Consumer Gradual Learning and Firm Non-stationary Pricing

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August 2025

Abstract

Consumers often gather information gradually before making a purchasing decision. This paper introduces a novel framework where firms adopt *non-stationary pricing strategies* – prices that evolve over time without being contingent on consumers' current valuation, which endogenously induces non-stationarity in the consumer's search problem. We show that non-stationary pricing strategies can outperform stationary ones. Under zero search costs, a perfectly patient firm's optimal price is arbitrarily close to constant, but with discounting, the firm may benefit from charging non-stationary prices. When search costs are positive, the optimal price is non-stationary even if the firm is perfectly patient. The price increases over time if the information is too noisy or the search cost is too high. The direction of price trajectories is more nuanced in other cases where consumers have a stronger incentive to search, with increasing prices for consumers with high or low initial valuation and decreasing prices for medium–value consumers.

* We thank J. Miguel Villas-Boas, Tat Chan, Francisco Cisternas, Anthony Dukes, Yuichiro Kamada, Tony Ke, Song Lin, Eddie Ning, Jiwoong Shin, Quitzé Valenzuela-Stookey, and Jungju Yu for constructive comments. All errors are our own. E-mail address: Wee Chaimanowong: chaimanowongw@gmail.com, Yunfei (Jesse) Yao: jesseyao@cuhk.edu.hk.

1 Introduction

Consumers often gather information gradually to reduce uncertainty about a product's value before making a purchasing decision. They might visit a seller's website, read reviews on the retailer's storefront, or search for review articles through search engines, yet these activities only partially resolve their uncertainty. Building on the seminal work of Weitzman (1979) and Wolinsky (1986), a rich literature has explored optimal consumer search strategies across multiple alternatives or product attributes (Moscarini and Smith, 2001; Armstrong et al., 2009; Branco et al., 2012; Ke and Villas-Boas, 2019; Zhong, 2022) and their implications for firms' strategies in information provision, pricing, product design, and advertising (Anderson and Renault, 2006; Villas-Boas, 2009; Bar-Isaac et al., 2010; Mayzlin and Shin, 2011; Chaimanowong and Ke, 2022). A critical dimension of these studies is pricing, with existing work often assuming exogenous prices, endogenous constant prices, or endogenous prices contingent on the consumers' current valuation. However, recent privacy regulations have disrupted firms' ability to track individual consumers in real time, making it challenging to tailor prices to evolving consumer beliefs. Even if a firm can track consumers' browsing behavior, it may be hard for the firm to know how consumers will interpret the information they see. For example, Tesla may be able to observe that a consumer clicks on an image of the interior design of the car, but may not know whether the consumer likes the large screen on Tesla or the traditional dashboard. This calls into question whether the firm can track the consumer's belief evolution process when the consumer is searching for information, and raises an important question: can firms benefit from dynamic pricing when consumer belief evolution is unobservable?

Without the ability to track consumers' valuation evolution, the only stationary pricing strategy is a constant price. This paper introduces a novel framework where firms adopt *non-stationary pricing strategies* – prices that evolve over time without being contingent on consumers' current valuation, which endogenously induces non-stationarity in the consumer's search problem. We address two key questions: (1) Is a stationary pricing strategy always optimal for a firm that cannot observe belief evolution? (2) If not, what are the characteristics of the optimal non-stationary pricing strategy?

Our findings challenge the conventional reliance on stationary pricing. We show that non-stationary pricing strategies can outperform stationary ones. We prove that a consumer can do almost as well by approximating any price which is sufficiently slow–moving by linear price if she is sufficiently myopic, which can be a building block for future research to simplify the strategy space of non-Markov problems. Given this result, by assuming that the consumer is sufficiently myopic and the price is linear and varies slowly, we show that, under zero search costs, a perfectly patient firm's optimal price is arbitrarily close to constant, but with discounting, the slope of the optimal price is bounded away from zero. When search costs are positive, the optimal price is non-stationary even if the firm is perfectly patient. The price increases over time if the information is too noisy or the search cost is too high. The direction of price trajector is more nuanced in other cases where consumers have a stronger incentive to search, with increasing prices for consumers with high or low initial valuation and decreasing prices for medium–value consumers.

By incorporating non-stationary strategies into a consumer search framework, we provide a theoretical advance in optimal control. Unlike most economic models, which impose stationarity for tractability, our results highlight that such restrictions may lead to sub-optimal outcomes. While a few earlier papers have explored non-stationarity in search problems driven by exogenous environment such as the finite horizon and the evolving distribution of rewards (Gilbert and Mosteller, 1966; Sakaguchi, 1978; Van den Berg, 1990; Smith, 1999; Kamada and Muto, 2015), we model endogenous non-stationarity arising from firms' strategic pricing in response to consumer gradual learning. To our knowledge, this is the first paper to study endogenous non-stationary pricing under consumer gradual learning, providing a foundational step toward understanding firms' non-stationary interventions in this context.

We also offer practical insights into how firms can adapt to privacy regulations. Non-stationary pricing leverages time – a freely available and regulation-resistant resource – as an information source for pricing decisions, reducing reliance on costly tracking technologies. This is especially relevant in light of privacy-driven disruptions, such as Apple's iPhone privacy upgrades, which cost publishers nearly $10 billion in ad revenue in 2021 alone.[1] Our work provides guidance on how firms can proactively adjust their pricing strategies to thrive in a privacy-conscious environment.

2 The Model

A firm offers a product with a marginal cost of g and chooses the price. A consumer decides whether to purchase it or not. The consumer's initial valuation is v_0, which is common knowledge. Before making a decision, she can gradually learn about various product attributes to update her belief about the product's value. We will focus on the learning processes that arise from the general non–linear optimal filtering framework (Liptser and Shiryaev, 2013, Chapter 8). Suppose that the total utility the consumer gets from consuming the product is given by an unobservable process $\{\pi_t\}_{t\geq 0}$. The consumer pays the flow search

[1] Source: https://www.businessinsider.com/apple-iphone-privacy-facebook-youtube-twitter-snap-lose-10-billion-2021-11.

cost cdt per dt time to learn about π_t by observing a process $\{S_t\}_{t\geq 0}$ which generates a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ such that $v_t := \mathbb{E}[\pi_t|\mathscr{F}_t]$ is a continuous martingale[2]. We will denote the valuation process as $\{v_s^{t,x}\}_{s\geq t}$ if we would like to emphasize the initial condition $v_t^{t,x} = x$, or as $\{v_t^x\}_{t\geq 0}$ if the initial condition $v_0^x = x$ is at $t = 0$, or simply as v_t when the initial value x is not so important. We assume that $\{v_s^{t,x}\}_{s\geq t}$ is the unique strong solution to:

$$dv_s^{t,x} = \mu(v_s^{t,x}, \pi_s)ds + \sigma(v_s^{t,x})dW_s, \qquad v_t^{t,x} = x, \tag{1}$$

where $\{W_t\}_{t\in\mathbb{R}_{\geq 0}}$ is the standard Brownian motion adapted to the filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in\mathbb{R}_{\geq 0}}, \mathbb{P})$. In particular, we have $\mathbb{E}[dv_t|\mathscr{F}_t] = 0$ and $\mathbb{E}[(dv_t)^2|\mathscr{F}_t] = \sigma(v_t)^2 dt$. We impose the following assumptions on $\mu(.)$ and $\sigma(.)$:

Assumption 1. *Let $\underline{\pi}, \bar{\pi} \in \mathbb{R} \cup \{\pm\infty\}$ be such that $\underline{\pi} \leq v_t \leq \bar{\pi}$ a.e., for all $t \in \mathbb{R}$, then we assume that:*

- $\mu(., \pi), \sigma(.) \in C^\infty(\underline{\pi}, \bar{\pi})$.
- $\sigma(x) > 0$ *for all* $x \in (\underline{\pi}, \bar{\pi})$.
- *The global Lipschitz condition holds:*

$$|\mu(x, \pi_t) - \mu(y, \pi_t)| + |\sigma(x) - \sigma(y)| \leq L|x - y|, \tag{2}$$

for some constant $L \geq 0$, for all $t \in \mathbb{R}$ and $x, y \in [\underline{\pi}, \bar{\pi}]$.

Note that $\underline{\pi}, \bar{\pi}$ represents the highest and lowest possible values of the product (i.e. of π_t) and we are open to the possibility that $\bar{\pi} = +\infty$, $\underline{\pi} = -\infty$. Given $\mathbf{x} = (t, x) \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}]$, we shall write $\mu(\mathbf{x})$ and $\sigma(\mathbf{x})$ instead of $\mu(x)$ and $\sigma(x)$ when it is more convenience to work in the 'vector' notation, even though $\mu(.)$ and $\sigma(.)$ do not depend directly on t.

Remark 1. *The assumption that $\{v_t\}_{t\geq 0}$ is a strong solution to the SDE (1) implies that $\{v_t\}_{t\geq 0}$ is a square–integrable martingale, i.e. $\mathbb{E}[v_t^2] < \infty$ for all $t \geq 0$. The global Lipschitz constant L plays the role in controlling the growth–rate of the square integral, in fact, we have $\mathbb{E}[v_t^2] = O(e^{L^2 t})$. This consider serves various technical purposes, such as to ensure that the expression (3) is well defined. A sufficient condition is $\sup_{\tau\in\mathcal{T}} \mathbb{E}\left[e^{-2r\tau} v_\tau^2\right] < \infty$, which holds as long as $L < \sqrt{r}$, as we shall implicitly assume to be the case for the remainder of this work. This assumption is far from restrictive as it covers many applications we considered, including when v_t is the standard Brownian motion ($L = 0$), when v_t is bounded (finite $\underline{\pi}, \bar{\pi}$), or when the consumer is myopic (r is large).*

[2] A sufficient condition for $\{v_t\}_{t\geq 0}$ to be martingale with respect to $\{\mathscr{F}_t\}_{t\geq 0}$ is if $\{\pi_t\}_{t\geq 0}$ is martingale with respect to the filtration generated by $\{(S_t, \pi_t)\}_{t\geq 0}$.

Previous works that study the firm's endogenous pricing strategy in the presence of consumer gradual learning either assume that the firm perfectly observes the consumer's valuation evolution processes (v_t is common knowledge between the consumer and the firm) and can condition the price on the consumer's current valuation, or that the firm charges a constant price over time. Suppose we define the state variable by the consumer's current valuation, as is standard in the literature. The firm's problem in the first scenario is to choose the optimal *Markov strategy*. This setup does not always fit real-world examples. Recent privacy regulations like GDPR and CCPA have disrupted firms' ability to track individual consumers in real time. Even if a firm can track consumers' browsing behavior, it may be hard for the firm to know how consumers will interpret the information they see. Moreover, in many offline settings, individual-level tracking is not feasible.

Without the ability to observe the consumer's valuation evolution and tailor prices based on v_t, the only stationary pricing strategy is a constant price. Is a stationary pricing strategy always optimal for a firm in such cases? The major innovation of this paper is to consider *non-stationary pricing strategies*, prices that evolve over time without being contingent on consumers' current valuation, which endogenously induces non-stationarity in the consumer's search problem. Formally, the firm can commit to a pricing scheme $p := \{p_t\}_{t\geq 0} \in \mathcal{P}$, where $\mathcal{P}$ is the set of *admissible* pricing strategies, a subset of smooth functions on $[0, \infty)$, $\mathcal{P} \subset C^\infty[0, \infty)$. This pricing strategy is a *non-Markov strategy* because p_t depends on history (time t) other than the current state v_t. It is widely known in optimal control that it is much harder to characterize *non-Markov strategies* than *Markov strategies*.

The consumer's search strategy consists of choosing an appropriate stopping time. Denote by $\mathcal{T}$ the set of all stopping times adapted to $\{\mathscr{F}_t\}_{t\in\mathbb{R}_{\geq 0}}$. We formalize the setup as a game with two players, a consumer ("Buyer" B) and a firm ("Seller" S), playing in the following sequence:

1. At $t = 0$, the firm commits to a pricing strategy $p \in \mathcal{P} \subset C^\infty[0, \infty)$.
2. At any $t > 0$, the consumer decides whether to purchase the product, exit, or search for more information.
3. The game ends when the consumer makes a purchase or exits.

The only knowledge the seller has about the consumer is their initial valuation, v_0. Importantly, when the consumer decides whether to purchase the product, exit, or keep searching at any given time, she takes into account both the current price and the future price trajectory. For any $p \in \mathcal{P}$ and $\tau \in \mathcal{T}$, we define [3]

$$\mathcal{V}^B(t, x; \tau, p) := \mathbb{E}\left[e^{-r(\tau-t)} \max\{v_\tau^{t,x} - p_\tau, 0\} - \int_t^\tau ce^{-r(s-t)}ds \mid \mathscr{F}_t\right] \tag{3}$$

and

$$\mathcal{V}^S(x; \tau, p) := \mathbb{E}\left[e^{-m\tau}(p_\tau - g) \cdot 1_{v_\tau^x \geq p_\tau}\right] \tag{4}$$

as the corresponding consumer's and firm's expected payoffs, respectively.

Commitment Assumption

We assume that the firm has dynamic commitment power. It would be a relatively strong assumption if the firm could track consumer valuation processes due to the hold-up problem. When new information (consumer's current valuation) arrives, the firm has an incentive to deviate from the pricing scheme announced at the beginning to extract more surplus from the consumer. Anticipating the firm's incentive to deviate, the consumer will not start searching without a commitment device. Ning (2021) shows that a firm without commitment power can address the hold-up problem by offering a "list price." In contrast, the firm does not need to offer a list price if it has dynamic commitment power. In such cases, whether the firm has commitment power will lead to qualitatively different results.

In our setting, the firm does not receive new information about consumer valuation v_t over time. In addition, the only new information the firm can learn at time t is whether the consumer has made a purchasing decision, which will not affect the firm's strategy because the game ends whenever the consumer purchases the product or exits. The firm does not have any new information during the game. Therefore, there is no hold-up problem and the firm does not have an incentive to deviate from the announced pricing strategy. Thus, the commitment assumption does not qualitatively affect the equilibrium outcome in this case. We make the assumption mainly for a cleaner analysis and presentation.

Solution Concept

We consider the following equilibrium concept.

Definition 1. *An ε–Subgame perfect Nash's equilibrium (ε–SPNE) consists of:*

$$(\{\tau^*[p] \in \mathcal{T}\}_{p \in \mathcal{P}}, p^* \in \mathcal{P})$$

[3] For simplicity, we use p to denote $\{p_t\}_{t\geq 0}$ whenever this does not cause confusion.

such that: for all $p \in \mathcal{P}$,

$$\mathcal{V}^B(t,x;\tau^*[p],p) \geq \mathcal{V}^B(t,x;\tau,p) - \varepsilon, \quad \forall \tau \in \mathcal{T},$$
$$\textit{and} \;\; \mathcal{V}^S(x;\tau^*[p^*],p^*) \geq \mathcal{V}^S(x;\tau^*[p],p) - \varepsilon, \quad \forall p \in \mathcal{P}.$$

The consumer's value function given the seller's pricing strategy p is:

$$V^B(t,x;p) := \sup_{\tau \in \mathcal{T}} \mathcal{V}^B(t,x;\tau,p). \tag{5}$$

When there is no ambiguity, we will compactly write $V^B(t,x) = V^B(t,x;p)$. Analogously, we define the seller's value function:

$$V^S(x) := \sup_{p \in \mathcal{P}} \mathcal{V}^S(x;\tau^*[p],p) \tag{6}$$

Our choice of the equilibrium concept is motivated by the greater analytical traceability of the problem via perturbation theory to the order of ε. For instance, we later solve for an analytical closed form of a myopic consumer's strategy to a slow moving pricing using linear approximation to the order of ε. For further discussion we refer to Assumption 2. There is also a technical reason for such an equilibrium concept as we do not need to be concerned about the existence of $\tau^*[p] \in \mathcal{T}$ or $p^* \in \mathcal{P}$ that achieve the supremum. In a certain case, it is possible to show that the firm's profit supremum can only be approached via a limit of an admissible pricing strategy (we did not require $\mathcal{P}$ to be closed in general).

3 Consumer's Strategy

The consumer faces an optimal stopping problem. She needs to determine the purchasing and quitting boundaries at any time. When the price is non-stationary, the consumer's purchasing and quitting boundaries are also time-contingent. This time-varying property makes her optimal stopping problem challenging even if we fix a pricing scheme. To illustrate the impact of non-stationary pricing on the consumer's problem, we first review the benchmark with constant price.

