Non-stationary Pricing and Search

Wee Chaimanowong

(Chinese University of Hong Kong)

Yunfei (Jesse) Yao*

(Chinese University of Hong Kong)

November 2025

* We thank J. Miguel Villas-Boas, Tat Chan, Francisco Cisternas, Anthony Dukes, Yuichiro Kamada, Tony Ke, Song Lin, Eddie Ning, Jiwoong Shin, Quitzé Valenzuela-Stookey, and Jungju Yu for constructive comments. We also thank seminar participants at the Berkeley Econ theory lunch. All errors are our own. E-mail address: Wee Chaimanowong: chaimanowongw@gmail.com, Yunfei (Jesse) Yao: jesseyao@cuhk.edu.hk.

Abstract

In the context of dynamic monopoly pricing with buyer learning, we study *non-stationary pricing strategies* – prices that evolve over time without being contingent on a buyer's current valuation – which endogenously induce non-stationarity in the buyer's search problem. We provide a full analytical characterization of the buyer's learning strategy and the seller's profit when the price changes linearly over time and the buyer's learning rate is constant. Under zero search costs, a perfectly patient seller's optimal price is arbitrarily close to constant. When search costs are positive, the optimal pricing strategy is non-stationary even if the seller is perfectly patient. Under a more general learning structure, we show that the seller can achieve the maximum possible profit by increasing the price as sharply as possible under some conditions. When such fast-rising pricing strategies are not feasible, we provide managerial recommendations on the direction of price evolution by demonstrating how sellers can improve profits by slowly changing the price from the optimal stationary level. The price increases over time when the information is too noisy or search costs are too high. When buyers are more incentivized to search, the price increases for buyers with high or low initial valuations and decreases for those with medium initial valuations. We also discuss the conditions under which a buyer will approximate any sufficiently slow-moving price with a linear price, opening the possibility for sellers to apply our theory to more general settings.

Keywords: Intertemporal pricing, learning, stochastic control, non-stationary strategies

1 Introduction

This paper studies dynamic monopoly pricing with buyer learning. Stokey (1979) originally considers the intertemporal price discrimination problem where a buyer (she) decides whether and when to buy a product from a monopolist seller (he), who chooses the pricing trajectory. Under the assumption that the buyer knows her valuation, which stays constant over time, Stokey (1979) shows that a constant price is optimal for the seller. In reality, buyers often have uncertainty about a product's value, and often gather noisy information gradually to reduce such uncertainty before making a purchasing decision. Building on the seminal work of Weitzman (1979) and Wolinsky (1986), a rich literature has explored optimal search strategies across multiple alternatives or product attributes and their implications for sellers' pricing strategies. Existing work on optimal pricing with buyer learning often assumes exogenous prices, endogenous constant prices (e.g., Branco et al., 2012), or endogenous prices contingent on the buyers' current valuation (e.g., Ning, 2021). However, recent privacy regulations have disrupted sellers' ability to track individual buyers in real time, making it challenging to tailor prices to evolving buyer beliefs. Even if a seller can track buyers' browsing behavior, it may be hard for the seller to know how buyers will interpret the information they see. For example, Tesla may be able to observe that a buyer clicks on an image of the interior design of the car, but may not know whether the buyer prefers the large screen on Tesla or the traditional dashboard. This calls into question whether the seller can track the evolution of the buyer's valuation when the buyer is searching for information, and raises an important question: can sellers benefit from dynamic pricing when the evolution of the buyer's valuation is unobservable?

Without the ability to track the evolution of the buyer's valuation, the only stationary pricing strategy is a constant price. This paper introduces a novel framework where sellers adopt *non-stationary pricing strategies* – prices that evolve over time without being contingent on the buyer's current valuation. Such strategies endogenously induce non-stationarity in the buyer's search problem. We address two key questions: (1) Is a stationary pricing strategy always optimal for a seller that cannot observe the evolution of a buyer's valuation? (2) If not, what are the characteristics of the optimal non-stationary pricing strategy?

Our findings challenge the conventional reliance on stationary pricing by showing that non-stationary pricing strategies can outperform stationary ones. We provide a full analytical characterization of the buyer's learning strategy and the seller's profit when the price changes linearly over time and the buyer's learning rate is constant. Under zero search costs, a perfectly patient seller's optimal price is arbitrarily close to constant. By contrast, with discounting, the seller may benefit from charging non-stationary prices. When search costs are positive, the optimal pricing strategy is non-stationary even if the seller is perfectly patient. Under a more general learning structure, we show that the seller can achieve the maximum possible profit by increasing the price as sharply as possible under some conditions. When such fast-rising pricing strategies are not feasible, we consider slow-moving linear pricing strategies. We derive analytical expressions for the seller's profit to the first order of the slope of the price. Based on this result, we provide managerial recommendations to sellers by demonstrating how they can improve profits by slowly changing the price from the optimal stationary level. The price increases over time if the information is too noisy or the search cost is too high. The direction of price trajectories is more nuanced in other cases where buyers have a stronger incentive to search, with increasing prices for buyers with high or low initial valuations and decreasing prices for buyers with a medium level of initial valuation. We also discuss the conditions under which a buyer will approximate any sufficiently slow-moving price with a linear price, opening the possibility for sellers to apply our theory to more general settings.

By incorporating non-stationary strategies into a search framework, this paper provides a theoretical advance in optimal control. Unlike most papers in the literature, which impose stationarity for tractability, our paper highlights that such restrictions may lead to sub-optimal outcomes. While a few papers have explored non-stationarity in search problems driven by exogenous environments, such as the finite horizon and the evolving distribution of rewards (Gilbert and Mosteller, 1966; Sakaguchi, 1978; Van den Berg, 1990; Smith, 1999; Kamada and Muto, 2015), we model endogenous non-stationarity arising from sellers' strategic pricing in response to buyer search, providing a foundational step toward understanding sellers' non-stationary interventions in this context.

We also offer managerial insights into how sellers can adapt to privacy regulations. Non-stationary pricing leverages time – a freely available resource – as an information source for pricing decisions. Firms do not need to invest heavily in the tracking technology. Hence, all the increased revenue due to non-stationary pricing becomes profit. This is especially relevant in light of privacy-driven disruptions. Apple's iPhone privacy upgrades cost publishers like Facebook, YouTube, Twitter, and Snap nearly 10 billion in ad revenue in 2021 alone because the increased privacy restrictions limited advertisers' ability to target consumers.[1] Privacy regulations can prevent firms from tracking consumers' demographic information, browsing behavior, and other characteristics, but cannot ban the use of time to which everyone has access. Our work guides sellers on how to adjust their pricing strategies to thrive in a privacy-conscious environment.

This paper fits into two strands of literature. It is related to the literature on optimal information acquisition (Moscarini and Smith, 2001; Armstrong et al., 2009; Branco et al., 2012; Ke and Villas-Boas, 2019; Ke and Lin, 2020; Guo, 2021; Zhong, 2022; Yuan et al., 2023, 2024; Ning et al., 2025; Urgun and Yariv, 2025; Wong, 2025) and its implications for sellers' strategies in pricing, information provision, product design, and advertising (Anderson and Renault, 2006; Villas-Boas, 2009; Bar-Isaac et al., 2010; Mayzlin and Shin, 2011; Guo and Zhang, 2012; Liu and Dukes, 2013; Branco et al., 2016; Dukes and Liu, 2016; Jerath and Ren, 2021; Villas-Boas and Yao, 2021; Chaimanowong and Ke, 2022; Ke et al., 2023; Lu, 2023; Zhong, 2023; Yao, 2024a,b; Chen et al., 2025; Zia and Kuksov, 2025). It also contributes to the literature on dynamic pricing with buyer learning, and more broadly, with strategic buyers (Levinthal and Purohit, 1989; Desai and Purohit, 1998; Villas-Boas, 2004; Bose et al., 2006; Su, 2007; Aviv and Pazgal, 2008; Chen and Zhang, 2009; Hviid and Shaffer, 2010; Jing, 2011; Deb, 2014; Sayedi, 2018; Gong et al., 2022; Huang and Gong, 2023; Gong and Huang, 2025; Ning and Zhou, 2025). This study adds to the existing literature by incorporating non-stationary pricing strategies into a buyer gradual learning framework. A closely related paper, Libgober and Mu (2021), also considers dynamic monopoly pricing with buyer learning. By focusing on maximizing profits against the worst-case information arrival process when the buyer's true valuation is fixed over time, it shows that a constant price leads to robustly optimal profit. In contrast, we consider scenarios where the seller knows the information arrival process but does not know the realization of the signal, and maximizes the expected profit. We show that a constant price may no longer be optimal in such cases.

[1] Source: https://www.businessinsider.com/apple-iphone-privacy-facebook-youtube-twitter-snap-lose-10-billion-2021-11.

2 The Model

A seller offers a product with marginal cost g and sets its price. A buyer then decides whether to purchase the product. The buyer's initial valuation is v_0, which is common knowledge. Before making a purchase decision, the buyer can gradually learn about various product attributes and update her belief about the product's value. The buyer's discount rate is r and the seller's discount rate is m. We focus on the learning processes that arise within the general non-linear optimal filtering framework (Liptser and Shiryaev, 2013, Chapter 8).

The buyer's total utility from consuming the product is given by an unobservable process $\{\pi_t\}_{t\geq 0}$. To learn about π_t, the buyer pays a flow search cost of c and observes a process $\{S_t\}_{t\geq 0}$, which generates a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ such that $v_t := \mathbb{E}[\pi_t|\mathscr{F}_t]$ is a continuous martingale.[2] We assume that $\{v_s^{t,x}\}_{s\geq t}$ is the unique strong solution to:

$$dv_s^{t,x} = \mu(v_s^{t,x}, \pi_s)ds + \sigma(v_s^{t,x})dW_s, \qquad v_t^{t,x} = x, \tag{1}$$

where $\{W_t\}_{t\in\mathbb{R}_{\geq 0}}$ is the standard Brownian motion adapted to the filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in\mathbb{R}_{\geq 0}}, \mathbb{P})$. We impose the following assumptions on the drift and diffusion coefficients, which ensure that the buyer's expected payoff is well-defined.

Assumption 1. *Let $\underline{\pi}, \bar{\pi} \in \mathbb{R}\cup\{\pm\infty\}$ be such that $\underline{\pi} \leq v_t \leq \bar{\pi}$ a.e., for all $t \in \mathbb{R}$. We assume that:*

- *$\mu(.,\pi), \sigma(.) \in C^\infty(\underline{\pi}, \bar{\pi})$, and $\sigma(x) > 0$ for all $x \in (\underline{\pi}, \bar{\pi})$.*
- *The global Lipschitz condition holds for some constant $L \geq 0$, for all $t \in \mathbb{R}$ and $x, y \in [\underline{\pi}, \bar{\pi}]$: $|\mu(x,\pi_t) - \mu(y,\pi_t)| + |\sigma(x) - \sigma(y)| \leq L|x-y|$.*

The constants $\underline{\pi}$ and $\bar{\pi}$ represent the highest and lowest possible values of the product, which is allowed to be unbounded. The global Lipschitz condition plays a role of controlling the growth rate of the square integral, $\mathbb{E}[v_t^2] = O(e^{L^2 t})$.

Previous work studying the seller's endogenous pricing strategy in the presence of buyer gradual learning assumes either that the seller perfectly observes the evolution of the buyer's valuation and

[2] We denote the valuation process as $\{v_s^{t,x}\}_{s\geq t}$ when emphasizing the initial condition $v_t^{t,x} = x$, and as $\{v_t^x\}_{t\geq 0}$ when the initial condition $v_0^x = x$ is specified at $t = 0$. When the initial value is not of central importance, we simply write v_t.

can condition the price on the buyer's current valuation (e.g., Ning, 2021), or that the seller charges a constant price over time (e.g., Branco et al., 2012). Suppose we define the state variable naturally by the buyer's current valuation. The seller's problem in the first scenario is to choose the optimal *stationary strategy*, because the strategy depends only on the current state and not on time. This setup does not always fit real-world examples. Recent privacy regulations have disrupted sellers' ability to track individual buyers in real time. Even if a seller can track buyers' browsing behavior, it may be hard for the seller to know how buyers will interpret the information they see.

Without the ability to observe the evolution of the buyer's valuation and thereby to tailor prices based on v_t, the only stationary pricing strategy is a constant price. Is a stationary pricing strategy always optimal for a seller in such cases? The major innovation of this paper is to consider *non-stationary pricing strategies* – prices that evolve over time without being contingent on a buyer's current valuation. Such pricing strategies endogenously induce non-stationarity in the buyer's search problem. Formally, the seller can commit to a pricing scheme $p := \{p_t\}_{t\geq 0} \in \mathcal{P}$, where $\mathcal{P}$ is the set of *admissible* pricing strategies, which is a subset of smooth functions on $[0, \infty)$, $\mathcal{P} \subset C^{\infty}[0, \infty)$. This pricing strategy is a *non-stationary strategy* because p_t depends explicitly on time.

The buyer's search strategy consists of choosing an appropriate stopping time. Denote by $\mathcal{T}$ the set of all stopping times adapted to $\{\mathscr{F}_t\}_{t\in\mathbb{R}_{\geq 0}}$. The timing of the game is as follows.

1. At $t = 0$, the seller commits to a pricing strategy $p \in \mathcal{P} \subset C^{\infty}[0, \infty)$.
2. At any $t \geq 0$, the buyer decides whether to purchase the product, quit, or search for more information.
3. The game ends when the buyer purchases the product or quits.

The only knowledge the seller has about the buyer is their initial valuation, v_0. Importantly, when the buyer decides whether to purchase the product, quit, or keep searching, she takes into account both the current price and the future price trajectory. For any pricing strategy $p \in \mathcal{P}$ and stopping time $\tau \in \mathcal{T}$, we define the buyer's and seller's expected payoffs as:

$$\mathcal{V}^B(t, x; \tau, p) := \mathbb{E}\left[e^{-r(\tau-t)} \max\{v_\tau^{t,x} - p_\tau, 0\} - \int_t^\tau ce^{-r(s-t)}ds \mid \mathscr{F}_t\right], \tag{2}$$

$$\mathcal{V}^S(x;\tau,p) := \mathbb{E}\left[e^{-m\tau}(p_\tau - g)\cdot 1_{v^x_\tau \geq p_\tau}\right]. \tag{3}$$

A sufficient condition for the buyer's expected payoff (2) to be well-defined is $\sup_{\tau\in\mathcal{T}} \mathbb{E}\left[e^{-2r\tau}v_\tau^2\right] < \infty$. This condition holds as long as the global Lipschitz constant L in Assumption 1 is less than $\sqrt{r}$. For the remainder of this work, we will assume $L < \sqrt{r}$, which is satisfied in many applications such as when v_t is the standard Brownian motion, when v_t is bounded, or when the buyer has a high discount rate.

Commitment Assumption

We assume that the seller has dynamic commitment power. Due to the hold-up problem, this would be a relatively strong assumption if the seller could track the evolution of the buyer's valuation. When new information (buyer's current valuation) arrives, the seller has an incentive to deviate from the pricing scheme announced at the beginning, in order to extract more surplus from the buyer. Anticipating the seller's incentive to deviate, the buyer will not start searching without a commitment device. In such cases, whether the seller has commitment power will lead to qualitatively different results.

In our setting, the seller does not receive new information about the buyer's valuation v_t over time. In addition, the only new information the seller can learn at time t is whether the buyer has made a purchasing decision, which will not affect the seller's strategy because the game ends whenever the buyer purchases the product or exits. The seller does not have any new information during the game. Therefore, there is no hold-up problem, and the seller does not have an incentive to deviate from the announced pricing strategy. Thus, the commitment assumption does not qualitatively affect the equilibrium outcome in this case. We make the assumption mainly for a cleaner analysis and presentation.

Solution Concept

Definition 1. *A subgame perfect ε-equilibrium (ε-SPE) consists of* $(\{\tau^*[p] \in \mathcal{T}\}_{p\in\mathcal{P}}, p^* \in \mathcal{P})$ *such that, for all* $p \in \mathcal{P}$, $\mathcal{V}^B(t,x;\tau^*[p],p) \geq \mathcal{V}^B(t,x;\tau,p) - \varepsilon, \quad \forall \tau \in \mathcal{T}$, *and* $\mathcal{V}^S(x;\tau^*[p^*],p^*) \geq \mathcal{V}^S(x;\tau^*[p],p) - \varepsilon, \quad \forall p \in \mathcal{P}$.

The buyer's value function given the seller's pricing strategy p is:

$$V^B(t,x;p) := \sup_{\tau \in \mathcal{T}} \mathcal{V}^B(t,x;\tau,p). \tag{4}$$

When there is no ambiguity, we will compactly write $V^B(t,x) = V^B(t,x;p)$. Analogously, we define the seller's value function as:

$$V^S(x) := \sup_{p \in \mathcal{P}} \mathcal{V}^S(x;\tau^*[p],p) \tag{5}$$

We work with subgame perfect ε-equilibrium rather than the usual subgame perfect equilibrium for tractability reasons. Specifically, it is challenging to characterize the buyer's search strategy given any non-stationary pricing strategies. To address this difficulty, a key idea of this paper is that, if a pricing strategy p is a small perturbation from a pricing strategy with a known solution, then the solution to p could also be a small perturbation from the known solution. The use of sub-game perfect ε-equilibrium allows us to formalize this idea using perturbation theory to the order of ε. The choice of ε can be very close to zero and does not drive the results.

3 Buyer's Strategy

The buyer faces an optimal stopping problem. When the price is non-stationary, her purchasing and quitting boundaries are also time-contingent. This time-varying property makes her optimal stopping problem challenging, even if we fix a pricing scheme. To illustrate the impact of non-stationary pricing on the buyer's problem, we first review the constant-price benchmark.

3.1 Benchmark: Stationary Pricing

When the price is constant, $p_t = p_0 \in \mathbb{R}$, the buyer's search strategy does not depend on time. In particular, we have a time-independent value function $V^B(t,x;p_0) = V_0^B(x;p_0)$, purchasing threshold $\bar{V}_t = \bar{V}[p_0]$, and quitting threshold $\underline{V}_t = \underline{V}[p_0]$. The value function of the buyer satisfies the Hamilton-Jacobi-Bellman (HJB) equation:

$$\frac{\sigma(x)^2}{2}\partial_x^2 V_0^B - rV_0^B - c = 0, \tag{6}$$

subject to the value-matching condition and smooth pasting conditions:

$$V_0^B(\bar{V}[p_0]; p_0) = \bar{V}[p_0] - p_0, \quad \partial_x V_0^B(\bar{V}[p_0]; p_0) = 1,$$
$$V_0^B(\underline{V}[p_0]; p_0) = 0, \quad \partial_x V_0^B(\underline{V}[p_0]; p_0) = 0.$$

We now consider two commonly used learning structures.

3.1.1 Product attributes learning (constant volatility)

We first consider the learning process studied in Ke et al. (2022), where a buyer gradually learns about the ground-truth value of a product, π_t, which evolves over time according to $d\pi_t = \sigma dW_t^\pi$. The buyer learns about $\{\pi_t\}_{t\geq 0}$ by observing the signal $\{S_t\}_{t\geq 0}$, where $dS_t := \pi_t dt + \sigma_S dW_t$. The constant σ_S represents the information quality of the signal or the amount of attention the buyer pays. In this case, the ground-truth volatility σ and the observation noise σ_S give the variance an asymptote of $\sigma\sigma_S > 0$. Assuming a normal prior belief $\pi_0 \sim \mathcal{N}(v_0, \sigma\sigma_S)$, the variance is constant over time and the valuation process has a constant volatility $dv_t = \frac{\sigma}{\sigma_S}(\pi_t - v_t)dt + \sigma dW_t$, or simply:[3]

$$dv_t = \sigma dW_t^v \tag{7}$$

by Lévy characterization, where $\{W_t^v\}_{t\geq 0}$ is a standard Brownian motion adapted to $\{\mathscr{F}_t\}_{t\geq 0}$. We write W_t^v as W_t hereinafter for convenience. Branco et al. (2012) have characterized the value function:

$$V_0^B(x; p_0) = \frac{c}{r}\left[\cosh\frac{\sqrt{2r}}{\sigma}(x - \underline{V} - p_0) - 1\right], \tag{8}$$

and the purchasing and quitting boundaries $\bar{V}[p_0] := p_0 + \bar{V}$, $\underline{V}[p_0] := p_0 + \underline{V}$, where:

$$\bar{V} := \sqrt{\frac{c^2}{r^2} + \frac{\sigma^2}{2r}} - \frac{c}{r}, \ \underline{V} := \left(\sqrt{\frac{c^2}{r^2} + \frac{\sigma^2}{2r}} - \frac{c}{r}\right) - \frac{\sigma}{\sqrt{2r}} \log\left(\sqrt{\frac{r\sigma^2}{2c^2}} + \sqrt{1 + \frac{r\sigma^2}{2c^2}}\right). \tag{9}$$

[3] An alternative interpretation that gives equivalent formula (Branco et al., 2012) is that a buyer gradually learns about various product attributes to update her belief about the product's value before making a purchase decision. Each attribute i has a ground-truth utility of x_i. The product's total expected utility relative to the outside option (which is normalized to zero), given t searched attributes, is $v_t := \sum_{i=0}^{t} x_i$. When there are an infinite number of attributes, each with a very small weight, v_t becomes a Brownian motion: $dv_t = \sigma dW_t^\pi$, for some constant σ.

3.1.2 Binary classification

Now suppose the product has a ground-truth value, which is a time-independent binary random variable $\pi_t = \pi \in \{0, 1\}$. A Bayesian decision-maker makes a purchase decision by learning whether the product has a *high* value ($\pi = \bar{\pi} = 1$) or a *low* value ($\pi = \underline{\pi} = 0$). Given the initial expectation $v_0 = \mathbb{E}[\pi|\mathscr{F}_0] \in [0, 1]$, the buyer can further learn the value of π by observing the signal $\{S_t\}_{t\geq 0}$, where $dS_t := \pi dt + \sigma_S dW_t$. Then, the valuation $v_t = \mathbb{E}[\pi|\mathscr{F}_t]$ is updated according to:

$$dv_t = \frac{v_t(1 - v_t)}{\sigma_S^2} \left[(\pi - v_t)dt + \sigma_S dW_t\right]. \tag{10}$$

The resulting free-boundary ODE problem has been considered in Ke and Villas-Boas (2019) in the non-discounting case: $r = 0$. With discounting, $r > 0$, the solution is given as follows:

$$V_0^B(x; p_0) = A_+ x^{m_+}(1 - x)^{m_-} + A_- x^{m_-}(1 - x)^{m_+} - \frac{c}{r}, \tag{11}$$

where $m_\pm := \frac{1\pm\sqrt{1+8r\sigma_S^2}}{2}$, $A_\pm := \frac{\bar{V}[p_0](1-\bar{V}[p_0])+(\bar{V}[p_0]-p_0+c/r)(\bar{V}[p_0]-m_\mp)}{(m_\pm-m_\mp)\bar{V}[p_0]^{m_\pm}(1-\bar{V}[p_0])^{m_\mp}}$, and $\bar{V}[p_0]$, $\underline{V}[p_0]$ are determined by:

$$\frac{\bar{V}[p_0](1 - \bar{V}[p_0]) + (\bar{V}[p_0] - p_0 + c/r)(\bar{V}[p_0] - m_\mp)}{\bar{V}[p_0]^{m_\pm}(1 - \bar{V}[p_0])^{m_\mp}} = \frac{(c/r)(\underline{V}[p_0] - m_\mp)}{\underline{V}[p_0]^{m_\pm}(1 - \underline{V}[p_0])^{m_\mp}} \tag{12}$$

3.1.3 Comparison Between Our Problem and the Benchmarks

Comparing the benchmarks with our problem, we can see that stationarity simplifies the problem significantly. In the benchmark model, the buyer's entire optimal stopping strategy can be summarized by **two unknowns**: $\bar{V}[p_0]$ and $\underline{V}[p_0]$. At any time, the buyer will purchase the product if her valuation reaches the constant purchasing threshold and will quit searching if her valuation reaches the constant quitting threshold. In contrast, in our main model, the buyer's optimal stopping strategy consists of **an infinite number of unknowns**. The buyer's decision at any time depends on the current price and the future trajectory of the prices. Knowing that prices change over time, the buyer's purchasing and quitting thresholds also evolve. These time-dependent thresholds significantly complicate our problem.

3.2 Buyer's Strategy under Non-Stationary Pricing

We consider the following set of admissible pricing strategies:

$$\mathcal{P} := \left\{p \in C^{\infty}[0,\infty) \mid p'_t + r(\bar{\pi} - p_t) + c > 0, p_t > \underline{\pi}, \text{ for all } t \in [0,\infty)\right\}.$$

The conditions on p_t and p'_t ensure that it is optimal for a buyer with $v_t = \bar{\pi}$ or $v_t = \underline{\pi}$ to make an immediate purchasing or quitting decision without searching. Note that any constant pricing policy $p_0 \in [\underline{\pi}, \bar{\pi}]$ is contained in $\mathcal{P}$, and that the condition on p'_t also controls the price growth rate, i.e., $\lim_{t\to\infty} e^{-rt}p_t = 0$. If the buyer's valuation is unbounded, $\bar{\pi} = +\infty$ and $\underline{\pi} = -\infty$, then the conditions always hold. In such cases, $\mathcal{P} = C^{\infty}[0,\infty)$. We will also work with the subset $\mathcal{P}_T \subset \mathcal{P}$ of strategies that are constant after some amount of time $T > 0$:

$$\mathcal{P}_T := \{p \in \mathcal{P} \mid p_t = p_T, \forall t \geq T\},$$

The idea is to first establish technical results for a finite $T > 0$, and then to take the limit $T \to \infty$ to establish the result for strategies in $\mathcal{P}$.

