Ω-Results for Exponential Sums Related to Maass Forms for $\mathrm{SL}(3, \mathbb{Z})$

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The article on which this summary is based on is in the final stage of preparation and will appear here shortly. This is joint work with Esa V. Vesalainen

• It is natural to consider exponential sums attached to cusp forms:

$$\sum_{x\le m\le x+\Delta} A(m,1)e(m\alpha), \tag{1}$$

where $\alpha \in \mathbb{R}$, $\Delta \ll x$, and $A(m,1)$ are Fourier coefficients of a Maass forms for the group $\mathrm{SL}(3,\mathbb{Z})$.

• One reason for this is that the exponential sums above contain all the information about the Fourier coefficients and thus provide a natural way to examine their behaviour.

• The study of Maass forms and their L-functions also quickly leads to exponential sums weighted by the corresponding Fourier coefficients of which the above linear sums are an important special case.

• Good estimates for short sums also provide a practical tool for reducing smoothing error thereby potentially leading to better results in questions from which such exponential sums arise.

• Furthermore, when Δ is small compared to x, the resulting short sums provide a natural analogue for the classical problems of analytic number theory studying various error terms, e.g. in the Dirichlet divisor problem or the Gauss circle problem, in short intervals.

• The relation between sums of the form (1) and classical number theoretic error terms is particularly clear in the case where the twist α is close to a reduced fraction h/k with a small denominator k. Furthermore, in this case the behaviour of such exponential sum is closely related to the behaviour of the sum at the fraction and hence it makes sense to study rationally additively twisted sums of Fourier coefficients.

• We are especially interested in sizes of such sums. As an example, for general twists the best known upper bound for long sums (i.e. the case where $\Delta \asymp x$) is $\ll_\varepsilon x^{3/4+\varepsilon}$ uniformly in $\alpha \in \mathbb{R}$ due to Miller.

• The moment estimates in the $\mathrm{GL}(2)$-situation, the square-root-cancellation heuristics, and the shape of Voronoi identities for rationally additively twisted sums related to $\mathrm{SL}(3, \mathbb{Z})$ Maass forms give rise to the conjecture that

$$\sum_{x \leqslant m \leqslant x+\Delta} A(m,1) e\left(\frac{mh}{k}\right) \ll_\varepsilon \min\left(\Delta^{1/2} x^\varepsilon, k^{1/2} x^{1/3}\right) \tag{2}$$

for every $\varepsilon > 0$.

Our first result is for the long sums.

Theorem (J.- Vesalainen, 2019)

Let $x \in [1, \infty[$, and let k prime so that $3 \leqslant k \ll x^{1/3-\delta}$ for some arbitrarily small fixed $\delta \in \mathbb{R}_+$. Then

$$\max_{h \in \mathbb{Z}_k^\times} \left| \sum_{m \leqslant x} A(m,1)\, e\left(\frac{mh}{k}\right) \right| = \Omega_\delta\left(k^{1/2} x^{1/3}\right),$$

where the maximum is taken over all reduced residue classes modulo k.

- Notice that $k^{1/2}x^{1/3}$ is the conjectured upper bound in (2) in the range $1 \leqslant k \ll x^{1/3}$. Thus the result is essentially optimal in the sense of conjecture (2).

For short sums we can prove the following result.

Theorem (J. - Vesalainen, 2019)

Let $x \in [1, \infty[$, let $k \geq 5$ prime, and suppose that $k^{3/2}x^{1/2+\delta} \ll \Delta \ll kx^{2/3}$ for some arbitrarily small fixed $\delta \in \mathbb{R}_+$. Then

$$\max_{h \in \mathbb{Z}_k^\times} \left| \sum_{x \leqslant m \leqslant x+\Delta} A(m,1)\, e\left(\frac{mh}{k}\right) \right| = \Omega_\delta \left(\Delta x^{-1/3} k^{-1/2}\right),$$

where the maximum is taken over all reduced residue classes modulo k.

• Notice that the result of this theorem is optimal in sense of the conjectural bound (2), up to x^ε, when $\Delta \asymp kx^{2/3}$. However, the smaller Δ is, this Ω-result is farther away from the expected upper bound.

