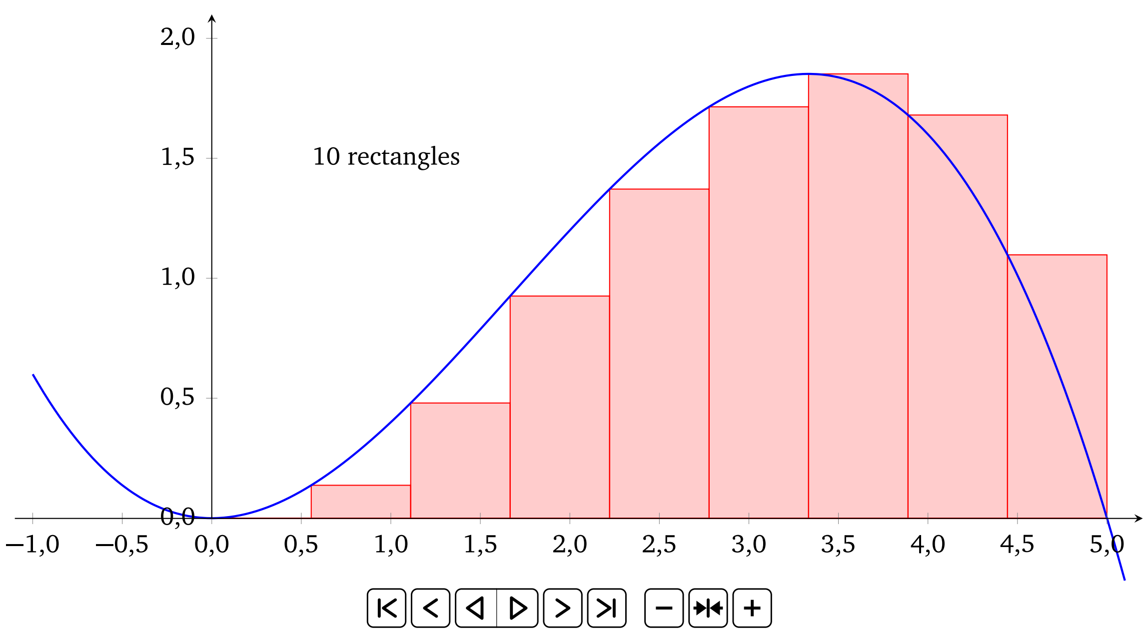
Numerical Integration Using The Riemann Left Sum Approach

Jesse op den Brouw

September 30, 2021



Please use a fully compliant PDF viewer like Acrobat Reader or Okular and press the play-button (▷) or click on the image. The function is:

$$f(x) = -\frac{1}{10}x^3 + \frac{1}{2}x^2 \tag{1}$$

1 Principle of operation

Integration using the Riemann left sum is one of the easiest method to understand. This method calculates the area under the function between values denoted by $x = a$ and $x = b$. Note that we assume the function is strictly non-negative.

When performing the integration, we divide the x axis between $x = a$ and $x = b$ in n equal parts. That is:

$$\Delta x = \frac{b-a}{n} \tag{2}$$

Of every part, we calculate the area:

$$A_{x*} = f(x*)\Delta x \qquad x* \text{ at a specific point between } a \text{ and } b \tag{3}$$

Now summing up all areas will give an approximation of the total area under the function $f(x)$. Note that we assume the function is strictly non-negative. Thus we have:

$$\int_a^b f(x)\,\mathrm{d}x \approx \sum_{i=0}^{n-1} f(x*)\Delta x \tag{4}$$

where $x*$ can be calculated with $a + i\Delta x$. Letting $n \to \infty$, where $\Delta x \to 0$, will give us an exact answer. So we have:

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(x*)\Delta x \tag{5}$$

2 Computation of the integral by a program

In any computer language that has computational powers, the area can be calculated by means of a program. For the historical reasons, we use the C programming language. We have to convert the summation so that we can program it in C. The summation can be written as:

$$A = \int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=0}^{n-1} f\underbrace{\left(a + k\cdot\underbrace{\frac{b-a}{n}}_{\Delta x}\right)}_{x*}\cdot\underbrace{\frac{b-a}{n}}_{\Delta x} \tag{6}$$

Here we have augmented $x*$ and Δx. Δx is a constant when n is chosen. So Δx is a constant in the program.

