Competition Math Class Material

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Sample Problems

This is a short collection of problems that you will be able to solve after taking this course in contest math. These problems range from AMC 8 to mid and high AMC 10 level, and you should be able to solve these with efficiency and accuracy after the course.

1 Problems

1.1 Introductory Level

1. Four siblings ordered an extra large pizza. Alex ate $\frac{1}{5}$, Beth $\frac{1}{3}$, and Cyril $\frac{1}{4}$ of the pizza. Dan got the leftovers. What is the sequence of the siblings in decreasing order of the part of pizza they consumed? (AMC 10B 2015 #4)

2. What are the sign and units digit of the product of all the odd negative integers strictly greater than -2015? (AMC 10B 2015 #10)

3. For a science project, Sammy observed a chipmunk and a squirrel stashing acorns in holes. The chipmunk hid 3 acorns in each of the holes it dug. The squirrel hid 4 acorns in each of the holes it dug. They each hid the same number of acorns, although the squirrel needed 4 fewer holes. How many acorns did the chipmunk hide? (AMC 10B 2012 #7)

4. Let X and Y be the following sums of arithmetic sequences:

$$\begin{aligned} X &= 10+12+14+\cdots+100, \\ Y &= 12+14+16+\cdots+102. \end{aligned}$$

   What is the value of $Y-X$? (AMC 10A 2011 #4)

5. A rectangle has a perimeter of 20 units. Find the largest possible area of such a rectangle.

1.2 Intermediate level

1. Which of the following equations does NOT have a solution?

   a) $(x+13)^2=0$
   b) $|6x-9|-1=0$
   c) $\sqrt{-x}-4=0$
   d) $|3x+7|+4=0$
   e) $|-2x|-7=3$

2. A fair coin is tossed six times in a row. What is the probability that there are the same number of heads as tails?

3. In rectangle $ABCD$ let M be the midpoint of AB and denote the intersection of MD and AC to be X. Given that $[AMX]=10$, compute $[ABCD]$. Note: $[ABC]$ denotes the area of $\triangle ABC$.

4. How many even integers are there between 200 and 700 whose digits are all different and come from the set $\{1,2,5,7,8,9\}$? (AMC 10A 2011 #13)

5. If x_1 and x_2 are the two solutions to the quadratic x^2+2x+6, then compute the value $\frac{1}{x_1}+\frac{1}{x_2}$.

1.3 Advanced Level

1. Positive integers a and b are such that the graphs of $y = ax + 5$ and $y = 3x + b$ intersect the x-axis at the same point. What is the sum of all possible x-coordinates of these points of intersection? (AMC 10A 2014 #21)

2. The largest divisor of $2,014,000,000$ is itself. What is its fifth largest divisor? (AMC 10B 2014 #12)

3. For how many integers x is the number $x^4 - 51x^2 + 50$ negative? (AMC 10B 2014 #20)

4. In $\triangle ABC$, $AB = 86$, and $AC = 97$. A circle with center A and radius AB intersects $\overline{BC}$ at points B and X. Moreover $\overline{BX}$ and $\overline{CX}$ have integer lengths. What is BC? (AMC 10A 2013 #23)

5. A 3 x 3 square is partitioned into 9 unit squares. Each unit square is painted either white or black with each color being equally likely, chosen independently and at random. The square is then rotated 90° clockwise about its center, and every white square in a position formerly occupied by a black square is painted black. The colors of all other squares are left unchanged. What is the probability the grid is now entirely black? (AMC 10A 2012 #20)

Diagnostic Test

You will have 40 minutes to complete this test. There will be problems from multiple areas of contest math that are from MATHCOUNTS® and the American Mathematics Competition series.
Note: The following problems are not necessarily in increasing difficulty. You may find it beneficial to skip some problems and come back to them later.

2 Problems

1. Twelve increased by three times a number is 48. What is that number?

2. For integers a,b, and k, we know that $a > 12$, $b < 20$, and $a < b$. If $b = 7k$, what is the value of k? (MATHCOUNTS 2009)

3. What is the value of x in the equation $6^{x+1} - 6^x = 1080$? (MATHCOUNTS 2009)

4. A circular pizza with a radius of 6 is cut along radii into three wedge-shaped slices. The measures of two of the central angles are $80°$ and $130°$. What is the number of square inches in the area of the largest slice. Express your answer in terms of π. (MATHCOUNTS 2002)

5. If $f(x) = x^3 + 6x + 10$, what is the remainder when $f(4) - f(1)$ is divided by 3?

6. At how many points do the graphs of the equations $y = x^3$ and $y = 4x^2 - 4x$ intersect?

7. A triangle has three different integer side lengths and a perimeter of 20 units. What is the maximum length of any one side? (MATHCOUNTS 2008)

8. In quadrilateral $ABCD$, $AB = 5$, $BC = 8$, and $CD = 20$. Angle B and angle C are both right angles. What is the length of segment AD? (MATHCOUNTS 2008)

9. Darla can do a job in eight hours while Lonnie can do the same job in six hours. Both Darla and Lonnie work three hours on the job. How much of the job is left to be finished? Express your answer as a common fraction. (MATHCOUNTS 2004)

10. What is the largest integer n such that $(1 + 2 + 3 + ... + n)^2 < 1^3 + 2^3 + ... + 7^3$? (MATHCOUNTS 2004)

11. What is the value of the following expression:
$1 - 3 + 5 - 7 + 9 - ... - 43 + 45 - 47 + 49$? (MATHCOUNTS 2005)

12. If n and m are integers and $n^2 + m^2$ is even, which of the following is impossible? (AMC 8 2014)

    a) n and m are even
    b) n and m are odd
    c) $n + m$ is even
    d) $n + m$ is odd
    e) none of these are impossible

13. A cube with 3-inch edges is to be constructed from 27 smaller cubes with 1-inch edges. Twenty-one of the cubes are colored red and 6 are colored white. If the 3-inch cube is constructed to have the smallest possible white surface area showing, what fraction of the surface area is white? (AMC 8 2014)

14. A fair coin is tossed 3 times. What is the probability of at least two consecutive heads? (AMC 8 2013)

15. A number of students from Fibonacci Middle School are taking part in a community service project. The ratio of 8^{th}-graders to 6^{th}-graders is $5:3$, and the the ratio of 8^{th}-graders to 7^{th}-graders is $8:5$. What is the smallest number of students that could be participating in the project? (AMC 8 2013)

16. A 1×2 rectangle is inscribed in a semicircle with longer side on the diameter. What is the area of the semicircle? (AMC 8 2013)

17. What is the sum of the digits of the decimal form of the product $2^{1999} \cdot 5^{2001}$? (AHSME 1999)

18. Jesse drives to Dallas at a speed of 40 miles per hour. He then returns on the same road going at 60 miles per hour. What was the average speed for the whole trip?

19. The ratio of the length to the width of a rectangle is $4:3$. If the rectangle has diagonal of length d, then the area may be expressed as kd^2 for some constant k. What is k? (AMC 10A 2015)

20. For how many integers x is the point $(x, -x)$ inside or on the circle of radius 10 centered at $(5, 5)$? (AMC 10B 2015)

3 Additional/Challenge Problems

1. Let a, b, and c be three distinct one-digit numbers. What is the maximum value of the sum of the roots of the equation $(x-a)(x-b)+(x-b)(x-c)=0$? (AMC 10B 2015)

2. What is the value of the expression $\frac{1}{\log_2 100!}+\frac{1}{\log_3 100!}+\frac{1}{\log_4 100!}+\cdots+\frac{1}{\log_{100} 100!}$? (AHSME 1998)

3. Suppose that a cows give b gallons of milk in c days. At this rate, how many gallons of milk will d cows give in e days? (AMC 10A 2014)

4. The product $(8)(888\ldots8)$, where the second factor has k digits, is an integer whose digits have a sum of 1000. What is k? (AMC 10A 2014)

5. A solid cube of side length 1 is removed from each corner of a solid cube of side length 3. How many edges does the remaining solid have? (AMC 10A 2013)

Combinatorics I

Welcome to the first class of our Combinatorics section. In Combinatorics I we will cover the foundations and basics of Combinatorics in contest math. We will cover the basics of permutations, combinations, and basic probability in this class.

4 Lecture I

What is Combinatorics?
Simply, Combinatorics is the way we count things!

4.1 Permutations and Combinations: Counting With and Without Order

How can we count the number of ways to choose k items from n total items if the order does matter? How about if the order doesn't matter?

Let's tackle the case where we do consider order first. How many ways can we choose the first item? n. How many ways can we choose the second item? $n-1$. And the third? $n-2$, Generalizing, we see that the number of ways to pick the k^{th} item is $n-k+1$. Thus, the total number of ways to choose k items would be $n(n-1)(n-2)\cdots(n-k+1)$.
The choosing of items where order does matter is a **permuation**. It is commonly denoted as $P(n,k)$ or ${}_nP_k$. Because ${}_nP_k = n(n-1)\cdots(n-k+1)$, we can rewrite this to derive a general formula for a permutation: ${}_nP_k = \frac{n!}{(n-k)!}$ where $n! = n(n-1)\cdots 1$.

Note: A factorial, denoted by $n!$, counts the number of ways to arrange n objects with order (in a line). So, if there are 4 people, there is a total of $4! = 24$ ways to line them up.

4.1.1 If there are 10 people in a club, how many ways are there to choose a president, vice president, secretary, and treasurer if one person can only hold one position?

This problem is basically asking for us to choose 4 people from a group of 10 where order does matter. Why? Because the positions are distinct. So, applying our knowledge of permutations, it is clear that there is a total of ${}_nP_k = 10*9*8*7 = \boxed{5040}$ ways to choose the people for the positions

Now, let's see how we can choose k objects from n total if the order they are chosen in doesn't matter. We already know that there are $\frac{n!}{(n-k)!}$ ways to choose k objects with order. Now we ask, how many ways could we have chosen the same k items? There are $k!$ ways to arrange the k objects, so there would be a total of $k!$ ways of choosing the given k objects in any order (To see this clearly, we can choose x, y, z in any of 3! ways, so we counted this set of three variables 6 times in our permutation count). Thus, each set of k objects we count is counted $k!$ times. So, it follows that we must divide our original count based on a permutation by $k!$ to account for no order. So, we have derived a formula for a **combination**, which is commonly denoted by $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

4.1.2 If we want to form an executive board of 4 people from a total of 10 people, how many ways can we do this?

Because the order we choose the four people does not matter, we will apply the concept of a combination. We seek to find $\binom{10}{4} = \frac{10!}{4!6!} = \boxed{210}$.

4.2 More Methods of Counting

4.2.1 If there are 5 different people and I want to arrange them in a line, but two people insist on standing next to each other, how many different arrangements are possible?

If these two people, call them Alice and Bob, insist on being next to each other, them make them a "superperson". Now, we have to arrange 4 items in a row: 3 people and 1 "superperson". So, there are 4! ways to arrange these 4 items in a line. Is 24 our final answer? No, because in the "superperson" group we can arrange Alice and Bob in 2 ways depending on who is "facing front". Thus, our final answer is $\boxed{48}$.

4.2.2 How many ways can 7 distinct people sit around a round table if rotations count as the same seating (Two arrangements are the same if every person has the same people sitting to their right and their left)?

Without loss of generality, we can number the "top" chair as 1 and in clockwise order, the rest of the chairs as 2,3,4,5,6, and 7. We might think that the answer is 7! because there are 7! ways to pick which people are in each chair number, but this method overcounts because we can rotate the people around the table. So, the following arrangements result in the same seating:

$$\begin{array}{c} 1\ 2\ 3\ 4\ 5\ 6\ 7 \\ ABCDEFG \\ BCDEFGA \\ CDEFGAB \\ DEFGABC \\ EFGABCD \\ FGABCDE \\ GABCDEF \end{array}$$

We can see that the same arrangement of people is counted 7 times, but why? Well, we can rotate person A to any of the seven seats and the arrangement will follow, with B next, C after that, and so on. So, in general, every seating arrangement is counted 7 times because it can rotate 7 ways around the circular table. Thus, our answer is $\frac{7!}{7} = 6! = \boxed{720}$.
Because this answer is a perfect factorial, we should seek another explanation to this question. If we choose any person and place him in an arbitrary position, there are 6! ways to arrange the other people in the circle around him. Because we arbitrarily placed this person in the circle, we eliminated overcounting since we are only counting the ways the other people can be seated around this certain person.

As an introduction to Stars and Bars, we will look at a more basic problem: How can we have k distinct positions chosen from n total people if repetition is allowed and order does matter (a person can hold more than one position? Well, we have n ways to choose the first item, n ways to choose the next item and so on. Thus, there are n^k ways to fill the k distinct positions.

4.2.3 If Jesse wants to write a sequence of A's, B's, and C's of length 10, how many ways can he do this?

Jesse has 3 choices for the first letter, 3 for the second letter, and so on. So, the total number of sequences he can write is $3^{10} = \boxed{59049}$.

4.3 Stars and Bars: Counting With Repetition and Without Order

Let's move on to stars and bars now. The question we are trying to answer is: How many ways can we build a cup of ice cream if we want a total of n scoops from k distinct flavors. In this case, the order of the scoops doesn't matter, so we approach this is a different way.

Crux move: We can imagine placing n stars in between $k-1$ bars so that the number of stars before the first bar represents the number of scoops of the first flavor, the number of stars between the first and second bar represents the number of scoops of the second flavor, and so on, until the number of stars after the last bar represents the number of scoops of the last flavor. It is clear that this method counts all cases once and only once, thus establishing a **one-to-one correspondence**!
The next question is, how can we count the number of ways to separate these n stars and $k-1$ bars? Well, we have a total of $n+k-1$ objects, so we have to choose where the $k-1$ bars go in the given $n+k-1$ available slots. For each of these arrangements, we get one outcome for the scoops of ice cream, thus completing our one-to-one correspondence.

4.3.1 How many ways can Dylan buy a total of 8 fruits if there are an unlimited supply of only apples, bananas, and oranges?

We first note that we have to have 2 bars and 8 stars so that we can separate the stars into three sections, with the first being apples, second being bananas, and the third being oranges (Does it matter which type of fruit corresponds with the first section, second section, and third section in this method of counting?). So, there are a total of $\binom{10}{2} = \boxed{45}$ ways to choose where the 2 bars go in a total of 10 available slots.

4.3.2 How many solutions in nonnegative integers are there to the equation $x + y + z = 10$?

This problem at first seems nearly impossible to solve without bashing or casework, but this problems turns out to be just a stars and bars problem!. If we have 2 bars and 10 stars and let the number of stars in front of the first bar be x, the number of stars in between the first and second bar be y, and the number of stars after the second bar be z, then we now have a stars and bars problem. So, we have a total of 12 objects and we need to choose where 2 bars go, giving us the answer $\binom{12}{2} = \boxed{66}$.

4.4 Some Useful Extensions of Stars and Bars

In the section above, we considered using stars and bars under the restriction that a section or type of ice cream could have 0 scoops. What would happen if we had to ensure that each flavor have at least one scoop?

4.4.1 How many ways can we build a cup of ice cream if we want a total of n scoops from k distinct flavors, and there is at least one scoop of each flavor? (k$\geq n$)

Solution 1: We can first distribute 1 star to each flavor and then proceed to count using the rest of the stars using the stars and bars method. So, we give each flavor 1 star, leaving us with $n-k$ stars left and still $k-1$ bars. Applying the stars and bars formula we obtain $\binom{(n-k)+(k-1)}{k-1} = \binom{n-1}{k-1}$ ways.

Solution 2: We can imagine the gaps between the scoops of ice cream or the "stars". It would look something like this:

$$* - * - * - * \cdots * - *$$

We can place the $k-1$ bars into any of the $n-1$ gaps between the stars with at most one bar in each gap so that we would end up with a representation for our ice cream flavors. This method works because we are insured that every 2 bars is separated by at least one star, meaning that there is at least one scoop of every flavor. Thus, we have $\binom{n-1}{k-1}$ ways to pick which gaps the bars go into.

Remark: From now on, we will have to think "outside the box" to find a clever way to solve Combinatorics problems.

4.4.2 How many solutions are there in nonnegative integers to the equation $x + y + z = 10$ where x, y, and z are all even?

At first glance, we see that this problem looks like it can be solved by stars and bars, but we don't have a clear idea of how to proceed. Here is where previous exposure and clever thinking comes in. To solve this, we use a basic fact in mathematics that all even integers can be written in the form $2k$, where k is an integer. So, we can write $x = 2k_1$, $y = 2k_2$, and $z = 2k_3$, where k_i is a nonnegative integer. Thus, our original equation can be rewritten as:

$$\begin{aligned} 2k_1 + 2k_2 + 2k_3 &= 10 \\ k_1 + k_2 + k_3 &= 5 \end{aligned}$$

Since now we have an equation that we want to solve in nonnegative integers, we know we can solve this using stars and bars. Thus, we can see that there are 5 stars and 2 bars, giving us an answer of $\binom{7}{2} = \boxed{21}$.

4.4.3 How many solutions are there in nonnegative integers to the equation $x + y + z = 11$ where x,y, and z are all odd?

We see that the problem wants us to solve for odd nonnegative integers, so we use the basic fact that odd integers can be represented by $2k+1$ where k is an integer. Thus, we can rewrite the variables as $x = 2k_1+1$, $y = 2k_2 + 1$, and $z = 2k_3 + 1$, where k_i is a nonnegative integer. So, our original equation becomes:

$$\begin{aligned} 2k_1 + 1 + 2k_2 + 1 + 2k_3 + 1 &= 11 \\ 2k_1 + 2k_2 + 2k_3 &= 8 \\ k_1 + k_2 + k_3 &= 4 \end{aligned}$$

Since now we have an equation that we want to solve in nonnegative integers, we know we can solve this using stars and bars. Thus, we can see that there are 4 stars and 2 bars, giving us an answer of $\binom{6}{2} = \boxed{15}$.

4.4.4 How many solutions are there in nonnegative integers to the inequality $x + y + z \leq 5$?

We can choose to think of this problem in two ways. The first way is to say that if $x + y + z \leq 5$, then $x + y + z = 5$ or $x + y + z = 4$,...,$x + y + z = 0$ because the sum of the variables must be a nonnegative integer. It is clear that none of these equations share a solution since the sums are all distinct. Thus, the number of solutions to $x + y + z = 5$ is $\binom{7}{2} = 21$, the number of solutions to $x + y + z = 4$ is $\binom{6}{2} = 15$,..., and the number of solutions to $x + y + z = 0$ is $\binom{2}{2} = 1$. So, the sum of the number of solutions is $\sum\limits_{i=2}^{7} \binom{n}{2} = \binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} + \binom{6}{2} + \binom{7}{2}$. We can proceed to compute this sum of binomials by hand, or we can apply the hockey stick identity. Either way, we get our final answer to be $\boxed{56}$.

Hockey Stick Identity: $\sum\limits_{i=k}^{n} \binom{n}{k} = \binom{k}{k} + \binom{k+1}{k} + \ldots + \binom{n}{k} = \binom{n+1}{k+1}$ (Just a neat formula that we will prove in Combinatorics II)

However, there is a solution to this problem that involves less casework and computation. There is something very special about the problem, and it is that there is a $\leq$ sign. So, it is motivated to add another variable to this problem to make the statement be:

$$x + y + z + n \quad = \quad 5$$

Why do we have an $=$ sign? Well, we are using the variable n as a placeholder. If we let $n \geq 0$, then we will always have $x + y + z \leq 5$, and now we have a one-to-one correspondence. So, since all the variables are nonnegative, we can simply apply stars and bars. With the addition of one extra variable, there are now 5 stars and 3 bars, giving us an answer of $\binom{8}{3} = \boxed{56}$.

