# Weekly Problems Archive

Competition Math Class

December 2015 -

## 1 December 2015

P1. 12/20-12/26: Find the number of multisubsets of 5 elements of $\{1,2,3,4,5,6\}$. A multisubset is defined as a subset in which an element can appear more than once. Ex: $\{1,1,1,2,3\}$ is a valid 5 element multisubset.
**Solution:** Let there be 5 stars and 5 bars such that when a star is in front of the first bar, it is a 1, and so on. Thus, there is a one-to-one correspondence with this counting method. Thus, the answer is $\binom{10}{2} = \boxed{252}$

P2. 12/27-1/2: When a polynomial $P(x)$ is divided by $x-19$, the remainder is 99. When $P(x)$ is divided by $x-99$, the remainder is 19. Find the remainder when $P(x)$ is divided by $(x-19)(x-99)$.
**Solution:** Let $P(x) = Q(x)(x-99)+19 = S(x)(x-19)+99$ because of the given conditions. It then follows that $P(99) = 19$ and $P(19) = 99$. We can also write $P(x) = R(x)(x-19)(x-99)+ax+b$, where $ax+b$ is the remainder. Plugging in 99 and 19, we get the system $99a+b = 19$ and $19a+b = 99$. Solving, we get that $a = -1$ and $b = 118$. Thus, the remainder is $\boxed{-x+118}$.

## 2 January 2016

P3. 1/3-1/9: Determine the smallest positive integer that leaves a remainder of 4 when divided by 7, a remainder of 5 when divided by 9, and a remainder of 6 when divided by 11.
**Solution:** If we let the integer be $x$, then the integer $2x$ will leave a remainder of 8, 10, and 12 when divided by 7, 9, and 11, respectively. This means that $2x$ leaves a remainder of 1 when divided by the 3 numbers. To find the smallest possible value of $x$, we take the least common multiple of 7, 9, and 11, add 1, and divide the result by 2. After trivial computation, our desired answer is $\boxed{347}$.

P4. 1/10-1/16: Given an equiangular hexagon $ABCDEF$ with $BC = 20$, $DE = 21$, $EF = 3$, and $FA = 25$, find $\frac{2AB}{CD}$.
**Solution:** Extend sides $BC$, $DC$, and $FA$ so that each pair meets at points $G$, $H$, and $I$. Because hexagon $ABCDEF$ is equiangular, we see that triangles $ABG$, $CDH$, $EFI$, and $GHI$ are all equilateral. Thus, $EF = FI = IE = 3$. Let $AB = BG = GA = x$. Then $IG = x+25+3 = x+28$. Because $IG = H$, $x+28 = DH+21+3$, so $DH = x+4 = HC$. Because $IG = GH = BG+BC+HC$, $x+28 = x+20+x+4$, so $x = 4$. Therefore, $\frac{2AB}{CD} = \frac{2*4}{4+4} = \boxed{1}$.

P5. 1/17-1/23: Given a quadratic function $f(x) = ax^2+bx+c$, if $f(2) = 17$, $f(6) = 33$, and $f(a) = 15$, find the product of all possible values of $a$.
**Solution:** We obtain the system of equations: $4a+2b+c = 17$, $36a+6b+c = 33$, $a^3+ab+c = 15$. Using the first two equations, we solve for $b$ and $c$ in terms of $a$ to get $b = 4-8a$ and $c = 9+12a$. Plugging these into the last equation, we get the following cubic in $a$: $a^3-8a^2+16a-6 = 0$. By Vieta's, the product of all possible values of $a$ is $\boxed{6}$.

P6. 1/24-1/30: What is the largest $n$ such that $3^n$ divides $1*3*5*...*97*99$?
**Solution:** First, we count the number of multiples of three that are in this set. The multiples of three range from $3*1$ to $3*33$, so there are a total of 17 powers in this preliminary count. Next, we take into account

the powers of 9, as we add another power of three. They range from $9 * 1$ to $9 * 11$, so there are 6 of these. Again, we need to count the powers of 27, and this comes out to be 2. Finally, there is one multiple of 81. So, in total, we have the maximum exponent of 3, $n = 16 + 7 + 2 + 1 = \boxed{26}$.

