Hauw’s Problem

-or-

Finding the best straight line for approaching a temperature rising curve

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In memory of colleague Hauw Khoe

Abstract

This document discusses the best straight line to approach the standard curve of the temperature rise of an infinitesimal object.

1 Introduction

Colleague Hauw Khoe has written a document for students of the XYZ[1] study programme. The students had to learn about the temperature rising of an (infinitesimal) object when exposed to a sudden temperature increase. Of course he suggested a function that described the course of the temperature. Everyone knows that this function harbors an exponential function and Hauw described this in his document. However, the lectures of the said study programme felt that this description was ”too difficult” for their students and preferred to use an approach in the form of a straight line. The students would understand this much better (Hauw suggested that this never could be the proper function, but his words were not heeded). Hauw opted for a straight line, starting at T_{min}, with a slope of $1/\tau$, up to T_{max}, after which the function further had a constant value of T_{max}.

2 Heating an object

Before we delve into the rest of the document, we should show how the temperature of an suddenly heated object elapses. The temperature of an object exposed to a sudden increase of temperature is, over time:

$$T_{\text{offical}}(t) = T_{min} + (T_{max} - T_{min}) \cdot (1 - \mathrm{e}^{-t/\tau}) \tag{1}$$

where T_{min} is the temperature of the object before exposure and T_{max} is the ultimate temperature of the object. The variable τ is the time constant. At $t = \tau$, the temperature has reached about 63 % of its final value, starting from T_{min} and ending at T_{max}. See Figure 1.

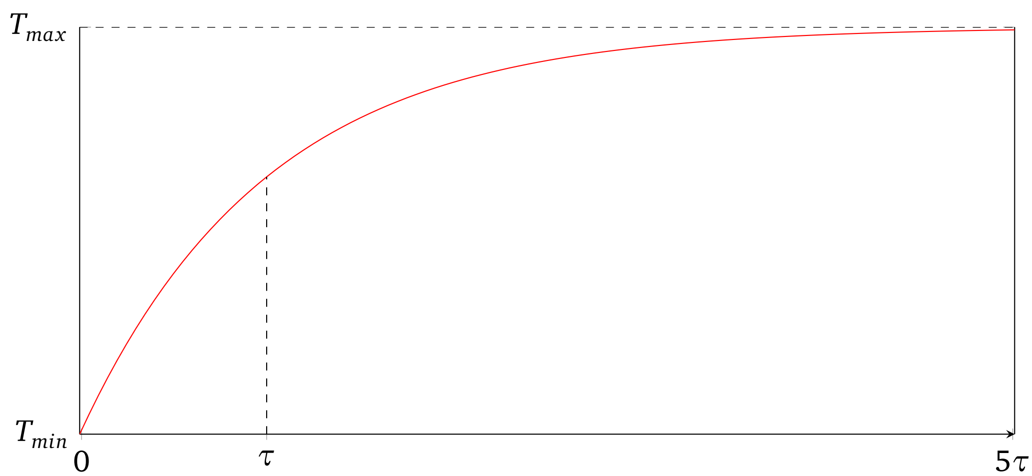

Figure 1: *Rising temperature of an object.*

[1] Not to be disclosed

Now removing the occurrences of T_{min} and T_{max} yields only the exponential function, which is of our main concern. The function is shown in Figure 2. The function will reach approximately 63 % of the end value at time τ. The function is:

$$T(t) = 1 - \mathrm{e}^{-t/\tau} \tag{2}$$

which leaves us with a value of 0.632120559 at $t = \tau$.

