Ω-RESULTS FOR EXPONENTIAL SUMS RELATED TO MAASS CUSP FORMS FOR $\mathrm{SL}_3(\mathbb{Z})$

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ABSTRACT. We obtain Ω-results for linear exponential sums with rational additive twists of small prime denominators weighted by Hecke eigenvalues of Maass cusp forms for the group $\mathrm{SL}_3(\mathbb{Z})$. In particular, our Ω-results match the expected conjectural upper bounds when the denominator of the twist is sufficiently small compared to the length of the sum. Non-trivial Ω-results for sums over short segments are also obtained. Along the way we produce lower bounds for mean squares of the exponential sums in question and also improve the best known upper bound for these sums in some ranges of parameters.

1. INTRODUCTION

Hecke eigenvalues of automorphic forms are mysterious objects of arithmetic importance and thus it is highly desirable to understand how these numbers $a(m)$ are distributed. Generally one expects significant randomness in their distribution and a manifestation of this belief is that the convolution sums

$$\sum_{x\leq m\leq x+\Delta} a(m)b(m), \tag{1.1}$$

with $\Delta \ll x$, should exhibit cancellation for various sequences $\{b(m)\}_m$. There is an extensive literature in the case of classical modular forms concerning estimates for such sums, see e.g. [24, 26, 13, 8, 9, 3, 14], for a wide variety of sequences $\{b(m)\}_m$. However, such sums have been less studied for higher rank forms. Especially interesting situation is the case where[1] $b(m) = e(m\alpha)$ for some fixed $\alpha \in \mathbb{R}$. These sums have long been connected to important questions in analytic theory, e.g. the shifted convolution problem [13, 27, 37], the subconvexity problem for twisted L-functions [1, 38], and the second moment of automorphic L-functions in the t-aspect [34]. Also, when Δ is small compared to x, the resulting short sums are closely related to the classical problems of studying various error terms, e.g. in the Dirichlet divisor problem or the Gauss circle problem, in short intervals. In the present article we shall investigate the extent of such cancellations in sums (1.1) when $a(m)$ are chosen to be the Hecke eigenvalues[2] $A(m,1)$ of a fixed Maass cusp form for the group $\mathrm{SL}_3(\mathbb{Z})$ and $b(m)$ are the exponential phases $e(m\alpha)$ with $\alpha \in \mathbb{R}$.

The relationship between exponential sums weighted by Hecke eigenvalues of automorhic forms and classical number theoretic error terms is particularly clear in the case where the twist α is close to a reduced fraction h/k with a small denominator k. Furthermore, in this case the behaviour of such exponential sum is closely related to the behaviour of the sum at the fraction and hence it makes studying rationally additively twisted linear exponential sums, that is the setting where $b(m) = e(mh/k)$ for a reduced fraction h/k, particularly interesting. This is the set-up we restrict ourselves in the present work. Finally, on the technical side, good estimates for short rationally twisted exponential sums also provide a practical tool for reducing smoothing error (see e.g. [26, 39]), which is beneficial in variety of settings where linear exponential sums arise.

For general linear twists the best known upper bound for long sums is

$$\sum_{m\leq x} A(m,1)e(m\alpha) \ll_\varepsilon x^{3/4+\varepsilon}$$

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[1] Throughout the paper $e(x)$ stands for $e^{2\pi i x}$.

[2] Fourier coefficients of GL_3 Maass cusp forms are indexed by pairs of natural numbers. It is known that if the form is a Hecke eigenform and it is normalised so that $A(1,1) = 1$, then the eigenvalue under the m^{th} Hecke operator is given by $A(m,1)$.

uniformly in $\alpha \in \mathbb{R}$ due to Miller [34] and sharper estimates are known when α is a rational number with small denominator [22]. For shorter sums very little is known in the higher rank setting. The moment estimates in the GL_2 situation [25, 5, 6, 39], the square-root-cancellation heuristics, and the shape of the truncated Voronoi identity for rationally additively twisted sums related to $\mathrm{SL}_3(\mathbb{Z})$ Maass cusp forms derived in [22] give rise to the conjectural bound

$$\sum_{x \leqslant m \leqslant x+\Delta} A(m,1) e\left(\frac{mh}{k}\right) \ll_\varepsilon \min\left(\Delta^{1/2} x^{\varepsilon}, k^{1/2} x^{1/3+\varepsilon}\right). \tag{1.2}$$

Our aim in this paper is to investigate what limitations there are for the extent of cancellation in rationally additively twisted sums attached to Maass cusp forms for the group $\mathrm{SL}_3(\mathbb{Z})$ by giving stronger evidence towards the conjectural bound (1.2). This is achieved in some ranges of parameters by establishing Ω-results.

The key tools used to study the rationally additively twisted exponential sums related to automorphic forms are the so-called Voronoi summation formulas. For example, classically (in the rank one setting) rationally additively twisted sums can be analysed using truncated Voronoi identities. These are reasonably sharp approximate formulations of full Voronoi summation formulas, which are essentially what one gets if one formally replaces the smooth cut-off function by a characteristic function of an interval in the Voronoi summation formula. Naturally, this formal substitution is analytically challenging as the Voronoi summation formulae typically require the test functions to be smooth enough. Nevertheless, in the classical setting twisted truncated Voronoi identities have been derived previously by Jutila [24] for the error term in the classical Dirichlet divisor problem and for sums involving Fourier coefficients of holomorphic cusp forms, and by Meurman [33] for sums involving Fourier coefficients of GL_2 Maass cusp forms. In higher rank a truncated Voronoi identity for the generalised divisor function $d_k(n)$ has been given in [16, (3.23)], and a truncated Voronoi identity for plain sums of coefficients of fairly general L-functions has been obtained in [10].

Rationally additively twisted Voronoi summation formulae have been implemented for GL_3 by Miller and Schmid [35] using the framework of automorphic distributions and for GL_n in [12] by more classical means. However, these types of formulas have limitations in higher rank. Namely, the convergence problems indicated above concerning replacing the smooth cut-off with a sharp cut-off get more difficult as the rank increases and this leads to quite large error terms. For example, the truncated GL_3 Voronoi identity in [22], cited in a corrected form in the appendix below, has an error term which is actually larger than the expected optimal upper bound for the corresponding sum.

The main feature in truncated Voronoi identities is the interplay between the length of the sum on the dual side (the so-called truncated Voronoi series) and the size of the error term. When considering sums with sharp cut-off, requiring the Voronoi series to be short forces the error term to be larger and likewise smaller error term requires longer Dirichlet polynomial on the dual side. The main drawback in the Voronoi identity of [22] is that even under the Ramanujan-Petersson conjecture for GL_3 Maass cusp forms the error term gives the dominant contribution. This feature makes it hard to analyse exponential sums beneficially using such summation formulas. Indeed, the truncated Voronoi identities mentioned above are not suitable for good pointwise bounds and to evaluate moments precisely one needs the dual sum to be sufficiently short, in which case the error term becomes too large. In the higher rank setting this highlights the significant technical challenges faced in proving sharp upper bounds or Ω-results for the sum in (1.2).

To circumvent these issues we take an alternate approach, building upon the paper [19] that studied a different question concerning obtaining upper bounds for the error term in the Rankin–Selberg problem. Essentially we relate sums with sharp cut-off to very specifically smoothed sums (the so-called Riesz weighted sums), where the smoothing is tailored specifically so that it is possible to connect sums with a sharp cut-off to these smoothed sums without major losses. For the latter sums very sharp Voronoi identities can be derived. Because of this, these new sums can be studied efficiently and this in turn yields information about the original exponential sum we are interested in.

To be more precise, our approach relies on considering Riesz weighted exponential sums roughly[3] given by

$$\widetilde{A}_a\left(x;\frac{h}{k}\right) := \frac{1}{a!}\sum_{m\leqslant x}{}' A(m,1)\, e\left(\frac{mh}{k}\right)(x-m)^a - \text{residue terms}, \tag{1.3}$$

where $a \geqslant 0$ is an integer. This is a very classical form of smoothing. For sufficiently large order a, Voronoi identities can be derived for these sums and they are better behaved than the Voronoi identities for sums with a sharp cut-off we are actually interested in (which essentially corresponds to the case $a = 0$). Obtaining these Voronoi identities in higher rank is of independent interest. It actually turns out that for Riesz weighted sums the Voronoi series on the dual side converges for $a \geq 2$ and so for these sums we do not need to truncate the Voronoi series unlike in the case $a = 1$ (or $a = 0$).

One strategy to obtain Ω-results is to prove lower bounds for the averaged second moment

$$\int\limits_X^{2X} \left|\sum_{m\leq x}{}' A(m,1)e\left(\frac{mh}{k}\right)\right|^2 \mathrm{d}x \approx \int\limits_X^{2X}\left|\widetilde{A}_0\left(x;\frac{h}{k}\right)\right|^2 \mathrm{d}x. \tag{1.4}$$

As discussed above, these moments are typically evaluated with the help of truncated Voronoi identities and this approach works well in the classical setting, but not so well in higher rank. Crucially for us, we will be able to evaluate second moments of $\widetilde{A}_a(x;h/k)$ for $a \geq 1$ using the Voronoi identities we establish, at least on average over $h \pmod k$. The asymptotic evaluation of the higher Riesz means can be converted into a lower bound for the second moment of the ordinary exponential sums (1.4) by elementary means explained in the following paragraph, which immediately implies an Ω-result for these sums.

The key property of the Riesz weighted sums is that

$$\int\limits_x^t \widetilde{A}_a\left(u;\frac{h}{k}\right)\mathrm{d}u = \widetilde{A}_{a+1}\left(t;\frac{h}{k}\right) - \widetilde{A}_{a+1}\left(x;\frac{h}{k}\right).$$

This may be combined with the trivial identity

$$\widetilde{A}_a\left(x;\frac{h}{k}\right) = \frac{1}{H}\int\limits_x^{x+H}\widetilde{A}_a\left(x;\frac{h}{k}\right)\mathrm{d}t,$$

and from these it quickly follows that

$$\widetilde{A}_{a+1}\left(x;\frac{h}{k}\right) = \frac{1}{H}\int\limits_x^{x+H}\left(\widetilde{A}_{a+1}\left(t;\frac{h}{k}\right) - \int\limits_x^t \widetilde{A}_a\left(u;\frac{h}{k}\right)\mathrm{d}u\right)\mathrm{d}t. \tag{1.5}$$

An analogous identity for Riesz weighted sums attached to the Dirichlet series coefficients of the Rankin-Selberg L-function was used in [19] to relate upper bounds for the error term in the Rankin-Selberg problem to upper bounds for its first Riesz mean. The novelty of the present work is to observe that (1.5) may further be used to relate moments of various different Riesz weighted sums.

Indeed, one may e.g. deduce (see the proof of Theorem 10.2) the estimate

$$\int\limits_X^{2X}\left|\widetilde{A}_1\left(x;\frac{h}{k}\right)\right|^2\mathrm{d}x \ll \frac{1}{H^2}\int\limits_X^{2X}\left|\widetilde{A}_2\left(x;\frac{h}{k}\right)\right|^2\mathrm{d}x + H\int\limits_X^{2X}\left|\widetilde{A}_0\left(x;\frac{h}{k}\right)\right|^2\mathrm{d}x$$

for any $H > 0$. The upshot is that, while we are unable to evaluate the mean square of $\widetilde{A}_0(x;h/k)$ directly, we may use our Voronoi identities to evaluate the other two moments asymptotically. Then choosing H optimally leads to a lower bound for the second moment of $\widetilde{A}_0(x;h/k)$. One can also apply the identity (1.5) directly to relate different sum $\widetilde{A}_a(x;h/k)$. An example of this is given e.g. in the proof Proposition 9.4. When studying short sums we need to work with even higher order Riesz weighted sums as for these sums

[3]For the explicit expression for the residue terms, see (7.1) below.

we are unable to evaluate the second moment of $\widetilde{A}_1(x+\Delta;h/k)-\widetilde{A}_1(x;h/k)$ precisely as the error term is larger compared to expected size of the sum in this situation.

To obtain Voronoi identities for the Riesz weighted exponential sums (1.3) in the case $a \geq 2$ we use standard arguments based on Perron's formula and analysing the resulting integral involving quotients of Gamma functions. However, we simplify this analysis by noting that these integrals are values of Meijer G-functions whose asymptotic behaviour is known. Establishing such formula in the case $a=1$ is more involved. The method used in the case $a \geq 2$ runs into difficulties due to lack of absolute convergence in certain sums and to circumvent this issue we need to implement some of the arguments in [25] to the rank two setting and develop them further.

The problem of estimating moments of exponential sums alluded above is also a classical theme by its own right. Upper bounds for such moments have been obtained in many different settings, see for example [18, 40, 22]. Moreover, in some cases even asymptotic behaviour is known [29, 21]. While there are some results concerning upper bounds in higher rank settings, as far as the author is aware of, no asymptotics or lower bounds for these moments are known unconditionally in higher rank. There are essentially two reasons that make the higher rank cases difficult. Firstly, the dual sum in the truncated GL_3 Voronoi identity has to be short enough so that its second moment can be evaluated asymptotically. This can be arranged, but unfortunately this forces the error on the dual side to be too large in order to obtain good results. The second reason is the presence of Kloosterman sums that appear on the dual side in the Voronoi identities when the sums are rationally additively twisted. In higher rank it is not even a priori clear that the expected main contribution in the second moment coming from the diagonal terms is positive. Note that such situation does not arise in the rank one setting where the exponential phases cancel each other. The strategy to overcome the first issue is described above. The problem with Kloosterman sums is resolved by averaging over the numerators $h \in \mathbb{Z}_k^\times$. In this way we will obtain lower bounds for the averaged mean squares

$$\sum_{h \in \mathbb{Z}_k^\times} \int_X^{2X} \left| \sideset{}{'}\sum_{x \leq m \leq x+\Delta} A(m,1) e\left(\frac{mh}{k}\right) \right|^2 \mathrm{d}x$$

in certain ranges of parameters involved (in particular for $\Delta = x$) when k is a prime (or $k=1$). For simplicity we restrict ourselves to prime denominators in the twists throughout the paper, but it is plausible that the argument could be made to work for more general k.

2. The main results

Our first main result shows the expected Ω-result for rationally additively twisted long sums when the denominator of the twist is sufficiently small. As far as we are aware of, this is the first time Ω-results have been obtained for rationally twisted sums in a higher rank setting. Although it is not explicitly mentioned in what follows, we assume throughout the paper that the underlying Maass cusp form is a Hecke eigenform and normalised so that $A(1,1)=1$. Our results should generalise for arbitrary Maass cusp forms, but we restrict ourselves to Hecke eigenforms to lighten the notation.

Theorem 2.1. *Let $x \in [1,\infty[$, and let k prime so that $k \ll_\varepsilon x^{1/3-\varepsilon}$. Then*

$$\max_{h \in \mathbb{Z}_k^\times} \left| \sum_{m \leqslant x} A(m,1)\, e\left(\frac{mh}{k}\right) \right| = \Omega\left(k^{1/2} x^{1/3}\right),$$

where the maximum is taken over all reduced residue classes modulo k.

Notice that $\ll_\varepsilon k^{1/2}x^{1/3+\varepsilon}$ is the conjectured upper bound in (1.2) for the range $k \ll x^{1/3}$. Thus the result of Theorem 2.1 is essentially optimal in the sense of conjecture (1.2).

This follows immediately from the following mean square result.

Theorem 2.2. *(Theorem 10.2) Let $X \in [1,\infty[$ and let k be a prime so that $k \ll_\varepsilon X^{1/3-\varepsilon}$. Then*

$$\sum_{h \in \mathbb{Z}_k^\times} \int_X^{2X} \left| {\sum_{m \le x}}' A(m,1) e\left(\frac{mh}{k}\right) \right|^2 \mathrm{d}x \gg X^{5/3} k^2.$$

As sketched in the introduction, the proof of the previous result relies on Voronoi summation formulas for both $\widetilde{A}_1(x;h/k)$ and $\widetilde{A}_2(x;h/k)$, but in Section 9 we shall also present an alternative proof for Theorem 2.1 that sidesteps the use of Voronoi summation formula for $\widetilde{A}_1(x;h/k)$.

The third main theorem establishes a non-trivial Ω-result for short sums of certain length.

Theorem 2.3. *Let $x \in [1,\infty[$, let k prime and suppose that $k^{3/2}x^{1/2} \ll \Delta \ll kx^{2/3}$ with sufficiently small implied constants. Then*

$$\max_{h \in \mathbb{Z}_k^\times} \left| \sum_{x \leqslant m \leqslant x+\Delta} A(m,1)\, e\left(\frac{mh}{k}\right) \right| = \Omega\left(\Delta x^{-1/3} k^{-1/2}\right),$$

where the maximum is taken over all reduced residue classes modulo k.

This is a direct consequence of the following mean square result.

Theorem 2.4. *(Theorem 13.1) Let $X \in [1,\infty[$ and let k be a prime so that $k^{3/2}X^{1/2} \ll \Delta \ll kX^{2/3}$ with sufficiently small implied constants. Then we have*

$$\sum_{h \in \mathbb{Z}_k^\times} \int_X^{2X} \left| {\sum_{x \leqslant m \leqslant x+\Delta}}' A(m,1)\, e\left(\frac{mh}{k}\right) \right|^2 \mathrm{d}x \gg \Delta^2 X^{1/3} k^{-1}.$$

Notice that the result of Theorem 2.3 is of the right order of magnitude in sense of the conjectural bound (1.2), up to X^ε, when $\Delta \asymp kX^{2/3}$. However, for smaller Δ this Ω-result is farther away from the expected upper bound.

The restriction to prime denominator is made mainly for simplicity so we could handle the sum over the divisors of k arising from the Voronoi identities for Riesz weighted exponential sums efficiently. Since we are interested in Ω-results this is not such a serious restriction. Note also that theorems above hold in the case $k = 1$, which can be easily seen by making simple cosmetic modifications to the proofs.

Likewise, we were unable to obtain Ω-results for an arbitrary numerator h coprime to k. The averaging over the reduced residue classes modulo k has to be included in order to be able to exract main terms when evaluating moments of the sums $\widetilde{A}_a(x;h/k)$. It seems that this cannot be achieved by our methods unless we perform such extra averaging. The presence of this is not so serious as we are mainly interested in limitations for obtaining bounds (1.2).

Finally, as a by-product of our analysis we improve the best known upper bound for rationally additively twisted sums with a small denominator.

Theorem 2.5. *Let $x \in [1,\infty[$, and let h and k be coprime integers so that $1 \leqslant k \ll x^{1/3}$. Then*

$$\sum_{m \leqslant x} A(m,1)\, e\left(\frac{mh}{k}\right) \ll_\varepsilon k^{3/4} x^{1/2+\vartheta/2+\varepsilon}.$$

Here $\vartheta \geqslant 0$ is the exponent towards the Ramanujan-Petersson conjecture for $\mathrm{SL}_3(\mathbb{Z})$ Maass cusp forms. It is widely expected that $\vartheta = 0$, but currently we only know that $\vartheta \leqslant 5/14$, see [28, Appendix 2]. Notice that this theorem does not require the assumption for the denominator k being a prime. Theorem 2.5 improves the best previously known upper bound [22, Corollary 3].

This paper is organised as follows. In Sections 4 and 6 we gather basic facts concerning additively rationally twisted L-functions and linear exponential sums on GL_3, respectively. In Section 5 we introduce Meijer G-functions relevant for the present work and study their asymptotic behaviour. Then we proceed to derive Voronoi summation formulas for the Riesz weighted sums $\widetilde{A}_a(x;h/k)$ in the next two Sections 7 (for $a \geq 2$)

and 8 (for $a = 1$). In Section 9 we evaluate the mean square of $\widetilde{A}_2(x; h/k)$ and give the first proof for Theorem 2.1. In the next section the mean square of $\widetilde{A}_1(x; h/k)$ is evaluated and another proof for Theorem 2.1 is deduced using Theorem 2.2. Theorem 2.5 is proved in the following section. The final two sections are devoted for the proofs of Theorems 2.3 and 2.4.