P7. 1/31-2/6: Find the remainder when $7^{10} + 9^{10}$ is divided by 64.
**Solution:** Notice that the expression is $(8-1)^{10} + (8+1)^{10}$. Using the binomial theorem and expanding, we find that all the terms are multiples of 64 except $-80 + 1 + 80 + 1$. Adding them yields the remainder, $\boxed{2}$.

# 3 February 2016

P8. 2/7-2/13: Determine the distance between the circumcenter and the incenter of a triangle with sides 3, 4 and 5.
**Solution:** Using standard techniques ($Area = rs$), we can compute the inradius to be 1 and the circumradius to be $\frac{5}{2}$ because the circumcenter lies on the midpoint of the hypotenuse of a right triangle. Using euler's formula, we find the distance to be $\sqrt{R(R-2r)} = \boxed{\frac{\sqrt{5}}{2}}$.
**Note:** Alternatively, we can find this length using euclidean and non-bash methods by drawing the inradius to the hypotenuse and using the pythagorean theorem.

P9. 2/14-2/20: Determine the number of integers with an even number of factors between 1 and 1000 inclusive.
**Solution:** A number has an even number of factors if and only if it is a square number. Thus, because there are 31 squares between 1 and 1000, the number of integers is $\boxed{31}$.

P10. 2/21-2/27: A fair coin is flipped 10 times. Find the number of sequences of these 10 flips such that there are no consecutive heads.
**Solution:** Assume such a sequence of $n$ flips ends in a tail. Then, we can have any string of flips of length $n-1$ before this tail. Now, assume the sequence of $n$ flips ends in a head. Then, we must have a tail directly to the left of the head, and after these two flips, we can add any sequence of $n-2$ that is valid. Because these two cases are mutually exclusive and add up to the total number of flips of length $n$, we obtain the recursion $F_n = F_{n-1} + F_{n-2}$. We can compute $F_1 = 2$ and $F_2 = 3$, so $F_{10} = \boxed{144}$.

P11. 2/28-3/5: Let triangle $ABC$ have lengths $AB = 13$, $BC = 14$, $CA = 15$. If $H$ is the intersection of the altitudes of triangle $ABC$ (the orthocenter), find the distance between the orthocenters of triangle $AHB$ and $AHC$.
**Solution:** Note that the orthocenters are just the vertices $C$ and $B$, respectively. Thus, the distance between them is just $BC = \boxed{14}$.

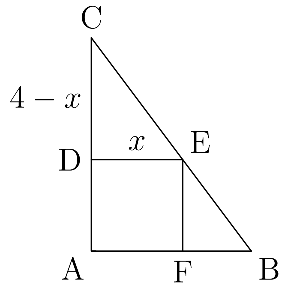
# 4 March 2016

P12. 3/6-3/12: How many trailing zeros does 100! have?
**Solution:** The answer is simply the number of factors of 10 in 100!, which is the number of factors of 5. This can be calculated to be $\lfloor \frac{100}{5} \rfloor + \lfloor \frac{100}{25} \rfloor = \boxed{24}$.

P13. 3/13-3/19: What is the sum of the first 100 triangular numbers?
**Solution:** We note that a triangular number is in the form $\frac{n(n+1)}{2}$, which is equivalent to $\binom{n+1}{2}$. Summing from 1 to 100, we get the sum to be $\binom{2}{2} + \binom{3}{2} + ... + \binom{101}{2}$, and by the hockey-stick identity, this is equal to $\binom{102}{3}$, which is $\boxed{171700}$.

P14. 3/20-3/26: A fair coin is tossed 15 times. Find the probability that the same number of heads and tails are flipped.
**Solution:** This can never be achieved. Thus, our answer is $\boxed{0}$.

P15. 3/27-4/2: A square is inscribed in a triangle with sides 3, 4, and 5. Find the area of the square.
**Solution:** We draw the following diagram and set up similar triangles. We have $AC = 4$, $AB = 3$, and $BC = 5$. Letting the side length of the square to be $x$, we have that $CD = 4 - x$ and $DE = x$. Since $DE//AB$, we know that $\triangle CDE \sim \triangle CAB$, giving us the ratio: $\frac{x}{4-x} = \frac{3}{4}$. Solving, we get $x = \frac{12}{7}$, so the area of the square is $x^2 = \boxed{\frac{144}{49}}$.




