ENGG 3390 Signal Processing
Exam Review

December 15, 2016

Contents

1 Study Tips

- Do the Problem Sets
- Recognize which parts of assigned problems are reasonable for an exam, and which are extensions
- Become familiar with the different forms of the Euler Identity:

$$e^{j\omega} = \cos\omega + j\sin\omega$$
$$\cos\omega = \frac{1}{2}\left(e^{j\omega} + e^{-j\omega}\right)$$
$$\sin\omega = \frac{1}{2j}\left(e^{j\omega} - e^{-j\omega}\right)$$

- Aim to know off-by-hand the very basic Fourier Transform / Series pairs:

$$\frac{\sin(Wt)}{\pi t} \iff \begin{cases} 1 & |\omega| \leq W \\ 0 & \text{else} \end{cases}$$
$$\cos(\omega_0 t) \iff \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

- (For other transform pairs:) Appendices A, C, and D from the textbook.

2 Fourier Transforms: Textbook Tools

Many of the problems in Problem Sets 4 & 5 employ the transform pairs and properties in Appendix C. Transform pairs must often be combined, using the properties in Table C.7. Here are some examples of this in action:

Example: 3.58 – Problem Set 5
Find the Fourier Transform $X(j\omega)$ of $x(t)$:

$$x(t) = \sin(2\pi t)e^{-t}\mu(t)$$

Solution
There are actually a number of methods for solving this problem. We'll show only one here.

First, we scan the Tables in Appendix C to see which basic transforms (#) we might use.

(1) We notice that $e^{-t}\mu(t)$ looks like one of the basic transforms in Table C.4:

$$x(t) = e^{-at}\mu(t) \iff X(j\omega) = \frac{1}{a + j\omega}$$

(2.a) Next, we could use $x(t)y(t) = \ldots$ from Table C.7, with $y(t) = \sin(2\pi t)$ but this involves convolution. This approach *would work*, however, we'll use another approach (which is often useful - so take note!)

(2.b) Instead, we invoke Euler, and write:

$$\sin(2\pi t) = \frac{1}{2j}\left[e^{+2\pi jt} - e^{-2\pi jt}\right]$$

Using this, let's expand $x(t)$:

$$\begin{aligned} x(t) &= \frac{1}{2j}\left[e^{+2\pi jt} - e^{-2\pi jt}\right]e^{-t}\mu(t) \\ &= \frac{1}{2j}e^{+2\pi jt}e^{-t}\mu(t) - \frac{1}{2j}e^{-2\pi jt}e^{-t}\mu(t) \end{aligned}$$

Cool. Now we have:

- two $e^{-t}\mu(t)$ terms ($a = 1$)
- a constant $\frac{1}{2j}$ in front of each
- a frequency shift in front of each: $e^{\pm 2\pi jt}$ ($\gamma = \pm 2\pi$)

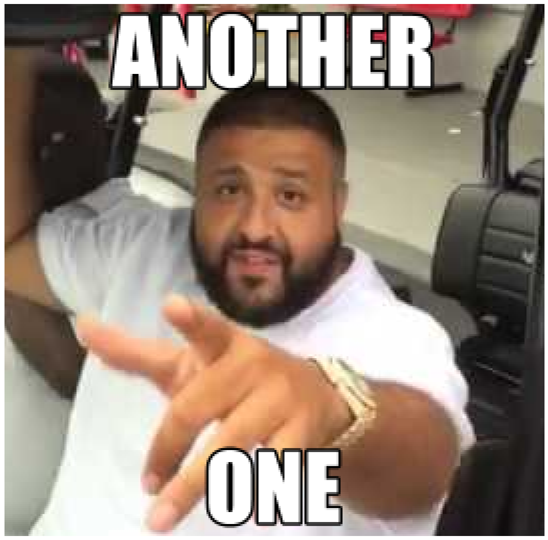
(3) From Table C.7 we recall the frequency shift property:

$$e^{\gamma jt}x(t) \iff X(j(\omega - \gamma))$$

So, we have all the pieces of the puzzle. Let's put them together, building off of (1), and from (3), $\omega \to \omega \mp 2\pi$:

$$X(j\omega) = \frac{1}{2j}\left[\frac{1}{1 + j(\omega - 2\pi)} - \frac{1}{1 + j(\omega + 2\pi)}\right]$$

Done!