Instead of directly finding the optimal stopping time $\tau^*[p] \in \mathcal{T}$ to the optimization problem (4), it is more convenient to characterize the buyer's *time-varying* purchasing and quitting thresholds, $\bar{V}_t[p]$ and $\underline{V}_t[p]$. The following results characterize some of their properties.

Proposition 1. *Let $p \in \mathcal{P}_T$ be a pricing strategy and let $h : \mathbb{R} \to \mathbb{R}_{\geq 0} \in \mathcal{P}_T$ be strictly increasing over $[0,T)$.*

1. *Suppose that $h_0 = 0$; then, at $t = 0$, the purchasing and quitting boundaries under the pricing strategy $\tilde{p} := p + h$ satisfy $\bar{V}_0[\tilde{p}] < \bar{V}_0[p]$, and $\underline{V}_0[\tilde{p}] \geq \underline{V}_0[p]$.*[4]

2. *Let $K \in \mathbb{R}$ be a constant; then, under the pricing strategy $\tilde{p} := p + Kh$, for any given $t \in [0,T)$, $\bar{V}_t[\tilde{p}] \searrow \max\{\tilde{p}_t, \underline{\pi}\}$, $\underline{V}_t[\tilde{p}] \nearrow \min\{\tilde{p}_t, \bar{\pi}\}$, as $K \to +\infty$ and $\bar{V}_t[\tilde{p}] \nearrow \bar{\pi}$, $\underline{V}_t[\tilde{p}] \searrow \underline{\pi}$, as $K \to -\infty$.*

[4] The inequality for the quitting boundary may not be strict. For example, when the search cost is zero, we can have $\underline{V}_t[p] = \underline{\pi}$ for all $p \in \mathcal{P}_T$. The result also implies that, if $\tilde{p} := p + h$ and $h : \mathbb{R} \to \mathbb{R}_{\leq 0} \in \mathcal{P}_T$ is strictly decreasing, then $\bar{V}_0[\tilde{p}] \geq \bar{V}_0[p]$, and $\underline{V}_0[\tilde{p}] < \underline{V}_0[p]$.

Proposition 1 implies that, compared to a constant price, an increasing price path with *the same initial price* changes the buyer's behavior at time zero. Anticipating a higher price in the future, the buyer expects a lower option value from searching and is more inclined to make a quicker decision. Therefore, she has a lower purchasing threshold and a higher quitting threshold at time zero. Similarly, compared to a constant price, a decreasing price path with the same initial price raises the buyer's purchasing threshold and reduces her quitting threshold at time zero, because the buyer expects a higher option value from searching. This observation that a buyer's decision depends not only on the local incentives, but also on the future incentives, plays an important role in shaping the seller's pricing strategy in the next section.

For a given $p \in \mathcal{P}_T$, the purchasing and quitting thresholds, as well as the buyer's value function can be determined by solving the corresponding free-boundary backward parabolic PDE initial-value problem: Find $V : \Omega \to \mathbb{R}$, and continuously differentiable functions $\bar{V}[p], \underline{V}[p] : \mathbb{R} \to [\underline{\pi}, \bar{\pi}]$ satisfying $\bar{V}_t[p] \geq \underline{V}_t[p]$, such that

$$\begin{cases} \dfrac{\sigma(x)^2}{2}\partial_x^2 V(t,x) + \partial_t V(t,x) - rV(t,x) - c = 0, & (t,x) \in \Omega \\ V(t, \bar{V}_t[p]) = \bar{V}_t[p] - p_t, & V(t, \underline{V}_t[p]) = 0, \\ \partial_x V(t, \bar{V}_t[p]) = 1, & \partial_x V(t, \underline{V}_t[p]) = 0, \\ V(T,x) = V_0^B(x; p_T), & \end{cases} \tag{13}$$

where $\Omega := \{(t,x) \in [0,T] \times [\underline{\pi}, \bar{\pi}] \mid \underline{V}_t[p] < x < \bar{V}_t[p]\}$.

This PDE connects us back to the constant price benchmark in §3.1, except that now we have the *moving* purchasing and quitting boundaries $\bar{V}_t[p]$ and $\underline{V}_t[p]$ instead of the constant purchasing and quitting boundaries $\bar{V}[p_0]$ and $\underline{V}[p_0]$. The second and third lines of (13) amount to the value-matching and the smooth-pasting conditions at the purchasing and quitting boundaries, respectively. Lemma 6 in the online Appendix shows that, if V satisfies (13) with the pricing policy $p \in \mathcal{P}_T$, such that $V(t,x) \geq \max\{x - p_t, 0\}$, and $p_t' + r(\bar{V}_t[p] - p_t) + c \geq 0$ for all $(t,x) \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}]$, then V coincides with the buyer's value function: $V^B = V$.

Solving (13) in full generality is beyond the scope of this research. For an arbitrary pricing policy $p \in \mathcal{P}_T$, there may not exist an analytical closed-form solution. However, if p is a small perturbation from a policy with a known solution, then the solution corresponding to p is expected to be a small perturbation from the known solution. So, we use perturbation theory to tackle the challenge of characterizing the buyer's search strategy given the seller's non-stationary pricing strategies.

Suppose we know that the value function $V^B(.,.;p)$ for a given $p \in \mathcal{P}_T$ is a solution to (13), and we would like to compute $V^B(.,.;p+\sqrt{\varepsilon}h)$ for some $h \in \mathcal{P}_T$ and a small $\varepsilon > 0$. The idea of perturbation theory is to proceed by writing $V^B(.,.;p+\sqrt{\varepsilon}h) = V_0(.,.) + V_1(.,.)\sqrt{\varepsilon} + V_2(.,.)\varepsilon + \cdots$, where $V_0(.,.) := V^B(.,.;p)$, and $\bar{V}_t[p+\sqrt{\varepsilon}h] = \bar{V}_{0,t} + \bar{V}_{1,t}\sqrt{\varepsilon} + \bar{V}_{2,t}\varepsilon + \cdots$, $\underline{V}_t[p+\sqrt{\varepsilon}h] = \underline{V}_{0,t} + \underline{V}_{1,t}\sqrt{\varepsilon} + \underline{V}_{2,t}\varepsilon + \cdots$, where $\bar{V}_{0,t} := \bar{V}_t[p], \underline{V}_{0,t} := \underline{V}_t[p]$. By substituting these expansions into (13) and comparing the $\varepsilon^{k/2}$ terms for $k = 1, 2, \cdots$, we can solve for $V_k, \bar{V}_k, \underline{V}_k$ because we know the value of $V_{k'}, \bar{V}_{k'}, \underline{V}_{k'}$ for $k' = 0, \cdots, k-1$. In the online appendix, we argue that such a technique is valid for all sufficiently small $\varepsilon > 0$. We assume this to be the case for the remainder of the work.

We will apply the perturbation technique to solve (13) up to the ε-order, to be consistent with the ε-equilibrium concept.[5] In other words, we stop at $k = 1$ in the process described, and have $V^B(t,x;p+\sqrt{\varepsilon}h) = V^B(t,x;p) + V_1(t,x)\sqrt{\varepsilon} + O(\varepsilon)$. Then, we can take the buyer's learning policy to be given by the corresponding boundaries $\bar{V}^{\varepsilon}_{\leq 1}[p+\sqrt{\varepsilon}h] = \bar{V}[p] + \bar{V}_1\sqrt{\varepsilon}$, and $\underline{V}^{\varepsilon}_{\leq 1}[p+\sqrt{\varepsilon}h] = \underline{V}[p] + \underline{V}_1\sqrt{\varepsilon}$. The following proposition provides a characterization of the $\sqrt{\varepsilon}$-order perturbed boundaries in terms of the zero-th order solution.

Proposition 2. *Let $p \in \mathcal{P}_T$ be a given pricing strategy such that the buyer's value function $V^B(.,.;p)$ is a solution to the PDE (13), which is C^∞-smooth on $\Omega := \{(t,x) \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}] \mid \underline{V}_t[p] < x < \bar{V}_t[p]\}$, with C^∞-smooth corresponding purchasing and quitting boundaries $\bar{V}[p], \underline{V}[p] : \mathbb{R} \to (\underline{\pi}, \bar{\pi})$.*[6] *Let $h \in \mathcal{P}_T$ be arbitrary; then, under the pricing strategy $\tilde{p} := p + \sqrt{\varepsilon}h$, we can find an ε-optimal value function taking the form:*

$$V^B(t,x;\tilde{p}) = V^B(t, x - \sqrt{\varepsilon}h_t; p) + \sqrt{\varepsilon}V_1^B(t,x) + O(\varepsilon), \tag{14}$$

[5] The online appendix provides a more thorough technical discussion of the connection between the problems (4) and (13), as well as further discussion of the validity of the perturbation technique.

[6] We impose the C^∞-smoothness assumptions for simplicity, which are not necessary conditions. The results hold as long as the classical solution to the $\sqrt{\varepsilon}$-order PDE boundary-value problem $V_1(t,x) \in C^{1,2}(\Omega)$ exists.

where $V_1^B(.,.) : \Omega \to \mathbb{R}$ *is given by:*

$$V_1^B(t,x) = -\mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}} h'_s e^{-r(s-t)} \partial_x V^B(s, v_s^{t,x}; p) ds | \mathscr{F}_t\right] \\ + \mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}} h_s e^{-r(s-t)} \sigma(v_s^{t,x}) \sigma'(v_s^{t,x}) \partial_x^2 V^B(s, v_s^{t,x}; p) ds | \mathscr{F}_t\right], \quad (15)$$

where $\tau_\Omega^{t,x} := \inf\{t' \geq t \mid (t', v_{t'}^{t,x}) \notin \Omega\}$ *is the exit time. The* ε*-optimal purchasing and quitting boundaries taking the form:*

$$\begin{aligned} \bar{V}[\tilde{p}] &= (\bar{V}[p] + \sqrt{\varepsilon} h) + \sqrt{\varepsilon} \bar{R} + O(\varepsilon) \\ \underline{V}[\tilde{p}] &= (\underline{V}[p] + \sqrt{\varepsilon} h) + \sqrt{\varepsilon} \underline{R} + O(\varepsilon) \end{aligned} \quad (16)$$

for functions $\bar{R} : \mathbb{R} \to \mathbb{R}$*, and* $\underline{R} : \mathbb{R} \to \mathbb{R}$ *given by:*

$$\bar{R}_t = -\frac{\partial_x V_1^B(t, \bar{V}_t[p])}{\partial_x^2 V^B(t, \bar{V}_t[p]; p)}, \qquad \underline{R}_t = -\frac{\partial_x V_1^B(t, \underline{V}_t[p])}{\partial_x^2 V^B(t, \underline{V}_t[p]; p)} \quad (17)$$

Corollary 1. *Under the setting of Proposition 2, if* $\sigma'(.) = O(\varepsilon)$ *(stable volatility) and* $h := K\tilde{h}$ *for some increasing* $\tilde{h} \in \mathcal{P}_T$ *and a constant* $K \in \mathbb{R} \setminus \{0\}$*, then* $\bar{S}_t := \bar{R}_t / K \leq 0$ *and* $\underline{S}_t := \underline{R}_t / K \geq 0$ *for all* $t \in \mathbb{R}$*.*

The next lemma shows that, for buyers who are not perfectly patient, any changes in price in the far future do not have much effect in the present. This enables us to extend our results for the buyer's ε-optimal response to the sender's pricing strategies from $p \in \mathcal{P}_T$ to an arbitrary $p \in \mathcal{P}$.

Lemma 1. *Let* $p, q \in \mathcal{P}$*; then,* $|V^B(t,x;p) - V^B(t,x;q)| \leq \max_{s \geq t} e^{-r(s-t)} |p_s - q_s|$ *for all* $(t,x) \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}]$*.*

In this paper, we pay special attention to pricing policies that are linear in time. Such linear pricing p does not belong to $\mathcal{P}_T$ for any $T > 0$. However, Lemma 1 indicates that this is not a problem. By choosing a sufficiently large T, an ε-optimal buyer will not differentiate between p and $p^T \in \mathcal{P}_T$. This enables us to utilize the theory we have developed so far for $\mathcal{P}_T$ on linear pricing.

Consideration of linear pricing may seem restrictive. However, the following proposition shows that, for myopic enough ε-optimal buyers, any pricing strategy that is sufficiently slow-moving can be approximated by linear pricing. Intuitively, unless the price changes very drastically in the far future, such as growing super-exponentially, myopic buyers do not look too far into the future, and any differentiable functions *look like* a linear function over any sufficiently short time interval.

Proposition 3. *(Near-optimality of linear price approximation) Let $p \in \mathcal{P}$ be an admissible pricing policy with $\sup_{t\in\mathbb{R}} |p''_t| \leq M$. At any $\mathbf{x} = (t, x) \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}]$, we consider the linear approximation pricing policy $l_{\mathbf{x}} \in \mathcal{P} : s \mapsto l_{\mathbf{x},s} := p_t + p'_t \cdot (s - t)$. Let the buyer's optimal learning strategy given the linear pricing $l_{\mathbf{x}}$ be $\tau^*[l_{\mathbf{x}}] \in \mathcal{T}$. If the buyer is sufficiently myopic: $r > e^{-1}\sqrt{2M/\varepsilon}$, then $\tau^*[l_{\mathbf{x}}]$ is also the buyer's ε-optimal stopping time under the p pricing strategy:*

$$\mathcal{V}^B(t, x; \tau^*[l_{\mathbf{x}}]; p) \geq V^B(t, x; p) - \varepsilon.$$

Proposition 3 indicates that ε-optimal buyers will respond to broader classes of non-linear pricing p as if they were linear under the following assumptions.

Assumption 2. *For a given $\varepsilon > 0$, we assume that the buyer is ε-optimal and sufficiently myopic (i.e., sufficiently large r), and that the seller adjusts the price slowly over time: $|p'_t| = O(\sqrt{\varepsilon})$, such that the conditions for Proposition 3 are satisfied.*

In practice, the above assumption holds if the buyers are relatively impulsive in their purchasing decisions, and if, due to regulation, the seller is restricted on how quickly he can change the price over time. It clarifies when we can apply our results under linear pricing to more general non-linear pricing strategies. For a given $p \in \mathcal{P}_T$ that satisfies the assumption, the buyer will derive the learning strategy from the linear pricing approximation:

$$t \mapsto p_0 + \sqrt{\varepsilon}Kt, \qquad \sqrt{\varepsilon}K := p'_0 = O(\sqrt{\varepsilon}). \tag{18}$$

The following corollary shows that linear perturbation is particularly simple.

Corollary 2. *Consider a linear pricing strategy $p : t \mapsto p_0 + \sqrt{\varepsilon}Kt \in \mathcal{P}$, where p_0 is a constant. Suppose that the constant price buyer's value function $V^B_0(.; p_0)$ is a solution to the PDE (13), which is C^∞-smooth on $\Omega := \{(t, x) \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}] \mid \underline{V}[p_0] < x < \bar{V}[p_0]\}$. Then, we can find an* *ε-optimal value function given by:*

$$V^B(t, x; p) = V_0^B(x - \sqrt{\varepsilon}Kt; p_0) + \sqrt{\varepsilon}V_1^B(t, x) + O(\varepsilon)$$

where $V_1^B(t, x) = V_{1,0}^B(x) + tV_{1,1}^B(x)$ *is linear in* t*, with* $V_{1,1}^B$ *being the unique solution to the ODE boundary-value problem:*

$$\frac{\sigma(x)^2}{2}\partial_x^2 V_{1,1}^B(x) - rV_{1,1}^B(x) + K\sigma(x)\sigma'(x)\partial_x^2 V_0^B(x; p_0) = 0, \quad V_{1,1}^B(\bar{V}[p_0]) = V_{1,1}^B(\underline{V}[p_0]) = 0, \tag{19}$$

and $V_{1,0}^B$ *being the unique solution to the ODE boundary-value problem:*

$$\frac{\sigma(x)^2}{2}\partial_x^2 V_{1,0}^B(x) - rV_{1,0}^B(x) + V_{1,1}^B(x) - K\partial_x V_0^B(x; p_0) = 0, \quad V_{1,0}^B(\bar{V}[p_0]) = V_{1,0}^B(\underline{V}[p_0]) = 0. \tag{20}$$

The ε-optimal purchasing and quitting boundaries: $\bar{V}_t[p] = \bar{V}[p_0] + \sqrt{\varepsilon}Kt + \sqrt{\varepsilon}\bar{R}_t + O(\varepsilon)$*, and* $\underline{V}_t[p] = \underline{V}[p_0] + \sqrt{\varepsilon}Kt + \sqrt{\varepsilon}\underline{R}_t + O(\varepsilon)$*, where*

$$\bar{R}_t = -\frac{\partial_x V_1^B(t, \bar{V}[p_0])}{\partial_x^2 V_0^B(\bar{V}[p_0]; p_0)} =: K\bar{S}_{0,0} + K\bar{S}_{0,1}t, \quad \underline{R}_t = -\frac{\partial_x V_1^B(t, \underline{V}[p_0])}{\partial_x^2 V_0^B(\underline{V}[p_0]; p_0)} =: K\underline{S}_{0,0} + K\underline{S}_{0,1}t \tag{21}$$

are linear in t*, for some constants* $\bar{S}_{0,0} \leq 0, \underline{S}_{0,0} \geq 0$*,* $\bar{S}_{0,1}, \underline{S}_{0,1}$*.*

We revisit the two learning processes considered in §3.1.1 and §3.1.2, obtaining closed-form expressions for the value function under the first learning process and perturbative solutions up to the ε-order under the second learning process.

Solution: Product attributes learning

In this example, the free-boundary PDE (13) can be solved exactly. This is because the probability measure of $\{v_s^{t,x}\}_{s\geq t}$ is x-translation-invariant when $\sigma(.)$ is a constant. Therefore, we can transform the original problem to a simpler problem where the price is fixed at p_0, while the buyer valuation process is a drifted Brownian motion $v_t = -Kt + \sigma W_t$. The transformed problem is stationary in time, with the corresponding HJB: $\frac{\sigma^2}{2}\partial_x^2 V(x) - K\partial_x V(x) - rV(x) - c = 0$. We first solve this HJB equation, and then make an inverse transformation back to the original problem.

Proposition 4. *Consider the product attribute learning process in §3.1.1. Given a linear pricing strategy $p : t \mapsto p_0 + Kt \in \mathcal{P}$, the buyer's value function is given by*

$$V^B(t,x) = A_1 e^{\frac{K-\sqrt{K^2+2r\sigma^2}}{\sigma^2}(x-p_0-Kt)} + A_2 e^{\frac{K+\sqrt{K^2+2r\sigma^2}}{\sigma^2}(x-p_0-Kt)} - \frac{c}{r} \tag{22}$$

with purchasing and quitting boundaries given by

$$\bar{V}_t = p_0 + \bar{V}[K] + Kt, \qquad \underline{V}_t = p_0 + \underline{V}[K] + Kt \tag{23}$$

where the constants $\bar{V}[K]$, $\underline{V}[K], A_1$, and A_2 are determined by boundary conditions.[7] *When the price is slow-moving, $p : t \mapsto p_0 + \sqrt{\varepsilon}Kt$, then we have:*

$$\bar{V}[\sqrt{\varepsilon}K] = \bar{V} + \sqrt{\varepsilon}\bar{R} + O(\varepsilon), \qquad \underline{V}[\sqrt{\varepsilon}K] = \underline{V} + \sqrt{\varepsilon}\underline{R} + O(\varepsilon), \tag{24}$$

where $\bar{V}, \underline{V}$ are given by (9), and

$$\underline{S} := \frac{\underline{R}}{K} = \left(\frac{\bar{V}-\underline{V}}{\sigma^2}\right)\left(\bar{V}+\frac{c}{r}\right) - \frac{1}{2r} = \frac{\sqrt{\frac{c^2}{r^2}+\frac{\sigma^2}{2r}}}{\sigma\sqrt{2r}} \log\left(\sqrt{\frac{r\sigma^2}{2c^2}} + \sqrt{1+\frac{r\sigma^2}{2c^2}}\right) - \frac{1}{2r} > 0,$$

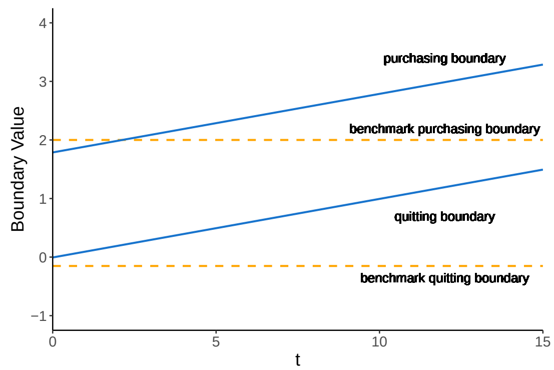
$$\bar{S} := \frac{\bar{R}}{K} = \underline{S} - \frac{1}{2r}\cdot\frac{\bar{V}-\underline{V}}{\bar{V}+c/r} = \frac{1/(\sigma\sqrt{2r})}{\sqrt{\frac{c^2}{r^2}+\frac{\sigma^2}{2r}}}\cdot\frac{c^2}{r^2}\log\left(\sqrt{\frac{r\sigma^2}{2c^2}} + \sqrt{1+\frac{r\sigma^2}{2c^2}}\right) - \frac{1}{2r} < 0.$$

Compared to the constant price benchmark, an increasing pricing scheme with the same initial price has two impacts on the purchasing threshold. On the one hand, the benefit of learning decreases because the buyer will have to pay more in the future if she receives positive information that makes her like the product more. Rationally anticipating this, the buyer has a lower incentive to search and is more inclined to purchase now, which reduces the purchasing threshold (captured by the negative $\sqrt{\varepsilon}\bar{R}$ term). On the other hand, a higher price makes the buyer less willing to purchase, which raises the purchasing threshold (captured by the positive $\sqrt{\varepsilon}Kt$ term). Because the first effect remains stable while the second effect increases over time, the purchasing threshold is lower than the benchmark threshold at the beginning but eventually exceeds the benchmark

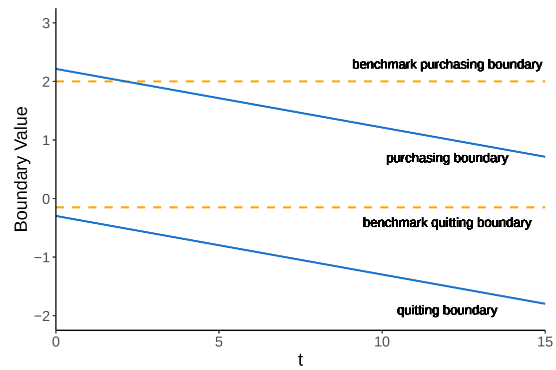
[7] See the system of equations (52) and (55) in the Appendix.

threshold as the price keeps increasing.

An increasing pricing scheme also has two impacts on the quitting threshold. Both a lower benefit of searching and a higher price make it more likely for the buyer to quit. So, the quitting threshold is always higher than the benchmark threshold. We also find that the buyer searches in a narrower region (smaller $\bar{V}_t - \underline{V}_t$) if the price increases rather than staying constant because of the lower benefit of searching. Figure 1a illustrates the purchasing and quitting boundaries in this case.



(a) Increasing price $K = 1$



(b) Decreasing price $K = -1$

Figure 1: Purchasing and quitting boundaries when $c = .2, p = 1, r = .1, \sigma = 1$, and $\epsilon = 0.01$. Benchmark: the boundaries under the optimal constant price.

A decreasing pricing scheme has the opposite impact on the purchasing and quitting thresholds. The purchasing threshold is higher than the benchmark threshold at the beginning because the buyer has a stronger incentive to search and is less inclined to purchase immediately. It eventually falls below the benchmark threshold as the price keeps decreasing. The quitting threshold is always lower than the benchmark threshold because the benefit of both searching and purchasing is higher, and the buyer searches in a broader region. Figure 1b illustrates it.

Solution: Binary classification

In this case, it is not possible to obtain an exact solution, hence we focus on the linear perturbation from a stationary price. With the help of Corollary 2, it is simple to obtain the buyer's value function, as well as the purchasing and quitting boundaries, in terms of the constant price parameters $\bar{V}[p_0]$, $\underline{V}[p_0]$, up to $O(\varepsilon)$, by evaluating the elementary integrals and solving the system of linear equations associated with the boundary conditions.