# 5 April 2016

P16. 4/3-4/9: How many elements of the set $\{1, 2, 3, ..., 100\}$ divide $100! + 1$?
**Solution:** We note that since all of the numbers in the set divide 100!, then if an element divides 100!+1, then it must divide 1. Thus, only $\boxed{1}$ element satisfies the condition.

P17. 4/10-4/16: For what integers $n$ is $\frac{n^2-n-7}{n-3}$ an integer?
**Solution:** Using long division, we find that the original fraction is equivalent to $n + 4 + \frac{5}{n-3}$. Because $n + 4$ is an integer, we need $n - 3$ to divide 5. Thus, $n - 3$ can be -5,-1,1, or 5, meaning that $n$ can be $\boxed{\text{-2,2,4,8}}$.

P18. 4/17-4/23: An equilateral triangle $ABC$ is inscribed in a circle. Let a random point on arc $BC$ not containing $A$ be $P$. Find the value $AP - BP - CP$.
**Solution:** Using Ptolemy's theorem for $ABPC$, we get $(AB)(CP)+(AC)(BP) = (BC)(AP)$, so $CP+BP = AP$, meaning that $AP - BP - CP = \boxed{0}$.

P19. 4/24-4/30: A bug starts at a vertex of an equilateral triangle. Every second, the bug must move to one of the adjacent vertices. What is the probability that the bug is at the vertex it started at at the end of 6 moves?
**Solution:** We note that the probability that the bug is at the starting vertex at any step is only non-zero if and only if the bug was not at the starting vertex the step before. Thus, if we let the probability that the bug is at the starting vertex at turn $n$ be $P_n$, then $P_n = \frac{1}{2}(1 - P_{n-1})$ because there is a $\frac{1}{2}$ chance that the bug moves to the starting vertex from another vertex. Using this recursion, we determine that $P_6 = \boxed{\frac{11}{32}}$.

# 6 May 2016

P20. 5/1-5/7: 13 marbles, 8 of which are red, 4 of which are blue, and 1 of which is white are randomly arranged in a line. What is the probability that the white ball appears after all of the red balls?
**Solution:** Consider the 9 locations where the 8 red balls and the 1 white ball are located. There is a $\boxed{\frac{1}{9}}$ chance that the white ball appears last.

P21. 5/8-5/14: A function $F$ is defined for all real numbers. If $F$ satisfies $F(x) + F(y) = F(xy)$, then what is $F(2016) + F(420)$?
**Solution:** Let $x = 2016$ and $y = 0$. Then, $F(2016)+F(0) = F(0)$, which means that $F(2016) = 0$. Similarly, we obtain that $F(420) = 0$ as well. Thus, $F(2016) + F(420) = \boxed{0}$.

P22. 5/15-5/21: Triangle $ABC$ has $AB = 6$, $AC = 9$, $BC = 11$. $D$ is on $AB$ and $AD = 4$, and $E$ is on $AC$ with $AE = 7$. Find the ratio of the areas of triangle $ADE$ and triangle $ABC$.
**Solution:** By the formula $K = \frac{1}{2}ab\sin C$, we get that the area of $ADE$ is $\frac{1}{2} * 4 * 7\sin A$. Similarly, we get that the area of $ABC$ to be $\frac{1}{2} * 9 * 11\sin A$, so the ratio of the areas of the two triangles is just $\frac{4*7}{9*11} = \boxed{\frac{28}{99}}$.

P23. 5/22-5/28: There exist positive integers $a$ and $b$ such that $\binom{6}{2} + 2\binom{6}{1} + \binom{6}{0} = \binom{a}{b}$. Determine all possible ordered pairs $(a, b)$.
**Solution:** We can view this as a committee choosing process. $\binom{6}{2}$ represents the case when 2 specific people are not in the committee, so we choose 2 from the remaining 6 people. $2\binom{6}{1}$ represents the case when one of the two specific people is chosen for the committee but the other is not. Finally, when both of the specific people are chosen, then there are $\binom{6}{0}$ ways to choose the rest. Another way to count choosing 2 people to form a committee from 8 people is $\binom{8}{2}$ or $\binom{8}{6}$. Thus, the possible ordered pairs are $\boxed{(8,2),\ (8,6)}$.