4.5 Complementary Counting

Sometimes when we approach a problem, the direct approach or casework approach is much harder than counting the **complement**, or opposite of what we are trying to count. For example:

4.5.1 How many 4-digit positive integers have at least one digit that is a 4 or a 5?

We could go ahead and use casework based on the number of 4s and the number of 5s, but that would take forever and could result in overcounting. What if we counted the numbers that don't have a 4 or a 5? Then, if we subtracted those from the total number of 4-digit numbers, then the numbers that would be left would have at least a 4 or a 5, our desired result. In order for a 4-digit number to not have a 4 or 5, then each of its digits can only be from the set $\{0, 1, 2, 3, 7, 8, 9\}$ with the exception of the leading digit, which can't be 0. Thus, there are $7 * 8 * 8 * 8 = 3584$ ways to form a 4-digit number without a 4 or 5. Is this our answer? No, because this is the opposite of what we are counting. We have to subtract this from $9 * 10 * 10 * 10$, which is the total number of 4-digit numbers, giving us our desired answer of $\boxed{5416}$ numbers.

4.5.2 How many ways are there for Alice, Bob, Catherine, Don, Ethan, Frank, and George to stand in a line if Alice and Bob refuse to be standing next to each other?

Instead of going through lots of cases where Alice and Bob are not next to each other, we can easily count the cases where they are next to each other and subtract from the total which is 7!. If Alice and Bob are together, we can treat them as a "supergroup", so now we have to arrange the 5 people and 1 "supergroup" which can be dont in 6! ways. However, Alice and be ahead of Bob or Bob can be ahead of Alice in their "supergroup", so we must multiply by 2 to get a total of $2 * 6!$ ways for Alice and Bob to be standing next to each other. Subtracting from 7!, we get that the total is $7! - 2 * 6! = 5 * 6! = \boxed{3600}$.

4.5.3 Let's visit problem 1.2.1 again: If there are 5 different people and I want to arrange them in a line, but two people insist on standing next to each other, how many different arrangements are possible?

Instead of a direct approach to counting, lets approach this problem trying to count the ways that the two people are not standing next to each other. Applying a sense of stars and bars, we have 2 bars, marking the two people that want to stand next to each other, and we have to place stars in the three sections. However, the middle section must have at least one star in order for them to not be next to each other. So, we have 2 people, or stars, left and 2 bars, giving us a total of $\binom{4}{2} = 6$ ways for the "identical" people to be placed. Because we treated the people as identical, we have to multiply by 3! for the ordering of the "stars" (the people that aren't the two) and multiply by 2! for the ordering of the ways the two people can be arranged, giving us a total of $6 * 6 * 2 = 72$ ways. But, we need to subtract this from 5! since we counted the complement. Thus our answer is $120 - 72 = \boxed{48}$, which was our answer to problem 1.2.1.

5 Problem Set I

1. There are 10 students in Mr. Feng's AP Calculus BC class. Mr. Feng wishes to form a committee for his class with a president, vice president, and treasurer. A student may hold more than one position, but no student can hold all the positions. How many ways can Mr. Feng's committee be formed?
2. 8 points are marked on a circle. How many distinct convex polygons of three or more sides can be drawn using some (and possibly all) of the eight points as vertices?
3. How many ways can the 6 students in Mr. Owen's chemistry class form 2 groups of 3 students?
4. How many ways can the 6 students in Mr. Owen's chemistry class form 2 groups of 3 people, if the first group is going on a field trip and the second group is staying at school? Is the answer the same as the previous question?
5. How many ways can I roll a total sum of 7 when I roll a die three times in a row?
6. How many ways can 7 people sit around a round table if two of the people insist that they have to sit next to each other?
7. How many orders of flips with 3 heads can I obtain if I flip a coin 6 times in a row?

8. If Dr. Flint wishes to choose a president, vice president, and secretary from a group of 8 students, but Zoheb won't serve in any position if Allan is serving in any position, how many ways can positions be chosen?
9. If Dr. Flint wishes to instead choose a committee of 4 people from her 8 students, and also designate a president for that committee, how many ways are there to form such a committee with a president?
10. How many ways can I write the number 6 as a sum of ordered, positive integers? Ex: 1+3+2, 3+1+1+1

6 Lecture II

6.1 An Introduction to Thinking Combinatorially

6.1.1 How can we count the number of subsets of $\{1, 2, 3, \ldots, n\}$?

A common way to approach a problem in Combinatorics is to start out with small cases and form a conjecture, later proving it. Let us find the number of subsets when $n = 3$:

$$\{\}\ \{1\}\ \{2\}\ \{3\}\ \{1, 2\}\ \{1, 3\}\ \{2, 3\}\ \{1, 2, 3\}$$

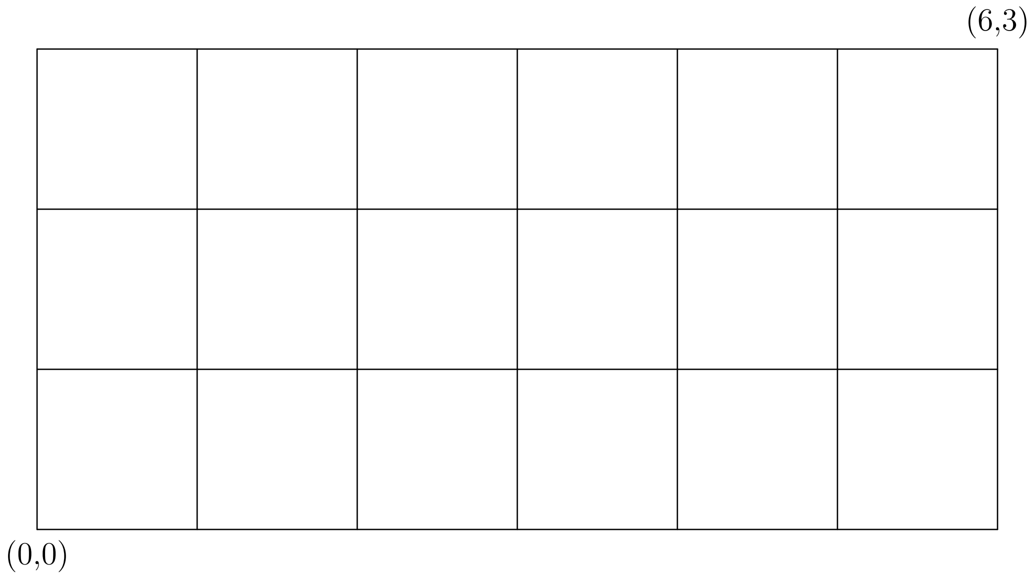
At first, we see that there are 8 total subsets, including the empty set, leading us to think that the total number of subsets might be 2^3 in this case. However, we cannot simply conclude this without proof. Let's count the number of subsets the number 1 is in: 4. The number of subsets the number 2 is in: 4. The number of subsets 3 is in: 4. Well, it seems that each number is in exactly half of the total number of subsets. Why? Well each of the elements in the subsets has two choices: either be in a certain subset, or not be in that subset. So, there is a $\frac{1}{2}$ chance that an element will be in any given subset, leading us to the proof that each element appears in half of the total subsets. However, we do not need this fact to conclude the proof; this is just a nifty fact regarding sets. In fact, we have stated the key idea for the proof already: Every element in the subset has two choices when we are building a subset, either be in it or not be in it. It is easy to see that every combination of "in the subset" or "not in the subset" for all the elements results in a distinct subset, and all of the possible subsets are covered in this way. Essentially, there are 2^n possible, distinct strings of "in the subset" or "not in the subset" for the list of elements. Thus, we have concluded our proof.

6.1.2 What formula can we derive with binomials/combinations from knowing the total number of subsets of $\{1, 2, 3, \ldots, n\}$ is 2^n?

The first way we counted the total number of subsets is by considering each element. Let us consider another way to count this. A subset of $\{1, 2, 3, \ldots, n\}$ can have 0,1,. . . ,n elements. How can we count the number of subsets with k items? We need to choose k elements from n total elements to make a valid subset, so it would be $\binom{n}{k}$. So, the total number of subsets would be $\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n}$. Thus, we derive:

$$\sum_{i=1}^{n} \binom{n}{k} \quad = \quad 2^n$$

6.1.3 One last lecture problem: How many ways are there to go from (0,0) to the point (6,3) by only moving along the lines of the grid and only moving to the right or up?



At first glance, the problem seems complicated and that there would be numerous cases to count. Let's try a few example sequences to get a feel for the problem. We can have the following sequences:

$$RRRUURRUR$$
$$UUURRRRRR$$
$$RURURRRUR$$
$$RRRRRRUUU$$
$$URURURRRR$$

Though there is no pattern in the specific sequences of the letters, but counting closely, we find that there is an **invariant**: The number of Rs is always 6 and the number of Us is always 3. Why is this true? Well, since we can only use Rs and Us in our path, we must take 3 steps up and 6 steps to the right total in order to reach the desired lattice point. Thus, our problem is reduced to finding the number of ways of arranging 6 Rs and 3 Us, which is simply $\binom{9}{3} = \boxed{84}$.

7 Problem Set II

1. If we we to count how many elements were listed total in all of the subsets of $\{1, 2, \ldots, 5\}$, what would we get? Can you generalize this for $\{1, 2, \ldots, n\}$?

2. Define an ascending number to be a number such that its digits are in increasing order from left to right. How many 3-digit ascending numbers are there? 4-digit? 5-digit?

3. Define a non-decreasing number to be a number such that its digits do not descend. For example, 1123445 is a non-decreasing number. How many 5-digit non-decreasing numbers are there?

4. Define a multisubset of $\{1, 2, \ldots, n\}$ to be a set consisting of only elements from $\{1, 2, \ldots, n\}$, but an element can occur more than once in the set. How many multisubsets of size 5 are there from the set $\{1, 2, 3, 4, 5, 6\}$?

5. In Mr. Liu's AP Statistics class, there are 15 students. At the beginning of class, each of the students shakes hands with every other student exactly once. How many handshakes take place?

6.* Jesse is given a set of numbers $S = \{1, 4, 9, 16, 25, 36\}$. If Jesse takes every two-element subset of S and writes it down on a sheet of paper, what will be the sum of all the numbers on his paper after he has gone through every two-element set?

7. 8 identical white cards and 3 identical yellow cards are arranged in a row such that no two yellow cards can be adjacent to another. Find the number of possible arrangements.

8. How many paths are there from (0,0) to (8,4) by only staying on the grid lines and moving up or to the right and passing through the point (4,1)?

9.* How many 12-letter arrangements of $AAAABBBBCCCC$ are there such that no As are in the first 4 letters, no Bs are in the second 4 letters, and no Cs are in the last 4 letters?

10.* How many pairs of subsets (A, B) from $\{1, 2, 3, 4, 5, 6\}$ are there such that B is a subset of A? (We define B as a subset of A if all the elements in B are in A. Ex: $\{1, 2\}$ is a subset of $\{1, 2, 4, 5, 23\}$)

Note: Starred problems denote challenge problems. They are solvable by techniques we have learned so far but take deeper thought. Hints to selected problems will be on the next page.

8 Hints to Selected Problems

1. The number of times 1 is counted is the number of subsets 1 is in. How can we count how many subsets contain 1?

3. Stars and Bars. How? Try it!

5. What does a handshake mean? How are people matched together?

6.* An element will be written down if it is (fill in the blank) than the other element.

9.* Start with the As. They can go anywhere in the last 8 spaces. Do casework on how many As are in each of the last two sections of 4.

10.* Each element has 4 choices in this problem (Why?). Which of these choices satisfy the condition?

Combinatorics II

In the second class covering Combinatorics, we will be covering numerous well-known and widely used methods and ideas in Combinatorics. In addition, basic proofs will be introduced to help prove useful formulas. **DO NOT TRY TO MEMORIZE THE FORMULAS! REMEMBER HOW THEY ARE DERIVED.**

9 Probability

Probability: How likely an event will occur, usually calculated by $\frac{\text{number of desired outcomes}}{\text{total outcomes}}$, though this method is usually not the best way.

THE PROBABILITY OF ANYTHING HAPPENING IS IN THE INTERVAL [0,1]. IT IS NEVER ABOVE 1.

The sum of the probabilities of all the possible outcomes of a situation should be 1.

Now, for some basic probability problems.

9.1 Lecture Problems

9.1.1 If I roll a die twice, what is the probability that I roll a sum of 7?

Well, the total possible outcomes is $6*6=36$ because there are two die. The ways we can obtain a sum of seven are:

$$(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)$$

So, we have a total of 7 possible outcomes, giving us a probability of $\boxed{\frac{7}{36}}$.

9.1.2 One bucket has 4 white orbs and 3 black orbs and a second bucket has 2 white orbs and 5 black orbs. What is the probability that if I take one orb from each bucket, I get exactly 1 black orb and 1 white orb?

There are two cases: Either I pick the white orb from the first bucket or I pick in from the second bucket. If I pick the white orb from the first bucket, there is a $\frac{4}{7}$ probability, and there is a $\frac{5}{7}$ probability of picking a black orb from the second bucket. If the white orb is picked in the second bucket, then there is $\frac{3}{7}$ probability of picking a black orb from the first bucket and a $\frac{2}{7}$ probability of picking a white orb from the second bucket. Thus, our total probability is $\frac{4}{7}*\frac{5}{7}+\frac{3}{7}*\frac{2}{7}=\boxed{\frac{26}{49}}$.

Alternatively, we could have used complementary counting here, noting that the condition is not satisfied if both orbs are white or if both orbs are black, having probabilities of $\frac{8}{49}$ and $\frac{15}{49}$ respectively, giving us a total of $\frac{23}{49}$. Subtracting from 1, we get the same answer, $\boxed{\frac{26}{49}}$.

9.1.3 What is the probability that if we roll 4 dice, a) all four dice will show the same value b) all four dice will show different values c) all four dice will show exactly three different values d) all four die will show exactly 2 different values?

Part A: If all four die have the same value, then all we need to choose is what number will appear on all four die, which can be done in 6 ways. Thus, the probability is $\frac{6}{6^4}=\boxed{\frac{1}{216}}$.

Part B: If all four die show different values, then we have to choose what four numbers will be displayed, a total of $\binom{6}{4}$, and how many ways we can arrange these numbers, which is 4!. Thus, the total is $15*24=360$, giving us a probability of $\frac{360}{6^4} = \boxed{\frac{5}{18}}$.

Part C: If exactly three different values are shown, we have to choose which three values will be shown, choose which value will appear twice, and then arrange the values. There are $\binom{6}{3}$ ways to choose the three values, $\binom{3}{1}$ ways to choose which one of these values wil appear twice, and $\binom{4}{2} * \binom{2}{1}$ ways to choose which of the four spots will be what value. So, our total desired outcome is $20 * 3 * 6 * 2 = 720$, leaving us with a probability of $\frac{720}{6^4} = \boxed{\frac{5}{9}}$.

Part C: With exactly two distinct values that show up on the die, there two cases: 2 of both number show up or 3 of one number show up and 1 of the other. For the first case, we have to choose which two values will show up and then calculate how many ways we can arrange them. There are $\binom{6}{2}$ ways to choose the two values and $\binom{4}{2}$ ways to choose the ordering of the values, giving us a total of $15 * 6 = 90$ ways. Thus, the probability is $\frac{90}{6^4} = \frac{5}{72}$. For the second case, there are 6 ways to choose the number that appears 3 times and 5 ways to choose which number appears once. Now, there are $\binom{4}{3}$ ways to arrange them, giving us a total of $6 * 5 * 4 = 120$ ways, and a probability of $\frac{120}{6^4} = \frac{5}{54}$. Summing these two probabilities, we get the final probability to be $\frac{210}{6^4} = \boxed{\frac{35}{216}}$.

Note: We can check our work by seeing if the probabilities from all 4 parts sum to 1 since these cases cover all possible outcomes of 4 dice rolls. Checking, we see that $\frac{1}{216} + \frac{5}{18} + \frac{5}{9} + \frac{35}{216} = \frac{1}{216} + \frac{60}{216} + \frac{120}{216} + \frac{35}{216} = \frac{216}{216} = \boxed{1}$.

9.2 Probability+Algebra=Pralgebra

9.2.1 In a unfair coin, the probability of flipping 2 heads and 1 tail is 5 times the probability of flipping 3 heads. What is the probability that any single toss will show heads?

At first, we may guess and check, but it is extremely hard to guess a certain probability. So, we apply a well-known tactic of setting the probability of heads as k. Thus, the probability of flipping tails is $1 - k$. Now, there are 3 ways to arrange a flip of 2 heads and 1 tails and 1 way to arrange a flip of 3 heads. Now we can set up this equation:

$$\begin{aligned} 3k^2(1-k) &= 4k(1-k)^2 \\ 3k &= 4(1-k) \\ 3k &= 4 - 4k \\ k &= \frac{4}{7} \end{aligned}$$

Thus, the probability of flipping heads with this coin is $\boxed{\frac{4}{7}}$.

9.2.2 If there are 10 balls in an urn, each either black or white, and the probability of drawing 2 balls at once and getting 1 ball of each color is $\frac{1}{5}$, then find all possible values of the number of black balls in the urn.

Let there be k black balls and $10 - k$ white balls. There are $k(10 - k)$ total ways to choose one of each ball, and there are $\binom{10}{2}$ ways to choose 2 balls from the 10. Thus, the problem equates to solving:

$$\begin{aligned} \frac{k(10-k)}{\binom{10}{2}} &= \frac{1}{5} \\ k(10-k) &= 9 \\ k^2 - 10k + 9 &= 0 \\ k &= 1, 9 \end{aligned}$$

Thus, there can be $\boxed{1}$ or $\boxed{9}$ black balls in the urn.

10 Binomial Theorem and Applications

10.1 The Theorem

The **binomial theorem** states that:

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n-1}x^1 y^{n-1} + \binom{n}{n}x^0 y^n$$

Why is this true? Well, imagine you had n $(x+y)$s like this:

$$(x+y)(x+y)(x+y)\ldots(x+y)$$

If we had the term with the power x^k, then k of the $(x+y)$ terms have been used up to obtain x^k, so there must be y^{n-k} for the y term. How can we choose which k of these $(x+y)$ terms we choose an x from? Well, it is simply $\binom{n}{k}$, thus showing why the theorem works.

10.2 In the expansion of $(x+2y)^5$, what is the coefficient of the x^2y^3 term?

First, the y would have to appear as $(2y)^3 = 8y^3$, giving us an 8 to the coefficient. Similarly, by combinations or the binomial theorem, the primary coefficient of this term is $\binom{5}{2} = 10$. Thus, the coefficient is $10 * 8 = \boxed{80}$.

10.3 Applications

One may ask, why is this theorem important? What can we do with this theorem?

10.3.1 Prove that $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = 2^n$

Well, we see that the items we are summing are all in the form $\binom{n}{k}$, hinting to us that we should use the binomial theorem. How? Let's set $x, y = 1$, so we get:

$$(1+1)^n = \binom{n}{0}1^n 1^0 + \binom{n}{1}1^{n-1}1^1 + \ldots + \binom{n}{n}1^0 1^n$$

Further simplifying:

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n}$$

Using the binomial theorem, we have found another way to prove that the number of subsets of $\{1, 2, \ldots, n\} = 2^n$

10.3.2 Prove that $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^n\binom{n}{n} = 0$

Once again, we see that we need to use the binomial theorem, but this time, we substitute $x = 1$ and $y = -1$ to get:

$$(1-1)^n = \binom{n}{0}1^n(-1)^0 + \binom{n}{1}1^{n-1}(-1)^1 + \ldots + \binom{n}{n}1^0(-1)^n$$

Simplifying, we get the desired result:

$$0 = \binom{n}{0} - \binom{n}{1} + \ldots + (-1)^n\binom{n}{n}$$

10.3.3 How many ordered sums of positive integers are there that add up to 6? Ex: 1+1+4, 3+2+1, 4+2

This problem can be simplified to many cases of stars and bars. Every ordered sum can be represented with stars and bars. Ex: 2+2+2 is represented by 6 stars and 2 bars, where each bar must go in a gap between the stars since the numbers must be positive. The number of bars we can have is at least 0 and at most 5 because there are 5 total gaps between the six stars. Since there are always 5 gaps between the stars, and we must place 0, 1, 2, 3, 4, or 5 bars in these gaps, with each gap having at most 1 bar, the total would be:

$$\binom{5}{0}+\binom{5}{1}+\binom{5}{2}+\binom{5}{3}+\binom{5}{4}+\binom{5}{5}$$

By the formula proved in 2.3.1, we get the total to be $\boxed{2^5}$.