Proposition 5. *Consider the binary classification learning process in §3.1.2 and a linear pricing strategy* $p : t \mapsto p_0 + \sqrt{\varepsilon}Kt \in \mathcal{P}$. *Let* $\bar{V}[p_0], \underline{V}[p_0] \in (0,1)$ *be the constant price* p_0 *purchasing and quitting boundaries as specified by the solution to (12). Define:* $u_\pm(x) := x^{m_\pm}(1-x)^{m_\mp}$. *There is an* ε*-optimal buyer learning strategy with the value function, purchasing, and quitting boundaries taking the form:* $V^B(t,x) = V_0^B(x - \sqrt{\varepsilon}Kt; p_0) + \sqrt{\varepsilon}V_1^B(t,x) + O(\varepsilon)$, $\bar{V}_t[p] = \bar{V}[p_0] + \sqrt{\varepsilon}Kt + \sqrt{\varepsilon}\bar{R}_t + O(\varepsilon)$, *and* $\underline{V}_t[p] = \underline{V}[p_0] + \sqrt{\varepsilon}Kt + \sqrt{\varepsilon}\underline{R}_t + O(\varepsilon)$, *respectively, where* $V_0^B(x;p_0) = A_+u_+(x) + A_-u_-(x) - \frac{c}{r}$ *is given by (11);* $V_1^B(t,x) := V_{1,0}^B(x) + tV_{1,1}^B(x)$ *where*

$$V_{1,0}^B(x) := \left(B_+ + \frac{2\sigma_S^2}{\sqrt{1+8r\sigma_S^2}} \int \frac{K\partial_x V_0^B(x;p_0) - V_{1,1}^B(x)}{x^{2-m_-}(1-x)^{2-m_+}} dx\right) u_+(x) + \left(B_- - \frac{2\sigma_S^2}{\sqrt{1+8r\sigma_S^2}} \int \frac{K\partial_x V_0^B(x;p_0) - V_{1,1}^B(x)}{x^{2-m_+}(1-x)^{2-m_-}} dx\right) u_-(x), \quad (25)$$

$$V_{1,1}^B(x) := \left(C_+ - \frac{4r\sigma_S^2 K}{\sqrt{1+8r\sigma_S^2}} \int \frac{1-2x}{x^{3-m_-}(1-x)^{3-m_+}} \left(V_0(x) + \frac{c}{r}\right) dx\right) u_+(x) + \left(C_- + \frac{4r\sigma_S^2 K}{\sqrt{1+8r\sigma_S^2}} \int \frac{1-2x}{x^{3-m_+}(1-x)^{3-m_-}} \left(V_0(x) + \frac{c}{r}\right) dx\right) u_-(x), \quad (26)$$

and $\bar{R}_t = K\bar{S}_{0,0} + K\bar{S}_{0,1}t$, $\underline{R}_t = K\underline{S}_{0,0} + K\underline{S}_{0,1}t$ *are given in terms of* $V_0^B(.,.;p_0)$, $V_{1,0}^B(.)$, *and* $V_{1,1}^B(.)$, *as in Corollary 2. The constants* $B_\pm$ *and* $C_\pm$ *are determined by the boundary conditions:* $V_{1,0}^B(\bar{V}[p_0]) = V_{1,0}^B(\underline{V}[p_0]) = 0$ *and* $V_{1,1}^B(\bar{V}[p_0]) = V_{1,1}^B(\underline{V}[p_0]) = 0$.

4 Seller's Strategy

4.1 Seller's Expected Payoff

The expected payoff for a seller implementing the pricing strategy $p \in \mathcal{P}$ is given by $\mathcal{V}^S(x;\tau^*[p],p)$, where $\tau^*[p] \in \mathcal{T}$ denotes the buyer's ε-optimal response to p. We will denote $\mathcal{V}^S(x;\tau^*[p],p)$ by $\mathcal{V}^S(x;p)$ hereinafter for simplicity. To characterize the optimal pricing, we need to compute $\mathcal{V}^S(x;p)$ for a given pricing strategy p.

4.1.1 Constant Price

In the simplest cases of a constant price, the seller's expected payoff can be derived from the properties of martingales.

Lemma 2. *Consider a constant pricing $p = p_0 \in \mathbb{R}$. Suppose that the purchasing and quitting boundaries $\bar{V}[p_0], \underline{V}[p_0] \in (\underline{\pi}, \bar{\pi})$ are finite. For any given $x \in [\underline{V}[p_0], \bar{V}[p_0]]$:*

1. *If $m = 0$, then $\mathcal{V}^S(x;p_0) = (p_0 - g)\frac{x-\underline{V}[p_0]}{\bar{V}[p_0]-\underline{V}[p_0]}$.*
2. *If the volatility is constant: $\sigma(x)^2 = \sigma^2$, then $\mathcal{V}^S(x;p_0) = (p_0 - g)\frac{\sinh\frac{\sqrt{2m}}{\sigma}(x-\underline{V}[p_0])}{\sinh\frac{\sqrt{2m}}{\sigma}(\bar{V}[p_0]-\underline{V}[p_0])}$.*

4.1.2 General Price

For a buyer with initial valuation x, let $\bar{V}[p], \underline{V}[p] : [0,\infty) \to \mathbb{R}$ denote the buyer's stopping boundaries corresponding to the $\tau^*[p]$ learning strategy and let $\Omega := \{(t,v) \in [0,\infty)\times[\underline{\pi},\bar{\pi}] \mid \underline{V}_t[p] < v < \bar{V}_t[p]\}$. We consider $U(s,v;t,x;p)$, the transition probability density of a particle starting from x at time t to some point v at a later time $s \geq t$, as described by the process $\{v_s^{t,x}\}_{s\geq t}$, without leaving the domain Ω. For any fixed $(s,v) \in \Omega$, $U(s,v;t,x;p)$ satisfies the Kolmogorov backward equation with absorbing boundary condition:

$$\begin{cases} \dfrac{\sigma(x)^2}{2}\partial_x^2 U(s,v;t,x;p) + \partial_t U(s,v;t,x;p) = 0, & (t,x) \in \Omega \\ U(s,v;t,\bar{V}_t[p];p) = 0, & U(s,v:t,\underline{V}_t[p];p) = 0 \\ U(s,v;t=s,x;p) = \delta(v-x) & \end{cases}, \tag{27}$$

where $\delta(v-x)$ denotes the Dirac-Delta distribution concentrated at v.

The existence and properties of the solution $U(s,v;t,x;p)$ depend on the smoothness conditions of $\bar{V}[p]$, $\underline{V}[p]$ (Friedman, 2008, Chapter 3). We assume all necessary conditions are satisfied so that the solution $U(s,v;t,x;p) \in C^{1,2}(\Omega)$ exists. The probability flux of the buyer hitting the moving purchasing boundary, and thus getting absorbed, at time s is $-\frac{1}{2}\partial_v\left[\sigma(v)^2 U(s,v;t,x;p)\right]|_{v=\bar{V}_s[p]} - \bar{V}_s'[p]\cdot U(s,\bar{V}_s;t,x;p) = -\frac{1}{2}\partial_v\left[\sigma(v)^2 U(s,v;t,x;p)\right]|_{v=\bar{V}_s[p]}$. In this equation, the term $\bar{V}_s'[p]$ is needed to take into account the boundary movement, which nevertheless vanishes because of the boundary condition: $U(s, \bar{V}_s; t, x; p) = 0$. Hence, for a pricing policy $p \in \mathcal{P}$, the seller's expected payoff from a buyer starting at time t with valuation x is:

$$\mathcal{V}^S(t, x; p) = -\frac{1}{2}\int_t^\infty e^{-m(s-t)}(p_s - g)\partial_v \left[\sigma(v)^2 U(s, v; t, x; p)\right]|_{v=\bar{V}_s[p]} ds, \quad \text{if } x \in (\underline{V}_t[p], \bar{V}_t[p]), \tag{28}$$

and $\mathcal{V}^S(t, x; p) = (p_t - g)1_{x \geq \bar{V}_t[p]}$ otherwise. We simply write $\mathcal{V}^S(x; p) = \mathcal{V}^S(t = 0, x; p)$ when $t = 0$.

If the seller's expected payoff is known to be given by some $\mathcal{V}^S_T(.) : [\underline{V}_T[p], \bar{V}_T[p]] \to \mathbb{R}$, then (28) and (27) imply that $\mathcal{V}^S$ satisfies the following backward parabolic PDE initial boundary value problem:

$$\begin{cases} \frac{\sigma(x)^2}{2}\partial_x^2 \mathcal{V}^S(t, x; p) + \partial_t \mathcal{V}^S(t, x; p) - m\mathcal{V}^S(t, x; p) = 0, & (t, x) \in \Omega \\ \mathcal{V}^S(t, \bar{V}_t[p]; p) = p_t - g, & \mathcal{V}^S_0(t, \underline{V}_t[p]; p) = 0 \text{ ,} \\ \mathcal{V}^S(T, x; p) = \mathcal{V}^S_T(x). \end{cases} \tag{29}$$

4.2 Constant Volatility with Linear Pricing

In this section, we focus on the learning process with constant volatility, as considered in §3.1.1, and focus on linear pricing strategies, $p : t \mapsto p_0 + Kt$. Because the pricing strategy can be characterized by the initial price p_0 and the slope K, we will write $\mathcal{V}^S(x; p_0, K) := \mathcal{V}^S(x; p)$.

Considering linear pricing from the seller's perspective is *not* without loss of generality, but it is sufficient to answer the economically relevant questions of whether a constant price is always optimal when the seller cannot track the evolution of the buyer's valuation, and what would be the profitable direction of a slow-moving price otherwise. In particular, denote by $\hat{p}_0 := \hat{p}_0(x)$ the optimal constant price given the buyer's initial valuation x. By computing $\frac{\partial \mathcal{V}^S}{\partial K}(x; p_0 = \hat{p}_0, K = 0)$, we can determine whether $K^* > 0$, $K^* < 0$, or $K^* = 0$, which characterizes (p_0^*, K^*), the optimal policy in some vicinity of $K = 0$. The above analysis provides normative guidance to a seller initially using optimal constant pricing $\hat{p}_0$ on how it can improve its profit via non-stationary pricing. The discussion of linear pricing also serves as a template for understanding pricing strategies in more general settings, where the space of admissible prices could include non-linear pricing strategies as long as Assumption 2 holds.

We can obtain a complete analytical description of both buyer's search strategy and the seller's expected payoff. The key idea is the following. Proposition 4 has computed the exact expressions for the purchasing and quitting boundaries, which are linear in t. In addition to (27), the transition probability $U(s, v; t, x; p)$ also satisfies the forward equation: $\frac{1}{2}\partial_v^2(\sigma(v)^2 U(s, v; t, x; p)) - \partial_s U(s, v; t, x; p) = 0$ with the initial condition $U(t, v; t, x; p) := \delta(v - x)$ and two absorbing boundaries $\bar{V}_t = \bar{V}_0 + Kt, \underline{V}_t = \underline{V}_0 + Kt$, where $\bar{V}_0 := p_0 + \bar{V}[K], \underline{V}_0 := p_0 + \underline{V}[K]$ are given in Proposition 4. When the learning process has constant volatility, $\sigma(v)^2 = \sigma^2$, the forward equation is simply the heat equation. The series solution to the heat equation with two fixed absorbing boundaries has been well-studied. We can use the Girsanov Theorem to transform such a solution to the solution to the equation with two linearly moving absorbing boundaries. By substituting the transformed solution into (28), we obtain a closed form expression for $\mathcal{V}^S(x; p_0, K)$.

We first consider the case with zero search costs, and then study the more interesting case with positive search costs.

4.2.1 Zero Search Costs

When the buyer has zero search costs, the continuation value of searching is positive, whereas the payoff from quitting is zero. In this case, she would never quit searching without purchasing the product. Therefore, her optimal search strategy is characterized by a single boundary, the purchasing boundary.

If the seller is perfectly patient, a purchase at any time yields the same payoff for him. Hence, he does not have an incentive to increase the price over time to push the buyer to make an early decision. In addition, the buyer's quitting threshold is $-\infty$. So, the seller has no incentive to reduce the price over time to prevent the buyer from quitting. In sum, the seller has little incentive to charge non-stationary prices. The following result shows that the optimal price is arbitrarily close to constant when the seller is perfectly patient. In contrast, the seller may benefit from charging non-stationary prices if he discounts the future.

Proposition 6. *Suppose the search cost is zero, $c = 0$.*

1. *When the seller is perfectly patient ($m = 0$), for any fixed initial price p_0, the seller can approach the profit supremum by choosing $K > 0$ as close to zero as possible, $V^S(x) = \sup_{K \searrow 0} \mathcal{V}^S(x; p_0, K) = 2p_0 + \sigma/\sqrt{2r} - g - x$.*

2. *When the seller discounts the future ($m > 0$) – in particular, when $m >> 0$ or $m \sim 0$ – the slope K of the optimal linear pricing is bounded away from zero.*

4.2.2 Positive Search Costs

When the buyer has a positive search cost, the continuation value of searching may be negative. Hence, both the purchasing and quitting boundaries are finite. We focus on the case where the seller is perfectly patient because, as the previous section suggests, the seller is more inclined to charge non-stationary prices if it discounts the future, and we will show in this section that even a perfectly patient seller will charge non-stationary prices.

Proposition 7. *Suppose the search cost is positive $c > 0$, and the seller is perfectly patient $m = 0$. The seller's expected profit from a buyer with initial valuation x is:*

$$\mathcal{V}^S(x; p_0, K) = \frac{p_0 - g + (\bar{V}_0 + x - 2\underline{V}_0)}{1 - \exp\left(+\frac{2K}{\sigma^2}(\bar{V}_0 - \underline{V}_0)\right)} - \frac{2(\bar{V}_0 - \underline{V}_0)}{\left(1 - \exp\left(+\frac{2K}{\sigma^2}(\bar{V}_0 - \underline{V}_0)\right)\right)^2}$$
$$- \frac{(p_0 - g + (\bar{V}_0 - x)) \exp\left(+\frac{2K}{\sigma^2}(x - \underline{V}_0)\right)}{1 - \exp\left(+\frac{2K}{\sigma^2}(\bar{V}_0 - \underline{V}_0)\right)} + \frac{2(\bar{V}_0 - \underline{V}_0) \exp\left(+\frac{2K}{\sigma^2}(x - \underline{V}_0)\right)}{\left(1 - \exp\left(+\frac{2K}{\sigma^2}(\bar{V}_0 - \underline{V}_0)\right)\right)^2}, \quad (30)$$

if $x \in (\underline{V}_0, \bar{V}_0)$ and $K \neq 0$, and $\mathcal{V}^S(x; p_0, K) = (p_0 - g)(x - \underline{V}_0)/(\bar{V}_0 - \underline{V}_0)$ if $x \in (\underline{V}_0, \bar{V}_0)$ and $K = 0$. $\mathcal{V}^S(x; p_0, K) = 0$ if $x \leq \underline{V}_0$. $\mathcal{V}^S(x; p_0, K) = p_0 - g$ if $x \geq \bar{V}_0$.

By taking the derivative of (30) at $K = 0$, one obtains equation (35), derived in the previous perturbative analysis. Unlike the no-search-cost case, here the slope K^* of the optimal pricing strategy can be bounded away from zero even if $m = 0$. From (30) we can find the optimal initial price $p_0^* = p_0^*(x, K)$ that maximizes $\mathcal{V}^S(x; ., K)$ for any $K \neq 0$ by solving $\frac{\partial \mathcal{V}^S}{\partial p_0}(x; p_0^*, K) = 0$:

$$p_0^*(x, K) := \frac{x + g}{2} + \frac{\sigma^2}{2K} - \frac{\underline{V}[K]}{2}\left(1 - \coth\frac{K}{\sigma^2}\left(\bar{V}[K] - \underline{V}[K]\right)\right)$$
$$- \frac{\bar{V}[K]}{2}\coth\frac{K}{\sigma^2}\left(\bar{V}[K] - \underline{V}[K]\right), \text{ for } x \in (\underline{V}_0, \bar{V}_0). \quad (31)$$

One can verify that $\lim_{K\to 0} p_0^*(x,K) = \hat{p}_0(x) = \frac{x+g-\underline{V}}{2}$ where we recall (Branco et al. (2012)) that

$$\hat{p}_0 = \hat{p}_0(x) = \begin{cases} \frac{x+g-\underline{V}}{2}, & \underline{V}+g < x < 2\bar{V} - \underline{V} + g \\ x - \bar{V}, & x \geq 2\bar{V} - \underline{V} + g \end{cases}, \tag{32}$$

is the optimal stationary price, and $\bar{V}, \underline{V}$ are given by (9). Because (30) is valid for all $K \in \mathbb{R}$, we can find the globally optimal linear pricing strategy $t \mapsto p_0 + Kt$ for each given x by maximizing $\mathcal{V}^S(x; p_0, K)$ over all $(p_0, K) \in \mathbb{R}^2$. The following Lemma further characterizes (p_0^*, K^*) when it lies in the perturbative regime, i.e. when it is known that $K^* \approx 0$.

Lemma 3. *Consider a perfectly patient seller ($m = 0$). Suppose that $r, \sigma, c > 0$; then, there exists $\varepsilon > 0$ sufficiently small such that the seller's profit maximizing linear pricing strategy (p_0^*, K^*) satisfies $|K^*| < \sqrt{\varepsilon}$ then either $p_0^* < \hat{p}_0, K^* \gtrsim 0$ or $p_0^* > \hat{p}_0, K^* \lesssim 0$.*

In order to keep track of $x \in (\underline{V}_0, \bar{V}_0)$, it is convenient to introduce the parameter: $q := \frac{x-\underline{V}-g}{2(\bar{V}-\underline{V})}$, the buyer's initial valuation relative to the purchasing and exiting boundaries under the optimal stationary price $\hat{p}_0$. Figure 2 illustrates the seller's expected payoff and the optimal (p_0^*, K^*) for representative choices of q and other parameters.

When q is not too high (the top-left and top-right of Figure 2), the global maximum (p_0^*, K^*) satisfies $K^* \approx 0$. The optimal strategy with increasing price is always coupled with a lower initial price, and the optimal strategy with decreasing price is always coupled with a higher initial price. This observation is consistent with Proposition 3. On the other hand, when q is sufficiently high, there is no longer a global maximum near $K = 0$ and the seller can achieve the supremum of the payoff using a fast-rising price $K \to +\infty$ (a take-it-or-leave-it offer) to force the high value buyer to make an immediate decision.

From Figure 2 we identify two important classes of non-stationary pricing strategies, namely, the fast-rising and the slow-moving pricing strategies. In §4.3 and §4.4 we will examine how these two classes of strategies apply to buyers under a more general learning process. In particular, we find in §4.3 that for all sufficiently high x the seller can achieve the payoff of $x - g$ which is the supremum over any arbitrary pricing strategies (linear or otherwise) with a fast-rising price. In §4.4 we show that the linear pricing strategies in the perturbative regime for a general class of learning

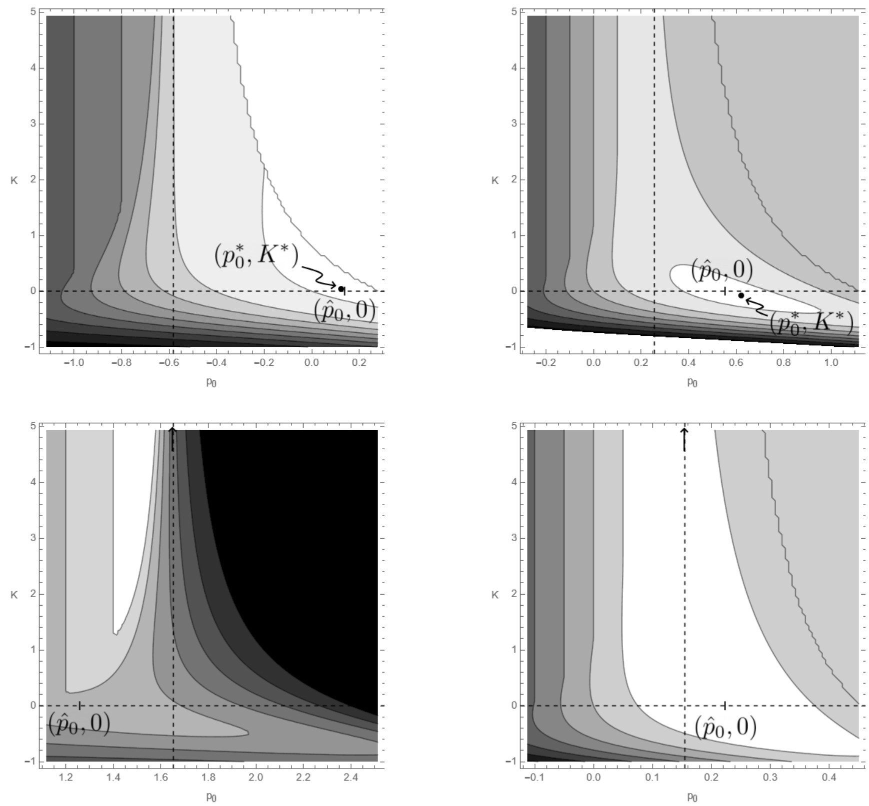


Figure 2: Contour plots of the seller's expected profit from the linear pricing strategy $t \mapsto p_0 + Kt$ when $r = \sigma = 1$. The dashed vertical line illustrates the initial valuation x and $\hat{p}_0$ denotes the optimal constant pricing.
Top-left figure: $q = 0.1$, $c = 0.2$, with the global maximum (p_0^*, K^*) in the perturbative regime where $p_0^* > \hat{p}_0$, $K^* \lesssim 0$. Top-right figure: $q = 0.4$, $c = 0.2$, with the global maximum (p_0^*, K^*) in the perturbative regime where $p_0^* < \hat{p}_0$, $K^* \gtrsim 0$. Bottom-left figure: $q = 0.9$, $c = 0.2$. Bottom-right figure: $q = 0.4$, $c = 0.8$.

processes bear a similar structure to the constant volatility model we consider here.

4.3 Fast-rising Price

In this section, we consider a non-stationary strategy of rapidly increasing the price. Specifically, when the buyer's initial valuation is sufficiently high, it is optimal for the seller to induce an immediate purchase by increasing the price as sharply as possible, which can be interpreted as a take-it-or-leave-it offer. In this case, we can characterize the supremum of the seller's payoff over a very general set of admissible pricing strategies under minimal assumptions. In particular, we neither require the buyer to be myopic nor require the price to be slow-moving, unlike in Assumption 2.

Proposition 8. *Let $h \in \mathcal{P}_T$ be an arbitrary pricing strategy strictly increasing over $[0, T)$ with $h_0 = 0$, and let $p_0 \in \mathbb{R}$ be a constant. Consider the pricing strategy $p = p_0 + Kh$ for $K \in \mathbb{R}$. Then,*

$$\lim_{K \to \infty} \mathcal{V}^S(x; p) = \begin{cases} p_0 - g, & \text{if } x > p_0 \\ 0, & \text{if } x \leq p_0 \end{cases}. \tag{33}$$

Further, suppose that, for all sufficiently high x and any given $\varepsilon > 0$, there exists an ε-optimal pricing strategy $\tilde{p} \in \mathcal{P}_T$ such that $\underline{V}_t[\tilde{p}] \geq g$ for all $t \in [0, \infty)$. Then,

$$V^S(x) = \sup_{p \in \mathcal{P}_T} \mathcal{V}^S(x; \tau^*[p], p) = x - g,$$

which can be approached by the sequence $\{p_n := p_{0,n} + K_n h \in \mathcal{P}_T\}_{n \in \mathbb{Z}_{\geq 0}}$, where $p_{0,n} \nearrow x$ and $K_n \to +\infty$.

If the buyer has positive search costs and constant volatility, then the condition $\underline{V}_t[\tilde{p}] \geq g$ holds as long as the buyer has a sufficiently high initial valuation x. In such cases, Branco et al. (2012) have shown that the optimal profit under stationary pricing is $x - \bar{V} - g$, which is lower than the upper bound $x - g$ attainable by non-stationary pricing. The intuition is that, by charging a static price, the seller may lose a buyer if he raises the price above the optimal constant price because the buyer will prefer to search for information at such prices. By contrast, charging a fast-rising price reduces the buyer's option value of searching and thus induces her to buy immediately at a higher price.

Note that the condition $\underline{V}_t[\tilde{p}] \geq g$ may never hold under some learning processes. For example, it never holds if the buyer has zero search costs and constant volatility because $\underline{V}_t[\tilde{p}] = -\infty$.

4.4 Slow-moving Linear Price

As the previous section shows, the condition for fast-rising pricing strategies to be optimal may not be satisfied. Even if the condition holds, such strategies of rapidly increasing the price may not be feasible due to regulatory or reputational considerations. Motivated by Proposition 3, this section focuses instead on the implementation of slow-moving linear pricing by the seller.

Specifically, we consider the following set of admissible pricing:

$$\mathcal{P}_{lin}^{\varepsilon} := \left\{ t \mapsto p_0 + \sqrt{\varepsilon} K t \mid p_0 \in \mathbb{R}, K \in [-1, +1] \right\} \subset C^{\infty}[0, \infty).$$

We denote the seller's expected payoff by $\mathcal{V}^S(x; p_0, K)$. The seller only needs to determine the optimal $(p_0, K) = (p_0^*, K^*)$. According to Proposition 3, the buyer's ε-optimal learning decision in response to any $p \in \mathcal{P}$ is entirely determined by the value of p_t and its slope p_t' at any given time t under Assumption 2. Because $|p_t'| = O(\sqrt{\varepsilon})$, we are able to utilize our linear perturbation framework. We have provided some justifications for considering linear prices in §4.2. Moreover, suppose that the consumer's valuation diffuses and is absorbed rapidly, or that the set of a more general admissible pricing strategies $\mathcal{P}$ faces some restrictions on the second derivative of its prices, then we can argue that any $p \in \mathcal{P}$ is approximately linear in the foreseeable future concerning the firm's ε-optimal profit. Then, by repeatedly applying the result in this section, the seller should be able to approximate the optimal non-linear price with a piece-wise linear price.

Similar to the reason in §4.2.2, we focus on the case of a perfectly patient seller. We start with the following result on linear perturbation from an arbitrary constant price p_0 under a general learning structure. Then, we discuss the application of this result to our two running examples of learning processes from §3.1.1 and §3.1.2.

