n)-1/2} \, dx$$

$$+ O(\Delta M^{-1/2-1/(2n)}).$$

Remark 24. *Notice that in the original article [A] there is a minor typo concerning the exponential phase $e(-(n-1)/8)$ in the previous theorem. In place of the term $n+3$ there should be $-(n-1)$ as stated in the formulation above.*

This generalises a result of Ernvall-Hytönen [23] who treated the case $n = 3$. In order to derive such a result we need to compute the asymptotics for the integrals appearing in the Voronoi summation formula for $\mathrm{SL}(n, \mathbb{Z})$ Maass cusp

forms discussed in the previous section. The idea of the proof is then to use this Voronoi-asymptotics. It follows from Theorem 17 that up to error terms, the left-hand side in Theorem 23 is given by

$$\int_M^{M+\Delta} g_m^\pm(x) w(x) e\left(\frac{d^{1/n}x}{M^{1-1/n}} \pm n x^{1/n} m^{1/n}\right) x^{1/2n-1/2} m^{1/2+1/2n} \mathrm{d}x,$$

where

$$g_m^\pm(x) := \sum_{\ell=0}^K c_\ell^\pm (xm)^{-\ell/n}.$$

By repeatedly integrating by parts, it follows that the terms coming from $g_m^+(\cdot)$ and $g_m^-(\cdot)$ for $d \neq m$, contribute $\ll 1$. Next, in $g_d^-(\cdot)$, the term $\ell = 0$ gives the main contribution. The larger values of ℓ can be easily estimated to be $\ll \Delta M^{-1/2-1/2n}$. This concludes the proof.

By partial summation, Theorem 23 yields the following corollary:

Corollary 25. *(Corollary 14 in [A]) Let $d \in \mathbb{Z}_+$ be fixed and such that $A(1,...,1,d) \neq 0$. Then there exists $\Delta \asymp M^{1-1/2n}$ such that*

$$\sum_{M \leq m \leq M+\Delta} A(m,1,...,1) e\left(\frac{d^{1/n}m}{M^{1-1/n}}\right) \gg M^{1/2}.$$

This corollary can be used to prove the following Ω-result for the plain sum of coefficients of sufficiently short intervals:

Theorem 26. *(Theorem 5 in [A]) Assume that $\Delta = o(M^{1/2-\vartheta-\varepsilon})$. Then*

$$\sum_{M \leqslant m \leqslant M+\Delta} A(m,1,\ldots,1) = \Omega(M^{1/(2n)-1/2}\Delta).$$

This can be seen as a generalisation of Ivić's result [48] showing that

$$\sum_{M \leq m \leq M+\Delta} a(m) = \Omega(\sqrt{\Delta})$$

when $M^\varepsilon \ll \Delta \ll M^{1/2-\varepsilon}$. Ernvall-Hytönen extended Ivić's result to cover the range $\Delta \asymp M^{1/2}$ in [21].

The final corollary of the resonance result concerns the determination of all the coefficients $A(m_1,\ldots,m_{n-1})$ from a sparse subset of coefficients. The following result is an immediate consequence of Theorem 23.

Corollary 27. *(Corollary 6 in [A]) Let M_1, M_2, ... be a sequence of positive real numbers tending to infinity, and let ε be an arbitrarily small positive real number. Write*

$$I = \mathbb{Z} \cap \bigcup_{\ell=1}^\infty \left[M_\ell, M_\ell + M_\ell^{1-1/n+\varepsilon}\right].$$

Then the Fourier coefficients $A(m,1,\ldots,1)$ with $m \in I$ uniquely determine all the Fourier coefficients $A(m_1,\ldots,m_{n-1})$.

This may be considered as a relative multiplicity one theorem (compare to results in [97, 4]).
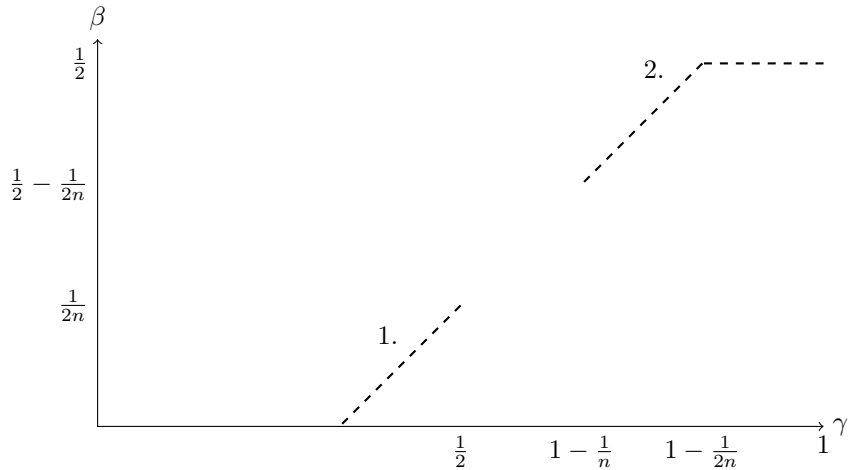

Figure 2. Various Ω-results for short exponential sums of the form $\Omega(M^\beta)$ over summation ranges $[M, M + M^\gamma]$ involving $\mathrm{GL}(n)$ Maass cusp forms coming from results of Section 5. The line segment 1 refers to Theorem 26 and the line segment 2 refers to Theorem 22. In this graph $\vartheta = 0$ for simplicity.

