

1 Measure Preserving System and Ergodicity

Definition 1 (Measure Preserving System). A **Measure Preserving System (MPS)** is a quadruple $(\Omega, \mathcal{B}, \mu, T)$ where $(\Omega, \mathcal{B}, \mu)$ is a measure space, and

1. $T : \Omega \rightarrow \Omega$ is a measurable transformation,
2. μ is T -invariance, i.e., $\mu(T^{-1}E) = \mu(E) \forall E \in \mathcal{B}$.

If μ is a probability measure, then the quadruple is called a **Probability Preserving System (PPS)**.

Remark 1. The transformation T need not be rigid body motions when $\Omega = \mathbb{R}^n$. Consider dividing up \mathbb{R}^n by grids into blocks and a transformation T that shuffles the blocks around. This is clearly a **MPS** when the space is equipped with Borel σ -algebra and Lebesgue measure.

Definition 2 (Ergodic). An **PPS** $(\Omega, \mathcal{B}, \mu, T)$ is said to be ergodic if $E \in \mathcal{B}$ is T -invariant, i.e., $E = T^{-1}(E)$, then either $\mu(E) = 0$ or $\mu(E) = 1$.

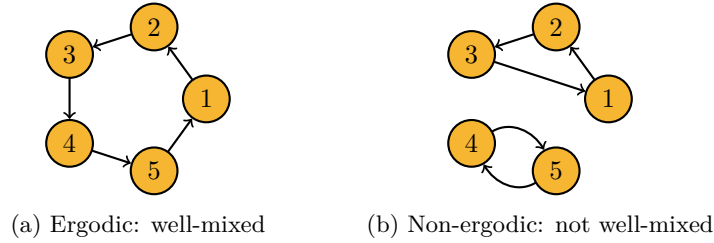


Figure 1: Ergodic v.s. Non-ergodic **PPS**.

Remark 2. An ergodic **PPS** is a system in which the only T -invariant subspaces are either negligible (measure zero) or the entire space itself. That is, an ergodic **PPS** is a system that is “well-mixed.” In **Figure 1**, two systems are shown where $\Omega = \{1, \dots, 5\}$ and transformation T is denoted by arrows. In **Figure 1b**, there are two non-trivial T -invariant subspaces that does not “mix” with each other; whereas in **Figure 1a**, every element in Ω are “mixed” with each other.

Lemma 1. Let $(\Omega, \mathcal{B}, \mu, T)$ be an **PPS**, then the following are equivalent:

1. the **PPS** is ergodic,
2. if $E \in \mathcal{B}$ and $\mu(E \triangle T^{-1}E) = 0$, then either $\mu(E) = 0$ or $\mu(E) = 1$,
3. if $f : \Omega \rightarrow \mathbb{R}$ is measurable and $f \circ T = f$ a.e., then f is constant a.e.

*This Introduction draws heavily from lecture note Sarig, 2023.

Proof. We show (1. \implies 2.), (2. \implies 3.), and (3. \implies 1.) respectively.

- (1. \implies 2.) Suppose there is a set $E_0 \in \mathcal{B}$ such that $E_0 = T^{-1}E_0$ and $\mu(E \triangle E_0) = 0$. By ergodicity, we have $\mu(E_0) = 0$ or $\mu(\Omega - E_0) = 0$. And since $\mu(E \triangle E_0) = 0$ implies $\mu(E) = \mu(E_0)$, we have $\mu(E) = 0$ or $\mu(\Omega - E) = 0$.

Now we construct E_0 . Consider the set $E_0 = \{\omega \in \Omega : T^{-k}(\omega) \in E \text{ i.o.}\}$. Clearly, E_0 is measurable and T -invariant. Also, we have $E \triangle E_0 \subseteq \bigcup_{k \geq 1} E \triangle T^{-k}E$. Hence, we have

$$\begin{aligned} \mu(E \triangle E_0) &\leq \sum_{k \geq 1} \mu(E \triangle T^{-k}E) \\ &\leq \sum_{k \geq 1} \sum_{j=0}^{k-1} \mu(T^{-j}E \triangle T^{-(j+1)}E) = \sum_{k \geq 1} k\mu(E \triangle T^{-1}E) \end{aligned}$$

where the last inequality is obtained by the fact that

$$\mu(A_1 \triangle A_2) \leq \mu(A_1 \triangle A_3) + \mu(A_3 \triangle A_2)$$

for all $A_i \in \mathcal{B}$. Since $\mu(E \triangle T^{-1}E) = 0$, we have that $\mu(E \triangle E_0) = 0$.