3. Notation

We use standard asymptotic notation. If f and g are complex-valued functions defined on some set, say $\mathcal{D}$, then we write $f \ll g$ to signify that $|f(x)| \leqslant C|g(x)|$ for all $x \in \mathcal{D}$ for some implicit constant $C \in \mathbb{R}_+$. The notation $O(g)$ denotes a quantity that is $\ll g$, and $f \asymp g$ means that both $f \ll g$ and $g \ll f$. We write $f = o(g)$ if g never vanishes in $\mathcal{D}$ and $f(x)/g(x) \longrightarrow 0$ as $x \longrightarrow \infty$. The notation $f = \Omega(g)$ means that $f \neq o(g)$. The letter ε denotes a positive real number, whose value can be fixed to be arbitrarily small, and whose value can be different in different instances in a proof. All implicit constants are allowed to depend on ε, on the implicit constants appearing in the assumptions of theorem statements, and on anything that has been fixed. When necessary, we will use subscripts $\ll_{\alpha,\beta,\ldots}, O_{\alpha,\beta,\ldots}$, etc. to indicate when implicit constants are allowed to depend on quantities $\alpha, \beta, \ldots$

As usual, complex variables are written in the form $s = \sigma + it$ with σ and t real, and we write $e(x)$ for $e^{2\pi i x}$. The subscript in the integral $\int_{(\sigma)}$ means that we integrate over the vertical line $\mathrm{Re}(s) = \sigma$. When $h \in \mathbb{Z}$ and $k \in \mathbb{Z}_+$ are coprime, then $\overline{h}$ denotes an integer such that $h\overline{h} \equiv 1 \,(\mathrm{mod}\ k)$. We also write $\langle \cdot \rangle$ for $(1 + |\cdot|^2)^{1/2}$. The notation $\sum'_{m \leqslant x}$ means the sum up to x with the last term halved if x is an integer. The notation $d(n)$ denotes the ordinary divisor function. Let us furthermore write

$$1_{a \equiv b\,(\ell)} := \begin{cases} 1 & \text{if } a \equiv b \pmod{\ell} \\ 0 & \text{otherwise} \end{cases}$$

Finally, for a positive integer k let us define the following averaging operator. Given a function $f\colon \mathbb{Z}_k^\times \longrightarrow \mathbb{C}$, we set

$$\mathop{\mathbb{E}}_{x \in \mathbb{Z}_k^\times} f(x) := \frac{1}{\varphi(k)} \sum_{x \in \mathbb{Z}_k^\times} f(x),$$

where φ is Euler's totient function, and $\mathbb{Z}_k^\times$ denotes the reduces residue classes modulo k, or a set of representatives of them.

4. Rationally additively twisted L-functions of GL_3 Maass cusp forms

We shall derive our Voronoi identities from the additively twisted L-function of the Maass cusp form in question. To this end, we quote some of its properties from Section 3 of [12]. We fix a Maass cusp form for $\mathrm{SL}_3(\mathbb{Z})$ with Fourier coefficients $A(m_1, m_2)$. Also, let us choose coprime integers h and k with k positive, as well as an index $j \in \{0, 1\}$. Then we have the additively rationally twisted version of the corresponding Godement–Jacquet L-function given by

$$L_j\left(s + j, \frac{h}{k}\right) := \sum_{m=1}^{\infty} \frac{A(m,1)}{m^s} \left(e\left(\frac{mh}{k}\right) + (-1)^j\, e\left(-\frac{mh}{k}\right) \right)$$

for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. This has an entire analytic extension and satisfies the functional equation

$$L_j\left(s + j, \frac{h}{k}\right) = i^{-j}\, k^{-3s+1}\, \pi^{3s-3/2}\, G_j(s + j)\, \widetilde{L}_j\left(1 - s - j, \frac{\overline{h}}{k}\right).$$

The Γ-factors are clumped together into the factor $G_j(s + j)$ given by

$$G_j(s + j) := \frac{\Gamma\left(\frac{1 - s + j + \alpha}{2}\right) \Gamma\left(\frac{1 - s + j + \beta}{2}\right) \Gamma\left(\frac{1 - s + j + \gamma}{2}\right)}{\Gamma\left(\frac{s + j - \alpha}{2}\right) \Gamma\left(\frac{s + j - \beta}{2}\right) \Gamma\left(\frac{s + j - \gamma}{2}\right)},$$

where in turn the complex constants α, β and γ are the Langlands parameters of the underlying Maass cusp form. We know from [11, Props. 6.3.1 & 12.1.9] and [12, Eq. (3.30)] that

$$\alpha+\beta+\gamma=0, \qquad \text{and} \qquad \max\left\{|\mathrm{Re}(\alpha)|,|\mathrm{Re}(\beta)|,|\mathrm{Re}(\gamma)|\right\}\leqslant\frac{1}{2}.$$

Finally, the Dirichlet series on the right-hand side of the functional equation is given by

$$\widetilde{L}_j\left(1-s-j,\frac{\overline{h}}{k}\right):=\sum_{d\mid k}\sum_{m=1}^{\infty}\frac{A(d,m)}{d^{1-2s}\,m^{1-s}}\left(S\left(\overline{h},m;\frac{k}{d}\right)+(-1)^j\,S\left(\overline{h},-m;\frac{k}{d}\right)\right)$$

for $\mathrm{Re}(s)<0$, and by an entire analytic continuation elsewhere. Here $S(a,b;c)$ is the usual Kloosterman sum.

From Stirling's formula, the functional equation above, and the Phragmén–Lindelöf principle, we obtain the convexity estimate

$$L_j\left(s+j,\frac{h}{k}\right)\ll_\varepsilon k^{3(1+\delta-\sigma)(1+\varepsilon)/2}\,\langle t\rangle^{3(1+\delta-\sigma)/2}$$

in the strip $-\delta\leqslant\sigma\leqslant1+\delta$ (see [22, p. 262]), where $\delta\in\mathbb{R}_+$ is arbitrary and fixed. Furthermore, it follows from this that, for each $j\in\{0,1\}$ and every $a\in\mathbb{Z}_+\cup\{0\}$, as well as for any coprime integers h and k with k positive,

$$L_j\left(-a+j,\frac{h}{k}\right)\ll_{a,\varepsilon} k^{3(1+2a)/2+\varepsilon}.$$

In particular,

$$L_j\left(0+j,\frac{h}{k}\right)\ll_\varepsilon k^{3/2+\varepsilon},\quad L_j\left(-1+j,\frac{h}{k}\right)\ll_\varepsilon k^{9/2+\varepsilon},\quad L_j\left(-2+j,\frac{h}{k}\right)\ll_\varepsilon k^{15/2+\varepsilon},$$

and

$$L_j\left(-3+j,\frac{h}{k}\right)\ll_\varepsilon k^{21/2+\varepsilon}.$$

When summing by parts we will repeatedly use the following consequences of the Rankin–Selberg theory saying that

$$\sum_{m\leqslant x}|A(d,m)|\ll_\varepsilon d^{\vartheta+\varepsilon}x \qquad \text{and} \qquad \sum_{m\leqslant x}|A(d,m)|^2\ll_\varepsilon d^{2\vartheta+\varepsilon}x, \tag{4.1}$$

uniformly in $d\in\mathbb{Z}_+$ and $x\in[1,\infty[$ (see e.g. [23]).

5. On certain Meijer G-functions and their asymptotics

Applying Perron's formula to the additively twisted L-function, shifting the line of integration to the left and then applying the additively twisted functional equation leads to integrals involving quotients of Gamma functions. We could derive the asymptotics we need in the same way as similar integrals were treated in earlier works, e.g [17, 34, 30, 7, 22]. However, here we prefer to follow the approach of [4], which observed that these types of integrals are actually Meijer G-functions, and thus one obtains the relevant asymptotics directly from the well-known asymptotic properties of the latter. Our main references for Meijer G-functions are [31, 32].

Let α, β and γ be the complex numbers as in the functional equation of the additively twisted L-function, and let $y\in\mathbb{Z}_+$, $a\in\mathbb{Z}_+$ and $j\in\{0,1\}$. We define $\mathscr{J}_{a,j}(y)$ to be the specific Meijer G-function of interest to us as

$$\begin{aligned}
&\mathscr{J}_{a,j}(y)\\
&:= G^{5,0}_{2,8}\left(y;\ \begin{matrix} 1, \frac{1}{2} \\ \frac{1+j+\alpha}{2}, \frac{1+j+\beta}{2}, \frac{1+j+\gamma}{2}, -\frac{a}{2}, \frac{1}{2}-\frac{a}{2}, 1-\frac{j-\alpha}{2}, 1-\frac{j-\beta}{2}, 1-\frac{j-\gamma}{2} \end{matrix}\right)\\
&= \frac{1}{2\pi i}\int\limits_{\mathscr{C}} Q(s)\, y^s \,\mathrm{d}s,
\end{aligned}$$

where $Q(s)$ denotes the Γ-quotient

$$Q(s) := \frac{\Gamma\Big(\frac{1+j+\alpha}{2}-s\Big)\,\Gamma\Big(\frac{1+j+\beta}{2}-s\Big)\,\Gamma\Big(\frac{1+j+\gamma}{2}-s\Big)\,\Gamma\Big(-\frac{a}{2}-s\Big)\,\Gamma\Big(\frac{1}{2}-\frac{a}{2}-s\Big)}{\Gamma\Big(s+\frac{j-\alpha}{2}\Big)\,\Gamma\Big(s+\frac{j-\beta}{2}\Big)\,\Gamma\Big(s+\frac{j-\gamma}{2}\Big)\,\Gamma(1-s)\,\Gamma\Big(\frac{1}{2}-s\Big)}.$$

The contour of integration $\mathscr{C}$ can be, say, a simple fractional line, which begins at $\sigma_0 - i\infty$ with a vertical half-line with abscissa σ_0 satisfying $\sigma_0 > 1/4 - a/6$, ends at $\sigma_0 + i\infty$ with another vertical half-line with the same abscissa σ_0, and having the property that all the poles of the Γ-factors in the numerator lie to the right of $\mathscr{C}$ [32, Subsect. 5.3.1]. For definiteness, we may select as $\mathscr{C}$ a fractional line $\mathscr{C}(\sigma_0, \sigma_1, \Lambda)$ connecting the points $\sigma_0 - i\infty$, $\sigma_0 - i\Lambda$, $\sigma_1 - i\Lambda$, $\sigma_1 + i\Lambda$, $\sigma_0 + i\Lambda$ and $\sigma_0 + i\infty$, in this order, where the second abscissa σ_1 is a real number satisfying $\sigma_1 < -a/2$ and Λ is a positive real number such that Λ is larger than the imaginary parts of $\alpha/2$, $\beta/2$ and $\gamma/2$.

We will not repeat the above parameters and instead write the above expression as $G^{5,0}_{2,8}(y;\ldots)$ with the understanding that the parameters are exactly as above. The asymptotics we wish to use is a special case of the case (4) of Theorem 2 in Section 5.10 of [31]. In our special case the result reads as follows.

Lemma 5.1. *Let $y \in \mathbb{R}_+$ be larger than some arbitrary fixed positive real constant, and let $j \in \{0,1\}$ and $a \in \mathbb{Z}_+$ be fixed. Then the above function $\mathscr{J}_{a,j}(y)$ has the asymptotic expansion*

$$\mathscr{J}_{a,j}(y) \sim -\frac{1}{2\sqrt{3\pi}} \sum_{\pm} \exp\Big(\pm 6i\, y^{1/6} \pm \frac{\pi i}{2}\,(a+j)\Big)\, y^{(1-a)/6} \sum_{\ell=0}^{\infty} M_\ell\, i^{\pm\ell}\, y^{-\ell/6},$$

for some complex coefficients M_0, M_1, M_2, $\ldots$ for which $M_0 = 1$, and which otherwise depend on α, β, γ, a, and j. In particular,

$$\mathscr{J}_{a,j}(y) = -\frac{1}{\sqrt{3\pi}}\, y^{(1-a)/6}\,\cos\Big(6\,y^{1/6} + \frac{\pi}{2}\,(a+j)\Big) + O(y^{-a/6}).$$

Proof. We apply the asymptotics (4) of Theorem 2 in Section 5.10 of [31] which says that the Meijer G-function $G^{5,0}_{2,8}(y;\ldots)$ has the asymptotic expansion

$$G^{5,0}_{2,8}(y;\ldots) \sim A^{5,0}_8\, H_{2,8}(y\,e^{3\pi i}) + \overline{A}^{5,0}_8\, H_{2,8}(y\,e^{-3\pi i}),$$

as $y \longrightarrow \infty$. In the notation of Section 5.7 in [31], we have

$$\sigma = 8 - 2 = 6 \qquad \text{and} \qquad \nu = 8 - 5 - 0 = 3.$$

We observe that our parameters satisfy the conditions (A) and (B) of Section 5.7 in [32] vacuously as in our case $n = 0$, and they satisfy the condition (C) simply because $1 - 1/2 = 1/2 \notin \mathbb{Z}$. In the notation of Theorem 5 in Section 5.7 of [32], we have

$$\Xi_1 = \frac{1+j+\alpha}{2} + \frac{1+j+\beta}{2} + \frac{1+j+\gamma}{2} - \frac{a}{2} + \frac{1}{2} - \frac{a}{2} + 1 - \frac{j-\alpha}{2} + 1 - \frac{j-\beta}{2} + 1 - \frac{j-\gamma}{2} = 5 - a,$$

and

$$\Lambda_1 = 1 + \frac{1}{2} = \frac{3}{2},$$

so that

$$\theta = \frac{1}{\sigma}\left(\frac{1}{2}(1-\sigma) + \Xi_1 - \Lambda_1\right) = \frac{1-a}{6}.$$

The asymptotics behaviour of $H_{2,8}$ is thus given by

$$H_{2,8}(y\, e^{\pm 3\pi i}) \sim i^{\pm(1-a)} \frac{(2\pi)^{5/2}}{\sqrt{6}} \exp(\mp 6i\, y^{1/6})\, y^{(1-a)/6} \sum_{\ell=0}^{\infty} M_\ell\, i^{\mp \ell}\, y^{-\ell/6},$$

as $y \longrightarrow \infty$, for some complex constants M_0, M_1, $M_2, \ldots$. In particular, $M_0 = 1$. Finally, in the notation of Subsection 5.9.2 in [31],

$$A_8^{5,0} = (-1)^\nu\, (2\pi i)^{-\nu} \exp\!\left(i\pi \left(0 - 1 + \frac{j-\alpha}{2} - 1 + \frac{j-\beta}{2} - 1 + \frac{j-\gamma}{2} \right)\right) = (2\pi)^{-3}\, i^{1-j},$$

and similarly,

$$\overline{A}_8^{5,0} = (2\pi)^{-3}\, i^{j-1}. \qquad \square$$

The Meijer G-function appears from the integral in Perron's formula. The following lemma connects the two.

Lemma 5.2. *Let $a \in \mathbb{Z}_+$, let $y \in \mathbb{R}_+$, and let σ_0 and σ_1 be real numbers such that $\sigma_0 > 1/4 - a/6$ and $\sigma_1 < -a/2$. Then*

$$\frac{1}{2\pi i} \int\limits_{\mathscr{C}(2\sigma_0, 2\sigma_1, 2\Lambda)} \frac{G_j(s+j)\, y^s\, \mathrm{d}s}{s\,(s+1)\cdots(s+a)} = -\,(-2)^{-a}\, \mathscr{J}_{a,j}(y^2).$$

Proof. Observing

$$\frac{1}{s\,(s+1)\cdots(s+a)} = \frac{\left(-\frac{1}{2}\right)^{a+1} \Gamma\!\left(-\frac{s}{2} - \frac{a}{2}\right) \Gamma\!\left(-\frac{s}{2} - \frac{a}{2} + \frac{1}{2}\right)}{\Gamma\!\left(-\frac{s}{2} + 1\right) \Gamma\!\left(-\frac{s}{2} + \frac{1}{2}\right)},$$

we see that

$$\begin{aligned}\frac{1}{2\pi i} \int\limits_{\mathscr{C}(2\sigma_0, 2\sigma_1, 2\Lambda)} \frac{G_j(s+j)\, y^s\, \mathrm{d}s}{s\,(s+1)\cdots(s+a)} &= -\,(-2)^{-a}\, \frac{1}{2\pi i} \int\limits_{\mathscr{C}(2\sigma_0, 2\sigma_1, 2\Lambda)} Q\!\left(\frac{s}{2}\right) \left(y^2\right)^{s/2} \frac{\mathrm{d}s}{2} \\ &= -\,(-2)^{-a}\, \mathscr{J}_{a,j}(y^2). \qquad \square\end{aligned}$$

In order to derive Voronoi identities with Riesz weights in the $a = 1$ case, we wish to differentiate the formula of case $a = 2$ termwise. The following lemma allows this.

Lemma 5.3. *Let $a \in \mathbb{Z}_+$, let $j \in \{0, 1\}$, and let $y \in \mathbb{R}_+$. Then*

$$\frac{\mathrm{d}}{\mathrm{d}y} \left(y^{a+1}\, \mathscr{J}_{a+1,j}(y^2) \right) = -2\, y^a\, \mathscr{J}_{a,j}(y^2).$$

Proof. Let σ_0 and σ_1 be real numbers such that $\sigma_0 > 1/4 - a/6$ and $\sigma_1 < -a/2 - 1/2$. Using Lemma 5.2 twice we may compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}y} \left(y^{a+1}\, \mathscr{J}_{a+1,j}(y^2) \right) &= \frac{\mathrm{d}}{\mathrm{d}y} \left(-\,(-2)^{a+1}\, \frac{1}{2\pi i} \int\limits_{\mathscr{C}(2\sigma_0, 2\sigma_1, 2\Lambda)} \frac{G_j(s+j)\, y^{s+a+1}\, \mathrm{d}s}{s\,(s+1)\cdots(s+a)\,(s+a+1)} \right) \\ &= (-2)\cdot(-1)\,(-2)^a\, \frac{1}{2\pi i} \int\limits_{\mathscr{C}(2\sigma_0, 2\sigma_1, 2\Lambda)} \frac{G_j(s+j)\, y^{s+a}\, \mathrm{d}s}{s\,(s+1)\cdots(s+a)} = -2\, y^a\, \mathscr{J}_{a,j}(y^2). \qquad \square\end{aligned}$$

6. Some known estimates for long linear exponential sums

We shall need estimates for long linear exponential sums involving Fourier coefficients of a GL_3 Maass cusp form. The following is Theorem 1.1 in [34].

Lemma 6.1. *Let $x \in [1, \infty[$. Then*

$$\sum_{m \leqslant x} A(m,1)\, e(m\alpha) \ll_\varepsilon x^{3/4+\varepsilon}$$

uniformly in $\alpha \in \mathbb{R}$.

On the other side of our Voronoi identities coefficients $A(d,m)$, where d divides the denominator of the twist, will also appear. Fortunately, it is easy to extend Miller's estimate for these coefficients with uniformity over d.

Lemma 6.2. *Let $x \in [1, \infty[$, $d \in \mathbb{Z}_+$ and $\alpha \in \mathbb{R}$. Then*

$$\sum_{m \leqslant x} A(m,d)\, e(m\alpha) \ll_\varepsilon d^{\vartheta+\varepsilon}\, x^{3/4+\varepsilon},$$

uniformly in both α and d.

Proof. As the underlying form is a normalised Hecke eigenform we have

$$A(m,d) = \sum_{\ell \mid (d,m)} \mu(\ell)\, A\left(1, \frac{d}{\ell}\right) A\left(\frac{m}{\ell}, 1\right)$$

for any $m, d \in \mathbb{Z}_+$, where μ is the Möbius function. Using this, we may estimate

$$\begin{aligned}\sum_{m \leqslant x} A(m,d)\, e(m\alpha) &= \sum_{m \leqslant x} \sum_{\ell \mid (d,m)} \mu(\ell)\, A\left(1, \frac{d}{\ell}\right) A\left(\frac{m}{\ell}, 1\right) e(m\alpha) \\ &= \sum_{\ell \mid d} \mu(\ell)\, A\left(1, \frac{d}{\ell}\right) \sum_{m \leqslant x/\ell} A(m,1)\, e(m\ell\alpha) \\ &\ll \sum_{\ell \mid d} \left(\frac{d}{\ell}\right)^{\vartheta+\varepsilon} \left(\frac{x}{\ell}\right)^{3/4+\varepsilon} \ll_\varepsilon d^{\vartheta+\varepsilon}\, x^{3/4+\varepsilon}.\end{aligned}$$

□

When α is a fraction with a small denominator, we have more precise estimates, which depend on ϑ. The following is Corollary 3 in [22].