Results of this type can be deduced from average results (in our case mean-square results) for the related exponential sums. These averaged results are in turn based on Voronoi-type identities for rationally additively twisted sums of Fourier coefficients. However, a plain Voronoi identity would be hard to work with. For example, the truncated Voronoi identity for $\mathrm{GL}(3)$ Maass forms has an error term which is actually larger than the expected best upper bound for the corresponding sum. To remedy this, we shall consider Riesz means given by

$$\widetilde{A}_a\left(x; \frac{h}{k}\right) = \frac{1}{a!} \sum_{m \leqslant x} A(m,1)\, e\left(\frac{mh}{k}\right) (x-m)^a - \text{explicit main term}$$

where $a \geqslant 0$ is an integer, which a classical form of smoothing. For large values of a, the Voronoi-type identities for these sums are better behaved and offer a firm grip on the Riesz means. Results on the Riesz means can then be used to prove results for plain exponential sums.

• The reason we restrict to the case of prime denominator in theorems above is so that we can handle efficiently the sum over the divisors of k arising from Voronoi-type identities for Riesz-weighted exponential sums. Actually theorems above also hold in the case $k = 1$, which can be easily seen by making obvious modifications to the proofs. Since we are interested in Ω-results this is not such a serious restriction.
• Likewise, our Ω-results do not hold for any h coprime to k. The maximum over the reduced residue classes modulo k has to be included in order to prove upper and lower bounds of the same order of magnitude for the mean-squares of Riesz-weighted sums $\widetilde{A}_a(x; h/k)$. This cannot be achieved unless we average over numerators coprime to k. The reason is that it allows us to handle correlations of Kloosterman sums arising from the Voronoi identity for $\widetilde{A}_a(x; h/k)$. Again, the presence of this is not so serious as we are interested in limitations for proving bounds (2), which are supposed to be uniform in h.

Identities of Voronoi-type will play a key role in the proofs of theorems above. By standard methods we can stablish the following identity for the Riesz weighted sums:

Proposition

Let $x \in [1, \infty[$, let $a \geqslant 2$ be an integer, and let h and k be coprime integers with $1 \leqslant k \ll x^{1/3}$. Suppose furthermore that k is a prime. Then

$$\widetilde{A}_a\left(x; \frac{h}{k}\right) = \frac{(-1)^a\, k^a\, x^{(2a+1)/3}}{2^{a+1}\,\pi^{a+1}\sqrt{3}} \sum_{d|k} \frac{1}{d^{(2a+1)/3}} \sum_{m=1}^{\infty} \frac{A(d,m)}{m^{(a+2)/3}} \sum_{\pm} i^{\pm a}\, S\left(\overline{h}, \pm m; \frac{k}{d}\right) \cdot e\left(\pm \frac{3\, d^{2/3}\, m^{1/3}\, x^{1/3}}{k}\right) + O\left(k^{(2a+3)/2}\, x^{2a/3}\right),$$

where $S(a, b; c)$ is the usual Kloosterman sum.

In the case $a = 1$, a careful analysis of the convergence properties of the Voronoi series in question allows us also to deduce the following identity, which does not follow from the methods used to prove the proposition on the previous slide.

Proposition

Let $x \in [1, \infty[$, let h and k be coprime integers with k prime and $1 \leq k \ll x^{1/3}$. If $N \in [1, \infty[$, and $N \gg k^{-3} x^3$, then

$$\widetilde{A}_1\left(x; \frac{h}{k}\right) = \frac{k\,x}{4\,\pi^2\,\sqrt{3}} \sum_{d|k} \frac{1}{d} \sum_{m \leqslant N} \frac{A(d,m)}{m} \sum_{\pm} (\mp i)\; S\left(\overline{h}, \pm m; \frac{k}{d}\right) \cdot e\left(\pm \frac{3\,d^{2/3}\,m^{1/3}\,x^{1/3}}{k}\right) + O\left(k^{5/2}\,x^{2/3+\varepsilon}\right).$$

From the truncated Voronoi identity for $\widetilde{A}_1(x; h/k)$ we get the following mean square-result.