6 Average behaviour of rationally twisted exponential sums in $\mathrm{GL}(3)$

The objective of Article [B] is to consider long rationally additively twisted exponential sums attached to Maass form for $\mathrm{SL}(3, \mathbb{Z})$, that is, sums of the type

$$\sum_{m \leq x} A(m, 1) e\left(\frac{mh}{k}\right) \tag{15}$$

with $(h, k) = 1$. The motivation was to obtain an analogue of Jutila's result regarding to an asymptotic formula for the mean square for such sums with holomorphic cusp form coefficients [61]. The first result concerns the average behaviour of this sum towards which we prove the following upper bound:

Theorem 28. *(Theorem 2 in [B]) Let h and k be positive integers such that $(h, k) = 1$. Then for $X \in [1, \infty[$ we have*

$$\int_1^X \left| \sum_{m \leq x} A(m, 1) \, e\left(\frac{mh}{k}\right) \right|^2 \, \mathrm{d}x \ll k^2 \, X^{5/3 + 2\vartheta + \varepsilon}.$$

This almost shows the expected upper bound $\ll k^{1/2} x^{1/3 + \varepsilon}$ (at least when $k \ll X^{1/3 - \varepsilon}$) for the sum (15) holds in the x-aspect on average if the Ramanujan-Petersson conjecture is assumed. Theorem 28 is an analogue of Jutila's result for holomorphic cusp forms [61]. Notice that Jutila gets an asymptotic formula whereas we only get an upper bound. The reason for this is that one of the error

terms in the truncated GL(3) Voronoi summation formula give a contribution larger than the expected main term which arises from the diagonal terms

The proof proceeds as follows. As usual, the first step is to apply the truncated Voronoi summation formula. Error terms coming from this are easily estimated to contribute the claimed upper bound. By opening the absolute square, the main term can be interpreted as $\sum_{m_1,d_1} \overline{\sum_{m_2,d_2}}$. We distinguish into diagonal terms $d_1^2 m_1 = d_2^2 m_2$ and off-diagonal terms $d_1^2 m_1 \neq d_2^2 m_2$. The diagonal terms can be handled by using the Cauchy-Schwarz inequality, the Weyl bound, the Rankin-Selberg theory, and partial summation, giving a total contribution $\ll k^{1+\varepsilon} X^{5/3}$. In the off-diagonal, we face integrals of the form

$$\int\limits_{X}^{2X} x^{2/3} \cos\left(\frac{6\pi d_1^{2/3} m_1^{1/3} x^{1/3}}{k}\right) \cos\left(\frac{6\pi d_2^{2/3} m_2^{1/3} x^{1/3}}{k}\right) \mathrm{d}x,$$

which are estimated by the first derivative test or by absolute values depending on the size of $d_2^2 m_2$ relative to the size of $d_1^2 m_1$. Using this together with the Rankin-Selberg estimate, Weil's bound for Kloosterman sums, and partial summation yields that the total off-diagonal contribution is $\ll k^2 X^{5/3+\vartheta+\varepsilon}$.

We remark that in the case of a trivial exponential twist ($h = 0, k = 1$), a truncated Voronoi summation formula involving Fourier coefficients of a Maass form for $\mathrm{SL}(n, \mathbb{Z})$ can be derived using arguments of Section 4. This together with techniques explained above can be used to produce upper bounds for the sum of these Fourier coefficients. However, the upper bounds our methods produce are weaker than what is currently known by using other techniques [9, 10, 80].

We also consider upper bounds for the sum (15). It is a simple matter to show from Theorem 28 that the following holds.

Corollary 29. *(Corollary 3 in [B]) Let $x \in [1, \infty[$, and let h and k be coprime integers with $1 \leqslant k \ll x^{2/3}$. Then,*

$$\sum_{m \leqslant x} A(m, 1)\, e\left(\frac{mh}{k}\right) \ll k^{1/2+\varepsilon}\, x^{2/3} + k\, x^{1/3+\vartheta+\varepsilon}.$$

Furthermore, when $\vartheta \leqslant 1/3$ and $k \ll x^{2/3-2\vartheta}$, we have

$$\sum_{m \leqslant x} A(m, 1)\, e\left(\frac{mh}{k}\right) \ll k^{3/4}\, x^{1/2+\vartheta/2+\varepsilon} + k^{9/8+3\vartheta/4}\, x^{1/4+3\vartheta^2/2+3\vartheta/4+\varepsilon}.$$

In particular, for $\vartheta = 0$ and $k \ll x^{2/3}$, we have

$$\sum_{m \leqslant x} A(m, 1)\, e\left(\frac{mh}{k}\right) \ll k^{3/4}\, x^{1/2+\varepsilon}.$$

Miller's [85] bound gives $\ll_\varepsilon x^{3/4+\varepsilon}$ for arbitrary real twists. Furthermore the dependence on the Laplace eigenvalue in the symmetric square lift-case was investigated by Li and Young [77]. Under the Ramanujan–Petersson conjecture $\vartheta = 0$, the above upper bound $\ll k^{3/4} x^{1/2+\varepsilon}$ is an improvement to Miller's bound when $k \ll x^{1/3}$.

7 Exponential sums related to classical Maass cusp forms

In Article [C] we study various aspects of exponential sums twisted by Fourier coefficients of classical Maass cusp forms. The motivation for this article was to check to what extent the best results proved for holomorphic cusp forms carry over to the setting of classical Maass cusp forms. In particular, in this case the Ramanujan-Petersson conjecture is not known and it is interesting to study how results depend on the exponent ϑ towards this conjecture. For certain ranges we can prove non-trivial upper bounds for short sums which are analogous to the results of Ernvall-Hytönen and Karppinen [26].

Theorem 30. *(Theorem 1 in [C]) Let $M \in [1, \infty[$ and let $\Delta \in [1, M]$ be such that $\Delta \ll M^{2/3}$. Then*

$$\sum_{M \leqslant n \leqslant M+\Delta} t(n)\,e(n\alpha) \ll \Delta^{1/6-\vartheta}\,M^{1/3+\vartheta+\varepsilon}$$

uniformly for $\alpha \in \mathbb{R}$. This is better than estimating via absolute values in the range $M^{2/(5+6\vartheta)} \ll \Delta \ll M^{2/3}$.

The idea of the proof is to introduce smoothing and then estimate smoothed sums. To this end, we create a smooth partition of unity on $]M, M+\Delta[$ in the following way. We start by defining points M_ℓ for $\ell \in \mathbb{Z}$ by setting

$$M_0 = M + \frac{\Delta}{2}$$

and then

$$M_{\pm\ell} = M + \frac{\Delta}{2} \pm \left(\frac{\Delta}{4} + \frac{\Delta}{8} + \cdots + \frac{\Delta}{2^{\ell+1}} \right).$$

Then pick functions $w_\ell \in C_c^\infty(\mathbb{R})$ such that $w_\ell \equiv 1$ on $[M_{2\ell}, M_{2\ell+1}]$, supported on $[M_{2\ell-1}, M_{2\ell+2}]$ and

$$w_\ell^{(\nu)}(x) \ll_\nu \left(\frac{\Delta}{4^{|\ell|}} \right)^{-\nu}$$

for $x \in [M_{2\ell-1}, M_{2\ell+2}]$. Furthermore, suppose that $w_\ell + w_{\ell+1} \equiv 1$ on $[M_{2\ell+1}, M_{2\ell+2}]$. Let $L \in \mathbb{Z}_+$ be such that $\Delta 4^{-L} \asymp M^{2/(5+6\vartheta)}$.