- (2. \implies 3.) Let f be a measurable function s.t. $f \circ T = f$ a.e. For any $y \in \mathbb{R}$, we have $[f > y] \triangle T^{-f > y} \subseteq [f \neq f \circ T]$, hence

$$\mu([f > y] \triangle T^{-f > y}) = 0.$$

By the assumption, either $\mu[f > y] = 0$ or $\mu[f \leq y] = 0$, i.e., either $f > y$ a.e. or $f \leq y$ a.e. Let $c := \sup\{y \in \mathbb{R} : f > y \text{ a.e.}\}$, then $f = c$ a.e.

- (3. \implies 1.) Let $E \in \mathcal{B}$ satisfies $E = T^{-1}(E)$. Consider $f = \mathbf{1}_E$. Since $f \circ T = f$, f is constant a.e., we have $f = 0$ a.e. or $f = 1$ a.e., implying that either $\mu(E) = 0$ or $\mu(E) = 1$. #

Remark 3. The third characterization is quite interesting and intuitive. Consider again [Figure 1b](#), where the *PPS* is equipped with probability space $\Omega = \{1, \dots, 5\}$, $\mathcal{B} = 2^\Omega$, and uniform μ . We can define the function f as follows:

$$f(\omega) = \begin{cases} 0 & \text{if } \omega \in \{1, 2, 3\}, \\ 1 & \text{otherwise.} \end{cases}$$

It is clear that $f \circ T = f$, but f is not constant on Ω . This is achievable since non-ergodicity means there are non-trivial T -invariant subspaces, and we can simply define f to be constant on each subspace. However, in [Figure 1a](#), since the system is “well-mixed,” this trick is not possible.

Definition 3 (Strong Mixing). An *PPS* $(\Omega, \mathcal{B}, \mu, T)$ is called strong mixing if for any $E, F \in \mathcal{B}$ we have

$$\mu(E \cap T^{-n}F) \rightarrow \mu(E)\mu(F) \quad \text{as } n \rightarrow \infty.$$

Remark 4. The fact that strong mixing is defined using the inverse map of T is not merely due to measure-theoretic technicalities. The interpretation is “no matter what obscure events F one chooses, it could have been from all over in the system entirely randomly, not just some specific part.” Thus, for any other event E , the “origins” of F must be as if independent of E .

Lemma 2. *Strong mixing implies ergodicity.*

Proof. Suppose $(\Omega, \mathcal{B}, \mu, \mathbb{T})$ is strong mixing. Let E be such that $E = \mathbb{T}^{-1}E$, then we have

$$\mu(E) = \mu(E \cap \mathbb{T}^{-n}E) \rightarrow \mu(E)^2 \quad \text{as } n \rightarrow \infty.$$

Clearly, $\mu(E) = \mu(E)^2$ implies $\mu(E)$ is either 0 or 1. #

2 Ergodicity Theorem

Theorem 1 (von Neumann's Ergodicity). *Let \mathcal{H} be a Hilbert space. Let $U : \mathcal{H} \rightarrow \mathcal{H}$ be a unitary operator. Let $\mathcal{I} = \{f \in \mathcal{H} : Uf = f\}$ denote the U -invariant subspace. Let $P : \mathcal{H} \rightarrow \mathcal{I}$ be the orthogonal projection onto \mathcal{I} . Then, for any $f \in \mathcal{H}$, we have*

$$\frac{1}{n} \sum_{i=1}^n U^i f \rightarrow Pf \quad \text{as } n \rightarrow \infty \quad (1)$$

in the norm induced by the inner product on \mathcal{H} .