3.1 Benchmark: Stationary Pricing

When the price is constant, $p_t = p_0 \in \mathbb{R}$, the consumer's search strategy does not depend on time. In particular, we have a time–independent value function $V^B(t,x;p_0) = V_0^B(x;p_0)$, purchasing threshold $\bar{V}_t = \bar{V}[p_0]$, and quitting threshold $\underline{V}_t = \underline{V}[p_0]$. The value function of the consumer satisfies the Hamilton–Jacobi–Bellman (HJB) equation:

$$\frac{\sigma(x)^2}{2}\partial_x^2 V_0^B - rV_0^B - c = 0 \tag{7}$$

subjects to the value matching condition and smooth pasting conditions:

$$\begin{aligned} V_0^B(\bar{V}[p_0]; p_0) &= \bar{V}[p_0] - p_0, \quad \partial_x V_0^B(\bar{V}[p_0]; p_0) = 1, \\ V_0^B(\underline{V}[p_0]; p_0) &= 0, \quad \partial_x V_0^B(\underline{V}[p_0]; p_0) = 0. \end{aligned}$$

We refer to Strulovici and Szydlowski (2015) for the derivation of the free–boundary ODE problem from the optimal stopping problem along with the results which guarantees the existence and uniqueness of the solution in our setting.[4] In particular, V_0^B is continuously differentiable for all $x \in \mathbb{R}$ and twice continuously differentiable for all $x \in \mathbb{R} \setminus \{\underline{V}[p_0], \bar{V}[p_0]\}$.

We now consider two specific learning structures commonly used in the literature.

3.1.1 Product attributes learning

We consider the learning process studied in Branco et al. (2012). The consumer gradually learns about various product attributes to update her belief about the product's value before making a purchase decision. We assume each attribute i has a ground–truth utility of x_i relative to the outside–option counterpart. The total product's utility relative to the outside option (which we normalized to zero) based on the t searched attributes is $\pi_t := \sum_{i=0}^{t} x_i$. When there are infinitely many attributes, each with a very small weight in value, π_t becomes a Brownian motion: $d\pi_t = \sigma dW_t^\pi$, for some constant σ. The consumer learns about $\{\pi_t\}_{t\geq 0}$ by observing the signal $\{S_t\}_{t\geq 0}$, where $dS_t := \pi_t dt + \sigma_S dW_t$, for some constant σ_S representing the information quality, or the amount of attention the consumer gives in learning. Assume the normal prior belief $\pi_0 \sim \mathcal{N}(v_0, \sigma\sigma_S)$, then we have $dv_t = \frac{\sigma}{\sigma_S}(\pi_t - v_t)dt + \sigma dW_t$, or simply:

$$dv_t = \sigma dW_t^v \tag{8}$$

by Lévy characterization, where $\{W_t^v\}_{t\geq 0}$ is a standard Brownian motion adapted to $\{\mathscr{F}_t\}_{t\geq 0}$. We shall write W_t^v simply as W_t for convenience, hereafter. Note that we have $\underline{\pi} = -\infty$ and

[4] Unlike in Strulovici and Szydlowski (2015) our $\mu(.)$ also depends on another unobservable process $\{\pi_t\}_{t\geq 0}$. One work–around is to consider the process $X_t := (v_t, \pi_t)$ which generates the full information filtration $\{\mathscr{F}_t^X\}_{t\geq 0}$, follows Strulovici and Szydlowski (2015), then take another expectation condition on $\{\mathscr{F}_t\}_{t\geq 0}$ at the end. Such detail is not so critical in this section as we will merely consider a few benchmark examples for motivation and we will later study the general version of this problem rigorously via the viscosity solutions framework.

$\bar{\pi} = +\infty$ in this case. Branco et al. (2012) characterize closed-form expressions for the value function:

$$V_0^B(x; p_0) = \frac{c}{r}\left[\cosh\frac{\sqrt{2r}}{\sigma}(x - \underline{V} - p_0) - 1\right], \tag{9}$$

and for the purchasing and quitting boundaries $\bar{V}[p_0] := p_0 + \bar{V}$, $\underline{V}[p_0] := p_0 + \underline{V}$, where:

$$\bar{V} := \sqrt{\frac{c^2}{r^2} + \frac{\sigma^2}{2r}} - \frac{c}{r}, \tag{10}$$

$$\underline{V} := \left(\sqrt{\frac{c^2}{r^2} + \frac{\sigma^2}{2r}} - \frac{c}{r}\right) - \frac{\sigma}{\sqrt{2r}}\log\left(\sqrt{\frac{r\sigma^2}{2c^2}} + \sqrt{1 + \frac{r\sigma^2}{2c^2}}\right). \tag{11}$$

3.1.2 Binary classification with Bayesian updating

We consider a product with the ground–truth value given by the time–independent binary random variable $\pi_t = \pi \in \{0, 1\}$. To make a purchase decision, the consumer must classify whether the product has a *high* value ($\pi = \bar{\pi} = 1$) or *low* value ($\pi = \underline{\pi} = 0$). Given the initial expectation is $v_0 = \mathbb{E}[\pi|\mathscr{F}_0] \in [0, 1]$, the consumer can further learn the value of π by observing the signal $\{S_t\}_{t\geq 0}$, where $dS_t := \pi dt + \sigma_S dW_t$. Then we have

$$dv_t = \frac{v_t(1 - v_t)}{\sigma_S^2}\left[(\pi - v_t)dt + \sigma_S dW_t\right]. \tag{12}$$

The resulting free–boundary ODE problem has been considered in Ke and Villas-Boas (2019) in the non–discounting case: $r = 0$. The solution is as follows:

$$V_0^B(x; p_0) = A_+ x^{m_+}(1 - x)^{m_-} + A_- x^{m_-}(1 - x)^{m_+} - \frac{c}{r} \tag{13}$$

where $m_\pm := \frac{1 \pm \sqrt{1 + 8r\sigma_S^2}}{2}$, $A_\pm := \frac{\bar{V}[p_0](1 - \bar{V}[p_0]) + (\bar{V}[p_0] - p_0 + c/r)(\bar{V}[p_0] - m_\mp)}{(m_\pm - m_\mp)\bar{V}[p_0]^{m_\pm}(1 - \bar{V}[p_0])^{m_\mp}}$, and $\bar{V}[p_0]$, $\underline{V}[p_0]$ can be solved from:

$$\frac{\bar{V}[p_0](1 - \bar{V}[p_0]) + (\bar{V}[p_0] - p_0 + c/r)(\bar{V}[p_0] - m_\mp)}{\bar{V}[p_0]^{m_\pm}(1 - \bar{V}[p_0])^{m_\mp}} = \frac{(c/r)(\underline{V}[p_0] - m_\mp)}{\underline{V}[p_0]^{m_\pm}(1 - \underline{V}[p_0])^{m_\mp}} \tag{14}$$

3.1.3 Comparison Between Our Problem and the Benchmarks

Comparing the benchmarks and our problem, we can see that stationarity simplifies the problem significantly. In the benchmark model, the consumer's entire optimal stopping strategy can be summarized by **two unknowns**: $\bar{V}[p_0]$ and $\underline{V}[p_0]$. The consumer will purchase the product at any time during the search if her valuation reaches the purchasing threshold and will quit searching at any time if her valuation reaches the quitting threshold. In contrast, the consumer's entire optimal stopping strategy consists of **an infinite number of unknowns**. Knowing that the price changes over time, the consumer's purchasing and quitting thresholds also evolve. She has different purchasing and quitting thresholds at different times. So, instead of pinning down a one-dimensional purchasing/quitting threshold, we need to determine a two-dimensional purchasing/quitting boundary. These time-dependent thresholds significantly complicate our problem.

3.2 Consumer's Strategy under Non–Stationary Pricing

The set of admissible pricing strategies we consider is:

$$\mathcal{P} := \{p \in C^{\infty}[0,\infty) \mid p_t' + r(\bar{\pi} - p_t) + c > 0, p_t > \underline{\pi}, \text{ for all } t \in [0,\infty)\}.$$

The conditions on p_t and p_t' ensure that it is optimal for a consumer with $v_t = \bar{\pi}$ or $v_t = \underline{\pi}$ to either immediately make a purchase or exit. Note that any constant pricing policy $p_0 \in [\underline{\pi}, \bar{\pi}]$ is in $\mathcal{P}$, and that the condition on p_t' also controls the price growth rate, i.e. $\lim_{t\to\infty} e^{-rt} p_t = 0$. For the learning process such that $\bar{\pi} = +\infty$, $\underline{\pi} = -\infty$, all the conditions are trivial, so we can take $\mathcal{P} := C^{\infty}[0,\infty)$. We will also work with the subset $\mathcal{P}_T \subset \mathcal{P}$ of strategies that are constant after some amount of time $T > 0$:

$$\mathcal{P}_T := \{p \in \mathcal{P} \mid p_t = p_T, \forall t \geq T\},$$

which is helpful in some existence and uniqueness arguments. The idea is to first establish technical results for a finite $T > 0$, then take the limit $T \to \infty$ to establish the result for strategies in $\mathcal{P}$. We start with the following intuitive characterization of $V^B(t, x; p)$:

Lemma 1. *Let $p \in \mathcal{P}_T$ be a pricing strategy.*

1. *$V^B(t, x; p)$ is monotonically increasing in x for any fixed t.*
2. *Let $q \in \mathcal{P}_T$ be another pricing strategy such that $q_t \geq p_t$ for all $t \in \mathbb{R}$, then $V^B(t, x; q) \leq V^B(t, x; p)$ for all fixed (t, x).*

Instead of directly finding the optimal $\tau^*[p] \in \mathcal{T}$ to the optimization problem (5) we consider the corresponding HJB equation:

$$H(t, x, V, \nabla V, \Delta V) = 0 \tag{15}$$

where $H : (\mathbb{R} \times [\underline{\pi}, \bar{\pi}]) \times \mathbb{R} \times \mathbb{R}^2 \times \mathcal{S}_2(\mathbb{R}) \to \mathbb{R}$ is given by

$$H\left(t, x, V, \nabla V, \Delta V\right) := \min\left\{c + rV - \partial_t V - \frac{\sigma(x)^2}{2}\partial_x^2 V, V - x + p_t, V\right\},$$

with $\mathcal{S}_2(\mathbb{R})$ denoted the space of 2×2 symmetric matrices. Since $p \in \mathcal{P}_T$ is only defined for $t \geq 0$, to discuss the solution on $\mathbb{R} \times [\underline{\pi}, \bar{\pi}]$, let us extends it by defining $p_t = p_0$ for all $t < 0$, analogous to how we defined $p_t = p_T$ for $t \geq T$. We consider the solution $V : \mathbb{R} \times [\underline{\pi}, \bar{\pi}] \to \mathbb{R}$ subject to the following asymptotic boundary conditions:

$$\begin{aligned} V(t,x) &= V_0^B(x; p_T),\ \forall t \geq T, & \lim_{t \to -\infty} V(t,x) &= V_0^B(x; p_0) \\ V(t,x) &= x - p_t,\ \forall x \geq \bar{V}_t[p], & V(t,x) &= 0,\ \forall x \leq \underline{V}_t[p] \end{aligned} \tag{16}$$

for some functions $\bar{V}[p], \underline{V}[p] : \mathbb{R} \to [\underline{\pi}, \bar{\pi}]$, depending on $p \in \mathcal{P}_T$ and $\bar{V}_t[p] \geq \underline{V}_t[p], \forall t \in \mathbb{R}$. The purchase and exit boundaries: $\bar{V}[p]$, and $\underline{V}[p]$, provide a simple characterization of the learning strategy. Note that by definition of $\mathcal{P}$, $\bar{V}[p]$ and $\underline{V}[p]$ are contained in $[\underline{\pi}, \bar{\pi}]$. The following results characterize some properties of the purchase and exit boundaries.

Proposition 1. *Let $p \in \mathcal{P}_T$ be a pricing strategy and $h : \mathbb{R} \to \mathbb{R}_{\geq 0} \in \mathcal{P}_T$ be strictly monotonically increasing over $[0, T)$.*

1. *Suppose that $h_0 = 0$, then at $t = 0$ the purchase and exit boundaries under the pricing strategy $\tilde{p} := p + h$ satisfies $\bar{V}_0[\tilde{p}] \leq \bar{V}_0[p]$, and $\underline{V}_0[\tilde{p}] \geq \underline{V}_0[p]$.*

2. *Let $K \in \mathbb{R}$ be a constant, then under the pricing strategy $\tilde{p} := p + Kh$ we have:*

$$\begin{aligned} \bar{V}_t[\tilde{p}] \searrow \max\{\tilde{p}_t, \underline{\pi}\}, \underline{V}_t[\tilde{p}] \nearrow \min\{\tilde{p}_t, \bar{\pi}\} \quad & \text{as } K \to +\infty \\ \bar{V}_t[\tilde{p}] \nearrow \bar{\pi}, \underline{V}_t[\tilde{p}] \searrow \underline{\pi} \quad & \text{as } K \to -\infty \end{aligned}$$

   for any given $t \in [0, T)$.

Remark 2. *The first part of Proposition 1 also implies that if $\tilde{p} := p + h$ and $h : \mathbb{R} \to \mathbb{R}_{\leq 0} \in \mathcal{P}_T$ is strictly monotonically decreasing then $\bar{V}_0[\tilde{p}] \geq \bar{V}_0[p]$, and $\underline{V}_0[\tilde{p}] \leq \underline{V}_0[p]$. This can be seen by switching the role of $\tilde{p}$ and p in the proposition's statement. We further note that the statement is specifically at $t = 0$ where $h_0 = 0$. It is not true that $\bar{V}_t[\tilde{p}] \leq \bar{V}_t[p]$ and $\underline{V}_t[\tilde{p}] \geq \underline{V}_t[p]$ for all $t \in \mathbb{R}$ given a monotonically increasing $h : \mathbb{R} \to \mathbb{R}_{\geq 0}$, for example see the explicit calculation in the case of linear pricing in Proposition 4. Intuitively, although the increasing price may have an immediate effects in forcing a high–value consumer to purchase instead of continue searching, such effects is short–term, if the price continues to increase for long enough then such a consumer will be hesitant due to the higher price.*

We need to establish the existence and uniqueness of the solution to (15) subjects to the boundary condition (16). Since the classical solution does not always exist, we will work with a relaxed notion of a *viscosity* solution:[5]

Definition 2. *Let $D \subset \mathbb{R}^n$ and $H : D \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}_n(\mathbb{R}) \to \mathbb{R}$ be a continuous function satisfying the* properness *condition: $H(\mathbf{x}, v, p, X) \geq H(\mathbf{x}, u, p, X)$ if $v \geq u$, and the* degenerate ellipticity *condition: $H(\mathbf{x}, v, p, X) \geq H(\mathbf{x}, v, p, Y)$ if $Y \geq X$.*

A continuous function $v : D \to \mathbb{R}$ is a viscosity subsolution *if for any $\mathbf{x}_0 \in D$ and any twice continuously differentiable function ϕ such that $\mathbf{x}_0$ is a local maximum of $v - \phi$ we have $H(\mathbf{x}_0, v(\mathbf{x}_0), \nabla\phi(\mathbf{x}_0), \Delta\phi(\mathbf{x}_0)) \leq 0$.*

A continuous function $v : D \to \mathbb{R}$ is a viscosity supersolution *if for any $\mathbf{x}_0 \in D$ and any twice continuously differentiable function ϕ such that $\mathbf{x}_0$ is a local minimum of $v - \phi$ we have $H(\mathbf{x}_0, v(\mathbf{x}_0), \nabla\phi(\mathbf{x}_0), \Delta\phi(\mathbf{x}_0)) \geq 0$.*

A continuous function $v : D \to \mathbb{R}$ is a viscosity *solution if it is both a viscosity subsolution and supersolution.*

The following result relates the consumer's value function V^B to the viscosity solution over the domain $D := \mathbb{R} \times [\underline{\pi}, \bar{\pi}]$.