Generalizing, we can see that the number of **compositions** of n is 2^{n-1}. You can also show this by saying that each of the $n-1$ gaps between the n stars can either have a bar or not, and each of these combinations results in a unique composition.

11 Combinatorial Identities

11.1 Pascal's Triangle and Variations

First, wee need to prove a basic, but important formula in Combinatorics.

11.1.1 Prove that $\binom{n}{k}=\binom{n}{n-k}$

This problem will be an introduction to a common way of proving formulas in Combinatorics. When we try to prove a formula or identity using committees, then we call this a **committee-forming argument**. How many ways are there to form a committee of k people from n total people? $\binom{n}{k}$. We can instead choose $n-k$ people so that they aren't on the committee, which can be done in $\binom{n}{n-k}$ ways. Thus, since these two methods both count the number of ways to form such a committee, we are done.

11.1.2 Prove that $\binom{n-1}{k}+\binom{n-1}{k-1}=\binom{n}{k}$ (Pascal's Identity)

Let's see the significance of the identity in Pascal's Triangle, where the terms in the n^{th} row are $\binom{n}{0}$, $\binom{n}{1}$,...,$\binom{n}{n}$:

$$\begin{array}{lccccccccc}
n=0: & & & & & 1 & & & & \\
n=1: & & & & 1 & & 1 & & & \\
n=2: & & & 1 & & 2 & & 1 & & \\
n=3: & & 1 & & 3 & & 3 & & 1 & \\
n=4: & 1 & & 4 & & 6 & & 4 & & 1
\end{array}$$

We see the identity at work here. For example:

$$\begin{array}{lccccccccc}
n=0: & & & & & 1 & & & & \\
n=1: & & & & 1 & & 1 & & & \\
n=2: & & & 1 & & 2 & & 1 & & \\
n=3: & & 1 & & 3 & & 3 & & 1 & \\
n=4: & 1 & & 4 & & 6 & & 4 & & 1
\end{array}$$

Letting $n = 4$ and $k = 2$ in Pascal's Identity, we get $\binom{3}{1} + \binom{3}{2} = \binom{4}{2}$. So, Pascal's identity shows that the sum of any two entries in Pascal's Triangle is equal to the entry below those two.

How can we prove the identity? Let's try using a committee argument again. Let's say we are trying to form a committee of k people from n total people again. This time we single out a single person, say Albert. There are two things that could happen to Albert: Either he is on the committee or he is not. If he is on the committee, then we need to choose $k-1$ other people from $n-1$ people left, giving us $\binom{n-1}{k-1}$. If Albert isn't on the committee, then we have to choose k people from $n-1$ people left, since we already accounted for Albert, giving us $\binom{n}{k-1}$. The total number of ways to form a committee is $\binom{n}{k}$, and equating the two ways of counting gives us the identity.

This way of counting the same thing twice is a very useful tactic for combinatorial proofs.

11.2 Other Identities

11.2.1 Prove that $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$

Once again, we attempt to use a committee forming argument, but this time there are extra variables instead of just binomials. Let's say that we are trying to form a committee of k people from n total people with a president included in the k people chosen. Let's attempt to count this in two ways. We can choose the k person committee first and then choose a president out of the k people, giving us $k\binom{n}{k}$, or we can choose the president first and then the $k-1$ remaining people to form the committee, giving us $n\binom{n-1}{k-1}$. Thus, we have:

$$\begin{aligned} n\binom{n}{k} &= k\binom{n-1}{k-1} \\ \binom{n}{k} &= \frac{n}{k}\binom{n-1}{k-1} \end{aligned} \tag{2}$$

And our proof is complete.

11.2.2 Prove that $\sum\limits_{i=k}^{n} \binom{i}{k} = \binom{k}{k} + \binom{k+1}{k} + \ldots + \binom{n}{k} = \binom{n+1}{k+1}$ (Hockey Stick Identity)

Suppose we have a set $\{1, 2, \ldots, n+1\}$ with $n+1$ elements. We wish to choose a $k+1$ element subset from it. Let's consider the largest element m of such a subset. Clearly, it must be at least $k+1$ and at most $n+1$. So, if m is $k+1$, we need to choose k other elements from the k remaining elements. If m is $k+2$, then we need to choose k other elements from the remaining $k+1$ elements.... If m is $n+1$, then we need to choose k other elements from the remaining n elements. Since all of these cases cover all the possible subsets, we can equate this sum with the total number of ways to choose such a subset, which is $\binom{n+1}{k+1}$. Thus, we get the desired identity:

$$\sum_{m=k+1}^{n+1} \binom{m-1}{k} = \binom{k}{k} + \binom{k+1}{k} + \ldots + \binom{n}{k} = \binom{n+1}{k+1}$$

11.2.3 Prove that $\sum\limits_{i=0}^{n} i\binom{n}{i} = 0\binom{n}{0} + 1\binom{n}{1} + \ldots + n\binom{n}{n} = n2^{n-1}$

To prove this combinatorially, let us try to count the number of ways choosing an committee of any size from n total people, and one of the committee members is the president. The RHS represents choosing the president first and then choosing any subset of the remaining $n-1$ people to form the rest of the committee. The LHS is a sum of binomials of the from $k\binom{n}{k}$, so this represents choosing a committee of size k and then choosing a president from that committee. So, since k ranges from 0 to n, we cover all possible sizes of committees with a president. Thus, the RHS and LHS count the same thing, and we are done.

Another way to prove this combinatorially is to view the LHS in terms of sets. There are $\binom{n}{k}$ ways to choose a subset of size k from $\{1, 2, \ldots, n\}$, and each of these sets has k elements. Thus, $k\binom{n}{k}$ counts the total number of elements over all the subsets of size k. Thus, as k ranges from 0 to n, which is the LHS, we count the total number of elements over all subsets of $\{1, 2, \ldots, n\}$. On the RHS, we consider each element. the number of subsets that include 1 is 2^{n-1}, and this is true for all n numbers. Thus, the total number of elements over all subsets is also $n2^{n-1}$, and our proof is complete.

We can also prove this algebraically. We can utilize the identity $\binom{n}{k} = \binom{n}{n-k}$ and letting $i\binom{n}{i} = 0\binom{n}{0} + 1\binom{n}{1} + \ldots + n-1\binom{n}{n-1} + n\binom{n}{n} = S$:

$$\begin{aligned} S &= 0\binom{n}{0} + 1\binom{n}{1} + \ldots + n-1\binom{n}{n-1} + n\binom{n}{n} \\ S &= 0\binom{n}{n} + 1\binom{n}{n-1} + \ldots + n-1\binom{n}{1} + n\binom{n}{0} \end{aligned}$$

Adding and combining coefficients for binomials:

$$\begin{aligned} 2S &= n\binom{n}{0} + n\binom{n}{1} + \ldots + n\binom{n}{n-1} + n\binom{n}{n} \\ &= n\left(\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n-1} + \binom{n}{n}\right) \\ &= n2^n \\ S &= n2^{n-1} \end{aligned}$$

Using previous identities, we complete the proof.

12 Recursion

12.1 Definition and Examples

We use recursion to define something in terms of previously defined items. For example, the Fibonacci Sequence is a recursion because it is defined as $F_n = F_{n-1} + F_{n-2}$.

12.1.1 How many sequences of flips of a coin (heads or tails) of length 10 have no two adjacent heads?

We consider any string of H and T of length n. If this string ends in T, then we can have any working string of length $n-1$ before it. If this string ends in H, then we must have a T before that, leaving us with TH preceded by any working string of length $n-2$. Thus, if we denote F_n as the number of working strings of length n. Because of the previous observations, we obtain the recursion:

$$F_n = F_{n-1} + F_{n-2}$$

Now, there are 2 strings of length 1 and 3 strings of length 2. Building a table, we seek to find F_{10}:

F_1	2
F_2	3
F_3	5
F_4	8
F_5	13
F_6	21
F_7	34
F_8	55
F_9	89
F_{10}	144

So, there are a total of $F_{10} = \boxed{144}$ strings.

12.1.2 How many ways are there to tile a 10x2 rectangle with 1x2 blocks?

Once again, we look at what can happen at the end of a length n tiling. If the last block is a vertical 2x1 block, then we can have any working tiling of length $n-1$ before that block. The only other option is two horizontal 1x2 blocks at the end, and in this case we can have any working string of length $n-2$ before those blocks. Thus, we derive the recursion $F_n = F_{n-1} + F_{n-2}$, with $F_1 = 1$ and $F_2 = 2$. Making a table:

F_1	1
F_2	2
F_3	3
F_4	5
F_5	8
F_6	13
F_7	21
F_8	34
F_9	55
F_{10}	89

So, there are $\boxed{89}$ total tilings.

12.1.3 Call a subset *cool* if there are no two consecutive numbers in it. Find the number of *cool* subsets of $\{1, 2, \ldots, 10\}$, if we are including the empty set. Ex: $\{1, 3, 5, 10\}$ is a *cool* subset, but $\{1, 5, 6, 9\}$ is not

Once again, we look to what happens with the last element of a set $\{1, 2, \ldots, n\}$. If we consider the element n, if it is in the subset, then we can add it to any previous *cool* subset of $\{1, 2, \ldots, n-2\}$. If the element n isn't in the subset, then we can have any *cool* subset of $\{1, 2, \ldots, n-1\}$. Once again, we have derived the recursion $F_n = F_{n-1} + F_{n-2}$, where F_n represents the number of *cool* subsets of $\{1, 2, \ldots, n\}$. Since $F_1 = 2$ and $F_2 = 3$, we build a table:

F_1	2
F_2	3
F_3	5
F_4	8
F_5	13
F_6	21
F_7	34
F_8	55
F_9	89
F_{10}	144

Thus, there are $\boxed{144}$ *cool* subsets of $\{1, 2, \ldots, 10\}$.

12.2 Cool Probability with Recursion

12.2.1 I lay down three cards in a row on a table. The middle card, before I do anything, is the Queen of Hearts. Every turn, I choose one of the two cards on the left or right side of the middle card, with equal probability, and switch that card to the middle, putting the middle card to the right or left position. What is the probability that, after 5 turns, the Queen of Hearts is in the middle?

In order to be in the middle on turn n, a card must be on one of the side positions on turn $n-1$. So, if we let P_n be the probability that the Queen of Hearts is in the middle on turn n, we derive that the probability it is in the middle after n turns is:

$$P_n = \frac{1}{2}(1 - P_{n-1})$$

This is true because there is a probability of $1 - P_{n-1}$ that the Queen of Hearts is not in the middle, and it has a probability of $\frac{1}{2}$ to be switched to the middle on turn n if it is on the side position. Since $P_1 = 0$ because the Queen of Hearts gets switched out the first move, we form a table:

P_1	0
F_2	$\frac{1}{2}$
F_3	$\frac{1}{4}$
F_4	$\frac{3}{8}$
F_5	$\frac{5}{16}$

So, the probability that the Queen of Hearts is in the middle after 5 moves is $\boxed{\frac{5}{16}}$.

13 Principle of Inclusion-Exclusion

The Principle of Inclusion-Exclusion (PIE) is used when we want to adjust for over counting. Key words that signal using PIE are: at least one, or.

13.1 A Few Problems

13.1.1 How many multiples of 2 or 5 are between 1 and 100?

Well, it seems like we can just count the number of multiples of 2 and the number of multiples of 5 and just add them right? Well, in fact, we over count some numbers. Let's list out some multiples to see this:

$$2, 4, 6, 8, \underline{10}, 12, 14, 16, 18, \underline{20}, 22, 24, \dots$$
$$5, \underline{10}, 15, \underline{20}, 25, 30, 35, 40, 45, 50, 55, 60, \dots$$

We see that 10, 20, 30,... are counted twice because they are multiples of 2 and 5, or rather, 10. So, when we counted the number of multiples of 2, we counted each multiple of 10 once. But when we counted the multiples of 5, we counted each multiple of 10 again. So, how can we fix this over count? We simply subtract the number of multiples of 10 because we need them to be only counted once. Thus, the way we count this is:

$$|\text{Number of multiples of 2}| + |\text{Number of multiples of 5}| - |\text{Number of multiples of 10}|$$

So, our final answer would be $50 + 20 - 10 = \boxed{60}$ numbers.

In general, when we have sets $A_1, A_2, A_3, \dots, A_n$, and we seek to find $|A_1 \cup A_2 \cup \dots \cup A_n|$, we can use the general formula: —A$_1 \cup A_2 \cup \dots \cup A_n| = |A_1| + |A_2| + \dots + |A_n|$
$- |A_1 \cap A_2| - |A_1 \cap A_2| - \dots - |A_{n-1} \cap A_n| + \dots$
$+ (-1)^{n-1}|A_1 \cap A_2 \cap \dots \cap A_n|$ Or, in summation notation:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \dots \; + (-1)^{n-1} |A_1 \cap \dots \cap A_n|$$

13.1.2 How many numbers from 1 to 100 are relatively prime to 2, 3, and 5?

We utilize PIE and complementary counting here. Since we are asked to find how many numbers have no factor of 2, 3, and 5, instead, we can count how many numbers are a multiple of 2, 3, or 5. By PIE, this is

$$\left\lfloor\frac{100}{2}\right\rfloor+\left\lfloor\frac{100}{3}\right\rfloor+\left\lfloor\frac{100}{5}\right\rfloor-\left\lfloor\frac{100}{6}\right\rfloor-\left\lfloor\frac{100}{10}\right\rfloor-\left\lfloor\frac{100}{15}\right\rfloor+\left\lfloor\frac{100}{30}\right\rfloor$$

$$\begin{aligned} &= 50+33+20-16-10-6+3 \\ &= 34 \end{aligned}$$

However, this is the complement, so our final answer is $100-34=\boxed{66}$ numbers.

14 Problem Set

1. In the expansion of $(2x-3y)^7$, what is the coefficient of x^4y^3?
2. How many positive integers not exceeding 2000 are multiples of 3 or 4 but not 5?
3. Show that for $0 \le m \le k \le n$, $\binom{n}{k}\binom{k}{m}=\binom{n}{m}\binom{n-m}{k-m}$
4. On a standard die one of the ldots is removed at random with each dot equally likely to be chosen. The die is then rolled. What is the probability that the top face has an odd number of ldots?
5. Bob is going up a staircase with 10 steps. If Bob can walk 1 step up or hop 2 steps up in one move, how many ways can Bob ascend the staircase. Note: Bob can technically hop 2 steps up on the ninth step
6. How many arrangements of $AAABBBCCC$ are there if no three consecutive letters can be the same?
7. How many ways are there to tile a 10x3 rectangle with 3x1 blocks?
8. Give a combinatorial proof for $\sum_{i=0}^{n}\binom{n}{i}^2=\binom{2n}{n}$
9. Call a number $prime-looking$ if it is composite but not divisible by 2, 3, or 5. There are 168 prime numbers less than 1000. How many $prime-looking$ numbers are there that are less than 1000?
10. Dylan flips 7 fair coins. If Dylan flips at least three tails, what is the probability that he flips at least two heads?
11. Ten women sit in 10 seats in a line. All of the 10 get up and then reseat themselves using all 10 seats, each sitting in the seat she was in before or a seat next to the one she occupied before. How many ways can the women be reseated in this manner?
12. What is the remainder when $6^{83}+8^{83}$ is divided by 49?
13. How many ways are there to distribute 7 **distinct** pieces of candy to three kids if each kid must have at least one candy?
14. A collection of 8 cubes consists of one cube with edge-length k for each integer k for $1 \le k \le 8$. A tower is to be built under the conditions that any cube may be at the bottom of the tower, and the cube immediately on top of a cube with edge-length k must have edge-length at most $k+2$ (You can't have a cube with edge-length 5 be on top of a cube with edge-length 2)

15 Hints to Selected Problems

3. Consider an executive committee within a committee.
8. Consider 2 groups of people and forming a committee out of the people from those groups.
9. Be careful. 2, 3, and 5 are prime but are divisible by 2, 3, or 5.
10. At most how many tails can show up if we have at least 2 heads?
11. Consider the woman that sat at one of the ends of the line. What options does she have? What is possible after she sits down?
12. $6 = 7 - 1$, $8 = 7 + 1$
13. Use PIE. Consider when at least one kid doesn't receive any candy.
14. In a previous, valid stack of $n - 1$ blocks, where can the block with edge-length n go?

Algebra I

In the first algebra class, we will go over numerous advanced techniques for factoring, problems dealing with quadratics, and ways of solving systems of equations. Remember to think outside of the box and try different things to solve algebra problems.