Theorem 1. *Consider a linear pricing strategy $p : t \mapsto p_0 + \sqrt{\varepsilon} K t \in \mathcal{P}$ of a perfectly patient seller $(m = 0)$ and the buyer's ε-optimal response according to Corollary 2. Let $\bar{V}[p_0], \underline{V}[p_0] \in (\underline{\pi}, \bar{\pi})$ be the purchasing and quitting boundaries corresponding to the constant price p_0 strategy. For all sufficiently small $\varepsilon > 0$, the seller's $\sqrt{\varepsilon}$-order expected payoff from the buyer with initial valuation $x \in (\underline{V}[p_0], \bar{V}[p_0])$ under the pricing strategy p is given by:*

$$\begin{aligned}
\mathcal{V}^S(x; p_0, K) = \frac{(p_0 - g)(x - \underline{V}[p_0])}{\bar{V}[p_0] - \underline{V}[p_0]} + \sqrt{\varepsilon} K \mathbb{E}\left[\tau_{\Omega}^x \cdot 1\left\{ v_{\tau_{\Omega}^x}^x \geq \bar{V}[p_0] \right\}\right] \\
- \sqrt{\varepsilon} K \cdot \frac{p_0 - g}{\bar{V}[p_0] - \underline{V}[p_0]} \left((1 + \underline{S}_{0,1}) \mathbb{E}\left[\tau_{\Omega}^x\right] + (\bar{S}_{0,1} - \underline{S}_{0,1}) \mathbb{E}\left[\tau_{\Omega}^x \cdot 1\left\{ v_{\tau_{\Omega}^x}^x \geq \bar{V}[p_0] \right\}\right] \right) \\
- \sqrt{\varepsilon} K \cdot \frac{p_0 - g}{\bar{V}[p_0] - \underline{V}[p_0]} \left(\underline{S}_{0,0} + (\bar{S}_{0,0} - \underline{S}_{0,0}) \mathbb{P}\left[v_{\tau_{\Omega}^x}^x \geq \bar{V}[p_0] \right] \right) + O(\varepsilon), \quad (34)
\end{aligned}$$

where $\tau_\Omega^x := \inf\{t \geq 0 \mid (t, v_t^x) \notin \Omega\}$ *is the stopping time,* $\Omega := \{(t,x) \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}] \mid \underline{V}[p_0] < x < \bar{V}[p_0]\}$*.*

The first term of the expression in (34) represents the seller's payoff from a constant price policy p_0. Below, we consider each of the following terms.

Change to the profit per purchase: $+\sqrt{\varepsilon} K \mathbb{E}\left[\tau_\Omega^x \cdot 1\left\{v_{\tau_\Omega^x}^x \geq \bar{V}[p_0]\right\}\right]$

This term is related to the expected change in price at the time of purchase. If the price is increasing over time ($K > 0$), the buyer will pay a price higher than the initial price p_0 if she ends up buying the product. If the price is decreasing over time ($K < 0$), the seller can only extract a lower profit if the buyer buys after searching.

Change to purchase probability due to rescaling of the search interval:

$$-\sqrt{\varepsilon} K \cdot \frac{p_0 - g}{\bar{V}[p_0] - \underline{V}[p_0]} \left(\underline{S}_{0,0} + (\bar{S}_{0,0} - \underline{S}_{0,0}) \mathbb{P}\left[v_{\tau_\Omega^x}^x \geq \bar{V}[p_0]\right]\right)$$

Expecting the price to change over time rather than stay constant, the buyer will adjust the search region. An increasing price trajectory shrinks the search region, whereas a decreasing price trajectory enlarges the search region. This economic force affects the search interval even at time 0 when the initial prices are identical. Consider an increasing price. Observing that $\mathbb{P}\left[v_{\tau_\Omega^x}^x \geq \bar{V}[p_0]\right]$ increases as the buyer's initial valuation becomes higher, and that $\bar{S}_{0,0} - \underline{S}_{0,0} \leq 0$, one can see that an increasing price increases the probability of purchase if the buyer has a high initial valuation, and reduces the probability of purchase if she has a low initial valuation.

Change to purchase probability due to moving boundaries and price:

$$\begin{aligned} &- \sqrt{\varepsilon} K \cdot \frac{p_0 - g}{\bar{V}[p_0] - \underline{V}[p_0]} \left((1 + \underline{S}_{0,1}) \mathbb{E}\left[\tau_\Omega^x\right] + (\bar{S}_{0,1} - \underline{S}_{0,1}) \mathbb{E}\left[\tau_\Omega^x \cdot 1\left\{v_{\tau_\Omega^x}^x \geq \bar{V}[p_0]\right\}\right]\right) \\ = &- \sqrt{\varepsilon} K \cdot \frac{p_0 - g}{\bar{V}[p_0] - \underline{V}[p_0]} \left(\mathbb{E}\left[\tau_\Omega^x\right] + \bar{S}_{0,1} \mathbb{E}\left[\tau_\Omega^x \cdot 1\left\{v_{\tau_\Omega^x}^x \geq \bar{V}[p_0]\right\}\right] + \underline{S}_{0,1} \mathbb{E}\left[\tau_\Omega^x \cdot 1\left\{v_{\tau_\Omega^x}^x \leq \underline{V}[p_0]\right\}\right]\right) \end{aligned}$$

The term $\mathbb{E}[\tau_\Omega^x]$ reflects that the probability of purchase is affected by the expected amount of price change over the entire search process. The buyer is less likely to make a purchase if the price increases over time, and is more likely to make a purchase if the price decreases over time. The term

$\bar{S}_{0,1}\mathbb{E}\left[\tau_{\Omega}^x \cdot 1\left\{v_{\tau_{\Omega}^x}^x \geq \bar{V}[p_0]\right\}\right]$ accounts for the fact that the purchasing boundary would have moved a certain distance by the time a buyer reaches the original purchasing boundary. Analogously, the term $\underline{S}_{0,1}\mathbb{E}\left[\tau_{\Omega}^x \cdot 1\left\{v_{\tau_{\Omega}^x}^x \leq \underline{V}[p_0]\right\}\right]$ captures that the quitting boundary has moved a certain distance by the time a buyer reaches the original quitting boundary.

The derivative $\frac{\partial \mathcal{V}^S}{\partial K}(x; p_0, K = 0)$

Theorem 1 allows us to compute the exact value of $\frac{\partial \mathcal{V}^S}{\partial K}(x; p_0, K = 0)$ at any arbitrary $x \in [\underline{V}[p_0], \bar{V}[p_0]]$ and p_0. For convenience, we denote by $q := \frac{x - \underline{V}[\hat{p}_0]}{\bar{V}[\hat{p}_0] - \underline{V}[\hat{p}_0]}$ the buyer's initial valuation relative to the purchasing and exiting boundaries under the optimal stationary price $\hat{p}_0$. The buyers with q near 1 are those who almost purchase without search, while those with q near 0 almost exit without search, under the optimal stationary price. We can equivalently consider $\frac{\partial \mathcal{V}^S}{\partial K}(x; p_0, K = 0)$ or $\frac{\partial \mathcal{V}^S}{\partial K}(q; p_0, K = 0)$.

Importantly, if $\frac{\partial \mathcal{V}^S}{\partial K}(q; p_0 = \hat{p}_0, K = 0)$ is non-zero for a given $q \in [0, 1]$, then the optimal K^* will be bounded away from 0. The seller can improve its expected profit by setting $K \gtrsim 0$ if $\frac{\partial \mathcal{V}^S}{\partial K}(q; p_0 = \hat{p}, K = 0) > 0$, and by setting $K \lesssim 0$ if $\frac{\partial \mathcal{V}^S}{\partial K}(q; p_0 = \hat{p}, K = 0) < 0$.

For the product attributes learning process in §3.1.1, we can analytically characterize the direction of a slow-moving price that improves the seller's expected profit over the optimal constant price. Substituting the buyer's linear perturbation solution from Proposition 4 into Theorem 1 leads to:

$$\frac{1}{\sqrt{\varepsilon}} \frac{\partial \mathcal{V}^S}{\partial K}(q; p_0 = \hat{p}_0, K = 0) = \frac{(\bar{V} - \underline{V})^2}{3\sigma^2}(1 - 2q)q(1 - q) - (\bar{S}q + \underline{S}(1 - q))q. \tag{35}$$

Because the sign of the above expression depends only on q, c/r, and σ^2/r, in Figure 3, we illustrate the direction of a slow-moving price that improves the seller's expected profit over the optimal constant price strategy as a function of q, c/r, and σ^2/r. Note that crossing the boundary from region IV to region III as q increases in Figure 3 corresponds to moving from the top-left to the top-right figure in Figure 2.

For the learning process of binary classification in §3.1.2, we can only evaluate $\hat{p}_0$ and $\bar{V}[\hat{p}_0]$, $\underline{V}[\hat{p}_0]$ numerically because $\bar{V}[p_0]$ and $\underline{V}[p_0]$ are only implicitly specified through a system of non-linear algebraic equations (12). Figure 4 presents the direction of a slow-moving price that improves the seller's expected profit over the optimal constant price.

Figures 3 and 4 provide managerial recommendations to sellers on how they can improve profits by slowly changing the price from the optimal stationary level. They are qualitatively similar. We classify each plot into four regions.

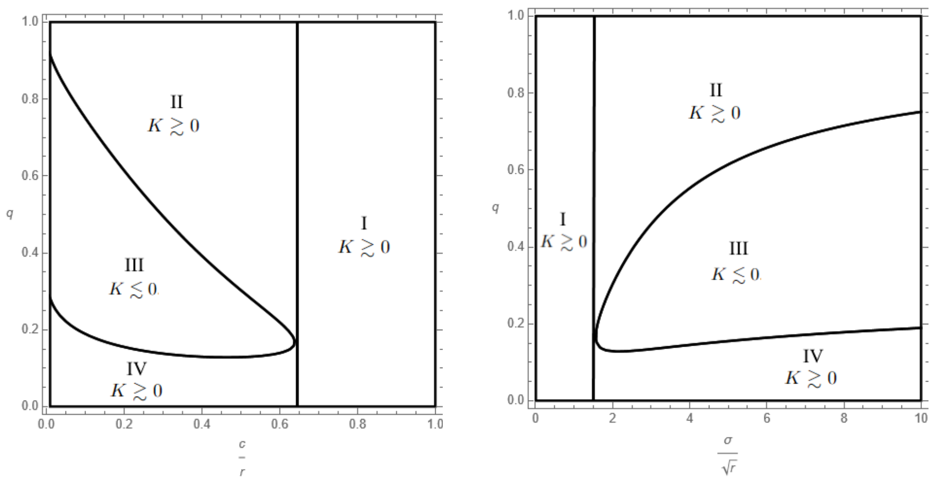


Figure 3: Direction of the price for a perfectly patient seller where the buyer follows the product attributes learning process, $\sigma^2/r = 1$ in the left plot, and $c/r = 1$ in the right plot.

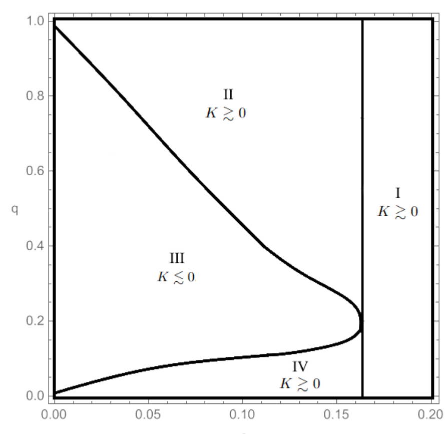


Figure 4: Direction of the price for a perfectly patient seller where the buyer follows the binary classification process, $g = 0.3$, $r = 1$, and $\sigma_S = 1$.

I (Low incentive to search): When the search cost c is too high, the buyer has a low incentive to search for information. The seller needs to give the buyer a high surplus to encourage her to search, which hurts the profit. So, it becomes more attractive for the seller to convince the buyer to purchase the product at the beginning, based on the initial valuation and the price trajectory. For any given initial price, by charging an increasing price over time, the seller lowers the purchasing threshold at the beginning by making it more desirable for the buyer to make an immediate decision. Compared with the stationary pricing strategy of charging a lower constant price, this non-stationary pricing strategy moves the purchasing threshold in the same direction without sacrificing the profit conditional on purchase. In other words, it increases the probability of purchase without reducing the profit per purchase.

II (High-valuation buyer): When the buyer has a high initial valuation, she is too valuable to lose from the seller's perspective. Therefore, the seller wants to increase the purchasing probability in this case. Moreover, a high-valuation buyer can earn a positive payoff from purchasing immediately, which decreases over time because of discounting. Thus, the seller also wants the buyer to make a quick purchase. An increasing pricing strategy reduces the benefits of searching and encourages the buyer to buy quickly and with a higher likelihood.

III (Medium-valuation buyer): When the buyer has a moderate interest in the product, an increase in price does not suffice to convince her to purchase quickly without learning much additional information. Instead, it reduces the benefit of searching because the buyer knows she will have to pay a higher price if she learns positive things. Therefore, an increasing price will lead to a quick exit rather than a quick purchase.

In this case, the seller can benefit from a decreasing price, which helps keep the buyer engaged in the search process, even if she receives some negative information early on. As a result, it increases the purchasing likelihood. Because of the moderate initial valuation, the seller can still obtain a decent profit at a lower price. This pricing strategy protects the seller from missing potentially valuable buyers.

IV (Low-valuation buyer): It is not worthwhile for the seller to reduce the price over time for two reasons. First, the profit from an immediate purchase is already low when the buyer has a low initial valuation. The seller will obtain an even lower profit from an eventual purchase if the buyer searches for a while and eventually buys from the seller at a lower price. Second, due to the low initial valuation, the buyer must accumulate a lot of positive information before purchasing. The purchasing probability will still be low even if the price is slightly reduced over time, and cannot offset the cost of a lower profit per purchase.

In this case, the seller quickly filters out many buyers by implementing an increasing pricing strategy. On the one hand, the loss from not making a deal with these people is limited due to the low profit per purchase and the low purchasing probability. On the other hand, the benefits of charging a higher price to the remaining buyers are high. Any buyers who do not quit despite the increasing price must have learned positive information and thus are more valuable to the seller.

5 Conclusion

We introduce a novel framework where sellers adopt non-stationary pricing strategies. Our findings challenge the conventional reliance on stationary pricing by showing that non-stationary pricing strategies can outperform stationary ones. We view this paper as a significant first step in understanding sellers' non-stationary interventions in the presence of buyer gradual learning.

This paper makes two main contributions. On the one hand, it provides new managerial insights for the firm. The primary goal of marketing is to reduce the cost and increase the return. Using time as the information source to guide pricing decisions is essentially free. Firms do not need to invest heavily in the tracking technology. Hence, all the increased revenue due to non-stationary pricing becomes profit. Such strategies are also immune to privacy regulations. Regulations can prevent firms from tracking consumers' demographic information, browsing behavior, and other characteristics, but cannot ban the use of time to which everyone has access.

On the other hand, it provides a theoretical advance in optimal control by incorporating non-stationary strategies into a buyer search framework. Unlike previous work, the non-stationarity in the buyer's search problem arises endogenously from sellers' strategic pricing in response to buyers' gradual learning. The vast majority of papers in marketing and economics restrict attention to stationary strategies. The most common reason is tractability rather than managerial justifications. Therefore, this restriction may not be without loss of generality and may cost firms "free dollars," as shown in this paper.

Appendix

Proof of Proposition 1. We start with the following intuitive characterization of the buyer's value function $V^B(t, x; p)$.

Lemma 4. *Let $p \in \mathcal{P}_T$ be a pricing strategy.*

1. *$V^B(t, x; p)$ is increasing in x for any fixed t. Moreover, if $V^B(t, x; p) > 0$, then we have a strict inequality: $V^B(t, x'; p) > V^B(t, x; p)$ for any $x' > x$.*

2. *Let $q \in \mathcal{P}_T$ be another pricing strategy such that $q_t \le p_t$ for all $t \in \mathbb{R}$; then, $V^B(t, x; q) \le V^B(t, x; p)$ for all fixed $(t, x) \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}]$. Moreover, if $q_t < p_t$ for all $t > 0$, and $V^B(t, x; q) > 0$ for any fixed $(t, x) \in \mathbb{R}_{\ge 0} \times [\underline{\pi}, \bar{\pi}]$, then we have a strict inequality: $V^B(t, x; q) < V^B(t, x; p)$.*

Proof. Part 1: Consider any $x, x' \in \mathbb{R}$, and suppose that $x' > x$. Let $\{v_s^{t,x}\}_{s \ge t}$ and $\{v_s^{t,x'}\}_{s \ge t}$ be the two strong solutions of the SDE (1), and we have $v_s^{t,x'} > v_s^{t,x}$ a.e., for all $s \ge t$. This can be seen by using the Lipschitz condition in Assumption 1 to analyze the difference process $d_s := v_s^{t,x'} - v_s^{t,x}$. It follows that $\mathcal{V}^B(t, x'; \tau, p) \ge \mathcal{V}^B(t, x; \tau, p)$ for all $\tau \in \mathcal{T}$. For any $\varepsilon > 0$, we can find $\tau_{t,x} \in \mathcal{T}$ such that $\mathcal{V}^B(t, x; \tau_{t,x}, p) \ge V(t, x; p) - \varepsilon$. Then, $V^B(t, x'; p) \ge \mathcal{V}^B(t, x'; \tau_{t,x}, p) \ge \mathcal{V}^B(t, x; \tau_{t,x}, p) \ge V(t, x; p) - \varepsilon$. Because $\varepsilon > 0$ is arbitrarily small, $V^B(t, x'; p) \ge V^B(t, x; p)$ as claimed. Further, if $V^B(t, x; p) > 0$, then either we can find $\tau_{t,x} \in \mathcal{T}$ such that $\mathbb{P}[\tau_{t,x} > t] > 0$ for any given $\varepsilon > 0$, or $V^B(t, x; p) = x - p_t$. In both cases, we have $\mathcal{V}^B(t, x'; \tau_{t,x}, p) > \mathcal{V}^B(t, x; \tau_{t,x}, p)$, which implies the strict inequality: $V^B(t, x'; p) > V^B(t, x; p)$ for any $x' > x$.

Part 2: Equation (2) implies that $\mathcal{V}^B(t, x; \tau, q) \le \mathcal{V}^B(t, x; \tau, p)$ for all $\tau \in \mathcal{T}$. Following similar arguments as in Part I, we get $V^B(t, x; q) \le V^B(t, x; p)$. Further, if $V^B(t, x; q) > 0$ then we can find $\tau_{t,x} \in \mathcal{T}$ such that $V^B(t, x; q) - \varepsilon \le \mathcal{V}^B(t, x; \tau_{t,x}, q)$ for any given $\varepsilon > 0$, and either $\mathbb{P}[\tau_{t,x} > t] > 0$ or $V^B(t, x; q) = x - q_t$. In both cases, we have $\mathcal{V}^B(t, x; \tau_{t,x}, q) < \mathcal{V}^B(t, x; \tau_{t,x}, p)$, proving $V^B(t, x; q) < V^B(t, x; p)$. □

After proving the above lemma, we now go back to prove the proposition.

Part 1: Lemma 4 implies that $V^B(0, x; \tilde{p}) \le V^B(0, x; p)$. In fact, we have $V^B(0, x; \tilde{p}) < V^B(0, x; p)$ at any x such that $V^B(0, x; \tilde{p}) > 0$. Meanwhile, we have $\max\{x - \tilde{p}_0, 0\} = \max\{x - p_0, 0\}$ from the assumption $h_0 = 0$. It follows that $\bar{V}_0[\tilde{p}] = \sup\left\{x \in \mathbb{R} \mid V^B(0, x; \tilde{p}) > \max\{x - p_0, 0\}\right\} < \sup\left\{x \in \mathbb{R} \mid V^B(0, x; p) > \max\{x - p_0, 0\}\right\} = \bar{V}_0[p]$, and similarly $\underline{V}_0[\tilde{p}] = \inf\left\{x \in \mathbb{R} \mid V^B(0, x; \tilde{p}) > \max\{x - p_0, 0\}\right\} \ge \inf\left\{x \in \mathbb{R} \mid V^B(0, x; p) > \max\{x - p_0, 0\}\right\} = \underline{V}_0[p]$, which proves the claim.

Part 2: Without the loss of generality, let's only consider $t = 0$ and h such that $h_0 = 0$, we can always redefine t and shift the x-axis by a constant, otherwise. Let us suppose for a contradiction that there exists $x \in [\underline{\pi}, \bar{\pi}]$ where it is optimal to continue learning for any $K \ge 0$, i.e. $V^B(0, x; \tilde{p}) > 0$ is bounded away from zero for all $K \ge 0$. In other words, for any $\varepsilon, \varepsilon' > 0$, we can choose $\{\tau[K]\}_{K \ge 0} \subset \mathcal{T}$ and $\delta > 0$ such that $\sup_{K \ge 0} \mathbb{E}[1_{\tau[K] < \delta}] < \varepsilon'^2$ and $V^B(0, x; \tilde{p}) \le \mathbb{E}\left[e^{-r\delta \wedge \tau[K]} V^B(\delta \wedge \tau[K], v^x_{\delta \wedge \tau[K]}; \tilde{p}) - \int_0^{\delta \wedge \tau[K]} c e^{-rs} ds\right] + \varepsilon\delta, \quad \forall K \ge 0.$

We can separate the expression above further into two terms corresponding to the events $\tau[K] \geq \delta$ and $\tau[K] < \delta$. Then, using Lemma 4: $V^B(t,x;\tilde{p}) \leq V^B(t,x;p)$ for all $(t,x) \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}]$, and $V^B(\delta, x; \tilde{p}) \leq V^B(\delta, x; p_\delta + Kh_\delta)$, where $p_\delta + Kh_\delta$ denotes a constant pricing policy (i.e. the buyer is better-off if $\tilde{p}_t$ stopped increasing after $t = \delta$), we obtain:

$$
\begin{aligned}
V^B(0,x;\tilde{p}) &\leq \mathbb{E}\left[e^{-r\tau[K]} V^B(\tau[K], v^x_{\tau[K]}; \tilde{p}) \cdot 1_{\tau[K]<\delta}\right] + \mathbb{E}\left[V^B(\delta, v^x_\delta; \tilde{p}) \cdot 1_{\tau[K]\geq\delta}\right] + \varepsilon\delta \\
&\leq \mathbb{E}\left[e^{-r\tau[K]} V^B(\tau[K], v^x_{\tau[K]}; p) \cdot 1_{\tau[K]<\delta}\right] + \mathbb{E}\left[V^B(\delta, v^x_\delta; p_\delta + Kh_\delta) \cdot 1_{\tau[K]\geq\delta}\right] + \varepsilon\delta.
\end{aligned}
$$

We bound the first term using the restriction on the growth-rate of the square-integral of $v^{t,x}_s$ (implied by Assumption 1) and the asymptotically linear condition in x for $V^B(t,x;p)$ (discussed in detail in the online appendix).

$$
\mathbb{E}\left[e^{-r\tau[K]} V^B(\tau[K], v^x_{\tau[K]}; p) \cdot 1_{\tau[K]<\delta}\right] \leq \sup_{\tau\in\mathcal{T}} \mathbb{E}\left[e^{-2r\tau} V^B(\tau, v^x_\tau; p)^2\right]^{1/2} \cdot \sup_{K\geq 0} \mathbb{E}\left[1_{\tau[K]<\delta}\right]^{1/2},
$$

where the first factor is finite, and the second factor is $< \varepsilon'$ by our choice of δ. For the second term, we know from the result on constant pricing policy value function that $V^B(\delta, v^x_\delta; p_\delta + Kh_\delta) = 0$ for all sufficiently large $K > 0$, giving a pointwise convergence of $V^B(\delta, v^x_\delta; p_\delta + Kh_\delta) \cdot 1_{\tau[K]\geq\delta}$ in the probability space. Thus, given any $\varepsilon'' > 0$, we can find a sufficiently large $K > 0$ such that $\mathbb{E}\left[V^B(\delta, v^x_\delta; p_\delta + Kh_\delta) \cdot 1_{\tau[K]\geq\delta}\right] < \varepsilon''$ by the Dominated Convergence Theorem. Overall, we have

$$
V^B(0,x;\tilde{p}) \leq \varepsilon' \cdot \sup_{\tau\in\mathcal{T}} \mathbb{E}\left[e^{-2r\tau} V^B(\tau, v^x_\tau; p)^2\right]^{1/2} + \varepsilon'' + \varepsilon\delta.
$$

Because $\varepsilon, \varepsilon', \varepsilon'' > 0$ are arbitrarily small, we conclude that $V^B(0,x;\tilde{p}) \leq 0$, a contradiction. Therefore, for any $x \in [\underline{\pi}, \bar{\pi}]$, for all sufficiently large $K \geq 0$, either it is optimal to purchase immediately ($x > \tilde{p}_0$), or exit immediately ($x < \tilde{p}_0$). It must be the case that: $\bar{V}_0[\tilde{p}] \searrow \max\{\tilde{p}_0, \underline{\pi}\}, \underline{V}_0[\tilde{p}] \nearrow \min\{\tilde{p}_0, \bar{\pi}\}$ as $K \to +\infty$.