Example: 4.18 – Problem Set 5
Given a filter with the impulse response $h(t)$ shown below, calculate the output of the system for input $x(t)$:

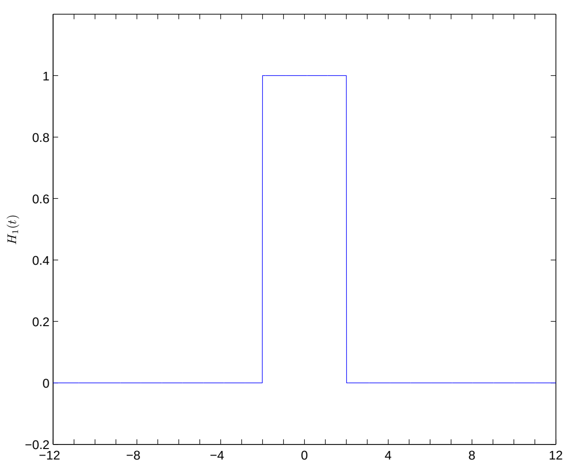
$$h(t) = \frac{\sin(\frac{\pi}{4}t)}{\pi t}\cos(7\pi t)$$
$$x(t) = \cos(2\pi t) + \sin(6\pi t)$$

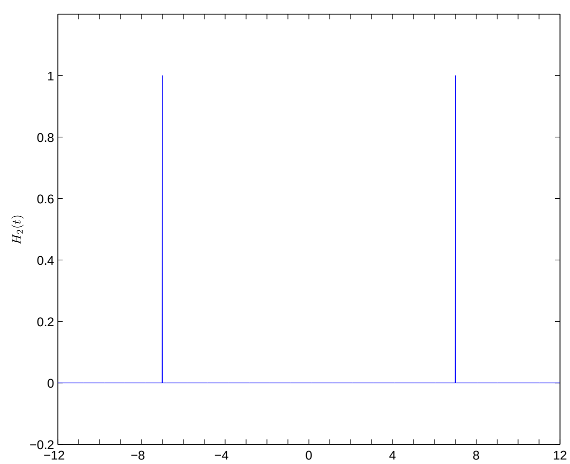
Solution
Don't panic! This looks nasty, but it's actually a cool result.

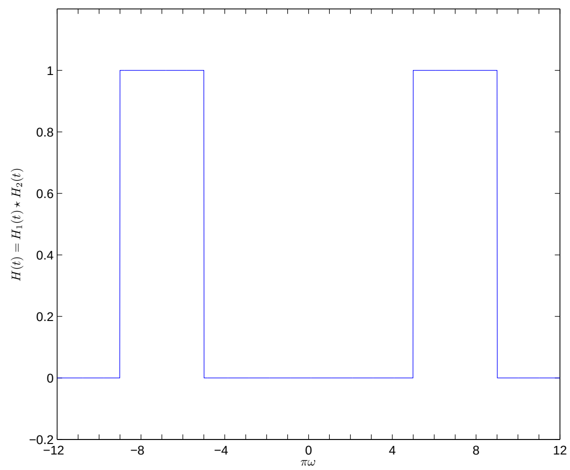
First, we observe that $h(t)$ can be written as $h(t) = h_1(t)h_2(t)$, where $h_1(t)$ and $h_2(t)$ can be found directly in Table C.4 and C.5:

$$h_1(t) = \frac{1}{\pi t}\sin(Wt) \iff H_1(j\omega) = \begin{cases} 1 & |\omega| \leq W \\ 0 & \text{else} \end{cases}, \qquad W = \frac{\pi}{4}$$
$$h_2(t) = \cos(\omega_0 t) \iff H_2(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

We can then use the multiplication-convolution property so that $H(j\omega) = H_1(j\omega) \star H_2(j\omega)$. In this case, the convolution isn't so bad. In fact, it's quite easy with some pictures:







At this point, you should recognize this filter as a bandpass filter. The multiplication of the **sinc** function by a **cos** at a frequency ω_0 creates a bandpass filter around that frequency ω_0. Neat! Moreover, the width of this passband is still controlled by the W of the **sinc**: $\pm 2\pi$, in this case.

Mathematically, we can write:

$$H(j\omega) = \begin{cases} 1 & |\omega| \in [5\pi, 9\pi] \\ 0 & \text{else} \end{cases}$$

from which it is easy to see the passband of the filter: 5π to 9π.

Now, returning to the problem at hand: recall we want to find $y(t) = h(t) \star x(t)$. Let's examine $x(t)$.

We see that it has two periodic components: a **cos** with $\omega_0 = 2\pi$ and a **sin** with $\omega_0 = 6\pi$. Since the passband is 5π to 9π, the **cos** dies while the **sin** passes straight through.

So, the output is simply

$$y(t) = \sin(6\pi t)$$

Done!