Lemma 6.3. *Let $x \in [1, \infty[$, and let h and k be coprime integers such that $1 \leqslant k \ll x^{2/3}$. Then*

$$\sum_{m \leqslant x} A(m,1)\, e\left(\frac{mh}{k}\right) \ll_\varepsilon k^{1/2+\varepsilon}\, x^{2/3} + k\, x^{1/3+\vartheta+\varepsilon}.$$

If in addition $\vartheta \leqslant 1/3$ and $k \ll x^{2/3-2\vartheta}$, then

$$\sum_{m \leqslant x} A(m,1)\, e\left(\frac{mh}{k}\right) \ll_\varepsilon k^{3/4}\, x^{1/2+\vartheta/2+\varepsilon} + k^{9/8+3\vartheta/4}\, x^{1/4+3\vartheta^2/2+3\vartheta/4+\varepsilon}.$$

In particular, if $\vartheta = 0$ and $k \ll x^{2/3}$, then

$$\sum_{m \leqslant x} A(m,1)\, e\left(\frac{mh}{k}\right) \ll_\varepsilon k^{3/4}\, x^{1/2+\varepsilon}.$$

7. Additively twisted Voronoi identities for Riesz means: the case $a \geqslant 2$

In this section we study the modified Riesz weighted sums

$$\widetilde{A}_{a,j}\left(x; \frac{h}{k}\right) := \frac{1}{a!} \sideset{}{'}\sum_{m \leqslant x} A(m,1) \left(e\left(\frac{mh}{k}\right) + (-1)^j\, e\left(-\frac{mh}{k}\right)\right) (x-m)^a$$
$$- \sum_{0 \leqslant \nu \leqslant a} \frac{(-1)^\nu\, x^{a-\nu}}{\nu!\,(a-\nu)!}\, L_j\left(-\nu + j, \frac{h}{k}\right),$$

where $x \in [0,\infty[$, $j \in \{0,1\}$, $a \in \mathbb{Z}_+ \cup \{0\}$, and h and k are coprime integers with k positive. Our goal in this section is to derive Voronoi-type identities for these sums when $a \geqslant 2$.

The classical Perron's formula, which is e.g. Theorem 1.4.4 in [2], has the following generalisation for Riesz means (see e.g. Chapter 5 in [36]), which is the starting point for proving Voronoi-type identities.

Lemma 7.1. *Let $\sigma \in \mathbb{R}_+$, and let $c\colon \mathbb{Z}_+ \longrightarrow \mathbb{C}$ be a sequence such that the Dirichlet series $\sum_{n=1}^\infty c(n)/n^\sigma$ converges absolutely. Then, for $x \in \mathbb{R}_+$, and for a non-negative integer a, we have*

$$\frac{1}{a!} \sideset{}{'}\sum_{n \leqslant x} c(n)\,(x-n)^a = \frac{1}{2\pi i} \int\limits_{(\sigma)} \left(\sum_{n=1}^\infty \frac{c(n)}{n^s}\right) \frac{x^{s+a}\,\mathrm{d}s}{s\,(s+1)\cdots(s+a)},$$

where the integration is over the vertical line where the real part is σ. When $a = 0$, the integral should be understood as the limit of $\int_{\sigma-iT}^{\sigma+iT}$ as $T \longrightarrow \infty$.

The main result of this section is the following theorem.

Proposition 7.2. *Let $x \in \mathbb{R}_+$, $j \in \{0,1\}$, $a \in \{2,3,\ldots\}$, and let h and k be coprime integers with k positive. Then*

$$\widetilde{A}_{a,j}\left(x; \frac{h}{k}\right) = i^{-j}\, \pi^{-3/2}\, k\, x^a\, (-1)^{a+1}\, 2^{-a}$$
$$\times \sum_{d \mid k} \sum_{m=1}^\infty \frac{A(d,m)}{dm} \left(S\left(\overline{h}, m; \frac{k}{d}\right) + (-1)^j\, S\left(\overline{h}, -m; \frac{k}{d}\right)\right) \mathscr{I}_{a,j}\left(\frac{\pi^6\, d^4\, m^2\, x^2}{k^6}\right).$$

Proof. Let us first fix some $\delta \in\,]0, 1/6[$. Perron's formula for Riesz means tells us that

$$\frac{1}{a!} \sideset{}{'}\sum_{m \leqslant x} A(m,1) \left(e\left(\frac{mh}{k}\right) + (-1)^j\, e\left(-\frac{mh}{k}\right)\right) (x-m)^a = \frac{1}{2\pi i} \int\limits_{(1+\delta)} L_j\left(s+j, \frac{h}{k}\right) \frac{x^{s+a}\,\mathrm{d}s}{s\,(s+1)\cdots(s+a)}.$$

Let $\mathscr{L}$ be the fractional line $\mathscr{C}(-\delta, -a-\delta, 2\Lambda)$ connecting the points $-\delta - i\infty$, $-\delta - 2i\Lambda$, $-a-\delta-2i\Lambda$, $-a-\delta+2i\Lambda$, $-\delta+2i\Lambda$ and $-\delta+i\infty$, in this order, where Λ is a fixed positive real number larger than the imaginary parts of $\alpha/2$, $\beta/2$ and $\gamma/2$. We shift the contour of integration from the vertical line $\mathrm{Re}(s) = 1+\delta$ to $\mathscr{L}$. This is justified by the convexity bound for $L_j(s+j, h/k)$ and the assumption $a \geqslant 2$. This leads to residue terms from the simple poles at the points $-a$, $-a+1$, …, 0. For each $\nu \in \{0,1,\ldots,a\}$ we get a residue term

$$\frac{(-1)^\nu\, x^{a-\nu}}{\nu!\,(a-\nu)!}\, L_j\left(-\nu+j, \frac{h}{k}\right).$$

Thus, we obtain

$$\widetilde{A}_{a,j}\left(x; \frac{h}{k}\right) = \frac{1}{2\pi i} \int\limits_{\mathscr{L}} L_j\left(s+j, \frac{h}{k}\right) \frac{x^{s+a}\,\mathrm{d}s}{s\,(s+1)\cdots(s+a)}.$$

Applying the additively twisted functional equation gives

$$\widetilde{A}_{a,j}\left(x; \frac{h}{k}\right) = \frac{1}{2\pi i} \int\limits_{\mathscr{L}} i^{-j}\, k^{-3s+1}\, \pi^{3s-3/2}\, G_j(s+j)\, \widetilde{L}_j\left(1-s-j; \frac{\overline{h}}{k}\right) \frac{x^{s+a}\,\mathrm{d}s}{s\,(s+1)\cdots(s+a)}.$$

Writing the L-function as a Dirichlet series and switching the order of summation and integration yields

$$\widetilde{A}_{a,j}\left(x;\frac{h}{k}\right) = i^{-j}\,k\,\pi^{-3/2}\,x^a \sum_{d\mid k}\sum_{m=1}^{\infty}\frac{A(d,m)}{dm}\left(S\left(\overline{h},m;\frac{k}{d}\right)+(-1)^j\,S\left(\overline{h},-m;\frac{k}{d}\right)\right)$$
$$\times\frac{1}{2\pi i}\int\limits_{\mathscr{L}} G_j(s+j)\left(\frac{\pi^3\,d^2\,mx}{k^3}\right)^s\frac{\mathrm{d}s}{s\,(s+1)\cdots(s+a)}.$$

Invoking Lemma 5.2 finishes the proof. □

Let us then study the Riesz means of order $a\in\mathbb{Z}_+\cup\{0\}$ defined as

$$\widetilde{A}_a\left(x;\frac{h}{k}\right) := \frac{1}{2}\left(\widetilde{A}_{a,0}\left(x;\frac{h}{k}\right)+\widetilde{A}_{a,1}\left(x;\frac{h}{k}\right)\right)$$
$$= \frac{1}{a!}\sideset{}{'}\sum_{m\leqslant x} A(m,1)\,e\left(\frac{mh}{k}\right)(x-m)^a$$
$$-\frac{1}{2}\sum_{0\leqslant\nu\leqslant a}\frac{(-1)^\nu\,x^{a-\nu}}{\nu!\,(a-\nu)!}\left(L_0\left(-\nu,\frac{h}{k}\right)+L_1\left(-\nu+1,\frac{h}{k}\right)\right), \tag{7.1}$$

where $x\in[0,\infty[$, and h and k are coprime integers with k positive. We observe at once that these sums have the pleasant property that

$$\int\limits_0^x \widetilde{A}_a\left(t;\frac{h}{k}\right)\mathrm{d}t = \widetilde{A}_{a+1}\left(x;\frac{h}{k}\right)-\widetilde{A}_{a+1}\left(0;\frac{h}{k}\right)$$
$$= \widetilde{A}_{a+1}\left(x;\frac{h}{k}\right)+\frac{(-1)^{a+1}}{2\cdot(a+1)!}\left(L_0\left(-(a+1);\frac{h}{k}\right)+L_1\left(-(a+1)+1;\frac{h}{k}\right)\right).$$

This allows us to later relate Riesz means of different orders, and in particular to turn our better control of higher order Riesz means to information about lower order Riesz means. Of course, similar identities hold for $\widetilde{A}_{a,j}$ with $j\in\{0,1\}$.

The previous proposition, together with Lemma 5.1, immediately yield the following corollary.

Corollary 7.3. *Let $x\in[1,\infty[$, let $a\geqslant 2$ be an integer, and let h and k be coprime integers with $1\leqslant k\ll x^{1/3}$. Then*

$$\widetilde{A}_a\left(x;\frac{h}{k}\right)$$
$$=\frac{(-1)^a\,k^a\,x^{(2a+1)/3}}{(2\pi)^{a+1}\sqrt{3}}\sum_{d\mid k}\frac{1}{d^{(2a+1)/3}}\sum_{m=1}^{\infty}\frac{A(d,m)}{m^{(a+2)/3}}\sum_{\pm} i^{\pm a}\,S\left(\overline{h},\pm m;\frac{k}{d}\right)e\left(\pm\frac{3\,d^{2/3}\,m^{1/3}\,x^{1/3}}{k}\right)$$
$$+O\left(k^{(2a+3)/2}\,d(k)\,x^{2a/3}\right).$$

We record here the following upper bound.

Corollary 7.4. *Let $x\in[1,\infty[$, $j\in\{0,1\}$, $a\in\{2,3,\ldots\}$, and let h and k be coprime integers with $1\leqslant k\ll x^{1/3}$. Then*

$$\widetilde{A}_{a,j}\left(x;\frac{h}{k}\right)\ll k^{(2a+1)/2}\,d(k)\,x^{(2a+1)/3}.$$

In particular,

$$\widetilde{A}_{2,j}\left(x;\frac{h}{k}\right)\ll k^{5/2}\,d(k)\,x^{5/3}\qquad\text{and}\qquad\widetilde{A}_{3,j}\left(x;\frac{h}{k}\right)\ll k^{7/2}\,d(k)\,x^{7/3}.$$

Proof. Estimating the infinite series in Proposition 7.2 by absolute values and using the asymptotics from Lemma 5.1 gives

$$\begin{aligned}\widetilde{A}_{a,j}\Big(x;\frac{h}{k}\Big) &\ll k\,x^a \sum_{d|k}\sum_{m=1}^{\infty} \frac{|A(d,m)|}{dm}\Big(\frac{k}{d}\Big)^{1/2} d(k)\Big(\frac{d^{2/3}\,m^{1/3}\,x^{1/3}}{k}\Big)^{1-a}\\ &\ll k^{a+1/2}\,d(k)\,x^{2a/3+1/3}\sum_{d|k} d^{\vartheta-5/6-2a/3} \ll k^{a+1/2}\,d(k)\,x^{2a/3+1/3}. \qquad \square\end{aligned}$$

8. ADDITIVELY TWISTED VORONOI IDENTITIES FOR RIESZ MEANS: THE CASE $a = 1$

In the process of deriving a Voronoi summation formula for $\widetilde{A}_1(x;h/k)$ we need a reasonable upper bound for $\widetilde{A}_{1,j}(x;h/k)$. After the Voronoi formula is derived, we will use it to deduce a better upper bound, which can then be feed back to the argument used to derive the Voronoi formula in the first place, leading to an improved error term in the said formula.

Lemma 8.1. *Let $x \in [1,\infty[$, let $j \in \{0,1\}$, and let h and k be coprime integers so that $1 \leqslant k \ll x^{1/3}$. Then*

$$\widetilde{A}_{1,j}\Big(x;\frac{h}{k}\Big) \ll_\varepsilon k^{5/4}\,x^{29/24+\varepsilon}.$$

Proof. We choose $H = k^{5/4}\,x^{11/24}$. Then $1 \leqslant H \ll x$, and using Lemma 6.1 and Corollary 7.4 gives

$$\begin{aligned}\widetilde{A}_{1,j}\Big(x;\frac{h}{k}\Big) &= \frac{1}{H}\int_x^{x+H} \widetilde{A}_{1,j}\Big(x;\frac{h}{k}\Big)\,\mathrm{d}t\\ &= \frac{1}{H}\int_x^{x+H} \widetilde{A}_{1,j}\Big(t;\frac{h}{k}\Big)\,\mathrm{d}t - \frac{1}{H}\int_x^{x+H}\int_x^{t} \widetilde{A}_{0,j}\Big(u;\frac{h}{k}\Big)\,\mathrm{d}u\,\mathrm{d}t\\ &= \frac{1}{H}\bigg(\widetilde{A}_{2,j}\Big(x+H;\frac{h}{k}\Big) - \widetilde{A}_{2,j}\Big(x;\frac{h}{k}\Big)\bigg) - \frac{1}{H}\int_x^{x+H}\int_x^{t} \widetilde{A}_{0,j}\Big(u;\frac{h}{k}\Big)\,\mathrm{d}u\,\mathrm{d}t\\ &\ll \frac{1}{H}\,k^{5/2}d(k)\,x^{5/3} + H\,x^{3/4+\varepsilon} \ll_\varepsilon k^{5/4}\,x^{29/24+\varepsilon},\end{aligned}$$

as desired. $\square$

The proof of the case $a = 1$ gives rise to the sums of the kind treated in the above lemma, but with coefficients $A(m,d)$ instead of $A(m,1)$. The following lemma deals with this minor complication.

Lemma 8.2. *Let $x \in [1,\infty[$, let $j \in \{0,1\}$, let h and k be coprime integers satisfying $1 \leqslant k \ll x^{1/3}$, and let $d \in \mathbb{Z}_+$ be so that $d|k$. Then*

$$\sideset{}{'}\sum_{m\leqslant x} A(m,d)\bigg(e\Big(\frac{mhd}{k}\Big) + (-1)^j\,e\Big(-\frac{mhd}{k}\Big)\bigg)(x-m) \ll d^{\vartheta+\varepsilon}\,k^{5/4}\,x^{29/24+\varepsilon}.$$

Proof. We may compute

$$
\begin{aligned}
&\sideset{}{'}\sum_{m\leqslant x} A(m,d)\left(e\Big(\frac{mhd}{k}\Big)+(-1)^j\,e\Big(-\frac{mhd}{k}\Big)\right)(x-m)\\
&=\sideset{}{'}\sum_{m\leqslant x}\sum_{\ell\mid(d,m)}\mu(\ell)\,A\Big(1,\frac{d}{\ell}\Big)\,A\Big(\frac{m}{\ell},1\Big)\left(e\Big(\frac{mhd}{k}\Big)+(-1)^j\,e\Big(-\frac{mhd}{k}\Big)\right)(x-m)\\
&=\sum_{\substack{\ell\mid d\\ \ell\leqslant x}}\mu(\ell)\,A\Big(1,\frac{d}{\ell}\Big)\,\ell\sideset{}{'}\sum_{m\leqslant x/\ell}A(m,1)\left(e\Big(\frac{\ell mhd}{k}\Big)+(-1)^j\,e\Big(-\frac{\ell mhd}{k}\Big)\right)\Big(\frac{x}{\ell}-m\Big)\\
&=\sum_{\substack{\ell\mid d\\ \ell\leqslant x}}\mu(\ell)\,A\Big(1,\frac{d}{\ell}\Big)\,\ell\left(\widetilde{A}_{1,j}\Big(\frac{x}{\ell};\frac{\ell hd}{k}\Big)+O\bigg(\Big(\frac{k}{(k,\ell)}\Big)^{3/2+\varepsilon}\cdot\frac{x}{\ell}\bigg)+O\bigg(\Big(\frac{k}{(k,\ell)}\Big)^{9/2+\varepsilon}\bigg)\right)\\
&\ll_\varepsilon\sum_{\substack{\ell\mid d\\ \ell\leqslant x}}\Big(\frac{d}{\ell}\Big)^{\vartheta+\varepsilon}\,\ell\left(k^{5/4}\Big(\frac{x}{\ell}\Big)^{29/24+\varepsilon}+k^{3/2+\varepsilon}\cdot\frac{x}{\ell}+k^{9/2+\varepsilon}\right),
\end{aligned}
$$

where we have used the fact that since $k \ll x^{1/3}$, also $k/\ell \ll (x/\ell)^{1/3}$, so that Lemma 8.1 is applicable. Finally, since $k \ll x^{1/3}$, we have

$$k^{3/2+\varepsilon}\,x + k^{9/2+\varepsilon} \ll_\varepsilon k^{5/4}\,x^{29/24+\varepsilon},$$

and the total contribution is $\ll_\varepsilon d^{\vartheta+\varepsilon}\,k^{5/4}\,x^{29/24+\varepsilon}$. □

We will need to handle various sums involving both Fourier coefficients and Kloosterman sums. Taking discrete Fourier transforms of the Kloosterman sums allows us to reduce things back to exponential sums weighted by just the Fourier coefficients.

Lemma 8.3. *Let $h \in \mathbb{Z}$ and $k \in \mathbb{Z}_+$ be coprime, and define*

$$\widehat{S}(h,\xi;k) := \frac{1}{k}\sum_{\ell=1}^{k} S(h,\ell;k)\,e\Big(-\frac{\ell\xi}{k}\Big)$$

for every $\xi \in \mathbb{Z}$. Then, for any $m,\xi \in \mathbb{Z}$, we have

$$S(h,m;k) = \sum_{\xi=1}^{k}\widehat{S}(h,\xi;k)\,e\Big(\frac{m\xi}{k}\Big),$$

and

$$\widehat{S}(h,\xi;k) \ll k^{1/2}\,d(k),$$

as well as

$$\sum_{\xi=1}^{k}\big|\widehat{S}(h,\xi;k)\big| \ll k\,d(k).$$

Proof. The first claim follows directly from the discrete Fourier inversion formula, and the second follows directly from Weil's bound. The third follows from the Cauchy–Schwarz inequality, the discrete Parseval identity and Weil's bound since

$$\begin{aligned}\sum_{\xi=1}^{k}\bigl|\widehat{S}(h,\xi;k)\bigr| &\leqslant k^{1/2}\sqrt{\sum_{\xi=1}^{k}\bigl|\widehat{S}(h,\xi;k)\bigr|^2}\\ &= k^{1/2}\sqrt{\frac{1}{k}\sum_{\ell=1}^{k}\bigl|S(h,\ell;k)\bigr|^2}\\ &\ll k^{1/2}\sqrt{\frac{1}{k}\cdot k\left(k^{1/2}\,d(k)\right)^2} \ll k\,d(k). \qquad \square\end{aligned}$$

The argument used in the proof of Proposition 7.2 runs into difficulties in the case $a = 1$ due to convergence issues, and so we need to proceed differently. Our next goal is to establish the case $a = 1$ essentially by differentiating the formula for $a = 2$ termwise. The requisite convergence considerations give a practical truncated version of the summation formula as a by-product.