Suppose that $K < 0$, consider any $x \in [\underline{\pi}, \bar{\pi}]$, we note that

$$
V^B(0,x;\tilde{p}) \geq \mathcal{V}^B(0,x;\delta,\tilde{p}) \geq e^{-r\delta}\mathbb{E}\left[\max\left\{v^x_\delta - p_\delta - Kh_\delta, 0\right\}\right] - c\delta \geq -e^{-r\delta}(Kh_\delta + p_\delta) - c\delta
$$

where δ denotes the simple policy of stopping exactly at some time $\delta > 0$ regardless of the valuation, and the first inequality follows from the sub-optimality of δ. So, $V^B(0,x;\tilde{p}) > 0$ for all sufficiently negative K, and it is optimal to continue searching: i.e. $\underline{V}_0[\tilde{p}] < x < \bar{V}_0[\tilde{p}], \forall x \in [\underline{\pi}, \bar{\pi}]$, proving $\bar{V}_0[\tilde{p}] \nearrow \bar{\pi}$ and $\underline{V}_0[\tilde{p}] \searrow \underline{\pi}$ as $K \to -\infty$. □

Proof of Proposition 2. We note that $V^B(t, x - \sqrt{\varepsilon}h_t; p)$ is simply the solution $V^B(t,x;p)$ shifted according to $\sqrt{\varepsilon}Kh$ which satisfies the value-matching and smooth-pasting conditions at $\bar{V}[p] + \sqrt{\varepsilon}h$ and $\underline{V}[p] + \sqrt{\varepsilon}h$, but does not satisfies the PDE, hence the $\sqrt{\varepsilon}V^B_1$ correction is needed. By adding $\sqrt{\varepsilon}V^B_1$ correction, we further need a $\sqrt{\varepsilon}$-order correction to the purchase and quitting boundaries

$\bar{V}[p]+\sqrt{\varepsilon}h$ and $\underline{V}[p]+\sqrt{\varepsilon}h$ which take the form (16). We find the equation for V_1^B by substituting the ansatz (14) into the PDE for $V^B(.,.;\tilde{p})$ and collecting the $\sqrt{\varepsilon}$-order terms:

$$\frac{\sigma(x)^2}{2}\partial_x^2 V_1^B(t,x)+\partial_t V_1^B(t,x)-rV_1^B(t,x)-h_t'\partial_x V^B(t,x;p)+h_t\sigma(x)\sigma'(x)\partial_x^2 V^B(t,x;p)=0. \quad (36)$$

To study $\bar{R}$ and $\underline{R}$ we analyze the boundary conditions of $V^B(.,.;\tilde{p})$ to the first-order in $\sqrt{\varepsilon}$. Note that $V^B(t,x-\sqrt{\varepsilon}h_t;p)$ automatically satisfies the value-matching conditions at $\bar{V}[\tilde{p}]$ and $\underline{V}[\tilde{p}]$, as we will conseller below, because $\partial_x V^B(t,\bar{V}_t[p];p)=1$ and $\partial_x V^B(t,\underline{V}_t[p];p)=0$. By substituting the ansatz (14) and (16) into the boundary conditions and comparing the $\sqrt{\varepsilon}$-order terms, we have:

$$\begin{aligned}
&V^B(t,\bar{V}_t[\tilde{p}];\tilde{p})=\bar{V}_t[\tilde{p}]-\tilde{p}_t\\
&\qquad\qquad\Longrightarrow\; V^B(t,\bar{V}_t[p]+\sqrt{\varepsilon}\bar{R}_t;p)+\sqrt{\varepsilon}V_1^B(t,\bar{V}_t[p])=\bar{V}_t[p]-p_t+\sqrt{\varepsilon}\bar{R}_t\\
&\qquad\qquad\qquad\qquad V_1^B(t,\bar{V}_t[p])=-\bar{R}_t\partial_x V^B(t,\bar{V}_t[p];p)+\bar{R}_t \;\Longrightarrow\; V_1^B(t,\bar{V}_t[p])=0. \quad (37)\\
&\partial_x V^B(t,\bar{V}_t[\tilde{p}];\tilde{p})=1 \;\Longrightarrow\; \partial_x V^B(t,\bar{V}_t[p]+\sqrt{\varepsilon}\bar{R}_t;p)+\sqrt{\varepsilon}\partial_x V_1^B(t,\bar{V}_t[p])=1\\
&\qquad\qquad\qquad\qquad \partial_x V_1^B(t,\bar{V}_t[p])=-\bar{R}_t\partial_x^2 V^B(t,\bar{V}_t[p];p) \;\Longrightarrow\; \bar{R}_t=-\frac{\partial_x V_1^B(t,\bar{V}_t[p])}{\partial_x^2 V^B(t,\bar{V}_t[p];p)}. \quad (38)\\
&V^B(t,\underline{V}_t[\tilde{p}];\tilde{p})=0 \;\Longrightarrow\; V^B(t,\underline{V}_t[p]+\sqrt{\varepsilon}\underline{R}_t;p)+\sqrt{\varepsilon}V_1^B(t,\underline{V}_t[p])=0\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Longrightarrow\; V_1^B(t,\underline{V}_t[p])=0. \quad (39)\\
&\partial_x V^B(t,\underline{V}_t[\tilde{p}];\tilde{p})=0 \;\Longrightarrow\; \partial_x V^B(t,\underline{V}_t[p]+\sqrt{\varepsilon}\underline{R}_t;p)+\sqrt{\varepsilon}\partial_x V_1^B(t,\underline{V}_t[p])=0\\
&\qquad\qquad\qquad\qquad \partial_x V_1^B(t,\underline{V}_t[p])=-\underline{R}_t\partial_x^2 V^B(t,\underline{V}_t[p];p) \;\Longrightarrow\; \underline{R}_t=-\frac{\partial_x V_1^B(t,\underline{V}_t[p])}{\partial_x^2 V^B(t,\underline{V}_t[p];p)}. \quad (40)
\end{aligned}$$

Because $p,h\in\mathcal{P}_T$, they are constant for all $t\geq T$. We have the terminal condition at any $T'\geq T$: $V^B(T',x;\tilde{p})=V_0^B(x;\tilde{p}_T)$ and $V^B(T',x;p)=V_0^B(x;p_T)$, giving the terminal condition for V_1^B:

$$V_1^B(T',x)=V_1^B(T,x)=\frac{1}{\sqrt{\varepsilon}}\left(V_0^B(x;p_T+\sqrt{\varepsilon}h_T)-V_0^B(x-\sqrt{\varepsilon}h_T;p_T)\right)+O(\sqrt{\varepsilon}). \quad (41)$$

We recognize the PDE (36) with (37), (39), and (41) as a backward parabolic (fixed) boundary-value problem. We may transform the problem into the more standard parabolic form for: $\widetilde{V}_1^B(t',x'):=V_1^B\left(T-t',\underline{V}_{T-t'}[p]+(\bar{V}_{T-t'}[p]-\underline{V}_{T-t'}[p])x'\right)$ on $\widetilde{\Omega}:=[0,\infty)\times[0,1]$ with smooth coefficients $(a_{ij}(.),b_i(.),c(.))$, according to our smoothness assumptions on $\bar{V}[p]$, $\underline{V}[p]$, and $\sigma(.)$. Because $-(h)_t'\partial_x V^B(.,.;p)+h_t\sigma(.)\sigma'(.)\partial_x^2 V^B(.,.;p)$ is assumed to be smooth on $\widetilde{\Omega}$, and $V_1^B(T,.)$ is smooth on $\{0\}\times[0,1]$, we can apply (Evans, 2022, Chapter 7.1, Theorem 7) or (Friedman, 2008, Chapter 3, §5, Corollary 2) to conclude the existence of the smooth solution $\widetilde{V}_1^B$ to the parabolic initial boundary-value problem. Transforming back to the original problem, we get the smooth solution $V_1^B(.,.)$. The solution is unique and admits a probabilistic expression via the semi-elliptic version of Feynman-Kac formula (Øksendal, 2003, Theorem 9.1.1):

$$V_1^B(t,x) = \mathbb{E}\left[e^{-r(T'-t)}V_1^B(T', v_{T'}^{t,x})\cdot 1\left\{\tau_\Omega^{t,x} \geq T'\right\}|\mathscr{F}_t\right]$$
$$- \mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}\wedge T'} h_s' e^{-r(s-t)}\partial_x V^B(s, v_s^{t,x}; p)ds|\mathscr{F}_t\right]$$
$$+ \mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}\wedge T'} h_s e^{-r(s-t)}\sigma(v_s^{t,x})\sigma'(v_s^{t,x})\partial_x^2 V^B(s, v_s^{t,x}; p)ds|\mathscr{F}_t\right], \quad (42)$$

The first term is upper-bounded by $\sup_{x\in[\underline{V}_T[p],\bar{V}_T[p]]} V_1^B(T,x)e^{-r(T'-t)} \to 0$ as $T' \to \infty$. Because $p, h \in \mathcal{P}_T$, we have that h_t, $V^B(t,x;p)$, and the boundaries $\bar{V}_t[p], \underline{V}_t[p]$ are constant in t for $t \geq T$. Meanwhile, $v_s^{t,x}$ is bounded inside $[\inf_{s\in[t,T]} \underline{V}_s[p], \sup_{s\in[t,T]} \bar{V}_s[p]]$. Therefore, the third and fourth terms are upper-bounded by some constant (which can be determined by the supremum of the absolute value of the integrand over the compact set $[t,T]\times[\inf_{s\in[t,T]} \underline{V}_s[p], \sup_{s\in[t,T]} \bar{V}_s[p]]$) multiple of $\int_t^\infty e^{-r(s-t)}ds = 1/r < \infty$. Taking the limit $T' \to \infty$ of (42) using the Dominated Convergence Theorem for the right-hand-side, and noting that the left-hand-side is independent of T', we obtain (15). $\square$

Proof of Corollary 1. Suppose that $h := K\tilde{h}$, where $\tilde{h} \in \mathcal{P}_T$ is increasing in t, and that $\sigma'(.) = O(\varepsilon)$. We define $\bar{S} := \bar{R}/K : \mathbb{R} \to \mathbb{R}$, and $\underline{S} := \underline{R}/K : \mathbb{R} \to \mathbb{R}$. It remains to show that $\bar{S}_t \leq 0$ and $\underline{S}_t \geq 0$. From our assumption that $\sigma'(.) = O(\varepsilon)$, we may ignore the third term in the $\sqrt{\varepsilon}$-order equation (15). Moreover, $\sigma'(.) = O(\varepsilon)$ implies $V_0^B(x; p_T + \varepsilon h_T) = V_0^B(x - \varepsilon h_T; p_T) + O(\varepsilon)$, hence we can also ignore the first term in (15). Because $V^B(t,.;p)$ is increasing in x from Lemma 4, it follows from the second term of (15) that $V_1^B(t,x)/K \leq 0$ for any $(t,x) \in \Omega$. In particular, $\partial_x V_1^B(t,\bar{V}_t[p])/K \geq 0$ and $\partial_x V_1^B(t,\underline{V}_t[p])/K \leq 0$.

Now, let us show that $\partial_x^2 V^B(t,\bar{V}_t[p];p) \geq 0$. Let $\mathbf{x}_0 = (t,\bar{V}_t[p]) \in \partial\Omega$ be a point on the purchasing boundary, then we can find sequences $\{\mathbf{x}_i^+ = (t_i, x_i^+)\}_{i=0}^\infty$ and $\{\mathbf{x}_i^- = (t_i, x_i^-)\}_{i=0}^\infty \subset \Omega$ converging to $\mathbf{x}_0$ such that $x_i^- \leq \bar{V}_{t_i}[p] \leq x_i^+$ for all $i \geq 0$. Because $V^B(.,.;p)$ is the viscosity solution, we have $c + rV^B(\mathbf{x}_i^+;p) - \partial_t V^B(\mathbf{x}_i^+;p) - \frac{\sigma(\mathbf{x}_i^+)^2}{2}\partial_x^2 V^B(\mathbf{x}_i^+;p) \geq 0$, while $c + rV^B(\mathbf{x}_i^-;p) - \partial_t V^B(\mathbf{x}_i^-;p) - \frac{\sigma(\mathbf{x}_i^-)^2}{2}\partial_x^2 V^B(\mathbf{x}_i^-;p) = 0$ for all $i \geq 0$. But $V^B(\mathbf{x}_i^+;p) = x_i^+ - p_{t_i}$, so $\frac{\sigma(\mathbf{x}_i^+)^2}{2}\partial_x^2 V^B(\mathbf{x}_i^+;p) = 0$, hence it follows from the continuous differentiability of $V^B(.,.;p)$ across the boundary $\partial\Omega$ that $\partial_x^2 V^B(t,\bar{V}_t[p];p) \geq 0$. Similarly, we have that $\partial_x^2 V^B(t,\underline{V}_t[p];p) \geq 0$.

It follows from (38) and (40) that the sign of $\bar{S}_t$ and $\underline{S}_t$ are opposite to the sign of $\partial_x V_1^B(t,\bar{V}_t[p])/K$ and $\partial_x V_1^B(t,\underline{V}_t[p])/K$, respectively. So, $\bar{S}_t \leq 0$ and $\underline{S}_t \geq 0$. $\square$

Proof of Lemma 1. Consider a fixed $(t,x) \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}]$, and suppose that $V^B(t,x;q) \leq V^B(t,x;p)$. For an arbitrary $\varepsilon > 0$, let $\tau_{t,x,\varepsilon}[p] \in \mathcal{T}$ be such that $\mathcal{V}^B(t,x;\tau_{t,x,\varepsilon}[p],p) \geq V^B(t,x;p) - \varepsilon$, then $V^B(t,x;q) \geq \mathcal{V}^B(t,x;\tau_{t,x,\varepsilon}[p],q) > \mathcal{V}^B(t,x;\tau_{t,x,\varepsilon}[p];p) - \max_{s\geq t} e^{-r(s-t)}|p_s - q_s| \geq V^B(t,x;p) - \max_{s\geq t} e^{-r(s-t)}|p_s - q_s| - \varepsilon$. Because $\varepsilon > 0$ is arbitrary, it must be the case that $V^B(t,x;p) \geq V^B(t,x;q) \geq V^B(t,x;p) - \max_{s\geq t} e^{-r(s-t)}|p_s - q_s|$. If $V^B(t,x;q) \geq V^B(t,x;p)$, then we simply switch the role of p,q and follow through with the above argument, hence we get that $|V^B(t,x;p) - V^B(t,x;q)| \leq \max_{s\geq t} e^{-r(s-t)}|p_s - q_s|$, which proves the result. $\square$

Proof of Proposition 3. Let $p^T, l^T_{\mathbf{x}} \in \mathcal{P}_T$ be given by some pricing strategies which coincide with $p, l_{\mathbf{x}}$ over $[0, T-\varepsilon]$ and constant for all $t \geq T$. By Lemma 1, we have $|V^B(t,x;p^T) - V^B(t,x;l^T_{\mathbf{x}})| \leq \max_{s\in[t,T]} e^{-r(s-t)}|p_s - l_{\mathbf{x},s}| \leq \max_{s\geq t} e^{-r(s-t)}|p_s - l_{\mathbf{x},s}|$. Because this inequality holds for all T, we conclude that $|V^B(t,x;p) - V^B(t,x;l_{\mathbf{x}})| \leq \max_{s\geq t} e^{-r(s-t)}|p_s - l_{\mathbf{x},s}|$. But from Taylor's Theorem, we have $|p_s - l_{\mathbf{x},s}| \leq \frac{M}{2}(s-t)^2$ for all $s \geq t$. It follows that $\max_{s\geq t} e^{-r(s-t)}|p_s - l_{\mathbf{x},s}| \leq \frac{M}{2}\max_{s\geq t}(s-t)^2 e^{-r(s-t)} = \frac{2M}{r^2}e^{-2}$. Therefore, $|V^B(t,x;p) - V^B(t,x;l_{\mathbf{x}})| < \varepsilon$ if $r > e^{-1}\sqrt{2M/\varepsilon}$. □

Proof of Corollary 2. Let p^T be given by p over $[0, T-\varepsilon]$, constant for all $t \geq T$, and some in-between smooth transition for $t \in (T-\varepsilon, T)$. We find the solution $V(.,.;p^T)$ of the free-boundary PDE initial value problem (13) corresponding to $p^T \in \mathcal{P}_T$, which coincides with the value function $V^B(.,.;p^T)$ according to Lemma 6 in the online appendix. Then, for all sufficiently large T,

$$\left|V\left(t,x;p^T\right) - V^B\left(t,x;p\right)\right| = \left|V^B\left(t,x;p^T\right) - V^B\left(t,x;p\right)\right| < \varepsilon \tag{43}$$

for an arbitrarily given $\varepsilon > 0$. This proves that the sequence of the solutions $\left\{V\left(.,.;p^T\right)\right\}_{T\geq 0}$ uniformly converges to the value function $V^B(.,.;p)$ of an infinite horizon pricing strategy p on any compact subset of $\mathbb{R} \times [\underline{\pi}, \bar{\pi}]$.

The existence and uniqueness of the ODE boundary value problems (19) and (20) follows from the standard theory (Agarwal et al., 2008, Lecture 40). In order to make use of Proposition 2, let us first fix a large $T \geq 0$ and consider $p^T = p_0 + \sqrt{\varepsilon}h^T \in \mathcal{P}_T$ where $h^T \in \mathcal{P}_T$ is given by $h^T_t = Kt$ for $t \in [0, T-\varepsilon]$, constant $h^T_t = KT$ for $t \geq T$, and some in-between smooth transition for $t \in (T-\varepsilon, T)$. We shall assume that $|(h^T)'_t| \leq 1$ for $t \in (T-\varepsilon, T)$. From Proposition 2, we have:

$$\begin{aligned} V^B_1(t,x;p^T) = -\mathbb{E}&\left[\int_t^{\tau^{t,x}_\Omega} (h^T)'_s e^{-r(s-t)}\partial_x V^B_0(v^{t,x}_s;p_0)ds|\mathscr{F}_t\right] \\ &+ \mathbb{E}\left[\int_t^{\tau^{t,x}_\Omega} h^T_s e^{-r(s-t)}\sigma(v^{t,x}_s)\sigma'(v^{t,x}_s)\partial^2_x V^B_0(v^{t,x}_s;p_0)ds|\mathscr{F}_t\right], \end{aligned} \tag{44}$$

where $\tau^{t,x}_\Omega := \inf\{t' \geq t \mid (t', v^{t,x}_{t'}) \notin \Omega\}$ is the stopping time. We note that $v^{t,x}_s$ is bounded inside $[\underline{V}[p_0], \bar{V}[p_0]]$, while $\int_0^\infty |h^T_s e^{-r(s-t)}|ds \leq Ke^{rt}/r^2$, and $\int_0^\infty |(h^T)'_s e^{-r(s-t)}|ds \leq Ke^{rt}/r$ by construction for all $T \geq 0$. Therefore, by taking the limit $T \to \infty$ of (44) and using Lemma 1 and inquality (43), we have $V^B_1(t,x;p^T) \to V^B_1(t,x)$. By applying the Dominated Convergence Theorem to the right-hand-side with $h^T_s \to Ks, (h^T)'_s \to K$, we obtain:

$$\begin{aligned} V^B_1(t,x) = -K\cdot\mathbb{E}&\left[\int_t^{\tau^{t,x}_\Omega} e^{-r(s-t)}\partial_x V^B_0(v^{t,x}_s;p_0)ds|\mathscr{F}_t\right] \\ &+ K\cdot\mathbb{E}\left[\int_t^{\tau^{t,x}_\Omega} se^{-r(s-t)}\sigma(v^{t,x}_s)\sigma'(v^{t,x}_s)\partial^2_x V^B_0(v^{t,x}_s;p_0)ds|\mathscr{F}_t\right] \end{aligned} \tag{45}$$

$$\begin{aligned}
\Rightarrow V_1^B(t,x) = &-K\cdot\mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}} e^{-r(s-t)}\partial_x V_0^B(v_s^{t,x};p_0)ds|\mathscr{F}_t\right]\\
&+K\cdot\mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}}(s-t)e^{-r(s-t)}\sigma(v_s^{t,x})\sigma'(v_s^{t,x})\partial_x^2 V_0^B(v_s^{t,x};p_0)ds|\mathscr{F}_t\right]\\
&+Kt\cdot\mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}} e^{-r(s-t)}\sigma(v_s^{t,x})\sigma'(v_s^{t,x})\partial_x^2 V_0^B(v_s^{t,x};p_0)ds|\mathscr{F}_t\right]\\
= &K\cdot\mathbb{E}\left[\int_0^{\tau_\Omega^x} se^{-rs}\sigma(v_s^x)\sigma'(v_s^x)\partial_x^2 V_0^B(v_s^x;p_0)ds|\mathscr{F}_0\right]-K\cdot\mathbb{E}\left[\int_0^{\tau_\Omega^x} e^{-rs}\partial_x V_0^B(v_s^x;p_0)ds|\mathscr{F}_0\right]\\
&+Kt\cdot\mathbb{E}\left[\int_0^{\tau_\Omega^x} e^{-rs}\sigma(v_s^x)\sigma'(v_s^x)\partial_x^2 V_0^B(v_s^x;p_0)ds|\mathscr{F}_0\right] =: V_1^B(0,x)+t\widetilde{V}_{1,1}^B(x). \quad (46)
\end{aligned}$$

The above expression is linear in t and the first two terms $V_1^B(0,x)$ and the factor $\widetilde{V}_{1,1}^B(x)$ of t in the last term are functions of x only. The boundary conditions $V_1^B(0,\bar{V}[p_0]) = V_1^B(0,\underline{V}[p_0]) = 0$ and $\widetilde{V}_{1,1}^B(\bar{V}[p_0]) = \widetilde{V}_{1,1}^B(\underline{V}[p_0]) = 0$ are evident from their probabilistic expression definitions. We recognize the probabilistic expression for $\widetilde{V}_{1,1}^B$ as that of the solution to (19), thus, we have $\widetilde{V}_{1,1}^B = V_{1,1}^B$. Define $V_{1,1}^B(x;\beta) := K\cdot\mathbb{E}\left[\int_0^{\tau_\Omega^x} e^{-\beta s}\sigma(v_s^x)\sigma'(v_s^x)\partial_x^2 V_0^B(v_s^x;p_0)ds\right]$, which satisfies:

$$\frac{\sigma(x)^2}{2}\partial_x^2 V_{1,1}^B(x;\beta) - \beta V_{1,1}^B(x;\beta) + K\sigma(x)\sigma'(x)\partial_x^2 V_0^B(x;p_0) = 0. \quad (47)$$

We can see that $V_{1,1}^B(x) := V_{1,1}^B(x;\beta = r)$, and that the first term of (46) is given by $-\partial_\beta V_{1,1}^B(x;\beta)|_{\beta=r}$. Therefore, $V_1^B(0,x) + \partial_\beta V_{1,1}^B(x;\beta)|_{\beta=r} = -K\cdot\mathbb{E}\left[\int_0^{\tau_\Omega^x} e^{-rs}\partial_x V_0^B(v_s^x;p_0)ds\right]$. Substituting this into the corresponding ODE of the probability expression on the RHS of the above, we get:

$$\begin{aligned}
&\frac{\sigma(x)^2}{2}\partial_x^2\left(V_1^B(0,x)+\partial_\beta V_{1,1}^B(x;\beta)|_{\beta=r}\right) - r\left(V_1^B(0,x)+\partial_\beta V_{1,1}^B(x;\beta)|_{\beta=r}\right) - K\partial_x V_0^B(x;p_0) = 0\\
&\qquad\qquad\implies \frac{\sigma(x)^2}{2}\partial_x^2 V_1^B(0,x) - rV_1^B(0,x) + V_{1,1}^B(x) - K\partial_x V_0^B(x;p_0) = 0,
\end{aligned}$$

where we used $V_{1,1}^B(x) = \frac{\sigma(x)^2}{2}\partial_x^2\partial_\beta V_{1,1}^B(x;\beta)|_{\beta=r} - \beta\partial_\beta V_{1,1}^B(x;\beta)|_{\beta=r}$, which is obtained by differentiating (47) at $\beta = r$. Note that although $V_0^B(x;p_0)$ depends on r, it does not depends on β, hence its β derivative vanishes. So, $V_1^B(0,.)$ satisfies the ODE (20) with the specified boundary conditions, and thus must coincide with $V_{1,0}$.