Lemma 2. *For a given $p \in \mathcal{P}_T$, the consumer's value function V^B is the unique viscosity solution to (15) subject to the asymptotic boundary conditions (16).*

Working directly with the viscosity solution via definition 2 can still be challenging, thus we alternatively consider the following free–boundary backward parabolic PDE boundary value problem: Find $V : \Omega \to \mathbb{R}$, and continuously differentiable functions $\bar{V}[p], \underline{V}[p] : \mathbb{R} \to [\underline{\pi}, \bar{\pi}]$ satisfying $\bar{V}_t[p] \geq \underline{V}_t[p]$, such that

$$\begin{cases} \dfrac{\sigma(x)^2}{2}\partial_x^2 V(t,x) + \partial_t V(t,x) - rV(t,x) - c = 0, & (t,x) \in \Omega \\ V(t, \bar{V}_t[p]) = \bar{V}_t[p] - p_t, & V(t, \underline{V}_t[p]) = 0, \\ \partial_x V(t, \bar{V}_t[p]) = 1, & \partial_x V(t, \underline{V}_t[p]) = 0, \\ V(T, x) = V_0^B(x; p_T), \end{cases} \tag{17}$$

where

$$\Omega := \{(t,x) \in (-\infty, T] \times [\underline{\pi}, \bar{\pi}] \mid \underline{V}_t[p] < x < \bar{V}_t[p]\}.$$

This PDE connects us back to the constant price benchmark calculations in §3.1, although now we have the *moving* purchase and exit boundaries $\bar{V}[p]$ and $\underline{V}[p]$ instead of

[5] Crandall et al. (1992) provides a detailed description of the viscosity solution.

the fixed counterpart back in §3.1. The second and the third lines of (17) amount to the value–matching and the smooth–pasting conditions at the purchase and exit boundaries, respectively. Given a solution V to (17) on Ω with the specified boundary conditions, we can extend it to $\widetilde{V}$, a function continuously differentiable on $\mathbb{R} \times [\underline{\pi}, \bar{\pi}]$, and twice continuously differentiable in x on $\mathbb{R} \times [\underline{\pi}, \bar{\pi}] \setminus \partial\Omega$, by defining $\widetilde{V}(t,x) = \max\{x - p_t, 0\}$ if $t \leq T$ and $x \notin (\underline{V}_t[p], \bar{V}_t[p])$, and $\widetilde{V}(t,x) = V_0^B(x; p_T)$ if $t > T$. This extension is rather natural, therefore, we will abuse the notation and simply refer to $\widetilde{V}$ as V. The upshot is that the solution V will coincides with the consumer's value function V^B, as we state formally below in Lemma 3. This justifies that the constant price benchmark solutions we solved in §3.1 are the viscosity solutions, hence the value functions of their respective consumer's problems.

Solving (17) in full generality is beyond the scope of this research. For an arbitrary given pricing policy $p \in \mathcal{P}_T$, it is likely that there exists no analytical closed–form viscosity solution. However, if p is a small perturbation from a nice policy with a known solution, then we expect the solution corresponding to p to be a small perturbation from the known solution. The PDE formulation of the problem (17) enables us to employ the perturbation theory.

Suppose that we know value function $V^B(.,.;p)$ for a given $p \in \mathcal{P}_T$ is a solution to (17), and we would like to compute $V^B(.,.;p + \sqrt{\varepsilon}h)$ for some $h \in \mathcal{P}_T$ and a small $\varepsilon > 0$. By Lemma 3, we aim to solve for the corresponding PDE solution $V(.,.;p + \sqrt{\varepsilon}h)$. The idea of perturbation theory is to proceed by writing $V(.,.;p + \sqrt{\varepsilon}h) = V_0(.,.) + V_1(.,.)\sqrt{\varepsilon} + V_2(.,.)\varepsilon + \cdots$, where $V_0(.,.) := V^B(.,.;p)$, and $\bar{V}_t[p + \sqrt{\varepsilon}h] = \bar{V}_{0,t} + \bar{V}_{1,t}\sqrt{\varepsilon} + \bar{V}_{2,t}\varepsilon + \cdots$, $\underline{V}_t[p + \sqrt{\varepsilon}h] = \underline{V}_{0,t} + \underline{V}_{1,t}\sqrt{\varepsilon} + \underline{V}_{2,t}\varepsilon + \cdots$, where $\bar{V}_{0,t} := \bar{V}_t[p], \underline{V}_{0,t} := \underline{V}_t[p]$. By substituting these expansions into (17) and comparing the $\varepsilon^{k/2}$ terms for $k = 1, 2, \cdots$, we can solve for $V_k, \bar{V}_k, \underline{V}_k$ using the knowledge of $V_{k'}, \bar{V}_{k'}, \underline{V}_{k'}$ for $k' = 0, \cdots, k-1$. In general, the validity of this procedure relies on the convergence with some positive radius of all the $\sqrt{\varepsilon}$–power series involved. The readers are welcome to accept this as an assumption and skip the technical detail in the next paragraph and the second part of Lemma 3. However, we also provide a self–contained comparison principle argument to formalize the process compatible with the ε–equilibrium concept, as we will state below in Lemma 3.

Suppose that $V^\varepsilon_{\leq k}(.,.;p + \sqrt{\epsilon}h) := V_0(.,.) + V_1(.,.)\sqrt{\varepsilon} + \cdots + V_k(.,.)\varepsilon^{k/2}$, $\bar{V}^\varepsilon_{\leq k,t}[p + \sqrt{\varepsilon}h] := \bar{V}_{0,t} + \bar{V}_{1,t}\sqrt{\varepsilon} + \cdots + \bar{V}_{k,t}\varepsilon^{k/2}$, and $\underline{V}^\varepsilon_{\leq k,t}[p + \sqrt{\varepsilon}h] := \underline{V}_{0,t} + \underline{V}_{1,t}\sqrt{\varepsilon} + \cdots + \underline{V}_{k,t}\varepsilon^{k/2}$, satisfies (17) over $\Omega^\varepsilon_{\leq k} := \{(t,x) \in (-\infty, T] \times [\underline{\pi}, \bar{\pi}] | \underline{V}^\varepsilon_{\leq k,t} < x < \bar{V}^\varepsilon_{\leq k,t}\}$ up to the $\varepsilon^{(k+1)/2}$–order. Note that although both the value–matching and smooth–pasting conditions are only satisfied to the $\varepsilon^{(k+1)/2}$–order, it is possible to find a twice continuously differentiable function $\chi : \mathbb{R} \times [\underline{\pi}, \bar{\pi}] \setminus \Omega^\varepsilon_{\leq k} \to \mathbb{R}$ which continuously differentiably transitions from $V^\varepsilon_{\leq k}$ at $\partial\Omega^\varepsilon_{\leq k}$ to $\max\{x - p_t, 0\}$ for all (t,x) some distance $R > 0$ away from $\Omega^\varepsilon_{\leq k}$, e.g. a smooth 'bump' function. In particular, we have $\chi = V^\varepsilon_{\le k}$ and $\nabla\chi = \nabla V^\varepsilon_{\le k}$ on $\partial\Omega^\varepsilon_{\le k}$, and additionally we require that $|\partial_t\chi(t,x) - p'_t - \sqrt{\varepsilon}h'_t| = O(\varepsilon^{(k+1)/2})$, $|\partial^2_x\chi| = O(\varepsilon^{(k+1)/2})$, and that the asymptotic boundary conditions (16) are met. We extend $V^\varepsilon_{\le k}$ to $\widetilde{V}^\varepsilon_{\le k}$, a function continuously differentiable on $\mathbb{R}\times[\underline{\pi},\bar{\pi}]$, and twice continuously differentiable in x on $\mathbb{R}\times[\underline{\pi},\bar{\pi}]\setminus\partial\Omega^\varepsilon_{\le k}$, by defining $\widetilde{V}^\varepsilon_{\le k}(t,x) = \chi(t,x)$ if $t \le T$ and $x \notin (\underline{V}^\varepsilon_{\le k,t}, \bar{V}^\varepsilon_{\le k,t})$, $\widetilde{V}^\varepsilon_{\le k}(t,x) = V^B_0(x; p_T + \sqrt{\varepsilon}h_T)$ if $t > T$, and $\widetilde{V}^\varepsilon_{\le k}(t,x) = V^\varepsilon_{\le k}(t,x)$ otherwise. We shall abuse the notation and simply refer to $\widetilde{V}^\varepsilon_{\le k}$ as $V^\varepsilon_{\le k}$.

Lemma 3. *Consider $p, h \in \mathcal{P}_T$ pricing strategies and a given $\varepsilon > 0$.*

1. *If V satisfies the free–boundary backward parabolic PDE boundary value problem (17) with the pricing policy $p \in \mathcal{P}_T$, such that $V(t,x) \ge \max\{x - p_t, 0\}$, and $p'_t + r(\bar{V}_t[p] - p_t) + c \ge 0$ for all $(t,x) \in \mathbb{R}\times[\underline{\pi},\bar{\pi}]$, then V is a viscosity solution to (15). In particular, the consumer's value function is given by $V^B = V$.*

2. *If $V^\varepsilon_{\le k}$ satisfies the free–boundary backward parabolic PDE boundary value problem (17) up to the $\varepsilon^{(k+1)/2}$–order with the pricing policy $p + \sqrt{\varepsilon}h \in \mathcal{P}_T$, such that $V^\varepsilon_{\le k}(t,x) \ge \max\{x - p_t, 0\} + O(\varepsilon^{(k+1)/2})$, and $p'_t + \sqrt{\varepsilon}h'_t + r(\bar{V}^\varepsilon_{\le k,t}[p + \sqrt{\varepsilon}h] - p_t - \sqrt{\varepsilon}h_t) + c \ge O(\varepsilon^{(k+1)/2})$ for all $(t,x) \in \mathbb{R}\times[\underline{\pi},\bar{\pi}]$, then $V^B = V^\varepsilon_{\le k} + O(\varepsilon^{(k+1)/2})$.*

We will use $k = 1$ in our application of perturbation theory, solving (17) up to the $\sqrt{\varepsilon}$–order, to be consistent with the ε–equilibrium concept. In other words, we will have

$$V^B(t,x; p + \sqrt{\varepsilon}h) = V^B(t,x;p) + V_1(t,x)\sqrt{\varepsilon} + O(\varepsilon),$$

and we can take the consumer's learning policy to be given by the corresponding boundaries $\bar{V}^\varepsilon_{\le 1}[p + \sqrt{\varepsilon}h] = \bar{V}[p] + \bar{V}_1\sqrt{\varepsilon}$, and $\underline{V}^\varepsilon_{\le 1}[p + \sqrt{\varepsilon}h] = \underline{V}[p] + \underline{V}_1\sqrt{\varepsilon}$.

Remark 3. *The conditions on p' and h' in Lemma 3 are sufficient conditions but may not be necessary conditions. They should not be too restrictive for us, especially when we study small perturbations from the known constant price solution and investigate the direction of consumers' reactions, which is a focus of this work. Typically, if our zero–th order perturbation for $p + \sqrt{\varepsilon}h$ is given by the consumer's value function: $V_0(.,.) = V^B(.,.;p)$, and the boundaries $\bar{V}_{0,t} = \bar{V}_t[p]$, $\underline{V}_{0,t} = \underline{V}_t[p]$, corresponding to p, then $p'_t + r(\bar{V}_t[p] - p_t) + c \ge 0$ by Lemma 2. Then it follows that $p'_t + \sqrt{\varepsilon}h'_t + r(V^\varepsilon_{\le k,t}[p + \sqrt{\varepsilon}h] - p_t - \sqrt{\varepsilon}h_t) + c \ge 0$ is satisfied for all sufficiently small $\varepsilon > 0$. To avoid unnecessary technical complications, for the remainder we shall assume that all the conditions for Lemma 3 has been taken care of whenever it is used.*

Remark 4. *Lemma 3 does not claim that* $\bar{V}[p+\sqrt{\varepsilon}h] = \bar{V}^{\varepsilon}_{\leq 1}[p+\sqrt{\varepsilon}h]+O(\varepsilon)$ *or* $\underline{V}[p+\sqrt{\varepsilon}h] = \underline{V}^{\varepsilon}_{\leq 1}[p+\sqrt{\varepsilon}h]+O(\varepsilon)$*, but this is where it is important to keep in mind that we are working with the* ε*–equilibrium concept. The solution* $V^{\varepsilon}_{\leq 1}$ *can be interpreted (up to* $O(\varepsilon)$*), via the probabilistic Feynman–Kac expression, as the expected discounted value of purchasing when the valuation process* $v^{t,x}_s$ *reaches* $\bar{V}^{\varepsilon}_{\leq 1,s}$ *and exiting when* $v^{t,x}_s$ *reaches* $\underline{V}^{\varepsilon}_{\leq 1,s}$*, under the flow cost c. Since* $V^B = V^{\varepsilon}_{\leq 1} + O(\varepsilon)$ *according to Lemma 3, the learning strategy characterized by* $\bar{V}^{\varepsilon}_{\leq 1}[p+\sqrt{\varepsilon}h]$ *and* $\underline{V}^{\varepsilon}_{\leq 1}[p+\sqrt{\varepsilon}h]$ *are considered* ε*–optimal. Given this understanding, we shall slightly abuse our notation for convenience by writing* $\bar{V}[p+\sqrt{\varepsilon}h] = \bar{V}^{\varepsilon}_{\leq 1}[p+\sqrt{\varepsilon}h]+O(\varepsilon)$ *and* $\underline{V}[p+\sqrt{\varepsilon}h] = \underline{V}^{\varepsilon}_{\leq 1}[p+\sqrt{\varepsilon}h]+O(\varepsilon)$*.*

The following proposition provides a characterization of the $\sqrt{\varepsilon}$–order perturbed boundaries in terms of the zero–th order solution.

Proposition 2. *Let* $p \in \mathcal{P}_T$ *be a given pricing strategy such that the consumer's value function* $V^B(.,.;p)$ *is a solution to the PDE (17) which is* C^∞*–smooth on* $\Omega := \{(t,x) \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}] \mid \underline{V}_t[p] < x < \bar{V}_t[p]\}$ *with the* C^∞*–smooth corresponding purchase and exit boundaries* $\bar{V}[p], \underline{V}[p] : \mathbb{R} \to (\underline{\pi}, \bar{\pi})$.[6] *Let* $h \in \mathcal{P}_T$ *be arbitrary, then under the pricing strategy* $\tilde{p} := p + \sqrt{\varepsilon}h$*, we can find an* ε*–optimal value function taking the form:*

$$V^B(t,x;\tilde{p}) = V^B(t, x - \sqrt{\varepsilon}h_t; p) + \sqrt{\varepsilon}V^B_1(t,x) + O(\varepsilon), \tag{18}$$

where $V^B_1(.,.) : \Omega \to \mathbb{R}$ *is given by:*

$$\begin{aligned} V^B_1(t,x) = &-\mathbb{E}\left[\int_t^{\tau^{t,x}_\Omega} h'_s e^{-r(s-t)} \partial_x V^B(s, v^{t,x}_s; p) ds | \mathscr{F}_t\right] \\ &+ \mathbb{E}\left[\int_t^{\tau^{t,x}_\Omega} h_s e^{-r(s-t)} \sigma(v^{t,x}_s)\sigma'(v^{t,x}_s) \partial^2_x V^B(s, v^{t,x}_s; p) ds | \mathscr{F}_t\right], \end{aligned} \tag{19}$$

where $\tau^{t,x}_\Omega := \inf\{t' \geq t \mid (t', v^{t,x}_{t'}) \notin \Omega\}$ *is the exit time. We can find the* ε*–optimal purchase and exit boundaries taking the form:*

$$\begin{aligned} \bar{V}[\tilde{p}] &= (\bar{V}[p] + \sqrt{\varepsilon}h) + \sqrt{\varepsilon}\bar{R} + O(\varepsilon) \\ \underline{V}[\tilde{p}] &= (\underline{V}[p] + \sqrt{\varepsilon}h) + \sqrt{\varepsilon}\underline{R} + O(\varepsilon) \end{aligned} \tag{20}$$

[6] We impose the C^∞–smoothness assumptions for simplicity, though they are not necessary conditions. As long as the classical solution to the $\sqrt{\varepsilon}$–order PDE boundary–value problem $V_1(t,x) \in C^{1,2}(\Omega)$ exists, then the results hold.

for functions $\bar{R} : \mathbb{R} \to \mathbb{R}$*, and* $\underline{R} : \mathbb{R} \to \mathbb{R}$ *given by:*

$$\bar{R}_t = -\frac{\partial_x V_1^B(t, \bar{V}_t[p])}{\partial_x^2 V^B(t, \bar{V}_t[p]; p)}, \qquad \underline{R}_t = -\frac{\partial_x V_1^B(t, \underline{V}_t[p])}{\partial_x^2 V^B(t, \underline{V}_t[p]; p)} \tag{21}$$

Corollary 1. *Under the setting of Proposition 2, if* $\sigma'(.) = O(\varepsilon)$ *(stable volatility), and* $h := K\tilde{h}$ *for some monotonically increasing* $\tilde{h} \in \mathcal{P}_T$ *and a constant* $K \in \mathbb{R} \setminus \{0\}$*, then* $\bar{S}_t := \bar{R}_t/K \le 0$ *and* $\underline{S}_t := \underline{R}_t/K \ge 0$ *for all* $t \in \mathbb{R}$*.*

We have shown in Lemma 3 that an $\sqrt{\varepsilon}$–order changes in p will results in $\sqrt{\varepsilon}$–order changes in the value of V, and the boundaries $\bar{V}[p], \underline{V}[p]$. The following result gives a more concrete upper–bounds:

Lemma 4. *Let* $p, q \in \mathcal{P}$ *then* $|V^B(t, x; p) - V^B(t, x; q)| \le \max_{s \ge t} e^{-r(s-t)} |p_s - q_s|$ *for all* $(t, x) \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}]$*.*

Lemma 4 shows that for consumers who are not perfectly patient, any changes in price in the far future do not have much effect in the present. This enables us to extend our consumer response results for $p \in \mathcal{P}_T$ to an arbitrary $p \in \mathcal{P}$. Note that, by definition, any $p \in \mathcal{P}$ satisfies the asymptotic condition, $\lim_{t\to\infty} e^{-rt} p_t = 0$. Let p^T be given by p over $[0, T - \varepsilon]$, constant for all $t \ge T$, and some in–between smooth transition for $t \in (T - \varepsilon, T)$. We find the solution $V(.,.; p^T)$ of the free–boundary PDE boundary value problem (17) corresponding to $p^T \in \mathcal{P}_T$, which coincides with the value function $V^B(.,.; p^T)$ according to Lemma 3. Then for all sufficiently large T we have

$$\left|V\left(t, x; p^T\right) - V^B\left(t, x; p\right)\right| = \left|V^B\left(t, x; p^T\right) - V^B\left(t, x; p\right)\right| < \varepsilon \tag{22}$$

for arbitrary given $\varepsilon > 0$. This proves that the sequence of viscosity solutions $\left\{V(.,.; p^T)\right\}_{T \ge 0}$ uniformly convergences to the value function $V^B(.,.; p)$ of an infinite horizon pricing strategy p on any compact subset of $\mathbb{R} \times [\underline{\pi}, \bar{\pi}]$.