Proposition 8.4. *Let $x \in [1,\infty[$, $j \in \{0,1\}$, and let h and k be positive integers with $1 \leqslant k \ll x^{1/3}$. Then*

$$\begin{aligned}\widetilde{A}_{1,j}\left(x;\frac{h}{k}\right) &= i^{-j}\,2^{-1}\,\pi^{-3/2}\,kx\\ &\times\sum_{d\mid k}\sum_{m=1}^{\infty}\frac{A(d,m)}{dm}\left(S\Bigl(\overline{h},m;\frac{k}{d}\Bigr)+(-1)^j\,S\Bigl(\overline{h},-m;\frac{k}{d}\Bigr)\right)\mathscr{I}_{1,j}\left(\frac{\pi^6\,d^4\,m^2\,x^2}{k^6}\right).\end{aligned}$$

Here the infinite series converges both boundedly and uniformly when x is restricted to a compact interval in $\mathbb{R}_+$. Furthermore, if $N \in [1,\infty[$, and $N \gg k^3$, then

$$\begin{aligned}\widetilde{A}_{1,j}\left(x;\frac{h}{k}\right) &= i^{-j}\,2^{-1}\,\pi^{-3/2}\,kx\\ &\times\sum_{d\mid k}\sum_{m\leqslant N}\frac{A(d,m)}{dm}\left(S\Bigl(\overline{h},m;\frac{k}{d}\Bigr)+(-1)^j\,S\Bigl(\overline{h},-m;\frac{k}{d}\Bigr)\right)\mathscr{I}_{1,j}\left(\frac{\pi^6\,d^4\,m^2\,x^2}{k^6}\right) + \textit{error},\end{aligned}$$

where the error is

$$\ll_\varepsilon k^{5/4+\varepsilon}\,x^{5/3}\,N^{\varepsilon-1/8}.$$

Proof. Our starting point is the formula

$$\begin{aligned}\widetilde{A}_{2,j}\left(x;\frac{h}{k}\right) &= -i^{-j}\,2^{-2}\,\pi^{-3/2}\,k\,x^2\\ &\times\sum_{d\mid k}\sum_{m=1}^{\infty}\frac{A(d,m)}{dm}\left(S\Bigl(\overline{h},m;\frac{k}{d}\Bigr)+(-1)^j\,S\Bigl(\overline{h},-m;\frac{k}{d}\Bigr)\right)\mathscr{I}_{2,j}\left(\frac{\pi^6\,d^4\,m^2\,x^2}{k^6}\right),\end{aligned}$$

and our plan is to differentiate this identity. The derivative of the left-hand side is of course simply $\widetilde{A}_{1,j}(x;h/k)$. On the other hand, Lemma 5.3 tells us that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\,\mathscr{I}_{2,j}\left(\frac{\pi^6\,d^4\,m^2\,x^2}{k^6}\right)\right) = -2x\,\mathscr{I}_{1,j}\left(\frac{\pi^6\,d^4\,m^2\,x^2}{k^6}\right),$$

and using this it is straightforward to check that, formally, the derivative of the right-hand side coincides with the right-hand side of the desired identity for $\widetilde{A}_{1,j}(x;h/k)$. From the analysis of the previous section it is clear that the above series for $\widetilde{A}_{2,j}(x;h/k)$ converges for any x. Thus, by standard arguments, it only remains to prove that the series for $\widetilde{A}_{1,j}(x;h/k)$ converges boundedly and uniformly, when x is restricted to a bounded interval.

We take arbitrary numbers $a, b \in [1, \infty[$ satisfying $a < b$ and $a \gg k^3$, and consider the partial sum

$$\Sigma(a,b) := kx \sum_{d|k} \sum_{a \leqslant m \leqslant b} \frac{A(d,m)}{dm} \left(S\left(\overline{h}, m; \frac{k}{d}\right) + (-1)^j \, S\left(\overline{h}, -m; \frac{k}{d}\right) \right) \mathscr{I}_{1,j}\left(\frac{\pi^6 \, d^4 \, m^2 \, x^2}{k^6} \right).$$

By Lemma 5.1, we have

$$\mathscr{I}_{1,j}\left(\frac{\pi^6 \, d^4 \, m^2 \, x^2}{k^6} \right) = -\frac{1}{2\sqrt{3\pi}} \sum_{\pm} \exp\left(\pm \frac{6i\pi \, d^{2/3} \, m^{1/3} \, x^{1/3}}{k} \pm \frac{\pi i}{2} \, (1+j) \right) + O\left(\left(d^{2/3} \, m^{1/3} \, x^{1/3} \, k^{-1} \right)^{-1} \right).$$

We may estimate the contribution of the O-term in $\Sigma(a,b)$ by

$$\ll kx \sum_{d|k} \sum_{a \leqslant m \leqslant b} \frac{|A(d,m)|}{dm} \left(\frac{k}{d} \right)^{1/2+\varepsilon} \left(d^{2/3} \, m^{1/3} \, x^{1/3} \, k^{-1} \right)^{-1} \ll_{\varepsilon} k^{5/2+\varepsilon} \, x^{2/3} \, a^{-1/3}.$$

We also write in each term of $\Sigma(a,b)$

$$S\left(\overline{h}, m; \frac{k}{d}\right) + (-1)^j \, S\left(\overline{h}, -m; \frac{k}{d}\right) = \sum_{\xi=1}^{k/d} \widehat{S}\left(\overline{h}, \xi; \frac{k}{d}\right) \left(e\left(\frac{m\xi d}{k} \right) + (-1)^j \, e\left(-\frac{m\xi d}{k} \right) \right).$$

Combining all these observations gives

$$\Sigma(a,b) = -\frac{1}{2\sqrt{3\pi}} \sum_{\pm} e\left(\pm \frac{\pi i}{2} \, (1+j) \right) \sum_{d|k} \frac{1}{d} \sum_{\xi=1}^{k/d} \widehat{S}\left(\overline{h}, \xi; \frac{k}{d}\right) kx \, \Sigma_{\pm}(a,b) + O(k^{5/2+\varepsilon} \, x^{2/3} \, a^{-1/3}),$$

where

$$\Sigma_{\pm}(a,b) := \sum_{a \leqslant m \leqslant b} \frac{A(d,m)}{m} \left(e\left(\frac{m\xi d}{k} \right) + (-1)^j \, e\left(-\frac{m\xi d}{k} \right) \right) e\left(\pm \frac{3 \, d^{2/3} \, m^{1/3} \, x^{1/3}}{k} \right).$$

To simplify the formulae below, we shall write, for each $\ell \in \{0, 1\}$, and all $x \in [1, \infty[$,

$$A_{\ell,j,d}\left(x; \frac{\xi}{k} \right) := \sum_{m \leqslant x} A(d,m) \left(e\left(\frac{m\xi d}{k} \right) + (-1)^j \, e\left(-\frac{m\xi d}{k} \right) \right) (x-m)^{\ell}.$$

We start the estimation of $\Sigma_{\pm}(a,b)$ by summing by parts getting

$$\begin{aligned} \Sigma_{\pm}(a,b) = {} & \frac{1}{t} \, e\left(\pm \frac{3 \, d^{2/3} \, t^{1/3} \, x^{1/3}}{k} \right) A_{0,j,d}\left(t; \frac{\xi}{k} \right) \Bigg]_{a}^{t=b} \\ & - \int_a^b A_{0,j,d}\left(t; \frac{\xi}{k} \right) \left(-\frac{1}{t^2} \pm \frac{2\pi i \, d^{2/3} \, x^{1/3}}{k \, t^{5/3}} \right) e\left(\pm \frac{3 \, d^{2/3} \, t^{1/3} \, x^{1/3}}{k} \right) \mathrm{d}t. \end{aligned}$$

By Lemma 6.2, the substitution term is $\ll_{\varepsilon} d^{\vartheta+\varepsilon} \, a^{\varepsilon-1/4}$, as is the contribution from the term involving $1/t^2$, and it only remains to estimate the term involving $t^{-5/3}$. We do this by integration by parts to get

$$\begin{aligned} & \int_a^b A_{0,j,d}\left(t; \frac{\xi}{k} \right) \frac{2\pi i \, d^{2/3} \, x^{1/3}}{k \, t^{5/3}} \, e\left(\pm \frac{3 \, d^{2/3} \, t^{1/3} \, x^{1/3}}{k} \right) \mathrm{d}t \\ & = A_{1,j,d}\left(t; \frac{\xi}{k} \right) \frac{2\pi i \, d^{2/3} \, x^{1/3}}{k \, t^{5/3}} \, e\left(\pm \frac{3 \, d^{2/3} \, t^{1/3} \, x^{1/3}}{k} \right) \Bigg]_{a}^{t=b} \\ & \quad - \int_a^b A_{1,j,d}\left(t; \frac{\xi}{k} \right) \left(-\frac{10 \, \pi i \, d^{2/3} \, x^{1/3}}{3 \, k \, t^{8/3}} \pm \frac{4 \, \pi^2 \, d^{4/3} \, x^{2/3}}{k^2 \, t^{7/3}} \right) e\left(\pm \frac{d^{2/3} \, t^{1/3} \, x^{1/3}}{k} \right) \mathrm{d}t. \end{aligned}$$

By Lemma 8.2, the substitution terms and the term involving $t^{-8/3}$ contribute

$$\ll_\varepsilon d^{2/3+\vartheta+\varepsilon}\, x^{1/3}\, k^{1/4}\, a^{\varepsilon-11/24},$$

whereas the contribution from the term involving $t^{-7/3}$ is

$$\ll_\varepsilon d^{4/3+\vartheta+\varepsilon}\, k^{-3/4}\, x^{2/3}\, a^{\varepsilon-1/8}.$$

Altogether, our estimate for $\Sigma_\pm(a,b)$ reads

$$\Sigma_\pm(a,b) \ll d^{\vartheta+\varepsilon}\, a^{\varepsilon-1/4} + d^{2/3+\vartheta+\varepsilon}\, x^{1/3}\, k^{1/4}\, a^{\varepsilon-11/24} + d^{4/3+\vartheta+\varepsilon}\, k^{-3/4}\, x^{2/3}\, a^{\varepsilon-1/8}.$$

Plugging this into our previous expression for $\Sigma(a,b)$ gives

$$\begin{aligned}\Sigma(a,b) &\ll \sum_{d\mid k}\frac{1}{d}\sum_{\xi=1}^{k/d}\left|\widehat{S}\left(\overline{h},\xi;\frac{k}{d}\right)\right| kx \\ &\qquad \times\left(d^{\vartheta+\varepsilon}\, a^{\varepsilon-1/4} + d^{2/3+\vartheta+\varepsilon}\, x^{1/3}\, k^{1/4}\, a^{\varepsilon-11/24} + d^{4/3+\vartheta+\varepsilon}\, k^{-3/4}\, x^{2/3}\, a^{\varepsilon-1/8}\right) \\ &\qquad + k^{5/2+\varepsilon}\, x^{2/3}\, a^{-1/3} \\ &\ll k^{2+\varepsilon}\, x\left(a^{\varepsilon-1/4} + x^{1/3}\, k^{1/4}\, a^{\varepsilon-11/24} + k^{-3/4}\, x^{2/3}\, a^{\varepsilon-1/8}\right) \\ &\qquad + k^{5/2+\varepsilon}\, x^{2/3}\, a^{-1/3} \\ &\ll_\varepsilon k^{2+\varepsilon}\, x\, a^{\varepsilon-1/4} + k^{9/4+\varepsilon}\, x^{4/3}\, a^{\varepsilon-11/24} + k^{5/4+\varepsilon}\, x^{5/3}\, a^{\varepsilon-1/8} + k^{5/2+\varepsilon}\, x^{2/3}\, a^{-1/3},\end{aligned}$$

and we are don with the first assertion. For the truncated formula we observe that

$$k^{2+\varepsilon}\, x\, a^{\varepsilon-1/4} \ll_\varepsilon k^{5/4+\varepsilon}\, x^{5/3}\, a^{\varepsilon-1/8},$$

and that

$$k^{5/2+\varepsilon}\, x^{2/3}\, a^{-1/3} \ll_\varepsilon k^{5/4+\varepsilon}\, x^{5/3}\, a^{\varepsilon-1/8},$$

as well as

$$k^{9/4+\varepsilon}\, x^{4/3}\, a^{\varepsilon-11/24} \ll_\varepsilon k^{5/4+\varepsilon}\, x^{5/3}\, a^{\varepsilon-1/8},$$

since $k \ll x^{1/3}$. Thus we have obtained

$$\Sigma(a,b) \ll_\varepsilon k^{5/4+\varepsilon} x^{5/3} a^{\varepsilon-1/8},$$

which immediately gives the truncated identity. □

From the above formula, we get directly a better upper bound for $\widetilde{A}_{1,j}(x;h/k)$.

Corollary 8.5. *Let $x \in [1,\infty[$, let $j \in \{0,1\}$, and let h and k be coprime integers with $1 \leqslant k \ll x^{1/3}$. Then*

$$\widetilde{A}_{1,j}\left(x;\frac{h}{k}\right) \ll_\varepsilon k^{3/2}\, x^{1+\varepsilon}.$$

Furthermore, if $d \in \mathbb{Z}_+$ and $d \mid k$, then also

$$\sum_{m\leqslant x} A(m,d)\left(e\left(\frac{mhd}{k}\right) + (-1)^j\, e\left(-\frac{mhd}{k}\right)\right)(x-m) \ll_\varepsilon d^{\vartheta+\varepsilon}\, k^{3/2}\, x^{1+\varepsilon}.$$

Proof. For the first estimate, we employ the second identity of Proposition 8.4 with the choice $N = \lfloor x^{16/3}k^{-2}\rfloor$, estimating all the terms by absolute values and using Lemma 5.1. The second estimate is proved in exactly the same way as Lemma 8.2. □

We can now feed this back into the estimation of $\Sigma(a,b)$ in the proof of Proposition 8.4, where we used the weaker estimate from Lemma 8.2 (which in turn was based on Lemma 8.1). The resulting estimate for $\Sigma_\pm(a,b)$ is

$$\Sigma_\pm(a,b) \ll_\varepsilon d^{\vartheta+\varepsilon}\, a^{\varepsilon-1/4} + d^{2/3+\vartheta+\varepsilon}\, k^{1/2}\, x^{1/3}\, a^{\varepsilon-2/3} + d^{4/3+\vartheta+\varepsilon}\, k^{-1/2}\, x^{2/3}\, a^{\varepsilon-1/3},$$

and the end result is

$$\Sigma(a,b) \ll k^{2+\varepsilon}\, x\, a^{\varepsilon-1/4} + k^{5/2+\varepsilon}\, x^{4/3}\, a^{\varepsilon-2/3} + k^{3/2+\varepsilon}\, x^{5/3}\, a^{\varepsilon-1/3} + k^{5/2+\varepsilon}\, x^{2/3}\, a^{-1/3}$$
$$\ll_\varepsilon k^{2+\varepsilon}\, x\, a^{\varepsilon-1/4} + k^{3/2+\varepsilon}\, x^{5/3}\, a^{\varepsilon-1/3}.$$

Thus we have obtained the following.

Corollary 8.6. *Let $x \in [1,\infty[$, $j \in \{0,1\}$, let h and k be positive integers with $1 \leqslant k \ll x^{1/3}$, and let $N \in [1,\infty[$ with $N \gg k^3$. Then*

$$\widetilde{A}_{1,j}\left(x; \frac{h}{k}\right) = i^{-j}\, 2^{-1}\, \pi^{-3/2}\, kx$$
$$\times \sum_{d \mid k} \sum_{m \leqslant N} \frac{A(d,m)}{dm} \left(S\left(\overline{h}, m; \frac{k}{d}\right) + (-1)^j\, S\left(\overline{h}, -m; \frac{k}{d}\right)\right) \mathscr{I}_{1,j}\left(\frac{\pi^6\, d^4\, m^2\, x^2}{k^6}\right) + \textit{error},$$

where the error is

$$\ll_\varepsilon k^{2+\varepsilon}\, x\, N^{\varepsilon-1/4} + k^{3/2+\varepsilon}\, x^{5/3}\, N^{\varepsilon-1/3}.$$

Proposition 8.4 and Corollary 8.6 yield the following result when combined with averaging over $j \in \{0,1\}$ and Lemma 5.1.

Corollary 8.7. *Let $x \in [1,\infty[$, $j \in \{0,1\}$, and let h and k be positive integers with $1 \leqslant k \ll x^{1/3}$. Then*

$$\widetilde{A}_1\left(x; \frac{h}{k}\right) = \frac{k\, x}{4\,\pi^2\sqrt{3}} \sum_{d \mid k} \frac{1}{d} \sum_{m=1}^{\infty} \frac{A(d,m)}{m} \sum_{\pm} (\mp i)\; S\left(\overline{h}, \pm m; \frac{k}{d}\right) e\left(\pm \frac{3\, d^{2/3}\, m^{1/3}\, x^{1/3}}{k}\right)$$
$$+ O_\varepsilon\left(k^{5/2+\varepsilon}\, x^{2/3}\right).$$

Here the infinite series converges both boundedly and uniformly when x is restricted to a compact interval in $\mathbb{R}_+$. Furthermore, if $N \in [1,\infty[$, and $N \gg k^{-3}\, x^3$, then

$$\widetilde{A}_1\left(x; \frac{h}{k}\right) = \frac{k\, x}{4\,\pi^2\sqrt{3}} \sum_{d \mid k} \frac{1}{d} \sum_{m \leqslant N} \frac{A(d,m)}{m} \sum_{\pm} (\mp i)\; S\left(\overline{h}, \pm m; \frac{k}{d}\right) e\left(\pm \frac{3\, d^{2/3}\, m^{1/3}\, x^{1/3}}{k}\right)$$
$$+ O_\varepsilon\left(k^{3/2+\varepsilon}\, x^{5/3+\varepsilon} N^{\varepsilon-1/3}\right).$$

9. Pointwise Ω-result for long sums on GL_3

Let us first gather some auxiliary results which will be useful in what follows. We start with results concerning Kloosterman sums.

Lemma 9.1. *Let k be a prime. Suppose also that m and n are integers. Then*

$$\sum_{a \in \mathbb{Z}_k^\times} S(a,m;k) = \begin{cases} 1 & \text{if } m \not\equiv 0 \pmod{k} \\ -k+1 & \text{if } m \equiv 0 \pmod{k} \end{cases}$$

as well as

$$\Sigma(m,n,k) := \sum_{a \in \mathbb{Z}_k^\times} S(a,m;k)\, S(a,n;k) = \begin{cases} k^2-k-1 & \text{if } m \equiv n \not\equiv 0 \pmod{k} \\ k-1 & \text{if } m \equiv n \equiv 0 \pmod{k} \\ -1 & \text{if } m \equiv 0 \text{ and } n \not\equiv 0 \pmod{k} \\ -1 & \text{if } m \not\equiv 0 \text{ and } n \equiv 0 \pmod{k} \\ -k-1 & \text{if } 0 \not\equiv m \not\equiv n \not\equiv 0 \pmod{k} \end{cases}$$

Proof. The first cases of both identities are treated in Chapter 4 of [20]. The rest are then fairly straightforward to prove simply by expanding the Kloosterman sums in terms of their definitions and taking advantage of the resulting geometric series. □

We will use the first derivative test repeatedly and record one of its formulations here. Results such as this are discussed for instance in Section 5.1 of [15].

Lemma 9.2. *Let $a, b \in \mathbb{R}$ with $a < b$, let $\lambda \in \mathbb{R}_+$, and let f be a real-valued continuously differentiable function on $]a, b[$ such that $|f'(x)| \geqslant \lambda$ for $x \in]a, b[$. Also, let g be a complex-valued continuously differentiable function on the interval $[a, b]$, and let $G \in \mathbb{R}_+$ be such that $g(x) \ll G$ for $x \in [a, b]$. Then*

$$\int_a^b g(x)\, e(f(x))\, \mathrm{d}x \ll \frac{G}{\lambda} + \frac{1}{\lambda} \int_a^b |g'(x)|\, \mathrm{d}x.$$

Now we are in a position to prove an asymptotic formula for the averaged mean-square for the long second order Riesz sums.

Theorem 9.3. *Suppose that $X \in [1, \infty[$ and that k is a prime such that $k \ll X^{1/3}$ with sufficiently small implied constant. Then we have*

$$\mathop{\mathbb{E}}_{h \in \mathbb{Z}_k^\times} \int_X^{2X} \left| \widetilde{A}_2\left(x; \frac{h}{k}\right) \right|^2 \mathrm{d}x = B(k)\, k^5\, X^{13/3} + O\left(k^6\, X^4\right),$$

where $B(k) \in \mathbb{R}_+$ and $B(k) \asymp 1$.