P24. 5/29-6/4: Given that $x$ is a positive real number, what is the minimum of $\frac{6}{x} + 9x$?
**Solution:** Using AM-GM, we see that the minimum is at $2\sqrt{54} = \boxed{6\sqrt{6}}$.

# 7 June 2016

P25. 6/5-6/11: Olivia has one fair coin and one unfair coin that shows heads with probability $\frac{2}{3}$. She picks one coin at random and flips it three times. Given that all three flips were heads, what is the probability that the coin she picked was the unfair one?
**Solution:** Using conditional probability, the probability that three heads show up is $\frac{1}{2} * \frac{1}{8} + \frac{1}{2} + \frac{8}{27} = \frac{91}{432}$. The probability that three heads show up and it is the unfair coin is $\frac{1}{2} * \frac{8}{27} = \frac{4}{27}$. Therefore, the desired probability is $\frac{\frac{4}{27}}{\frac{91}{432}} = \boxed{\frac{64}{91}}$.

P26. 6/12-6/18: The number $23481x6545$ is divisible by 99 for some $x$. Determine $x$.
**Solution:** We note that the number must be divisible by 9, so by the divisibility rule for 9, $2+3+4+8+1+x+6+5+4+5 = 38+x$ must be divisible by 9. Hence, the only possible one-digit number for $x$ is 7. Testing 234816545, it is also divisible by 11. Thus, $\boxed{7}$ is our desired digit.

P27. 6/19-6/25: Compute the probability that in a seven game series, one team wins in exactly four games, given that each team has a probability of $\frac{1}{2}$ of winning each game.
**Solution:** We don't care about what happens in the first game, but in the next three games, the team that won the first game must win three more games in a row. Thus, the desired probability is $(\frac{1}{2})^3 = \boxed{\frac{1}{8}}$.

P28. 6/26-7/2: If $x^2 - 3x + 1 = 0$, compute the value of $x^2 + \frac{1}{x^2}$.
**Solution:** Since $x \neq 0$, we can move the $-3x$ to the right side and divide by $x$ on both sides, giving $x + \frac{1}{x} = 3$. Squaring this equation gives $x^2 + \frac{1}{x^2} + 2 = 9$. Thus, $x^2 + \frac{1}{x^2} = \boxed{7}$.

# 8 July 2016

P29. 7/3-7/9: A binary string is a string of 1s and 0s. Consider the set of all binary strings of length 5. How many of these strings have more 1s than 0s in them?
**Solution:** We note that we can create pairs of binary strings within the $2^5 = 32$ total strings of length 5. We take any string and change its 1s to 0s and its 0s to 1s and pair this new string with the original. Thus, we have 16 total pairs, and in each pair there exists exactly one string that has more 1s than 0s. So, our desired answer is $\boxed{16}$.

P30. 7/10-7/16: Determine the value of $k$ such that $x^2 + kx + 17 = 0$ has exactly one solution.
**Solution:** In order for this to occur, the discriminant of the quadratic function must be equal to 0, so

$k^2 - 4 \cdot 17 = 0$. Thus, $k = \boxed{2\sqrt{17}}$.

P31. 7/17-7/23: Determine the value of the infinite sum: $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + ...$
**Solution:** Using partial fraction decomposition, we see that $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$, so our infinite sum becomes: $\frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + ...$ We now note that all of the terms cancel out except for 1, so the answer is $\boxed{1}$.

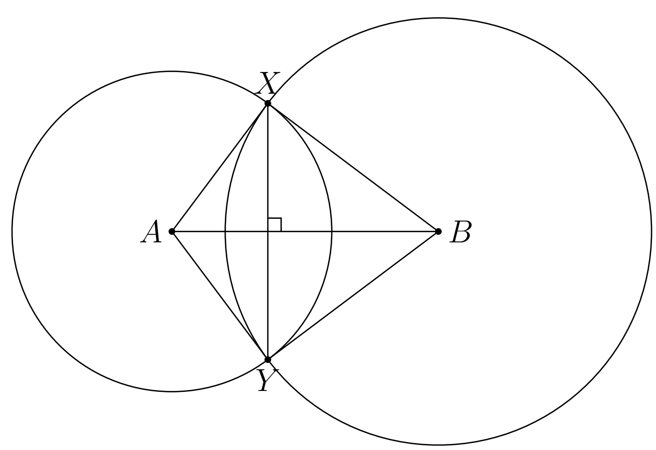
P32. 7/24-7/30: In a right triangle, the altitude to the hypotenuse divides the hypotenuse into two segments of length 4 and 9. Find the area of the right triangle.
**Solution:** By similar triangles, we can deduce that the height of the right triangle to the hypotenuse is $\sqrt{4 \cdot 9} = 6$. Thus, the area of the triangle is $\frac{6 \cdot 13}{2} = \boxed{39}$.