The function shown on the first page is:

$$f(x) = -\frac{1}{10}x^3 + \frac{1}{2}x^2 \tag{7}$$

The program to calculate the area under the function is shown below. The program starts with the definition of the function that accepts an x and returns an y. It is easy to adapt this function to suit your needs. In the function `main` we iterate over the number of points (or areas) and sum all the computed areas.

```
#include <stdio.h>

double f(double x) {
   return -0.1*x*x*x + 0.5*x*x;
}

int main(void) {

    int k;          /* for iteration of sum */
    int n = 1000;   /* number of points */

    double a = 0.0; /* start point */
    double b = 5.0; /* end point */

    double deltax = (b - a) / n;

    double sum = 0.0;

    for (k = 0; k < n; k++) {
        sum = sum + f(a + k*deltax) * deltax;
    }

    printf("Left rule Riemann sum = %.20e\n", sum);

    return 0;
}
```

Calculating the integral using this program yields 5.208328. The exact answer is 125/24 which is $5{,}208\overline{3}$. (The 3 is repeated indefinitely.)

3 Negative values of the function

If a function exhibits negative y values, the corresponding areas are designated as negative. So the complete answer to the integral is:

$$A_{total} = A_{positive} + A_{negative} \tag{8}$$

where $A_{negative}$ is negative.

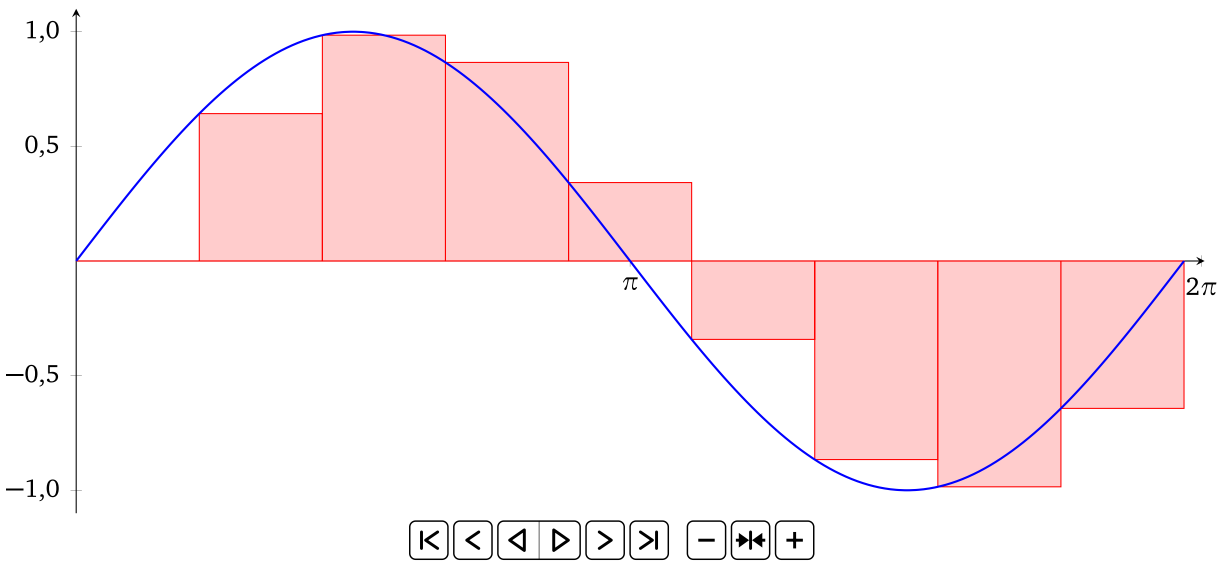
4 Applications in electrical engineering

One application is the average value of a sine wave voltage. The average voltage is 0 V. This is shown in the figure below. When integrating over a full period of the sine wave the positive areas equals the negative areas, hence the total area is 0.