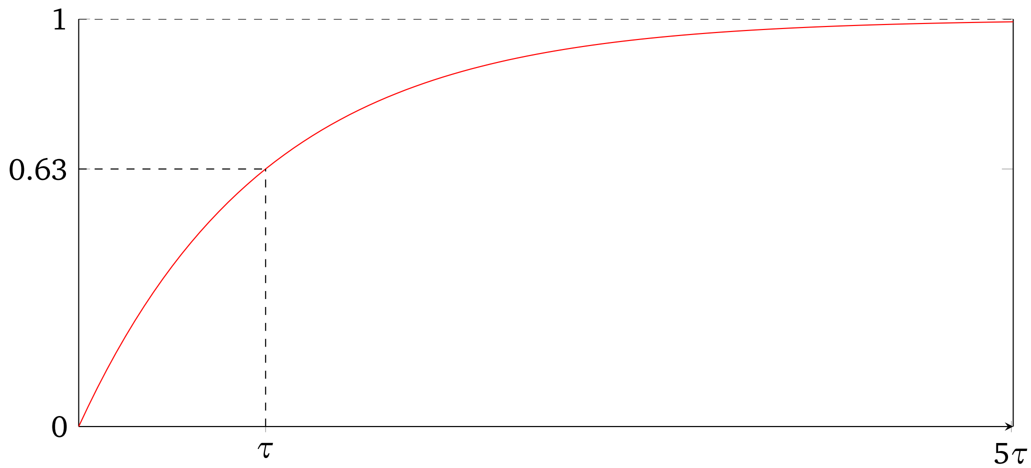

Figure 2: *Relative rising temperature of an object.*

3 Suggested function by Hauw

Hauw suggested a straight line function with of slope of $1/\tau$ which will reach the end value at time τ. After that, the function stays at the end value. See Figure 3. So Hauw's function is:

$$H(t) = \begin{cases} t/\tau & \text{for } 0 \le t \le \tau \\ 1 & \text{for } \tau \le t < \infty \end{cases} \tag{3}$$

Obviously, there is a discrepancy between Hauw's function and the real function as shown in Figure 3. The maximum discrepancy is denoted by Δy and is

$$1 - (1 - \mathrm{e}^{-1}) = 0.367879441 \qquad (t = \tau) \tag{4}$$

The deviation between Hauw's function and the relative rising of the temperature can be seen in Figure 4. As can be seen, the maximun deviation is at $t = \tau$.

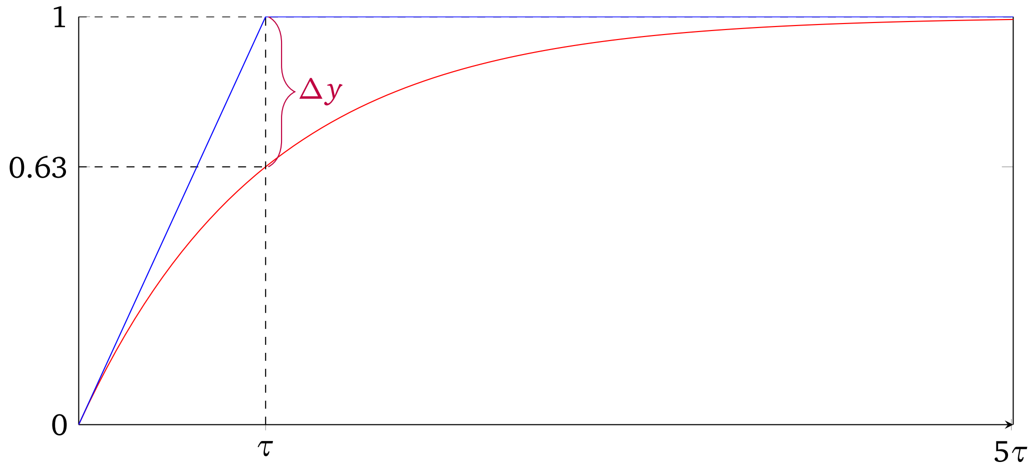

Figure 3: *Hauw's function $H(t)$ and relative rising temperature of an object $T(t)$.*

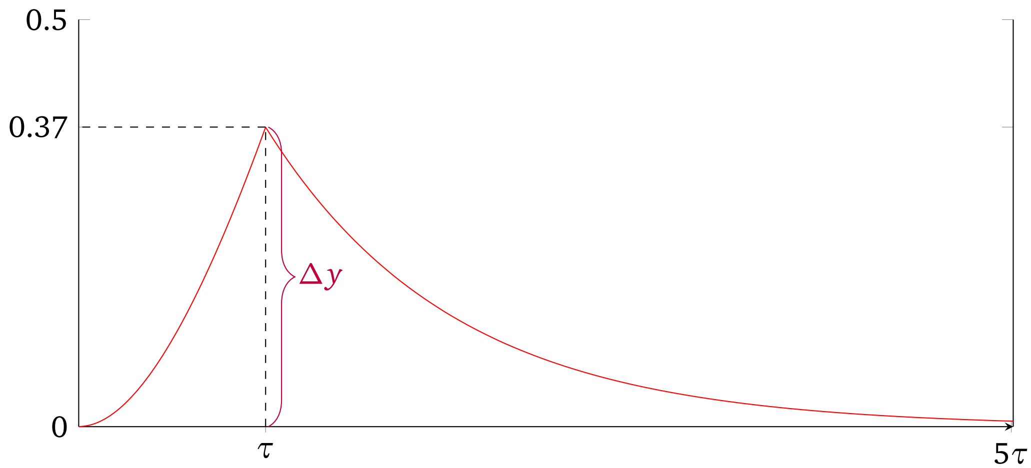

Figure 4: *Difference between Hauw's function $H(t)$ and relative rising of the temperature of an object $T(t)$.*