P33. 7/31-8/6: If a standard 6-sided dice is tossed 20 times, what is the expected value of the sum of the 20 numbers that land face up?
**Solution:** For each toss, the expected value of the number that lands face up is $\frac{1+2+3+4+5+6}{6} = \frac{21}{6} = \frac{7}{2}$. However, there are a total of 20 tosses, meaning that the expected value of the sum of these 20 tosses is $20(\frac{7}{2}) = \boxed{70}$.

# 9 August 2016

P34. 8/7-8/13: What is the remainder when $1^3 + 2^3 + 3^3 + ...10^3$ is divided by 11?
**Solution:** By the sum of cubes formula, $1^3 + 2^3 + 3^3 + ... + 10^3 = \left(\frac{(10)(11)}{2}\right)^2 = 55^2$. Thus, the remainder when $55^2$ is divided by 11 is $\boxed{0}$.
**Solution II:** Using modular arithmetic, $1^3 + 2^3 + 3^3 + ... + 10^3 \equiv 1^3 + 2^3 + 3^3 + ... + (-3)^3 + (-2)^3 + (-1)^3 \pmod{11}$. We then see that the sum of cubes is symmetric in modulo 11, so the terms cancel out, giving us a remainder of $\boxed{0}$.

P35. 8/14-8/20: Consider a line segment $\overline{AB}$ with length 5. There exist two points $X$ and $Y$ that are 3 units away from $A$ and 4 units away from $B$. Compute the distance $XY$.
**Solution:** By definition, $X$ and $Y$ are the intersection points of two circles: a circle with radius 3 and centered at A and a circle with radius 4 centered at B. Using congruent triangles, we can easily show that $XY \perp AB$ and $AB$ bisects $XY$. Thus, $ABX$ and $ABY$ are 3-4-5 right triangles as shown in the diagram below, so $XY$ is just twice the height to the hypotenuse of a 3-4-5 right triangle. The height is $\frac{12}{5}$, so $XY = \boxed{\frac{24}{5}}$.




P36. 8/21-8/27: Determine the sum of the squares of all the factors of 24.
**Solution:** Every square of a factor of 24 has the form $2^{2a}3^{2b}$, where $a$ and $b$ are nonnegative integers. So, since the prime factorization of 24 is $2^3 \cdot 3$, the sum of the squares of all the factors of 24 can be represented

by the product: $(1+2^2+2^4+2^6)(1+3^2)$. This is because when we expand this product, we get the sum of all possible numbers of the form $2^{2a}3^{2b}$, where $a = 0, 1, 2, 3$ and $b = 0, 1$. Thus, the desired sum is $85*10 = \boxed{850}$.

P37. 8/28-9/3: Compute the value of $\sin 69 \cos 21 + \sin 21 \cos 69$.
**Solution:** We note that this is the expansion of $\sin 69 + 21$ using the sine addition formula. Thus, the answer is $\sin 90 = \boxed{1}$.

# 10 September 2016

P38. 9/4-9/10: Fred drops a bouncy ball from 1 $m$ above the ground. Every bounce, the ball bounces up to $\frac{2}{3}$ the height of the previous bounce. On which bounce does the ball reach less than 20 $cm$?
**Solution:** On the $n^{th}$ bounce, the ball bounces to $(\frac{2}{3})^n$ meters high. So, we are just solving the inequality $(\frac{2}{3})^n < \frac{1}{5}$. We note that when $n = 4$, $(\frac{2}{3})^4 = \frac{16}{81}$, which is just a tiny bit smaller than $\frac{16}{80} = \frac{1}{5}$. However, when $n = 3$, $(\frac{2}{3})^3 = \frac{8}{27}$ is larger than $\frac{1}{5}$.So, the ball reaches less than 20 $cm$ when $n = \boxed{4}$, or on the fourth bounce.