We can obtain the ε-optimal purchasing and quitting boundaries (21) directly from (17) in Proposition 2. Below, we provide a more careful argument. we have shown that there exists a search strategy under a linear pricing strategy $p: t\mapsto p_0+\sqrt{\varepsilon}Kt$ such that the payoff $V_{\leq 1}^{\varepsilon}(t,x) = V_0^B(x-\sqrt{\varepsilon}Kt;p_0) + \sqrt{\varepsilon}(V_{1,0}^B(x)+tV_{1,1}^B(x))$ is within $O(\varepsilon)$ of the optimal payoff $V^B(t,x;p)$. But then, $V_{\leq 1}^{\varepsilon}(t,x)$ must satisfy the value-matching and smooth-pasting conditions at the corresponding purchase and exit boundaries to the $O(\varepsilon)$ order. Otherwise, an improvement of order $O(\sqrt{\varepsilon})$ can be made by a deviation at the boundaries, which is a contradiction. Following the proof of Proposition 2 (i.e., substituting $V^{\varepsilon}_{\leq 1}$ into (37)-(40)), one can solve for $\bar{R}$ and $\underline{R}$ and obtain (21). $\square$

Proof of Proposition 4. In the special case of linear pricing $t \mapsto p_t := p_0 + Kt$ the value function takes the form (22) over Ω as we can directly check that it satisfies the PDE of (13). Let's define $K_{\pm} := \frac{K \pm \sqrt{K^2+2r\sigma^2}}{\sigma^2}$ for convenience. The purchase and quitting boundaries ansatz take the form (23). We determine the unknown $A_1, A_2, \bar{V}[K]$, and $\underline{V}[K]$ from the boundary conditions

$$V^B(t, \bar{V}_t) = \bar{V}_t - p_t \implies A_1 e^{K_- \bar{V}[K]} + A_2 e^{K_+ \bar{V}[K]} - c/r = \bar{V}[K] \tag{48}$$

$$\partial_x V^B(t, \bar{V}_t) = 1 \implies A_1 K_- e^{K_- \bar{V}[K]} + A_2 K_+ e^{K_+ \bar{V}[K]} = 1 \tag{49}$$

$$V^B(t, \underline{V}_t) = 0 \implies A_1 e^{K_- \underline{V}[K]} + A_2 e^{K_+ \underline{V}[K]} - c/r = 0 \tag{50}$$

$$\partial_x V^B(t, \bar{V}_t) = 0 \implies A_1 K_- e^{K_- \underline{V}[K]} + A_2 K_+ e^{K_+ \underline{V}[K]} = 0 \tag{51}$$

From (50) and (51) we find that

$$A_1 = \frac{c}{r} \left(\frac{K_+}{K_+ - K_-} \right) e^{-K_- \underline{V}[K]}, \quad A_2 = \frac{c}{r} \left(\frac{K_-}{K_- - K_+} \right) e^{-K_+ \underline{V}[K]}. \tag{52}$$

Substituting (52) back into (49), we obtain the equation for $(\bar{V}[K] - \underline{V}[K])$:

$$e^{K_+(\bar{V}[K] - \underline{V}[K])} - e^{K_-(\bar{V}[K] - \underline{V}[K])} = \frac{r}{c} \cdot \frac{K_- - K_+}{K_- K_+}, \tag{53}$$

we note that the LHS is an increasing function, hence the solution always exists. Finally, we find $\bar{V}[K]$ by substituting (52) back into (48) and simplify:

$$\bar{V}[K] = \frac{1}{K_-} + \frac{c}{r} \left(e^{K_+(\bar{V}[K] - \underline{V}[K])} - 1 \right) \tag{54}$$

from this it is simple to find $\underline{V}[\sqrt{\varepsilon} K]$. Equation (53) and (54) is equivalent to the following non-linear system of equations:

$$\begin{cases} e^{\frac{K+\sqrt{K^2+2r\sigma^2}}{\sigma^2}(\bar{V}[K] - \underline{V}[K])} - e^{\frac{K-\sqrt{K^2+2r\sigma^2}}{\sigma^2}(\bar{V}[K] - \underline{V}[K])} & = \dfrac{\sqrt{K^2+2r\sigma^2}}{c} \\ \dfrac{c}{r} \left(e^{\frac{K+\sqrt{K^2+2r\sigma^2}}{\sigma^2}(\bar{V}[K] - \underline{V}[K])} - 1 \right) - \bar{V}[K] & = \dfrac{K + \sqrt{K^2+2r\sigma^2}}{2r} \end{cases} \tag{55}$$

When the price is slow-moving, we replace K with $\sqrt{\varepsilon} K \sim 0$, we may obtain a simple expression for $\bar{V}[\sqrt{\varepsilon} K]$ and $\underline{V}[\sqrt{\varepsilon} K]$ up to the $O(\varepsilon)$ order. We substituting the ansatz (24) into (53), (54), and comparing the zeroth-order and $O(\sqrt{\varepsilon})$ terms we get the claimed expression for $\bar{S} := \bar{R}/K, \underline{S} := \underline{R}/K$. The signs of $\bar{S}$ and $\underline{S}$ followed from the Proposition 2. $\square$

Proof of Proposition 5. The solution (25) and (26) to (20) and (19) can be obtained using standard

ODE solving techniques such as the "variation of parameters". The rest of the results are taken care of by Corollary 2. $\square$

Proof of Lemma 2. Part 1: Assumption 1 implies that $\sigma(x) \geq \underline{\sigma}, \forall x \in [\underline{V}[p_0], \bar{V}[p_0]]$, for some constant $\underline{\sigma} > 0$. Using the Dubins–Schwarz theorem, by computing the survival probability of the standard Brownian motion from the Heat equation series solution, we have

$$\mathbb{P}\left[\tau^*[p_0] > T\right] \leq C \cdot e^{-\frac{\pi^2 \underline{\sigma}^2}{2(\bar{V}[p_0]-\underline{V}[p_0])^2}\cdot T}, \text{ for some constant } C,\ \forall T > 0. \tag{56}$$

This implies that $\mathbb{P}[\tau^*[p_0] < \infty] = 1$ and $\mathbb{E}[\tau^*[p_0]] < \infty$. Therefore, $\mathbb{P}[v^x_{\tau^*[p_0]} \geq \bar{V}[p_0]] + \mathbb{P}[v^x_{\tau^*[p_0]} \leq \bar{V}[p_0]] = 1$. Because $\{v^x_{t\wedge\tau^*[p_0]}\}_{t\geq 0}$ is a uniformly integrable martingale, by the Martingale Stopping Theorem, $x = v^x_0 = \mathbb{E}[v^x_{\tau^*[p_0]}] = \bar{V}[p_0]\mathbb{P}[v^x_{\tau^*[p_0]} \geq \bar{V}[p_0]] + \underline{V}[p_0]\mathbb{P}[v^x_{\tau^*[p_0]} \leq \bar{V}[p_0]]$. If $m = 0$ then $\mathcal{V}^S(x; p_0) = (p_0 - g)\mathbb{P}[v^x_{\tau^*[p_0]} \geq \bar{V}[p_0]]$. One can prove the first part by solving the system of linear equations for $\mathbb{P}[v^x_{\tau^*[p_0]} \geq \bar{V}[p_0]]$.

Part 2: We consider $v_t = \sigma W_t$, where $\{W_t\}_{t\geq 0}$ is a standard Brownian motion. Then, $\mathcal{V}^S(x; p_0) = (p_0 - g)\mathbb{E}[e^{-m\tau^*[p_0]} \cdot 1\{v^x_{\tau^*[p_0]} = \bar{V}[p_0]\}]$, which can be evaluated using the standard technique involving Martingale Stopping Theorem (see Karatzas and Shreve (2012)). $\square$

Proof of Proposition 6. Part 1: From the first equation of (55), when $c \searrow 0$, the RHS becomes large which means $\bar{V}[K] - \underline{V}[K]$ becomes large, and the LHS is $\sim e^{\frac{K+\sqrt{K^2+2r\sigma^2}}{\sigma^2}(\bar{V}[K]-\underline{V}[K])}$. Therefore, the second equation of (55) together with (23) gives: $\bar{V}_t = p_0 + Kt + \frac{\sqrt{K^2+2r\sigma^2}-K}{2r}$ and $\underline{V}_t = -\infty$. Therefore, we only have one linearly moving boundary $\bar{V}_t$. Let's assume throughout also that $p_0 \geq g$. The solution $U(t, v) := U(t, v; t = 0, x; p)$ to the heat equation with the single linearly moving absorbing boundary with initial condition $U(t = 0, v) = \delta(v - x), x \leq \bar{V}_0$, is well-known: $U(t, v) = \frac{\exp[-\frac{K}{\sigma^2}(v-x-Kt)-\frac{K^2}{2\sigma^2}t]}{\sigma\sqrt{2\pi t}}[e^{-\frac{(v-Kt-x)^2}{2t\sigma^2}} - e^{-\frac{(v-Kt+x-2\bar{V}_0)^2}{2t\sigma^2}}]$. Therefore, the purchase probability flux is $-\frac{\sigma^2}{2}\partial_v U(t, \bar{V}_t) = \frac{\bar{V}_0-x}{\sigma\sqrt{2\pi t^3}}\exp[-\frac{(\bar{V}_t-x)^2}{2t\sigma^2}]$.

It is now straightforward to compute the expected seller's payoff at $t = 0$:

$$\begin{aligned}
\mathcal{V}^S(x; p_0, K) &:= -\frac{\sigma^2}{2}\int_0^\infty e^{-ms}(p_s - g)\partial_v U(s, \bar{V}_s)ds \\
&= \left(p_0 - g + \frac{K}{\sqrt{2m\sigma^2 + K^2}}\left(p_0 - x + \frac{\sqrt{K^2 + 2r\sigma^2} - K}{2r}\right)\right) \\
&\qquad \times \exp\left(-\left(\frac{K + \sqrt{2m\sigma^2 + K^2}}{\sigma^2}\right)\left(p_0 - x + \frac{\sqrt{K^2 + 2r\sigma^2} - K}{2r}\right)\right),
\end{aligned} \tag{57}$$

for $x \leq \bar{V}_0$, and $\mathcal{V}^S(x; p_0, K) = p_0 - g$ if $x > \bar{V}_0$.

When $m = 0$, we have $\mathcal{V}^S(x; p_0, K) = (2p_0 - g - x + \frac{\sqrt{K^2+2r\sigma^2}-K}{2r})\exp[-\frac{2K}{\sigma^2}(p_0 - x + \frac{\sqrt{K^2+2r\sigma^2}-K}{2r})]$, if $K > 0$, $\mathcal{V}^S(x; p_0, K) = p_0 - g$ if $K = 0$, and $\mathcal{V}^S(x; p_0, K) = x - g - \frac{\sqrt{K^2+2r\sigma^2}-K}{2r}$ if $K < 0$. For any fixed p_0, we can approach the supremum $2p_0 + \frac{\sigma}{\sqrt{2r}} - g - x \geq p_0 - g$ of $\mathcal{V}^S$ by choosing $K \gtrsim 0$

as close to 0 as possible.

Part 2: For $m > 0$, the optimal K is now bounded from 0. This is because

$$\begin{aligned}&\frac{\partial \mathcal{V}^S}{\partial K}(x; p_0, K = 0)\\ =&e^{-\frac{\sqrt{2m}}{\sigma}\left(p_0 - x + \frac{\sigma}{\sqrt{2r}}\right)} \cdot \left[\frac{p_0 - x + \sigma/\sqrt{2r}}{\sigma\sqrt{2m}} - (p_0 - g)\left(\frac{p_0 - x + \frac{\sigma}{\sqrt{2r}} - \frac{\sigma}{r}\sqrt{m/2}}{\sigma^2}\right)\right],\end{aligned}$$

which is strictly positive for sufficiently small and sufficiently large $m > 0$. □

Proof of Proposition 7. The standard solution U_0 to the heat equation with 2 absorbing non-moving boundaries at $\bar{V}_0 := p_0 + \bar{V}[K]$, $\underline{V}_0 := p_0 + \underline{V}[K]$, and the initial condition $U_0(0, v) = \delta(v - x)$ is given by Karatzas and Shreve (2012):

$$U_0(t, v) = \frac{1}{\sigma\sqrt{2\pi t}} \sum_{k=-\infty}^{+\infty} \left[e^{-\frac{(v - x + 2k(\bar{V}_0 - \underline{V}_0))^2}{2t\sigma^2}} - e^{-\frac{(v + x - 2\underline{V}_0 + 2k(\bar{V}_0 - \underline{V}_0))^2}{2t\sigma^2}}\right]. \tag{58}$$

By the standard application of Girsanov Theorem, if $\{W_t\}_{t=0}^{\infty}$ is the standard Brownian process on $(\Omega, \mathscr{F}, \Sigma, \mathbb{P})$ then $\{x + \sigma W_t\}_{t=0}^{\infty}$ is the Brownian process with drift starting at x on $(\Omega, \mathscr{F}, \Sigma, \mathbb{Q})$ where $\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathscr{F}_t} = \exp\left(-\frac{K}{\sigma}W_t - \frac{K^2}{2\sigma^2}t\right)$. Consequently, we have that the solution U to the heat equation with moving boundaries $\bar{V}_t, \underline{V}_t$ are given by: $U(t, v)dv = \exp[-\frac{K}{\sigma^2}(v - x - Kt) - \frac{K^2}{2\sigma^2}t]U_0(t, v - Kt)dv$. So, the purchase probability flux is:

$$-\frac{\sigma^2}{2}\partial_v U(t, \bar{V}_t) = \sum_{k=-\infty}^{+\infty} \frac{(2k+1)(\bar{V}_0 - \underline{V}_0) - (x - \underline{V}_0)}{\sigma\sqrt{2\pi t^3}} e^{\frac{2Kk}{\sigma^2}(\bar{V}_0 - \underline{V}_0)} e^{-\frac{((2k+1)(\bar{V}_0 - \underline{V}_0) - (x - \underline{V}_0) + Kt)^2}{2t\sigma^2}}. \tag{59}$$

The term-by-term differentiation is justified at $v = \bar{V}_t$ for any fixed $x \in (\underline{V}_0, \bar{V}_0)$ because $0 < |\bar{V}_0 - x| < |\bar{V}_0 - \underline{V}_0|$, hence the series representation of $U_0(t, v - Kt)$, and the derivative series both converge absolutely and uniformly for all v in some neighborhoods of $\bar{V}_t$ and $t \in [0, \infty)$. We now compute the seller's expected profit.

Claim 1. *The seller's expected profit from the buyer initially at $x \in (\underline{V}_0, \bar{V}_0)$ is:*

$$\begin{aligned}\mathcal{V}^S(x; p_0, K) = &\frac{\left(p_0 - g - \frac{K}{\sqrt{2m\sigma^2 + K^2}}(\bar{V}_0 + x - 2\underline{V}_0)\right) e^{-\frac{K + \sqrt{2m\sigma^2 + K^2}}{\sigma^2}(\bar{V}_0 - x)}}{1 - e^{-\frac{2\sqrt{2m\sigma^2 + K^2}}{\sigma^2}(\bar{V}_0 - \underline{V}_0)}}\\ &+ \frac{\frac{2K}{\sqrt{2m\sigma^2 + K^2}}(\bar{V}_0 - \underline{V}_0)e^{-\frac{K + \sqrt{2m\sigma^2 + K^2}}{\sigma^2}(\bar{V}_0 - x)}}{\left(1 - e^{-\frac{2\sqrt{2m\sigma^2 + K^2}}{\sigma^2}(\bar{V}_0 - \underline{V}_0)}\right)^2}\\ &- \frac{\left(p_0 - g - \frac{K}{\sqrt{2m\sigma^2 + K^2}}(\bar{V}_0 - x)\right) e^{-\frac{K + \sqrt{2m\sigma^2 + K^2}}{\sigma^2}(\bar{V}_0 - \underline{V}_0)} e^{+\frac{K - \sqrt{2m\sigma^2 + K^2}}{\sigma^2}(x - \underline{V}_0)}}{1 - e^{-\frac{2\sqrt{2m\sigma^2 + K^2}}{\sigma^2}(\bar{V}_0 - \underline{V}_0)}}\end{aligned}$$

$$
-\frac{\frac{2K}{\sqrt{2m\sigma^2+K^2}}(\bar{V}_0-\underline{V}_0)e^{-\frac{K+\sqrt{2m\sigma^2+K^2}}{\sigma^2}(\bar{V}_0-\underline{V}_0)}e^{+\frac{K-\sqrt{2m\sigma^2+K^2}}{\sigma^2}(x-\underline{V}_0)}}{\left(1-e^{-\frac{2\sqrt{2m\sigma^2+K^2}}{\sigma^2}(\bar{V}_0-\underline{V}_0)}\right)^2}, \quad (60)
$$

if $m > 0$ *or* $K \neq 0$*, and* $\mathcal{V}^S(x;p_0,K) = (p_0-g)\left(\frac{x-\underline{V}_0}{\bar{V}_0-\underline{V}_0}\right)$ *if* $m = 0, K = 0$*. On the other hand, if* $x \leq \underline{V}_0$ *then* $\mathcal{V}^S(x;p_0,K) = 0$*, and if* $x \geq \bar{V}_0$ *then* $\mathcal{V}^s(x;p_0,K) = p_0 - g$*.*

Proof. Let us only consider the non-trivial case when $x \in (\underline{V}_0, \bar{V}_0)$. The result when $m = 0, K = 0$ has been covered by Lemma 2. For the cases where $m > 0$ or $K \neq 0$, we compute $\mathcal{V}^S(x;p_0,K)$ by substituting (59) into (28), after switching the order of summation and integration, which can be justified by Fubini's theorem when $m > 0$ or $K \neq 0$, the resulting infinite series is a standard geometric series which can easily be evaluated to gives (60). □

In the limit $\underline{V}_0 \to -\infty$, (60) reduces to (57) we previously studied. Unlike in the single boundary case, in the presence of the quitting boundary, the expected seller's profit is not only continuous at $K = 0$, but also differentiable, even when $m = 0$, as we will show below. Consider the case of $m = 0$. According to (60), $\mathcal{V}^S(x;p_0,K<0)|_{m=0}$ is given by (30), and:

$$
\begin{aligned}
&\mathcal{V}^S(x;p_0,K>0)|_{m=0} = \\
&\qquad \frac{(p_0-g-(\bar{V}_0+x-2\underline{V}_0))\exp\left(-\frac{2K}{\sigma^2}(\bar{V}_0-x)\right)}{1-\exp\left(-\frac{2K}{\sigma^2}(\bar{V}_0-\underline{V}_0)\right)} + \frac{2(\bar{V}_0-\underline{V}_0)\exp\left(-\frac{2K}{\sigma^2}(\bar{V}_0-x)\right)}{\left(1-\exp\left(-\frac{2K}{\sigma^2}(\bar{V}_0-\underline{V}_0)\right)\right)^2} \\
&\qquad - \frac{(p_0-g-(\bar{V}_0-x))\exp\left(-\frac{2K}{\sigma^2}(\bar{V}_0-\underline{V}_0)\right)}{1-\exp\left(-\frac{2K}{\sigma^2}(\bar{V}_0-\underline{V}_0)\right)} - \frac{2(\bar{V}_0-\underline{V}_0)\exp\left(-\frac{2K}{\sigma^2}(\bar{V}_0-\underline{V}_0)\right)}{\left(1-\exp\left(-\frac{2K}{\sigma^2}(\bar{V}_0-\underline{V}_0)\right)\right)^2}. \quad (61)
\end{aligned}
$$

Both (30) and (61) are valid expressions for all $K \neq 0$, and with some works, we can show them to be equal for all $K \neq 0$. This proves $\mathcal{V}^S(x;p_0,K)$ is given by (30) for all $K \neq 0$. □

Proof of Lemma 3. One can see that $\frac{\partial p_0^*}{\partial K}(x, K = 0) = \frac{1}{12r\sigma}[3\sigma - 3\sqrt{\frac{2c^2}{r}+\sigma^2}\sinh^{-1}\sqrt{\frac{r\sigma^2}{2c^2}} - \sigma(\sinh^{-1}\sqrt{\frac{r\sigma^2}{2c^2}})^2] \leq 0$, where the inequality is strict everywhere except when $r\sigma^2/c^2 = 0$. Given that $r\sigma^2/c^2 > 0$, we can find a sufficiently small $\varepsilon > 0$ such that $p_0^*(x,.)$ is a decreasing function for $K \in [-1,+1]$. Any local maximum point of $\mathcal{V}^S(x;.,.)$ would take the form $(p_0^*(x;K^*),K^*)$ where $K^* := \arg\max_K \mathcal{V}^S(x;p_0^*(x,K),K)$. Hence, for all sufficiently small $\varepsilon > 0$, (p_0^*, K^*) in $\mathcal{P}_{lin}^{\varepsilon}$ either satisfies $p_0^* < \hat{p}_0, K^* \gtrsim 0$, or $p_0^* > \hat{p}_0, K^* \lesssim 0$. □

Proof of Proposition 8. Equation (33) in the proposition follows from an application of Proposition 1 to the pricing strategy $p := p_0 + Kh$. It follows that as $K \to \infty$, the corresponding purchase and quitting boundaries $\bar{V}_t[p_0+Kh]$ and $\underline{V}_t[p_0+Kh]$ will monotonically decrease and increase toward $p_0 + Kh_t$, respectively. If $x > p_0$ then only the purchasing boundary $\bar{V}_0[p_0+Kh]$ will reach x as $K \to \infty$, giving the seller the payoff $p_0 - g$. Likewise, for $x \leq p_0$ only $\underline{V}_0[p_0+Kh]$ will reach x as $K \to \infty$ giving the seller the payoff 0.

Further, suppose that x is sufficiently high such that we can find a seller's ε-optimal pricing strategy $\tilde{p} \in \mathcal{P}_T$ satisfying $\underline{V}_t[\tilde{p}] > g$ for all $t \in [0, \infty)$. Let $\tau^*[\tilde{p}] \in \mathcal{T}$ denotes the corresponding ε-optimal buyer's stopping time to the pricing strategy $\tilde{p}$. It follows that

$$\begin{aligned}\mathcal{V}^S(x; \tau^*[\tilde{p}], \tilde{p}) &= \mathbb{E}\left[e^{-m\tau^*[\tilde{p}]}(p_{\tau^*[\tilde{p}]} - g) \cdot 1_{v_{\tau^*[\tilde{p}]} \geq \tilde{p}_{\tau^*[\tilde{p}]}} \mid v_0 = x\right] \\ &\leq \mathbb{E}\left[\left(\tilde{p}_{\tau^*[\tilde{p}]} - g\right) \cdot 1_{v_{\tau^*[\tilde{p}]} \geq \tilde{p}_{\tau^*[\tilde{p}]}} \mid v_0 = x\right] \leq \mathbb{E}\left[v_{\tau^*[\tilde{p}]} - g \mid v_0 = x\right] = x - g. \quad (62)\end{aligned}$$

The first inequality follows from removing the discounting factor. The second inequality follows by noting that if v_t hits the purchasing boundary $\bar{V}_t[\tilde{p}]$ first we would have $v_{\tau^*[\tilde{p}]} - g \geq \tilde{p}_{\tau^*[p]} - g$, and if v_t hits the quitting boundary first we would have $v_{\tau^*[\tilde{p}]} < \tilde{p}_{\tau^*[\tilde{p}]}$, so $v_{\tau^*[\tilde{p}]} - g = \underline{V}_{\tau^*[\tilde{p}]}[\tilde{p}] - g \geq 0 = (p_{\tau^*[\tilde{p}]} - g) \cdot 1_{v_{\tau^*[\tilde{p}]} \geq \tilde{p}_{\tau^*[\tilde{p}]}}$. The final equality followed from the Martingale stopping theorem because $\left|v_{t \wedge \tau^*[\tilde{p}]}\right|$ is bounded by $\max_{s \in [0,\infty)}\{|\underline{V}_s[\tilde{p}]|, |\bar{V}_s[\tilde{p}]|\} = \max_{s \in [0,T]}\{|\underline{V}_s[\tilde{p}]|, |\bar{V}_s[\tilde{p}]|\}$ where the latter is finite because both boundaries are continuous over $[0, T]$ and are constant over $[T, \infty)$ by the definition of $\mathcal{P}_T$. So $x - g \geq V^S(x) - \varepsilon$ for any arbitrary $\varepsilon > 0$, hence we conclude that $V^S(x) = x - g$. The claim that this supremum can be approached by the sequence $\{p_n := p_{0,n} + K_n h \in \mathcal{P}_T\}_{n \in \mathbb{Z}_{\geq 0}}$ follows from (33). □

Proof of Theorem 1. Let us first fix a large $T \geq 0$, and let us assume that $\mathcal{V}^S_T(.) := \mathcal{V}^S(T, .; p) : [\underline{V}_T[p], \bar{V}_T[p]] \to \mathbb{R}$ is known and can be used as the terminal condition. We would like to solve the PDE initial boundary value problem (29) up to the ε-order. The idea is similar to the proof of Proposition 2 but simpler Because the boundaries $\bar{V}_t[p] = \bar{V}[p_0] + \sqrt{\varepsilon}Kt + \sqrt{\varepsilon}\bar{R}_t$, and $\underline{V}_t[p] = \underline{V}[p_0] + \sqrt{\varepsilon}Kt + \sqrt{\varepsilon}\underline{R}_t$ are already determined for us by Corollary 2. The claim is that if $\mathcal{V}^S(., .; p)$ solves (29) exactly for the given $\bar{V}[p], \underline{V}[p]$, and if $\mathcal{V}^\varepsilon_{\leq k}(., .; p)$ solves (29) up to the $\varepsilon^{(k+1)/2}$-order with the same given $\bar{V}[p], \underline{V}[p]$, then by comparing their corresponding Feynman-Kac expressions, we have $\mathcal{V}^S = \mathcal{V}^\varepsilon_{\leq k} + O(\varepsilon^{(k+1)/2})$. We omit further details, and proceed with $k = 1$ to obtain the seller's expected payoff up to the ε-order.