16 Factoring

16.1 Basic Factoring and Expansion Identities

Let's take a look at some basic factoring identities that are used widely

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

$$x^2 - y^2 = (x+y)(x-y)$$

$$x^3 + y^3 = (x+y)(x^2 - xy + y^2)$$

$$x^3 - y^3 = (x-y)(x^2 + xy + y^2)$$

$$x^n + y^n = (x+y)(x^{n-1} - x^{n-2}y + \ldots + y^{n-1}), \text{where n is odd}$$

$$x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \ldots + y^{n-1})$$

$$(x+y+z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$$

$$(x+y+z)^3 = x^3 + y^3 + z^3 + 3x^2y + 3xy^2 + 3y^2z + 3yz^2 + 3z^2x + 3zx^2 + 6xyz$$

16.2 Factoring Problems

16.2.1 Factor $x^4 - 3x^2y^2 + y^4$

We can attempt to complete the square:

$$\begin{aligned} x^4 - 3x^2y^2 + y^4 &= (x^2 - y^2)^2 - x^2y^2 \\ &= (x^2 + xy - y^2)(x^2 - xy - y^2) \end{aligned}$$

16.2.2 Factor $(x+y)(x-y) - 4(y+1)$

First, we expand and then try to complete the square:

$$\begin{aligned} (x+y)(x-y) - 4(y+1) &= x^2 - y^2 - 4y - 4 \\ &= x^2 - (y+2)^2 \\ &= (x+y+2)(x-y-2) \end{aligned}$$

16.2.3 Factor $4(x^2 + x - y^2) + 1$

Once again, we see quadratics and nice coefficients, so we complete the square:

$$\begin{aligned} 4(x^2 + x - y^2) + 1 &= 4x^2 + 4x - 4y^2 + 1 \\ &= (2x+1)^2 - (2y)^2 \\ &= (2x + 2y + 1)(2x - 2y + 1) \end{aligned}$$

16.2.4 Factor $a(a - 4b) + 4(b^2 - 1)$

Expanding, but this time completing the square in two variables:

$$\begin{aligned} a(a-4b) + 4(b^2 - 1) &= a^2 - 4ab + 4b^2 - 4 \\ &= (a - 2b)^2 - 2^2 \\ &= (a - 2b + 2)(a - 2b - 2) \end{aligned}$$

16.2.5 Factor $x^2 - y^2 + 2(x + 3y - 4)$

In this problem, we complete the square 2 times:

$$\begin{aligned} x^2 - y^2 + 2(x + 3y - 4) &= x^2 + 2x - y^2 + 6y - 8 \\ &= (x+1)^2 - (y-3)^2 \\ &= (x + y - 2)(x - y + 4) \end{aligned}$$

16.2.6 Factor $x^4 + x^2 + 1$

Completing the square:

$$\begin{aligned} x^4 + x^2 + 1 &= (x^2+1)^2 - x^2 \\ &= (x^2 + x + 1)(x^2 - x + 1) \end{aligned}$$

16.2.7 Factor 100010001

We notice that $100010001 = 100^4 + 100^2 + 1$, so we can use the result from the previous problem:

$$\begin{aligned} 100010001 &= 100^4 + 100^2 + 1 \\ &= (100^2 + 100 + 1)(100^2 - 100 + 1) \\ &= ((100+1)^2 - 100)(9901) \\ &= ((100 + 10 + 1)(100 - 10 + 1))(9901) \\ &= 111 * 91 * 9901 \\ &= 3 * 7 * 13 * 37 * 9901 \end{aligned}$$

16.2.8 Prove that $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$

Let's try to write $x^3 + y^3 + z^3$ in terms of other **symmetric polynomials**:

$$\begin{aligned} x^3 + y^3 + z^3 &= (x+y+z)^3 - 3(x^2y + xy^2 + y^2z + yz^2 + z^x + zx^2) - 6xyz \\ &= (x+y+z)^3 - 3[(x+y+z)(xy+yz+zx) - 3xyz] - 6xyz \\ &= (x+y+z)^3 - 3(x+y+z)(xy+yz+zx) + 3xyz \\ x^3 + y^3 + z^3 - 3xyz &= (x+y+z)^3 - 3(x+y+z)(xy+yz+zx) \\ &= (x+y+z)[(x+y+z)^2 - 3xy + yz + zx)] \\ &= (x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx) \end{aligned}$$

16.2.9 Factor $x^4 + 2x^3 + 2x^2 + 2x + 1$

Using the fact that $(x+1)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1$, we manipulate to get:

$$\begin{aligned} x^4 + 2x^3 + 2x^2 + 2x + 1 &= (x+1)^4 - (2x^3 + 4x^2 + 2x) \\ &= (x+1)^4 - 2x(x^2 + 2x + 1) \\ &= (x+1)^4 - 2x(x+1)^2 \\ &= (x+1)^2((x+1)^2 - 2x) \\ &= (x+1)^2(x^2+1) \end{aligned}$$

17 Algebraic Expressions, Equations, and Manipulations

17.1 Expressions and Manipulations

17.1.1 If $x + \frac{1}{x} = 3$, then find the values of $x^2 + \frac{1}{x^2}$,$x^3 + \frac{1}{x^3}$, and $x^4 + \frac{1}{x^4}$

Because we see an x^2 term in $x^2 + \frac{1}{x^2}$, we square the given equation to see if something good happens:

$$\begin{aligned} (x + \frac{1}{x})^2 &= x^2 + 2x * \frac{1}{x} + \frac{1}{x^2} \\ &= x^2 + \frac{1}{x^2} + 2 \end{aligned}$$

Thus, it follows that $x^2 + \frac{1}{x^2} = 9 - 2 = \boxed{7}$.

Now, to find $x^3 + \frac{1}{x^3}$, we try to multiply two given expressions together:

$$\begin{aligned} (x + \frac{1}{x})(x^2 + \frac{1}{x^2}) &= x^3 + x * \frac{1}{x^2} + x^2 * \frac{1}{x} + \frac{1}{x^3} \\ &= x^3 + \frac{1}{x^3} + x + \frac{1}{x} \\ &= x^3 + \frac{1}{x^3} + 3 \end{aligned}$$

So, we get that $x^3 + \frac{1}{x^3} = 7 * 3 - 3 = \boxed{18}$.

To get an x^4 term, we can square the expression we found for $x^2 + \frac{1}{x^2}$:

$$\begin{aligned} (x^2 + \frac{1}{x^2})^2 &= x^4 + 2x^2 * \frac{1}{x^2} + \frac{1}{x^4} \\ &= x^4 + \frac{1}{x^4} + 2 \end{aligned}$$

Thus, $x^4 + \frac{1}{x^4} = 7^2 = 2 = \boxed{47}$.

17.1.2 Given that $a + b = 5$ and $ab = 2$, find the value of $a^3 + b^3$

We could try to expand $(a+b)^3$ and substitute, but instead, we will use the nifty factorization:

$$a^3 + b^3 = (a+b)^3 - 3ab(a+b)$$

So, plugging in our givens, $a^3 + b^3 = 3^3 - 3 * 5 * 3 = 27 - 45 = \boxed{\text{-18}}$.

17.2 Equations

17.2.1 If x, y are integers, find all solutions to $xy = x + y + 6$

We move the $x + y$ to the left side and then use **Simon's Favorite Factoring Trick (SFFT)**:

$$\begin{aligned} xy - x - y &= 6 \\ (x-1)(y-1) &= 7 \end{aligned}$$

Now, since $x - 1$ and $y - 1$ must also both be integers, they must be integers that multiply to 7, giving us the possibilities $(x-1, y-1) = (1, 7), (7, 1), (-7, -1), (-1, -7)$. Solving each of these cases, we get all the solutions: $(x, y) = (2, 8), (8, 2), (-6, 0), (0, -6)$.

17.2.2 Find all possible ordered triplets of reals (x, y, z) such that $x^2+y^2+z^2-2x+8y-6z = -50$

We can complete the square for all the variables on the LHS, giving us:

$$\begin{aligned} (x-1)^2 + (y+4)^2 + (z-3)^2 &= -50 + 1 + 16 + 9 \\ &= -24 \end{aligned}$$

Since x, y, and z are reals, then $x^2 \geq 0$, $y^2 \geq 0$, $z^2 \geq 0$. Since they are reals, the sum of the squares of three reals cannot be negative, giving us **no solutions**.

18 Vieta's Formulas

We can let $P(x)$ be a polynomial of degree n, so $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, where the coefficient of x^i is a_i and $a_n \neq 0$. By the Fundamental Theorem of Algebra, we can also write $P(x) = a_n(x - r_1)(x - r_2)\ldots(x - r_n)$, where r_i are the roots of $P(x)$. We thus have that

$$a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = a_n(x - r_1)(x - r_2)\ldots(x - r_n)$$

Expanding out the right hand side gives us

$$a_n x^n - a_n(r_1 + r_2 + \ldots + r_n)x^{n-1} + a_n(r_1 r_2 + r_1 r_3 + \ldots + r_{n-1}r_n)x^{n-2} + \ldots + (-1)^n a_n r_1 r_2 \ldots r_n$$

We now have two different expressions for $P(x)$. These must be equal. However, the only way for two polynomials to be equal for all values of x is for each of their corresponding coefficients to be equal. So, starting with the coefficient of x^n, we see that

$$\begin{aligned} a_n &= a_n \\ a_{n-1} &= -a_n(r_1 + r_2 + \ldots + r_n) \\ a_{n-2} &= a_n(r_1 r_2 + r_1 r_3 + \ldots + r_{n-1}r_n) \\ &\vdots \\ a_0 &= (-1)^n a_n r_1 r_2 \ldots r_n \end{aligned}$$

If we denote σ_k as the kth symmetric sum, then we can write those formulas more compactly as $\sigma_k = (-1)^k \cdot \frac{a_{n-k}}{a_n}$, for $1 \leq k \leq n$.

18.1 Introductory Problems

18.1.1 If a, b are the roots of $x^2 + 2x + 3$, then find $a + b$, ab, $a^2 + b^2$, and $\frac{1}{a} + \frac{1}{b}$

By Vieta's, we know that $a + b = \boxed{\text{-2}}$ and $ab = \boxed{3}$.
By writing $a^2 + b^2 = (a+b)^2 - 2ab$, we get that $a^2 + b^2 = 4 - 6 = \boxed{\text{-2}}$.
Since $\frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab}$, then $\frac{1}{a} + \frac{1}{b} = \boxed{-\frac{2}{3}}$.

18.1.2 If a, b, and c are the roots of $x^3 - 2x^2 + 3$, then find $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$

Expanding the desired expression, we get

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{ab + bc + ca}{abc}$$

However, by Vieta's, we know that $ab + bc + ca = 0$. Thus, the whole expression is $\boxed{0}$.

19 Problem Set

1. Factor $x^3 + 3x^2 + 3x - 7$
2. Prime Factorize 160401
3. Solve for integers x, y: $2xy + x + y = 5$
4. If a, b, and c are the roots of $x^3 + 10x + 42$, then determine $(a + b)(b + c)(c + a)$
5. If a, b, and c are the roots of $x^3 - 4x^2 - 6x + 4269$, then find the value of $a(1+b+c)+b(1+c+a)+c(1+a+b)$
6. Solve in real numbers the equation $x^3 - 3x^2 + 3x + 3 = 0$
7. Simplify the expression $\frac{x^2+4y^2-z^2+4xy}{x^2-4y^2-z^2+4yz}$
8. Let a b, c, d be real numbers such that $a + b + 2ab = 3$, $b + c + 2bc = 4$, and $c + d + 2cd = -5$. Find $d + a + 2ad$

20 Challenge Problems

1. Solve in real numbers the system of equations

$$\begin{aligned} x + y &= 2z \\ x^3 + y^3 &= 2z^3 \end{aligned}$$

2. Factor $x^5 + x + 1$

Algebra II

21 Quadratics

Before we dive into some problems, we need to understand the fundamentals of quadratics. There are three main ways to express a quadratic:

$$\begin{aligned} y &= ax^2+bx+c,\ a \neq 0 \\ y &= a(x-h)^2+k,\ \text{where } (h,k) \text{ is the vertex} \\ y &= a(x-x_1)(x-x_2),\ \text{where } x_1,\, x_2 \text{ are the roots} \end{aligned}$$

In addition, we must know the quadratic formula and its properties, which we will illustrate in the following section.

21.1 Proving the Quadratic Formula

Let the quadratic be in the form $ax^2+bx+c=0$.
Moving c to the other side, we obtain

$$ax^2+bx=-c$$

Dividing by a and adding $\frac{b^2}{4a^2}$ to both sides yields

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=-\frac{c}{a}+\frac{b^2}{4a^2}$$

Factoring the LHS (Completing the square) gives

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

An equation in this form can be solved, yielding

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Now, we have a formula that we can use to find the roots of any quadratic, but what does the quadratic formula tell us?

21.2 The Discriminant

The discriminant of a quadratic is b^2-4ac, or the expression inside the square root of the quadratic formula. There are three things that the discriminant can be.

1. $b^2-4ac>0$
   This means that there are two **real** solutions to the quadratic.

2. $b^2-4ac=0$
   This means that there is one **real** solution to the quadratic.

3. $b^2-4ac<0$
   This means that there are two **complex** solutions to the quadratic.

21.2.1 How many real solutions are there to the equation $x^6 = 1$?

Factoring using difference of squares and the cubic factoring formulas gives:

$$\begin{aligned} x^6 - 1 &= 0 \\ (x^3+1)(x^3-1) &= 0 \\ (x+1)(x^2-x+1)(x-1)(x^2+x+1) &= 0 \end{aligned}$$

$x = 1, -1$ are clearly solutions to the equation, but how do we know that there are no more? Well, we use the quadratic formula on $x^2 - x + 1$ and $x^2 + x + 1$ and get that their discriminants are both -3, and it follows that $\boxed{x = 1, -1}$ are the only solutions.

21.2.2 For what values of a does $x^2 + ax + 6$ have no real solutions?

Using the discriminant, we must have that

$$\begin{gathered} a^2 - 24 < 0 \\ (a + 2\sqrt{6})(a - 2\sqrt{6}) < 0 \end{gathered}$$

Solving this quadratic inequality, we get that $\boxed{a \in (-2\sqrt{6}, 2\sqrt{6})}$.

21.2.3 For what values of a is $\frac{x+2}{ax^2+100}$ defined for all values of x?

In order for the expression to be defined, then there must be no x so that value in the denominator is 0. Thus, this means that $ax^2 + 100 = 0$ must have no solution.

$$\begin{gathered} 0^2 - 400a < 0 \\ -a < 0 \\ a > 0 \end{gathered}$$

However, we must note that we are assuming the denominator is quadratic when using the quadratic formula. So, if $a = 0$, then the denominator also has no roots that make it 0. Thus, if $\boxed{a \geq 0}$, then the given expression is defined for all x.

21.3 Cool Extension

21.3.1 Find the values of a such that $y = ax$ is tangent to the circle $(x-3)^2 + y^2 = 4$

For $y = ax$ to be tangent, then there must be exactly one intersection point, or in other terms,

$$\begin{aligned} (x-3)^2 + (ax)^2 &= 4 \\ x^2 - 6x + 9 + a^2x^2 &= 4 \\ (a^2+1)x^2 - 6x + 5 &= 0 \end{aligned}$$

has exactly one real root. This occurs when the discriminant is equal to 0, so

$$\begin{aligned} 36 - 4 * 5(a^2+1) &= 0 \\ 9 - 5(a^2+1) &= 0 \\ 5a^2 &= 4 \\ a^2 &= \frac{4}{5} \\ a &= \pm\frac{2\sqrt{5}}{5} \end{aligned}$$

Thus, when $\boxed{a = \pm\frac{2\sqrt{5}}{5}}$, the line $y = ax$ is tangent to the circle.

22 Polynomials

22.1 Rational Root Theorem

A polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ with integral coefficients is given, $a_n \neq 0$. The Rational Root Theorem states that if $P(x)$ has a rational root $r = \pm\frac{p}{q}$ with p, q relatively prime positive integers, p is a divisor of a_0 and q is a divisor of a_n.

22.1.1 Proof

Given $\frac{p}{q}$ is a rational root of a polynomial $f(x) = a_n x^n + x_{n-1}x^{n-1} + \ldots + a_0$, where the a_n's are integers, we wish to show that $p|a_0$ and $q|a_n$. Since $\frac{p}{q}$ is a root,

$$0 = a_n \left(\frac{p}{q}\right)^n + \ldots + a_0$$

Multiplying by q^n, we have:

$$0 = a_n p^n + a_{n-1}p^{n-1}q + \ldots + a_0 q^n$$

Examining this in modulo p, we have $a_0 q^n \equiv 0 \pmod{p}$. Because q and p are relatively prime, $p|a_0$. With the same logic, but with modulo q, we have $q|a_n$, and we are done.

22.1.2 Find all real roots of $x^3 - 5x^2 + 2x + 8 = 0$

Using the rational root theorem, we test $x = -1$, and with synthetic division, we get $(x+1)(x^2 - 6x + 8) = 0$. By factoring, we get $(x+1)(x-2)(x-4) = 0$, meaning our real roots are $\boxed{\text{-1,2,4}}$.

In general, when we are presented a problem in which we need to find solutions or a problem where we need to factor, we always either try to factor, manipulate, or use the rational root theorem. Using the rational root theorem can usually give us a root to a large degree polynomial that we wish to find the roots of.

22.2 Remainder Theorem

The theorem states that, given a polynomial $P(x)$, the remainder when $P(x)$ is divided by $x - a$ is the value $P(a)$. What does this mean? Well, if we plug in a to a polynomial and it returns a value of 0, then we know that a is a root of the polynomial. This is a quick way to check if certain numbers are roots of any given polynomial.

22.2.1 Proof

By polynomial division, we can write:

$$P(x) \quad = \quad (x-a)Q(x) + R(x), \text{ where } deg(R) = 0$$

Plugging in $x = a$ to the equation:

$$\begin{aligned} P(a) &= (a-a)Q(a) + R(a) \\ P(a) &= R(a) \end{aligned}$$

Thus, $P(a)$ is the remainder when $P(x)$ is divided by $x - a$.

22.2.2 Find the remainder when $x^{100} + 1$ is divided by $x + 1$

We could compute the remainder using repeated polynomial division, but it is much simpler due to the remainder theorem. To find the remainder, all we need to do is plug in -1 to $P(x) = x^{100} + 1$. Thus, the remainder is $(-1)^{100} + 1 = \boxed{2}$.

22.3 Division and Remainders With Polynomials

22.3.1 Find the remainder when $P(x) = x + x^3 + x^9 + x^{27} + x^{81} + x^{243}$ is divided by $x^2 - 1$

We can write:

$$P(x) = (x^2 - 1)Q(x) + R(x)$$

Since the degree of R must be less than or equal to 1, we can substitute $R(x) = ax + b$:

$$P(x) = (x^2 - 1)Q(x) + ax + b$$

By plugging $x = 1, -1$ into the equation above, we get:

$$P(1) = 6 = a + b$$
$$P(-1) = -6 = -a + b$$

Solving this linear system, we get that $a = 6$ and $b = 0$. Thus, the remainder is $\boxed{6x}$.

22.3.2 A polynomial $P(x)$ leaves a remainder of 99 when divided by $x - 19$ and a remainder of 19 when divided by $x - 99$. Find the remainder when $P(x)$ is divided by $(x - 19)(x - 99)$

Using the two givens, we can construct three equations:

$$\begin{aligned} P(x) &= (x - 19)Q_1(x) + 99 \\ P(x) &= (x - 99)Q_2(x) + 19 \\ P(x) &= (x - 19)(x - 99)Q_3(x) + R(x) \end{aligned}$$

Once again, substituting $R(x) = ax + b$ and plugging in $x = 19, 99$ to the third equation gives:

$$\begin{aligned} P(19) &= 19a + b \\ P(99) &= 99a + b \end{aligned}$$

Since $P(19) = 99$ and $P(99) = 19$ by the remainder theorem, we get:

$$\begin{aligned} 19a + b &= 99 \\ 99a + b &= 19 \end{aligned}$$

Solving the linear system, we get that $a = -1$ and $b = 118$, giving us that the remainder is $\boxed{-x + 118}$.

23 Solving Equations

23.1 Simon's Favorite Factoring Trick

23.1.1 Find all solutions in integers to $xy = x + y + 2$

Moving the $x + y$ to the LHS gives:

$$\begin{aligned} xy - x - y &= 2 \\ (x - 1)(y - 1) &= 3 \end{aligned}$$

In order to solve this, we note that both $x - 1$ and $y - 1$ are integers, so they must be integer factors of 3. So, we can have $(x - 1, y - 1) = (3, 1), (1, 3), (-1, -3), (-3, -1)$. Thus, our solutions are $(x, y) =$ $\boxed{\text{(4,2), (2,4), (0,-2), (-2,0)}}$.

23.1.2 Find $3x^2y^2$ if x and y are integers such that $y^2 + 3x^2y^2 = 30x^2 + 517$

Using SFFT:

$$(3x^2+1)(y^2-10) = 517-10$$

Because 507=$3*13^2$, and $3x^2+1$ cannot be a multiple of 3 or 169, then $3x^2+1=13$, so $x^2=4$. Similarly, $y^2-10=39$, so $y^2=49$. Thus, $3x^2y^2 = 3*4*49 = \boxed{588}$.

23.2 Cool Substitutions and Manipulations

23.2.1 Solve the equation $x(x+1)(x+2)(x+3) = 24$

With wishful thinking, we multiply $x(x+3)$ and $(x+1)(x+2)$ because of a sort of symmetry, giving us:

$$(x^2+3x)(x^2+3x+2) = 24$$

Substituting $y = x^2+3x$, we get:

$$\begin{aligned} y(y+2) &= 24 \\ y^2+2y-24 &= 0 \\ y &= 4, -6 \end{aligned}$$

So, we must have that $y = 4, -6 = x^2+3x$. So, we are left with two quadratics, $x^2+3x=4$ and $x^2+3x=-6$. Solving this two, we get that the solutions are $\boxed{\text{-4, 1, } \frac{-3+\sqrt{15}i}{2}, \frac{-3-\sqrt{15}i}{2}}$.