P39. 9/11-9/17: There exists a real number $n$ such that $3x - 5y = 9$ and $nx + 7y = 17$ do not intersect in the euclidean plane. Find $n$.
**Solution:** The slopes of the two lines must be the same. The slope of the first line is $\frac{3}{5}$ and the slope of the second line is $\frac{-n}{7}$. Setting these two equal gives that $n = \boxed{-\frac{21}{5}}$.

P40. 9/18-9/24: Given that $a$ is a positive integer and that $\frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} + ... = \frac{1}{16}$, determine $a$.
**Solution:** We can see that this is an infinite geometric sequence with first term $\frac{1}{a}$ and common ratio $\frac{1}{a}$. The sum of an infinite geometric sequence formula tells us that $\frac{1}{a-1} = \frac{1}{16}$. Thus, $a = \boxed{17}$.

P41. 9/25-10/1: What is the smallest positive integer that has 5 factors?
**Solution:** We see that the integer must be in the form $p^4$, where $p$ is a prime number. Thus, because the smallest prime number is 2, the smallest positive integer that has 4 factors is $\boxed{16}$.

# 11 October 2016

P42. 10/2-10/8: Eric has an unbiased coin and a standard 6 sided dice. Eric flips the coin first, and if he flips heads, then he rolls the dice. If he flips tails , then he he loses. If he rolls a 1, he also loses. What is the probability that Eric wins?
**Solution:** The only possible way Eric wins is if he flips a head and then flips a number that is not a 1. The probability of the former is $\frac{1}{2}$, and the probability of the latter is $\frac{5}{6}$. Thus, the probability of winning is $\boxed{\frac{5}{12}}$.

P43. 10/9-10/15: Bob chooses a real number between 0 and 100, and Joe chooses a real number between 0 and 1000. What is the probability that Joe chooses a bigger number than Bob?
**Solution:** If Joe chooses a number from 100 to 1000, he automatically has a bigger number than Bob, and this occurs with a probability of $\frac{9}{10}$. However, if Joe chooses a number from 0-100, there is exactly a $\frac{1}{2}$ chance that Joe chooses a bigger number than Bob because of symmetry. Hence, the probability in this case is $\frac{1}{20}$. Thus, the total probability is $\boxed{\frac{19}{20}}$.

P44. 10/16-10/22: What is the largest prime factor of $2011! + 2012! + 2013!$?
**Solution:** We can factor the given expression as follows: $2011!(1 + 2012 + 2012 \cdot 2013)$. We can then further simplify this as $2011!(2013)^2$. Because 2013 is divisible by 3 and 2011 is a prime number, the largest prime factor of the expression has to be $\boxed{2011}$.

P45. 10/23-10/29: If the sum of two numbers is 4 and the product of the two numbers is 5, what is the sum of the reciprocals of the two numbers?

**Solution:** The two sentences give us that $a+b=4$ and $ab=5$. Hence, $\frac{a+b}{ab}=\frac{1}{a}+\frac{1}{b}=\frac{4}{5}$ by division. Thus, our desired answer is $\boxed{\frac{4}{5}}$.

P46. 10/30-11/5: A square is inscribed in a circle, and Bob throws a dart at the figure. What is the probability that the dart lands outside the square but inside the circle, given that it hits the figure?
**Solution:** The area of the region outside the square but inside the circle can be given as $\pi r^2-2r^2$, if the radius of the circle is $r$. The area of the circle is $\pi r^2$, so the probability is just $\frac{\pi-2}{\pi}$, which can be simplified to $\boxed{1\text{-}\frac{2}{\pi}}$.

# 12 November 2016

P47. 11/6-11/12: In the island chain of CROTH, the residents give each of their islands distinct 5 letter names. However, each name can only use the letters $a$ and $b$. So far, each of their islands have exactly 3 $a$'s in their names. In addition, no names with 3 $a$'s are left. How many possible names remain?
**Solution:** The residents have used up exactly $\binom{5}{3}$ names because they used up all of the names that have 3 $a$'s. Now, the total number of names is $2^5=32$ because each letter has precisely two choices. Hence, there are $32-10=\boxed{22}$ names left.