Lemma

Let $x \in [1, \infty[$ and suppose that k is a prime so that $5 \leqslant k \ll X^{1/3}$. Then we have

$$\frac{1}{k-1} \sum_{h \in \mathbb{Z}_k^\times} \int_X^{2X} \left| \widetilde{A}_1 \left(x; \frac{h}{k} \right) \right|^2 \mathrm{d}x = C(k) \cdot X^3 k^3 + O\left(k^{4+\varepsilon} X^{8/3} \right),$$

where $C(k) \in \mathbb{R}_+$ with $C(k) \asymp 1$.

As a consequence of the above lemma we obtain lower bounds for certain averaged mean-squares.

Corollary

Suppose that k is a prime so that $5 \leqslant k \ll X^{1/3-\delta}$ for some arbitrarily small fixed $\delta \in \mathbb{R}_+$. Then the lower bounds

$$\frac{1}{k-1} \sum_{h \in \mathbb{Z}_k^\times} \int_X^{2X} \left| \widetilde{A}_1 \left(x; \frac{h}{k} \right) \right|^2 \mathrm{d}x \gg X^3 k^3$$

and

$$\frac{1}{k-1} \sum_{\mathbb{Z}_k^\times} \int_X^{2X} \left| \widetilde{A}_0 \left(x; \frac{h}{k} \right) \right|^2 \mathrm{d}x \gg X^{5/3} k$$

hold.

The first bound is immediate from the previous lemma. For the latter bound the idea is to related higher order Riesz means to lower order ones. The main observation is that we can bound the mean square of $\widetilde{A}_1(x; h/k)$ from the above in terms of mean squares of $\widetilde{A}_2(x; h/k)$ and $\widetilde{A}_0(x; h/k)$. Knowing a lower bound for the averaged mean square of $\widetilde{A}_1(x; h/k)$ and an upper bound for the averaged mean square of $\widetilde{A}_2(x; h/k)$ then leads to the desired conclusion.

Indeed, for any $H > 0$, which is chosen later, the main observation is that

$$\frac{1}{k-1} \sum_{h \in \mathbb{Z}_k^\times} \int_X^{2X} \left| \widetilde{A}_1\left(x; \frac{h}{k}\right) \right|^2 \mathrm{d}x$$

$$= \frac{1}{H^2(k-1)} \sum_{h \in \mathbb{Z}_k^\times} \int_X^{2X} \left| \int_x^{x+H} \widetilde{A}_1\left(x; \frac{h}{k}\right) \mathrm{d}t \right|^2 \mathrm{d}x$$

$$= \frac{1}{H^2(k-1)} \sum_{h \in \mathbb{Z}_k^\times} \int_X^{2X} \left| \int_x^{x+H} \left(\widetilde{A}_1\left(t; \frac{h}{k}\right) - \int_x^t \widetilde{A}_0\left(u; \frac{h}{k}\right) \mathrm{d}u \right) \mathrm{d}t \right|^2 \mathrm{d}x$$

$$\ll \frac{1}{H^2(k-1)} \sum_{h \in \mathbb{Z}_k^\times} \int_X^{2X} \left| \int_x^{x+H} \widetilde{A}_1\left(t; \frac{h}{k}\right) \mathrm{d}t \right|^2 \mathrm{d}x$$

$$+ \frac{1}{H^2(k-1)} \sum_{h \in \mathbb{Z}_k^\times} \int_X^{2X} \left| \int_x^{x+H} \int_x^t \widetilde{A}_0\left(u; \frac{h}{k}\right) \mathrm{d}u \, \mathrm{d}t \right|^2 \mathrm{d}x.$$

By using the upper bound for the integral involving $\widetilde{A}_1(x; h/k)$, which follows from the relevant Voronoi identity together with the first derivative test, and the first statement of the corollary, the second claim follows after some elementary arguments. This yields the first main result.

Turning to short sums, the $\widetilde{A}_a(x; h/k)$-Voronoi identity leads to the bounds

$$\frac{1}{k-1} \sum_{h \in \mathbb{Z}_k^\times} \int_X^{2X} \left| \widetilde{A}_2\left(x+\Delta; \frac{h}{k}\right) - \widetilde{A}_2\left(x; \frac{h}{k}\right) \right|^2 \mathrm{d}x \asymp \Delta^2 X^3 k^3, \qquad (3)$$

and

$$\frac{1}{k-1} \sum_{h \in \mathbb{Z}_k^\times} \int_X^{2X} \left| \widetilde{A}_3\left(x+\Delta; \frac{h}{k}\right) - \widetilde{A}_3\left(x; \frac{h}{k}\right) \right|^2 \mathrm{d}x \asymp \Delta^2 X^{13/3} k^5 \qquad (4)$$

in the range $k^{3/2} X^{1/2+\delta} \ll \Delta \ll X^{2/3} k$. By using similar ideas as before, it is possible to convert these into Ω-results.