Next, observe that

$$\sum_{M \leq n \leq M+\Delta} t(n)e(n\alpha) = \sum_{\ell=-L}^{L} \sum_{n \in \mathbb{Z}} t(n)e(n\alpha)w_\ell(n)$$

$$+ \sum_{M \leq n \leq M+\Delta} t(n)e(n\alpha) \left(1 - \sum_{\ell=-L}^{L} w_\ell(n) \right) \quad (16)$$

Therefore we are reduced to consider the weighted sum

$$\sum_{n \in \mathbb{Z}} t(n)e(n\alpha)w(n)$$

with a weight function $w \in C_c^\infty(\mathbb{R}^+)$ supported on $[M, M + \Delta]$ and satisfying $w^{(\ell)}(x) \ll \Delta^{-\ell}$. The reason for introducing smoothing is that it allows us to win an extra factor Δ^{-1} every time when integrating by parts the exponential integrals arising from the application of the Voronoi summation formula. For smoothed sums, we have the following theorem.

Theorem 31. *(Theorem 18 in [C]) Let us be given a small $\varepsilon \in \mathbb{R}_+$, and let $\delta \in \mathbb{R}_+$ satisfy $\delta \ll \varepsilon$ with a sufficiently small implicit constant. Let $M \in [1, \infty[$, and let $\Delta \in [1, M]$ with $\Delta \gg M^\delta$. Furthermore, let $\alpha \in \mathbb{R}$, and let $h \in \mathbb{Z}$, $k \in \mathbb{Z}_+$ and $\eta \in \mathbb{R}$ be such that*

$$\alpha = \frac{h}{k} + \eta, \quad (h, k) = 1, \quad k \leqslant K, \quad |\eta| \leqslant \frac{1}{kK},$$

where $K = \Delta^{1/2-\delta}$.

1. If $\eta \ll \Delta^{-1+\delta}$, then

$$\sum_{M \leqslant n \leqslant M+\Delta} t(n)\, e(n\alpha)\, w(n) \ll_\delta \Delta^{1/6}\, M^{1/3+\varepsilon}.$$

2. If $\Delta^{-1+\delta} \ll \eta$ and $k^2\, \eta^2\, M < 1/2$, then

$$\sum_{M \leqslant n \leqslant M+\Delta} t(n)\, e(n\alpha)\, w(n) \ll_\delta 1.$$

3. If $\Delta^{-1+\delta} \ll \eta \ll M\, \Delta^{-2}$, $k^2\, \eta^2\, M \gg 1$ and $k^2\, \eta\, M\, \Delta^{-1+2\delta} \ll 1$, then

$$\sum_{M \leqslant n \leqslant M+\Delta} t(n)\, e(n\alpha)\, w(n) \ll_\delta 1 + k^{-1/2}\, \Delta\, M^{-1/4} \left(k^2\, \eta^2\, M\right)^{\vartheta - 1/4 + \varepsilon}.$$

4. If $\Delta^{-1+\delta} \ll \eta \ll M\, \Delta^{-2}$, $k^2\, \eta^2\, M \gg 1$ and $k^2\, \eta\, M\, \Delta^{-1+2\delta} \gg 1$, then

$$\sum_{M \leqslant n \leqslant M+\Delta} t(n)\, e(n\alpha)\, w(n) \ll_\delta \left(k^2\, \eta^2\, M\right)^{\vartheta} \Delta^{1/6}\, M^{1/3+\varepsilon}.$$

Before sketching the proof of Theorem 31, we explain how to finish the proof of Theorem 30 by using it. We assume that $\Delta \gg M^{2/(5+6\vartheta)}$ as otherwise the theorem follows by estimating trivially. Let $\alpha = h/k + \eta$ be any Farey approximation of α of order $\Delta^{1/2-\delta}$. As $\Delta^{-1/2+\delta} \ll M\Delta^{-2}$, we have $|\eta| \ll M/\Delta^2$. Therefore, applying Theorem 31 yields

$$\sum_{n \in \mathbb{Z}} t(n)e(n\alpha)w_\ell(n) \ll \left(\frac{\Delta}{4^{|\ell|}}\right)^{1/6-\vartheta} M^{1/3+\vartheta+\varepsilon}$$

uniformly on ℓ. Thus

$$\sum_{\ell=-L}^{L} \sum_{n \in \mathbb{Z}} t(n)e(n\alpha)w_\ell(n) \ll \Delta^{1/6-\vartheta} M^{1/3+\vartheta+\varepsilon}.$$

The last thing to observe is that the second summand on the right-hand side of (16) is $\ll M^{2/(5+6\vartheta)} M^{\vartheta+\varepsilon} \ll \Delta^{1/6-\vartheta} M^{1/3+\vartheta+\varepsilon}$ by using the assumption $\Delta 4^{-L} \asymp M^{2/(5+6\vartheta)}$ and estimating by absolute values.

Let us now explain how to prove Theorem 31. Writing $\alpha = h/k + \eta$ produces a rational exponential twist and we can use Voronoi summation formula to obtain

$$
\sum_{M \leq n \leq M + \Delta} t(n) e(n\alpha) w(n)
$$

$$
= \frac{C}{k} \sum_{n=1}^{\infty} t(n) e\left(-\frac{n\overline{h}}{k}\right) \int_{M}^{M+\Delta} \frac{k^{1/2}}{n^{1/4} x^{1/4}}
$$

$$
\cdot \sum_{\pm} (\pm 1) e\left(\pm \frac{2\sqrt{nx}}{k} \mp \frac{1}{8}\right) g_{\pm}(x; n, k) e(\eta x) w(x)\, \mathrm{d}x + O(1),
$$

where C is a real constant and

$$
g_{\pm}(x; n, k) := 1 + \sum_{\ell=1}^{K} c_{\ell}^{\pm} k^{\ell} n^{-\ell/2} x^{-\ell/2},
$$

for some constants $c_{\ell}^{\pm}$. Now we split into two cases depending on whether $\eta \ll \Delta^{-1+\delta}$ or $\eta \gg \Delta^{-1+\delta}$. In both cases we further split the sum over n into low-frequency terms $n \leq X$ and high-frequency terms $n \geq X$ for some suitable X.