In particular, if $x \in [1, \infty[$ and $k \in \mathbb{Z}_+$ so that $k \ll X^{1/3}$ with sufficiently small implied constant, then we have

$$\max_{h \in \mathbb{Z}_k^\times} \left| \widetilde{A}_2\left(x; \frac{h}{k}\right) \right| = \Omega\left(k^{5/2}\, x^{5/3}\right).$$

Proof. Note that the Ω-result follows immediately from the moment result. For the first statement, using Corollary 7.3 the second moment is

$$\begin{aligned}
&\mathop{\mathbb{E}}_{h \in \mathbb{Z}_k^\times} \int_X^{2X} \left| \widetilde{A}_2\left(x; \frac{h}{k}\right) \right|^2 \mathrm{d}x \\
&= \frac{k^4}{192\,\pi^6} \mathop{\mathbb{E}}_{h \in \mathbb{Z}_k^\times} \int_X^{2X} x^{10/3} \left| \sum_{d \mid k} \sum_{m=1}^\infty \frac{A(d,m)}{d^{5/3}\, m^{4/3}} \sum_\pm S\left(\overline{h}, \pm m; \frac{k}{d}\right) e\left(\pm \frac{3\, d^{2/3}\, m^{1/3}\, x^{1/3}}{k}\right) \right|^2 \mathrm{d}x \\
&\quad - \frac{k^2}{8\pi^3\sqrt{3}} \operatorname{Re} \mathop{\mathbb{E}}_{h \in \mathbb{Z}_k^\times} \int_X^{2X} x^{5/3} \sum_{d \mid k} \sum_{m=1}^\infty \frac{A(d,m)}{d^{5/3}\, m^{4/3}} \\
&\qquad\qquad \times \sum_\pm S\left(\overline{h}, \pm m; \frac{k}{d}\right) e\left(\pm \frac{3\, d^{2/3}\, m^{1/3}\, x^{1/3}}{k}\right) \cdot O(k^{7/2}\, x^{4/3})\, \mathrm{d}x \\
&\quad + O(k^7\, X^{11/3}).
\end{aligned}$$

Once we have proved that the first term on the right-hand side is $\asymp k^5\, X^{13/3}$, it immediately follows from the Cauchy–Schwarz inequality that the mixed term integral is

$$\ll \sqrt{k^5\, X^{13/3}}\, \sqrt{k^7\, X^{11/3}} \ll k^6\, X^4,$$

as required. Also, since $k \ll X^{1/3}$, we have $k^7\, X^{11/3} \ll k^6\, X^4$.

In the first integral, we expand $|\Sigma|^2$ as $\Sigma\, \overline{\Sigma}$. The diagonal terms, i.e. those terms where $d_1^2\, m_1 = d_2^2\, m_2$ and where the signs in sums $\sum_\pm$ are chosen to be equal, give the contribution

$$\frac{k^4}{192\pi^6} \sum_{d_1 \mid k} \sum_{d_2 \mid k} \mathop{\sum_{m_1=1}^\infty \sum_{m_2=1}^\infty}_{d_1^2 m_1 = d_2^2 m_2} \frac{A(d_1, m_1)\, \overline{A(d_2, m_2)}}{d_1^{5/3}\, d_2^{5/3}\, m_1^{4/3}\, m_2^{4/3}} \mathop{\mathbb{E}}_{h \in \mathbb{Z}_k^\times} \sum_\pm S\left(\overline{h}, \pm m_1; \frac{k}{d_1}\right) S\left(\overline{h}, \pm m_2; \frac{k}{d_2}\right) \int_X^{2X} x^{10/3}\, \mathrm{d}x.$$

We will examine the contribution of the different pairs (d_1, d_2) separately at first. The terms where $d_1 = d_2 = 1$ contribute

$$\frac{k^4}{96\pi^6}\sum_{m=1}^{\infty}\frac{\left|A(1,m)\right|^2}{m^{8/3}}\left(\frac{k^2-k-1}{k-1}1_{m\not\equiv 0\,(k)}+1_{m\equiv 0\,(k)}\right)\int_X^{2X} x^{10/3}\,\mathrm{d}x \tag{9.1}$$

by Lemma 9.1.

Similarly, the terms with $d_1 = d_2 = k$ contribute

$$\frac{k^4}{96\pi^6}\sum_{m=1}^{\infty}\frac{\left|A(k,m)\right|^2}{k^{10/3}m^{8/3}}\int_X^{2X} x^{10/3}\,\mathrm{d}x \tag{9.2}$$

and the terms where $d_1d_2 = k$ contribute

$$-\frac{k^4}{48\pi^6}\mathrm{Re}\left(\sum_{m=1}^{\infty}\frac{A(1,k^2m)\overline{A(k,m)}}{k^{13/3}m^{8/3}}\right)\int_X^{2X} x^{10/3}\,\mathrm{d}x. \tag{9.3}$$

Our aim is to show that these three terms sum up to $B(k)k^5X^{13/3}$, where $B(k) \in \mathbb{R}$ with $B(k) \asymp 1$. The upper bound is straightforward to establish just by using the known estimates for the Fourier coefficients (see (4.1)), but the lower bound requires some work. Towards this, observe first that by the arithmetic-geometric mean inequality, the absolute value of (9.3) is bounded from the above by

$$\begin{aligned}
&\frac{k^4}{96\pi^6}\sum_{m=1}^{\infty}\frac{\left|A(1,k^2m)\right|^2}{k^{16/3}m^{8/3}}\int_X^{2X} x^{10/3}\,\mathrm{d}x+\frac{k^4}{96\pi^6}\sum_{m=1}^{\infty}\frac{\left|A(k,m)\right|^2}{k^{10/3}m^{8/3}}\int_X^{2X} x^{10/3}\,\mathrm{d}x\\
&\leq \frac{k^4}{96\pi^6}\sum_{m=1}^{\infty}\frac{\left|A(1,m)\right|^2}{m^{8/3}}\int_X^{2X} x^{10/3}\,\mathrm{d}x+\frac{k^4}{96\pi^6}\sum_{m=1}^{\infty}\frac{\left|A(k,m)\right|^2}{k^{10/3}m^{8/3}}\int_X^{2X} x^{10/3}\,\mathrm{d}x.
\end{aligned} \tag{9.4}$$

Notice that the latter term in (9.4) is exactly (9.2). Thus, using the identity $1_{m\not\equiv 0(k)} = 1 - 1_{m\equiv 0(k)}$, a simple computation shows that the sum of (9.1), (9.2) and (9.3) is bounded from the below by

$$\frac{k^4}{96\pi^6}\cdot\frac{k^2-2k}{k-1}\sum_{m=1}^{\infty}\frac{|A(m,1)|^2}{m^{8/3}}\left(1-1_{m\equiv 0\,(k)}\right)\int_X^{2X} x^{10/3}\,\mathrm{d}x.$$

The first term in the parenthesis gives the required lower bound $\gg k^5X^{13/3}$ using partial summation, whereas the contribution of the second term gives a smaller contribution, say, $\ll k^{11/3}X^{13/3}$ just by trivial estimation along with the pointwise bound $A(mk,1) \ll_\varepsilon (mk)^{\vartheta+\varepsilon}$.

The contribution coming from the terms $d_1^2m = d_2^2n$, where the signs in $\sum_\pm$ are chosen to be distinct, can be estimated by using the first-derivative test together with Weil's bound as

$$\begin{aligned}
&\ll k^4\sum_{\pm}\sum_{d_1|k}\sum_{d_2|k}\sum_{m=1}^{\infty}\sum_{\substack{n=1\\ d_1^2m=d_2^2n}}^{\infty}\frac{\left|A(d_1,m)\overline{A(d_2,n)}\right|}{d_1^{5/3}d_2^{5/3}m^{4/3}n^{4/3}}S\left(\overline{h},\pm m;k/d_1\right)S\left(\overline{h},\mp n;k/d_2\right)\\
&\qquad\times\int_X^{2X} x^{10/3}\,e\!\left(\pm\frac{6\pi(d_1^2mx)^{1/3}}{k}\right)\mathrm{d}x\\
&\ll k^4\sum_{m=1}^{\infty}\frac{\left|A(1,m)\right|^2}{m^{8/3}}k\,X^{10/3}\,\frac{X^{2/3}k}{m^{1/3}}\ll X^4k^6,
\end{aligned}$$

as the largest contribution comes from the terms where $d_1 = d_2 = 1$ (when $d_1d_2 \neq 1$ one gets a smaller contribution by estimating trivially).

In the same spirit, the off-diagonal terms contribute, using again the first derivative test and Weil's bound,

$$\ll k^4 \sum_{\pm} \sum_{d_1|k} \sum_{d_2|k} \sum_{m=1}^{\infty} \frac{|A(d_1,m)|}{m^{4/3} d_1^{5/3}} \sum_{\substack{n=1 \\ d_1^2 m \neq d_2^2 n}}^{\infty} \frac{|A(d_2,n)|}{n^{4/3} d_2^{5/3}} S\left(\overline{h}, \pm m; \frac{k}{d_1}\right) S\left(\overline{h}, \pm n; \frac{k}{d_2}\right)$$

$$\times \int_X^{2X} x^{10/3}\, e\left(\frac{\pm 3\pi (m x d_1^2)^{1/3} \pm 3\pi (n x d_2^2)^{1/3}}{k}\right) \mathrm{d}x$$

$$\ll k^4 \sum_{d_1|k} \sum_{d_2|k} \sum_{m=1}^{\infty} \frac{|A(d_1,m)|}{m^{4/3} d_1^{5/3}} \sum_{\substack{n=1 \\ d_1^2 m \neq d_2^2 n}}^{\infty} \frac{|A(d_2,n)|}{n^{4/3} d_2^{5/3}} \cdot \frac{X^4 k}{\left|d_1^{2/3} m^{1/3} - d_2^{2/3} n^{1/3}\right|} S\left(\overline{h}, \pm m; \frac{k}{d_1}\right) S\left(\overline{h}, \pm n; \frac{k}{d_2}\right)$$

$$\ll X^4 k^5 \sum_{d_1|k} \sum_{d_2|k} \sum_{m=1}^{\infty} \frac{|A(d_1,m)|}{m^{4/3} d_1^{5/3}} \sum_{d_2 n > d_1 m} \frac{n^{\vartheta+\varepsilon} d_2^{\vartheta+\varepsilon}}{n^{4/3}} \cdot \frac{n^{2/3} d_2^{1/3}}{d_2 n - d_1 m} S\left(\overline{h}, \pm m; \frac{k}{d_1}\right) S\left(\overline{h}, \pm n; \frac{k}{d_2}\right)$$

$$\ll X^4 k^6 \sum_{m=1}^{\infty} \frac{|A(1,m)|}{m^{4/3}} \sum_{n>m} \frac{1}{(n-m)^{1+\varepsilon}} \ll X^4 k^6.$$

and we are done. □

Next we present the first proof for Theorem 2.1. The idea is to relate upper bounds of different Riesz weighted sums. We first connect the sizes of $\widetilde{A}_1(x; h/k)$ and $\widetilde{A}_2(x; h/k)$.

Proposition 9.4. *Let k be a prime number. Let $\gamma \in [1,2]$ and $\eta \in [1,2]$ be such that*

$$\widetilde{A}_1\left(x; \frac{h}{k}\right) \ll x^{\gamma} k^{\eta}.$$

for $k \ll x^{1/3}$. Then we have

$$\widetilde{A}_2\left(x; \frac{h}{k}\right) \ll x^{7/6+\gamma/2} k^{7/4+\eta/2}.$$

Furthermore, we have

$$\max_{h \in \mathbb{Z}_k^{\times}} \left|\widetilde{A}_1\left(x; \frac{h}{k}\right)\right| = \Omega\left(x k^{3/2}\right)$$

for any prime $k \ll x^{1/3}$ with sufficiently small implied constant.

Proof. For any $H > 0$ we can write

$$\widetilde{A}_2\left(x; \frac{h}{k}\right) = \frac{1}{H} \int_x^{x+H} \widetilde{A}_2\left(x; \frac{h}{k}\right) \mathrm{d}t = \frac{1}{H} \int_x^{x+H} \widetilde{A}_2\left(t; \frac{h}{k}\right) \mathrm{d}t - \frac{1}{H} \int_x^{x+H} \int_x^t \widetilde{A}_1\left(u; \frac{h}{k}\right) \mathrm{d}u\, \mathrm{d}t. \tag{9.5}$$

Substituting to the first term the Voronoi identity for $\widetilde{A}_2(x; h/k)$, we may estimate it by the first derivative test as

$$\frac{1}{H} \int_x^{x+H} \widetilde{A}_2\left(t; \frac{h}{k}\right) \mathrm{d}t \ll \frac{1}{H} x^{7/3} k^{7/2} + k^{7/2} x^{4/3}.$$

Now, if $\widetilde{A}_1(x, h/k) \ll x^{\gamma} k^{\eta}$, then the second term on the right-hand side of (9.5) is $\ll H\, x^{\gamma}\, k^{\eta}$, and choosing $H = x^{7/6-\gamma/2} k^{7/4-\eta/2}$ immediately gives

$$\widetilde{A}_2\left(x; \frac{h}{k}\right) \ll x^{7/6+\gamma/2} k^{7/4+\eta/2}.$$

To prove the second statement, note that from Theorem 9.3 and (9.5) it follows that

$$\Omega\left(x^{5/3} k^{5/2}\right) = \max_{h \in \mathbb{Z}_k^{\times}} \left|\widetilde{A}_2\left(x; \frac{h}{k}\right)\right| \ll \frac{1}{H} x^{7/3} k^{7/2} + H \max_{u \in [x, x+H]} \max_{h \in \mathbb{Z}_k^{\times}} \left|\widetilde{A}_1\left(u; \frac{h}{k}\right)\right|.$$

Thus for any $H \leq x$ with $x^{2/3}k = o(H)$ we have

$$\max_{h \in \mathbb{Z}_k^\times} \left| \widetilde{A}_1\left(x; \frac{h}{k}\right) \right| = \Omega\left(H^{-1} x^{5/3} k^{5/2}\right),$$

from which the claim follows. □

Lemma 9.5. *Let $x \in [1, \infty[$, $H \in [1, x]$, and $k \ll x^{1/3}$ prime. Then*

$$\frac{1}{H} \int_x^{x+H} \widetilde{A}_1\left(t; \frac{h}{k}\right) \mathrm{d}t \ll H^{-1}\, x^{5/3} k^{5/2}.$$

Proof. We have

$$\frac{1}{H} \int_x^{x+H} \widetilde{A}_1\left(t; \frac{h}{k}\right) \mathrm{d}t = \frac{1}{H}\left(\widetilde{A}_2\left(x+H; \frac{h}{k}\right) - \widetilde{A}_2\left(x; \frac{h}{k}\right)\right),$$

and by Corollary 7.4, this is $\ll H^{-1}\, x^{5/3} k^{5/2}$, where there is no ε in the exponent of k as the denominator is a prime. □

Proposition 9.4 together with the first part of the following result proves Theorem 2.1, a result which is formulated as the latter part.

Theorem 9.6. *Let $\alpha \in \,]0, 1]$ and $\beta \in]0, 1]$ be such that*

$$\sum_{m \leqslant x} A(m, 1) e\left(\frac{mh}{k}\right) \ll x^\alpha k^\beta \tag{9.6}$$

for $x \in [1, \infty[$. Suppose that k is a prime so that $k \ll x^{1/3}$. Then

$$\widetilde{A}_1\left(x; \frac{h}{k}\right) \ll x^{5/6+\alpha/2} k^{5/4+\beta/2}.$$

Furthermore, when $x \longrightarrow \infty$, we have

$$\max_{h \in \mathbb{Z}_k^\times} \left| \sum_{m \leqslant x} A(m, 1) e\left(\frac{mh}{k}\right) \right| = \Omega\left(x^{1/3} k^{1/2}\right)$$

for any prime $k \ll x^{1/3}$ with sufficiently small implied constant.

Proof. We may write

$$\widetilde{A}_1\left(x; \frac{h}{k}\right) = \frac{1}{H} \int_x^{x+H} \widetilde{A}_1\left(x; \frac{h}{k}\right) \mathrm{d}t = \frac{1}{H} \int_x^{x+H} \widetilde{A}_1\left(t; \frac{h}{k}\right) \mathrm{d}t - \frac{1}{H} \int_x^{x+H} \int_x^{t} \widetilde{A}_0\left(u; \frac{h}{k}\right) \mathrm{d}u\, \mathrm{d}t. \tag{9.7}$$

For the first term, we already know by Lemma 9.5 that

$$\frac{1}{H} \int_x^{x+H} \widetilde{A}_1\left(t; \frac{h}{k}\right) \mathrm{d}t \ll H^{-1}\, x^{5/3} k^{5/2}.$$

On the other hand, from (9.6) we certainly have $\widetilde{A}_0(x; h/k) \ll x^\alpha k^\beta$ and

$$\frac{1}{H} \int_x^{x+H} \int_x^{t} \widetilde{A}_0\left(u; \frac{h}{k}\right) \mathrm{d}u\, \mathrm{d}t \ll H\, x^\alpha k^\beta.$$

Choosing $H = x^{5/6-\alpha/2} k^{5/4-\beta/2}$ gives

$$\widetilde{A}_1\left(x; \frac{h}{k}\right) \ll x^{5/6+\alpha/2} k^{5/4+\beta/2}.$$

To prove the second statement, note that from Proposition 9.4 and (9.7) it follows that

$$\Omega\left(xk^{3/2}\right) = \max_{h\in\mathbb{Z}_k^\times}\left|\widetilde{A}_1\left(x;\frac{h}{k}\right)\right| \ll \frac{1}{H}\,x^{5/3}k^{5/2} + H \max_{u\in[x,x+H]}\max_{h\in\mathbb{Z}_k^\times}\left|\widetilde{A}_0\left(u;\frac{h}{k}\right)\right|.$$

Thus for any $H \le x$ with $x^{2/3}k = o(H)$ we have

$$\max_{h\in\mathbb{Z}_k^\times}\left|\widetilde{A}_0\left(x;\frac{h}{k}\right)\right| = \Omega\left(H^{-1}xk^{3/2}\right),$$

from which the claim follows as the residue term in $\widetilde{A}_0(x;h/k)$ has size $O(k^{3/2+\varepsilon})$. □

Corollary 9.7. *The exponent pairs $(\alpha,\beta) = (1,0)$, $(\alpha,\beta) = (3/4+\varepsilon,0)$, $(\alpha,\beta) = (2/3+\varepsilon,1/2)$ and $(\alpha,\beta) = (1/2+\varepsilon,3/4)$, each of which either holds or holds under some assumptions, in the previous estimate lead to*

$$\widetilde{A}_1\left(x;\frac{h}{k}\right) \ll x^{4/3}k^{5/4}, \qquad \ll_\varepsilon x^{29/24+\varepsilon}k^{5/4}, \qquad \ll_\varepsilon x^{7/6+\varepsilon}k^{3/2}, \qquad \text{and} \qquad \ll_\varepsilon x^{13/12+\varepsilon}k^{13/8},$$

respectively.

10. Second moments of long sums

Using the truncated Voronoi identity for $\widetilde{A}_1(x;h/k)$ we deduce the following mean square-result.

Theorem 10.1. *Let $x\in[1,\infty[$ and suppose that k is a prime so that $k \ll_\varepsilon X^{1/3-\varepsilon}$. Then we have*

$$\mathop{\mathbb{E}}_{h\in\mathbb{Z}_k^\times}\int_X^{2X}\left|\widetilde{A}_1\left(x;\frac{h}{k}\right)\right|^2\mathrm{d}x = C(k)\cdot X^3k^3 + O_\varepsilon\left(k^4\,X^{8/3+\varepsilon}\right),$$

where $C(k)\in\mathbb{R}_+$ with $C(k)\asymp 1$.

Proof. We apply Corollary 8.7 with the choice $N = \lfloor k^{-3}X^3\rfloor$ to get

$$\begin{aligned}
&\mathop{\mathbb{E}}_{h\in\mathbb{Z}_k^\times}\int_X^{2X}\left|\widetilde{A}_1\left(x;\frac{h}{k}\right)\right|^2\mathrm{d}x\\
&= \frac{k^2}{48\,\pi^4}\mathop{\mathbb{E}}_{h\in\mathbb{Z}_k^\times}\int_X^{2X} x^2\left|\sum_{d|k}\sum_{m\le N}\frac{A(d,m)}{d\,m}\sum_{\pm}(\mp i)S\left(\overline{h},\pm m;\frac{k}{d}\right)\,e\left(\pm\frac{3\,d^{2/3}\,m^{1/3}\,x^{1/3}}{k}\right)\right|^2\mathrm{d}x\\
&\quad+\frac{k}{4\pi^2\sqrt{3}}\,\mathrm{Re}\mathop{\mathbb{E}}_{h\in\mathbb{Z}_k^\times}\int_X^{2X} x\sum_{d|k}\sum_{m\le N}\frac{A(d,m)}{d\,m}\\
&\qquad\times\sum_{\pm}(\mp i)S\left(\overline{h},\pm m;\frac{k}{d}\right)\,e\left(\pm\frac{3\,d^{2/3}\,m^{1/3}\,x^{1/3}}{k}\right)\cdot O(k^{5/2}\,x^{2/3})\,\mathrm{d}x\\
&\quad+O_\varepsilon(k^5\,X^{7/3+2\varepsilon}).
\end{aligned}$$

Once we have proved that the first term on the right-hand side is $\asymp k^3\,X^3$, it immediately follows from the Cauchy–Schwarz inequality that the mixed term integral is

$$\ll \sqrt{k^3\,X^3}\,\sqrt{k^5\,X^{7/3+2\varepsilon}} \ll k^4\,X^{8/3+\varepsilon}.$$

Also, as $k \ll_\varepsilon X^{1/3-\varepsilon}$, we have $k^5X^{7/3+2\varepsilon} \ll_\varepsilon k^4X^{8/3+\varepsilon}$, as required.