• Indeed, estimates (3) and (4) imply a lower bound for the short sum of the first order Riesz mean as in the case of long sums. Similarly, this together with (3) leads to the Ω-result for the exponential sum we are interested in.

• We need to consider higher order Riesz means as the error term in the $\widetilde{A}_1$-Voronoi identity is too large to yield non-trivial results in the case of short sums.

Finally, as a by-product of our methods we can improve the best known upper bound for rationally additively twisted sums with a small denominator.

Theorem (J. - Vesalainen, 2019)

Let $x \in [1, \infty[$, and let h and k be coprime integers so that $1 \leqslant k \ll x^{1/3}$. Then

$$\sum_{m \leqslant x} A(m,1)\, e\left(\frac{mh}{k}\right) \ll k^{3/4} x^{1/2+\vartheta/2+\varepsilon}.$$

Notice that this theorem does not require the assumption that the denominator is a prime. Here $\vartheta \geqslant 0$ is the exponent towards the Ramanujan-Petersson conjecture for $\mathrm{SL}(3, \mathbb{Z})$ Maass forms. It is widely expected that $\vartheta = 0$, but currently we only know that $\vartheta \leqslant 5/14$. This theorem improves the best previously known upper bounds.

To prove this, observe that we have

$$\sum_{m\leqslant x} A(m,1)\, e\left(\frac{mh}{k}\right) = \widetilde{A}_0\left(x;\frac{h}{k}\right) + O(k^{3/2+\varepsilon}) + O(x^{\vartheta+\varepsilon}),$$

and so it is enough to prove the desired bound for $\widetilde{A}_0(x;h/k)$.
Let $H \in [1,\infty[$ satisfy $H \ll x$. We shall choose H in the end. Estimating a short exponential sum by absolute values allows us to get

$$\begin{aligned}
&\int\limits_x^{x+H} \widetilde{A}_0\left(t;\frac{h}{k}\right)\,\mathrm{d}t - H\,\widetilde{A}_0\left(x;\frac{h}{k}\right) \\
&= \int\limits_x^{x+H} \left(\widetilde{A}_0\left(t;\frac{h}{k}\right) - \widetilde{A}_0\left(x;\frac{h}{k}\right)\right)\mathrm{d}t \\
&\ll H^2\, x^{\vartheta+\epsilon},
\end{aligned}$$

so that

$$H\,\widetilde{A}_0\left(x;\frac{h}{k}\right)=\int_x^{x+H}\widetilde{A}_0\left(t;\frac{h}{k}\right)\,\mathrm{d}t+O(H^2\,x^{\vartheta+\varepsilon}).$$

We know from that

$$\widetilde{A}_1\left(x;\frac{h}{k}\right)\ll k^{3/2}\,x^{1+\varepsilon}$$

by the relevant Voronoi identity.
This bound gives

$$\begin{aligned}\widetilde{A}_0\left(x;\frac{h}{k}\right)&=\frac{1}{H}\int_x^{x+H}\widetilde{A}_0\left(t;\frac{h}{k}\right)\,\mathrm{d}t+O(H\,x^{\vartheta+\varepsilon})\\&=\frac{1}{H}\left(\widetilde{A}_1\left(x+H;\frac{h}{k}\right)-\widetilde{A}_1\left(x;\frac{h}{k}\right)\right)+O(H\,x^{\vartheta+\varepsilon})\\&\ll\frac{1}{H}\,k^{3/2}\,x^{1+\varepsilon}+H\,x^{\vartheta+\varepsilon}.\end{aligned}$$

Choosing $H = k^{3/4}\,x^{(1-\vartheta)/2}$ yields

$$\widetilde{A}\left(x; \frac{h}{k}\right) \ll k^{3/4}\,x^{1/2+\vartheta/2+\varepsilon},$$

finishing the proof of the final result.