Suppose first that $\eta \ll \Delta^{-1+\delta}$. In this case we choose $X = k^2 M \Delta^{3\delta-2}$. When $n > X$ we repeatedly integrate by parts to see that the contribution coming from these values of n is $\ll 1$. For low-frequency terms we localise the terms in the summation range by making a dyadic split, i.e. we write the low-frequency contribution as

$$
k^{-1/2} \sum_{\substack{L \leq X/2 \\ \text{dyadic}}} \int_{M}^{M+\Delta} x^{-1/4} w(x)
$$

$$
\cdot \sum_{L \leq n \leq 2L} t(n) n^{-1/4} g_{\pm}(x; n, k) e\left(\pm \frac{2\sqrt{nx}}{k} - \frac{n\overline{h}}{k} + \eta x\right)\, \mathrm{d}x.
$$

The sum inside the integral can be conveniently estimated by a result of Karppinen [64] for non-linear sums (see also [63]) to be $\ll L^{5/12} M^{1/6+\varepsilon/2} k^{-1/3}$ and therefore the total contribution of low-frequency terms is $\ll \Delta^{1/6} M^{1/3+\varepsilon/2+5\delta/4}$ by trivial estimation, finishing the first case.

In the complementary case $\eta \gg \Delta^{-1+\delta}$, we again split the range of summation over n into low-frequency terms $n \leq X$ and high-frequency terms $n > X$, but this time with $X = k^2 \eta^2 M$. With this choice, the high-frequency terms contribute again $\ll 1$ by repeatedly integrating by parts as before. The contribution of the low-frequency terms is easily seen to be $\ll 1$ if $X \ll 1$, settling case 2. If this is not the case, we partition the range $n \leq 2X$ into two sets: those with $|n - X| \geq W$ and those with $|n - X| < W$, where $W = k^2 M \eta \Delta^{-1+2\delta}$. The idea is that the assumption $|n - X| \geq W$ assures that

$$
\frac{\mathrm{d}}{\mathrm{d}x}\left(\pm \frac{2\sqrt{nx}}{k} + \eta x\right) \gg \Delta^{-1+2\delta},
$$

and hence the terms in the first set can be seen to contribute $\ll 1$ again by using integration by parts. If $W \ll 1$, the remaining terms contribute

$\ll k^{-1/2}(k^2\eta^2 M)^{\vartheta-1/4+\varepsilon}\Delta M^{-1/4}$ which is obtained by estimating by absolute values, finishing the case 3. If $W \gg 1$ and $\eta \gg \Delta^{-1+\delta}$ we use an analogue of the non-linear estimate of Ernvall-Hytönen and Karppinen, see [26, Theorem 5.5] or [C,Theorem 16] establishing the case 4.

As an application we can use the above theorem to reduce the smoothing error, thus obtaining the following estimates for the rationally additively twisted sums. Previously similar bounds have been obtained for the plain sums of coefficients (i.e. in the case $h = 0$ and $k = 1$) by Hafner and Ivić [37] and Lü [81].

Theorem 32. *(Theorem 4 in [C]) Let $M \in [1, \infty[$, $h \in \mathbb{Z}$, $k \in \mathbb{Z}_+$ and $(h, k) = 1$. Also, let $\delta \in]0, 1/2[$ and assume that $k \ll M^{1/2-\delta}$. Then*

$$\sum_{n \leqslant M} t(n)\, e\left(\frac{nh}{k}\right) \ll_\delta k^{2/3}\, M^{1/3+\vartheta/3+\varepsilon}.$$

When $M^{3/(5+6\vartheta)-1/2+\vartheta} \ll k \ll M^{5/18+\vartheta/3}$, we have

$$\sum_{n \leqslant M} t(n)\, e\left(\frac{nh}{k}\right) \ll k^{(1-6\vartheta)/(4-6\vartheta)}\, M^{3/(8-12\vartheta)+\varepsilon}.$$

Similarly, for $M^{5/18+\vartheta/3} \ll k \ll M^{1/2-\delta}$, we have

$$\sum_{n \leqslant M} t(n)\, e\left(\frac{nh}{k}\right) \ll_\delta k^{2/3}\, M^{7/27+\vartheta/9+\varepsilon}.$$

Let us elaborate a bit. We choose a smooth weight function supported in $[M, M+\Delta]$, being identically one in $[M+U, M+\Delta-U]$ for some optimally chosen $U < M/2$, and satisfying $w(x)^\nu \ll U^{-\nu}$ for $\nu \in \{0, 1, 2\}$. The starting point is the observation that estimating by absolute values and using the pointwise approximation towards the Ramanujan-Petersson conjecture, we have

$$\sum_{M \leq n \leq M+\Delta} t(n)e\left(\frac{nh}{k}\right)$$

$$= \sum_{M \leq n \leq M+\Delta} t(n)e\left(\frac{nh}{k}\right)(1-w(n)) + \sum_{M \leq n \leq M+\Delta} t(n)e\left(\frac{nh}{k}\right)w(n)$$

$$\ll UM^{\vartheta+\varepsilon} + \sum_{M \leq n \leq M+\Delta} t(n)e\left(\frac{nh}{k}\right)w(n).$$

For the smoothed sum we can prove the bound

$$\sum_{M \leq n \leq M+\Delta} t(n)e\left(\frac{nh}{k}\right)w(n) \ll k^{1/2}X^{1/4}M^{1/4} + k^{3/2}X^{-1/4}M^{3/4}U^{-1} \quad (17)$$

for any $X \gg 1$. This follows by standard arguments. We apply the relevant Voronoi summation formula which gives rise to exponential integrals of the form

$$\int_M^{M+\Delta} w(x)k^{1/2}n^{-1/4}x^{-1/4}\left(1 + ckn^{-1/2}x^{-1/2}\right)e\left(\pm\frac{2\sqrt{nx}}{k}\right)\,\mathrm{d}x$$

for some constant c. Integrating by parts twice gives an upper bound

$$\ll k^{5/2} n^{-5/4} M^{3/4} U^{-1}.$$

Notice that this is an effect of the special weight function. If we had the assumption $\ll \Delta^{-\nu}$ for the derivatives of w, we would get a weaker bound $\ll k^{5/2} n^{-5/4} M^{3/4} U^{-2} \Delta$. By using this the high-frequency terms $n > X$ contribute the second term. Low-frequency terms are estimated by using the first derivative test giving the first term on the right-hand side of (17). Now the theorem follows by choosing U and X appropriately in different ranges of k.