Proof. Clearly, Equation 1 holds when $f \in \mathcal{I}$. Let $\mathcal{J} := \{g - Ug : g \in \mathcal{H}\}$. Suppose $f \in \mathcal{J}$, we have

$$\langle f, h \rangle = \langle g, h \rangle - \langle Ug, h \rangle = \langle g, h \rangle - \langle g, Uh \rangle = 0 \quad \forall h \in \mathcal{I}.$$

Hence, $Pf = 0$. Furthermore, we have

$$\left\| \frac{1}{n} \sum_{i=1}^n U^i f \right\|_2 = \frac{1}{n} \|Ug - U^{n+1}g\|_2 \leq \frac{1}{n} \|2g\|_2 \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Therefore, Equation 1 holds for \mathcal{J} .

We claim that Equation 1 also holds for the closure of \mathcal{J} , denoted by $\bar{\mathcal{J}}$. Suppose $f \in \bar{\mathcal{J}}$, then $\forall \varepsilon > 0 \exists g \in \mathcal{J}$ s.t. $\|f - g\|_2 < \varepsilon$. Choose N such that $\left\| \frac{1}{n} \sum_{i=1}^n U^i g \right\|_2 < \varepsilon$ $\forall n > N$. Then, we have

$$\left\| \frac{1}{n} \sum_{i=1}^n U^i f \right\|_2 \leq \left\| \frac{1}{n} \sum_{i=1}^n U^i (f - g) \right\|_2 + \left\| \frac{1}{n} \sum_{i=1}^n U^i g \right\|_2 \leq 2\varepsilon.$$

Thus, the claim holds.

Lastly, we claim that $\bar{\mathcal{J}}^\perp = \mathcal{I}$. Suppose $f \perp \bar{\mathcal{J}}$, we have that $\forall g \in \mathcal{H}$,

$$0 = \langle f, g - Ug \rangle = \langle f, g \rangle - \langle f, Ug \rangle \implies \langle f, g \rangle = \langle U^* f, g \rangle.$$

This implies that $f = Uf$ a.e., i.e., $f \in \mathcal{I}$. Conversely, we have shown that if $f \in \mathcal{J}$, then $f \perp \mathcal{I}$. Hence, the claim holds. And since $\mathcal{H} = \bar{\mathcal{J}} \oplus \mathcal{I}$, Equation 1 holds for all $f \in \mathcal{H}$. #

Remark 5. *If we think about Theorem 1 in linear algebra terms, it lends itself to an intuitive understanding. In Figure 2, we consider \mathbb{R}^3 space with U being a rotation along the z -axis. Clearly, a rotation is an unitary transformation, and its corresponding invariant subspace is exactly the z -axis. Consider an arbitrary vector f and its transformations $U^i f$. As $U^i f$ rotates along the z -axis, it is clear that their "average" converges to Pf , the orthogonal projection of f onto the z -axis. One can also easily see that vectors of the form $f - Uf$ is orthogonal to the z -axis.*

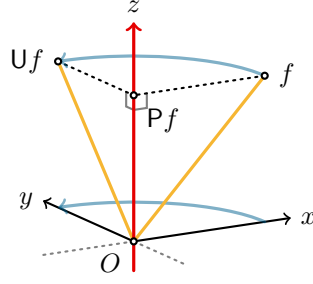


Figure 2: Visualization of **Theorem 1**.

Corollary 1.1. Let $(\Omega, \mathcal{B}, \mu, \mathbb{T})$ be an **PPS**. If $f \in L^2(\Omega)$, then

$$\frac{1}{n} \sum_{i=1}^n f \circ \mathbb{T}^i \xrightarrow{L^2} \bar{f} \quad \text{as } n \rightarrow \infty$$

where $\bar{f} \in L^2(\Omega)$ is \mathbb{T} -invariant. Furthermore, if $(\Omega, \mathcal{B}, \mu, \mathbb{T})$ is ergodic, then

$$\bar{f} = \int_{\Omega} f d\mu.$$

Proof. Note that $L^2(\Omega)$ is a Hilbert space. Let $Uf := f \circ \mathbb{T}$. Since $\|f \circ \mathbb{T}\|_2 = \|f\|_2$ for $f \in L^2(\Omega)$, U is unitary. Hence, the statement is a direct implication of **Theorem 1**. Furthermore, if the **PPS** is ergodic, then by **Lemma 1**, the \mathbb{T} -invariant subspace \mathcal{I} contains only constant functions. Therefore, the orthogonal projection of f onto \mathcal{I} is its expectation. #