23.2.2 Solve the equation $x^3 + (x+1)^3 + (x+2)^3 + (x+3)^3 = 0$

Factoring the LHS as a sum of cubes, we get:

$$\begin{aligned} (x+x+3)(x^2-x(x+3)+(x+3)^2)+(x+1+x+2)((x+1)^2-(x+1)(x+2)+(x+2)^2) &= 0 \\ (2x+3)(x^2-x^2-3x+x^2+6x+9)+(2x+3)(x^2+2x+1-x^2-3x-2+x^2+4x+4) &= 0 \\ (2x+3)(x^2+3x+9)+(2x+3)(x^2+3x+3) &= 0 \\ (2x+3)(2x^2+6x+12) &= 0 \\ (2x+3)(x^2+3x+6) &= 0 \end{aligned}$$

Solving the quadratic, we get that the solutions are $x = \boxed{-\frac{3}{2}, \frac{-3+\sqrt{15}i}{2}, \frac{-3-\sqrt{15}i}{2}}$.

23.2.3 Solve the equation $x^2 + 3x - 16 = \sqrt{4x^2+12x+32}$

Substituting $y = \sqrt{x^2+3x+8}$, we get:

$$\begin{aligned} y^2-24 &= 2y \\ y^2-2y-24 &= 0 \\ y &= 6, -4 \end{aligned}$$

We can then solve 2 more quadratic equations for x to obtain the four solutions.

23.2.4 Let a be a real number. Find the real numbers $x_1, x_2, \ldots, x_n$ knowing that

$$\begin{aligned} x_1^2 + ax_1 + (\frac{a-1}{2})^2 &= x_2 \\ x_2^2 + ax_2 + (\frac{a-1}{2})^2 &= x_3 \\ &\cdots \\ x_n^2 + ax_n + (\frac{a-1}{2})^2 &= x_1 \end{aligned}$$

Adding the given equations together:

$$\begin{aligned} x_1^2+(a-1)x_1+(\frac{a-1}{2})^2+x_2^2+(a-1)x_2+(\frac{a-1}{2})^2+\ldots+x_n^2+(a-1)x_n+(\frac{a-1}{2})^2 &= 0 \\ \left(x_1+\frac{a-1}{2}\right)^2+\left(x_2+\frac{a-1}{2}\right)^2+\ldots+\left(x_n+\frac{a-1}{2}\right)^2 &= 0 \end{aligned}$$

Since all the x_i are real, then it follows that $\boxed{x_1=x_2=\ldots=x_n=\frac{1-a}{2}}$.

23.3 Using Graphs to Solve Equations

23.3.1 For what values of a will $||x-2|-4|=a$ have four real solutions?

We know what the graph of $y=|x-2|-4$ looks like: an absolute value graph shifted w units to the right and 4 units down, with x-intercepts at 6 and -2 and minimum point at (2,-4). Now, when we take the graph of $y=||x-2|-4|$, we take the part of the graph $y=|x-2|-4$ in the interval $[-2,6]$ and flip it up across the x-axis, with the point $(2,4)$ being the "peak" of the middle section of the graph. It is now clear that if $y=a$ intersects the graph at 4 points, then $\boxed{a\in(0,4)}$.

24 Problem Set

DO NOT BE SAD IF YOU CAN NOT SOLVE ALL OF THESE PROBLEMS. They are well above mid-AMC 10 level, and the latter problems are up to par with mid to high AIME problems.

1. Solve the equation $\frac{2x}{2x^2-5x+3}+\frac{13x}{2x^2+x+3}=6$
2. Solve the equation $(3x+1)(4x+1)(6x+1)(12x+1)=5$
3. Two different prime numbers between 4 and 18 are chosen. When their sum is subtracted from their product, which of the following numbers could be obtained?

   a 21

   b) 60

   c) 119

   d) 180

   e) 231
4. Solve the equation $\frac{8^x+27^x}{12^x+18^x}=\frac{7}{6}$
5. Determine if it is possible each of the polynomials $P(x)=ax^2+bx+c$, $Q(x)=bx^2+cx+a$, and $R(x)=cx^2+ax+b$ to have two real roots if all a,b,c are positive reals.
6. Find the remainder when $P(x)$ is divided by $(x-2)(x-3)$ if $P(x)$ leaves a remainder of 5 when divided by $x-2$ and a remainder of 6 when divided by $x-3$.
7. Find the value of a for which $y=ax$ is tangent to the graph of $y=\sqrt{x}$.
8. Solve the equation $2^x+3^x-4^x+6^x-9^x=1$
9. Find all real solutions (x,y) of $y^4+4y^2x-11y^2+4xy-8y+8x^2-40x+52=0$
10. Solve the equation $\sqrt[4]{x-2}+\sqrt[4]{3-x}=1$
11. Solve the equation $\sqrt{5-x}=5-x^2$

25 Hints

1. Divide both the numerator and denominator of each fraction by x, and then see if you can make any good substitutions
2. Group in a nice way
3. SFFT
4. Factor and use the sum of cubes formula for the numerator
5. Use the discriminant
6. MEH
7. Use the discriminant
8. Set $2^x = a$ and $3^x = b$ and see what you can do from there
9. This is a quadratic equation in x
10. Set $\sqrt[4]{x-2} = a$ and $\sqrt[4]{x-3} = b$
11. Solve for 5

Geometry I

In addition to the AMC 8 practice today, we have a short collection of geometry problems that you can work on after class or for homework

26 Sample Problems

1. Two right circular cylinders have the same volume. The radius of the second cylinder is 10% more than the radius of the first. What is the relationship between the heights of the two cylinders? (AMC 10 2015)

2. The ratio of the length to the width of a rectangle is $4 : 3$. If the rectangle has diagonal of length d, then the area may be expressed as kd^2 for some constant k. What is k? (AMC 10 2015)

3. The two legs of a right triangle, which are altitudes, have lengths $2\sqrt{3}$ and 6. How long is the third altitude of the triangle? (AMC 10 2014)

4. Trapezoid $ABCD$ has parallel sides $\overline{AB}$ of length 33 and $\overline{CD}$ of length 21. The other two sides are of lengths 10 and 14. The angles at A and B are acute. What is the length of the shorter diagonal of $ABCD$? (AMC 10 2014)

5. Two sides of a triangle have lengths 10 and 15. The length of the altitude to the third side is the average of the lengths of the altitudes to the two given sides. How long is the third side? (AMC 10 2013)

6. In triangle ABC, medians AD and CE intersect at P, $PE = 1.5$, $PD = 2$, and $DE = 2.5$. What is the area of $AEDC$? (AMC 10 2013)

7. Externally tangent circles with centers at points A and B have radii of lengths 5 and 3, respectively. A line externally tangent to both circles intersects ray AB at point C. What is BC? (AMC 10 2012)

8. Two equilateral triangles are contained in square whose side length is $2\sqrt{3}$. The bases of these triangles are the opposite side of the square, and their intersection is a rhombus. What is the area of the rhombus? (AMC 10 2012)

9. Square $EFGH$ has one vertex on each side of square $ABCD$. Point E is on $\overline{AB}$ with $AE = 7 \cdot EB$. What is the ratio of the area of $EFGH$ to the area of $ABCD$? (AMC 10 2011)

10. Rectangle $ABCD$ has $AB = 6$ and $BC = 3$. Point M is chosen on side AB so that $\angle AMD = \angle CMD$. What is the degree measure of $\angle AMD$? (AMC 10 2011)

11. Rhombus $ABCD$ has side length 2 and $\angle B = 120$. Region R consists of all points inside the rhombus that are closer to vertex B than any of the other three vertices. What is the area of R? (AMC 10 2011)

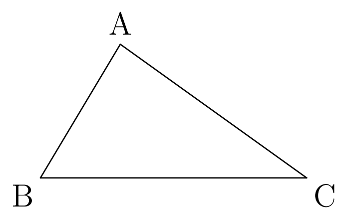
Geometry II

In this class, we will look into some of the most commonly used techniques in geometry in math contests. Some of these topics include area, polygons, similar triangles, and circles.

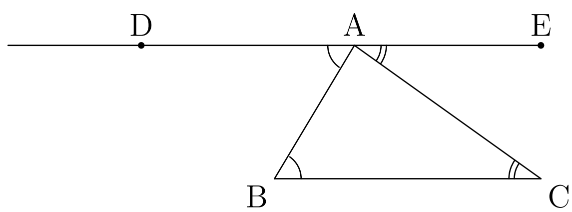
27 Lecture

27.1 Polygons

27.1.1 The sum of the angles in a triangle is $180°$



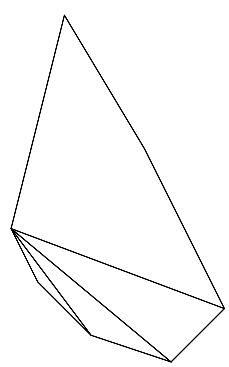
Let's construct a line parallel to BC through A.



Using alternate interior angles, $\angle DAB = \angle ABC$ and $\angle EAC = \angle ACB$. Thus, since D, A, and E are collinear, $\angle DAB + \angle BAC + \angle EAC = 180°$. Substituting the latter set of equations, we get the desired sum $\angle ABC + \angle BAC + \angle ACB = 180°$.

27.1.2 The sum of the angles in a convex n-gon is $(n-2)180°$

We can construct $n-2$ triangles in any n-gon in the following fasion:

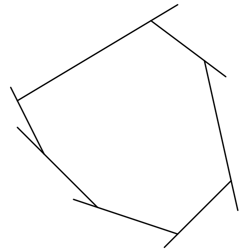


So, since a triangle's angles sum to $180°$, we are done.

27.1.3 The number of diagonals in a convex polygon is $\frac{n(n-3)}{2}$

If we choose any vertex, we can pick any other vertex to connect this vertex to to create a diagonal in $n-3$ ways (this is because we can't choose the two adjacent vertices and the vertex itself). Since each vertex is counted twice (we can count A_1A_2 in two ways: using A_1 as the "vertex" and using A_2 as the "vertex"), we divide $n(n-3)$ by two to get the desired formula.

27.1.4 The sum of the exterior angles of any convex polygon is $360°$



We can see that the sum of the exterior angles can be written as

$$\begin{aligned}(180° - A_1) + (180° - A_2) + \ldots + (180° - A_n) \\ &= 180°n - (A_1 + A_2 + \ldots + A_n) \\ &= 180°n - (n-2)180° \\ &= 360°\end{aligned}$$

27.2 Triangles

27.2.1 The medians of a triangle intersect at the centroid, and the centroid splits the medians into two parts with length ratio $1:2$.

27.2.2 The perpendicular bisectors intersect at the circumcenter, which is equidistant from each of the triangle's three vertices. This means that the circumcenter is the center of a circle that passes through all three vertices of a triangle.

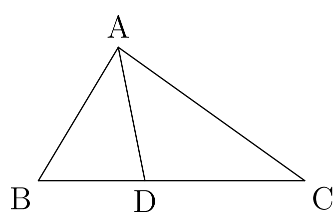
27.2.3 The angle bisectors intersect at the incenter, which is equidistant from each of the triangle's sides. It is the center of a circle inscribed inside the triangle (tangent to all three sides).

27.2.4 The altitudes intersect at the orthocenter

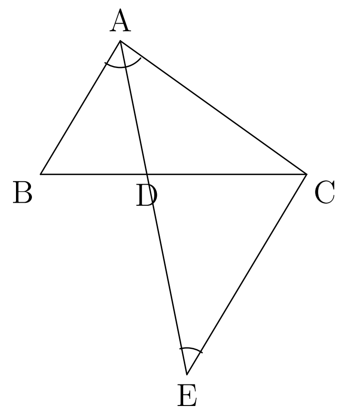
27.2.5 Congruency Theorems: SSS, SAS, ASA, AAS

27.2.6 Similarity Theorems: AA, SAS, SSS

27.2.7 In a triangle with angle bisector AD, then $\frac{AB}{AC} = \frac{BD}{CD}$ (Angle-Bisector Theorem)



Let us construct a line parallel to AB and let the intersection of that line and the extension of AD be E.



We notice that by alternate interior angles, $\angle BAD = \angle CAD = \angle DEC$. This means that $\triangle ACE$ is isosceles, so $AC = CE$. Since $\angle BAD = \angle DEC$ and $\angle ADB = \angle CDE$, we get that $\triangle ADB$ is similar to $\triangle EDC$. From this, we get the ratio $\frac{AB}{CE} = \frac{BD}{CD}$. Plugging in $CE = AC$ we get the desired relation $\frac{AB}{AC} = \frac{BD}{CD}$.

27.2.8 Similar triangles and figures postulate

Firstly, if we have two figures that are similar, then **all of their corresponding parts are in the same ratio**.
Secondly, if we have two similar figures, say with their sides in the ratio $1:2$, then **their areas are in the ratio of 1:4**. We can extend this further to claim that if we have two similar $3-D$ solids and the ratio of their sides is $\frac{m}{n}$, then the ratio of their volumes is $\frac{m^3}{n^3}$.

27.2.9 Heron's Formula: $K = \sqrt{s(s-a)(s-b)(s-c}$, where s is the semiperimeter

27.3 Circles

27.3.1 The radius drawn to the point of tangency of a tangent line to a circle is perpendicular to the tangent line at that point

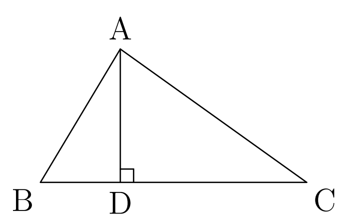
27.3.2 If two tangent lines drawn from the same point, say A, are tangent to the circle at points B and C, then $AC = AB$.

27.3.3 A central angle of a circle has the same measure as the measure of its intercepted arc

27.3.4 An inscribed angle of a circle has measure that is half the measure of its intercepted arc (This means that if an angle intercepts an arc bounded by a diameter, then its measure is $90°$).

27.4 Trig

27.4.1 The area of a triangle can be expressed as $\frac{1}{2}ab\sin C$



We can write the area of the triangle as $\frac{1}{2}(BC)(AD)$. Since $\sin C = \frac{AD}{AC}$, we can rewrite this as $AD = AC\sin C$. Substituting this back into the equation for the area of a triangle, we get that $[ABC] = \frac{1}{2}(BC)(AC)\sin C = \frac{1}{2}ab\sin C$.

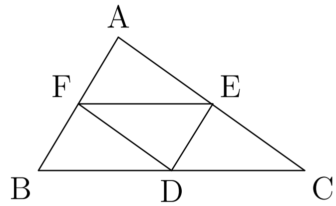
27.4.2 Law of Sines: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$**, where** R **is the circumradius.**

27.4.3 Law of Cosines: $c^2 = a^2 + b^2 - 2ab\cos C$

27.4.4 If $a^2 + b^2 > c^2$**, then** ABC **is an acute triangle. If** $a^2 + b^2 < c^2$**, then** ABC **is an obtuse triangle. This follows from the Law of Cosines.**

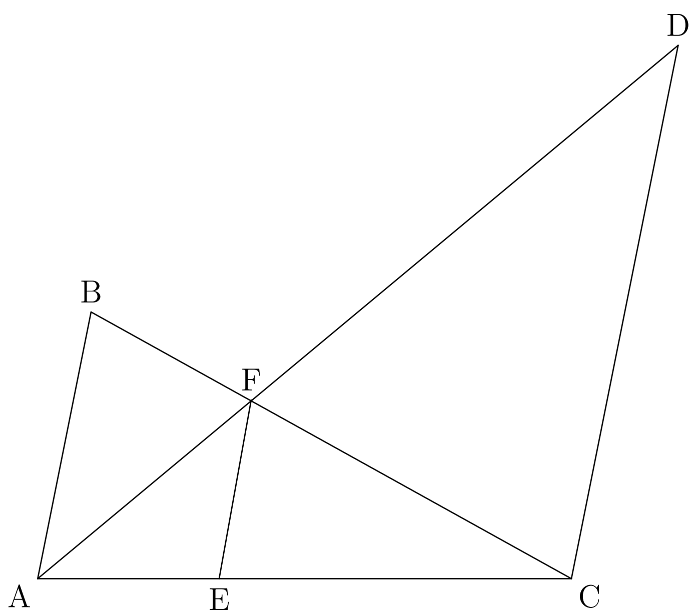
27.5 Some Select Problems

27.5.1 Prove that the medial triangle of any triangle has area $\frac{1}{4}$ **of the original triangle.**



Because EF, ED, DF are midsegments of the triangle, they are parallel to BC, AB, AC, respectively. Thus, we can easily observe that $AEDF$, $CEFD$, $BFED$ are all parallelograms, giving us that the angles in DEF are corresponding to the angles in ABC, meaning that those two triangles are similar. In addition, since EF is a midsegment, its length is $\frac{1}{2}$ of BC, and this is true for the rest of the sides. Thus, the ratio of similarity between the two triangles is $1:2$, meaning that the ratio of the areas is $1:4$, and this concludes the proof.

27.5.2 If the lines AB **and** CD **are parallel,** AD **and** BC **intersect at** E**, a line is drawn through** E **that is parallel to** AB **and** CD**, and the intersection of that line and** AC **is** D**, then prove that** $\frac{1}{AB} + \frac{1}{CD} = \frac{1}{EF}$



Because we are given three parallel lines, we find that $\triangle CEF$ is similar to $\triangle CAB$, and $\triangle AEF$ is similar to $\triangle ACD$. Thus, we derive the following equalities:

$$\frac{EF}{AB} = \frac{CE}{CA}$$
$$\frac{EF}{CD} = \frac{EA}{CA}$$

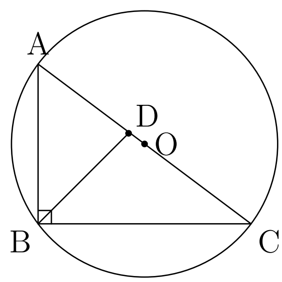
Adding the two equations, we get:

$$\begin{aligned} \frac{EF}{AB} + \frac{EF}{CD} &= \frac{CA}{CA} \\ &= 1 \end{aligned}$$

because $EA + CE = CA$. Dividing through by EF finishes our proof:

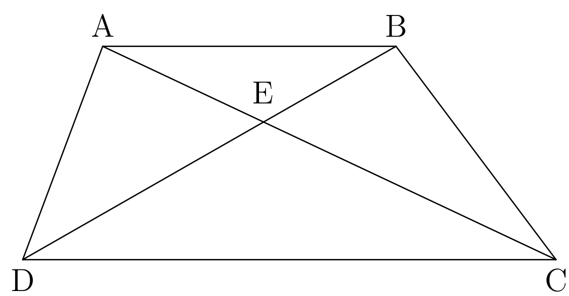
$$\frac{1}{AB} + \frac{1}{CD} = \frac{1}{EF}$$

27.5.3 If $\triangle ABC$ is a right triangle with right angle at B, $AB = 3$, $BC = 4$, and $AC = 5$, then what is the distance between the foot of the angle bisector from B to the circumcenter of $\triangle ABC$?



Because $\angle B = 90°$, AC is the diameter of the circumcircle of ABC because of a previous theorem regarding to inscibed angles in the lecture portion of this handout. In particular, O, the circumcenter, must lie on AC and is the midpoint of AC because it is the center. Thus, $AO = \frac{5}{2}$.
Now, by the angle bisector theorem, $\frac{AB}{BC} = \frac{AD}{DC}$. Because $AB = 3$ and $BC = 4$, then $\frac{AD}{CD} = \frac{3}{7}$. In other words, $\frac{AD}{AC} = \frac{3}{7}$, so $AD = \frac{15}{7}$.
The distance we seek to find is $DO = AO - AD$, so plugging in our values gives us that $AD = \boxed{\frac{5}{14}}$.