Let us then discuss on long sums. In [118] Wilton proved the bound

$$\sum_{n \le M} a(n) e(n\alpha) \ll M^{1/2} \log M$$

uniformly in $\alpha \in \mathbb{R}$, for long exponential sums involving Fourier coefficients of holomorphic cusp forms. The logarithm was later removed by Jutila [62].

An analogous result for Wilton's upper bound for classical Maass cusp forms is a theorem of Epstein, Hafner and Sarnak [19, 36]. By the Rankin-Selberg result (11) only the logarithm can be removed, and in the spirit of Jutila's result for holomorphic cusp forms we prove that this is possible.

Theorem 33. *(Theorem 7 in [C]) We have*

$$\sum_{n \le M} t(n) e(n\alpha) \ll M^{1/2},$$

uniformly in $\alpha \in \mathbb{R}$.

A key intermediate result for achieving this is the following approximate functional equation which is of independent interest. Write

$$T(M, \Delta; \alpha) := \sum_{M \le n \le M+\Delta} t(n) e(n\alpha).$$

Theorem 34. *(Theorem 6 in [C]) Let $\alpha \in \mathbb{R}$ have the rational approximation $\alpha = \frac{h}{k} + \eta$, where h and k are coprime integers with $1 \le k \le M^{1/4}$ and $|\eta| \le k^{-1} M^{-1/4}$. Furthermore, let $M \in [1, \infty[$ and $\Delta \in [1, M]$. If $k^2 \eta^2 M \gg 1$, then*

$$\frac{T(M, \Delta; \alpha)}{M^{1/2}} = \frac{T(k^2 \eta^2 M, k^2 \eta^2 \Delta; \beta)}{(k^2 \eta^2 M)^{1/2}} + O\big((k^2 \eta^2 M)^{\vartheta/2 - 1/12 + \varepsilon}\big),$$

where $\beta := -\dfrac{\overline{h}}{k} - \dfrac{1}{k^2 \eta}$.

The first approximate functional equation for linear exponential sums involving the divisor function is due to Wilton [119]. Later this was extended to cover additive rational twists of the divisor function by Jutila [60] and then generalised for the Fourier coefficients of holomorphic cusp forms also by Jutila [62] with an unspecified constant a in the error term in place of $-1/12$. Ernvall-Hytönen showed that one can take a to be $-1/12$ [20]. Let us first discuss the proof of Theorem 34 and then the removal of the logarithm. First, we need some notation. Let $M_{-1} = M - JU$, $M_1 = M + \Delta$ and $M_2 = M + \Delta + JU$, where

$J \in \mathbb{Z}_+$ is a large integer and $U = M^{1/2}\eta^{-1/2}(k^2\eta^2 M)^d$. Let $\eta_J(\cdot)$ be a special smooth weight function defined by the convolution

$$\eta_J := \underbrace{\frac{1}{U}\chi_{[0,U]} * \cdots * \frac{1}{U}\chi_{[0,U]}}_{J\ \text{times}} *\chi_{[M_1,M_2-JU]}.$$

Notice that then η_J is identically one in $[M, M+\Delta]$. We start by writing

$$\sum_{M \le n \le M+\Delta} t(n)e(n\alpha) = \sum_{M_{-1} \le n \le M_2} t(n)e(n\alpha)\eta_J(n)$$
$$- \sum_{M_{-1} < n \le M} t(n)e(n\alpha)\eta_J(n) - \sum_{M_1 < n \le M_2} t(n)e(n\alpha)\eta_J(n). \tag{18}$$

The last two terms on the right-hand side contribute

$$\sum_{M_{-1} \le n < M} t(n)\, e(\alpha n)\, \eta_J(n) + \sum_{M_1 < n \le M_2} t(n)\, e(\alpha n)\, \eta_J(n)$$
$$\ll M^{1/2} \left(k^2\, \eta^2\, M\right)^{\vartheta/2 - 1/12 + \varepsilon} \tag{19}$$

by partial summation assuming an upper bound for short sums of certain length (Lemma 22 in [C]) which can be proved using techniques similar to those used in the proof of Theorem 30. Let us proceed to the first term on the right-hand side of (18). Voronoi summation formula and standard asymptotics of Bessel-functions lead to

$$\sum_{M_{-1} \le n \le M_2} t(n)e(n\alpha)\eta_J(n) = O(1) + \frac{C'}{k}\sum_{n=1}^{\infty} t(n)e\left(-\frac{n\overline{h}}{k}\right)\int_{M_{-1}}^{M_2}\frac{k^{1/2}}{n^{1/4}x^{1/4}}$$
$$\cdot \sum_{\pm}(\pm 1)e\left(\pm\frac{2\sqrt{nx}}{k} \mp \frac{1}{8}\right)g_{\pm}(x;n,k)e(\eta x)\eta_J(x)\,\mathrm{d}x,$$

where

$$g_{\pm}(x;n,k) = 1 + \sum_{\ell=1}^{K} c_\ell^{\pm} k^\ell n^{-\ell/2} x^{-\ell/2}$$

as before.

The terms involving $g_+(x;n,k)$ contribute $\ll 1$ by repeated integration by parts. The main terms come from certain integrals involving $g_-(x;n,k)$.

