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1 Statement of the Theorem

Definition 1 (Preference Relation)**.** Let $\mathcal{X}$ denote a set of alternatives. A binary relation $\succsim$ over $\mathcal{X}$ is said to be a *preference relation* if it satisfies the following properties:
1. Completeness: $\forall\{x,y\} \subseteq \mathcal{X}$, $x \not\succsim y$ implies $y \succsim x$.
2. Transitivity: $\forall\{x,y,z\} \subseteq \mathcal{X}$, $x \succsim y$ and $y \succsim z$ implies $x \succsim z$.

Definition 2 (Social Choice Model (SCM))**.** A *social choice model* consists of the following:
1. Alternatives: A set of alternatives $\mathcal{X}$.
2. Individuals: A set of finite individuals $\mathcal{N}$ with $|\mathcal{N}| = n$.
3. Preference Profile: An n-tuple of individual preferences $P = (\succsim_1, ..., \succsim_n)$ over alternatives $\mathcal{X}$. Denote the space of all possible preference profiles as $\mathcal{P}$.
4. Social Welfare Function: A Social Welfare Function (SWF), denoted by φ, is a function that assigns a preference relation $\succsim_P$ to every possible preference profile P. We write the output of $\varphi(P)$ as $\succsim_P$.

Remark 1. Something to note with SWF:
1. SWF is a way to aggregate preference to form an "overall preference" of the individuals. Note that this means for any profile P, φ cannot simply spit-out a single element in $\mathcal{X}$ and declare it to be 'the' choice. The output has to be a preference relation, which is able to compare every element in the alternatives $\mathcal{X}$.
2. We require SWF to be a function, i.e., it has to produce a social preference for any possible preference profile. This properties is sometimes referred in the literature as *unrestricted domain.*

Naturally, we wish an SWF to satisfy certain properties we deem "fair" or "obvious."

Axiom 1 (Pareto Efficiency (PE))**.** An SWF is said to satisfy *PE* if for any profile $P = (\succsim_i)_{i\in\mathcal{N}}$ such that $x \succsim_i y$ $\forall i \in \mathcal{N}$, we have $x \succsim_P y$.

Remark 2. That is to say, if everyone prefers x to y, then the aggregated social preference should also reflect that.

Axiom 2 (Independent of Irrelevant Alternatives (IIA))**.** An SWF is said to satisfy *IIA* if for any two profiles $P = (\succsim_i)_{i\in\mathcal{N}}$ and $P' = (\succsim_i')_{i\in\mathcal{N}}$ such that $x \succsim_i y$ iff $x \succsim_i' y$, we have $x \succsim_P y$ iff $x \succsim_{P'} y$.

Remark 3. That is, if two preference profiles ranks alternatives x and y the same way, then the aggregate preferences should rank x and y the same way, regardless of how other alternatives are ranked in the two profiles.

Theorem 1 (Arrow's Impossibility Theorem)**.** Under an SCM with $|\mathcal{X}| \geq 3$, any SWF φ that satisfies PE and IIA is dictatorial, that is, $\exists i^* \in \mathcal{N}$ such that $\varphi(P) = \succsim_{i^*}$ $\forall P \in \mathcal{P}$.

2 Proof of the Theorem

Definition 3 (Decisiveness)**.** Under an SCM, a group of individuals $\mathcal{G} \subseteq \mathcal{N}$ is said to be *decisive* over a pair of alternatives $\{x,y\} \subseteq \mathcal{X}$ if $x \succsim_P y$ whenever preference profile P satisfies $x \succsim_i y$ $\forall i \in \mathcal{G}$.

Lemma 1.1 (Globally Decisive Group)**.** Suppose an SWF satisfies PE and IIA. If a group $\mathcal{G}$ is decisive over a pair $\{x,y\} \subseteq \mathcal{X}$, then the group is decisive over every pair in $\mathcal{X}$.

Proof. Let $\{a,b\} \subseteq \mathcal{X}$ be different from $\{x,y\}$. Suppose that in a certain profile P, we have $a \succsim_i x$ and $y \succsim_i b$ $\forall i \in \mathcal{N}$. By PE, we must have $a \succsim_P x$ and $y \succsim_P b$. Further suppose that in profile P, we have $x \succsim_i y$ $\forall i \in \mathcal{G}$. Then, since $\mathcal{G}$ is decisive over $\{x,y\}$, we have $x \succsim_P y$. Since $\succsim_P$ must be transitive, we have $a \succsim_P b$. Notice that for the individuals outside of $\mathcal{G}$, the preference relation between a and b is unspecified under P. Consider another preference profile P' where the preference relation between $\{a,b\}$ is the same as P for all individuals, but the preferences over other alternatives, including $\{x,y\}$, are arbitrary. By IIA, we must have $a \succsim_{P'} b$ since $a \succsim_P b$. Hence, the group $\mathcal{G}$ is also decisive over the pair $\{a,b\}$. Similarly, we can consider pairs $\{x,b\}$ or $\{a,y\}$ and conclude that $\mathcal{G}$ is decisive over those pairs. ■

Remark 4. By Lemma 1.1, decisiveness over any pair entails decisiveness over every pair. Hence, we will simply refer to groups as *decisive* without specifying the pair of alternatives.

Lemma 1.2 (Contraction of Decisive Group)**.** Suppose an SWF satisfies PE and IIA. If $\mathcal{G}$ is decisive (with more than one individual), then a proper subset of $\mathcal{G}$ is also decisive.

Proof. Partition the group $\mathcal{G}$ into two non-empty sub-groups: $\mathcal{G}_1$ and $\mathcal{G}_2$. Suppose that we have a profile P such that $x \succsim_i y$ and $x \succsim_i z$ $\forall i \in \mathcal{G}_1$, also $x \succsim_i y$ and $z \succsim_i y$ $\forall i \in \mathcal{G}_2$. Since $\mathcal{G}$ is decisive, we have $x \succsim_P y$. Consider two cases:
- Suppose $z \succsim_P x$, then by transitivity we have $z \succsim_P y$. Notice that no assumption about the preference relation over $\{y,z\}$ is made apart from the individuals in $\mathcal{G}_2$. Hence, by IIA, $\mathcal{G}_2$ is decisive since $\mathcal{G}_2$ is decisive over $\{y,z\}$.
- Suppose $z \not\succsim_P x$, then by completeness we have $x \succsim_P z$. Similarly, no assumption about the preference relation over $\{x,z\}$ is made apart from the individuals in $\mathcal{G}_1$. Hence, by IIA, $\mathcal{G}_1$ is decisive.

Therefore, between $\mathcal{G}_1$ and $\mathcal{G}_2$, one of which must be decisive. ■

Proof of Arrow's Impossibility Theorem. By PE, all individuals as a group $\mathcal{N}$ is decisive. By Lemma 1.2, we known that a proper subset of $\mathcal{N}$ is, too, decisive. Since $\mathcal{N}$ is finite, this process of 'contracting' the decisive group terminates when the decisive group contains only one individual, the dictator. ■

[*]This note draws heavily from Maskin and Sen (2014) *The Arrow Impossibility Theorem* and Rubinstein (2006) *Lecture Notes in Microeconomic Theory.* This note is essetionally a quick summary of some of the results mentioned in both for my own reference.