27.5.4 Let $ABCD$ be a trapezoid with AB parallel to CD and let the diagonals intersect at E. Prove that $[AED] = [BEC]$



We first notice that $\triangle ADC$ and $\triangle BCD$ have the same height to their shared base, DC, because AB and CD are parallel. Thus, since they have the same base and the same height, $[ACD] = [BCD]$. We know that:

$$\begin{aligned} [AED] &= [ACD] - [DEC] \\ [BEC] &= [BCD] - [DEC] \end{aligned}$$

It then follows that $[AED] = [BEC]$.

28 Problems

1. Triangle ABD is a right triangle with right angle at B. On AD there is a point C for which $AC = CD$ and $AB = BC$. Find $\angle DAB$.
2. If a triangle has sides 40, 60, and 80, then the shortest altitude is K times the longest altitude. Find the value of K.
3. A regular polygon has each interior angle measuring $176°$. Find the number of sides of this regular polygon.
4. The medians of a right triangle which are drawn from the vertices of the acute angles are 5 and $\sqrt{40}$. Find the length of the hypotenuse.
5. Find the area of a rhombus with a side of length 13 and one diagonal of length 24.
6. The length of a rectangular picture is three times its width. The picture is surrounded by a frame which is 4 inches wide. If the perimeter of the outside frame is 96 inches, what is the length of the picture, in inches?
7. What is the length of the common external tangent segment of two externally tangent circles whose radii are 8 and 11?
8. Given that $ABCDEF$ is a regular hexagon with side length 6, find the area of triangle BCE.
9. What is the maximum amount of right angles an octagon can have?
10. In a regular polygon $ABCDE\cdots$, we have $\angle CAD = 12°$. How many sides does this polygon have?
11. In a rectangle, interior point E is chosen at random. Prove that the sum of the areas of triangles AEB and EDC is the same regardless of where in $ABCD$ point E is chosen.
12. In a rectangle, an interior point E is chosen. Prove that $AE^2 + CE^2 = BE^2 + DE^2$ (British Flag Theorem)
13. Find a formula for the area of a regular hexagon given its side length.
14. Prove that if we connect, in order, the midpoints of the sides of any quadrilateral, then we form a parallelogram.
15. Given that I is the incenter of $\triangle ABC$ and $AB = AC = 5$, and $BC = 8$, find the distance AI.
16. Given a circle inscribed in a triangle (the incircle), prove that the lengths of the tangents closest to A, B, and C have lengths $s - a$, $s - b$, and $s - c$, respectively.
17. A cow is tied to the corner of a 20 foot by 15 foot shed with a 30 foot rope. Find her total grazing area.
18. A park is in the shape of a regular hexagon 2 km on a side. Starting at a corner, Alice walks along the perimeter of the park for a distance of 5 km. How many kilometers is she from her starting point?
19. The median to a 10 cm side of a triangle has length 9cm and is perpendicular to a second median of the triangle. Find the exact value in centimeters of the length of the third median.
20. Given a right triangle with side lengths 3, 4, and 5, find the distance between the incenter and the circumcenter.

Final Test

1. How many ways are there to seat 7 people in a row of 7 chairs if JT Landry and John Graass won't sit next to each other?

2. Solve the following equation for real numbers x, y, and z:

$$x^2 + 4x + y^2 + 6y + z^2 + 8z + 10 = -19$$

3. If $x^3 + \frac{1}{x^3} = -2$, find all possible values of $x + \frac{1}{x}$

4. Find the number of ordered pairs (x,y) where x and y are integers and $x + 4xy + 4y = 15^2$

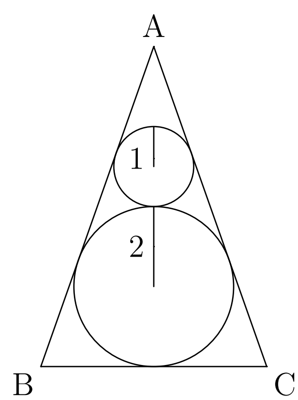
5. Given that the polynomial $x^4 + 2x^3 + 3x^2 + 4x + 5$ has roots a, b, c, and d, find the monic polynomial of degree 4 that has roots bcd, acd, abd, and abc.

6. Let F be a point in the interior of $\triangle ABC$ such that $\angle AFB = \angle BFC = \angle CFA = 120°$. Suppose $\angle A = 60°$, $AF = 4$, and $BF = 5$. Find the perimeter of $\triangle ABC$. Hint: Find the length of CF first.

7. Find the smallest positive integer that has factors which sum to 2016.

8. Let a *cow* set be defined as a subset of the set $\{1, 2, ..., 15\}$ in which no two terms are consecutive integers. Find the number of *cow* sets, including the empty set.

9. A circle of radius 1 is tangent to a circle of radius 2. The sides of $\triangle ABC$ are tangent to the circles as shown, and the sides $\overline{AB}$ and $\overline{AC}$ are congruent. What is the area of $\triangle ABC$? (AMC 10)



10. Let the ordered set $A = \{a_1, a_2, ..., a_8\}$ be a permutation of the first 8 odd positive integers. Find the number of permutations of the set such that $a_1a_2 > a_3a_4 > a_5a_6 > a_7a_8$.

BONUS (1 point) : Which organelle is commonly known as "the powerhouse of the cell?"

Solutions to Problem Sets

29 Combinatorics I, Problem Set I

1. There are 10 students in Mr. Feng's AP Calculus BC class. Mr. Feng wishes to form a committee for his class with a president, vice president, and treasurer. A student may hold more than one position, but no student can hold all the positions. How many ways can Mr. Feng's committee be formed?

   Solution: We can count the number of ways to form a committee and then subtract the number of ways that are not allowed, or the number of ways that one student holds all the positions. So, we have $10^3 = 1000$ total ways to fill the three distinct positions, but there are 10 ways for one person to hold all three positions. Thus, our answer is $1000 - 10 = \boxed{990}$.

2. 8 points are marked on a circle. How many distinct convex polygons of three or more sides can be drawn using some (and possibly all) of the eight points as vertices?

   Solution: If the polygon has k vertices, then we have to choose k vertices out of the 8, and each of these will result in a distinct convex polygon. Thus, we seek to find: $\sum_{i=3}^{8} \binom{8}{i}$. We see that this sum is $\binom{8}{3} + \binom{8}{4} + \ldots + \binom{8}{8}$. We then write this as $\left(\binom{8}{0} + \binom{8}{1} + \ldots + \binom{8}{8}\right) - \binom{8}{0} - \binom{8}{1} - \binom{8}{2} = 2^8 - 1 - 8 - 28 = 256 - 37 = \boxed{219}$.

3. How many ways can the 6 students in Mr. Owen's chemistry class form 2 groups of 3 students?

   Solution: We can choose the three students in one group in $\binom{6}{3}$ ways, and the rest of the students are in the second group. But, we double count everything because the two groups we form are distinct, so choosing a, b, and c first and d, e, and f second is the same thing as choosing d, e, and f first. So, we have to divide by 2 to account for this. Our final answer is $\frac{\binom{6}{3}}{2} = \boxed{10}$.

4. How many ways can the 6 students in Mr. Owen's chemistry class form 2 groups of 3 people, if the first group is going on a field trip and the second group is staying at school? Is the answer the same as the previous question?

   Solution: This time, the two groups are distinct because they are doing different things. So, there is no overcount this time, and our answer is just $\binom{6}{3} = \boxed{20}$.

5. How many ways can I roll a total sum of 7 when I roll a die three times in a row?

   Solution: We can think of this as a stars and bars problem with 7 stars and 2 bars. The number of stars in the first section is the number we roll in the first toss, the number of stars in the second section is the number we roll in the second toss, and the number of stars in the third section is the number we roll in the third toss. Since each roll results in an integer from $[1, 6]$, we are essentially solving $x + y + z = 7$, where x, y, and z are positive integers. We have 6 slots between the stars to put 2 bars, giving us an answer of $\binom{6}{2} = \boxed{15}$.

6. How many ways can 7 people sit around a round table if two of the people insist that they have to sit next to each other?

   Solution: Once again, we can think of these two people as one unit. So, we are now arranging 6 units around a round table, which can be done in 5! ways. However, the two people in the big unit can seated in 2! ways, giving us a final answer of $5! * 2! = \boxed{240}$.

7. How many orders of flips with 3 heads can I obtain if I flip a coin 6 times in a row?

Solution: We are given that exactly three heads occur. So, we can just choose 3 spots out of six total spots to put the three heads, and the rest of the spaces will be tails. So, our answer is $\binom{6}{3} = \boxed{20}$.

8. If Dr. Flint wishes to choose a president, vice president, and secretary from a group of 8 students, but Zoheb won't serve in any position if Allan is serving in any position, how many ways can positions be chosen?

   Solution: The only cases where the conditions are not met is if Allan and Zoheb are both serving in a position. So, there are a total of $8 * 7 * 6 = 336$ ways to choose the people for the positions. If Allan and Zoheb are both serving in a position, then we have 3 ways to choose the position for Allan, 2 ways to choose the position for Zoheb, and 6 ways to fill the last open position, giving us a total of $3 * 2 * 6 = 36$ invalid choices. So, our final answer is $336 - 36 = \boxed{300}$.

9. If Dr. Flint wishes to instead choose a committee of 4 people from her 8 students, and also designate a president for that committee, how many ways are there to form such a committee with a president?

   Solution: We can choose the committee in $\binom{8}{4}$ ways, and with those 4 people, we have 4 ways to choose who is president. Thus, our answer is $4\binom{8}{4} = \boxed{280}$.

10. How many ways can I write the number 6 as a sum of ordered, positive integers? Ex: 1+3+2, 3+1+1+1

    Solution: We can think of this as a stars and bars problem, needing to separate 6 stars into groups that have at least one star in them. We can have at most 6 parts and 5 bars and at least 1 part or 0 bars. So, since there are 5 spaces between the 6 stars, we want the sum: $\sum_{i=0}^{5} \binom{5}{i} = 2^5 = \boxed{32}$.

30 Combinatorics I, Problem Set II

1. If we we to count how many elements were listed total in all of the subsets of $\{1, 2, \ldots, 5\}$, what would we get? Can you generalize this for $\{1, 2, \ldots, n\}$?

   Solution: For any element, let's say 1, it is 2^4 subsets because there are 2^4 subsets of $\{2, 3, 4, 5\}$. This means that 1 appears 2^4 times total. Similarly, we can use this process to obtain that there are $5 * 2^4 = \boxed{80}$ total elements in all of the subsets. For a subset of size n, we could use the same reasoning to obtain a total of $\boxed{\text{n}*2^{n-1}}$.

2. Define an ascending number to be a number such that its digits are in increasing order from left to right. How many 3-digit ascending numbers are there? 4-digit? 5-digit?

   Solution: First, we can't pick 0 because that would mean zero would be the leftmost digit, which is impossible. Now, we notice that if we pick any 3 numbers from 1 to 9, there is exactly one way to form an ascending number. Thus, the total number of 3-digit ascending numbers is $\binom{9}{3} = \boxed{84}$. Similarly, the number of 4-digit ascending numbers is $\binom{9}{4} = \boxed{126}$. The number of 5-digit ascending numbers is $\binom{9}{5} = \boxed{126}$.

3. Define a non-decreasing number to be a number such that its digits do not descend. For example, 1123445 is a non-decreasing number. How many 5-digit non-decreasing numbers are there?

   Solution: Once again, we notice that if we take any set of numbers from 1 to 9, we can only form one non-decreasing number. Thus, our problem is now counting the number of ways to pick 5 not necessarily distinct numbers from 1 to 9. We can do this with stars and bars: 5 stars and 8 bars, representing 5 total numbers and 9 different places the stars can be. So, the total is $\binom{13}{5} = \boxed{1287}$.

4. Define a multisubset of $\{1, 2, \ldots, n\}$ to be a set consisting of only elements from $\{1, 2, \ldots, n\}$, but an element can occur more than once in the set. How many multisubsets of size 5 are there from the set

$\{1,2,3,4,5,6\}$?

Solution: This problem is similar to the previous problem. We want to choose a total of 5 not necessarily distinct numbers from 1 to 6. We can achieve this with stars and bars: 5 stars and 5 bars, representing five total numbers chosen and 6 different regions the stars can be in. So, our total is $\binom{10}{5} = \boxed{252}$.

5. In Mr. Liu's AP Statistics class, there are 15 students. At the beginning of class, each of the students shakes hands with every other student exactly once. How many handshakes take place?

   Solution: Each handshake is when two people are chosen from 15. Thus, there are $\binom{15}{2} = \boxed{105}$ ways to choose 2 people, which is exactly the number of handshakes that take place.

6.* Jesse is given a set of numbers $S = \{1,4,9,16,25,36\}$. If Jesse takes every two-element subset of S and writes it down on a sheet of paper, what will be the sum of all the numbers on his paper after he has gone through every two-element set?

   Solution: We see that if a number is in a subset, it will be written down if and only if the other number is less than it. So, there are 5 subsets where 36 is the largest element, 4 where 25 is the largest, 3 where 16 is the largest, 2 where 9 is the largest, and 1 where 4 is the largest. Thus, our sum is $5*36+4*25+3*16+2*9+1*4 = \boxed{350}$.

7. 8 identical white cards and 3 identical yellow cards are arranged in a row such that no two yellow cards can be adjacent to another. Find the number of possible arrangements.

   Solution: We can think of this problem as a stars and bars problem, where we try to pick three of the spaces between the 9 total spaces between or outside the white cards to put the yellow cards. So, our answer is $\binom{9}{3} = \boxed{84}$.

8. How many paths are there from (0,0) to (8,4) by only staying on the grid lines and moving up or to the right and passing through the point (4,1)?

   Solution: We can solve this problem by just counting the number of paths from (0,0) to (4,1) and from (4,1) to (8,4). The number of paths from (0,0) to (4,1) is $\binom{5}{1} = 5$, and the number of paths from (4,1) to (8,4) is $\binom{7}{3} = 35$. So, our total is $5*35 = \boxed{175}$.

9.* How many 12-letter arrangements of $AAAABBBBCCCC$ are there such that no As are in the first 4 letters, no Bs are in the second 4 letters, and no Cs are in the last 4 letters?

   Solution: We see that the As can only go in the last 2 blocks of 4. Let's say k of them go to the second block of 4. This means that the $4-k$ rest go to the last block of 4. Now, the Bs must fill up the rest of the last block because no C can go there. Since there are $4-k$ As already there, we need k Bs to go there. So, the remaining $4-k$ Bs go to the first block of 4. For the Cs, their positions are already fixed by the positions of the As and Bs. Now, there are $\binom{4}{k}$ ways to place the k As in the second group and $\binom{4}{4-k} = \binom{4}{k}$ ways to place the $4-k$ As in the last block. Finally, there are $\binom{4}{4-k} = \binom{4}{k}$ ways to place the $4-k$ Bs in the first group. Thus, our total is $\binom{4}{k}^3$. Thus, we seek the sum $\sum_{k=0}^{4} \binom{4}{k}^3 = \boxed{346}$.

10.* How many pairs of subsets (A,B) from $\{1,2,3,4,5,6\}$ are there such that B is a subset of A? (We define B as a subset of A if all the elements in B are in A. Ex: $\{1,2\}$ is a subset of $\{1,2,4,5,23\}$)

    Solution: If we take any element k, it has 4 choices:

$$k \text{ is in } A, k \text{ is in } B \tag{3}$$

$$k \text{ is in } A, k \text{ is not in } B \quad (4)$$
$$k \text{ is not in } A, k \text{ is in } B \quad (5)$$
$$k \text{ is not in } A, k \text{ is not in } B \quad (6)$$

For B to be a subset of A, only options (1), (2), and (4) allow for the condition to be held. So, each element has 3 choices that work, giving us a total of $3^6 = \boxed{729}$.

31 Combinatorics II, Problem Set I

1. In the expansion of $(2x - 3y)^7$, what is the coefficient of x^4y^3?

   Solution: By the binomial theorem, the x^4y^3 term looks like $\binom{7}{3}(2x)^4(-3y)^3 = 35 * 16 * -27x^4y^3 = -15120x^4y^3$. So, the coefficient is $\boxed{-15120}$.

2. How many positive integers not exceeding 2000 are multiples of 3 or 4 but not 5?

   Solution: There are $\lfloor \frac{2000}{3} \rfloor + \lfloor \frac{2000}{4} \rfloor - \lfloor \frac{2000}{12} \rfloor = 1000$ numbers not exceeding 2000 that are multiples of 3 or 4. Now, we must subtract the multiples of 15 or 20. This is $\lfloor \frac{2000}{15} \rfloor + \lfloor \frac{2000}{20} \rfloor - \lfloor \frac{2000}{60} \rfloor = 200$. Thus, there are exactly $\boxed{800}$ such integers.

3. Show that for $0 \leq m \leq k \leq n$, $\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$

   Solution: The LHS represents the number of ways to choose a committee of size k from n people and then choosing a subcommittee of m people from the committee of k people. The RHS represents first choosing m people for the subcommittee and then choosing the rest of the $k - m$ people who are not in the subcommittee but are in the committee from the remaining $n - m$ people. Since both of these count the same thing, we are done.

4. On a standard die one of the dots is removed at random with each dot equally likely to be chosen. The die is then rolled. What is the probability that the top face has an odd number of dots?

   Solution: Let's use casework here. The probability that a dot is removed from a face with an even number of dots is $\frac{4}{7}$. The probability that we then roll a face with an odd number of dots is $\frac{2}{3}$, giving us a total of $\frac{8}{21}$ in this case. The probability that a dot is removed from a face with an odd number of dots is $\frac{3}{7}$. The probability that we then roll a face with an odd number of dots is $\frac{1}{3}$. The probability in this case would then be $\frac{1}{7}$. Thus, summing the two probabilities, we get the final probability of $\frac{11}{21}$.

5. Bob is going up a staircase with 10 steps. If Bob can walk 1 step up or hop 2 steps up in one move, how many ways can Bob ascend the staircase. Note: Bob can technically hop 2 steps up on the ninth step

   Solution: We see that to get to the n^{th} step, we can either jump up 1 from the $n - 1^{st}$ step or jump 2 from the $n - 2^{nd}$ step. Thus, we derive the recursion $F_n = F_{n-1} + F_{n-2}$. We know $F_1 = 1$ and $F_2 = 2$, so building this recurison we get that $F_9 = 89$ and $F_{10} = 144$. However, our answer is not just F_{10} because we can also jump 2 steps from the 9^{th} step. Thus, our total would be $F_{10} + F_9 = 89 + 144 = \boxed{233}$.

6. How many arrangements of $AAABBBCCC$ are there if no three consecutive letters can be the same?

   Solution: We can proceed with complementary counting and PIE. We seek to find the number of arrangements where at least three consecutive letters are the same. If the three As are next to each other, then there would be $\frac{7!}{3!3!} = 140$ arrangements. The number of arrangements is symmetric for 3 Bs and 3 Cs, so we have a total of 420. However, we need to subtract the cases where two or more sets of consecutive letters are present. If 3 As and 3 Bs are adjacent, then we have $\frac{5!}{3!} = 20$ ways to arrange them. Once again, this is symmetric for the three possibilities, giving us a total of 60. Finally, we must add back the case where all three sets of consecutive letters are present. This can be done in $3! = 6$ ways. Thus, our total is $120 - 60 + 6 = \boxed{66}$.