Let c be a positive constant so that the estimate

$$x\eta - \frac{2\sqrt{nx}}{k} \gg \frac{\sqrt{nM}}{k}$$

holds when $n > cN$, where $N = k^2\,\eta^2\,M$ (and more generally $N_i = k^2\,\eta^2 M_i$ for $i \in \{-1,1,2\}$). A direct application of partial integration gives

$$\sum_{n>cN} t(n)\, n^{-1/4}\int_{M_{-1}}^{M_2} k^{-1/2}\, x^{-1/4}\, e\left(x\eta - \frac{2\sqrt{nx}}{k}\right) g_-(x;n,k)\, \eta_J(x)\,\mathrm{d}x \ll 1.$$

For the terms with $n \leqslant cN$ we split $g_-(x; n, k)$ into two parts, 1 and $g_-(x; n, k) - 1$, and estimate corresponding terms differently.

For the second term, using the second derivative test, we get

$$\int_{M_{-1}}^{M_2} x^{-1/4} e\left(x\eta - \frac{2\sqrt{nx}}{k}\right) (g_-(x; n, k) - 1)\, \eta_J(x)\, \mathrm{d}x \ll \frac{k^{3/2}}{n^{3/4}}.$$

Therefore we have

$$\sum_{n \leqslant cN} t(n)\, n^{-1/4} \int_{M_{-1}}^{M_2} k^{-1/2}\, x^{-1/4}\, e\left(x\eta - \frac{2\sqrt{nx}}{k}\right) (g_-(x; n, k) - 1)\, \eta_J(x)\, \mathrm{d}x$$

$$\ll k \sum_{n \leqslant cN} \frac{|t(n)|}{n} \ll k\, N^\varepsilon = k\, (k^2\, \eta^2\, M)^\varepsilon.$$

The remaining terms are treated using the first saddle point lemma [61, Theorem 2.2.]. Here we use the properties of the special weight function η_J. For $1 \leqslant n < cN$, we get

$$\int_{M_{-1}}^{M_2} e\left(x\eta - \frac{2\sqrt{nx}}{k}\right) \eta_J(x)\, x^{-1/4}\, \mathrm{d}x = \xi(n) \cdot \frac{\sqrt{2}\, n^{1/4}}{\sqrt{k}\, \eta}\, e\left(-\frac{n}{k^2\eta} + \frac{1}{8}\right) + \text{error},$$

where ξ is a certain weight function satisfying some specific properties. Especially, $\xi(\cdot) \equiv 1$ on $[N, N_1]$, ξ' is piecewise continuously differentiable and we have $\xi'(\cdot) \ll (k^2\, \eta^2\, U)^{-1}$ where the derivative exists. Estimation of the error terms in the saddle-point theorem is easily done by using partial summation.

The main term on the right-hand side produces the total contribution

$$\frac{1}{k\eta} \sum_{N \leqslant n \leqslant N_1} t(n)\, e\left(-\frac{n\overline{h}}{k} - \frac{n}{k^2\eta}\right) + \frac{1}{k\eta} \sum_{N_{-1} \leqslant n < N} t(n)\, \xi(n)\, e\left(-\frac{n\overline{h}}{k} - \frac{n}{k^2\eta}\right)$$

$$+ \frac{1}{k\eta} \sum_{N_1 < n \leqslant N_2} t(n)\, \xi(n)\, e\left(-\frac{n\overline{h}}{k} - \frac{n}{k^2\eta}\right). \tag{20}$$

By using Theorem 30, we get that that the last two summands in (20) contribute

$$\frac{1}{k\eta} \left(\sum_{N_{-1} \leqslant n < N} + \sum_{N_1 < n \leqslant N_2} \right) t(n)\, \xi(n)\, e\left(-\frac{n\overline{h}}{k} - \frac{n}{k^2\eta}\right)$$

$$\ll M^{1/2}\, (k^2\, \eta^2\, M)^{\vartheta/2 - 1/12 + d/6 - d\vartheta + \varepsilon}.$$

Hence, we have

$$\sum_{M \leq n \leq M+\Delta} t(n)e(n\alpha) = \frac{1}{k\eta} \sum_{N \leq n \leq N_1} t(n)e(n\beta) + O(M^{1/2}(k^2\eta^2 M)^{1/2 - Jd + \vartheta + \varepsilon})$$

$$+ O(M^{1/2}(k^2\eta^2 M)^{\varepsilon + d - 1/12 + \vartheta/2}) + O(k(k^2\eta^2 M)^\varepsilon).$$

This proves the approximate functional equation by choosing J to be sufficiently large depending on d, and letting $d \in \mathbb{R}_+$ to be arbitrarily small.

Regarding the proof of the logarithm removal, the first step is to consider the case where α is close to a rational point with a small denominator. The key is the following lemma.

Lemma 35. *(Lemma 24 in [C]) Assume $\alpha = h/k + \eta$ with $(h, k) = 1$, $1 \leq k \leq M^{1/4}$, $|\eta| \leq k^{-1} M^{-1/4}$ and $k^2 \eta^2 M < 1/2$. Then*

$$\sum_{M \leq n \leq 2M} t(n) e(n\alpha) \ll k^{(1-6\vartheta)/(4-6\vartheta)} M^{3/(8-12\vartheta)+\varepsilon}.$$

This is proved easily by using partial summation, Theorem 32, facts about the asymptotic behaviour of the J-Bessel function and the first derivative test. For $k^2 \eta^2 M \gg 1$, one applies the approximate functional equation and gets

$$\frac{1}{M^{1/2}} \sum_{M \leq n \leq 2M} t(n) e(n\alpha) = \frac{1}{(k^2 \eta^2 M)^{1/2}} \sum_{k^2 \eta^2 M \leq n \leq 2k^2 \eta^2 M} t(n) e(n\beta)$$
$$+ O((k^2 \eta^2 M)^{\vartheta/2 - 1/12 + \varepsilon}).$$

Write $\beta = \frac{h_1}{k_1} + \eta_1$ with $(h_1, k_1) = 1$, $1 \leq k_1 \leq (k^2 \eta^2 M)^{1/4}$ and $|\eta_1| \leq k_1^{-1} (k^2 \eta^2 M)^{-1/4}$. If $k_1^2 \eta_1^2 (k^2 \eta^2 M) < 1/2$, then the right-hand side is $\ll 1$ by Lemma 35. Otherwise apply the approximate functional equation until we are in the situation to apply the lemma or until the sums become shorter than a constant. Notice that the length of the new sum is at most a square root of the previous one. This proves the theorem.