Consider $p := p_0 + \sqrt{\varepsilon}h$, where $h_t = Kt$. We propose the perturbation ansatz:

$$\begin{aligned}\mathcal{V}^S(t, x; p) &= \mathcal{V}^S_0\left(\frac{\bar{V}[p_0] - \underline{V}[p_0]}{\bar{V}_t[p] - \underline{V}_t[p]}(x - \underline{V}_t[p]) + \underline{V}[p_0]; p_0\right) + \sqrt{\varepsilon}\mathcal{V}^S_1(t, x) + O(\varepsilon) \\ &= \mathcal{V}^S_0\left((1 - \sqrt{\varepsilon}r_{1,t})x - \sqrt{\varepsilon}r_{0,t}; p_0\right) + \sqrt{\varepsilon}\mathcal{V}^S_1(t, x) + O(\varepsilon), \quad (63)\end{aligned}$$

where, in the second equality, we expanded the argument of $\mathcal{V}^S_0(.; p_0)$ to the first order in $\sqrt{\varepsilon}$ and we define

$$
\begin{aligned}
r_{1,t} &:= \frac{\bar{R}_t - \underline{R}_t}{\bar{V}[p_0] - \underline{V}[p_0]} = K \cdot \frac{\bar{S}_{0,0} - \underline{S}_{0,0} + (\bar{S}_{0,1} - \underline{S}_{0,1})t}{\bar{V}[p_0] - \underline{V}[p_0]} \\
r_{0,t} &:= h_t + \underline{R}_t - r_{1,t}\underline{V}[p_0] \\
&= K \cdot \left(\underline{S}_{0,0} - \frac{\bar{S}_{0,0} - \underline{S}_{0,0}}{\bar{V}[p_0] - \underline{V}[p_0]} \underline{V}[p_0] \right) + Kt \cdot \left(1 + \underline{S}_{0,1} - \frac{\bar{S}_{0,1} - \underline{S}_{0,1}}{\bar{V}[p_0] - \underline{V}[p_0]} \underline{V}[p_0] \right),
\end{aligned}
\tag{64}
$$

where $\bar{S}_{0,0}, \bar{S}_{0,1}, \underline{S}_{0,0}, \underline{S}_{0,1}$ are as defined in Corollary 2. The first term represents a naive rescaling of the constant price solution according to the buyer's response moving boundaries. Substituting the ansatz into the PDE (29) and collect the $\sqrt{\varepsilon}$-terms, we obtain the PDE for $\mathcal{V}_1^S$, $\frac{\sigma(x)^2}{2}\partial_x^2 \mathcal{V}_1^S(t,x) + \partial_t \mathcal{V}_1^S(t,x) - m\mathcal{V}_1^S(t,x) + \left(\sigma(x)\sigma'(x)(r_{1,t}x + r_{0,t}) - \sigma(x)^2 r_{1,t}\right)\partial_x^2 \mathcal{V}_0^S(x;p_0) - (r'_{1,t}x + r'_{0,t})\partial_x \mathcal{V}_0^S(x;p_0) = 0$, along with the boundary conditions up to the ε-order:

$$
\begin{aligned}
&\mathcal{V}^S(t, \bar{V}_t[p]; p) = p_t - g \implies \mathcal{V}_0^S(\bar{V}[p_0]; p_0) + \sqrt{\varepsilon}\mathcal{V}_1^S(t, \bar{V}[p_0]) + O(\varepsilon) = p_0 + \sqrt{\varepsilon} h_t - g \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\implies \mathcal{V}_1^S(t, \bar{V}[p_0]) = h_t \\
&\mathcal{V}^S(t, \underline{V}_t[p]; p) = 0 \implies \mathcal{V}_0^S(\underline{V}[p_0]; p_0) + \sqrt{\varepsilon}\mathcal{V}_1^S(t, \underline{V}[p_0]) + O(\varepsilon) = 0 \implies \mathcal{V}_1^S(t, \underline{V}[p_0]) = 0,
\end{aligned}
$$

and finally the terminal condition at T, $\mathcal{V}^S(T,x;p) = \mathcal{V}_T^S(x)$, gives $\mathcal{V}_1^S(T,x) = \frac{1}{\sqrt{\varepsilon}}\left(\mathcal{V}_T^S(x) - \mathcal{V}^S((1-\sqrt{\varepsilon}r_{1,T})x - \sqrt{\varepsilon}r_{0,T}; p_0)\right) + O(\sqrt{\varepsilon})$.

We can reverse the time-axis, then apply (Evans, 2022, Chapter 7, Theorem 7) or (Friedman, 2008, Chapter 3, §5, Corollary 2) to conclude the existence of the smooth solution $\mathcal{V}_1^S$ to the parabolic initial boundary-value problem with fixed boundaries $\bar{V}[p_0], \underline{V}[p_0]$. The solution is unique and admits the following expression via the semi-elliptic version of Feynman-Kac Formula (Øksendal, 2003, Theorem 9.1.1):

$$
\begin{aligned}
\mathcal{V}_1^S(t,x) &= \mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}\wedge T} e^{-m(s-t)}\left(\sigma(v_s^{t,x})\sigma'(v_s^{t,x})(r_{1,s}v_s^{t,x} + r_{0,s}) - \sigma(v_s^{t,x})^2 r_{1,s}^T\right)\partial_x^2 \mathcal{V}_0^S(v_s^{t,x}; p_0) ds | \mathscr{F}_t\right] \\
&+ \mathbb{E}\left[e^{-m(T-t)}\mathcal{V}_1^S(T, v_T^{t,x}) \cdot 1\left\{\tau_\Omega^{t,x} \geq T\right\} | \mathscr{F}_t\right] + \mathbb{E}\left[h_{\tau_\Omega^{t,x}} e^{-m(\tau_\Omega^{t,x}-t)} \cdot 1\left\{v_{\tau_\Omega^{t,x}}^{t,x} \geq \bar{V}[p_0], \tau_\Omega^{t,x} < T\right\} | \mathscr{F}_t\right] \\
&- \mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}\wedge T} e^{-m(s-t)}(r'_{1,s}v_s^{t,x} + r'_{0,s})\partial_x \mathcal{V}_0^S(v_s^{t,x}; p_0) ds | \mathscr{F}_t\right].
\end{aligned}
\tag{65}
$$

When $m = 0$, we have from Lemma 2 that $\mathcal{V}_0^S(x;p_0) = (p_0 - g)\frac{x - \underline{V}[p_0]}{\bar{V}[p_0] - \underline{V}[p_0]}$. This means that the first term of (65) vanishes and (65) simplified to:

$$
\mathcal{V}_1^S(t,x) = \mathbb{E}\left[\mathcal{V}_1^S(T, v_T^{t,x}) \cdot 1\left\{\tau_\Omega^{t,x} \geq T\right\} | \mathscr{F}_t\right] + \mathbb{E}\left[h_{\tau_\Omega^{t,x}} \cdot 1\left\{v_{\tau_\Omega^{t,x}}^{t,x} \geq \bar{V}[p_0], \tau_\Omega^{t,x} < T\right\} | \mathscr{F}_t\right]
$$

$$- \frac{p_0 - g}{\bar{V}[p_0] - \underline{V}[p_0]} \mathbb{E}\left[\int_t^{\tau_\Omega^{t,x} \wedge T'} (r'_{1,s} v_s^{t,x} + r'_{0,s}) ds | \mathscr{F}_t \right], \quad (66)$$

We can further simplify the third term as follows:

$$\begin{aligned}
&- \frac{p_0 - g}{\bar{V}[p_0] - \underline{V}[p_0]} \mathbb{E}\left[\int_t^{\tau_\Omega^x \wedge T} (r'_{1,s} v_s^x + r'_{0,s}) ds | \mathscr{F}_t \right] \\
&\quad = - \frac{p_0 - g}{\bar{V}[p_0] - \underline{V}[p_0]} \mathbb{E}\left[\int_t^{\tau_\Omega^x \wedge T} d(r_{1,s} v_s^x) - \int_t^{\tau_\Omega^x \wedge T} r_{1,s} dv_s^x + \int_t^{\tau_\Omega^x \wedge T} r'_{0,s} ds | \mathscr{F}_t \right] \\
&\qquad = - \frac{p_0 - g}{\bar{V}[p_0] - \underline{V}[p_0]} \left(\mathbb{E}\left[r_{1,\tau_\Omega^x \wedge T} v^x_{\tau_\Omega^x \wedge T} + r_{0,\tau_\Omega^x \wedge T} | \mathscr{F}_t \right] - (r_{1,t} x - r_{0,t}) \right).
\end{aligned}$$

We used Ito's Lemma in the first equality. For the second equality, note that $\{v_t^x\}_{t \geq 0}$ is a square-integrable martingale, hence we know that $\int_0^{\tau_\Omega^x} r_{1,s} dv_s^x$ is a continuous square-integrable martingale, therefore its expectation vanishes.

Because h is linear in t, $\mathcal{V}_1^S(T, x)$ is at most linear in T. We have seen from (64) that $r_{1,t}$, $r_{0,t}$ are linear in t. Meanwhile, $v_s^{t,x}$ is bounded inside $[\underline{V}[p_0], \bar{V}[p_0]]$. Therefore, the first term is upper-bounded by $\sup_{x \in [\underline{V}[p_0], \bar{V}[p_0]]} \mathcal{V}_1^S(T, x) \mathbb{P}\left[\tau_\Omega^{t,x} \geq T | \mathscr{F}_t \right] \to 0$ as $T \to \infty$ because $\mathbb{P}\left[\tau_\Omega^{t,x} \geq T | \mathscr{F}_t \right] \to 0$ exponentially according to equation (56). Both the second and third terms are upper-bounded by some constant multiple of $\mathbb{E}[\tau_\Omega^{t,x} | \mathscr{F}_t] < \infty$. Taking the limit $T \to \infty$ using the Dominated Convergence Theorem, then setting $t = 0$, we obtain the expression for $\mathcal{V}_1^S(0, x)$. Substituting the expression for $\mathcal{V}_1^S(0, x)$ into the perturbative expansion (63) at $t = 0$, we obtain $\mathcal{V}^S(x; p_0, K) = (p_0 - g) \frac{x - \underline{V}[p_0]}{\bar{V}[p_0] - \underline{V}[p_0]} + \sqrt{\varepsilon} K \mathbb{E}\left[\tau_\Omega^x \cdot 1\left\{ v^x_{\tau_\Omega^x} \geq \bar{V}[p_0] \right\} \right] - \frac{\sqrt{\varepsilon}(p_0 - g)}{\bar{V}[p_0] - \underline{V}[p_0]} \mathbb{E}\left[r_{1,\tau_\Omega^x} v^x_{\tau_\Omega^x} + r_{0,\tau_\Omega^x} \right] + O(\varepsilon)$, which leads to (34) after some simplifications and substitution of (64). □

References

Agarwal, R. P., O'Regan, D., et al. (2008). *An introduction to ordinary differential equations.* Springer.

Anderson, S. P. and Renault, R. (2006). Advertising content. *American Economic Review*, 96(1):93–113.

Armstrong, M., Vickers, J., and Zhou, J. (2009). Prominence and consumer search. *The RAND Journal of Economics*, 40(2):209–233.

Aviv, Y. and Pazgal, A. (2008). Optimal pricing of seasonal products in the presence of forward-looking consumers. *Manufacturing & service operations management*, 10(3):339–359.

Bar-Isaac, H., Caruana, G., and Cuñat, V. (2010). Information gathering and marketing. *Journal of Economics & Management Strategy*, 19(2):375–401.

Bose, S., Orosel, G., Ottaviani, M., and Vesterlund, L. (2006). Dynamic monopoly pricing and herding. *The RAND Journal of Economics*, 37(4):910–928.

Branco, F., Sun, M., and Villas-Boas, J. M. (2012). Optimal search for product information. *Management Science*, 58(11):2037–2056.

Branco, F., Sun, M., and Villas-Boas, J. M. (2016). Too much information? information provision and search costs. *Marketing Science*, 35(4):605–618.

Chaimanowong, W. and Ke, T. T. (2022). A simple micro-founded model of repeat buying based on continuous information tracking. *Available at SSRN 3954939*.

Chen, Y. and Zhang, Z. J. (2009). Dynamic targeted pricing with strategic consumers. *International Journal of Industrial Organization*, 27(1):43–50.

Chen, Z., Shi, M., and Zhong, Z. (2025). Predictive accuracy, consumer search, and personalized recommendation. *working paper*.

Crandall, M. G., Ishii, H., and Lions, P.-L. (1992). User's guide to viscosity solutions of second order partial differential equations. *Bulletin of the American mathematical society*, 27(1):1–67.

Deb, R. (2014). Intertemporal price discrimination with stochastic values. *University of Toronto*, pages 891–928.

Desai, P. and Purohit, D. (1998). Leasing and selling: Optimal marketing strategies for a durable goods firm. *Management Science*, 44(11-part-2):S19–S34.

Dukes, A. and Liu, L. (2016). Online shopping intermediaries: The strategic design of search environments. *Management Science*, 62(4):1064–1077.

Evans, L. C. (2022). *Partial differential equations*, volume 19. American Mathematical Society.

Friedman, A. (2008). *Partial differential equations of parabolic type*. Courier Dover Publications.

Gilbert, J. P. and Mosteller, F. (1966). Recognizing the maximum of a sequence. *Journal of the American Statistical Association*, 61(313):35–73.

Gong, Z. and Huang, J. (2025). Limited time offer and consumer search. *Management Science*, 71(9):7692–7706.

Gong, Z., Huang, J., and Chen, Y. (2022). What the past tells about the future: historical prices in the durable goods market. *Management Science*, 68(12):8857–8871.

Guo, L. (2021). Endogenous evaluation and sequential search. *Marketing Science*, 40(3):413–427.

Guo, L. and Zhang, J. (2012). Consumer deliberation and product line design. *Marketing Science*, 31(6):995–1007.

Huang, J. and Gong, Z. (2023). A model of two learning processes. *Available at SSRN 5137259*.

Hviid, M. and Shaffer, G. (2010). Matching own prices, rivals'prices or both? *The Journal of Industrial Economics*, 58(3):479–506.

Jerath, K. and Ren, Q. (2021). Consumer rational (in) attention to favorable and unfavorable product information, and firm information design. *Journal of Marketing Research*, 58(2):343–362.

Jing, B. (2011). Social learning and dynamic pricing of durable goods. *Marketing Science*, 30(5):851–865.

Kamada, Y. and Muto, N. (2015). Multi-agent search with deadline. *working paper*.

Karatzas, I. and Shreve, S. (2012). *Brownian motion and stochastic calculus*, volume 113. Springer Science & Business Media.

Ke, T. T. and Lin, S. (2020). Informational complementarity. *Management Science*, 66(8):3699–3716.

Ke, T. T., Shin, J., and Yu, J. (2023). A model of product portfolio design: Guiding consumer search through brand positioning. *Marketing Science*, 42(6):1101–1124.

Ke, T. T., Tang, W., Villas-Boas, J. M., and Zhang, Y. P. (2022). Parallel search for information in continuous time—optimal stopping and geometry of the pde. *Applied Mathematics & Optimization*, 85(2):3.

Ke, T. T. and Villas-Boas, J. M. (2019). Optimal learning before choice. *Journal of Economic Theory*, 180:383–437.

Levinthal, D. A. and Purohit, D. (1989). Durable goods and product obsolescence. *Marketing Science*, 8(1):35–56.

Libgober, J. and Mu, X. (2021). Informational robustness in intertemporal pricing. *The Review of Economic Studies*, 88(3):1224–1252.

Liptser, R. S. and Shiryaev, A. N. (2013). *Statistics of random processes: I. General theory*, volume 5. Springer Science & Business Media.

Liu, L. and Dukes, A. (2013). Consideration set formation with multiproduct firms: The case of within-firm and across-firm evaluation costs. *Management Science*, 59(8):1871–1886.

Lu, M. Y. (2023). Content marketing: Why do firms handicap themselves with brand-neutral in a competitive environment. *working paper*.

Mayzlin, D. and Shin, J. (2011). Uninformative advertising as an invitation to search. *Marketing science*, 30(4):666–685.

Moscarini, G. and Smith, L. (2001). The optimal level of experimentation. *Econometrica*, 69(6):1629–1644.

Ning, Z. E. (2021). List price and discount in a stochastic selling process. *Marketing Science*, 40(2):366–387.

Ning, Z. E., Villas-Boas, J. M., and Yao, Y. (2025). Search fatigue, choice deferral, and closure. *Marketing Science*.

Ning, Z. E. and Zhou, Z. (2025). Managing consumer attention to diverse information sources in product diffusion. *working paper*.

Øksendal, B. (2003). *Stochastic differential equations*. Springer.

Sakaguchi, M. (1978). When to stop: randomly appearing bivariate target values. *Journal of the Operations Research Society of Japan*, 21(1):45–58.

Sayedi, A. (2018). Pricing in a duopoly with observational learning. *Available at SSRN 3131561.*

Smith, L. (1999). Optimal job search in a changing world. *Mathematical Social Sciences*, 38(1):1–9.

Stokey, N. L. (1979). Intertemporal price discrimination. *The Quarterly Journal of Economics*, 93(3):355–371.

Su, X. (2007). Intertemporal pricing with strategic customer behavior. *Management Science*, 53(5):726–741.

Urgun, C. and Yariv, L. (2025). Contiguous search: Exploration and ambition on uncharted terrain. *Journal of Political Economy*, 133(2):522–567.

Van den Berg, G. J. (1990). Nonstationarity in job search theory. *The review of economic studies*, 57(2):255–277.

Villas-Boas, J. M. (2004). Price cycles in markets with customer recognition. *RAND Journal of Economics*, pages 486–501.

Villas-Boas, J. M. (2009). Product variety and endogenous pricing with evaluation costs. *Management Science*, 55(8):1338–1346.

Villas-Boas, J. M. and Yao, Y. (2021). A dynamic model of optimal retargeting. *Marketing Science*, 40(3):428–458.

Weitzman, M. L. (1979). Optimal search for the best alternative. *Econometrica: Journal of the Econometric Society*, pages 641–654.

Wolinsky, A. (1986). True monopolistic competition as a result of imperfect information. *The Quarterly Journal of Economics*, 101(3):493–511.

Wong, Y. F. (2025). Forward-looking experimentation of correlated alternatives. *Theoretical Economics*, 20(3):883–909.

Yao, Y. (2024a). Dynamic persuasion and strategic search. *Management Science*, 70(10):6778–6803.

Yao, Y. J. (2024b). Invitation to search or purchase? optimal multi-attribute advertising. *Optimal Multi-attribute Advertising (August 23, 2024).*

Yong, J. and Zhou, X. Y. (2012). *Stochastic controls: Hamiltonian systems and HJB equations*, volume 43. Springer Science & Business Media.

Yuan, M., Zhu, Y., and Dukes, A. J. (2024). Search prominence among shopping platforms. *Available at SSRN 4898746.*

Yuan, M., Zhu, Y., Xu, L., and Guan, X. (2023). Search prominence in a distribution channel. *working paper.*

Zhong, W. (2022). Optimal dynamic information acquisition. *Econometrica*, 90(4):1537–1582.

Zhong, Z. (2023). Platform search design: The roles of precision and price. *Marketing Science*, 42(2):293–313.

Zia, M. and Kuksov, D. (2025). Consumer search and product line length: The role of the consumer-product fit distribution. *Marketing Science*, 44(4):802–819.

Online Appendix for Non-stationary Pricing and Search

Viscosity Solution and Perturbation Theory

As a standard practice in optimal stopping theory, rather than directly finding the optimal $\tau^*[p] \in \mathcal{T}$ to the optimization problem (4), it is often more analytically tractable to consider the corresponding Hamilton-Jacobi-Bellman (HJB) equation:

$$H(t, x, V, \nabla V, \Delta V) = 0, \tag{67}$$

where $H : (\mathbb{R} \times [\underline{\pi}, \bar{\pi}]) \times \mathbb{R} \times \mathbb{R}^2 \times \mathcal{S}_2(\mathbb{R}) \to \mathbb{R}$ is given by $H(t, x, V, \nabla V, \Delta V) := \min\left\{c + rV - \partial_t V - \frac{\sigma(x)^2}{2}\partial_x^2 V, V - x + p_t, V\right\}$, with $\mathcal{S}_2(\mathbb{R})$ denoted the space of 2×2 symmetric matrices. Because $p \in \mathcal{P}_T$ is only defined for $t \geq 0$, to discuss the solution on $\mathbb{R} \times [\underline{\pi}, \bar{\pi}]$, we extend it by defining $p_t = p_0$ for all $t < 0$. We consider the solution $V : \mathbb{R} \times [\underline{\pi}, \bar{\pi}] \to \mathbb{R}$ subject to the following asymptotic boundary conditions:

$$\begin{aligned} &V(t, x) = V_0^B(x; p_T),\ \forall t \geq T, \qquad &&\lim_{t \to -\infty} V(t, x) = V_0^B(x; p_0) \\ &V(t, x) = x - p_t,\ \forall x \geq \bar{V}_t[p], \qquad &&V(t, x) = 0,\ \forall x \leq \underline{V}_t[p] \end{aligned} \tag{68}$$

for some functions $\bar{V}[p], \underline{V}[p] : \mathbb{R} \to [\underline{\pi}, \bar{\pi}]$, depending on $p \in \mathcal{P}_T$ and $\bar{V}_t[p] \geq \underline{V}_t[p], \forall t \in \mathbb{R}$. The purchase and quitting boundaries, $\bar{V}[p]$, and $\underline{V}[p]$, provide a simple characterization of the learning strategy. By definition of $\mathcal{P}$, the range of $\bar{V}[p]$ and $\underline{V}[p]$ are contained in $[\underline{\pi}, \bar{\pi}]$. We need to establish the existence and uniqueness of the solution to (67) subjects to (68). Because the classical solution does not always exist, we will work with a relaxed notion of a *viscosity* solution:[1]

Definition 2. *Let $D \subset \mathbb{R}^n$ and $H : D \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}_n(\mathbb{R}) \to \mathbb{R}$ be a continuous function satisfying the* properness *condition: $H(\mathbf{x}, v, p, X) \geq H(\mathbf{x}, u, p, X)$ if $v \geq u$, and the* degenerate ellipticity *condition: $H(\mathbf{x}, v, p, X) \geq H(\mathbf{x}, v, p, Y)$ if $Y \geq X$.*

A continuous function $v : D \to \mathbb{R}$ is a viscosity subsolution *(*supersolution*) if for any $\mathbf{x}_0 \in D$ and any twice continuously differentiable function ϕ such that $\mathbf{x}_0$ is a local maximum (minimum) of $v - \phi$ we have $H(\mathbf{x}_0, v(\mathbf{x}_0), \nabla\phi(\mathbf{x}_0), \Delta\phi(\mathbf{x}_0)) \leq (\geq) 0$.*

A continuous function $v : D \to \mathbb{R}$ is a viscosity *solution if it is both a viscosity subsolution and supersolution.*

The following existence and uniqueness result relates the buyer's value function V^B to the viscosity solution over the domain $D := \mathbb{R} \times [\underline{\pi}, \bar{\pi}]$.

Lemma 5. *For a given $p \in \mathcal{P}_T$, the buyer's value function V^B is the unique viscosity solution to (67) subject to the asymptotic boundary conditions (68).*

[1] Crandall et al. (1992) provides a detailed description of the viscosity solution.