Remark 6. The sum still converges without ergodicity, but this is not very useful, since we do not know the form of \bar{f} . With ergodicity, we know \bar{f} is a constant function. Hence, we can simply pick any starting point $\omega \in \Omega$ and compute $f \circ \mathbb{T}^i(\omega)$ along the way to obtain the same constant \bar{f} , without worrying about the entire functional space.

3 Markov Chains under the Language of Ergodicity

Now we want to rephrase what we already understand about Markov chains under the language of von Neumann's Ergodicity Theorem.

Definition 4 (Subshift of Finite Type). Let \mathcal{S} with be a finite set of states. Let $A = (A_{ij})_{\mathcal{S} \times \mathcal{S}}$ be a adjacency matrix with $A_{ij} \in \{0, 1\}$ and without rows or columns be entirely zero. Let Ω_A be defined as the space of possible sequences

$$\Omega_A := \{\omega = (\omega_0, \omega_1, \dots) : A_{\omega_i, \omega_{i+1}} = 1 \ \forall i\}.$$

A **Subshift of Finite Type (SFT)** with states \mathcal{S} with adjacency matrix A is the triple $(\Omega_A, d, \mathbb{T})$ where $d(\omega, \omega') = 2^{-\min\{k: \omega_k \neq \omega'_k\}}$ is a metric and $\mathbb{T}(\omega_0, \omega_1, \dots) = (\omega_1, \omega_2, \dots)$ is a shift transformation.

Remark 7. The topology generated by the metric $d(\cdot, \cdot)$ is equivalent to the topology generated by sets of the form

$$[s_0, \dots, s_{n-1}] := \{\omega \in \Omega_A : \omega_i = s_i \ \forall i \in \{0, \dots, n-1\}\}.$$

This form are sometimes referred to as “cylinders.” This is the product topology on $S^{\mathbb{N}}$ when S is given the discrete topology.

Remark 8. The space Ω is simply all possible sequencing of \mathcal{S} that is permitted by A . Consider again [Figure 1](#), now with the arrows denoting the adjacency matrix A , then [Figure 1a](#) and [Figure 1b](#) generates different Ω_A ’s. What we want next is a way to say “how likely is any given sequence realized/observed.”

Definition 5 (Markov Measure). Given a transition matrix P , i.e., a matrix with each row consisting weights summing to one, and a probability vector π , we construct a measure μ by letting

$$\mu[s_0, \dots, s_{n-1}] := \pi_{s_0} P_{s_0 s_1} \cdots P_{s_{n-2} s_{n-1}}.$$

Remark 9. The definition of Markov measure extends to a unique probability measure on Ω_A with product topology via Carathéodory extension theorem.

Lemma 3. μ is T -invariant iff π is stationary w.r.t. P .

Proof. Note that μ is T -invariant iff $\mu[\cdot, s] = \mu[s] \ \forall s = (s_0, \dots, s_{n-1})$. That is,

$$\begin{aligned} \sum_{t \in \mathcal{S}} \pi_t P_{ts_0} P_{s_0 s_1} \cdots P_{s_{n-2} s_{n-1}} &= \pi_{s_0} P_{s_0 s_1} \cdots P_{s_{n-2} s_{n-1}} \quad \forall s \\ \iff \sum_{t \in \mathcal{S}} \pi_t P_{ts_0} &= \pi_{s_0} \quad \forall s_0 \in \mathcal{S}. \end{aligned} \quad \#$$

Remark 10. With [Lemma 3](#), we established that the system corresponding to the [SFT](#) is a measure preserving system as long as the Markov measure is constructed with a stationary distribution. It remains to ask the question: When is such system ergodic? When is such system strong mixing? From the intuition established previously, we know that...