Another application of the approximate functional equation is the following non-trivial upper bound for slightly longer short sums than in Theorem 30 above.

Theorem 36. *(Theorem 8 in [C]) Let $M \in [1, \infty[$ and $\Delta \in [1, M]$ with $M^{2/3} \ll \Delta \ll M^{3/4}$, and let $\alpha \in \mathbb{R}$. Then*

$$\sum_{M \leq n \leq M + \Delta} t(n) \, e(n\alpha) \ll M^{3/8 + (3+12\vartheta)/(32+48\vartheta) + \varepsilon} + \Delta \, M^{-1/4 + 3\vartheta/(32+48\vartheta) + \varepsilon}.$$

8 Short sums involving Fourier coefficients of Hecke-Maass cusp forms for $\mathrm{SL}(n, \mathbb{Z})$

In Article [D] we derive an asymptotic formula for the mean square of a sum of Fourier coefficients of Hecke-Maass cusp forms over certain short intervals in the general $\mathrm{GL}(n)$-situation assuming the generalised Lindelöf hypothesis for the L-function attached to the underlying Maass cusp form in the t-aspect and a slightly improved upper bound concerning the exponent towards the Ramanujan-Petersson conjecture. Previously, an analogous result has been established by Jutila [59] for sums of Fourier coefficients of holomorphic cusp forms unconditionally and for the error term in the Dirichlet divisor problem for the k-fold divisor function by Lester [75] under the assumption of the Lindelöf hypothesis. We will follow Lester's strategy which differs from the one by Jutila. The assumption for the underlying Maass cusp form f being a Hecke form is needed in order to derive asymptotics for the sum $\sum_{m \leq x} |A(m, 1, ..., 1)|^2$.

Theorem 37. *(Theorem 1 in [D]) Let f be a Hecke-Maass cusp form for $\mathrm{SL}(n,\mathbb{Z})$ normalised so that $A(1,...,1) = 1$. Assume the generalised Lindelöf hypothesis for $L(s,f)$ in the t-aspect and that the exponent towards the Ramanujan-Petersson conjecture satisfies $\vartheta < 1/2 - 1/n$. Furthermore, suppose that $2 \leq L \ll X^{1/(n(n-1))-\varepsilon}$ for some small fixed $\varepsilon > 0$. Then we have*

$$\frac{1}{X} \int_X^{2X} \left| \sum_{x \leq m \leq x + x^{1-1/n}/L} A(m,1,...,1) \right|^2 \mathrm{d}x \sim C_f \cdot \frac{X^{1-1/n}}{L},$$

where

$$C_f := \frac{\left(2^{1-1/n} - 1\right)}{(2n-1)} \cdot r_f \cdot H_f(1).$$

Here r_f is the residue of the Rankin-Selberg L-function, $L(s, f \times \widetilde{f})$ attached to f, at $s = 1$. It is given by

$$r_f = \frac{4\pi^{n^2/2}}{n\Gamma(f)} \|f\|^2.$$

For the proof of this, see Appendix A in [71]. Furthermore,

$$\Gamma(f) := \prod_{1 \leq j \leq n} \Gamma\left(\frac{1 + 2\Re(\lambda_j(f))}{2}\right) \prod_{1 \leq j < k \leq n} \left| \Gamma\left(\frac{1 + \lambda_j(f) + \overline{\lambda_k(f)}}{2}\right) \right|^2,$$

where $\lambda_j(f)$, $j = 1,...,n$, are the Langlands parameters of the form f. Finally, $H_f(1)$ is given by

$$H_f(1) := \prod_p P_n(\alpha_p(f), \overline{\alpha_p(\widetilde{f})}, p^{-1}),$$

where $P_n(\cdot, \cdot, \cdot)$ is a certain polynomial of the Satake parameters of f and its dual form $\widetilde{f}$.

The idea behind the proof of Theorem 37 is to approximate the mean square of the sum

$$\sum_{x \leq m \leq x + x^{1-1/n}/L} A(m,1,...,1)$$

by the mean square of the expression

$$P\left(x + \frac{x^{1-1/n}}{L}; \theta\right) - P(x; \theta),$$

where

$$P(x; \theta) := \frac{x^{1/2-1/2n}}{\pi\sqrt{n}} \sum_{m \leq X^\theta} \frac{A(1,...,1,m)}{m^{1/2+1/2n}} \cos\left(2\pi n \sqrt[n]{mx} + \frac{n-3}{4}\pi\right)$$

for some $0 < \theta \leq 1$.

Notice that this is the main term coming from the truncated Voronoi summation for the sum of the Fourier coefficients. Unfortunately the pointwise bounds for the error term are too large. However, we can prove that on average this error is sufficiently small assuming the generalised Lindelöf hypothesis.

The key observation is to use the elementary identity

$$S^2 = Q^2 + (S-Q)^2 + 2Q(S-Q)$$

to see that

$$\frac{1}{X}\int_X^{2X}\left|\sum_{x \le m \le x + x^{1-1/n}/L} A(m,1,...,1)\right|^2 \mathrm{d}x \tag{21}$$

$$= \frac{1}{X}\int_X^{2X}\left|P\left(x + \frac{x^{1-1/n}}{L};\theta\right) - P(x,\theta)\right|^2 \mathrm{d}x$$

$$+ \frac{1}{X}\int_X^{2X}\left|\sum_{x \le m \le x + x^{1-1/n}/L} A(m,1,...,1) - \left(P\left(x + \frac{x^{1-1/n}}{L};\theta\right) - P(x,\theta)\right)\right|^2 \mathrm{d}x$$

$$+ \text{ error.} \tag{22}$$

To treat the first term on the right-hand side, we write

$$P\left(x + \frac{x^{1-1/n}}{L};\theta\right) - P(x;\theta)$$

$$= \underbrace{P\left(x + \frac{x^{1-1/n}}{L};\theta\right) - P\left(\left(\sqrt[n]{x} + \frac{1}{nL}\right)^n;\theta\right)}_{=:I_1(x,L;\theta)} + \underbrace{P\left(\left(\sqrt[n]{x} + \frac{1}{nL}\right)^n;\theta\right) - P(x;\theta)}_{=:I_2(x,L;\theta)}.$$

The idea here is that $I_2(x,L;\theta)$ is easier to handle than the original difference and intuitively $I_1(x,L;\theta)$ should be small on average, which turns out to be the case when $0 < \theta < 1/(n-1+2n\vartheta)$ and $\vartheta < 1/2 - 1/n$. By using standard manipulations, one sees that

$$\frac{1}{X}\int_X^{2X}|I_2(x,L;\theta)|^2 \mathrm{d}x \sim \frac{X^{1-1/n}}{L} \cdot C_f.$$

The other terms give a smaller contribution by using the first derivative test and the Cauchy-Schwarz inequality.