4 Proposed function

We propose a moderation of Hauw's function. We introduce a variable α to control the slope of the straight line. Note that τ is constant for a certain object.

$$P(t) = \begin{cases} t/\alpha\tau & \text{for } 0 \leq t \leq \alpha\tau \\ 1 & \text{for } \alpha\tau \leq t < \infty \end{cases} \tag{5}$$

Note that is a non-continuous function. Hauw's function $H(t)$, the rising temperature function $T(t)$ and the proposed function $P(t)$ are shown in Figure 5. Note that $P(t)$ and $T(t)$ intersect at $t = z\tau$. Note that z depends on α.

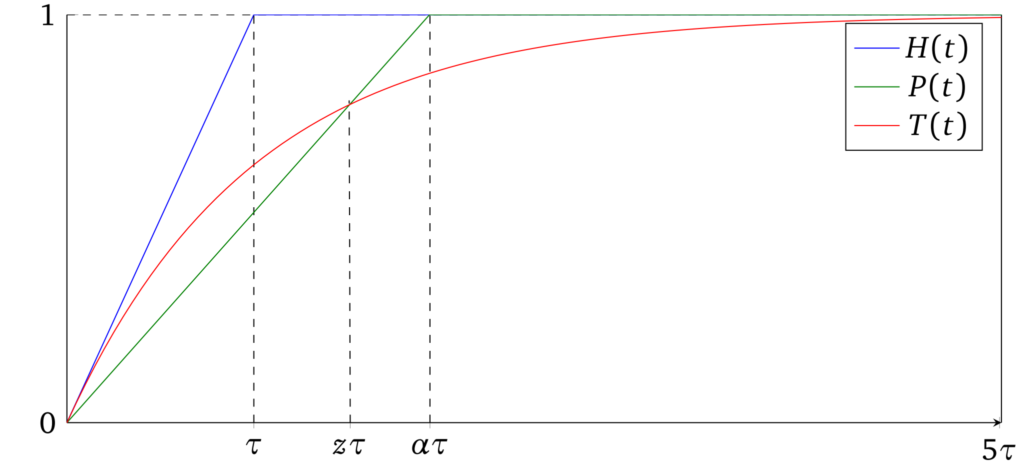

Figure 5: *Showing Hauw's function $H(t)$ (blue), temperature rising function $T(t)$ (red) and the proposed function $P(t)$ (green).*

Obviously, we can see that $\alpha \geq 1$ since $\alpha = 1$ for $P(t)$, $P(t)$ is equal to $H(t)$. We need to find an α where the difference between $P(t)$ and $T(t)$ is minimal. We denote this function $G(t)$. This difference is shown in Figure 6. So:

$$G(t) = P(t) - T(t) \tag{6}$$

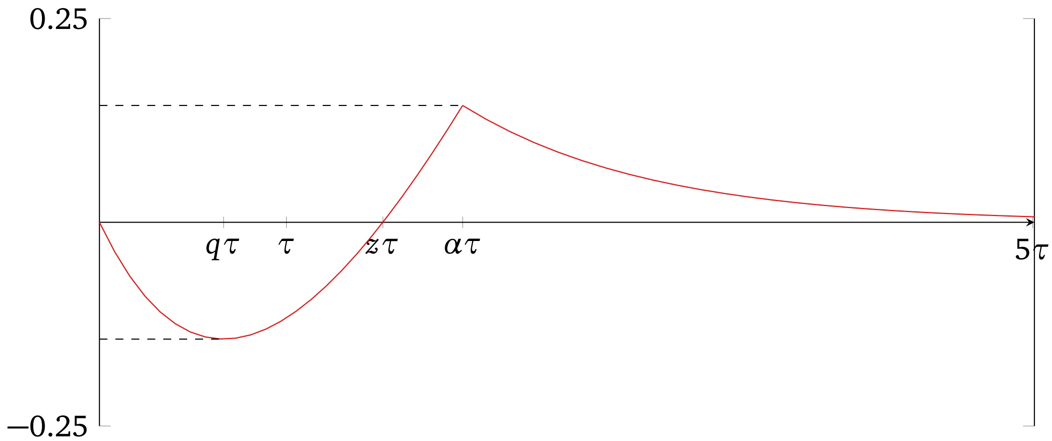

Figure 6: *Function $G(t)$, the difference between $P(t)$ and $T(t)$.*

We split $G(t)$ in two helper functions to describe the flow:

$$G_1(t) = P(t) - T(t) = t/\alpha\tau - (1 - \mathrm{e}^{-t/\tau}) \qquad 0 \leq t \leq \alpha\tau \tag{7}$$
$$G_2(t) = P(t) - T(t) = 1 - (1 - \mathrm{e}^{-t/\tau}) \qquad \alpha\tau \leq t < \infty \tag{8}$$

For function $G_1(t)$ there is a minimin at $t = q\tau$. We need to find this minimum. So we have:

$$\frac{\mathrm{d}G_1(t)}{\mathrm{d}t} = 0 \quad \longrightarrow \quad \frac{1}{\alpha\tau} - \frac{\mathrm{e}^{-t/\tau}}{\tau} = 0 \quad \longrightarrow \quad t = \tau \ln \alpha \tag{9}$$

so $q = \ln \alpha$. Now we fill in this result into $G_1(t)$ we get:

$$G_1(t)\Big|_{t=\tau \ln \alpha} = \frac{1}{\alpha} \ln \alpha + \frac{1}{\alpha} - 1 \tag{10}$$

Note that this value is *negative* for $\alpha > 1$. Also note τ is disappeared in this function. So the point of the lowest y value of $G_1(t)$ solely depends on α.

The maximum value of $G(t)$ is where $G_1(t)$ meets $G_2(t)$. This is by definition at $t = \alpha\tau$. We could use either function, because the result is the same:

$$G_1(t)\Big|_{t=\alpha\tau} = G_2(t)\Big|_{t=\alpha\tau} \tag{11}$$

Using $G_2(t)$ we find:

$$G_2(t)\Big|_{t=\alpha\tau} = 1 - (1 - \mathrm{e}^{-t/\tau})\Big|_{t=\alpha\tau} = \mathrm{e}^{-\alpha\tau/\tau} = \mathrm{e}^{-\alpha} \tag{12}$$

Note that this value is *positive*. Also note that τ has disappeared in this function.

Now the optimum value for α is when the deviations of $G_1(t)$ equals $G_2(t)$ so neither deviation is greater or smaller than the other. Because the maximum of $G_1(t)$ is negative, we have to change the sign. Changing the sign of $G_1(t)$ we find:

$$1 - \frac{1}{\alpha} - \frac{1}{\alpha} \ln \alpha = \mathrm{e}^{-\alpha} \tag{13}$$

This non-linear function can only be solved by approximation, so we get:

$$\begin{aligned} \alpha_1 &= 0.42663268550049 \\ \alpha_2 &= 1.94237704854534 \end{aligned} \tag{14}$$

The value of α_1 is not applicable because $\alpha \geq 1$. So the only correct result is α_2. Using this value we can calculate z and q (see Figure 6).

For z we find:

$$
\begin{aligned}
z &\longrightarrow \{P(t) = T(t)\}\Big|_{\substack{\alpha=\alpha_2 \\ t=z\tau}} \\
&\longrightarrow \frac{z\tau}{\alpha_2\tau} = 1 - \mathrm{e}^{-z\tau/\tau} \\
&\longrightarrow \frac{z}{\alpha_2} = 1 - \mathrm{e}^{-z} \\
&\longrightarrow z = 1.51574
\end{aligned} \tag{15}
$$

and for q we find:

$$
q = \ln \alpha_2 = 0.663912506 \tag{16}
$$

For the maximum deviation, we find using $G_2(t)$:

$$
\delta = e^{-\alpha_2} = 0.143362764 \tag{17}
$$

5 Conclusion

The best approach for the function with the minimum deviation of $P(t)$ from $T(t)$ is:

$$
P(T) = \begin{cases} \frac{1}{1.94237704854534\tau} \cdot t & \text{for } 0 \leq t \leq 1.94237704854534\tau \\ 1 & \text{for } 1.94237704854534\tau \leq t \leq \infty \end{cases} \tag{18}
$$

A good approximation would be to set the constant in the denominator to 2 for easy computations.

$$
P(T) = \begin{cases} \frac{1}{2\tau} \cdot t & \text{for } 0 \leq t \leq 2\tau \\ 1 & \text{for } 2\tau \leq t \leq \infty \end{cases} \tag{19}
$$