- Ergodic means that the system is “well-mixed,” i.e., all the states should be able to be reached from any other state. This corresponds to the idea of irreducibility.
- Strong mixing means that the “origin” of any set could be all over the system entirely randomly, i.e., we shouldn’t observe a set that follows some path. This corresponds to the idea of aperiodicity.

It would turn out that our intuitions are spot on in describing what kinds of [SFT](#) are ergodic and strong mixing.

Definition 6 (Irreducible). A transition matrix P is called irreducible if $\forall a, b \in \mathcal{S} \ \exists s_1, \dots, s_{n-1} \in \mathcal{S} \text{ s.t. } P_{a, s_1} \cdots P_{s_{n-1}, b} > 0$. We denote this as $a \xrightarrow{n} b$.

Lemma 4. If P is irreducible, then $\gcd\{n \in \mathbb{N} : a \xrightarrow{n} a\}$ is independent of a .

Proof. Suppose $p_a = \gcd\{n : a \xrightarrow{n} a\}$ and $p_b = \gcd\{n : b \xrightarrow{n} b\}$ for $a, b \in \mathcal{S}$. Since P is irreducible, we have $a \xrightarrow{m_1} b \xrightarrow{m_2} a$ for some $m_1, m_2 \in \mathbb{N}$ and $m_b \in \{n : b \xrightarrow{n} b\}$. Since $p_a \mid m_1 + m_b + m_2$ and $p_b \mid m_b$, we must have $p_a \mid p_b$. Similarly, we have $p_b \mid p_a$ by swapping a and b . Therefore, $p_a = p_b$. $\#$

Definition 7 (Period). The period of an P is $\gcd\{n : a \xrightarrow{n} a\}$. An irreducible P is aperiodic if $\gcd\{n : a \xrightarrow{n} a\} = 1$.

Remark 11. 7 is well-defined since Lemma 4 guarantees that as long as P is irreducible, then the period is independent of state.

Theorem 2 (Ergodicity for Markov Chains). Let P be a transition matrix and let P_{ab}^n denote the (a, b) -th element of P^n . Let π be a stationary distribution of P . Then,

1. if P is irreducible, then as $n \rightarrow \infty$,

$$\frac{1}{n} \sum_{i=1}^n P_{ab}^i \rightarrow \pi_b \quad \forall a, b \in \mathcal{S}.$$

2. if P is irreducible and aperiodic, then as $n \rightarrow \infty$,

$$P_{ab}^n \rightarrow \pi_b \quad \forall a, b \in \mathcal{S}.$$

Proof. Omitted. See, e.g., Chapter 1.5 in Sarig, 2023 for a proof. $\#$

Corollary 2.1. Let $(\Omega_A, \mathcal{B}, \mu, T)$ be the **PPS** corresponding to the **SFT** (Ω_A, d, T) with Markov measure μ under transition matrix P . If P is irreducible, then the **PPS** is ergodic. Furthermore, if P is also aperiodic, then the **PPS** is strong mixing.

Proof. First, note that for all cylinders $[a] = [a_0, \dots, a_{n_a-1}]$ and $[b] = [b_0, \dots, b_{n_b-1}]$ and $k > n_a$, we have

$$\begin{aligned} \mu([a] \cap T^{-k}[b]) &= \mu\left(\bigsqcup_{c \in \mathcal{C}_{k-n_a}} [a, c, b]\right) \\ &= \mu[a] \left(\sum_{c \in \mathcal{C}_{k-n_a}} P_{a_{n_a-1}, c_0} \cdots P_{c_{k-n_a-1}, b_0} \right) \frac{\mu[b]}{\pi_{b_0}} \\ &= \mu[a] \mu[b] \frac{P_{a_{n_a-1}, b_0}^{k-n_a}}{\pi_{b_0}} \end{aligned} \tag{2}$$

where $\mathcal{C}_\ell := \{c = (c_0, \dots, c_{\ell-1}) : [a, c, b] \neq \emptyset\}$. By Theorem 2, we have that

$$\frac{1}{n} \sum_{k=n_a+1}^n \mu([a] \cap T^{-k}[b]) = \mu[a] \mu[b] \frac{1}{\pi_{b_0}} \left(\frac{1}{n} \sum_{k=n_a+1}^n P_{a_{n_a-1}, b_0}^{k-n_a} \right) \rightarrow \mu[a] \mu[b].$$