To prove that

$$\frac{1}{X}\int_X^{2X}\left|\sum_{x \le m \le x + x^{1-1/n}/L} A(m,1,...,1) - \left(P\left(x + \frac{x^{1-1/n}}{L};\theta\right) - P(x,\theta)\right)\right|^2 \mathrm{d}x$$

is smaller than the mean square of $I_2(x,L;\theta)$ we proceed as follows. First we simply estimate it to be

$$\ll \frac{1}{X}\int_X^{2X}\left|\sum_{m \le x + x^{1-1/n}/L} A(m,1,...,1) - P\left(x + \frac{x^{1-1/n}}{L};\theta\right)\right|^2 \mathrm{d}x$$

$$+ \frac{1}{X}\int_X^{2X}\left|\sum_{m \le x} A(m,1,...,1) - P(x;\theta)\right|^2 \mathrm{d}x.$$

As the treatment of these terms is identical, we focus on the latter one. The second step is to express the sum of Fourier coefficients as an integral by using the truncated Perron's formula:

$$\sum_{m \le x} A(m, 1, ..., 1) = \frac{1}{2\pi i} \int_{1+\delta-iX}^{1+\delta+iX} L(s, f) x^s \frac{\mathrm{d}s}{s} + O\left(X^{1/2-1/n+\varepsilon}\right),$$

uniformly for $X \le x \le 2X$, for some $\delta > 0$, where $L(s, f)$ is the L-function attached to the underlying Maass cusp form f. Here we have also used the assumption $\vartheta < 1/2 - 1/n$. Then we shift the line segment of integration from $\sigma = 1 + \delta$ to $\sigma = 1/2$ with an admissible error (under the assumption of the generalised Lindelöf hypothesis). On the line $\sigma = 1/2$ we split the line segment of integration into three parts

$$\int_{1/2-iX}^{1/2+iX} L(s, f) x^s \frac{\mathrm{d}s}{s} = \int_{1/2-iY}^{1/2+iY} L(s, f) x^s \frac{\mathrm{d}s}{s} + \frac{1}{2\pi i} \int_{1/2-iX}^{1/2-iY} L(s, f) x^s \frac{\mathrm{d}s}{s}$$
$$+ \frac{1}{2\pi i} \int_{1/2+iY}^{1/2+iX} L(s, f) x^s \frac{\mathrm{d}s}{s},$$

where $Y \asymp X^{(1+\theta)/n}$.

Shifting the line segment of integration to the line $\sigma = -\delta$ in the first term on the right-hand side and evaluating the integral over the line segment $[-\delta - iY, -\delta + iY]$ using techniques similar to those in Section 5 shows that

$$\frac{1}{2\pi i} \int_{1/2-iY}^{1/2+iY} L(s, f) x^s \frac{\mathrm{d}s}{s} = P(x; \theta) + O\left(X^{1/2-(1+\theta)/2n+\varepsilon}\right)$$

for $0 < \theta < 1/(n - 1 + 2n\vartheta)$.

Thus,

$$\sum_{m \le x} A(m, 1, ..., 1) - P(x; \theta)$$
$$= \int_{1/2-iX}^{1/2-iY} L(s, f) x^s \frac{\mathrm{d}s}{s} + \int_{1/2+iY}^{1/2+iX} L(s, f) x^s \frac{\mathrm{d}s}{s} \, \mathrm{d}x + O\left(X^{1/2-(1+\theta)/2n+\varepsilon}\right),$$

from which we get

$$\frac{1}{X} \int_X^{2X} \left| \sum_{m \le x} A(m, 1, ..., 1) - P(x; \theta) \right|^2 \mathrm{d}x$$
$$\ll \frac{1}{X} \int_X^{2X} \left| \int_{1/2-iX}^{1/2-iY} L(s, f) x^s \frac{\mathrm{d}s}{s} + \int_{1/2+iY}^{1/2+iX} L(s, f) x^s \frac{\mathrm{d}s}{s} \right|^2 \mathrm{d}x$$
$$+ X^{1-(1+\theta)/n+\varepsilon}$$

for $0 < \theta < 1/(n - 1 + 2n\vartheta)$. One can show that under the assumption of the generalised Lindelöf hypothesis the first term on the right-hand side is $\ll X^{1-(1+\theta)/n+\varepsilon}$, which finishes the proof. Using similar techniques we can also prove the following theorem.

Theorem 38. *(Theorem 3 in [D]) Let f be a Hecke-Maass cusp form for* $\mathrm{SL}(n, \mathbb{Z})$ *normalised so that $A(1, ..., 1) = 1$. Suppose that $X^{1-1/n+\varepsilon} \ll \Delta \ll X^{1-\varepsilon}$ for some small fixed $\varepsilon > 0$ and that the generalised Lindelöf hypothesis for $L(s, f)$ holds in the t-aspect. Suppose also that the exponent towards the Ramanujan-Petersson conjecture satisfies $\vartheta < 1/2 - 1/n$. Then we have*

$$\frac{1}{X} \int_X^{2X} \left| \sum_{x \leq m \leq x+\Delta} A(m, 1, ..., 1) \right|^2 \, \mathrm{d}x \sim B_f \cdot X^{1-1/n},$$

where

$$B_f := \frac{1}{\pi^2} \cdot \frac{2^{2-1/n} - 1}{2n - 1} \sum_{m=1}^{\infty} \frac{|A(m, 1, ..., 1)|^2}{m^{1+1/n}}.$$

The fact that B_f is finite follows from (9) and partial summation. This is a partial generalisation to the results of Ivić [47], Jutila [59], and Vesalainen [112] in the higher rank setting and an analogue of Lester's result [75] for cusp forms.

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