P48. 11/13-11/19: A 6 sided dice has faces of 1, 3, 5, 7, 9, and 11. Another dice has $n$ faces with values of 1, 2, 3,..., $n$. The expected value when rolling each die once is the same for both die. What is the value of $n$?
**Solution:** The expected value of one roll of the first dice is just $\frac{1+3+5+7+9+11}{6}=6$. The expression for the expected value of the second dice is $\frac{1+2+...+n}{n}=\frac{n(n+1)}{2n}=\frac{n+1}{2}$. Thus, we equate the two expected values to obtain that $n=\boxed{11}$.

P49. 11/20-11/26: At what time, to the nearest minute, between 3:00 and 4:00 do the hour and minute hands of an analog clock meet?
**Solution:** Each minute, the minute hand moves $6°$ and the hour hand moves $\frac{1}{2}°$. So, we can set up the following equation: $90+\frac{1}{2}x=6x$. Solving gives us that $x=\frac{180}{11}$, so to the nearest integer, $x=16$. Thus, the two hands overlap at $\boxed{3{:}16}$.

P50. 11/27-12/3: How many times during a 24-hour day do the hour and minute hands of an analog clock overlap?
**Solution:** It seems obvious that during every hour period, the hour and minute hand overlap exactly once. However, during the 11:00-1:00 period, the hour and minute hand meet exactly once at 12:00, so we have to subtract 2 from 24. Thus, the desired answer is $\boxed{22}$.

# 13 December 2016

P51. 12/4-12/10: The polynomial $x^3+ax^2-10x+10$ has three real roots. The quintic polynomial $x^5+4x^4+cx^3+dx^2+ex+f$ has the same roots as the polynomial above with an additional two roots: 1 and -1. What is the value of $f$?
**Solution:** By Vieta's, the product of the three unknown roots is $-10$. Thus, the product of the five total roots must be 10. Hence, by Vieta's again, $f=\boxed{\text{-}10}$.

P52. 12/11-12/17: The locus of points in three dimensions that are between 3 and 5 units from a fixed point forms a solid with a volume $V$. Find $V$.
**Solution:** The volume is just a sphere of radius 5 with a hollow spherical center of radius 3. So, the volume can be expressed as $\frac{4}{3}\pi(5^3-3^3)$. Simplifying, we get that $V=\boxed{\frac{392\pi}{3}}$.

P53. 12/18-12/24: Abe, Babe, Cabe, and Dabe want to sit together in a row of four chairs, but Abe

and babe refuse to sit next to each other, and Cabe and Dabe refuse to sit next to each other. How many ways are there for the four of them to sit in a row?
**Solution:** We can proceed by casework and symmetry. If Abe sits on the very left end, then Babe must sit in the third chair from the left. Then, Cabe and Dabe have 2 choices of where to sit. If Abe sits on the second chair from the left, then Babe must sit on the rightmost chair. Cabe and Dabe have two choices. Thus, there are 4 total choices, but we have to multiply by two for when Abe sits on the right two chairs. Thus, our answer is $\boxed{8}$.

P54. 12/25-12/31: $y$ is inversely proportional to the cube of $x$. Given that the ordered pair $(2, 2)$ satisfies the relation, what is $y$ when $x = 1$?
**Solution:** The given relation can be expressed as $yx^3 = a$ for some constant $a$. Thus, we can plug in the known ordered pair to get that $a = 16$. Thus, when $x = 1$, $y = \boxed{16}$.

# 14 January 2017

P55. 1/1-1/7: Of the whole numbers from 1 to 999 inclusive, how many times is the digit 1 written?
**Solution:** In each digit place (ones, tens, and hundreds), 1 is used $10^2$ times. Thus, it is written $\boxed{300}$ times.

P56. 1/8-1/14: A sphere with the same radius as a cylinder has the same volume as the cylinder. What is the ratio of the cylinder's height to its radius?
**Solution:** Let the radius of the sphere be $r$ and let the radius of the cylinder be $r$. We can equate: $\frac{4}{3}\pi r^3 = \pi r^2 h$. By manipulating, we get that $\frac{h}{r} = \boxed{\frac{4}{3}}$.