In the first integral, we expand $|\Sigma|^2$ as $\Sigma\overline{\Sigma}$. The diagonal terms, i.e. those terms where $d_1^2 m = d_2^2 n$ and where the signs in sums $\sum_{\pm}$ are chosen to be equal, give the contribution

$$\frac{k^2}{48\pi^4}\sum_{d_1|k}\sum_{d_2|k}\sum_{\substack{m\leq N\\ d_1^2m=d_2^2n}}\sum_{n\leq N}\frac{A(d_1,m)\,\overline{A(d_2,n)}}{d_1\,d_2\,m\,n}\mathop{\mathbb{E}}_{h\in\mathbb{Z}_k^\times}\sum_{\pm}S\Big(\overline{h},\pm m;\frac{k}{d_1}\Big)\,S\Big(\overline{h},\pm n;\frac{k}{d_2}\Big)\int_X^{2X}x^2\,\mathrm{d}x.$$

We will examine the contribution of the different pairs (d_1,d_2) separately at first. The terms where $d_1 = d_2 = 1$ contribute

$$\frac{k^2}{24\pi^4}\sum_{m\leq N}\frac{|A(1,m)|^2}{m^2}\left(\frac{k^2-k-1}{k-1}1_{m\not\equiv 0\,(k)}+1_{m\equiv 0\,(k)}\right)\int_X^{2X}x^2\,\mathrm{d}x \tag{10.1}$$

by Lemma 9.1.

Similarly, the terms with $d_1 = d_2 = k$ contribute

$$\frac{k^2}{24\pi^4}\sum_{m\leq N}\frac{|A(k,m)|^2}{k^2m^2}\int_X^{2X}x^2\,\mathrm{d}x \tag{10.2}$$

and the terms where $d_1d_2 = k$ contribute

$$-\frac{k^2}{12\pi^4}\mathrm{Re}\left(\sum_{m\leq N/k^2}\frac{A(1,k^2m)\overline{A(k,m)}}{k^3m^2}\right)\int_X^{2X}x^2\,\mathrm{d}x. \tag{10.3}$$

Our aim is to show that these three terms (10.1), (10.2), and (10.3), sum up to $C(k)k^3X^3$, where $C(k)\in\mathbb{R}$ with $C(k)\asymp 1$. The argument is simlar as in the proof of Theorem 9.3.

The upper bound of the right order of magnitude follows from estimating the Fourier coefficients pointwise. For the lower bound, observe first that by using the arithmetic-geometric mean inequality the absolute value of (10.3) is bounded from the above by

$$\begin{aligned}&\frac{k^2}{24\pi^4}\sum_{m\leq N/k^2}\frac{\left|A(1,k^2m)\right|^2}{k^4m^2}\int_X^{2X}x^2\,\mathrm{d}x+\frac{k^2}{24\pi^4}\sum_{m\leq N/k^2}\frac{|A(k,m)|^2}{k^2m^2}\int_X^{2X}x^2\,\mathrm{d}x\\ &\leq\frac{k^2}{24\pi^4}\sum_{m\leq N/k^2}\frac{\left|A(1,k^2m)\right|^2}{k^4m^2}\int_X^{2X}x^2\,\mathrm{d}x+\frac{k^2}{24\pi^4}\sum_{m\leq N}\frac{|A(k,m)|^2}{k^2m^2}\int_X^{2X}x^2\,\mathrm{d}x.\end{aligned} \tag{10.4}$$

Notice that the latter term in (10.4) is exactly (10.2). Hence the sum of (10.1), (10.2) and (10.3) is bounded from the below by

$$\frac{k^2}{24\pi^4}\cdot\frac{k^2-2k}{k-1}\sum_{m\leq N}\frac{|A(m,1)|^2}{m^2}\left(1-1_{m\equiv 0\,(k)}\right)\int_X^{2X}x^2\,\mathrm{d}x.$$

This gives the lower bound $\gg k^3X^3$ similarly as in the proof of Theorem 9.3.

The contribution arising from the terms $d_1^2 m = d_2^2 n$ where the signs in the sum $\sum_\pm$ are chosen to be distinct is by the first derivative test and Weil's bound given by

$$\begin{aligned}
&\ll k^2 \sum_\pm \sum_{d_1|k} \sum_{d_2|k} \sum_{m\le N} \sum_{\substack{n\le N\\ d_1^2 m = d_2^2 n}} \frac{\left|A(d_1,m)\overline{A(d_2,n)}\right|}{d_1 d_2 mn} S\left(\overline{h}, \pm m; k/d_1\right) S\left(\overline{h}, \mp n; k/d_2\right) \\
&\qquad \times \int_X^{2X} x^2\, e\left(\frac{\pm 6\pi (d_1^2 m x)^{1/3}}{k}\right) \mathrm{d}x \\
&\ll k^2 \sum_{m\le N} \frac{|A(1,m)|^2}{m^2} k\, X^2 \frac{X^{2/3} k}{m^{1/3}} \ll X^{8/3} k^4
\end{aligned}$$

as in the proof of Theorem 9.3.

Next we study the off-diagonal-contribution in the same sum, which is given, up to a constant, by

$$\begin{aligned}
&k^2 \sum_\pm \sum_{d_1|k} \sum_{d_2|k} \sum_{\substack{m\le N\\ d_1^2 m\ne d_2^2 n}} \sum_{n\le N} \frac{A(m,d_1)\overline{A(n,d_2)}}{mnd_1d_2} \mathop{\mathbb{E}}_{h\in\mathbb{Z}_k^\times} S\left(\overline{h}, \pm m; \frac{k}{d_1}\right) S\left(\overline{h}, \pm n; \frac{k}{d_2}\right) \\
&\qquad \times \int_X^{2X} x^2 e\left(\pm \frac{3\pi (mxd_1^2)^{1/3}}{k} \mp \frac{3\pi (nxd_2^2)^{1/3}}{k}\right) \mathrm{d}x.
\end{aligned}$$

Using the first derivative test, the integral can be estimated in this situation from the above by

$$\ll \frac{X^{8/3} k}{\left|d_1^{2/3} m^{1/3} - d_2^{2/3} n^{1/3}\right|}.$$

Thus the off-diagonal contribution is by Weil's bound

$$\begin{aligned}
&\ll X^{8/3} k^4 \sum_{d_1|k} \sum_{d_2|k} \sum_{m\le N} \frac{|A(d_1,m)|}{md_1} \sum_{\substack{m\leqslant n\\ d_1^2 m\ne d_2^2 n}} \frac{|A(d_2,n)|}{d_2 n} \cdot \frac{1}{\left|d_1^{2/3} m^{1/3} - d_2^{2/3} n^{1/3}\right|} \\
&\ll_\varepsilon X^{8/3} k^4 \sum_{d_1|k} \sum_{m\le N} \frac{|A(d_1,m)|}{md_1} \sum_{d_2|k} \sum_{m<n} \frac{n^{-1/3+\vartheta+\varepsilon} d_2^{-2/3}}{|m-n|} \ll X^{8/3} k^4,
\end{aligned}$$

finishing the proof. □

As a consequence of the above theorem we obtain lower bounds for certain averaged mean-squares. Theorem 2.1 follows immediately from the latter bound.

Theorem 10.2. *Suppose that k is a prime so that $k \ll_\varepsilon X^{1/3-\varepsilon}$. Then the lower bounds*

$$\mathop{\mathbb{E}}_{h\in\mathbb{Z}_k^\times} \int_X^{2X} \left|\widetilde{A}_1\left(x; \frac{h}{k}\right)\right|^2 \mathrm{d}x \gg X^3 k^3 \quad \text{and} \quad \mathop{\mathbb{E}}_{h\in\mathbb{Z}_k^\times} \int_X^{2X} \left|\widetilde{A}_0\left(x; \frac{h}{k}\right)\right|^2 \mathrm{d}x \gg X^{5/3} k$$

hold.

Proof. The first bound is immediate from the previous theorem. For the latter bound, the main observation is that we can bound the mean square of $\widetilde{A}_1(x; h/k)$ from above in terms of mean squares of $\widetilde{A}_2(x; h/k)$ and $\widetilde{A}_0(x; h/k)$. Knowing a lower bound for the averaged mean square of $\widetilde{A}_1(x; h/k)$ and an upper bound for the averaged mean square of $\widetilde{A}_2(x; h/k)$ then leads to the desired conclusion.

Indeed, for any $0 < H \leq x$, which is chosen later, we compute

$$\begin{aligned}
&\mathop{\mathbb{E}}_{h \in \mathbb{Z}_k^\times} \int_X^{2X} \left| \widetilde{A}_1\left(x; \frac{h}{k}\right) \right|^2 \mathrm{d}x \\
&= \frac{1}{H^2} \mathop{\mathbb{E}}_{h \in \mathbb{Z}_k^\times} \int_X^{2X} \left| \int_x^{x+H} \widetilde{A}_1\left(x; \frac{h}{k}\right) \mathrm{d}t \right|^2 \mathrm{d}x \\
&= \frac{1}{H^2} \mathop{\mathbb{E}}_{h \in \mathbb{Z}_k^\times} \int_X^{2X} \left| \int_x^{x+H} \left(\widetilde{A}_1\left(t; \frac{h}{k}\right) - \int_x^t \widetilde{A}_0\left(u; \frac{h}{k}\right) \mathrm{d}u \right) \mathrm{d}t \right|^2 \mathrm{d}x \\
&\ll \frac{1}{H^2} \mathop{\mathbb{E}}_{h \in \mathbb{Z}_k^\times} \int_X^{2X} \left| \int_x^{x+H} \widetilde{A}_1\left(t; \frac{h}{k}\right) \mathrm{d}t \right|^2 \mathrm{d}x + \frac{1}{H^2} \mathop{\mathbb{E}}_{h \in \mathbb{Z}_k^\times} \int_X^{2X} \left| \int_x^{x+H} \int_x^t \widetilde{A}_0\left(u; \frac{h}{k}\right) \mathrm{d}u \, \mathrm{d}t \right|^2 \mathrm{d}x.
\end{aligned}$$

By simply integrating, the first term on the previous line is clearly

$$\ll \frac{1}{H^2} \mathop{\mathbb{E}}_{h \in \mathbb{Z}_k^\times} \int_X^{2X} \left| \widetilde{A}_2\left(x; \frac{h}{k}\right) \right|^2 \mathrm{d}x.$$

The latter term can be estimated as follows. First note that

$$\int_x^{x+H} \int_x^t \widetilde{A}_0\left(u; \frac{h}{k}\right) \mathrm{d}u \, \mathrm{d}t = \int_x^{x+H} \widetilde{A}_0\left(u; \frac{h}{k}\right) (u - x) \, \mathrm{d}u \ll H \int_x^{x+H} \left| \widetilde{A}_0\left(u; \frac{h}{k}\right) \right| \mathrm{d}u.$$

Using this together with the Cauchy-Schwarz inequality yields

$$\begin{aligned}
&\frac{1}{H^2} \mathop{\mathbb{E}}_{h \in \mathbb{Z}_k^\times} \int_X^{2X} \left| \int_x^{x+H} \int_x^t \widetilde{A}_0\left(u; \frac{h}{k}\right) \mathrm{d}u \, \mathrm{d}t \right|^2 \mathrm{d}x \\
&\ll \mathop{\mathbb{E}}_{h \in \mathbb{Z}_k^\times} \int_X^{2X} \left(\int_x^{x+H} \left| \widetilde{A}_0\left(u; \frac{h}{k}\right) \right| \mathrm{d}u \right)^2 \mathrm{d}x \\
&\ll H \mathop{\mathbb{E}}_{h \in \mathbb{Z}_k^\times} \int_X^{2X} \int_x^{x+H} \left| \widetilde{A}_0\left(u; \frac{h}{k}\right) \right|^2 \mathrm{d}u \, \mathrm{d}x \\
&\ll H^2 \mathop{\mathbb{E}}_{h \in \mathbb{Z}_k^\times} \int_X^{2X} \left| \widetilde{A}_0\left(x; \frac{h}{k}\right) \right|^2 \mathrm{d}x.
\end{aligned}$$

Thus we have shown that

$$\mathop{\mathbb{E}}_{h \in \mathbb{Z}_k^\times} \int_X^{2X} \left| \widetilde{A}_1\left(x; \frac{h}{k}\right) \right|^2 \mathrm{d}x \ll \frac{1}{H^2} \mathop{\mathbb{E}}_{h \in \mathbb{Z}_k^\times} \int_X^{2X} \left| \widetilde{A}_2\left(x; \frac{h}{k}\right) \right|^2 \mathrm{d}x + H^2 \mathop{\mathbb{E}}_{h \in \mathbb{Z}_k^\times} \int_X^{2X} \left| \widetilde{A}_0\left(x; \frac{h}{k}\right) \right|^2 \mathrm{d}x.$$

As the second moment of $\widetilde{A}_2(x; h/k)$ averaged over h modulo k is $\asymp X^{13/3} k^5$ when $k \ll_\varepsilon X^{1/3-\varepsilon}$, this implies

$$\mathop{\mathbb{E}}_{h \in \mathbb{Z}_k^\times} \int_X^{2X} \left| \widetilde{A}_1\left(x; \frac{h}{k}\right) \right|^2 \mathrm{d}x \ll \frac{X^{13/3} k^5}{H^2} + H^2 \mathop{\mathbb{E}}_{h \in \mathbb{Z}_k^\times} \int_X^{2X} \left| \widetilde{A}_0\left(x; \frac{h}{k}\right) \right|^2 \mathrm{d}x.$$

We optimise the choice of H by equating two terms on the right-hand side, which means choosing

$$H = \frac{X^{13/12}k^{5/4}}{\sqrt[4]{\mathbb{E}_{h\in\mathbb{Z}_k^\times} \int_X^{2X} \left|\widetilde{A}_0\left(x; \frac{h}{k}\right)\right|^2 \,\mathrm{d}x}}.$$

By the previous theorem, the mean-square of $\widetilde{A}_1(x; h/k)$ averaged over h modulo k is $\asymp X^3k^3$ when $k \ll_\varepsilon X^{1/3-\varepsilon}$, and hence we conclude that

$$X^3k^3 \ll X^{13/6}k^{5/2}\sqrt{\mathop{\mathbb{E}}_{h\in\mathbb{Z}_k^\times} \int_X^{2X} \left|\widetilde{A}_0\left(x; \frac{h}{k}\right)\right|^2 \,\mathrm{d}x},$$

yielding the second bound. $\square$

11. An improved pointwise bound for long sums with rational twists

In this section we improve the best known upper bound for rationally additively twisted sums with sufficiently small denominators.

Theorem 11.1. *Let $x \in [1, \infty[$, and let h and k be coprime integers with $1 \leqslant k \ll x^{1/3}$. Then*

$$\sum_{m\leqslant x} A(m,1)\, e\left(\frac{mh}{k}\right) \ll_\varepsilon k^{3/4}\, x^{1/2+\vartheta/2+\varepsilon}.$$

Proof. We have

$$\sum_{m\leqslant x} A(m,1)\, e\left(\frac{mh}{k}\right) = \widetilde{A}_0\left(x; \frac{h}{k}\right) + O(k^{3/2+\varepsilon}) + O(x^{\vartheta+\varepsilon}),$$

and so it is enough to prove the desired bound for $\widetilde{A}_0(x; h/k)$.

Let $H \in [1, \infty[$ satisfy $H \ll x$, which will be chosen at the end. Estimating a short exponential sum by absolute values yields

$$\int_x^{x+H} \widetilde{A}_0\left(t; \frac{h}{k}\right) \mathrm{d}t - H\, \widetilde{A}_0\left(x; \frac{h}{k}\right) = \int_x^{x+H} \left(\widetilde{A}_0\left(t; \frac{h}{k}\right) - \widetilde{A}_0\left(x; \frac{h}{k}\right)\right) \mathrm{d}t \ll H^2\, x^{\vartheta+\epsilon},$$

so that

$$H\, \widetilde{A}_0\left(x; \frac{h}{k}\right) = \int_x^{x+H} \widetilde{A}_0\left(t; \frac{h}{k}\right) \mathrm{d}t + O(H^2\, x^{\vartheta+\varepsilon}).$$

We know from Corollary 8.5 that

$$\widetilde{A}_1\left(x; \frac{h}{k}\right) \ll_\varepsilon k^{3/2}\, x^{1+\varepsilon}.$$

This bound gives

$$\begin{aligned}
\widetilde{A}_0\left(x; \frac{h}{k}\right) &= \frac{1}{H} \int_x^{x+H} \widetilde{A}_0\left(t; \frac{h}{k}\right) \mathrm{d}t + O(H\, x^{\vartheta+\varepsilon}) \\
&= \frac{1}{H}\left(\widetilde{A}_1\left(x+H; \frac{h}{k}\right) - \widetilde{A}_1\left(x; \frac{h}{k}\right)\right) + O(H\, x^{\vartheta+\varepsilon}) \\
&\ll_\varepsilon \frac{1}{H}\, k^{3/2}\, x^{1+\varepsilon} + H\, x^{\vartheta+\varepsilon}.
\end{aligned}$$

Choosing $H = k^{3/4}\,x^{(1-\vartheta)/2}$ yields

$$\widetilde{A}_0\left(x;\frac{h}{k}\right) \ll_\varepsilon k^{3/4}\,x^{1/2+\vartheta/2+\varepsilon},$$

as desired. □

12. Short second moment for Riesz weighted sums

The goal of this section is to compute the short second moment of Riesz weighted sums for $a=2$ and $a=3$. Let us start by listing some consequences of the first derivative test.

Lemma 12.1. *Let $\Xi, U, T \in [0,X]$ and $k, m, n, d_1, d_2 \in \mathbb{Z}_+$. Then*

$$\int_X^{X+\Xi} x^{(4a+2)/3}\, e\left(\pm\frac{3\left(d_1^{2/3}m^{1/3}\,(x+T)^{1/3}+d_2^{2/3}n^{1/3}\,(x+U)^{1/3}\right)}{k}\right)\mathrm{d}x \ll \frac{X^{(4a+4)/3}k}{d_1^{2/3}m^{1/3}+d_2^{2/3}n^{1/3}}.$$

Lemma 12.2. *Let $\Xi, U, T \in [0,X]$ and $k, m, n, d_1, d_2 \in \mathbb{Z}_+$ with $U \leqslant T$ and $d_1^2 m < d_2^2 n$. Then*

$$\int_X^{X+\Xi} x^{(4a+2)/3}\, e\left(\pm\frac{3\left(d_1^{2/3}m^{1/3}\,(x+T)^{1/3}-d_2^{2/3}n^{1/3}\,(x+U)^{1/3}\right)}{k}\right)\mathrm{d}x \ll \frac{X^{(4a+4)/3}k}{d_2^{2/3}n^{1/3}-d_1^{2/3}m^{1/3}}.$$

Lemma 12.3. *Let $\Xi, T \in\,]0,X]$ and $k, m, n, d_1, d_2 \in \mathbb{Z}_+$ with $d_1^2 m < d_2^2 n$, $d_1 \geqslant d_2$, as well as $m \leqslant X/(6T)$. Then*

$$\int_X^{X+\Xi} x^{(4a+2)/3}\, e\left(\pm\frac{3\left(d_1^{2/3}m^{1/3}\,x^{1/3}-d_2^{2/3}n^{1/3}\,(x+T)^{1/3}\right)}{k}\right)\mathrm{d}x \ll \frac{X^{(4a+4)/3}\,k\,n^{2/3}\,d_2^{-2/3}}{n-m}. \tag{12.1}$$

The first two lemmas are straightforward to prove, but the third one requires some explanation. The derivative of the phase function is, up to a sign, given by

$$\frac{d_1^{2/3}m^{1/3}}{x^{2/3}k}-\frac{d_2^{2/3}n^{1/3}}{(x+T)^{2/3}k}=\frac{d_1^{2/3}m^{1/3}\,(x+T)^{2/3}-d_2^{2/3}n^{1/3}\,x^{2/3}}{(x+T)^{2/3}\;x^{2/3}\,k}.$$

The denominator is of course $\asymp X^{4/3}k$, and the numerator is, by using the identity $x^3-y^3=(x-y)(x^2+xy+y^2)$, given by

$$\frac{\left(d_1^2 m - d_2^2 n\right)x^2+2\,x\,T d_1^2 m + d_1^2 m\,T^2}{\left(d_1^{2/3}m^{1/3}\,(x+T)^{2/3}\right)^2+d_1^{2/3}m^{1/3}\,d_2^{2/3}n^{1/3}\,x^{2/3}\,(x+T)^{2/3}+\left(d_2^{2/3}n^{1/3}\,x^{2/3}\right)^2}.$$

In the numerator, the denominator is given by $\asymp d_2^{4/3}n^{2/3}\,X^{4/3}$ and the numerator is

$$\asymp \left(d_2^2 n - d_1^2 m\right)X^2.$$

Now (12.1) follows immediately from Lemma 9.2.