In our work, we pay special attention to the pricing policies linear in time. Of course, such linear pricing p does not belong to $\mathcal{P}_T$ for any $T > 0$, however, this is not a problem according to Lemma 4. By choosing a sufficiently large T, an ε–optimal consumer will not differentiate between p and $p^T \in \mathcal{P}_T$. This enables us to utilize the theory we have developed so far for $\mathcal{P}_T$ on linear pricing.

As it turns out, when p is linear in t, the consumer's value function admits an analytic closed–form under some simple learning settings, such as when $\{v_t\}_{t \ge 0}$ is a vanilla Brownian motion. It is also simpler to analyze the firm's strategies restricted to the space of linear pricing. The fact that the space of linear pricing is much smaller than the general pricing space also simplifies the problem, especially when searching for the firm's optimal pricing strategy later in §4.

Consideration of linear pricing may seem restrictive, but the following proposition, which is an application of Lemma 4, shows that for myopic enough ε–optimal consumers, any pricing strategies which is sufficiently slow–moving can be approximated by linear pricing. Intuitively, unless the price changes very drastically in the far future such as growing super–exponentially, the myopic consumers do not look too far into the future, and over any sufficiently short time interval any differentiable functions *look like* a linear function.

Proposition 3. *(Almost optimality of linear price approximation) Let $p \in \mathcal{P}$ be an admissible pricing policy with $\sup_{t\in\mathbb{R}} |p''_t| \le M$. At any $\mathbf{x} = (t, x) \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}]$ we consider the linear approximation pricing policy $l_{\mathbf{x}} \in \mathcal{P} : s \mapsto l_{\mathbf{x},s} := p_t + p'_t \cdot (s-t)$.*[7] *Let the consumer's optimal learning strategy given the linear pricing $l_{\mathbf{x}}$ be $\tau^*[l_{\mathbf{x}}] \in \mathcal{T}$. If the consumer is sufficiently myopic: $r > e^{-1}\sqrt{2M/\varepsilon}$, then $\tau^*[l_{\mathbf{x}}]$ is also the consumer's ε–optimal stopping time under the p pricing strategy:*

$$\mathcal{V}^B(t, x; \tau^*[l_{\mathbf{x}}]; p) \ge V^B(t, x; p) - \varepsilon.$$

The following shows that linear perturbation is particularly simple.

Corollary 2. *Consider a linear pricing strategy $p : t \mapsto p_0 + \sqrt{\varepsilon}Kt \in \mathcal{P}$, where p_0 is a constant. Suppose that the constant price consumer's value function $V_0^B(.; p_0)$ is a solution to the PDE (17) which is C^∞–smooth on $\Omega := \{(t, x) \in \mathbb{R}\times[\underline{\pi}, \bar{\pi}] \mid \underline{V}[p_0] < x < \bar{V}[p_0]\}$, where $\bar{V}[p_0], \underline{V}[p_0] \in (\underline{\pi}, \bar{\pi})$ are the corresponding constant purchase and exit boundaries. Then we can find an ε–optimal value function given by:*

$$V^B(t, x; p) = V_0^B(x - \sqrt{\varepsilon}Kt; p_0) + \sqrt{\varepsilon}V_1^B(t, x) + O(\varepsilon)$$

where $V_1^B(t, x) = V_{1,0}^B(x) + tV_{1,1}^B(x)$ is linear in t with $V_{1,1}^B$ the unique solution to the ODE boundary–value problem:

$$\frac{\sigma(x)^2}{2}\partial_x^2 V_{1,1}^B(x) - rV_{1,1}^B(x) + K\sigma(x)\sigma'(x)\partial_x^2 V_0^B(x; p_0) = 0, \quad V_{1,1}^B(\bar{V}[p_0]) = V_{1,1}^B(\underline{V}[p_0]) = 0 \tag{23}$$

and $V_{1,0}^B$ the unique solution to the ODE boundary –value problem:

$$\frac{\sigma(x)^2}{2}\partial_x^2 V_{1,0}^B(x) - rV_{1,0}^B(x) + V_{1,1}^B(x) - K\partial_x V_0^B(x; p_0) = 0, \quad V_{1,0}^B(\bar{V}[p_0]) = V_{1,0}^B(\underline{V}[p_0]) = 0. \tag{24}$$

[7] It is also possible to apply Lemma 4 to the constant price approximation, i.e. we assume $\sup_{t\in R} |p'_t| \le M$ and consider $p_0 \in \mathcal{P} : t \mapsto p_0$ for all $t \in \mathbb{R}$. We have $\mathcal{V}^B(t, x; \tau^*[p_0]; p) \ge V^B(t, x; p) - \varepsilon$ if $r > e^{-1}M/\varepsilon$. In other words, if the consumer is very myopic, $r = O(1/\varepsilon)$, then every pricing $p \in \mathcal{P}$ can be treated as constant, which is a trivial result.

We can find the ε–optimal purchase and exit boundaries: $\bar{V}_t[p] = \bar{V}[p_0] + \sqrt{\varepsilon}Kt + \sqrt{\varepsilon}\bar{R}_t + O(\varepsilon)$, *and* $\underline{V}_t[p] = \underline{V}[p_0] + \sqrt{\varepsilon}Kt + \sqrt{\varepsilon}\underline{R}_t + O(\varepsilon)$, *where*

$$\bar{R}_t = -\frac{\partial_x V_1^B(t, \bar{V}[p_0])}{\partial_x^2 V_0^B(\bar{V}[p_0]; p_0)} =: K\bar{S}_{0,0} + K\bar{S}_{0,1}t, \quad \underline{R}_t = -\frac{\partial_x V_1^B(t, \underline{V}[p_0])}{\partial_x^2 V_0^B(\underline{V}[p_0]; p_0)} =: K\underline{S}_{0,0} + K\underline{S}_{0,1}t$$

are linear in t, for some constants $\bar{S}_{0,0}, \bar{S}_{0,1}$, $\underline{S}_{0,0}, \underline{S}_{0,1}$.

We now revisit the two consumer learning processes considered in §3.1.1 and §3.1.2 under linear pricing. We can obtain the closed-form expression for the value function given the first learning process and obtain the perturbative solution up to the ε–order given the second learning process.

Solution: Product attributes learning

The learning process in §3.1.1 is a rare example where the free–boundary PDE (17) can be solved exactly, which leads to the exact value function according to the first part of Lemma 3. The main reason is that $\sigma(.)$ is a constant in this case, and thus, the probability measure of $\{v_s^{t,x}\}_{s\geq t}$ is x–translation invariant. Therefore, we can transform the original problem to a simpler problem where the price is fixed at p_0 while the consumer valuation process is a drifted Brownian motion $v_t = -\sqrt{\varepsilon}Kt + \sigma W_t$. The transformed problem is stationary in time, with the corresponding HJB

$$\frac{\sigma^2}{2}\partial_x^2 V(x) - \sqrt{\varepsilon}K\partial_x V(x) - rV(x) - c = 0.$$

Therefore, the free–boundary problem (17) can be solved in this case by first solving the HJB above, and then making an inverse transformation back to the original problem.

Proposition 4. *Consider the consumer's learning process as in §3.1.1. Under a linear pricing strategy $p : t \mapsto p_0 + \sqrt{\varepsilon}Kt \in \mathcal{P}$, the consumer's value function is given by*

$$V^B(t,x) = A_1 e^{\frac{\sqrt{\varepsilon}K - \sqrt{\varepsilon K^2 + 2r\sigma^2}}{\sigma^2}(x - p_0 - \sqrt{\varepsilon}Kt)} + A_2 e^{\frac{\sqrt{\varepsilon}K + \sqrt{\varepsilon K^2 + 2r\sigma^2}}{\sigma^2}(x - p_0 - \sqrt{\varepsilon}Kt)} - \frac{c}{r} \tag{25}$$

with purchase and exit boundaries given by

$$\bar{V}_t = p_0 + \bar{V}[\sqrt{\varepsilon}K] + \sqrt{\varepsilon}Kt, \qquad \underline{V}_t = p_0 + \underline{V}[\sqrt{\varepsilon}K] + \sqrt{\varepsilon}Kt \tag{26}$$

where the constants $\bar{V}[\sqrt{\varepsilon}K]$, $\underline{V}[\sqrt{\varepsilon}K], A_1,$ and A_2 are determined by boundary conditions *in the appendix. To the $\sqrt{\varepsilon}$–order, $\bar{V}[\sqrt{\varepsilon}K]$ and $\underline{V}[\sqrt{\varepsilon}K]$ take the following analytical form,*

$$\bar{V}[\sqrt{\varepsilon}K] = \bar{V} + \sqrt{\varepsilon}\bar{R} + O(\varepsilon), \qquad \underline{V}[\sqrt{\varepsilon}K] = \underline{V} + \sqrt{\varepsilon}\underline{R} + O(\varepsilon), \tag{27}$$

where

$$\underline{S} := \frac{\underline{R}}{K} = \left(\frac{\bar{V} - \underline{V}}{\sigma^2}\right)\left(\bar{V} + \frac{c}{r}\right) - \frac{1}{2r} = \frac{\sqrt{\frac{c^2}{r^2} + \frac{\sigma^2}{2r}}}{\sigma\sqrt{2r}} \log\left(\sqrt{\frac{r\sigma^2}{2c^2}} + \sqrt{1 + \frac{r\sigma^2}{2c^2}}\right) - \frac{1}{2r} > 0$$

$$\bar{S} := \frac{\bar{R}}{K} = \underline{S} - \frac{1}{2r} \cdot \frac{\bar{V} - \underline{V}}{\bar{V} + c/r} = \frac{1/(\sigma\sqrt{2r})}{\sqrt{\frac{c^2}{r^2} + \frac{\sigma^2}{2r}}} \cdot \frac{c^2}{r^2} \log\left(\sqrt{\frac{r\sigma^2}{2c^2}} + \sqrt{1 + \frac{r\sigma^2}{2c^2}}\right) - \frac{1}{2r} < 0.$$

and $\bar{V}, \underline{V}$ are given by (10), (11), respectively.

Compared to the result of Proposition 2, $\bar{R}$ and $\underline{R}$ are constant in this case. Compared to the constant price benchmark, an increasing pricing scheme ($K > 0$) with the same initial price has two impacts on the purchasing threshold. On the one hand, the benefit of learning becomes lower because the consumer needs to pay more in the future if she receives positive information and likes the product more. Rationally anticipating this, the consumer has a lower incentive to search and is more inclined to purchase now, reducing the purchasing threshold (captured by the negative $\sqrt{\varepsilon}K\bar{S}$ term). On the other hand, a higher price makes the consumer less willing to purchase, raising the purchasing threshold (captured by the positive $\sqrt{\varepsilon}Kt$ term). Since the first effect remains stable while the second effect increases over time, the purchasing threshold is lower than the benchmark threshold at the beginning but eventually exceeds the benchmark threshold as the price keeps increasing.

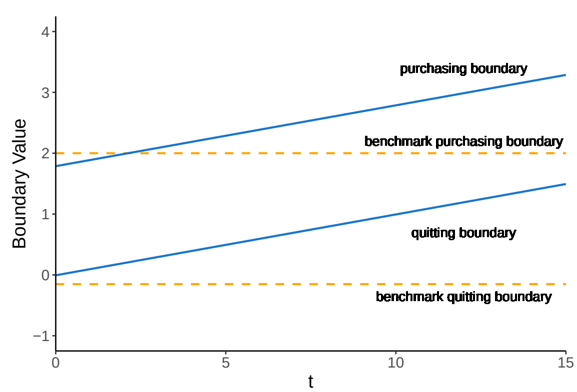

Figure 1: Purchasing and quitting boundaries when $c = .2, p = 1, r = .1, \sigma = 1, \epsilon = 0.01$, and $K = 1$.

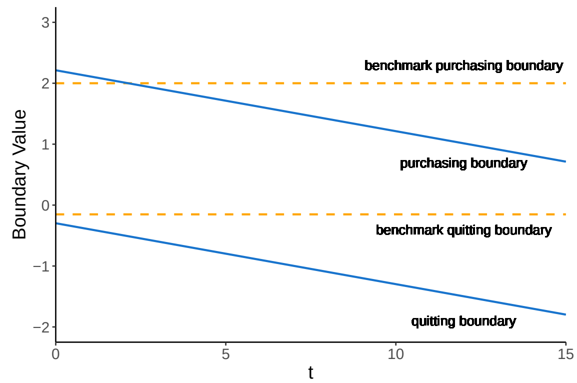

Figure 2: Purchasing and quitting boundaries when $c = .2, p = 1, r = .1, \sigma = 1, \epsilon = 0.01$, and $K = -1$.

An increasing pricing scheme also has two impacts on the quitting threshold. Both a lower benefit of searching and a higher price make it more likely for the consumer to quit. So, the quitting threshold is always higher than the benchmark threshold. We also find that the consumer searches in a narrower region (smaller $\bar{V}_t - \underline{V}_t$) if the price increases rather than staying constant because of the lower benefit of searching. Figure 1 illustrates the purchasing and quitting boundaries in this case, under both non-stationary pricing and constant price.

A decreasing pricing scheme ($K < 0$) has the opposite impact on the purchasing and quitting thresholds. The purchasing threshold is higher than the benchmark threshold at the beginning because the consumer has a stronger incentive to search and is less inclined to purchase immediately. It eventually falls below the benchmark threshold as the price keeps decreasing. The quitting threshold is always lower than the benchmark threshold because the benefit of both searching and purchasing is higher. Also, the consumer searches in a broader region. Figure 2 illustrates the purchasing and quitting boundaries in this case, under both non-stationary pricing and constant price.

Solution: Binary classification with Bayesian updating

We revisit the learning process in §3.1.2 under a linear pricing strategy. This problem cannot be solved exactly, but with the help of Corollary 2 it is possible to obtain up to the ε–order an analytically closed–form of the consumer's value function, the purchase and the exit boundaries, in terms of the constant price parameters $\bar{V}[p_0]$, $\underline{V}[p_0]$. However, such an analytical closed–form is lengthy and will be omitted; nevertheless, they can be obtained by evaluating the elementary integrals and solving the system of linear equations associated with the boundary conditions, which is summarized in the following proposition.