7. How many ways are there to tile a 10x3 rectangle with 3x1 blocks?

   Solution: We notice that any tiling must end with either a vertical 3x1 block or three horizontal 1x3 blocks on top of each other. Thus, we derive the recursion $F_n = F_{n-1} + F_{n-3}$. Creating a table with $F_1 = F_2 = 1$ and $F_3 = 2$:

F_1	1
F_2	1
F_3	2
F_4	3
F_5	4
F_6	6
F_7	9
F_8	13
F_9	19
F_{10}	28

   Thus, there are $\boxed{28}$ tilings.

8. Give a combinatorial proof for $\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$

   Solution: Let us consider two separate groups of n people each and we want to choose n total people from these two groups. For the first group, we can take 0, 1, ..., n people, and correspondingly, n, $n-1$, ..., 0 people from the second group. So, we get the sum $\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \ldots + \binom{n}{n}\binom{n}{0}$. Using the identity $\binom{n}{k} = \binom{n}{n-k}$, we achieve the LHS of the original equality: $\binom{n}{0}\binom{n}{0} + \binom{n}{1}\binom{n}{1} + \ldots + \binom{n}{n}\binom{n}{n}$. Because the total number of ways to choose n people from $2n$ total people, we obtain the RHS to be $\binom{2n}{n}$.

9. Call a number $prime - looking$ if it is composite but not divisible by 2, 3, or 5. There are 168 prime numbers less than 1000. How many $prime - looking$ numbers are there that are less than 1000?

   Solution: Our plan is to count the total number of composite numbers less than 1000 and divisible by 2, 3, or 5 and also the primes under 1000 and subtract that total from 999. If we count the total number of multiples of 2, 3, or 5 then we have the total number of composite numbers that are divisible by 2, 3, or 5 plus the primes 2, 3, and 5. The number of multiples of 2, 3, or 5 under 1000 is $\lfloor \frac{999}{2} \rfloor + \lfloor \frac{999}{3} \rfloor + \lfloor \frac{999}{5} \rfloor - \lfloor \frac{999}{6} \rfloor - \lfloor \frac{999}{10} \rfloor - \lfloor \frac{999}{15} \rfloor + \lfloor \frac{999}{30} \rfloor = 499 + 333 + 199 - 166 - 99 - 66 + 33 = 733$. However, we must take out 2, 3, and 5 because they are counted in the 168. Also, we must subtract out 1 because it doesn't fit the condition as well. So, we seek $999 - 730 - 168 - 1 = \boxed{100}$.

10. Dylan flips 7 fair coins. If Dylan flips at least three tails, what is the probability that he flips at least two heads?

    Solution: If Dylan flips at least two heads, then he must flip at most 5 tails. So, he can only flip 3, 4, or 5 tails. The total number of possible flips is 3, 4, 5, 6, or 7 tails. Thus, our probability is $\frac{\binom{7}{3}+\binom{7}{4}+\binom{7}{5}}{\binom{7}{3}+\binom{7}{4}+\binom{7}{5}+\binom{7}{6}+\binom{7}{7}} = \boxed{\frac{91}{99}}$.

11. Ten women sit in 10 seats in a line. All of the 10 get up and then reseat themselves using all 10 seats, each sitting in the seat she was in before or a seat next to the one she occupied before. How many ways can the women be reseated in this manner?

Solution: Let's consider the woman at the very left of the seats. She has 2 choices: sit in her own seat, or sit in the seat one away from the left end. If she sits in her own seat, then we have any valid reseating of the remaining 9 people since the woman next to her now becomes the "end" of the line. If she sits in the seat one away from the left end, then the leftmost seat must be filled by the woman who was seated in the seat one away from the left end. Now, we are left with a valid arrangement of 8 people since the woman who was seated two seats away from the left end is now the "end". Thus, we derive the recurrence $F_n = F_{n-1} + F_{n-2}$. Since $F_1 = 1$ and $F_2 = 2$, we get that $F_{10} = \boxed{144}$.

12. What is the remainder when $6^{83} + 8^{83}$ is divided by 49?

    Solution: We see that we can make the given expression be $(7-1)^{83} + (7+1)^{83}$. So, when we expand using the binomial theorem, we get something like:

    $$[7^{83} - \binom{83}{1}7^{82} + \ldots - \binom{83}{81}7^2 + \binom{83}{82}7 - 1] + [7^{83} + \binom{83}{1}7^{82} + \ldots + \binom{83}{81}7^2 + \binom{83}{82}7 + 1]$$

    When we take this expression modulo 49, or when we divide by 49 and try to find the remainder, all the terms to the left of $\binom{83}{82}7$ are divisible by 49 because they have at least 7^2 in them. So, we are left to only find the remainder when $\binom{83}{82}7 - 1 + \binom{83}{82}7 + 1$ is divided by 49. Simplifying, we see that the expression is just $83 + 83 = 166$, and this is $\boxed{19}$ modulo 49, or has a remainder of 19 when divided by 49.

13. How many ways are there to distribute 7 **distinct** pieces of candy to three kids if each kid must have at least one candy?

    Solution: We attempt to do this problem using PIE. Each candy has 3 options, so our baseline total is 3^7. Now, we subtract the ways that one kid doesn't receive any candy, and this can be done in $3 * 2^7$ ways since there are 3 ways to choose the kid and 2 choices for each candy afterwards. We have overcounted and need to add back the ways that 2 kids don't receive any candy. This can be done in $3 * 1^7$ ways by similar reasoning. Thus, our desired total is $3^7 - 3 * 2^7 + 3 = \boxed{1806}$.

14. A collection of 8 cubes consists of one cube with edge-length k for each integer k for $1 \leq k \leq 8$. A tower is to be built under the conditions that any cube may be at the bottom of the tower, and the cube immediately on top of a cube with edge-length k must have edge-length at most $k+2$ (You can't have a cube with edge-length 5 be on top of a cube with edge-length 2)

    Solution: We notice that, given a valid stack of $k-1$ cubes, we can put a cube of side length k in three ways: on top of the cube with side $k-2$, on top of the cube with side $k-1$, or at the very bottom of the stack. Thus, we derive the recursion $F_n = 3F_{n-1}$. Since $F_1 = 1$, and $F_2 = 2$, we get that $F_8 = 3^6 * 2 = \boxed{1458}$.

32 Algebra I

1. Factor $x^3 + 3x^2 + 3x - 7$

   Solution: We see that $(x+1)^3 = x^3 + 3x^2 + 3x + 1$, so we can write the original expression as:

   $$\begin{aligned} x^3 + 3x^2 + 3x - 7 &= (x+1)^3 - 8 \\ &= (x-1)((x+1)^2 + 2(x+1) + 4) \\ &= (x-1)(x^2 + 2x + 1 + 2x + 2 + 4) \\ &= (x-1)(x^2 + 4x + 7) \end{aligned}$$

2. Prime Factorize 160401

Solution: We see that $160401 = 160000+400+1 = 20^4+20^2+1$. Using the factorization $x^4+x^2+1 = (x^2+x+1)(x^2-x+1)$, we get that $160401 = (400+20+1)(400-20+1) = 421*381 = \boxed{\text{3*127*421}}$.

3. Solve for integers x, y: $2xy+x+y=5$

   Solution: Multiplying the original equation by 2 and then applying SFFT:

$$\begin{aligned} 4xy+2x+2y &= 10 \\ (2x+1)(2y+1) &= 11 \end{aligned}$$

   So, we have only the possibilities of $(2x+1, 2y+1) = (1,11), (11,1), (-1,-11), (-11,-1)$, so $(x,y) = \boxed{\text{(0,5), (5,0), (-1,-6), (-6,-1)}}$.

4. If a, b, and c are the roots of $x^3+10x+42$, then determine $(a+b)(b+c)(c+a)$

   Solution: Using Vieta's formulas, we know that $a+b+c=0$, so we can rewrite the original expression as $(-a)(-b)(-c) = -abc$. Now, we also know that $abc = -42$ by Vieta's, so the expression is equal to $\boxed{42}$.

5. If a, b, and c are the roots of $x^3-4x^2-6x+4269$, then find the value of $a(1+b+c)+b(1+c+a)+c(1+a+b)$

   Solution: The expression we are asked to find can be simplified to $2(ab+bc+ca)+a+b+c$. Using Vieta's, we know that $a+b+c=4$ and $ab+bc+ca=-6$, so the expression is equal to $2*-6+4 = \boxed{-8}$.

6. Solve in real numbers the equation $x^3-3x^2+3x+3=0$

   Solution: We know that $(x-1)^3 = x^3-3x^2+3x-1$, so $(x-1)^3+4=0$. Since x has to be real, then $x-1 = \sqrt[3]{-4}$, so $x = \boxed{1+\sqrt[3]{-4}}$.

7. Simplify the expression $\frac{x^2+4y^2-z^2+4xy}{x^2-4y^2-z^2+4yz}$

   Solution: Completing the square on the numerator, we get $(x+2y)^2-z^2$, and completing the square on the denominator, we get $x^2-(z-2y)^2$. Thus, our fraction becomes $\frac{(x+2y+z)(x+2y-z)}{(x-2y+z)(x+2y-z)} = \boxed{\frac{x+2y+z}{x-2y+z}}$.

8. Let a b, c, d be real numbers such that $a+b+2ab=3$, $b+c+2bc=4$, and $c+d+2cd=-5$. Find $d+a+2ad$

   Solution: Multiplying the equations by 2 and factoring using SFFT, we get:

$$\begin{aligned} (2a+1)(2b+1) &= 7 \\ (2b+1)(2c+1) &= 9 \\ (2c+1)(2d+1) &= -9 \end{aligned}$$

   Dividing the first two equations, we get $\frac{2a+1}{2c+1} = \frac{7}{9}$. Then, multiplying this equation by the third equation in the latter system, we get $(2a+1)(2d+1) = -7$, and this means that, by expanding, $2d+2a+4ad = -8$, so $d+a+da = \boxed{-4}$.

33 Algebra I, Challenge Problems

1. Solve in real numbers the system of equations

$$\begin{aligned} x+y &= 2z \\ x^3+y^3 &= 2z^3 \end{aligned}$$

Solution: Factoring, we get:

$$\begin{aligned}
x^3 + y^3 &= (x+y)(x^2 - xy + y^2) \\
2z^3 &= 2z(x^2 - xy + y^2) \\
x^2 - xy + y^2 &= z^2 \text{ or } z = 0
\end{aligned}$$

From $z = 0$, we get our first family of equations. If $z = 0$, then $x + y = 0$, and we get the family of solutions $(x, y, z) = (x, -x, 0)$.
If $x^2 - xy + y^2 = z^2$, then we can rewrite $x^2 - xy + y^2$ as $(x + y)^2 - 3xy = 4z^2 - 3xy$, so substituting this into the latter equation that we got, we know that $x^2 - xy + y^2 = 4z^2 - 3xy = z^2$, and it follows that $z^2 = xy$. Finally, $x^2 - xy + y^2 = z^2 = xy$, so $(x - y)^2 = 0$, and since they are reals, $x = y$. This, our second family of solutions is $(x, y, z) = (x, x, x)$.

2. Factor $x^5 + x + 1$

   Solution: We can add and then subtract x^2 to create some nice expressions:

$$\begin{aligned}
x^5 + x + 1 &= x^5 - x^2 + x^2 + x + 1 \\
&= x^2(x^3 - 1) + x^2 + x + 1 \\
&= x^2(x - 1)(x^2 + x + 1) + x^2 + x + 1 \\
&= (x^3 - x^2 + 1)(x^2 + x + 1)
\end{aligned}$$

34 Algebra II

1. Solve the equation $\frac{2x}{2x^2-5x+3} + \frac{13x}{2x^2+x+3} = 6$

   Solution: Dividing both the numerator by x, we get:

$$\frac{2}{2x + \frac{3}{x} - 5} + \frac{13}{2x + \frac{3}{x} + 1} = 6$$

   Substituting $y = 2x + \frac{3}{x}$, we can get the following:

$$\frac{2}{y - 5} + \frac{13}{y + 1} = 6$$

   This then turns into a quadratic in y and we can solve for y and then solve for x after that.

2. Solve the equation $(3x + 1)(4x + 1)(6x + 1)(12x + 1) = 5$

   Solution: Multiplying the first and last term and the middle two terms:

$$(36x^2 + 15x + 1)(24x^2 + 10x + 1) = 5$$

   Now, substituting $y = 12x^2 + 5x$, we get $(3y + 1)(2y + 1) = 5$, which is a quadratic that we can easily solve.

3. Two different prime numbers between 4 and 18 are chosen. When their sum is subtracted from their product, which of the following numbers could be obtained?

   a 21

   b) 60

   c) 119

d) 180

e) 231

Solution: This can be expressed as:

$$\begin{aligned} pq - p - q &= x \\ (p-1)(q-1) &= x+1 \end{aligned}$$

Since $p-1$ and $q-1$ are both even, then $x+1$ is even, meaning x is odd. So, testing $x = 119$ works since $(13-1)(11-1) = 119+1$.

4. Solve the equation $\frac{8^x+27^x}{12^x+18^x} = \frac{7}{6}$

Solution: Factoring:

$$\begin{aligned} \frac{8^x + 27^x}{12^x + 18^x} &= \frac{(2^x)^3 + (3^x)^3}{6^x(2^x+3^x)} \\ &= \frac{(2^x+3^x)(4^x - 6^x + 9^x)}{6^x(2^x+3^x)} \\ &= \frac{4^x - 6^x + 9^x}{6^x} \\ &= \left(\frac{2}{3}\right)^x + \left(\frac{3}{2}\right)^x - 1 \\ \frac{13}{6} &= \left(\frac{2}{3}\right)^x + \left(\frac{3}{2}\right)^x \end{aligned}$$

Letting $\left(\frac{2}{3}\right)^x = y$, then we have $y + \frac{1}{y} = \frac{13}{6}$, so $6y^2 - 13y + 6 = 0$, or $(3y-2)(2y-3) = 0$. Thus, $y = \frac{2}{3}, \frac{3}{2}$. This means that $\left(\frac{2}{3}\right)^x = \frac{2}{3}, \frac{3}{2}$, and it follows that $x = \boxed{1, -1}$.

5. Determine if it is possible each of the polynomials $P(x) = ax^2 + bx + c$, $Q(x) = bx^2 + cx + a$, and $R(x) = cx^2 + ax + b$ to have two real roots if all a, b, c are positive reals.

Solution: Let's assume that all three of the quadratics have two real roots. Then, we must have

$$\begin{aligned} &b^2 > 4ac \\ &c^2 > 4ab \\ &a^2 > 4bc \end{aligned}$$

Since all of these inequalities have positive reals on each side, we can multiply them together to get $a^2b^2c^2 > 64a^2b^2c^2$, which is clearly false. Thus, they can not all have two real roots.

6. Find the remainder when $P(x)$ is divided by $(x-2)(x-3)$ if $P(x)$ leaves a remainder of 5 when divided by $x-2$ and a remainder of 6 when divided by $x-3$.

Solution: We can easily solve a system by letting the remainder be of the form $ax+b$ just like in problem 2.3.2

7. Find the value of a for which $y = ax$ is tangent to the graph of $y = \sqrt{x}$.

Solution: If it is tangent, then there must be only one solution to the quadratic $ax = \sqrt{x}$, or $a^2x^2 = x$.The discriminant of this quadratic is just 1 since $c = 0$, so this means that there exists no such a.

8. Solve the equation $2^x + 3^x - 4^x + 6^x - 9^x = 1$

   Solution: If we let $2^x = a$ and $3^x = b$, then we get the following equation:

$$\begin{aligned} a + b - a^2 + ab - b^2 &= 1 \\ a^2 + b^2 - a - b - ab + 1 &= 0 \\ a^2 - (b+1)a + b^2 - b + 1 &= 0 \end{aligned}$$

   Using this as a quadratic in a, we use the quadratic formula and eventually get

$$a = \frac{b + 1 \pm (b-1)\sqrt{3}i}{2}$$

   We note that in order for a to be real, then $b = 1$, so $3^x = 1$, meaning that $x = \boxed{0}$.

9. Find all real solutions (x, y) of $y^4 + 4y^2x - 11y^2 + 4xy - 8y + 8x^2 - 40x + 52 = 0$

   Solution: We write this as a quadratic with respect to x. The discriminant turns out to be $-16(y^2 - y - 2)^2$. Since this is always non positive, the only way we can have a real solution for x is if the discriminant is 0. Thus, $y^2 - y - 2 = 0$, or $y = 2, = 1$, giving us the solutions $\boxed{(1,2),\ (\frac{5}{2},-1)}$.

10. Solve the equation $\sqrt[4]{x-2} + \sqrt[4]{3-x} = 1$

   Solution: If we let $\sqrt[4]{x-2} = a$ and $\sqrt[4]{3-x} = b$, then we can derive the following system:

$$\begin{aligned} a + b &= 1 \\ a^4 + b^4 &= 1 \end{aligned}$$

   From the second equation, we can get $a^4 + b^4 = (a^2 + b^2)^2 - 2a^2b^2 = ((a+b)^2 - 2ab)^2 - 2a^2b^2 = (1 - 2ab)^2 - 2a^2b^2 = 2a^2b^2 - 4ab + 1 = 1$. This means that $ab(ab - 2) = 0$, so $ab = 0, 2$. From here, we can write $ab = \sqrt[4]{(x-2)(3-x)} = 0, 2$ and solve these two quadratics to get that $x = \boxed{2,\ 3}$.

11. Solve the equation $\sqrt{5 - x} = 5 - x^2$

   Solution: Squaring both sides gives $5 - x = 5^2 - 10x^2 + x^4$. Seeing this as a quadratic in 5 gives us that

$$5 = \frac{2x^2 + 1 \pm 2x - 1}{2}$$

   From here, we can get the solutions for x and check if they are extraneous. Three of them are, giving us the only solution $x = \boxed{\frac{1-\sqrt{17}}{2}}$.

35 Geometry I

1. Two right circular cylinders have the same volume. The radius of the second cylinder is 10% more than the radius of the first. What is the relationship between the heights of the two cylinders? (AMC 10 2015)

   Solution: Let the radius and height of the first cylinder be r_1 and h_1, respectively. Let the radius and the height of the second cylinder be $1.1r_1$ and h_2, respectively, because the radius is 10% more than r_1. Using the formula for the volume of a cylinder and equating, we get:

$$\begin{aligned} r_1^2 h_1 &= (1.1r_1)^2 h_2 \\ r_1^2 h_1 &= 1.21 r_1^2 h_2 \\ h_1 &= 1.21 h_2 \end{aligned}$$

Thus, the height of the first cylinder is $\boxed{21\%}$ greater than the height of the second cylinder.

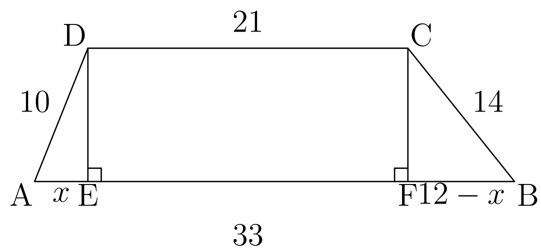
2. The ratio of the length to the width of a rectangle is $4:3$. If the rectangle has diagonal of length d, then the area may be expressed as kd^2 for some constant k. What is k? (AMC 10 2015)

   Solution: Let us, without loss of generality, set the side lengths of the rectangle as $3x$ and $4x$ so that the length of the diagonal would be $5x$. So, since the area of the rectangle is $12x^2$, we are trying to write $12x^2$ in terms of $25x^2$, which is d^2. It is clear that $12x^2 = \frac{12}{25} * 25x^2$, so we have that $k = \boxed{\frac{12}{25}}$.