Proof of Lemma 5. The proof that the value function V^B is a viscosity solution to (67) is standard (e.g. see Yong and Zhou (2012)) and we shall omit the details. The fact that V^B satisfies the boundary condition (68) is enforced by the definition of $\mathcal{P}_T$. It remains for us to check the uniqueness of the viscosity solution to (67) with boundary condition (68). This is mostly an application of the comparison principle (Crandall et al., 1992, Theorem 3.3). However, unlike in the standard setup, our domain is unbounded, therefore we provide the details for completeness. For convenience, in the following we will use $A_1(\mathbf{x}, V, \nabla V, \Delta V) := c + rV - \partial_t V - \frac{\sigma(\mathbf{x})^2}{2}\partial_x^2 V$, $A_2(\mathbf{x}, V, \nabla V, \Delta V) := V - x + p_t$, $A_3(\mathbf{x}, V, \nabla V, \Delta V) := V$, so that $H := \min_{i=1,2,3} A_i$.[2]

Let $u : \mathbb{R} \times [\underline{\pi}, \bar{\pi}] \to \mathbb{R}$ and $v : \mathbb{R} \times [\underline{\pi}, \bar{\pi}] \to \mathbb{R}$ be viscosity subsolution and supersolution to (67), respectively, and suppose that $\lim_{t\to\pm\infty}(u - v) \leq 0$, $\lim_{x\to\underline{\pi}}(u - v) \leq 0$, and $\lim_{x\to\bar{\pi}}(u - v) \leq 0$. We claim that $u \leq v$ everywhere on $\mathbb{R} \times [\underline{\pi}, \bar{\pi}]$. To show this let us assume the contrary that there exists $\hat{\mathbf{x}} \in \mathbb{R} \times (\underline{\pi}, \bar{\pi})$ such that $u(\hat{\mathbf{x}}) - v(\hat{\mathbf{x}}) = \max_{\mathbf{x}\in\mathbb{R}\times[\underline{\pi},\bar{\pi}]}(u(\mathbf{x}) - v(\mathbf{x})) > 0$. Consider the function: $w_\alpha(\mathbf{x}, \mathbf{y}) := u(\mathbf{x}) - v(\mathbf{y}) - (\alpha/2)\|\mathbf{x} - \mathbf{y}\|_2^2$ for some constant $\alpha \geq 0$. The assumption on the boundary conditions of u and v implies that for any $\alpha \geq 0$, there exists a local maximum $(\mathbf{x}_\alpha, \mathbf{y}_\alpha) \in (\mathbb{R} \times [\underline{\pi}, \bar{\pi}])^2$ of w_α, and by (Crandall et al., 1992, Lemma 3.1):

$$\lim_{\alpha\to\infty} \alpha\|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2^2 = 0, \quad \lim_{\alpha\to\infty}\left(u(\mathbf{x}_\alpha) - v(\mathbf{x}_\alpha) - \frac{\alpha}{2}\|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2^2\right) = u(\hat{\mathbf{x}}) - v(\hat{\mathbf{x}}).$$

By our assumption, we can find $\delta > 0$ such that $u(\mathbf{x}_\alpha) - v(\mathbf{y}_\alpha) \geq \delta$ for all $\alpha \geq 0$. We can apply (Crandall et al., 1992, Theorem 3.2) because $\mathbb{R} \times [\underline{\pi}, \bar{\pi}]$ is locally compact, and we can find $X, Y \in \mathcal{S}_2(\mathbb{R})$ such that

$$-3\alpha\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3\alpha\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \tag{69}$$

with $\mathbf{x}_\alpha$ a local maximum of $u(\mathbf{x}) - \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha)^\intercal(\mathbf{x} - \mathbf{x}_\alpha) - \frac{1}{2}(\mathbf{x} - \mathbf{x}_\alpha)^\intercal X(\mathbf{x} - \mathbf{x}_\alpha)$ and $\mathbf{y}_\alpha$ a local minimum of $v(\mathbf{y}) - \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha)^\intercal(\mathbf{y} - \mathbf{y}_\alpha) - \frac{1}{2}(\mathbf{y} - \mathbf{y}_\alpha)^\intercal Y(\mathbf{y} - \mathbf{y}_\alpha)$. Because u and v are subsolution and supersolution, respectively, we have:

$$H(\mathbf{x}_\alpha, u(\mathbf{x}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X) \leq 0 \leq H(\mathbf{y}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), Y). \tag{70}$$

$$\begin{aligned}
(69) \Rightarrow\ & A_1(\mathbf{y}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), Y) - A_1(\mathbf{x}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X) \\
&= \frac{\sigma(\mathbf{x}_\alpha)^2}{2}X_{xx} - \frac{\sigma(\mathbf{y}_\alpha)^2}{2}Y_{xx} = \begin{pmatrix} \sigma(\mathbf{x}_\alpha) & \sigma(\mathbf{y}_\alpha) \end{pmatrix}\begin{pmatrix} X_{xx} & 0 \\ 0 & -Y_{xx} \end{pmatrix}\begin{pmatrix} \sigma(\mathbf{x}_\alpha) \\ \sigma(\mathbf{y}_\alpha) \end{pmatrix} \\
&\leq 3\alpha\begin{pmatrix} \sigma(\mathbf{x}_\alpha) & \sigma(\mathbf{y}_\alpha) \end{pmatrix}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}\begin{pmatrix} \sigma(\mathbf{x}_\alpha) \\ \sigma(\mathbf{y}_\alpha) \end{pmatrix} = 3\alpha(\sigma(\mathbf{x}_\alpha) - \sigma(\mathbf{y}_\alpha))^2 \leq 3\alpha L^2\|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2^2
\end{aligned}$$

[2] Given $\mathbf{x} = (t, x) \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}]$, we will sometimes write $\mu(\mathbf{x})$ and $\sigma(\mathbf{x})$ instead of $\mu(x)$ and $\sigma(x)$ when it is more convenient to use vector notation, even though $\mu(.)$ and $\sigma(.)$ do not explicitly depend on t.

where we used the global Lipschitz condition in Assumption 1 for σ in the last inequality. Similarly, we can check that

$$A_2(\mathbf{y}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), Y) - A_2(\mathbf{x}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X) \leq \|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2 + |p_{t_x} - p_{t_y}| \\ \leq \left(1 + \max_{t\in[0,T]} |p'_t|\right) \|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2,$$

and $A_3(\mathbf{y}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), Y) - A_3(\mathbf{x}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X) = 0$. Let $\omega(r) := \max\left\{3L^2, 1 + \max_{t\in[0,T]} |p'_t|\right\} \cdot r$ and $i^* := \operatorname{argmin}_{i=1,2,3} A_i(\mathbf{x}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X)$,

$$\begin{aligned}
&H(\mathbf{y}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), Y) - H(\mathbf{x}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X) \\
&\qquad \leq A_{i^*}(\mathbf{y}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), Y) - A_{i^*}(\mathbf{x}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X) \\
&\qquad\qquad \leq \omega\left(\alpha\|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2^2 + \|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2\right). \\
&\Rightarrow 0 < \min\{1, r\}\delta \leq \min\{1, r\}(u(\mathbf{x}_\alpha) - v(\mathbf{y}_\alpha)) \\
&\qquad \leq H(\mathbf{x}_\alpha, u(\mathbf{x}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X) - H(\mathbf{x}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X) \\
&\qquad = H(\mathbf{x}_\alpha, u(\mathbf{x}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X) - H(\mathbf{y}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), Y) \\
&\qquad + H(\mathbf{y}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), Y) - H(\mathbf{x}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X) \\
&\qquad\qquad \leq \omega\left(\alpha\|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2^2 + \|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2\right) \qquad (71)
\end{aligned}$$

for all $\alpha \geq 0$, where we used (70) to replace the first two terms with zero in the last inequality. By taking the $\alpha \to \infty$ limit, $\omega\left(\alpha\|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2^2 + \|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2\right) \to 0$, while the inequality above specifies that it is bounded away from zero by $\min\{1, r\}\delta$, which is a contradiction. In other words, we have $u \leq v$ over the entire $\mathbb{R} \times [\underline{\pi}, \bar{\pi}]$. Therefore, if $u : \mathbb{R} \times [\underline{\pi}, \bar{\pi}] \to \mathbb{R}$ and $v : \mathbb{R} \times [\underline{\pi}, \bar{\pi}] \to \mathbb{R}$ are both viscosity solution to (67) with the specified boundary conditions: $\lim_{t\to\pm\infty}(u - v) = 0$, $\lim_{x\to\underline{\pi}}(u - v) = 0$, and $\lim_{x\to\bar{\pi}}(u - v) = 0$, then $u = v$ over the entire $\mathbb{R} \times [\underline{\pi}, \bar{\pi}]$. □

Working directly with the viscosity solution via Definition 2 can still be challenging, thus we alternatively consider the following free-boundary backward parabolic PDE initial-value problem: Find $V : \Omega \to \mathbb{R}$, and continuously differentiable functions $\bar{V}[p], \underline{V}[p] : \mathbb{R} \to [\underline{\pi}, \bar{\pi}]$ satisfying $\bar{V}_t[p] \geq \underline{V}_t[p]$, such that

$$\begin{cases}
\dfrac{\sigma(x)^2}{2}\partial_x^2 V(t,x) + \partial_t V(t,x) - rV(t,x) - c = 0, & (t,x) \in \Omega \\
V(t, \bar{V}_t[p]) = \bar{V}_t[p] - p_t, & V(t, \underline{V}_t[p]) = 0, \\
\partial_x V(t, \bar{V}_t[p]) = 1, & \partial_x V(t, \underline{V}_t[p]) = 0, \\
V(T, x) = V_0^B(x; p_T), &
\end{cases} \qquad (72)$$

where

$$\Omega := \{(t,x) \in (-\infty, T] \times [\underline{\pi}, \bar{\pi}] \mid \underline{V}_t[p] < x < \bar{V}_t[p]\}.$$

Any *exact* solution to the problem (72), subject to some additional mild conditions, will also be the viscosity solution of the HJB (67) and thus the consumer's value function by Lemma 5. Meanwhile, (72) enables us to use a perturbation technique to obtain the consumer value function up to any $\varepsilon^{(k+1)/2}$-order by *solving (72) up to the* $\varepsilon^{(k+1)/2}$*-order.* This is because the solution up to the $\varepsilon^{(k+1)/2}$-order of (72), subject to some additional mild conditions, also agree with the viscosity solution to (67) up to the $\varepsilon^{(k+1)/2}$-order.

Given a solution V to (72) on Ω with the specified boundary conditions, we can extend it to $\widetilde{V}$, a function continuously differentiable on $\mathbb{R} \times [\underline{\pi}, \bar{\pi}]$, and twice continuously differentiable in x on $\mathbb{R} \times [\underline{\pi}, \bar{\pi}] \setminus \partial\Omega$, by defining $\widetilde{V}(t,x) = \max\{x - p_t, 0\}$ if $t \leq T$ and $x \notin (\underline{V}_t[p], \bar{V}_t[p])$, and $\widetilde{V}(t,x) = V_0^B(x; p_T)$ if $t > T$. This extension is rather natural, therefore, we will abuse the notation and simply refer to $\widetilde{V}$ as V. We will state formally in Lemma 6 that the solution V will coincide with the buyer's value function V^B. This justifies that the constant price benchmark solutions in §3.1 are the viscosity solutions, and therefore the value functions of their respective buyer's problems.

Suppose we know that $V^B(.,.;p)$ for a given $p \in \mathcal{P}_T$ is a solution to (72), and we would like to compute $V^B(.,.;p + \sqrt{\varepsilon}h)$ for some $h \in \mathcal{P}_T$ and a small $\varepsilon > 0$ up to the $\varepsilon^{(k+1)/2}$-order. By Lemma 6, we aim to solve for the corresponding solution $V(.,.;p + \sqrt{\varepsilon}h)$ to (13) up to the $\varepsilon^{(k+1)/2}$-order. The idea of perturbation theory is to consider the solution ansatz

$$
\begin{aligned}
V^{\varepsilon}_{\leq k}(.,.) = V^{\varepsilon}_{\leq k}(.,.;p + \sqrt{\epsilon}h) &:= V_0(.,.) + V_1(.,.)\sqrt{\varepsilon} + \cdots + V_k(.,.)\varepsilon^{k/2} \\
\bar{V}^{\varepsilon}_{\leq k,t} = \bar{V}^{\varepsilon}_{\leq k,t}[p + \sqrt{\varepsilon}h] &:= \bar{V}_{0,t} + \bar{V}_{1,t}\sqrt{\varepsilon} + \cdots + \bar{V}_{k,t}\varepsilon^{k/2} \\
\underline{V}^{\varepsilon}_{\leq k,t} = \underline{V}^{\varepsilon}_{\leq k,t}[p + \sqrt{\varepsilon}h] &:= \underline{V}_{0,t} + \underline{V}_{1,t}\sqrt{\varepsilon} + \cdots + \underline{V}_{k,t}\varepsilon^{k/2},
\end{aligned}
$$

where $V_0(.,.) := V_0^B(.,.;p)$, $\bar{V}_{0,t} := \bar{V}_t[p]$, and $\underline{V}_{0,t} := \underline{V}_t[p]$, satisfying (72) over $\Omega^{\varepsilon}_{\leq k} := \{(t,x) \in (-\infty, T] \times [\underline{\pi}, \bar{\pi}] | \underline{V}^{\varepsilon}_{\leq k,t} < x < \bar{V}^{\varepsilon}_{\leq k,t}\}$ up to the $\varepsilon^{(k+1)/2}$-order, i.e.

$$
\begin{cases}
\dfrac{\sigma(x)^2}{2}\partial_x^2 V^{\varepsilon}_{\leq k}(t,x) + \partial_t V^{\varepsilon}_{\leq k}(t,x) - rV^{\varepsilon}_{\leq k}(t,x) - c = O(\varepsilon^{(k+1)/2}), \quad (t,x) \in \Omega^{\varepsilon}_{\leq k} \\
V^{\varepsilon}_{\leq k}(t, \bar{V}^{\varepsilon}_{\leq k,t}) = \bar{V}^{\varepsilon}_{\leq k,t} - (p_t + \sqrt{\varepsilon}h_t) + O(\varepsilon^{(k+1)/2}), \\
V^{\varepsilon}_{\leq k}(t, \underline{V}^{\varepsilon}_{\leq k,t}) = O(\varepsilon^{(k+1)/2}), \\
\partial_x V^{\varepsilon}_{\leq k}(t, \bar{V}^{\varepsilon}_{\leq k,t}) = 1 + O(\varepsilon^{(k+1)/2}), \\
\partial_x V^{\varepsilon}_{\leq k}(t, \underline{V}^{\varepsilon}_{\leq k,t}) = O(\varepsilon^{(k+1)/2}), \\
V^{\varepsilon}_{\leq k}(T,x) = V_0^B(x;p_T) + O(\varepsilon^{(k+1)/2})
\end{cases}.
$$

By substituting the expression of $V^{\varepsilon}_{\leq k}(.,.), \bar{V}^{\varepsilon}_{\leq k,t}$, and $\underline{V}^{\varepsilon}_{\leq k,t}$ into (72) and comparing the $\varepsilon^{k'/2}$ terms for $k' = 1, 2, \cdots, k$, we can solve for $V_{k'}, \bar{V}_{k'}, \underline{V}_{k'}$ using the knowledge of $V_{k''}, \bar{V}_{k''}, \underline{V}_{k''}$ for $k'' = 0, \cdots, k' - 1$. Although both the value-matching and smooth-pasting conditions are only satisfied up to the $\varepsilon^{(k+1)/2}$-order, it is possible to find a twice continuously differentiable function

$\chi : \mathbb{R} \times [\underline{\pi}, \bar{\pi}] \setminus \Omega^{\varepsilon}_{\leq k} \to \mathbb{R}$ which continuously differentiably transitions from $V^{\varepsilon}_{\leq k}$ at $\partial\Omega^{\varepsilon}_{\leq k}$ to $\max\{x - p_t, 0\}$ for all (t, x) some distance $R > 0$ away from $\Omega^{\varepsilon}_{\leq k}$, e.g. a smooth 'bump' function. In particular, we have $\chi = V^{\varepsilon}_{\leq k}$ and $\nabla\chi = \nabla V^{\varepsilon}_{\leq k}$ on $\partial\Omega^{\varepsilon}_{\leq k}$. We also require that $|\partial_t\chi(t, x) - p'_t - \sqrt{\varepsilon}h'_t| = O(\varepsilon^{(k+1)/2})$, $|\partial^2_x\chi| = O(\varepsilon^{(k+1)/2})$, and that the asymptotic boundary conditions (68) are met. We extend $V^{\varepsilon}_{\leq k}$ to $\widetilde{V}^{\varepsilon}_{\leq k}$, a function continuously differentiable on $\mathbb{R} \times [\underline{\pi}, \bar{\pi}]$, and twice continuously differentiable in x on $\mathbb{R} \times [\underline{\pi}, \bar{\pi}] \setminus \partial\Omega^{\varepsilon}_{\leq k}$, by defining $\widetilde{V}^{\varepsilon}_{\leq k}(t, x) = \chi(t, x)$ if $t \leq T$ and $x \notin (\underline{V}^{\varepsilon}_{\leq k,t}, \bar{V}^{\varepsilon}_{\leq k,t})$, $\widetilde{V}^{\varepsilon}_{\leq k}(t, x) = V^B_0(x; p_T + \sqrt{\varepsilon}h_T)$ if $t > T$, and $\widetilde{V}^{\varepsilon}_{\leq k}(t, x) = V^{\varepsilon}_{\leq k}(t, x)$ otherwise. We abuse the notation and refer to $\widetilde{V}^{\varepsilon}_{\leq k}$ as $V^{\varepsilon}_{\leq k}$.

Lemma 6. *Consider pricing strategies $p, h \in \mathcal{P}_T$ and a given $\varepsilon > 0$.*

1. *If V satisfies the free-boundary backward parabolic PDE initial-value problem (72) with the pricing policy $p \in \mathcal{P}_T$, such that $V(t, x) \geq \max\{x - p_t, 0\}$, and $p'_t + r(\bar{V}_t[p] - p_t) + c \geq 0$ for all $(t, x) \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}]$, then V is a viscosity solution to (67). In particular, the buyer's value function is given by $V^B = V$.*

2. *If $V^{\varepsilon}_{\leq k}$ satisfies the free-boundary backward parabolic PDE initial-value problem (72) up to the $\varepsilon^{(k+1)/2}$-order with the pricing policy $p + \sqrt{\varepsilon}h \in \mathcal{P}_T$, such that $V^{\varepsilon}_{\leq k}(t, x) \geq \max\{x - p_t, 0\} + O(\varepsilon^{(k+1)/2})$, and $p'_t + \sqrt{\varepsilon}h'_t + r(\bar{V}^{\varepsilon}_{\leq k,t}[p + \sqrt{\varepsilon}h] - p_t - \sqrt{\varepsilon}h_t) + c \geq O(\varepsilon^{(k+1)/2})$ for all $(t, x) \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}]$, then $V^B = V^{\varepsilon}_{\leq k} + O(\varepsilon^{(k+1)/2})$.*

The conditions on p' and h' in Lemma 6 should not be particularly restrictive. For example, if we can check that $\lim_{x \nearrow \bar{V}_t[p]} \partial^2_x V(t, x) \geq 0$ or $\lim_{x \nearrow \bar{V}^{\varepsilon}_{\leq k,t}[p+\sqrt{\varepsilon}h]} \partial^2_x V^{\varepsilon}_{\leq k,t}(t, x) \geq O(\varepsilon^{(k+1)/2})$, then the conditions on p' and h' are automatically satisfied. In the context of our work, these conditions are easily satisfied when we focus on small perturbations from the known constant price solution and investigate the direction of buyers' reactions. Typically, if our zero-th order perturbation for $p + \sqrt{\varepsilon}h$ is given by the buyer's value function: $V_0(.,.) = V^B(.,.; p)$, and the boundaries $\bar{V}_{0,t} = \bar{V}_t[p]$, $\underline{V}_{0,t} = \underline{V}_t[p]$, corresponding to p, then $p'_t + r(\bar{V}_t[p] - p_t) + c \geq 0$ by Lemma 5. Then it follows that $p'_t + \sqrt{\varepsilon}h'_t + r(V^{\varepsilon}_{\leq k,t}[p + \sqrt{\varepsilon}h] - p_t - \sqrt{\varepsilon}h_t) + c \geq 0$ is satisfied for all sufficiently small $\varepsilon > 0$. To avoid unnecessary technical complications, for the remainder we shall assume that all the conditions in Lemma 6 are satisfied whenever it is used.

For compatibility with the ε-equilibrium concept, we will only use Lemma 6 with $k = 1$ in all of our applications. However, it is important to remark that Lemma 6 does not directly claim that: $\bar{V}[p + \sqrt{\varepsilon}h] = \bar{V}^{\varepsilon}_{\leq 1}[p + \sqrt{\varepsilon}h] + O(\varepsilon)$ or $\underline{V}[p + \sqrt{\varepsilon}h] = \underline{V}^{\varepsilon}_{\leq 1}[p + \sqrt{\varepsilon}h] + O(\varepsilon)$, instead, some extra cares is needed which we outline as follows. The solution $V^{\varepsilon}_{\leq 1}$ can be interpreted (up to $O(\varepsilon)$), via the probabilistic Feynman-Kac expression, as the expected discounted value of purchasing when the valuation process $v^{t,x}_s$ reaches $\bar{V}^{\varepsilon}_{\leq 1,s}$ and exiting when $v^{t,x}_s$ reaches $\underline{V}^{\varepsilon}_{\leq 1,s}$, under the flow cost c. Because $V^B = V^{\varepsilon}_{\leq 1} + O(\varepsilon)$ according to Lemma 6, the learning strategy characterized by $\bar{V}^{\varepsilon}_{\leq 1}[p + \sqrt{\varepsilon}h]$ and $\underline{V}^{\varepsilon}_{\leq 1}[p + \sqrt{\varepsilon}h]$ are considered ε-optimal. Given this understanding, we shall slightly abuse our notation for convenience by writing $\bar{V}[p + \sqrt{\varepsilon}h] = \bar{V}^{\varepsilon}_{\leq 1}[p + \sqrt{\varepsilon}h] + O(\varepsilon)$ and $\underline{V}[p + \sqrt{\varepsilon}h] = \underline{V}^{\varepsilon}_{\leq 1}[p + \sqrt{\varepsilon}h] + O(\varepsilon)$.

Proof of Lemma 6. Part 1: Let such a solution V to (13) be given. Because we have assumed $V(t,x) \geq \max\{x - p_t, 0\}$ and $p_t' + r(\bar{V}_t[p] - p_t) + c \geq 0$, $V - \max\{x - p_t, 0\} \geq 0$ for all $\mathbf{x} \in \Omega$, and $c + rV - \partial_t V - \frac{\sigma(\mathbf{x})^2}{2}\partial_x^2 V \geq 0$ for all $\mathbf{x} \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}] \setminus \Omega$. By the value-matching, the smooth pasting conditions, and the assumption that $p \in \mathcal{P}_T$ is smooth, we have that V is continuously differentiable[3]. Moreover, V is twice continuously differentiable in x on $\mathbb{R} \times [\underline{\pi}, \bar{\pi}] \setminus \partial\Omega$, as it is a (classical) solution to the PDE on Ω, and $\max\{x - p_t, 0\}$ is twice continuously differentiable in x on $\mathbb{R} \times [\underline{\pi}, \bar{\pi}] \setminus \Omega$. Therefore, we have $H(\mathbf{x}, V, \nabla V, \Delta V) = 0$ classically on $\mathbb{R} \times [\underline{\pi}, \bar{\pi}] \setminus \partial\Omega$. Thus, for any twice continuously differentiable ϕ and any $\mathbf{x}_0 \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}]$, we have $\nabla\phi(\mathbf{x}_0) = \nabla V(\mathbf{x}_0)$, and we can find $\{\mathbf{x}_i\}_{i=0}^{\infty} \subset \mathbb{R} \times [\underline{\pi}, \bar{\pi}] \setminus \partial\Omega$ converging to $\mathbf{x}_0$. If $\mathbf{x}_0$ is a local maximum of $V - \phi$ then $\partial_x^2\phi(\mathbf{x}_0) \geq \lim_{i\to\infty} \partial_x^2 V(\mathbf{x}_i)$ which implies $H(\mathbf{x}_0, V(\mathbf{x}_0), \nabla\phi(\mathbf{x}_0), \Delta\phi(\mathbf{x}_0)) \leq \lim_{i\to\infty} H(\mathbf{x}_i, V(\mathbf{x}_i), \nabla V(\mathbf{x}_i), \Delta V(\mathbf{x}_i)) = 0$. Similarly, if $\mathbf{x}_0$ is a local minimum of $V - \phi$ then $\partial_x^2\phi(\mathbf{x}_0) \leq \lim_{i\to\infty} \partial_x^2 V(\mathbf{x}_i)$ which implies $H(\mathbf{x}_0, V(\mathbf{x}_0), \nabla\phi(\mathbf{x}_0), \Delta\phi(\mathbf{x}_0)) \geq \lim_{i\to\infty} H(\mathbf{x}_i, V(\mathbf{x}_i), \nabla V(\mathbf{x}_i), \Delta V(\mathbf{x}_i)) = 0$.

Part 2: Repeat the argument from the previous part with the perturbed pricing policy $p + \sqrt{\varepsilon}h \in \mathcal{P}_T$, we have that $H(\mathbf{x}, V_{\leq k}^{\varepsilon}, \nabla V_{\leq k}^{\varepsilon}, \Delta V_{\leq k}^{\varepsilon}) = O(\varepsilon^{(k+1)/2})$ classically on $\mathbb{R} \times [\underline{\pi}, \bar{\pi}] \setminus \partial\Omega_{\leq k}^{\varepsilon}$. Moreover, for any twice continuously differentiable ϕ and any $\mathbf{x}_0 \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}]$, if $\mathbf{x}_0$ is a local maximum of $V_{\leq k}^{\varepsilon} - \phi$ then $H(\mathbf{x}_0, V_{\leq k}^{\varepsilon}, \nabla\phi, \Delta\phi) \leq O(\varepsilon^{(k+1)/2})$, and if $\mathbf{x}_0$ is a local minimum of $V_{\leq k}^{\varepsilon} - \phi$ then $H(\mathbf{x}_0, V_{\leq k}^{\varepsilon}, \nabla\phi, \Delta\phi) \geq O(\varepsilon^{(k+1)/2})$. Because $V_{\leq k}^{\varepsilon}$ satisfies the same asymptotic boundary conditions as the value function V^B, we can repeat the comparison principle argument in the proof of Lemma 5. In particular, setting $u := V_{\leq k}^{\varepsilon}, v := V^B$ we have (70) becomes $H(\mathbf{x}_\alpha, V_{\leq k}^{\varepsilon}, \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X) + O(\varepsilon^{(k+1)/2}) \leq 0 \leq H(\mathbf{y}_\alpha, V^B, \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), Y)$, which means (71) becomes $\min\{1, r\}(V_{\leq k}^{\varepsilon}(\mathbf{x}_\alpha) - V^B(\mathbf{y}_\alpha)) \leq \omega(\alpha\|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2^2 + \|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2) + O(\varepsilon^{(k+1)/2})$. Taking the limit $\alpha \to \infty$, we find that $\sup_{\mathbf{x} \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}]} \left(V_{\leq k}^{\varepsilon}(\mathbf{x}) - V^B(\mathbf{x})\right) \leq O(\varepsilon^{(k+1)/2})$, in other words: $V_{\leq k}^{\varepsilon}(\mathbf{x}) \leq V^B(\mathbf{x}) + O(\varepsilon^{(k+1)/2})$. On the other hand, setting $u := V^B, v := V_{\leq k}^{\varepsilon}$ yields $V^B(\mathbf{x}) \leq V_{\leq k}^{\varepsilon}(\mathbf{x}) + O(\varepsilon^{(k+1)/2})$, thus we have $V^B(\mathbf{x}) = V_{\leq k}^{\varepsilon}(\mathbf{x}) + O(\varepsilon^{(k+1)/2})$. ☐

[3] To get the continuity of t derivative across the boundary, consider the defining equation: $V^B(t, \bar{V}_t[p]; p) = \bar{V}_t[p] - p_t$. Differentiating with respect to t gives: $\bar{V}_t'[p] \cdot \partial_x V^B(t, \bar{V}_t[p]; p) + \partial_t V^B(t, \bar{V}_t[p]; p) = \bar{V}_t'[p] - p_t'$, or $\partial_t V^B(t, \bar{V}_t[p]; p) = -p_t'$. Similarly, we have $\partial_t V^B(t, \underline{V}_t[p]; p) = 0$