Proposition 5. *Consider the consumer's binary classification process as in §3.1.2 under a linear pricing strategy* $p : t \mapsto p_0 + \sqrt{\varepsilon} K t \in \mathcal{P}$*. Let* $\bar{V}[p_0], \underline{V}[p_0] \in (0,1)$ *are the constant price* p_0 *purchase and exit boundaries as specified by the solution to (14). For convenience, let us define:* $u_\pm(x) := x^{m_\pm}(1-x)^{m_\mp}$*. According to Corollary 2 there is an* ε*-optimal consumer learning strategy with the value function, purchase, and exit boundaries taking the form:*

$$V^B(t,x) = V_0^B(x - \sqrt{\varepsilon}Kt; p_0) + \sqrt{\varepsilon} V_1^B(t,x) + O(\varepsilon),$$

$\bar{V}_t[p] = \bar{V}[p_0] + \sqrt{\varepsilon}Kt + \sqrt{\varepsilon}\bar{R}_t + O(\varepsilon)$*, and* $\underline{V}_t[p] = \underline{V}[p_0] + \sqrt{\varepsilon}Kt + \sqrt{\varepsilon}\underline{R}_t + O(\varepsilon)$*, respectively. Where* $V_0^B(x; p_0) = A_+ u_+(x) + A_- u_-(x) - \frac{c}{r}$ *is given by (13);* $V_1^B(t,x) := V_{1,0}^B(x) + tV_{1,1}^B(x)$ *where*

$$V_{1,0}^B(x) := \left(B_+ + \frac{2\sigma_S^2}{\sqrt{1+8r\sigma_S^2}} \int \frac{K\partial_x V_0^B(x;p_0) - V_{1,1}^B(x)}{x^{2-m_-}(1-x)^{2-m_+}} dx \right) u_+(x) + \left(B_- - \frac{2\sigma_S^2}{\sqrt{1+8r\sigma_S^2}} \int \frac{K\partial_x V_0^B(x;p_0) - V_{1,1}^B(x)}{x^{2-m_+}(1-x)^{2-m_-}} dx \right) u_-(x), \quad (28)$$

$$V_{1,1}^B(x) := \left(C_+ - \frac{4r\sigma_S^2 K}{\sqrt{1+8r\sigma_S^2}} \int \frac{1-2x}{x^{3-m_-}(1-x)^{3-m_+}} \left(V_0(x) + \frac{c}{r} \right) dx \right) u_+(x) + \left(C_- + \frac{4r\sigma_S^2 K}{\sqrt{1+8r\sigma_S^2}} \int \frac{1-2x}{x^{3-m_+}(1-x)^{3-m_-}} \left(V_0(x) + \frac{c}{r} \right) dx \right) u_-(x), \quad (29)$$

and $\bar{R}_t = K\bar{S}_{0,0} + K\bar{S}_{0,1}t$*,* $\underline{R}_t = K\underline{S}_{0,0} + K\underline{S}_{0,1}t$ *are given in terms of* $V_0^B(.,.;p_0)$*,* $V_{1,0}^B(.)$*, and* $V_{1,1}^B(.)$ *as in Corollary 2. The constants* $B_\pm$ *and* $C_\pm$ *are determined by the boundary conditions:* $V_{1,0}^B(\bar{V}[p_0]) = V_{1,0}^B(\underline{V}[p_0]) = 0$ *and* $V_{1,1}^B(\bar{V}[p_0]) = V_{1,1}^B(\underline{V}[p_0]) = 0$*, respectively.*

3.3 Generalizability of the Results

We have shown in the previous section that we can obtain many nice results under a linear pricing strategy with a small slope of $\sqrt{\varepsilon}$–order. Proposition 3 indicates that ε–optimal consumers will respond to broader classes of non–linear pricing p as if they were linear under the assumptions below:

Assumption 2. *For a given* $\varepsilon > 0$*, we assume that:*

- *The consumer is* ε*–optimal,*
- *The consumer is sufficiently myopic (sufficiently large* $r >> 0$*),*

- *The firm adjusts the price slowly over time:* $|p_t'| = O(\sqrt{\varepsilon})$*,*

such that the conditions for Proposition 3 are satisfied.

Assumption 2 ensures the validity of our perturbation technique and the consistency with the ε–equilibrium concept. It clarifies when we can apply the linear pricing results in our work to more general non–linear pricing strategies. For a given price function $p \in \mathcal{P}_T$ that satisfies Assumption 2, the consumer will derive the learning strategy from the linear pricing approximation based on Proposition 3:

$$t \mapsto p_0 + \sqrt{\varepsilon} K t, \qquad \sqrt{\varepsilon} K := p_0' = O(\sqrt{\varepsilon}). \tag{30}$$

4 Seller's Strategy

4.1 Seller's Expected Payoff

The expected payoff for the firm implementing the pricing strategy $p \in \mathcal{P}$ with marginal cost g is given by $\mathcal{V}^S(x; \tau^*[p], p)$, where $\tau^*[p] \in \mathcal{T}$ denotes the consumer's ε–optimal response to p. We will denote $\mathcal{V}^S(x; \tau^*[p], p)$ by $\mathcal{V}^S(x; p)$ hereafter for simplicity. In order to pin down the optimal pricing, we need to be able to calculate $\mathcal{V}^S(x; p)$ for a given pricing strategy p.

4.1.1 Constant Price

In the simplest cases of a constant price, the seller's expected payoff can be derived from the standard properties of martingales.

Lemma 5. *Consider a constant pricing $p = p_0 \in \mathbb{R}$. Suppose that the constant purchase and exit boundaries $\bar{V}[p_0], \underline{V}[p_0] \in (\underline{\pi}, \bar{\pi})$ are finite. For any given $x \in [\underline{V}[p_0], \bar{V}[p_0]]$:*

1. *If $m = 0$, then $\mathcal{V}^S(x; p_0) = (p_0 - g)\frac{x - \underline{V}[p_0]}{\bar{V}[p_0] - \underline{V}[p_0]}$.*
2. *If the volatility is constant: $\sigma(x)^2 = \sigma^2$, then $\mathcal{V}^S(x; p_0) = (p_0 - g)\frac{\sinh \frac{\sqrt{2m}}{\sigma}(x - \underline{V}[p_0])}{\sinh \frac{\sqrt{2m}}{\sigma}(\bar{V}[p_0] - \underline{V}[p_0])}$.*

In this case, $\tau^*[p_0]$ is the exit time from the interval $[\underline{V}[p_0], \bar{V}[p_0]]$. Assumption 1 that $\sigma(x) \geq \underline{\sigma}$ for some constant $\underline{\sigma} > 0$, for all $x \in [\underline{V}[p_0], \bar{V}[p_0]]$, plays an important role to ensure that the consumer will almost surely either purchase or quit within a finite amount of time. Specifically, using Dubins–Schwarz theorem, by computing the survival probability of the standard Brownian motion from the Heat equation series solution, we can derive the following probability tail bound:

$$\mathbb{P}\left[\tau^*[p_0] > T\right] \leq C \cdot \exp\left(-\frac{\pi^2 \underline{\sigma}^2}{2(\bar{V}[p_0] - \underline{V}[p_0])^2} \cdot T\right), \tag{31}$$

for some constant C, for any $T > 0$. This implies that $\mathbb{E}[\tau^*[p_0]] < \infty$.

4.1.2 General Price

For a consumer with initial valuation x, let $\bar{V}[p], \underline{V}[p] : [0, \infty) \to \mathbb{R}$ denotes the consumer's decision boundaries corresponding to the $\tau^*[p]$ learning strategy and let $\Omega := \{(t, v) \in [0, \infty) \times \mathbb{R} \mid \underline{V}_t[p] < v < \bar{V}_t[p]\}$. We consider $U(s, v; t, x)$, the transition probability density of a particle starting from x at time t to some point v at later time $s \geq t$ as described by the process $\{v_s^{t,x}\}_{s \geq t}$ without leaving the domain Ω. It is known that $U(s, v; t, x)$ for any fixed $(s, v) \in \Omega$, satisfies the Kolmogorov backward equation in (t, x) with absorbing boundary condition:

$$\begin{cases} \dfrac{\sigma(x)^2}{2} \partial_v^2 U(s, v; t, x) + \partial_t U(s, v; t, x) = 0, & (t, x) \in \Omega \\ U(s, v; t, \bar{V}_t[p]) = 0, & U(s, v : t, \underline{V}_t[p]) = 0 \, , \\ U(s, v; t = s, x) = \delta(v - x) & \end{cases} \tag{32}$$

where $\delta(v-x)$ denotes the Dirac–Delta distribution concentrated at v. When it is clear from the context, we may denote $U(t, v; t_0 = 0, x)$ simply as $U(t, v)$.

Remark 5. *Alternatively, $U(s, v; t, x)$ for any fixed $(t, x) \in \Omega$ satisfies the Kolmogorov* forward *equation (a.k.a. the Fokker–Planck equation) in (s, v) with absorbing boundary condition:*

$$\begin{cases} \dfrac{1}{2} \partial_v^2 \left[\sigma(v)^2 U(s, v; t, x)\right] - \partial_t U(s, v; t, x) = 0, & (s, v) \in \Omega \\ U(s, \bar{V}_s[p]; t, x) = 0, & U(s, \underline{V}_s[p]; t, x) = 0 \, \cdot \\ U(s = t, v; t, x) = \delta(v - x) & \end{cases} \tag{33}$$

The existence and properties of the solution $U(s, v; t, x)$ depend on the smoothness conditions of $\bar{V}[p]$, $\underline{V}[p]$ (see (Friedman, 2008, Chapter 3)). We assume all necessary conditions are satisfied so that the solution $U(s, v; t, x) \in C^{1,2}(\Omega)$ exists. The probability flux of the consumer hitting the moving purchase boundary, thus getting absorbed, at time s is:

$$-\frac{1}{2} \partial_v \left[\sigma(v)^2 U(s, v; t, x)\right] |_{v = \bar{V}_s[p]} - \bar{V}_s'[p] \cdot U(s, \bar{V}_s; t, x) = -\frac{1}{2} \partial_v \left[\sigma(v)^2 U(s, v; t, x)\right] |_{v = \bar{V}_s[p]}.$$

In the above equation, the term $\bar{V}'_s[p]$ is needed to take into account the boundary movement, which nevertheless vanishes because of the boundary condition: $U(s, \bar{V}_s; t, x) = 0$. Hence, for a pricing policy $p \in \mathcal{P}$, the seller's expected payoff from a consumer starting at time t with valuation x is:

$$\mathcal{V}^S(t, x; p) = -\frac{1}{2} \int_t^\infty e^{-m(s-t)}(p_s - g)\partial_v \left[\sigma(v)^2 U(s, v; t, x)\right] |_{v=\bar{V}_s[p]} ds, \text{ if } x \in (\underline{V}_t[p], \bar{V}_t[p]), \tag{34}$$

and $\mathcal{V}^S(t, x; p) = (p_t - g)1_{x \geq \bar{V}_t[p]}$ otherwise.

If $p \in \mathcal{P}_T$, then (34) and (32) imply that $\mathcal{V}^S$ satisfies the following backward parabolic PDE initial boundary value problem:

$$\begin{cases} \frac{\sigma(x)^2}{2}\partial_x^2 \mathcal{V}^S(t, x; p) + \partial_t \mathcal{V}^S(t, x; p) - m\mathcal{V}^S(t, x; p) = 0, & (t, x) \in \Omega \\ \mathcal{V}^S(t, \bar{V}_t[p]; p) = p_t - g, & \mathcal{V}_0^S(t, \underline{V}_t[p]; p) = 0 \text{ ,} \\ \mathcal{V}^S(T, x; p) = \mathcal{V}_0^S(x; p_T) \end{cases} \tag{35}$$

where $\mathcal{V}_0^S(x; p_T)$ denotes the firm's payoff under the constant price policy: $p_t = p_T$ for all $t \geq T$. $\mathcal{V}_0^S$ is time–independent and is determined by the ODE: $\frac{\sigma^2(x)}{2}\partial_x^2 \mathcal{V}_0^S - m\mathcal{V}_0^S = 0$. An important case is $t = 0$, in which we write $\mathcal{V}^S(x; p) = \mathcal{V}^S(t = 0, x; p)$. When the price is close to being constant, we can solve for the seller's payoff perturbatively as follows.

Proposition 6. *Consider a pricing strategy $p := p_0 + \sqrt{\varepsilon} h \in \mathcal{P}_T$ where $p_0 \in \mathbb{R}$ is a constant, and $h \in \mathcal{P}_T$. Suppose that $\bar{V}[p_0], \underline{V}[p_0] \in (\underline{\pi}, \bar{\pi})$ are the purchase and exit boundaries corresponding to the constant price p_0 strategy, and $\bar{R}, \underline{R} : \mathbb{R} \to \mathbb{R}$ are the consumer's $\sqrt{\varepsilon}$–order responses to p as given in Proposition 2. Let $\varepsilon > 0$ be sufficiently small such that $\bar{V}_t[p] = (\bar{V}[p_0] + \sqrt{\varepsilon} h_t) + \sqrt{\varepsilon}\bar{R}_t \in (\underline{\pi}, \bar{\pi})$, and $\underline{V}_t[p] = (\underline{V}[p_0] + \sqrt{\varepsilon} h_t) + \sqrt{\varepsilon}\underline{R}_t \in (\underline{\pi}, \bar{\pi})$ for all $t \in [0, T]$. Then the seller's expected payoff from the consumer with initial valuation $x \in (\underline{V}[p_0], \bar{V}[p_0])$ under the pricing strategy p up to the ε–order is given by:*

$$\mathcal{V}^S(x; p) = \mathcal{V}_0^S\left((1 - \sqrt{\varepsilon} r_{1,0})x - \sqrt{\varepsilon} r_{0,0}; p_0\right) + \sqrt{\varepsilon}\mathcal{V}_1^S(0, x) + O(\varepsilon), \tag{36}$$

where $r_{1,t} := \frac{\bar{R}_t - \underline{R}_t}{\bar{V}[p_0] - \underline{V}[p_0]}$ and $r_{0,t} := h_t + \underline{R}_t - r_{1,t}\underline{V}[p_0]$, and $\mathcal{V}_1^S(.,.) : \Omega := \{(t, x) \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}] \mid \underline{V}[p_0] < x < \bar{V}[p_0]\} \to \mathbb{R}$ is given by:

$$\begin{aligned} \mathcal{V}_1^S(0, x) = \mathbb{E}\left[h_{\tau_\Omega^x} e^{-m\tau_\Omega^x} \cdot 1\left\{v_{\tau_\Omega^x}^x \geq \bar{V}[p_0]\right\}\right] - \mathbb{E}\left[\int_0^{\tau_\Omega^x} e^{-ms}(r'_{1,s} v_s^x + r'_{0,s})\partial_x \mathcal{V}_0^S(v_s^x; p_0) ds\right] \\ + \mathbb{E}\left[\int_0^{\tau_\Omega^x} e^{-ms}\left(\sigma(v_s^x)\sigma'(v_s^x)(r_{1,s} v_s^x + r_{0,s}) - \sigma(v_s^x)^2 r_{1,s}\right)\partial_x^2 \mathcal{V}_0^S(v_s^x; p_0) ds\right] \end{aligned} \tag{37}$$

where $\tau_\Omega^x := \inf\{t \geq 0 \mid (t, v_t^x) \notin \Omega\}$ *is the exit time.*

4.2 Forced–Purchase Strategy

In this section, we study a certain case where we can characterize the supremum of the seller's payoff over a very general set of admissible pricing strategies under minimal assumptions.[8] Namely, when the buyer's initial valuation is sufficiently high, it is optimal for the seller to force an immediate purchase by increasing the price as sharply as possible (i.e. take–it–or–leave–it offer). This presents another way for the firm to take advantage of non–stationary pricing strategies.

Proposition 7. *Let* $h \in \mathcal{P}_T$ *be an arbitrary pricing strategy strictly increasing over* $[0, T)$ *with* $h_0 = 0$*, and let* $p_0 \in \mathbb{R}$ *be a constant. Consider the pricing strategy* $p = p_0 + Kh$ *for* $K \in \mathbb{R}$*, then*

$$\lim_{K\to\infty} \mathcal{V}^S(x; p) = \begin{cases} p_0 - g, & \text{if } x > p_0 \\ 0, & \text{if } x \leq p_0 \end{cases}. \tag{38}$$

Further, suppose that for all sufficiently high x *and any given* $\varepsilon > 0$ *there exists an* ε*–optimal pricing strategy* $\tilde{p} \in \mathcal{P}_T$ *such that* $\underline{V}_t[\tilde{p}] \geq g$ *for all* $t \in [0, \infty)$*. Then*

$$V^S(x) = \sup_{p\in\mathcal{P}_T} \mathcal{V}^S(x; \tau^*[p], p) = x - g$$

which can be approached by the sequence

$$\{p_n := p_{0,n} + K_n h \in \mathcal{P}_T\}_{n\in\mathbb{Z}_{\geq 0}} \tag{39}$$

where $p_{0,n} \nearrow x$ *and* $K_n \to +\infty$*.*

Condition $\underline{V}_t[\tilde{p}] \geq g$ is crucial for the result. It ensures that selling at the price $p_t = v_t$ is always profitable for the seller if he can hypothetically observe the consumer valuation v_t at any given time. If the consumer's willingness-to-pay is greater than the production cost, then selling the product at the willingness-to-pay is the best possible outcome for the seller. Proposition 7 shows how the seller can achieve such best-case scenario via non–stationary pricing. One can see that this condition is always satisfied if $g \leq \underline{\pi}$. Under a positive search cost $c > 0$, and a constant volatility σ, Branco et al. (2012) shows that for all sufficiently high x $(\geq 2\bar{V} - \underline{V} + g)$, the optimal static pricing strategy is $\hat{p}_0 = x - \bar{V}$. In this case,

[8] Unlike in Assumption 2, we neither require the consumer to be myopic nor require the price to be slow–moving in this section.

$\mathcal{V}^S(x;\hat{p}_0,K=0)=x-\bar{V}-g$, which is lower than the upper–bound $x-g$ approachable by non–stationary pricing. While a static price seller risks losing the customer by raising the price above $\hat{p}_0=x-\bar{V}$, a non–stationary price seller uses the threat of rapid price increase to force a high–value customer to immediately accept the price $p_0=x$.