P57. 1/15-1/21: A number greater than one is a perfect cube, fourth power, and fifth power. What is the smallest possible number?
**Solution:** We know this number must be of the form $a^3$, $a^4$, and $a^5$, so we can take the LCM of 3, 4, and 5 to get that the number must be in the form $a^{60}$. Thus, the smallest number that satisfies this is $\boxed{2^{60}}$.

P58. 1/22-1/28: For how many positive integers $n$ less than 100 is $\frac{2^n}{n}$ also an integer?
**Solution:** For this condition to hold, it must be necessary that $n$ itself be a power of 2. Thus, because there are 6 powers of 2 between 1 and 100, the answer is $\boxed{6}$.

P59. 1/29-2/4: For how many bases $b$ with $b$ between 1 and 100 is 11 base $b$ a perfect square (in base 10)?
**Solution:** We want $b + 1$ to be a perfect square, so it must equal the square of 2, 3, 4,..., 10 in order for $b$ to satisfy the given restrictions. Thus, there exist exactly $\boxed{9}$ values of $b$.

# 15 February 2017

P60. 2/5-2/11: How many square numbers between 1 and 1000 end in an odd digit?
**Solution:** To end in an odd digit, the number must be odd itself. Thus, because there are 16 odd numbers whose squares are less than 1000, the desired answer is $\boxed{16}$.

P61. 2/12-2/18: The lines $2x + 3y = 1$, $3x + 4y = 1$, and $4x + 5y = a$ intersect at one point. Find $a$.
**Solution:** Subtracting the first two equations, we see that $x + y = 0$. So, it must be true that $3x + 4y + (x + y) = 1 + 0$, which means that $4x + 5y = 1$. Thus, $a = \boxed{1}$.

P62. 2/19-2/25: Martin is creating a string of letters with the letters $A$, $B$, $C$, and $D$. However, he wishes his string to satisfy the property that no two letters are the same. How many strings of 5 letters can Martin create?
**Solution:** For the first letter, Martin can choose any of the four letters, so he has four choices. For every

subsequent letter, he has three choices: any other letter besides the previous letter. Thus, the total number of strings is $4 \cdot 3^4 = \boxed{324}$.

P63. 2//26-3/4: A right prism has a base that is a regular polygon with $n$ sides, with each side having length 2. If the height of the prism is 12 and the lateral surface area of the prism is 1440, find $n$.
**Solution:** The lateral surface area of the prism comprises of $n$ rectangles, each with dimensions 2 by 12. Hence, the lateral surface area can be expressed as $24n$. Hence, we solve the equation $24n = 1440$ to obtain that $n = \boxed{60}$.

## 16 March 2017

P64. 3/5-3/11: Two different parallel lines are constructed in the same plane. One line has 4 points on it while the other has 5 points on it. All 20 segments connecting points from one parallel line to the other are drawn. How many intersections lie strictly within the region between the two parallel lines, assuming that no three line segments are concurrent?
**Solution:** We note that if we pick any two points on one parallel line and any two points on the other parallel line, there is exactly one intersection that lies strictly between the two lines. Thus, for every two points chosen one line and every two points chosen on the other line, there is exactly one intersection because no intersection is counted twice. Thus, the total number of intersections is $\binom{4}{2}\binom{5}{2} = \boxed{60}$.

P65. 3/12-3/18: A semicircle with radius 5 and diameter $\overline{AB}$ has points $C$ and $D$ on its circumference such that $C$ and $D$ are equally spaced. Compute $AB^2 + AC^2 + AD^2$.
**Solution:** By symmetry, $AC = BD$, so we need to compute $AB^2 + BD^2 + AD^2$. By the Pythagorean Theorem, $AD^2 + BD^2 = AB^2$, so the summation is simply $2AB^2$, which is just $\boxed{200}$.

P66. 3/19-3/25: The repeat of a two-digit number $\overline{ab}$ is defined to be the number $\overline{abab}$. For example, the repeat of 21 is 2121. For how many two-digit numbers is the repeat of that number a perfect square?
**Solution:** We note that if a two-digit number is $n$, then the repeat can be expressed as $101n$. However, 101 is prime, so if $101n$ were a perfect square, $n$ would have to be a multiple of 101. This is clearly impossible as $n$ is a two-digit number. Hence, there are $\boxed{0}$ numbers.