Let us then proceed to the proof of the short mean square result for Riesz weighted exponential sums related to Maass cusp forms.

Theorem 12.4. *Let $X \in [1,\infty[$, k be a prime with $k \ll x^{1/3}$, and let $\Delta, \Xi \in [1,X]$ satisfy $X^{1/2}k^{3/2} \ll \Delta \leqslant c\,X^{2/3}k$, where c is a sufficiently small (depending on the underlying Maass cusp form) fixed positive real constant, and $X^2k^3 \ll \Xi\,\Delta^2$. Then, for X large enough,*

$$\mathop{\mathbb{E}}_{h\in\mathbb{Z}_k^\times}\int_X^{X+\Xi}\left|\widetilde{A}_2\left(x+\Delta;\frac{h}{k}\right)-\widetilde{A}_2\left(x;\frac{h}{k}\right)\right|^2\mathrm{d}x \asymp \Xi\,\Delta^2\,X^2k^3,$$

and

$$\mathop{\mathbb{E}}_{h\in\mathbb{Z}_k^\times}\int_X^{2X}\left|\widetilde{A}_3\left(x+\Delta;\frac{h}{k}\right)-\widetilde{A}_3\left(x;\frac{h}{k}\right)\right|^2\mathrm{d}x\asymp\Xi\,\Delta^2\,X^{10/3}k^5.$$

Proof. We shall actually prove a more general result concerning $\widetilde{A}_a(x;h/k)$ for any integer $a\geqslant 2$ satisfying $a\equiv 0 \pmod 3$ or $a\equiv 2 \pmod 3$. Our starting point is the formula

$$\begin{aligned}\widetilde{A}_a\left(x+\Delta;\frac{h}{k}\right)-\widetilde{A}_a\left(x;\frac{h}{k}\right)&=\frac{(-1)^ak^a}{(2\pi)^{a+1}\sqrt{3}}\sum_{\pm}i^{\pm a}\sum_{d|k}\sum_{m=1}^{\infty}\frac{A(d,m)}{m^{(a+2)/3}d^{(2a+1)/3}}S\left(\overline{h},\pm m;\frac{k}{d}\right)\\&\times\left((x+\Delta)^{(2a+1)/3}\,e\biggl(\pm\frac{3(md^2(x+\Delta))^{1/3}}{k}\biggr)-x^{(2a+1)/3}\,e\biggl(\pm\frac{3(md^2x)^{1/3}}{k}\biggr)\right)\\&+O\left(X^{2a/3}k^{(2a+3)/2}\right),\end{aligned}$$

which follows from Corollary 7.3. Note that $d(k)$ does not appear in the error term as we assume k to be prime.

Exchanging here the factor $(x+\Delta)^{(2a+1)/3}$ to $x^{(2a+1)/3}$ costs $\ll\Delta\,X^{(2a-2)/3}k^{(2a+1)/2}$, which is $\ll X^{2a/3}k^{(2a+3)/2}$ provided that $\Delta\ll kX^{2/3}$. Thus,

$$\begin{aligned}\widetilde{A}_a\left(x+\Delta;\frac{h}{k}\right)-\widetilde{A}_a\left(x;\frac{h}{k}\right)&=\frac{(-1)^ax^{(2a+1)/3}k^a}{(2\pi)^{a+1}\sqrt{3}}\sum_{\pm}i^{\pm a}\sum_{d|k}\sum_{m=1}^{\infty}\frac{A(d,m)}{m^{(a+2)/3}d^{(2a+1)/3}}S\left(\overline{h},\pm m;\frac{k}{d}\right)\\&\times\left(e\biggl(\pm\frac{3(md^2(x+\Delta))^{1/3}}{k}\biggr)-e\biggl(\pm\frac{3(md^2x)^{1/3}}{k}\biggr)\right)\\&+O\left(X^{2a/3}k^{(2a+3)/2}\right).\end{aligned}$$

Now the second moment may be expanded as

$$\begin{aligned}&\mathop{\mathbb{E}}_{h\in\mathbb{Z}_k^\times}\int_X^{X+\Xi}\left|\widetilde{A}_a\left(x+\Delta;\frac{h}{k}\right)-\widetilde{A}_a\left(x;\frac{h}{k}\right)\right|^2\mathrm{d}x\\&=\frac{k^{2a}}{(2\pi)^{2a+2}\cdot 3}\int_X^{X+\Xi}x^{(4a+2)/3}\mathop{\mathbb{E}}_{h\in\mathbb{Z}_k^\times}\left|\sum_{\pm}i^{\pm a}\sum_{d|k}\sum_{m=1}^{\infty}\frac{A(d,m)}{m^{(a+2)/3}d^{(2a+1)/3}}S\left(\overline{h},\pm m;\frac{k}{d}\right)(e(\ldots)-e(\ldots))\right|^2\mathrm{d}x\\&-\frac{(-1)^ak^a}{2^a\pi^{a+1}\sqrt{3}}\mathrm{Re}\int_X^{X+\Xi}x^{(2a+1)/3}\sum_{\pm}i^{\pm a}\sum_{d|k}\sum_{m=1}^{\infty}\frac{A(d,m)}{m^{(a+2)/3}d^{(2a+1)/3}}\mathop{\mathbb{E}}_{h\in\mathbb{Z}_k^\times}S\left(\overline{h},\pm m;\frac{k}{d}\right)\\&\qquad\times(e(\ldots)-e(\ldots))\cdot O\left(X^{2a/3}k^{(2a+3)/2}\right)\mathrm{d}x+O\left(\Xi\,X^{4a/3}k^{2a+3}\right).\end{aligned}\tag{12.2}$$

First, $\Xi\,X^{4a/3}k^{2a+3}\ll\Xi\,\Delta^2\,X^{(4a-2)/3}k^{2a-1}$ provided that $\Delta\gg X^{1/3}k^2$. Second, once we have proved that the first term is $\asymp\Xi\,\Delta^2\,X^{(4a-2)/3}k^{2a-1}$, then the middle term is by the Cauchy–Schwarz inequality

$$\ll\sqrt{\Xi\,\Delta^2\,X^{(4a-2)/3}k^{2a-1}}\sqrt{\Xi\,X^{4a/3}k^{2a+3}}\ll\Xi\,\Delta\,X^{(4a-1)/3}\,k^{2a+1},$$

and this is $\ll\Xi\Delta^2\,X^{(4a-2)/3}k^{2a-1}$ provided that $\Delta\gg X^{1/3}k^2$. Thus, we can focus on examining the first term on the right-hand side of (12.2). For $\lambda\in\{0,\Delta\}$ we write

$$s_\lambda^+:=i^a\sum_{d|k}\sum_{m=1}^{\infty}\frac{A(d,m)}{m^{(a+2)/3}d^{(2a+1)/3}}e\biggl(\frac{3(md^2(x+\lambda))^{1/3}}{k}\biggr)S\left(\overline{h},m;\frac{k}{d}\right)$$

and

$$s_\lambda^- := i^{-a} \sum_{d|k} \sum_{m=1}^{\infty} \frac{A(d,m)}{m^{(a+2)/3} d^{(2a+1)/3}} e\left(-\frac{3(md^2(x+\lambda))^{1/3}}{k}\right) S\left(\overline{h}, -m; \frac{k}{d}\right),$$

so that, up to a constant, the integral under study takes the shape

$$\begin{aligned}
&k^{2a} \int_X^{X+\Xi} x^{(4a+2)/3} \mathop{\mathbb{E}}_{h\in\mathbb{Z}_k^\times} \left| \sum_{\pm} i^{\pm a} \sum_{d|k} \sum_{m=1}^{\infty} \frac{A(d,m)}{m^{(a+2)/3} d^{(2a+1)/3}} S\left(\overline{h}, \pm m; \frac{k}{d}\right) (e(\ldots) - e(\ldots)) \right|^2 \mathrm{d}x \\
&= k^{2a} \int_X^{X+\Xi} x^{(4a+2)/3} \mathop{\mathbb{E}}_{h\in\mathbb{Z}_k^\times} \left| \left(s_\Delta^+ - s_0^+\right) + \left(s_\Delta^- - s_0^-\right) \right|^2 \mathrm{d}x
\end{aligned}$$

$$= k^{2a} \sum_{\pm} \int_X^{X+\Xi} x^{(4a+2)/3} \mathop{\mathbb{E}}_{h\in\mathbb{Z}_k^\times} \left| s_\Delta^\pm - s_0^\pm \right|^2 \mathrm{d}x + 2k^{2a} \mathrm{Re} \int_X^{X+\Xi} x^{(4a+2)/3} \mathop{\mathbb{E}}_{h\in\mathbb{Z}_k^\times} \left(s_\Delta^+ - s_0^+\right) \overline{\left(s_\Delta^- - s_0^-\right)} \mathrm{d}x, \tag{12.3}$$

where we sum over both choices of the sign $\pm$. Furthermore, we split each of the sums $s_\lambda^\pm$ into parts

$$s_\lambda^{\pm,\leqslant} := i^{\pm a} \sum_{d|k} \sum_{m\leqslant N_0} \ldots \quad \text{and} \quad s_\lambda^{\pm,>} := i^{\pm a} \sum_{d|k} \sum_{m>N_0} \ldots,$$

where we set $N_0 := X/(6\Delta)$.

Now the first integral on the right-hand side of (12.3) is

$$\begin{aligned}
&k^{2a} \sum_{\pm} \int_X^{X+\Xi} x^{(4a+2)/3} \mathop{\mathbb{E}}_{h\in\mathbb{Z}_k^\times} \left| s_\Delta^{\pm,\leqslant} - s_0^{\pm,\leqslant} \right|^2 \mathrm{d}x \\
&+ k^{2a} \sum_{\pm} \int_X^{X+\Xi} x^{(4a+2)/3} \mathop{\mathbb{E}}_{h\in\mathbb{Z}_k^\times} \left| s_\Delta^{\pm,>} - s_0^{\pm,>} \right|^2 \mathrm{d}x \\
&+ 2k^{2a} \mathrm{Re} \sum_{\pm} \int_X^{X+\Xi} x^{(4a+2)/3} \mathop{\mathbb{E}}_{h\in\mathbb{Z}_k^\times} \left(s_\Delta^{\pm,\leqslant} - s_0^{\pm,\leqslant}\right) \overline{\left(s_\Delta^{\pm,>} - s_0^{\pm,>}\right)} \mathrm{d}x.
\end{aligned} \tag{12.4}$$

Of the three new integrals we treat the last one first. It is given by

$$\begin{aligned}
&2k^{2a} \mathrm{Re} \sum_{\pm} \int_X^{X+\Xi} x^{(4a+2)/3} \left(s_\Delta^{\pm,\leqslant} - s_0^{\pm,\leqslant}\right) \overline{\left(s_\Delta^{\pm,>} - s_0^{\pm,>}\right)} \mathrm{d}x \\
&= k^{2a} \sum_{\pm} \int_X^{X+\Xi} x^{(4a+2)/3} \left(\overline{\left(s_\Delta^{\pm,\leqslant} - s_0^{\pm,\leqslant}\right)} \left(s_\Delta^{\pm,>} - s_0^{\pm,>}\right) + \left(s_\Delta^{\pm,\leqslant} - s_0^{\pm,\leqslant}\right) \overline{\left(s_\Delta^{\pm,>} - s_0^{\pm,>}\right)} \right) \mathrm{d}x \\
&\ll k^{2a} \sum_{\pm} \sum_{d_1|k} \sum_{m\leq N_0} \frac{A(d_1,m)}{m^{(a+2)/3} d_1^{(2a+1)/3}} S\left(\overline{h}, \pm m; \frac{k}{d_1}\right) \sum_{d_2|k} \sum_{n>N_0} \frac{\overline{A(d_2,n)}}{n^{(a+2)/3} d_2^{(2a+1)/3}} S\left(\overline{h}, \pm n; \frac{k}{d_2}\right) \\
&\times \int_X^{X+\Xi} x^{(4a+2)/3} \left(e\left(\mp \frac{3(x+\Delta)^{1/3}\left(m^{1/3}d_1^{2/3} - n^{1/3}d_2^{2/3}\right)}{k}\right) - e\left(\mp \frac{3\left((x+\Delta)^{1/3}m^{1/3}d_1^{2/3} - x^{1/3}n^{1/3}d_2^{2/3}\right)}{k}\right) \right. \\
&\qquad \left. - e\left(\mp \frac{3\left(x^{1/3}m^{1/3}d_1^{2/3} - (x+\Delta)^{1/3}n^{1/3}d_2^{2/3}\right)}{k}\right) + e\left(\mp \frac{3x^{1/3}\left(m^{1/3}d_1^{2/3} - n^{1/3}d_2^{2/3}\right)}{k}\right) \right) \mathrm{d}x.
\end{aligned}$$

As $n > m$ and k is a prime, the diagonal contribution $d_1^2 m = d_2^2 n$ can only arise when $d_1 = k$ and $d_2 = 1$ in the integrals corresponding to the first and fourth exponential term on the right-hand size of the previous display. Hence their total contribution is

$$\begin{aligned}
&\ll k^{2a} \sum_{m \le N_0} \frac{|A(k,m)|}{m^{(a+2)/3} k^{(2a+1)/3}} \cdot \frac{\left|A\left(1,k^2 m\right)\right|}{\left(k^2 m\right)^{(a+2)/3}} \cdot k^{1/2} \Xi X^{(4a+2)/3} \\
&\ll k^{3\vartheta + 2a/3 - 7/6} \Xi X^{(4a+2)/3+\varepsilon} \sum_{m \le N_0} \frac{1}{m^{(2a+4)/3 - 2\vartheta}} \\
&\ll_\varepsilon k^{3\vartheta + 2a/3 - 7/6} \Xi X^{(4a+2)/3+\varepsilon},
\end{aligned}$$

where we have applied the pointwise bound $|A(m,n)| \ll (mn)^{\vartheta+\varepsilon}$ in the penultimate step.

The diagonal contribution in the integrals corresponding to the two other exponentials are estimated similarly, noting that by the first derivative test

$$\int_X^{X+\Xi} x^{(4a+2)/3} e\left(\mp \frac{3 m^{1/3} d^{2/3} \left((x+\Delta)^{1/3} - x^{1/3}\right)}{k}\right) \mathrm{d}x \ll \frac{X^{(4a+7)/3}\, k}{\Delta\, m^{1/3} d^{2/3}}.$$

We may estimate the off-diagonal $d_1^2 m \neq d_2^2 n$ by Lemma 12.2 or Lemma 12.3 depending on the term in question, and Weil's bound as

$$\begin{aligned}
&\ll k^{2a+2} \sum_{d_1 \mid k} \sum_{m \leqslant N_0} \frac{|A(d_1,m)|}{m^{(a+2)/3} d_1^{(2a+1)/3+1/2}} \sum_{d_2 \mid k} \sum_{\substack{n > N_0 \\ d_1^2 m \neq d_2^2 n}} \frac{|A(d_2,n)|}{n^{(a+2)/3} d_2^{(2a+1)/3+1/2}} \cdot \frac{X^{(4a+4)/3}\, n^{2/3}}{d_1^{2/3} m - d_2^{2/3} n} \\
&\ll_\varepsilon k^{2a+2} X^{(4a+4)/3} \sum_{d_1 \mid k} \sum_{m \leqslant N_0} \frac{|A(d_1,m)|}{m^{(a+2)/3} d_1^{(2a+1)/3+1/2}} \sum_{d_2 \mid k} \sum_{n > N_0} \frac{n^{\vartheta - a/3 + \varepsilon} d_2^{\vartheta - 5/6 - 2a/3 + \varepsilon}}{(n-m)^{1+\varepsilon}} \\
&\ll k^{2a+2}\, X^{(4a+4)/3}\, N_0^{\vartheta - a/3}
\end{aligned}$$

and this is $\ll k^{2a-1} \Xi\, \Delta^2\, X^{(4a-2)/3}$, provided that

$$k^3\, X^2 \left(\frac{X}{\Delta}\right)^{\vartheta - a/3} \ll \Xi\, \Delta^2,$$

which certainly holds by the assumption as $\vartheta < 1/2$. Note also that the second integral on the right-hand side of (12.3) can be treated using similar arguments, giving the contribution $\ll k^{2a-1} \Xi \Delta^2 X^{(4a-2)/3}$ using Lemma 12.1 to bound the exponential integral.

Let us next treat the middle term in (12.4). Using the triangle inequality it is clearly enough to estimate the integral

$$k^{2a} \int_X^{X+\Xi} x^{(4a+2)/3} \left|s_\lambda^{\pm,>}\right|^2 \mathrm{d}x.$$

We expand the square to see that the integral is bounded from the above by the expression

$$\begin{aligned}
k^{2a} \sum_{d_1 \mid k} \sum_{d_2 \mid k} \sum_{m > N_0} \sum_{n > N_0} \frac{|A(d_1,m)\, A(d_2,n)|}{m^{(a+2)/3} n^{(a+2)/3} d_1^{(2a+1)/3} d_2^{(2a+1)/3}} S\left(\overline{h}, \pm m; \frac{k}{d_1}\right) S\left(\overline{h}, \pm n; \frac{k}{d_2}\right) \\
\times \int_X^{X+\Xi} x^{(4a+2)/3} e\left(\pm \frac{3\,(x+\lambda)^{1/3} \left(m^{1/3} d_1^{2/3} - n^{1/3} d_2^{2/3}\right)}{k}\right) \mathrm{d}x.
\end{aligned}$$

We treat the diagonal and off-diagonal terms separately. Let us first concentrate on the diagonal, which correspond to the terms where $d_1^2 m_1 = d_2^2 m_2$. These contribute

$$\begin{aligned}
&\ll k^{2a+1} \sum_{m>N_0} \frac{|A(1,m)|^2}{m^{(2a+4)/3}} \cdot \Xi\, X^{(4a+2)/3} \\
&\ll k^{2a+1}\, \Xi\, X^{(4a+2)/3}\, N_0^{-(2a+1)/3} \\
&\ll k^{2a+1} \Xi\, X^{(2a+1)/3}\, \Delta^{(2a+1)/3}
\end{aligned}$$

by estimating trivially, which contribute $\ll \Xi\, \Delta^2\, X^{(4a-2)/3} k^{2a-1}$, provided that $k^6 \Delta^{2a-5} \ll X^{2a-3}$. This certainly holds as the assumptions imply that $k \ll X^{1/3}$.

Similarly, the off-diagonal terms contribute, using Lemma 12.2 and Weil's bound,

$$\begin{aligned}
&\ll k^{2a+1} \sum_{d_1|k} \sum_{d_2|k} \sum_{m>N_0} \frac{|A(d_1,m)|}{m^{(a+2)/3} d_1^{(2a+1)/3}} \sum_{n>m} \frac{|A(d_2,n)|}{n^{(a+2)/3} d_2^{(2a+1)/3}} \cdot \frac{X^{(4a+4)/3}\, n^{2/3} k}{d_1^{2/3} n - d_2^{2/3} m} \\
&\ll_\varepsilon k^{2a+2} X^{(4a+4)/3} \sum_{d_1|k} \sum_{m>N_0} \frac{|A(d_1,m)|}{m^{(a+2)/3} d_1^{(2a+1)/3+1/2}} \sum_{d_2|k} \sum_{n>m} \frac{n^{\vartheta-a/3+\varepsilon} d_2^{\vartheta-2/3}}{(n-m)^{1+\varepsilon}} \\
&\ll k^{2a+2} X^{(4a+4)/3} \sum_{d|k} \sum_{m>N_0} \frac{|A(d,m)|}{m^{2a/3+2/3-\vartheta-\varepsilon} d^{2a/3+3/2-\vartheta-\varepsilon}} \\
&\ll k^{2a+2} X^{(4a+4)/3}\, N_0^{-2a/3+1/3+2\vartheta} \\
&\ll k^{2a+2} X^{(4a+4)/3} \left(\frac{X}{\Delta}\right)^{2\vartheta+1/3-2a/3},
\end{aligned}$$

which is $\ll \Xi\, \Delta^2\, X^{(4a-2)/3} k^{2a-1}$, provided that

$$k^3\, X^2 \left(\frac{X}{\Delta}\right)^{2\vartheta+1/3-2a/3} \ll \Xi\, \Delta^2,$$

which in turn holds thanks to the assumptions as $\vartheta < 1/2$.