However, in some consumer search models, it is possible that the condition cannot be satisfied for any arbitrary high x. For example, in §4.3 we consider the model with $c=0$, a constant volatility σ, and $m=0$. Then $\underline{V}_t[p_0]=-\infty$, and the consumer valuation process hits $\bar{V}[p_0]$ with probability 1. Therefore, it is optimal to restrict to static pricing strategies where we can set an arbitrary high p_0, and we get $V^S(x)=+\infty$. Even if we impose an artificial constraint $p_0<x$, we still find that the optimal linear pricing strategy is to set the slope $K\gtrsim 0$ as close to 0 as possible and achieve $\lim_{p_0\nearrow x,K\searrow 0}\mathcal{V}^S(x;p_0,K)=x-g+\frac{\sigma}{\sqrt{2r}}>x-g$. This example demonstrates how Proposition 7 breaks down when $\underline{V}_t[p]=-\infty<g$. Intuitively, a higher profit than $x-g$ can be obtained since the consumer will never exit given $c=0$.

4.3 Direction of Price Evolution for a Forward–looking Seller

Although the forced–purchase strategy presented in §4.2 is simple and intuitive, it is not always applicable. For example, it is only useful for sufficiently high-value consumers. Moreover, it may not be practical for the firm to increase the price drastically due to regulatory or reputational considerations. In this section, we focus on the implementation of a slow–moving linear pricing by the firm. Specifically, the set of admissible pricing is:

$$\mathcal{P}^{\varepsilon}_{lin}:=\left\{t\mapsto p_0+\sqrt{\varepsilon}Kt \mid p_0\in\mathbb{R},K\in[-1,+1]\right\}\subset C^{\infty}[0,\infty).$$

Within $\mathcal{P}^{\varepsilon}_{lin}$, we denote the expected payoff by $\mathcal{V}^S(x;p_0,K)$, and the firm only needs to determine the optimal $(p_0,K)=(p_0^*,K^*)$. Considering linear pricing from the firm's perspective is not without loss of generality, but it is sufficient to answer the economically relevant question of whether the constant price is always optimal when the firm cannot track the consumer's belief evolution process, and what would be the profitable direction of price evolution otherwise. In particular, denote by $\hat{p}_0:=\hat{p}_0(x)$ the optimal constant price given the consumer's initial valuation x. By computing $\frac{\partial\mathcal{V}^S}{\partial K}(x;p_0=\hat{p}_0,K=0)$, we can determine whether $K^*>0$, $K^*<0$, or $K^*=0$, which characterizes (p_0^*,K^*), the optimal policy in some vicinity of $K=0$. The above analysis provides normative guidance to a firm initially using an optimal constant pricing $\hat{p}_0$ on how it can improve its profit with non–stationary pricing.

The discussion of linear pricing also serves as a template for understanding pricing strategies in more general settings where $\mathcal{P}$ could include non–linear pricing strategies as long as Assumption 2 holds. With these assumptions, the consumer ε–optimal learning decision to any $p \in \mathcal{P}$ is entirely determined by the value of p_t and its slope p'_t at any given time t according to Proposition 3, and since $|p'_t| = O(\sqrt{\varepsilon})$ we are able to utilize our linear perturbation framework. In practice, such assumptions hold if the firm's consumers are impatient and impulsive in their purchasing decisions, and due to some regulations, the firm is restricted on how quickly it can change the price over time. We will elaborate on this in section §4.5.

We focus on the case where the firm is perfectly patient ($m = 0$). We start with the following result on linear perturbation from an arbitrary constant price p_0, an important special case of Proposition 6.

Theorem 1. *Consider a linear pricing $p : t \mapsto p_0 + \sqrt{\varepsilon}Kt \in \mathcal{P}$ implementation of a forward–looking seller with $m = 0$, where p_0 is a constant. Suppose that $\bar{V}[p_0], \underline{V}[p_0] \in (\underline{\pi}, \bar{\pi})$ are the purchase and exit boundaries corresponding to the constant price p_0 strategy. Then for all sufficiently small $\varepsilon > 0$, the seller's expected payoff from the consumer with initial valuation $x \in (\underline{V}[p_0], \bar{V}[p_0])$ under the pricing strategy p up to the ε–order is given by:*

$$
\begin{aligned}
\mathcal{V}^S(x; p_0, K) = {} & \frac{(p_0 - g)(x - \underline{V}[p_0])}{\bar{V}[p_0] - \underline{V}[p_0]} + \sqrt{\varepsilon}K\mathbb{E}\left[\tau^x_\Omega \cdot 1\left\{v^x_{\tau^x_\Omega} \geq \bar{V}[p_0]\right\}\right] \\
& - \sqrt{\varepsilon}K \cdot \frac{p_0 - g}{\bar{V}[p_0] - \underline{V}[p_0]}\left((1 + \underline{S}_{0,1})\mathbb{E}\left[\tau^x_\Omega\right] + (\bar{S}_{0,1} - \underline{S}_{0,1})\mathbb{E}\left[\tau^x_\Omega \cdot 1\left\{v^x_{\tau^x_\Omega} \geq \bar{V}[p_0]\right\}\right]\right) \\
& - \sqrt{\varepsilon}K \cdot \frac{p_0 - g}{\bar{V}[p_0] - \underline{V}[p_0]}\left(\underline{S}_{0,0} + (\bar{S}_{0,0} - \underline{S}_{0,0})\mathbb{P}\left[v^x_{\tau^x_\Omega} \geq \bar{V}[p_0]\right]\right) + O(\varepsilon) \qquad (40)
\end{aligned}
$$

where $\tau^x_\Omega := \inf\{t \geq 0 \mid (t, v^x_t) \notin \Omega\}$ is the exit time, $\Omega := \{(t, x) \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}] \mid \underline{V}[p_0] < x < \bar{V}[p_0]\}$, and the constants $\bar{S}_{0,0}, \bar{S}_{0,1}$, $\underline{S}_{0,0}, \underline{S}_{0,1}$ determine the $\sqrt{\varepsilon}$–order consumer's response to p as given in Corollary 2.

The first term of the expression in (40) represents the seller's payoff from the constant price policy p_0. Below, we consider each of the following terms, where the second term affects the profit per purchase and the last two terms affect the probability of purchase.

Change to the profit per purchase:

$$
+\sqrt{\varepsilon}K\mathbb{E}\left[\tau^x_\Omega \cdot 1\left\{v^x_{\tau^x_\Omega} \geq \bar{V}[p_0]\right\}\right]
$$

This term is related to the expected change in price (relative to the initial price) at the time of purchase. If the price is increasing over time, the consumer will pay a price higher than the initial price p_0 if she ends up buying the product, reflected by the non-negative value of this term. If the price is decreasing over time, the seller can only extract a lower profit if the consumer buys after searching, reflected by the non-positive value of this term. Note that $\mathbb{E}\left[\tau_\Omega^x \cdot 1\left\{v_{\tau_\Omega^x}^x \geq \bar{V}[p_0]\right\}\right]$ satisfies the ODE $\frac{1}{2}\sigma(x)^2 w''(x) = -\frac{x-\underline{V}[p_0]}{\bar{V}[p_0]-\underline{V}[p_0]}$ with $w(\bar{V}[p_0]) = w(\underline{V}[p_0]) = 0$. Solving the ODE with the boundary condition gives us:

$$\begin{aligned}\mathbb{E}\left[\tau_\Omega^x \cdot 1\left\{v_{\tau_\Omega^x}^x \geq \bar{V}[p_0]\right\}\right] = &\frac{2(x-\underline{V}[p_0])}{(\bar{V}[p_0]-\underline{V}[p_0])^2}\int_{\underline{V}[p_0]}^{\bar{V}[p_0]} \frac{(\bar{V}[p_0]-z)(z-\underline{V}[p_0])}{\sigma(z)^2}dz \\ &- \frac{2}{\bar{V}[p_0]-\underline{V}[p_0]}\int_{\underline{V}[p_0]}^{x} \frac{(x-z)(z-\underline{V}[p_0])}{\sigma(z)^2}dz.\end{aligned}$$

Change to purchase probability due to rescaling of the search interval:

$$-\sqrt{\varepsilon}K \cdot \frac{p_0-g}{\bar{V}[p_0]-\underline{V}[p_0]}\left(\underline{S}_{0,0} + (\bar{S}_{0,0}-\underline{S}_{0,0})\mathbb{P}\left[v_{\tau_\Omega^x}^x \geq \bar{V}[p_0]\right]\right).$$

Expecting the price to change over time rather than stay constant, the consumer will adjust the search region. An increasing price trajectory shrinks the search region, whereas a decreasing price trajectory enlarges the search region. This economic force affects the search interval even at time 0 when the prices are identical for the cases of time-varying price and constant price. Consider an increasing price. Observing that $\mathbb{P}\left[v_{\tau_\Omega^x}^x \geq \bar{V}[p_0]\right]$ increases as the consumer's initial valuation becomes higher, and that $\bar{S}_{0,0}-\underline{S}_{0,0} \leq 0$ according to Proposition 1, one can see that an increasing price raises the consumer's probability of purchase if she has a high initial valuation (near the purchasing boundary), and reduces the consumer's probability of purchase if she has a low initial valuation (near the quitting boundary). This term can be evaluated using the fact that $\mathbb{P}\left[v_{\tau_\Omega^x}^x \geq \bar{V}[p_0]\right] = \frac{x-\underline{V}[p_0]}{\bar{V}[p_0]-\underline{V}[p_0]}$, as shown in the proof of Lemma 5.

Change to purchase probability due to moving boundaries and price:

$$\begin{aligned}&-\sqrt{\varepsilon}K \cdot \frac{p_0-g}{\bar{V}[p_0]-\underline{V}[p_0]}\left((1+\underline{S}_{0,1})\mathbb{E}\left[\tau_\Omega^x\right] + (\bar{S}_{0,1}-\underline{S}_{0,1})\mathbb{E}\left[\tau_\Omega^x \cdot 1\left\{v_{\tau_\Omega^x}^x \geq \bar{V}[p_0]\right\}\right]\right) \\ =&-\sqrt{\varepsilon}K \cdot \frac{p_0-g}{\bar{V}[p_0]-\underline{V}[p_0]}\left(\mathbb{E}\left[\tau_\Omega^x\right] + \bar{S}_{0,1}\mathbb{E}\left[\tau_\Omega^x \cdot 1\left\{v_{\tau_\Omega^x}^x \geq \bar{V}[p_0]\right\}\right] + \underline{S}_{0,1}\mathbb{E}\left[\tau_\Omega^x \cdot 1\left\{v_{\tau_\Omega^x}^x \leq \underline{V}[p_0]\right\}\right]\right).\end{aligned}$$

The first term in the bracket, $\mathbb{E}[\tau_\Omega^x]$, reflects that the probability of purchase is affected by the expected amount of price change over the entire search process. The consumer is less likely to make a purchase if the price increases over time, and is more likely to make a purchase if the price decreases over time. The second term in the bracket, $\bar{S}_{0,1}\mathbb{E}\left[\tau_\Omega^x \cdot 1\left\{v_{\tau_\Omega^x}^x \geq \bar{V}[p_0]\right\}\right]$, accounts for the fact that the purchasing boundary has moved a certain distance by the time a consumer reaches the original purchasing boundary. Analogously, third term in the bracket, $\underline{S}_{0,1}\mathbb{E}\left[\tau_\Omega^x \cdot 1\left\{v_{\tau_\Omega^x}^x \leq \underline{V}[p_0]\right\}\right]$, accounts for the fact that the quitting boundary has moved a certain distance by the time a consumer reaches the original quitting boundary.

To evaluate the above formula, it remains to compute $\mathbb{E}[\tau_{\Omega}^{x}]$, which satisfies the ODE $\frac{1}{2}\sigma(x)^2 w''(x) = -1$ with $w(\bar{V}[p_0]) = w(\underline{V}[p_0]) = 0$. Solving the ODE with the boundary condition gives us:

$$\mathbb{E}[\tau_{\Omega}^{x}] = \frac{2(x-\underline{V}[p_0])}{\bar{V}[p_0]-\underline{V}[p_0]}\int_{\underline{V}[p_0]}^{\bar{V}[p_0]}\frac{\bar{V}[p_0]-z}{\sigma(z)^2}dz - 2\int_{\underline{V}[p_0]}^{x}\frac{x-z}{\sigma(z)^2}dz.$$

The derivative $\frac{\partial \mathcal{V}^S}{\partial K}(x; p_0, K=0)$

Theorem 1 allows us to compute the exact value of $\frac{\partial \mathcal{V}^S}{\partial K}(x; p_0, K=0)$ at any arbitrary $x \in [\underline{V}[p_0], \bar{V}[p_0]]$ and p_0, in terms of the consumer's $\sqrt{\varepsilon}$–order response characterized by the parameters: $\underline{V}[p_0], \bar{V}[p_0], \underline{S}_{0,0}, \bar{S}_{0,0}, \underline{S}_{0,1}, \bar{S}_{0,1}$. For convenience, we use $q := \frac{x-\underline{V}[\hat{p}_0]}{\bar{V}[\hat{p}_0]-\underline{V}[\hat{p}_0]}$ to denote the consumer's initial valuation relative to the purchasing and exiting boundaries under the optimal static price $\hat{p}_0$. Since there is a bijection between x and q, we can equivalently consider $\frac{\partial \mathcal{V}^S}{\partial K}(x; p_0, K=0)$ or $\frac{\partial \mathcal{V}^S}{\partial K}(q; p_0, K=0)$.

Importantly, if $\frac{\partial \mathcal{V}^S}{\partial K}(q; p_0=\hat{p}_0, K=0)$ is non-zero for a given $q \in [0,1]$, then the optimal K^* will be bounded away from 0. The seller can improve its expected profit by setting $K \gtrsim 0$ if $\frac{\partial \mathcal{V}^S}{\partial K}(q; p_0=\hat{p}, K=0) > 0$, and by setting $K \lesssim 0$ if $\frac{\partial \mathcal{V}^S}{\partial K}(q; p_0=\hat{p}, K=0) < 0$.

For the product attributes learning process in §3.1.1 with constant volatility σ, it is known that the optimal static price is given by:

$$\hat{p}_0 = \hat{p}_0(x) = \begin{cases} \frac{x+g-\underline{V}}{2}, & \underline{V}+g < x < 2\bar{V}-\underline{V}+g \\ x-\bar{V}, & x \geq 2\bar{V}-\underline{V}+g \end{cases}.$$

where $\bar{V}, \underline{V}$ are given by (10), (11), respectively. In this case, we also find that $q = \frac{x-\underline{V}-g}{2(\bar{V}-\underline{V})}$. Substituting the consumer's linear perturbation solution from Proposition 4 into Theorem 1 leads to the following result:

$$\frac{1}{\sqrt{\varepsilon}}\frac{\partial \mathcal{V}^S}{\partial K}(q; p_0=\hat{p}_0, K=0) = \frac{(\bar{V}-\underline{V})^2}{3\sigma^2}(1-2q)q(1-q) - (\bar{S}q + \underline{S}(1-q))q. \tag{41}$$

Since the sign of the above expression depends only on q, c/r, and σ^2/r, in Figure 3, we illustrate the direction of price evolution that improves the seller's expected profit over the optimal constant price strategy as a function of q, c/r, and σ^2/r.

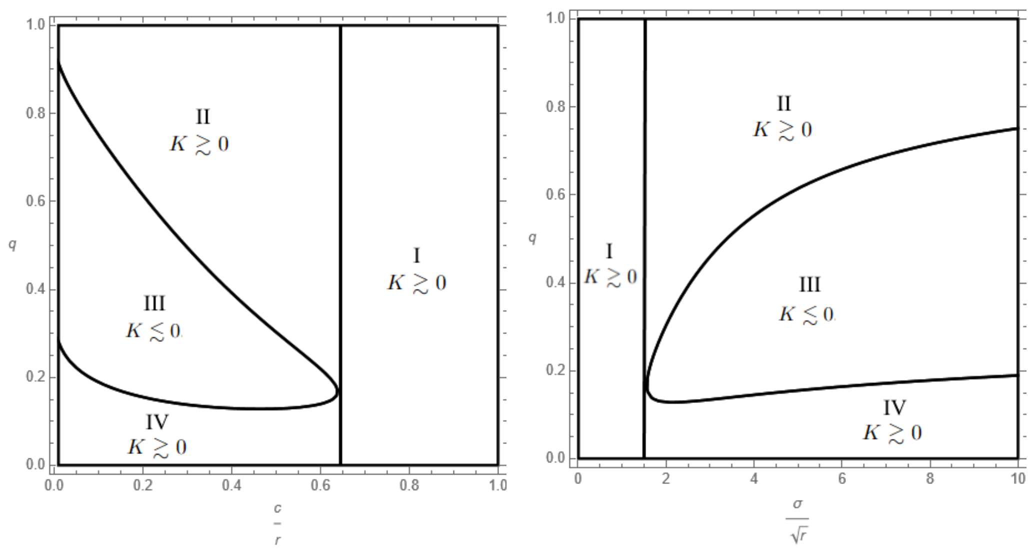

Figure 3: Region plots of a forward–looking ($m = 0$) seller's expected profit improvement direction in the vicinity of $K \sim 0$ where the consumer follows the product attributes learning process. $\sigma^2/r = 1$ for the left plot, and $c/r = 1$ for the right plot.