3 Bilinear Transform

Bilinear Transforms also look scary but aren't, once you work out a simplification technique. We'll point it out in this example:

Example:
What is the discrete time filter $H(z)$ which has the same frequency response as $H(s)$ shown? Assume a sampling period $T = 1$ sec.

$$H(s) = \frac{s}{s^2 - 3s + 3}$$

Solution
First, we define the substitution:

$$s = \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) = \frac{2}{T}\left(\frac{z-1}{z+1}\right) = 2\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right).$$

Next, we substitute and expand:

$$H(z) = \frac{2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}{\left(2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right)^2 - 3\left(2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) + 3}$$

Now, here's the simplification technique: we multiply all terms by the denominator of the substitution fraction $\left(1+z^{-1}\right)$ as many times as the highest power (so it no longer appears in any denominators). Now the simplification should be straightforward algebra.

$$\begin{aligned}
H(z) &= \frac{2\left(1-z^{-1}\right)\left(1+z^{-1}\right)}{\left(2\left(1-z^{-1}\right)2\left(1-z^{-1}\right)\right)-3\left(2\left(1-z^{-1}\right)\left(1+z^{-1}\right)\right)+3\left(\left(1+z^{-1}\right)\left(1+z^{-1}\right)\right)} \\
&= \frac{2\left(1-z^{-2}\right)}{4\left(1-2z^{-1}+z^{-2}\right)+6\left(1-z^{-2}\right)+3\left(1+2z^{-1}+z^{-2}\right)} \\
&= \frac{2-2z^{-2}}{13-2z^{-1}+z^{-2}}
\end{aligned}$$

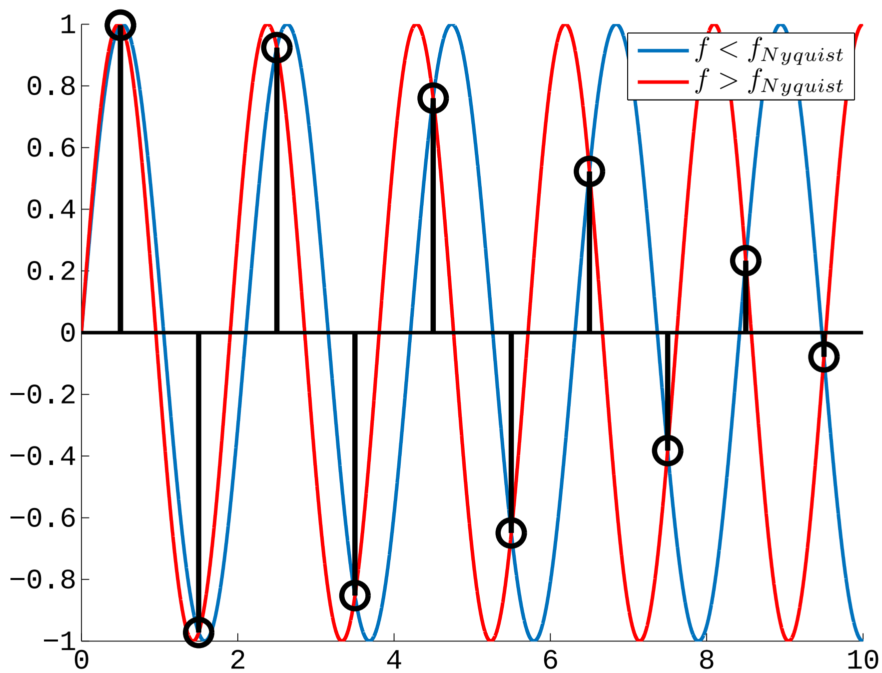
Done!

Note: We could be asked a similar problem in the other direction (i.e. find $H(s)$ given $H(z)$), for which we have the rearrangement:

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \qquad \Longleftrightarrow \qquad z^{-1} = \frac{1-\frac{T}{2}s}{1+\frac{T}{2}s} \qquad \Longleftrightarrow \qquad z = \frac{1+\frac{T}{2}s}{1-\frac{T}{2}s}$$

4 Aliasing

Finally, we talked briefly about aliasing: the inability to distinguish two signals from one another due to sampling. Below, we show an example of this, since the samples in black could come from either sine wave.



In general, for a signal with frequency f, the frequencies of its aliases are defined as (confirm this for yourself!),

$$f_{alias} = |f - Nf_s|, \qquad N = 1, 2, 3, \ldots$$

This then yields a reflection of the frequency spectrum of sampled signals about the Nyquist frequency (and also all $(N + 1/2)$ multiples). Components above the Nyquist frequency then *appear* as lower frequency components, confounding the observed signal (i.e. we always want to avoid aliasing!).