3. The two legs of a right triangle, which are altitudes, have lengths $2\sqrt{3}$ and 6. How long is the third altitude of the triangle? (AMC 10 2014)

   Solution: By the Pythagorean Theorem, the length of the third side is $\sqrt{12+36} = 4\sqrt{3}$. We can find the area of the triangle to be $6\sqrt{3}$ using the lengths of the two legs. Thus, we know the length of the third side and the area of the triangle, so we can find the length of the height to that side, which is $\boxed{3}$.

4. Trapezoid $ABCD$ has parallel sides $\overline{AB}$ of length 33 and $\overline{CD}$ of length 21. The other two sides are of lengths 10 and 14. The angles at A and B are acute. What is the length of the shorter diagonal of $ABCD$? (AMC 10 2014)



   Solution: We draw in $DE = CF$ as heights in the trapezoid, so we can let $AE = x$ and $BF = 12 - x$. We can set up the system of equations:

$$\begin{aligned} 100 - x^2 &= 196 - (12-x)^2 \\ 100 - x^2 &= 196 - (144 - 24x + x^2) \\ 100 - x^2 &= 52 + 24x - x^2 \\ 24x &= 48 \\ x &= 2 \end{aligned}$$

   This means that the height is $\sqrt{96} = 4\sqrt{6}$. We see that the shorter diagonal would be part of the right triangle with legs CF and AF, so using the Pythagorean Theorem, $AC^2 = 529 + 96 = 625$, so $AC = \boxed{25}$.

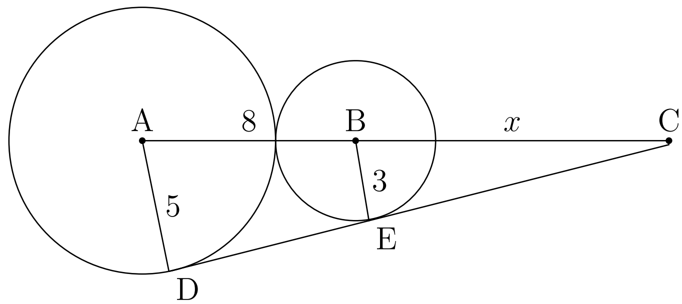
5. Two sides of a triangle have lengths 10 and 15. The length of the altitude to the third side is the average of the lengths of the altitudes to the two given sides. How long is the third side? (AMC 10 2013)

   Solution: Letting the heights to the two known side lengths be h_1 and h_2, we get the equation, using the triangle's area, $10h_1 = 15h_2$, so $h_2 = \frac{2}{3}h_1$. So, the height to the third side is the average of h_1 and $\frac{2}{3}h_1$, which is $\frac{5}{6}h_1$. Now, we can once again and use the area of the triangle to get the equation $10h_1 = x * \frac{5}{6}h_1$, where x is the length of the third side. Solving, we get that $x = \boxed{12}$.

6. In triangle ABC, medians AD and CE intersect at P, $PE = 1.5$, $PD = 2$, and $DE = 2.5$. What is the area of $AEDC$? (AMC 10 2013)

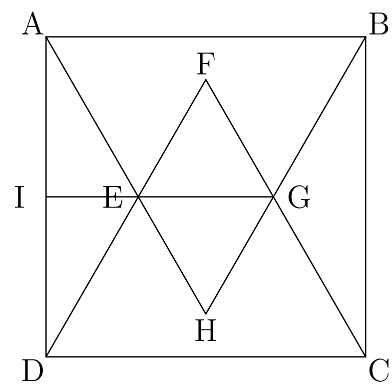
   Solution: Note that triangle DPE is a right triangle, and that the four angles that have point P are all right angles. Using the fact that the centroid (P) divides each median in a $2:1$ ratio, $AP = 4$ and $CP = 3$. Quadrilateral $AEDC$ is now just four right triangles. The area is $\frac{4\cdot1.5+4\cdot3+3\cdot2+2\cdot1.5}{2} = \boxed{13.5}$

7. Externally tangent circles with centers at points A and B have radii of lengths 5 and 3, respectively. A line externally tangent to both circles intersects ray AB at point C. What is BC? (AMC 10 2012)



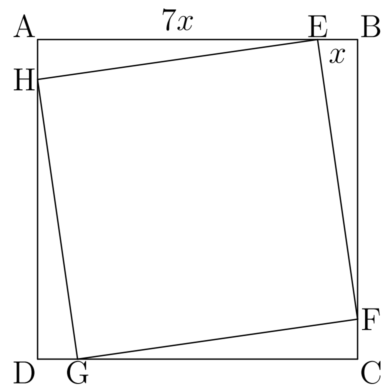
Solution: Let D and E be the points of tangency on circles A and B with line CD. $AB = 8$. Also, let $BC = x$. As $\angle ADC$ and $\angle BEC$ are right angles (a radius is perpendicular to a tangent line at the point of tangency) and both triangles share $\angle ACD$, $\triangle ADC \sim \triangle BEC$. From this we can get a proportion:
$\frac{BC}{AC} = \frac{BE}{AD} \rightarrow \frac{x}{x+8} = \frac{3}{5} \rightarrow 5x = 3x + 24 \rightarrow x = \boxed{12}$

8. Two equilateral triangles are contained in square whose side length is $2\sqrt{3}$. The bases of these triangles are the opposite side of the square, and their intersection is a rhombus. What is the area of the rhombus? (AMC 10 2012)



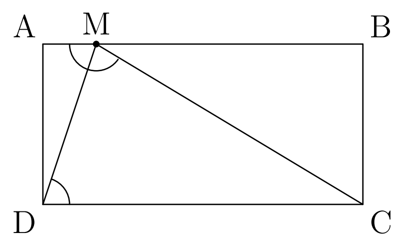
We note that $\triangle IAE$ is a $30-60-90$ right triangle, and since the figure is "symmetrical", we see that $AI = \sqrt{3}$. Thus, $AE = 2$. By the givens, $AH = 2\sqrt{3}$, so this means that $EH = 2\sqrt{3} - 2$. Because $EFGH$ is a rhombus with angles $60°$ and $120°$, then $\triangle EHG$ and $\triangle EFG$ are equilateral triangles with side lengths $2\sqrt{3} - 2$. Using the area formula for an equilateral triangle, we get that the area of one of the equilateral triangles is $\frac{\sqrt{3}}{4}(2\sqrt{3} - 2)^2 = 4\sqrt{3} - 6$. Thus, the area of two equilateral triangles is $\boxed{8\sqrt{3} - 12}$.

9. Square $EFGH$ has one vertex on each side of square $ABCD$. Point E is on $\overline{AB}$ with $AE = 7 \cdot EB$. What is the ratio of the area of $EFGH$ to the area of $ABCD$? (AMC 10 2011)



Solution: We can simply let $AE = 7x$ and $BE = x$. By symmetry, $BF = 7x$, so we can use the Pythagorean Theorem in $\triangle EBF$ to get $EF = \sqrt{50}x$. Thus, $[EFGH] = 50x^2$ and $[ABCD] = 64x^2$, so the ratio of the areas is $\boxed{\frac{25}{32}}$.

10. Rectangle $ABCD$ has $AB = 6$ and $BC = 3$. Point M is chosen on side AB so that $\angle AMD = \angle CMD$. What is the degree measure of $\angle AMD$? (AMC 10 2011)



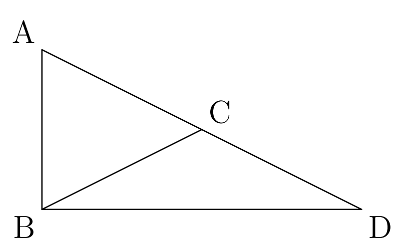
Solution: We can use alternate interior angles to get that $\angle AMD = \angle DMC = \angle MDC$. Thus, $\triangle MDC$ is isosceles with $MC = DC = 6$. Thus, we see that $\triangle MBC$ is a $30-60-90$ triangle because $MC = 6$ and $BC = 3$. This means that $\angle BMC = 30°$. So, $\angle AMD = \frac{1}{2}(180° - 30°) = \boxed{75°}$.

11. Rhombus $ABCD$ has side length 2 and $\angle B = 120$. Region R consists of all points inside the rhombus that are closer to vertex B than any of the other three vertices. What is the area of R? (AMC 10 2011)

Solution: See $http://artofproblemsolving.com/wiki$
Key point: The perpendicular bisector of a segment is the locus of all points equidistant from the two given endpoints of that segment

36 Geometry II

1. Triangle ABD is a right triangle with right angle at B. On AD there is a point C for which $AC = CD$ and $AB = BC$. Find $\angle DAB$.



Solution: We know that the median to the hypotenuse is the same length as half of the length of the hypotenuse. In particular $AC = DC = BC$. However, we are also given that $AB = BC$, so we have that $AC = CB = BA$, or $\triangle ACB$ is equilateral. Thus, $\angle DAB = \boxed{60°}$.

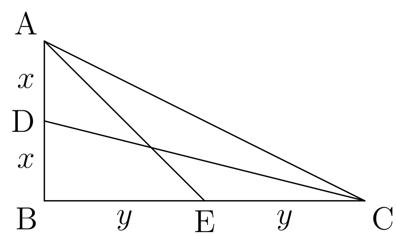
2. If a triangle has sides 40, 60, and 80, then the shortest altitude is K times the longest altitude. Find the value of K.

   Solution: Let the area of the triangle be A, and let the length of the altitude to the side of 80 be h_1 and let the length of the altitude to the side of 40 be h_2. Thus, we can write $A = \frac{1}{2}80h_1 = \frac{1}{2}40h_2$, so it follows that $2h_1 = h_2$, so $K = \boxed{\frac{1}{2}}$.

3. A regular polygon has each interior angle measuring 176°. Find the number of sides of this regular polygon.

   Solution: We know that the measure of each exterior angle is $180° - 176° = 4°$. Because the sum of the exterior angles must sum to 360°, then the polygon must have $\frac{360}{4} = \boxed{90}$ sides.

4. The medians of a right triangle which are drawn from the vertices of the acute angles are 5 and $\sqrt{40}$. Find the length of the hypotenuse.



   Solution: Let D, E be the midpoints of the legs of the right triangle. We see that we can construct the following system of equations:

$$\begin{aligned} x^2 + 4y^2 &= 40 \\ 4x^2 + y^2 &= 25 \end{aligned}$$

   Now, we could solve the system of equations for x and y, but we see that we want the length of the hypotenuse, which is $\sqrt{4x^2 + 4y^2}$. Thus, from the previous system, we can add the two equations to get $5x^2 + 5y^2 = 65$, so $4x^2 + 4y^2 = 52$. This means that the length of the hypotenuse is $\boxed{2\sqrt{13}}$.

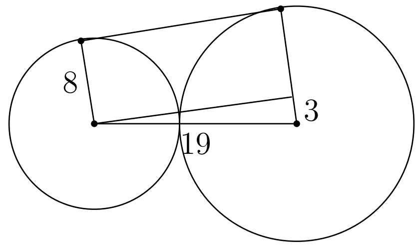
5. Find the area of a rhombus with a side of length 13 and one diagonal of length 24.

   Solution: Since the diagonals of a rhombus are perpendicular and bisect each other, then we can focus in on one of the right triangles formed by the diagonals. The hypotenuse of one of these triangles is the side length of the rhombus, which is 13. One of the legs is half of the known diagonal, which is 12. So, this means that the remaining diagonal has length $5 * 2 = 10$. Now, the area of a rhombus is half the product of its diagonals, which is $\boxed{120}$.

6. The length of a rectangular picture is three times its width. The picture is surrounded by a frame which is 4 inches wide. If the perimeter of the outside frame is 96 inches, what is the length of the picture, in inches?

   Solution: If we let the sides of the picture be $3x$ and x, then the picture frame will have dimensions of $3x + 8$ and $x + 8$. Thus, the perimeter, $8x + 32$ is equal to 96. Solving, we get that $x = 8$, so the length of the picture is $\boxed{24}$.

7. What is the length of the common external tangent segment of two externally tangent circles whose radii are 8 and 11?



Solution: If we draw the perpendicular from the center of one circle to the radius of the other, we form a right triangle and a rectangle. We see that the sides of the right triangle are 19 and 3 because the radii of the circles are 8 and 11. Thus, the length of the other leg of the right triangle, which is also the length of the tangent, is $\sqrt{352} = \boxed{4\sqrt{22}}$.

8. Given that $ABCDEF$ is a regular hexagon with side length 6, find the area of triangle BCE.

   Solution: We note that $\angle BCE$ is a right angle. In addition, we can use inscribed angles to find that $\triangle BCE$ is actually a 30-60-90 triangle. Thus, since $BC = 6$, $CE = 6\sqrt{3}$, so the area is $\boxed{18\sqrt{3}}$.

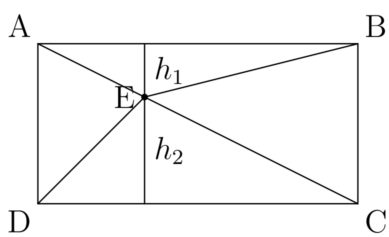
9. What is the maximum amount of right angles an octagon can have?

   Solution: Let's say that there are k right angles in an octagon. Thus, there are $8 - k$ remaining angles. Each of these angles have to be less than $180°$. So, the maximum sum of the remaining angles is $1440° - 180°k$. However, the sum of all the angles in the octagon is $1080°$, and the sum of the right angles is $90°$. Thus, we can establish the inequality $1080° - 90°k < 1440° - 180°k$. Thus means that $k < 4$. Thus, the greatest value of k is $\boxed{3}$.

10. In a regular polygon $ABCDE\cdots$, we have $\angle CAD = 12°$. How many sides does this polygon have?

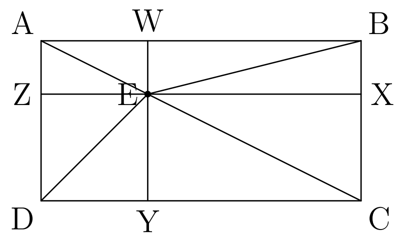
    Solution: Because the given angle is an inscribed angle, then the measure of arc CD is $24°$. This means that one side of the polygon corresponds to $\frac{24}{360} = \frac{1}{15}$ of the total circular arc. Thus, there are $\boxed{15}$ sides in the regular polygon.

11. In a rectangle, interior point E is chosen at random. Prove that the sum of the areas of triangles AEB and EDC is the same regardless of where in $ABCD$ point E is chosen.



    Solution: We can write $[AEB] = \frac{1}{2}ABh_1$ and $[EDC] = \frac{1}{2}DCh_2$. Because $DC = AB$, the sum of the areas of the two triangles is $\frac{1}{2}AB(h_1 + h_2)$. Now we see that $h_1 + h_2$ is actually just AD, so we can simplify this to $\frac{1}{2}AB * AD$, and this is always the same for any point E because it is just half the area of the rectangle.

12. In a rectangle, an interior point E is chosen. Prove that $AE^2 + CE^2 = BE^2 + DE^2$ (British Flag Theorem)



Solution: We can use the pythagorean theorem multiple times to get:

$$\begin{aligned} AZ^2 + AW^2 &= AE^2 \\ BW^2 + BX^2 &= BE^2 \\ CX^2 + CY^2 &= CE^2 \\ DY^2 + DZ^2 &= DE^2 \end{aligned}$$

Now, we see that $AE^2 + CE^2 = AZ^2 + AW^2 + CX^2 + CY^2 = BX^2 + DY^2 + DZ^2 + BW^2$, and this is exactly $BE^2 + DE^2$, so we are done.

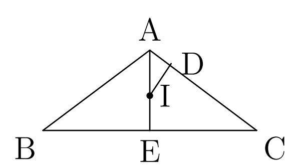
13. Find a formula for the area of a regular hexagon given its side length.

    Solution: We can draw the center of the hexagon and draw the lines from the center to the vertices. This splits the hexagon into equilateral triangles (why?). Thus, each equilateral triangle has side length s and therefore has area $\frac{\sqrt{3}}{4}s^2$. This means that six of these triangles with have area $\boxed{\frac{3\sqrt{3}}{2}s^2}$.

14. Prove that if we connect, in order, the midpoints of the sides of any quadrilateral, then we form a parallelogram.

    Solution: By the midsegment theorem, two of the sides of the "new" quadrilateral are parallel to AC and the other two sides are parallel to BD. Thus, this "new" quadrilateral must be a parallelogram by definition.

15. Given that I is the incenter of $\triangle ABC$ and $AB = AC = 5$, and $BC = 8$, find the distance AI.



    Solution: We see that the height is 3. If we draw the line from the incenter to the point of tangency of the incircle to AC, we get the line segment ID which is perpendicular to AC. To find the length of the inradius, ID, we use the formula $Area = rs$. Plugging in, we get that $ID = \frac{4}{3}$. We use similar triangles ADI and AEC to get the proportion $\frac{AI}{ID} = \frac{AC}{EC}$. Using this proportion, we get that $AI = \boxed{\frac{5}{3}}$.

16. Given a circle inscribed in a triangle (the incircle), prove that the lengths of the tangents closest to A, B, and C have lengths $s - a$, $s - b$, and $s - c$, respectively.

    Hint: Use equal tangents

17. A cow is tied to the corner of a 20 foot by 15 foot shed with a 30 foot rope. Find her total grazing area.

    Solution: To get this, notice that the cow has a grazing area of a 3/4's circle with a 30 foot rope tether. This area is $\frac{900\pi}{4}$ or 675π

However, the cow has more than just that much! Let's say he goes wandering along the side of the shed that is 15 feet long, such that his rope is touching the side, and it's stretched all the way. We see the cow has 15 extra feet of rope that goes past the 15 foot side. (15+15=30 feet). The cow can then make a 90 degree turn to touch the side that is 20 feet long. This creates a quarter circle of radius 15, so the area of that region is $\frac{225\pi}{4}$ or 56.25π

Similarly, the cow can wander along the side that is 20 feet long such that his rope touches the side and the rope is stretched out all the way. The cow has 10 feet of rope left not touching the side once he stretches out all the way.He can make a 90 degree turn such that he touches the side that has a length of 15 ft. Since the cow has 10 ft of free rope, and he can make a 90 degree quarter turn until she's stopped, he has $\frac{10^2\pi}{4}$ or 25π square feet of area. Adding these areas up, we get $25\pi+56.25\pi+675\pi = \boxed{756.25\pi}$

18. A park is in the shape of a regular hexagon 2 km on a side. Starting at a corner, Alice walks along the perimeter of the park for a distance of 5 km. How many kilometers is she from her starting point?

    Solution: Let the starting point be A. Thus, Alice ends up at the midpoint of CD. Let this point be X. We see that when we draw BC, it is perpendicular to CD by angle chasing. So, we just need to compute AC, which we can do because ABC is isosceles and we can split it into two 30-60-90 triangles, giving us that $AC = 2\sqrt{3}$. Thus, we have $CX = 1$ and $AC = 2\sqrt{3}$ as legs of the right triangle ACX. Thus, the length $AX = \boxed{\sqrt{13}}$.

19. The median to a 10 cm side of a triangle has length 9cm and is perpendicular to a second median of the triangle. Find the exact value in centimeters of the length of the third median.

    Answer: $\boxed{3\sqrt{13}}$
    Hint: Use the fact that the medians divide each other in the ratio 1 : 2 and the median to a hypotenuse lengths.

20. Given a right triangle with side lengths 3, 4, and 5, find the distance between the incenter and the circumcenter.

    Answer: $\boxed{\frac{\sqrt{5}}{2}}$
    Hint: Where is the circumcenter in a right triangle? Try to construct a right triangle with the incenter, circumcenter, and point of tangency of the incircle and the hypotenuse. You will need to find the inradius and use the $s - a$ formula.