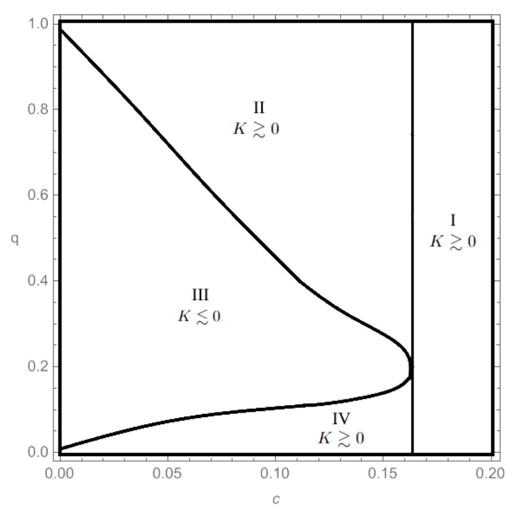

Figure 4: Region plots of a forward–looking ($m = 0$) seller's expected profit improvement direction in the vicinity of $K \sim 0$. We assume $g = 0.3$ production cost for the seller, and the consumer follows the binary classification process with $r = 1$, $\sigma_S = 1$ noise level.

For the learning process of binary classification with Bayesian updating in §3.1.2, we can substitute the consumer's linear perturbation solution given by Proposition 5 into Theorem 1. Although it is possible to obtain the closed–form expression for $\frac{\partial \mathcal{V}^S}{\partial K}(x; p_0 = \hat{p}_0, K = 0)$ in terms of $\underline{V}[\hat{p}_0]$ and $\underline{V}[\hat{p}_0]$ this way, the expression is extremely long and not so interesting, hence will be omitted. On the other hand, the static price p_0 decision boundaries $\bar{V}[p_0]$ and $\underline{V}[p_0]$ are only implicitly specified through a system of non–linear algebraic equations as discussed in §3.1.2. Consequently, it is only possible for us to evaluate $\hat{p}_0$ and $\bar{V}[\hat{p}_0]$, $\underline{V}[\hat{p}_0]$ numerically. We present in Figure 4 the numerical region plot of the expected profit improvement direction.

Figures 3 and 4 are qualitatively similar. In particular, we can classify each plot into four regions.

I (Low incentive to search) When the search cost c is too high, the consumer has a low incentive to search for information. The firm needs to give the consumer a high surplus to encourage her to search, which hurts its profit. So, it becomes more attractive for the firm to convince the consumer to purchase the product at the beginning, based on the initial valuation and the expected price trajectory. For any given initial price, by charging an increasing price over time, the firm lowers the purchasing threshold at the beginning by making it more desirable for the consumer to make an immediate decision. Compared with the stationary pricing strategy of charging a lower constant price, this non-stationary pricing strategy moves the purchasing threshold in the same direction (downwards) without sacrificing the profit conditional on purchase. In other words, it increases the probability of purchase without reducing the profit per purchase.

II (High–value consumer) When the consumer has a high initial valuation, she is too valuable to lose from the firm's perspective. Therefore, the firm wants to increase the purchasing probability in this case. Moreover, a high–value consumer can earn a positive payoff from purchasing immediately, which decreases over time because of discounting. Thus, the firm also wants the consumer to make a quick purchase. An increasing pricing strategy reduces the benefits of searching and encourages the consumer to buy quickly and with a higher likelihood.

III (Medium–value consumer) When the consumer has a moderate interest in the product, an increase in price does not suffice to convince the consumer to purchase quickly without learning much additional information. Instead, it reduces the benefit of searching because the consumer knows she has to pay a higher price if she learns positive things. Therefore, an increasing price will lead to a quick exit rather than a quick purchase.

The firm can benefit from reducing the price gradually in this case. A decreasing price helps the firm keep the consumer engaged in the search process even if she receives some negative information early on. It increases the purchasing likelihood. Because of the moderate initial valuation, the firm can still obtain a decent profit at a lower price. This pricing strategy protects the firm from missing potentially valuable consumers.

IV (Low–value consumer) By charging a decreasing price over time, the firm can keep the consumer engaged in the search process even if she receives some negative information early on. However, it is not worth it for the firm to reduce the price over time for two reasons. First, the profit from an immediate purchase is already low when the consumer has a low initial valuation. The firm will obtain an even lower profit from purchasing if the consumer searches for a while and eventually buys at a lower price. Second, the consumer must accumulate a lot of positive information before purchasing due to the low initial valuation. The purchasing probability will still be low even if the price slightly reduces over time, and cannot offset the cost of a lower profit per purchase.

In this case, the firm quickly filters out many consumers by implementing an increasing pricing strategy. On the one hand, the loss from not converting these people is limited due to the low profit per purchase and the low purchasing probability. On the other hand, the benefits of charging a higher price to the remaining consumers are high. Any consumers who do not quit despite the increasing price must have learned positive information and are more valuable to the firm.

4.4 Constant Volatility with Linear Pricing

When the consumer follows the product attributes learning process in §3.1.1 with constant volatility $\sigma > 0$, one can derive the seller's payoff $\mathcal{V}^S(x; p_0, K)$ exactly under any linear pricing strategy $p : t \mapsto p_0 + \sqrt{\varepsilon}Kt \in \mathcal{P}^{\varepsilon}_{lin}$, for any $\varepsilon > 0$ and $p_0 \in \mathbb{R}$, $K \in [-1, +1]$. The key reason is that, in this case, Proposition 4 has shown that the resulting purchasing and quitting boundaries are linear in t. The series solution to the heat equation $U(s, v; t, x)$ with two absorbing fixed boundaries is well–studied, and we can use the Girsanov Theorem to transform such a solution to the one with two linearly moving absorbing boundaries. By substituting this solution into (34), we can obtain a closed form expression for $\mathcal{V}^S(x; p_0, K)$.

Below, we first consider the special case with zero search costs, $c = 0$. Then we study the more interesting case with positive search costs.

Zero Search Costs

When the consumer has zero search costs, the continuation value of searching is positive, whereas the payoff from quitting is zero. Therefore, she would never quit searching without purchasing the product. It implies that her quitting boundary is $-\infty$. Therefore, her optimal search strategy is characterized by a single boundary, the purchasing boundary $\bar{V}[p]$.

If the firm is perfectly patient, it will not have a direct incentive to speed up the consumer's decision-making process. A purchase at any time gives the firm the same payoff.

Hence, it does not have a strong incentive to increase the price over time to push the consumer to make an early decision. In addition, the firm will charge a sufficiently high price such that the consumer's payoff from purchasing the product is negative initially. Therefore, discounting does not reduce the consumer surplus if she delays the decision by searching for more information. Even if the price does not decrease over time, the consumer will keep searching for information because she has nothing to lose. So, the firm has no incentive to reduce the price over time to prevent the consumer from quitting. In sum, in this case, the firm has little incentive to charge non-stationary prices. The following proposition shows that the optimal price is arbitrarily close to constant when the firm is perfectly patient. In contrast, the firm may benefit from charging non-stationary prices if it discounts the future.

Proposition 8. *Suppose the search cost is zero, $c = 0$.*

1. *When the firm is perfectly patient ($m = 0$), for any fixed initial price p_0 the firm can approach the profit supremum by choosing $K > 0$ as close to zero as possible, i.e.:*

$$V^S(x) = \sup_{K \searrow 0} \mathcal{V}^S(x; p_0, K) = 2p_0 + \frac{\sigma}{\sqrt{2r}} - g - x.$$

   If p_0 is not fixed, then it is optimal for the firm to set p_0 as large as possible.

2. *When the firm is not perfectly patient ($m > 0$), in particular, when $m >> 0$ or $m \sim 0$, then the slope K of the optimal linear pricing is bounded away from zero.*

Positive Search Costs

We now consider the case with a positive search cost. In this case, the continuation value of searching may be negative. Hence, both the purchase and exit boundaries are finite. We focus on the case of a perfectly patient firm ($m = 0$) because we will show that the firm will charge non-stationary prices even in this case, and as we can see from the zero-search-cost case, the firm is more inclined to charge non-stationary prices if it discounts the future ($m > 0$).[9]

Proposition 9. *Suppose the search cost is positive $c > 0$, and the firm is perfectly patient $m = 0$. The firm's expected profit from a consumer whose initial valuation is x is:*

$$\mathcal{V}^S(x; p_0, K) = \frac{p_0 - g + (\bar{V}_0 + x - 2\underline{V}_0)}{1 - \exp\left(+\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0 - \underline{V}_0)\right)} - \frac{2(\bar{V}_0 - \underline{V}_0)}{\left(1 - \exp\left(+\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0 - \underline{V}_0)\right)\right)^2}$$

[9] We also derive the expected profit when $m > 0$ in equation (74) in the online appendix.

$$-\frac{(p_0-g+(\bar{V}_0-x))\exp\left(+\frac{2\sqrt{\varepsilon}K}{\sigma^2}(x-\underline{V}_0)\right)}{1-\exp\left(+\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0-\underline{V}_0)\right)}+\frac{2(\bar{V}_0-\underline{V}_0)\exp\left(+\frac{2\sqrt{\varepsilon}K}{\sigma^2}(x-\underline{V}_0)\right)}{\left(1-\exp\left(+\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0-\underline{V}_0)\right)\right)^2} \quad (42)$$

if $x\in(\underline{V}_0,\bar{V}_0)$ *and* $K\neq 0$, *and* $\mathcal{V}^S(x;p_0,K)=(p_0-g)\left(\frac{x-\underline{V}_0}{\bar{V}_0-\underline{V}_0}\right)$ *if* $x\in(\underline{V}_0,\bar{V}_0)$ *and* $K=0$. $\mathcal{V}^S(x;p_0,K)=0$ *if* $x\leq\underline{V}_0$. $\mathcal{V}^S(x;p_0,K)=p_0-g$ *if* $x\geq\bar{V}_0$.

It is straightforward to derive (41) we have previously obtained from the perturbative analysis by taking the derivative of (42) at $K=0$. Indeed, by expanding (42) to the $\sqrt{\varepsilon}$–order, we would arrive at (40) specialized to the consumer search model with a constant $\sigma>0$. Unlike when $c=0$, here the slope K^* of the optimal pricing strategy can be bounded away from zero even if $m=0$.

So far, we know that $\hat{p}_0=\frac{x+g-\underline{V}}{2}$ maximizes $\mathcal{V}^S(x;,.,K=0)$, and we discussed K in the vicinity of 0 by analyzing the derivative of $\mathcal{V}^S(x;\hat{p}_0,K)$ at $K=0$, where $\hat{p}_0$. The analytical expression (42) enable us to find the optimal initial price $p_0^*=p_0^*(x,K)$ that maximizes $\mathcal{V}^S(x;.,K)$ for any $K\neq 0$ by solving $\frac{\partial\mathcal{V}^S}{\partial p_0}(x;p_0^*,K)=0$, the result is:

$$\begin{aligned} p_0^*(x,K):=\frac{x+g}{2}+\frac{\sigma^2}{2\sqrt{\varepsilon}K}-\frac{\underline{V}[\sqrt{\varepsilon}K]}{2}\left(1-\coth\frac{\sqrt{\varepsilon}K}{\sigma^2}\left(\bar{V}[\sqrt{\varepsilon}K]-\underline{V}[\sqrt{\varepsilon}K]\right)\right)\\ -\frac{\bar{V}[\sqrt{\varepsilon}K]}{2}\coth\frac{\sqrt{\varepsilon}K}{\sigma^2}\left(\bar{V}[\sqrt{\varepsilon}K]-\underline{V}[\sqrt{\varepsilon}K]\right)\end{aligned}$$

for $x\in(\underline{V}_0,\bar{V}_0)$ and we can check that $\lim_{K\to 0}p_0^*(x,K)=\hat{p}_0(x)$.

Lemma 6. *Consider a forward–looking seller* ($m=0$). *Suppose that* $r,\sigma,c>0$, *then there exists* $\varepsilon>0$ *sufficiently small such that the seller's profit maximizing strategy* (p_0^*,K^*) *in* $\mathcal{P}^{\varepsilon}_{lin}$ *either satisfies* $p_0^*<\hat{p}_0,K^*\gtrsim 0$, *or* $p_0^*>\hat{p}_0,K^*\lesssim 0$.

Lemma 6 characterizes the profit maximization pricing strategy in the linear perturbative regime $\mathcal{P}^{\varepsilon}_{lin}$ for the consumer under the constant volatility learning process. Namely, the optimal strategy with increasing price always couple with a lower initial price, and the optimal strategy with decreasing price always couple with a higher initial price. Examples of both types of the strategies can be found from Figure 3. In particular, at the boundary between regions III and VI, $(p_0^*,K^*)=(\hat{p},0)$ is the maximum point in $\mathcal{P}^{\varepsilon}_{lin}$ for all sufficiently small $\varepsilon>0$. Due to the continuity of $\mathcal{V}^S(.;.,.)$ and $p_0^*(.,.)$, by choosing a slightly higher q, we obtain an example of a maximum point with $p_0^*>\hat{p}_0,K^*\lesssim 0$, while choosing a slightly lower q, we obtain an example of a maximum point with $p_0^*<\hat{p}_0,K^*\gtrsim 0$. A similar argument can be made at the boundary between regions II and III.

Lastly, we note that the closed–form formula (42) gives us a unified picture of the forced–purchase strategy studied in §4.2 together with the perturbative analysis in §4.3, at least for the case of linear pricing under constant volatility σ. Since (42) is valid for all $\sqrt{\varepsilon}K \in \mathbb{R}$, it is relatively simple to find the globally optimal linear pricing strategy $t \mapsto p_0 + Kt$ for each given x by maximizing $\mathcal{V}^S(x; p_0, K)$ over all $(p_0, K) \in \mathbb{R}^2$.[10] We show results for some representative choices of x and other parameters in Figure 5. As discussed in §4.2, the firm can approach the payoff supremum $x - g$ via rapid price increase when for any given $\varepsilon > 0$ there exists an ε–optimal pricing strategy $\tilde{p}$ satisfying $\underline{V}_t[\tilde{p}] \geq g$ for all $t \in [0, \infty)$. Intuitively, this occurs when x or c are sufficiently high, which is confirmed by the bottom–row of Figure 5. On the other hand, the top–row of Figure 5 shows the global maximum lies in the perturbative regime when x is not too high.

4.5 Generalizability of the Results

We have studied two types of non–stationary pricing strategies: The force–purchase strategy, where the price rapidly increases to force a high–value buyer to make an immediate purchase, and the linear perturbation strategies, where the slope is constrained to some neighborhood of zero. Since the first type of strategies requires minimal assumptions, we will elaborate on the generalizability of the second type of strategies in this section.

Firstly, in practice, sellers may not know precisely the consumer's initial valuation x. Instead, they may only know the distribution of the consumer's initial valuation, denoted by ϕ. In such cases, the seller's expected profit under the strategy $p \in \mathcal{P}$ is:

$$\mathcal{V}^S(\phi; p) := \int_{\mathbb{R}} \mathcal{V}^S(x; \tau^{x*}[p], p)\phi(x)dx,$$

where we denoted by $\tau^{x*}[p] \in \mathcal{T}$ the ε–optimal stopping time for the consumer with initial valuation x.[11] Consider a perfectly patient seller. To find out the direction of price evolution that improves the profits from charging a constant price p_0, we can integrate the linear perturbation result (40) in Theorem 1 against the distribution ϕ.

Secondly, let us expand the set of admissible pricing strategies from the linear one $\mathcal{P}^{\varepsilon}_{lin}$ to a more general one:

$$\mathcal{P}^{\varepsilon,M} := \left\{ p \in C^{\infty}[0,\infty) \mid \sup_{t \in \mathbb{R}_{>0}} |p'_t| \leq \sqrt{\varepsilon}, \sup_{t \in \mathbb{R}_{>0}} |p''_t| \leq M \right\}$$

[10] In other words, we study the global maximum over $\mathcal{P}^{\varepsilon}_{lin}$ for any arbitrary large $\varepsilon > 0$. For notational convenience, we absorb the coefficient $\sqrt{\varepsilon}$ into K and allow K to span the domain $\mathbb{R}$ in the following.

[11] Note that the optimal learning strategy of the consumers at any initial valuation x can be characterized by the same purchase and exit boundaries $\bar{V}[p], \underline{V}[p] : \mathbb{R} \to [\underline{\pi}, \bar{\pi}]$.