This means that:

$$\int_0^{2\pi} \sin(x)\,\mathrm{d}x = 0 \tag{9}$$

An illustration of the summation is shown in the figure below.



Another application is the average power dissipated in a resistor when applying a sine wave voltage. Finding the average power is calculated by:

$$P_{avg} = \frac{1}{T}\int_0^{T} u(t)\cdot i(t)\,\mathrm{d}t \tag{10}$$

Since a resistor introduces no phase shift between the voltage over and the current through the resistor, we can write:

$$P_{avg} = \frac{1}{T}\int_0^{T} \hat{u}\sin t \cdot \hat{\imath}\sin t\ \mathrm{d}t \tag{11}$$

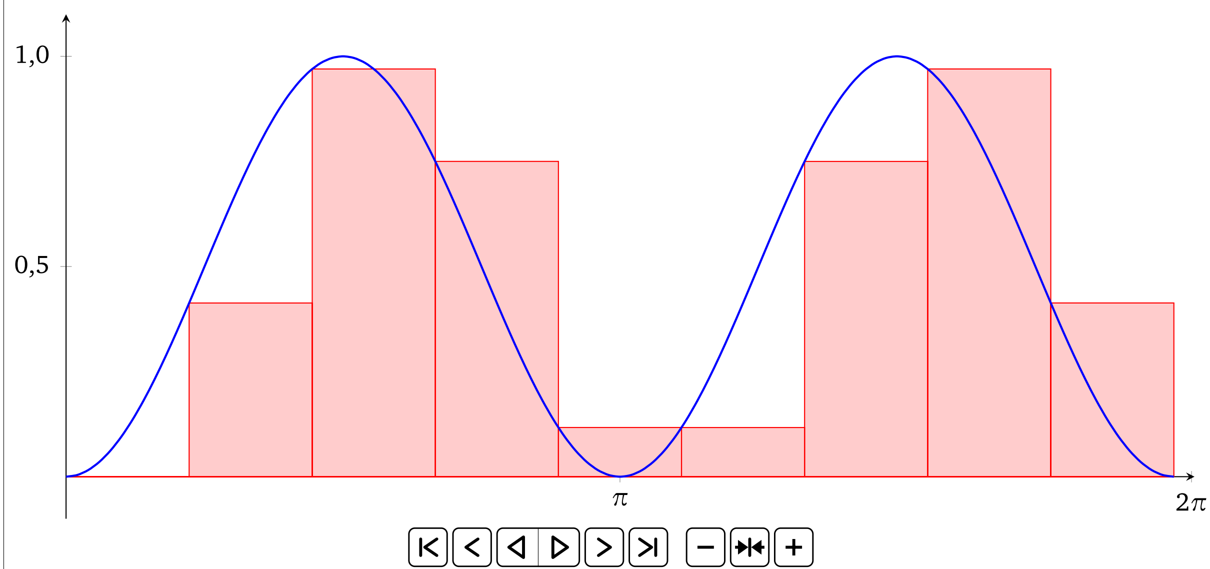
where $\hat{u}$ and $\hat{i}$ are the maximum values of the voltage and the current. Now $\hat{u}$ and $\hat{\imath}$ are constant over time, so we can put them before the integral and we can rewrite the function:

$$P_{avg} = \hat{u}\hat{\imath}\frac{1}{T}\int_0^T \sin^2 t \; \mathrm{d}t \tag{12}$$

Since by definition $T = 2\pi$ for one period, we can write:

$$P_{avg} = \hat{u}\hat{\imath}\frac{1}{2\pi}\int_0^{2\pi} \sin^2 t \; \mathrm{d}t \tag{13}$$

Now we can plot the function and integrate it by the Riemann left rule:



The summation by 1000 steps equals 3.141593, the exact value is π. Using the constant before the integral we get:

$$P_{avg} = \hat{u}\hat{\imath}\frac{1}{2\pi}\int_0^{2\pi} \sin^2 t \; \mathrm{d}t = \hat{u}\hat{\imath}\frac{1}{2\pi}\pi = \frac{1}{2}\hat{u}\hat{\imath} \tag{14}$$