Now only one integral remains, namely

$$k^{2a} \int_X^{X+\Xi} x^{(4a+2)/3} \mathop{\mathbb{E}}_{h \in \mathbb{Z}_k^\times} \left| s_\Delta^{\pm,\leqslant} - s_0^{\pm,\leqslant} \right|^2 \mathrm{d}x,$$

where we have for simplicity dropped the constant $(2\pi)^{-(2a+2)}/3$ in front. When we again multiply out $|\Sigma|^2 = \Sigma\, \overline{\Sigma}$, we obtain from the off-diagonal terms by Lemma 12.2 or Lemma 12.3 depending on the term in question, and Weil's bound a contribution

$$\begin{aligned}
&\ll k^{2a+2} \sum_{d_1|k} \sum_{d_2|k} \sum_{m\leqslant N_0} \sum_{m<n\leqslant N_0} \frac{|A(d_1,m)| \cdot |A(d_2,n)|}{m^{(a+2)/3}\, n^{(a+2)/3} d_1^{2a/3+5/6} d_2^{2a/3+5/6}} \cdot \frac{X^{(4a+4)/3}\, n^{2/3}}{d_1^{2/3} n - d_2^{2/3} m} \\
&\ll_\varepsilon k^{2a+2} X^{(4a+4)/3} \sum_{d_1|k} \sum_{d_2|k} \sum_{m\leqslant N_0} \frac{|A(d_1,m)|}{m^{(a+2)/3} d_1^{2a/3+5/6}} \sum_{m<n\leqslant N_0} \frac{n^{\vartheta-a/3+\varepsilon} d_2^{-2/3-5/6+\vartheta}}{(n-m)^{1+\varepsilon}} \\
&\ll k^{2a+2} X^{(4a+4)/3} \sum_{d|k} \sum_{m\leqslant N_0} \frac{|A(d,m)|}{m^{2a/3+2/3-\vartheta-\varepsilon} d^{2a/3+3/2-\vartheta-\varepsilon}} \ll X^{(4a+4)/3} k^{2a+2},
\end{aligned}$$

which is $\ll \Xi\, \Delta^2\, X^{(4a-2)/3} k^{2a-1}$ provided that $k^3 X^2 \ll \Xi\, \Delta^2$.

To evaluate the diagonal contribution we set $N_1 := \ell\, X^2 k^3\, \Delta^{-3}$ with a constant ℓ so small that

$$\frac{m^{1/3}\left((x+\Delta)^{1/3} - x^{1/3}\right)}{k} \leqslant \frac{1}{12},$$

say, for each $m \leqslant N_1$. Here ℓ depends on the implicit constant in the upper bound $\Delta \ll X^{2/3}k$. We will pose a lower bound for ℓ as well, in a moment. Clearly $N_1 < N_0$ for X sufficiently large, as $\Delta \gg X^{1/2}k^{3/2}$.

By estimating trivially, the higher frequencies $m \geq N_1$ in the diagonal terms contribute

$$\begin{aligned} &\ll k^{2a+1} \sum_{N_1 < m \leqslant N_0} \frac{|A(1,m)|^2}{m^{(2a+4)/3}} \,\Xi\, X^{(4a+2)/3} \\ &\ll k^{2a+1} N_1^{-2a/3-1/3}\,\Xi\, X^{(4a+2)/3} \ll \Xi\,\Delta^{2a+1}, \end{aligned}$$

which is $\ll \Xi\,\Delta^2\, X^{(4a-2)/3}k^{2a-1}$, thanks to the assumption $\Delta \ll X^{2/3}k$.

In the remaining smaller frequency diagonal terms, we may use Lemma 9.1 to see that these terms contribute

$$\begin{aligned} &k^{2a+1} \sum_{\pm} \sum_{d_1|k} \sum_{d_2|k} \sum_{m\leqslant N_1} \sum_{\substack{n\leqslant N_1 \\ d_1^2 m = d_2^2 n}} \frac{A(d_1,m)\overline{A(d_2,n)}}{m^{(a+2)/3} n^{(a+2)/3} d_1^{(2a+1)/3} d_2^{(2a+1)/3}} \\ &\times \int_X^{X+\Xi} x^{(4a+2)/3} \left| e\left(\pm \frac{3m^{1/3} d_1^{2/3}\left((x+\Delta)^{1/3} - x^{1/3}\right)}{k}\right) - 1\right|^2 \mathrm{d}x. \end{aligned} \tag{12.5}$$

We note that this equals $S_1 + S_2 + S_3$, where

$$S_1 := k^{2a+1} \sum_{\pm} \sum_{m\leq N_1} \frac{|A(1,m)|^2}{m^{(2a+4)/3}} \int_X^{X+\Xi} x^{(4a+2)/3} \left| e\left(\pm \frac{3m^{1/3}\left((x+\Delta)^{1/3} - x^{1/3}\right)}{k}\right) - 1\right|^2 \mathrm{d}x$$

$$S_2 := k^{2a+1} \sum_{\pm} \sum_{m\leq N_1} \frac{|A(k,m)|^2}{m^{(2a+4)/3} k^{(4a+2)/3}} \int_X^{X+\Xi} x^{(4a+2)/3} \left| e\left(\pm \frac{3m^{1/3}k^{2/3}\left((x+\Delta)^{1/3} - x^{1/3}\right)}{k}\right) - 1\right|^2 \mathrm{d}x$$

$$\begin{aligned} S_3 := 2k^{2a+1} &\sum_{\pm} \mathrm{Re} \sum_{m\leq N_1/k^2} \frac{A(1,k^2m)\overline{A(k,m)}}{(k^2m)^{(a+2)/3} m^{(a+2)/3} k^{(2a+1)/3}} \\ &\times \int_X^{X+\Xi} x^{(4a+2)/3} \left| e\left(\pm \frac{3m^{1/3}k^{2/3}\left((x+\Delta)^{1/3} - x^{1/3}\right)}{k}\right) - 1\right|^2 \mathrm{d}x. \end{aligned}$$

Our next objective is to show that $S_1 + S_2 + S_3 \asymp \Xi\Delta^2 X^{(4a-2)/3}k^{2a-1}$. The upper bound follows simply by estimating trivially using known growth properties of the Fourier coefficients. Towards the lower bound, note that from the Arithmetic-Geometric mean inequality we have

$$2 \cdot \frac{|A(1,k^2m)\overline{A(k,m)}|}{(k^2m)^{(a+2)/3} m^{(a+2)/3} k^{(2a+1)/3}} \leq \frac{|A(1,k^2m)|^2}{(k^2m)^{(2a+4)/3}} + \frac{|A(k,m)|^2}{k^{(4a+2)/3} m^{(2a+4)/3}}$$

and so

$$\begin{aligned} |S_3| \leq\ & k^{2a+1} \sum_{\pm} \sum_{m\leq N_1/k^2} \frac{|A(1,k^2m)|^2}{(k^2m)^{(2a+4)/3}} \int_X^{X+\Xi} x^{(4a+2)/3} \left| e\left(\pm \frac{3m^{1/3}k^{2/3}\left((x+\Delta)^{1/3} - x^{1/3}\right)}{k}\right) - 1\right|^2 \mathrm{d}x \\ &+ k^{2a+1} \sum_{\pm} \sum_{m\leq N_1/k^2} \frac{|A(k,m)|^2}{k^{(4a+2)/3} m^{(2a+4)/3}} \int_X^{X+\Xi} x^{(4a+2)/3} \left| e\left(\pm \frac{3m^{1/3}k^{2/3}\left((x+\Delta)^{1/3} - x^{1/3}\right)}{k}\right) - 1\right|^2 \mathrm{d}x. \end{aligned}$$

From this it follows that

$$S_1+S_2+S_3 \gg k^{2a+1} \sum_{\pm} \sum_{m\leq N_1} \frac{|A(m,1)|^2}{m^{(2a+4)/3}} \left(1 - 1_{m\equiv 0\,(k^2)}\right) \int_X^{X+\Xi} x^{(4a+2)/3} e\left(\pm \frac{3m^{1/3}\left((x+\Delta)^{1/3} - x^{1/3}\right)}{k}\right) \mathrm{d}x.$$

Estimating trivially the terms involving $1_{m\equiv 0\,(k^2)}$ contribute $\ll \Xi\Delta^2 X^{(4a-2)/3}k^{2a-1}$.

Using the estimate $|e(\alpha)-1| \asymp |\alpha|$ the contribution corresponding to the first term in the parenthesis is

$$\begin{aligned}
&\gg k^{2a+1} \sum_{m\le N_1} \frac{|A(1,m)|^2}{m^{(2a+4)/3}} \cdot \Xi\, X^{(4a+2)/3} \left(\frac{m^{1/3} d_1^{2/3}\,\Delta}{X^{2/3}k}\right)^2 \\
&\asymp \Xi\,\Delta^2\, X^{(4a-2)/3}\, k^{2a-1} \sum_{m\leqslant N_1} \frac{|A(1,m)|^2}{m^{(2a+2)/3}} \\
&\asymp \Xi\,\Delta^2\, X^{(4a-2)/3}\, k^{2a-1}.
\end{aligned}$$

We conclude that the expression (12.5) is $\asymp \Xi\,\Delta^2\, X^{(4a-2)/3}\, k^{2a-1}$. The statements of the theorem follow by specialising to the cases $a=2$ and $a=3$.

□

13. Ω-result for short sums of Fourier coefficients

Choosing $\Xi = X$ in Theorem 12.4 leads to the bounds

$$\mathop{\mathbb{E}}_{h\in\mathbb{Z}_k^\times} \int_X^{2X} \left|\widetilde{A}_2\left(x+\Delta;\frac{h}{k}\right) - \widetilde{A}_2\left(x;\frac{h}{k}\right)\right|^2 \mathrm{d}x \asymp \Delta^2\, X^3 k^3, \tag{13.1}$$

and

$$\mathop{\mathbb{E}}_{h\in\mathbb{Z}_k^\times} \int_X^{2X} \left|\widetilde{A}_3\left(x+\Delta;\frac{h}{k}\right) - \widetilde{A}_3\left(x;\frac{h}{k}\right)\right|^2 \mathrm{d}x \asymp \Delta^2\, X^{13/3} k^5 \tag{13.2}$$

in the range $k^{3/2}X^{1/2} \ll \Delta \ll X^{2/3}k$. Using similar ideas as before, these can be used to deduce lower bounds for short sums of the Fourier coefficients on average when one also averages over the numerator of the exponential phase, which will in turn yield an Ω-result for short sums.

Theorem 13.1. *Let $X \in [1,\infty[$, let k be a prime and suppose that $k^{3/2}\,X^{1/2} \ll \Delta \ll kX^{2/3}$ with sufficiently small implied constants. Then we have*

$$\mathop{\mathbb{E}}_{h\in\mathbb{Z}_k^\times} \int_X^{2X} \left|\widetilde{A}_0\left(x+\Delta;\frac{h}{k}\right) - \widetilde{A}_0\left(x;\frac{h}{k}\right)\right|^2 \mathrm{d}x \gg \Delta^2\, X^{1/3} k^{-1}.$$

Proof. We begin by proving a lower bound for the mean square of a short first order Riesz sum $\widetilde{A}_1(x+\Delta;h/k) - \widetilde{A}_1(x;h/k)$. Similarly as in the proof of Theorem 10.2, we have for any $H>0$ that

$$\begin{aligned}
&\mathop{\mathbb{E}}_{h\in\mathbb{Z}_k^\times} \int_X^{2X} \left|\widetilde{A}_2\left(x+\Delta;\frac{h}{k}\right) - \widetilde{A}_2\left(x;\frac{h}{k}\right)\right|^2 \mathrm{d}x \\
&\ll \frac{1}{H^2} \mathop{\mathbb{E}}_{h\in\mathbb{Z}_k^\times} \int_X^{2X} \left|\widetilde{A}_3\left(x+\Delta;\frac{h}{k}\right) - \widetilde{A}_3\left(x;\frac{h}{k}\right)\right|^2 \mathrm{d}x + H^2 \mathop{\mathbb{E}}_{h\in\mathbb{Z}_k^\times} \int_X^{2X} \left|\widetilde{A}_1\left(x+\Delta;\frac{h}{k}\right) - \widetilde{A}_1\left(x;\frac{h}{k}\right)\right|^2 \mathrm{d}x \\
&\ll \frac{1}{H^2}\,\Delta^2\, X^{13/3}k^5 + H^2 \mathop{\mathbb{E}}_{h\in\mathbb{Z}_k^\times} \int_X^{2X} \left|\widetilde{A}_1\left(x+\Delta;\frac{h}{k}\right) - \widetilde{A}_1\left(x;\frac{h}{k}\right)\right|^2 \mathrm{d}x,
\end{aligned}$$

where the last estimate follows from (13.2).

Balancing terms on the right-hand side by choosing

$$H = \frac{\Delta^{1/2}\, X^{13/12} k^{5/4}}{\sqrt[4]{\mathbb{E}_{h\in\mathbb{Z}_k^\times} \int_X^{2X} \left| \widetilde{A}_1\left(x+\Delta; \frac{h}{k}\right) - \widetilde{A}_1\left(x; \frac{h}{k}\right) \right|^2 \mathrm{d}x}}$$

yields

$$\begin{aligned} k^3 \Delta^2\, X^3 &\ll \mathop{\mathbb{E}}_{h\in\mathbb{Z}_k^\times} \int_X^{2X} \left| \widetilde{A}_2\left(x+\Delta; \frac{h}{k}\right) - \widetilde{A}_2\left(x; \frac{h}{k}\right) \right|^2 \mathrm{d}x \\ &\ll \Delta\, X^{13/6} k^{5/2} \sqrt{\mathop{\mathbb{E}}_{h\in\mathbb{Z}_k^\times} \int_X^{2X} \left| \widetilde{A}_1\left(x+\Delta; \frac{h}{k}\right) - \widetilde{A}_1\left(x; \frac{h}{k}\right) \right|^2 \mathrm{d}x}, \end{aligned}$$

where the leftmost estimate follows from (13.1).

Consequently,

$$\mathop{\mathbb{E}}_{h\in\mathbb{Z}_k^\times} \int_X^{2X} \left| \widetilde{A}_1\left(x+\Delta; \frac{h}{k}\right) - \widetilde{A}_1\left(x; \frac{h}{k}\right) \right|^2 \mathrm{d}x \gg \left(\frac{\Delta^2\, X^3\, k^3}{\Delta\, X^{13/6}\, k^{5/2}} \right)^2 = \Delta^2\, X^{5/3}\, k, \tag{13.3}$$

giving a lower bound for short first order Riesz sum.

We now turn into estimating the sum appearing in the statement of the theorem. Again, arguing as in the proof of Theorem 10.2 and using (13.1) gives

$$\begin{aligned} &\mathop{\mathbb{E}}_{h\in\mathbb{Z}_k^\times} \int_X^{2X} \left| \widetilde{A}_1\left(x+\Delta; \frac{h}{k}\right) - \widetilde{A}_1\left(x; \frac{h}{k}\right) \right|^2 \mathrm{d}x \\ &\ll \frac{1}{H^2} \mathop{\mathbb{E}}_{h\in\mathbb{Z}_k^\times} \int_X^{2X} \left| \widetilde{A}_2\left(x+\Delta; \frac{h}{k}\right) - \widetilde{A}_2\left(x; \frac{h}{k}\right) \right|^2 \mathrm{d}x + H^2 \mathop{\mathbb{E}}_{h\in\mathbb{Z}_k^\times} \int_X^{2X} \left| \widetilde{A}_0\left(x+\Delta; \frac{h}{k}\right) - \widetilde{A}_0\left(x; \frac{h}{k}\right) \right|^2 \mathrm{d}x \\ &\ll \frac{1}{H^2}\, \Delta^2\, X^3\, k^3 + H^2 \mathop{\mathbb{E}}_{h\in\mathbb{Z}_k^\times} \int_X^{2X} \left| \widetilde{A}_0\left(x+\Delta; \frac{h}{k}\right) - \widetilde{A}_0\left(x; \frac{h}{k}\right) \right|^2 \mathrm{d}x. \end{aligned}$$

In this case, choosing

$$H = \frac{\Delta^{1/2}\, X^{3/4} k^{3/4}}{\sqrt[4]{\mathbb{E}_{h\in\mathbb{Z}_k^\times} \int_X^{2X} \left| \widetilde{A}_0\left(x+\Delta; \frac{h}{k}\right) - \widetilde{A}_0\left(x; \frac{h}{k}\right) \right|^2 \mathrm{d}x}}$$

yields (together with (13.3))

$$\begin{aligned} k\, \Delta^2\, X^{5/3} &\ll \mathop{\mathbb{E}}_{h\in\mathbb{Z}_k^\times} \int_X^{2X} \left| \widetilde{A}_1\left(x+\Delta; \frac{h}{k}\right) - \widetilde{A}_1\left(x; \frac{h}{k}\right) \right|^2 \mathrm{d}x \\ &\ll k^{3/2} \Delta\, X^{3/2} \sqrt{\mathop{\mathbb{E}}_{h\in\mathbb{Z}_k^\times} \int_X^{2X} \left| \widetilde{A}_0\left(x+\Delta; \frac{h}{k}\right) - \widetilde{A}_0\left(x; \frac{h}{k}\right) \right|^2 \mathrm{d}x}, \end{aligned}$$

and hence

$$\mathop{\mathbb{E}}_{h\in\mathbb{Z}_k^\times} \int_X^{2X} \left| \widetilde{A}_0\left(x+\Delta; \frac{h}{k}\right) - \widetilde{A}_0\left(x; \frac{h}{k}\right) \right|^2 \mathrm{d}x \gg \left(\frac{\Delta^2\, X^{5/3}\, k}{\Delta\, X^{3/2}\, k^{3/2}} \right)^2 = \Delta^2\, X^{1/3}\, k^{-1}.$$

This completes the proof. □

APPENDIX: CORRECTIONS TO [22]

In [22], there was a wrong factor d instead of the correct $d^{1/3}$ in Theorem 1. This was corrected in [23], which also described how the proofs of Theorem 2 and Corollary 3 need to be modified in order to obtain the same conclusions. Alas, the computation of the asymptotics for the Meijer G-function revealed another mistake in the proof of Theorem 1 of [22]. Namely, when $j = 1$, the Γ-quotient expression used to approximate the Γ-quotient coming from the functional equation of the twisted L-function of the underlying Maass cusp form is correct only in the upper half-plane; in the lower half-plane the factor i^j should read $(-i)^j$, thereby changing Theorem 1, though fortunately not really affecting the proofs of Theorem 2 and Corollary 3 apart from obvious cosmetic changes. The corrected formulation of Theorem 1 of [22] reads as follows.

Theorem 13.2. *Let $x, N \in [2, \infty[$ with $N \ll x$, and let h and k be coprime integers with $1 \leqslant k \leqslant x$ and $k \ll (Nx)^{1/3}$, the latter having a sufficiently small implicit constant depending on the underlying Maass cusp form. Then we have*

$$\begin{aligned}
&\sum_{m \leqslant x} A(m,1)\, e\left(\frac{mh}{k}\right) \\
&\qquad = \frac{x^{1/3}}{2\pi\sqrt{3}} \sum_{d \mid k} \frac{1}{d^{1/3}} \sum_{d^2\, m \leqslant N_k} \frac{A(d,m)}{m^{2/3}} \sum_{\pm} S\left(\overline{h}, \pm m; \frac{k}{d}\right) e\left(\pm \frac{3\, d^{2/3}\, m^{1/3}\, x^{1/3}}{k}\right) \\
&\qquad\qquad + O(k\, x^{2/3+\vartheta+\varepsilon}\, N^{-1/3}) + O(k\, x^{1/6+\varepsilon}\, N^{1/6+\vartheta}).
\end{aligned}$$

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