Now suppose that \mathbf{P} is irreducible. Let $E \in \mathcal{B}$ be invariant, then $\exists [\mathbf{a}_1], \dots, [\mathbf{a}_m]$ s.t. $\mu(E \triangle \bigsqcup_{j=1}^m [\mathbf{a}_j]) < \varepsilon$.¹ Then,

$$\mu(E) = \mu(E \cap \mathbb{T}^{-k} E) = \sum_{i=1}^m \sum_{j=1}^m \mu([\mathbf{a}_i] \cap \mathbb{T}^{-k} [\mathbf{a}_j]) \pm 2\varepsilon.$$

Taking average over k , by the fact we mentioned above, we have

$$\begin{aligned} \mu(E) &= \sum_{i=1}^m \sum_{j=1}^m \left(\frac{1}{n} \sum_k \mu([\mathbf{a}_i] \cap \mathbb{T}^{-k} [\mathbf{a}_j]) \right) \pm 2\varepsilon \\ &\rightarrow \sum_{i=1}^m \sum_{j=1}^m \mu[\mathbf{a}_i] \mu[\mathbf{a}_j] \pm 2\varepsilon = \left(\sum_{i=1}^m \mu[\mathbf{a}_i] \right)^2 \pm 2\varepsilon = (\mu(E) \pm \varepsilon)^2 \pm 2\varepsilon \end{aligned}$$

as $n \rightarrow \infty$. Therefore, take $\varepsilon \downarrow 0$ and we have that $\mu(E) = \mu(E)^2$, which implies either $\mu(E) = 0$ or $\mu(E) = 1$. Hence, we have ergodicity.

Now suppose further that \mathbf{P} is aperiodic. By [Theorem 2](#) and (2), we have $\mu([\mathbf{a}] \cap \mathbb{T}^{-k} [\mathbf{b}]) \rightarrow \mu[\mathbf{a}] \mu[\mathbf{b}]$ as $k \rightarrow \infty$. By a similar argument with approximation using cylinders, we obtain the desired result. $\#$

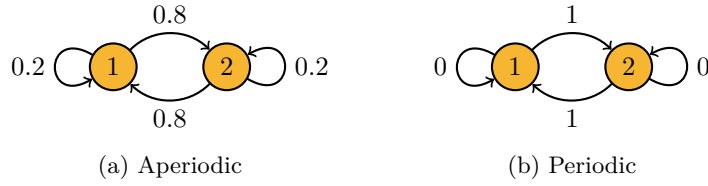


Figure 3: Markov Chains under different transition matrix \mathbf{P} .

Remark 12. [Figure 3](#) shows two Markov chains with same adjacency \mathbf{A} but different transition \mathbf{P} . One can easily check both chains represented in [Figure 3a](#) and [Figure 3b](#) share the same invariant distribution $\pi = (0.5, 0.5)$. However, notice that [Figure 3b](#) has period 2. Hence, \mathbf{P}_{11}^n converges to 0.5 in [Figure 3a](#) but fails to converge in [Figure 3b](#).

Acronyms

MPS	Measure Preserving System. 1
PPS	Probability Preserving System. 1 , 2 , 4 , 6
SFT	Subshift of Finite Type. 4-6

References

Sarig, O. (2023). Lecture notes on ergodic theory. <https://www.weizmann.ac.il/math/sarigo>

¹Such $[\mathbf{a}_1], \dots, [\mathbf{a}_m]$ always exists. We can check that

$$\mathcal{B}' := \left\{ E \in \mathcal{B} : \forall \varepsilon > 0 \exists [\mathbf{a}_1], \dots, [\mathbf{a}_n] \text{ s.t. } \mu \left(E \triangle \bigsqcup_{j=1}^n [\mathbf{a}_j] \right) < \varepsilon \right\}$$

forms a σ -algebra. And since \mathcal{B}' contains the cylinders, we have $\mathcal{B} = \mathcal{B}'$.