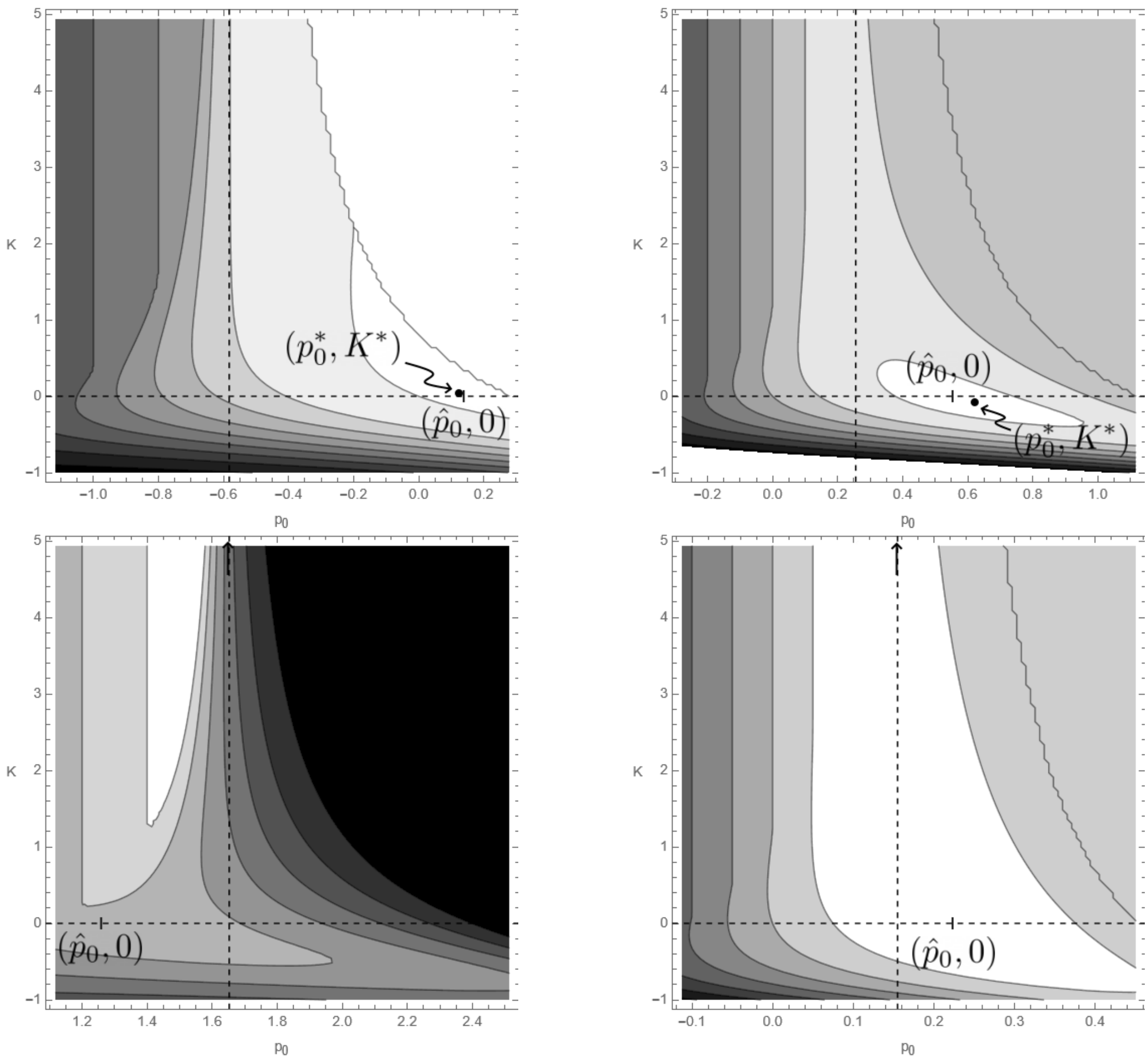

Figure 5: Contour plots of the firm's expected profit from the linear pricing strategy $t \mapsto p_0 + Kt$ using (42) with $r = \sigma = \varepsilon = 1$ at a few fixed x (a few fixed q). In all plots, the horizontal line marks the $K = 0$ axis, while the vertical line shows the value of x. The optimal forced–purchase strategy can be approached asymptotically along the left–side of the vertical line in each plot. The optimal constant pricing $\hat{p}_0$ is shown in all plots. The top–left shows the plot with $q = 0.1$, $c = 0.2$, with the global maximum (p_0^*, K^*) in the perturbative regime given by $p_0^* > \hat{p}_0$, $K^* \lesssim 0$. The top–right shows the plot with $q = 0.4$, $c = 0.2$, with the global maximum (p_0^*, K^*) in the perturbative regime given by $p_0^* < \hat{p}_0$, $K^* \gtrsim 0$. The bottom–left and the bottom–right shows the plot with $q = 0.9$, $c = 0.2$ and $q = 0.4$, $c = 0.8$, respectively. In both cases, the supremum of the firm's payoff is $x - g$, and can be approached by the forced–purchase strategy, asymptotically along the left–side of the vertical line.

for some $\varepsilon > 0$ and $M > 0$. The expanded set includes non-linear pricing strategies that are not *too* far away from linear strategies. In practice, sellers may be restricted by regulations or reputational concerns in how quickly they can change the price over time, indicating that $\varepsilon > 0$ cannot be too large. According to Proposition 3, a myopic consumer with $r > e^{-1}\sqrt{2M/\varepsilon}$ can make an ε–optimal learning decision at any time t by approximating the pricing strategy $p \in \mathcal{P}^{\varepsilon,M}$ with the linear pricing $l_{\mathbf{x}} : s \mapsto p_t + p'_t \cdot (s-t)$, i.e., the learning decision at any time t is entirely determined by p_t and p'_t.

From the seller's perspective, suppose that the search process is very informative, $\sigma(x)^2 \geq \underline{\sigma}^2$, for all x over the relevant learning region and for some constant $\underline{\sigma}^2 >> 0$. Suppose also that the search cost is sufficiently expensive $c >> 0$ to keep the purchase and exit boundaries bounded. In such cases, the consumer updates her valuation and reaches a purchasing decision quickly. We can argue that it is also ε–optimal for the seller to approximate any $p \in \mathcal{P}^{\varepsilon,M}$ with a linear pricing.

Let $\tau^*[l] \in \mathcal{T}$ denotes the ε–optimal consumer's stopping time corresponding to $p \in \mathcal{P}^{\varepsilon,M}$ by the linear approximationi $l : t \mapsto p_0 + p'_0 t$, then for any $\delta > 0$ we have:

$$V^S(x) = \sup_{p \in \mathcal{P}^{\varepsilon,\varepsilon'}} \Bigg[\mathbb{E}\left[e^{-m\tau^*[l]}(p_{\tau^*[l]} - g) \cdot 1\left\{ v^x_{\tau^*[l]} \geq p_{\tau^*[l]}, \tau^*[l] < \delta \right\} \right] + e^{-m\delta} \mathcal{V}^S\left(U(\delta, .; 0, x); p_{.+\delta} \right) \Bigg], \quad (43)$$

where $p_{.+\delta} : t \mapsto p_{t+\delta}$ is the corresponding time–shifted pricing strategy, and $U(\delta, .; 0, x)$ is the transition probability density (satisfying equation (32)). Given a finite $M > 0$, we can choose $\delta = O(\sqrt{\varepsilon})$ so that the first term of (43) can be approximated with a linear pricing l up to an order of some $\varepsilon > 0$, i.e.,

$$\mathbb{E}\left[e^{-m\tau^*[l]}(p_{\tau^*[l]} - l_{\tau^*[l]}) \cdot 1\left\{ v^x_{\tau^*[l]} \geq p_{\tau^*[l]}, \tau^*[l] < \delta \right\} \right] < \frac{1}{2} M \delta^2 = O(\varepsilon).$$

The survival probability $\mathbb{P}[\tau^*[l] > \delta]$ can be upper–bounded by the survival probability $O\left(\frac{1}{\underline{\sigma}\sqrt{\delta}}\right)$ of the Brownian motion with volatility $\underline{\sigma}$ and a single absorbing boundary. Suppose that $\underline{\sigma}^2 >> 0$, then we may argue that the second term of (43) is $O(\varepsilon)$. In conclusion, for all sufficiently large $\underline{\sigma}^2 >> 0$, it is ε–optimal for the seller to only plan for the $\delta = O(\sqrt{\varepsilon})$ time ahead using the linear perturbation theory for $\mathcal{P}^{\varepsilon}_{lin}$ we studied to specify $l : t \mapsto p_0 + Kt$, i.e. specifying (p^*_0, K^*).

After time δ, the seller can repeat the process to further improve profit by maximizing $\mathcal{V}^S(\phi^1; p_{.+\delta})$, where $\phi^1 := U(\delta, .; 0, x)$, over the next δ–period using a linear approximation.

More generally, the seller can specifies the ε–optimal pricing strategy over the interval $t \in [k\delta, (k+1)\delta)$ for any $k \geq 1$ using the linear approximation $l^{k+1} : t \mapsto l^k_{k\delta} + K^{k+1} \cdot (t - k\delta) \in \mathcal{P}^{\varepsilon}_{lin}$ which can be specified from Theorem 1 integrated against ϕ^k. By patching together the linear segments, we obtain the piecewise linear pricing strategy which should approximate the optimal $p \in \mathcal{P}^{\varepsilon,M}$[12]. We leave further analysis of the pricing dynamic based on this outline for future research.

5 Conclusion

This paper introduces a novel framework where firms adopt non-stationary pricing strategies. Our finding challenges the conventional reliance on stationary pricing, and shows that non-stationary pricing strategies can outperform stationary ones. We provide a theoretical advance in optimal control by incorporating non-stationary strategies into a consumer search framework. Unlike previous works, the non-stationarity in the consumer's search problem arises endogenously from firms' strategic pricing in response to consumer gradual learning.

Appendix

Proof of Lemma 1. Part 1:

Consider any $x, x' \in \mathbb{R}$, and suppose that $x' > x$. Let $\{v^{t,x}_s\}_{s\geq t}$ and $\{v^{t,x'}_s\}_{s\geq t}$ be the two strong solutions of the SDE (1), and we have $v^{t,x'}_s > v^{t,x}_s$ a.e., for all $s \geq t$. This can be seen by using the Lipschitz condition (2) to analyze the difference process $d_s := v^{t,x'}_s - v^{t,x}_s$. It follows that $\mathcal{V}^B(t, x'; \tau, p) \geq \mathcal{V}^B(t, x; \tau, p)$ for all $\tau \in \mathcal{T}$. For any $\varepsilon > 0$, we can find $\tau_{t,x} \in \mathcal{T}$ such that $\mathcal{V}^B(t, x; \tau_{t,x}, p) \geq V(t, x; p) - \varepsilon$. Then

$$V^B(t, x'; p) \geq \mathcal{V}^B(t, x'; \tau_{t,x}, p) \geq \mathcal{V}^B(t, x; \tau_{t,x}, p) \geq V(t, x; p) - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrarily small, hence $V^B(t, x'; p) \geq V^B(t, x; p)$ as claimed.

Part 2:

It is clear from (3) that $\mathcal{V}^B(t, x; \tau, q) \leq \mathcal{V}^B(t, x; \tau, p)$ for all $\tau \in \mathcal{T}$, hence following the similar argument as in the previous part, we get $V^B(t, x; q) \leq V^B(t, x; p)$ as claimed. $\square$

Proof of Proposition 1. Part 1:

[12] Although the assumption of Proposition 3 is not entirely satisfied for piecewise linear pricing strategies, due to the lack of second derivatives, this is not really an issue as long as we impose the constrain: $(K^{k+1} - K^k)/\delta \leq M$ for all $k \geq 1$. Intuitively, linear approximation is a better approximation to a piecewise linear function than a non–linear function.

From Lemma 1, we already know that $V^B(0,x;\tilde{p}) \leq V^B(0,x;p)$. Meanwhile, we have $\max\{x-\tilde{p}_0,0\} = \max\{x-p_0,0\}$ from the assumption that $h_0 = 0$. It follows that:

$$\begin{aligned}\bar{V}_0[\tilde{p}] &= \sup\left\{x \in \mathbb{R} \mid V^B(0,x;\tilde{p}) > \max\{x-p_0,0\}\right\} \\ &\leq \sup\left\{x \in \mathbb{R} \mid V^B(0,x;p) > \max\{x-p_0,0\}\right\} = \bar{V}_0[p]\end{aligned}$$

and similarly:

$$\begin{aligned}\underline{V}_0[\tilde{p}] &= \inf\left\{x \in \mathbb{R} \mid V^B(0,x;\tilde{p}) > \max\{x-p_0,0\}\right\} \\ &\geq \inf\left\{x \in \mathbb{R} \mid V^B(0,x;p) > \max\{x-p_0,0\}\right\} = \underline{V}_0[p],\end{aligned}$$

proving the claim.

Part 2:

Without the loss of generality, let's only consider $t = 0$ and h such that $h_0 = 0$, we can always redefine t and shift the x–axis by a constant, otherwise.

Let us suppose for a contradiction that there exists $x \in [\underline{\pi}, \bar{\pi}]$ where it is optimal to continue learning for any $K \geq 0$, i.e. $V^B(0,x;\tilde{p}) > 0$ is bounded away from zero for all $K \geq 0$. In other words, for any $\varepsilon, \varepsilon' > 0$, we can choose $\{\tau[K]\}_{K\geq 0} \subset \mathcal{T}$ and $\delta > 0$ such that $\sup_{K\geq 0} \mathbb{E}[1_{\tau[K]<\delta}] < \varepsilon'^2$ and:

$$V^B(0,x;\tilde{p}) \leq \mathbb{E}\left[e^{-r\delta\wedge\tau[K]} V^B(\delta\wedge\tau[K], v^x_{\delta\wedge\tau[K]};\tilde{p}) - \int_0^{\delta\wedge\tau[K]} ce^{-rs}ds\right] + \varepsilon\delta, \quad \forall K \geq 0.$$

We can separate the expression above further into two terms corresponding to the events $\tau[K] \geq \delta$ and $\tau[K] < \delta$. Then, using Lemma 1: $V^B(t,x;\tilde{p}) \leq V^B(t,x;p)$ for all $(t,x) \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}]$, and $V^B(\delta,x;\tilde{p}) \leq V^B(\delta,x;p_\delta + Kh_\delta)$, where $p_\delta + Kh_\delta$ denotes a constant pricing policy (i.e. the consumer is better–off if $\tilde{p}_t$ stopped increasing after $t = \delta$), we obtain:

$$\begin{aligned}V^B(0,x;\tilde{p}) &\leq \mathbb{E}\left[e^{-r\tau[K]} V^B(\tau[K], v^x_{\tau[K]};\tilde{p}) \cdot 1_{\tau[K]<\delta}\right] + \mathbb{E}\left[V^B(\delta, v^x_\delta;\tilde{p}) \cdot 1_{\tau[K]\geq\delta}\right] + \varepsilon\delta \\ &\leq \mathbb{E}\left[e^{-r\tau[K]} V^B(\tau[K], v^x_{\tau[K]};p) \cdot 1_{\tau[K]<\delta}\right] + \mathbb{E}\left[V^B(\delta, v^x_\delta;p_\delta + Kh_\delta) \cdot 1_{\tau[K]\geq\delta}\right] + \varepsilon\delta.\end{aligned}$$

We bound the first term using Remark 1 on the growth–rate of the square–integral of the process $v^{t,x}_s$, and (16) the asymptotically linear condition in x for $V^B(t,x;p)$, we have:

$$\mathbb{E}\left[e^{-r\tau[K]} V^B(\tau[K], v^x_{\tau[K]};p) \cdot 1_{\tau[K]<\delta}\right] \leq \sup_{\tau\in\mathcal{T}} \mathbb{E}\left[e^{-2r\tau} V^B(\tau, v^x_\tau;p)^2\right]^{1/2} \cdot \sup_{K\geq 0} \mathbb{E}\left[1_{\tau[K]<\delta}\right]^{1/2},$$

where the first factor is finite, and the second factor is $< \varepsilon'$ by our choice of δ. For the second term, we know from the result on constant pricing policy value function that $V^B(\delta, v_\delta^x; p_\delta + Kh_\delta) = 0$ for all sufficiently large $K > 0$, giving a pointwise convergence of $V^B(\delta, v_\delta^x; p_\delta + Kh_\delta) \cdot 1_{\tau[K] \geq \delta}$ in the probability space. Thus, given any $\varepsilon'' > 0$, we can find a sufficiently large $K > 0$ such that $\mathbb{E}\left[V^B(\delta, v_\delta^x; p_\delta + Kh_\delta) \cdot 1_{\tau[K] \geq \delta}\right] < \varepsilon''$ by the Dominated Convergence Theorem. Overall, we have

$$V^B(0, x; \tilde{p}) \leq \varepsilon' \cdot \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[e^{-2r\tau} V^B(\tau, v_\tau^x; p)^2\right]^{1/2} + \varepsilon'' + \varepsilon\delta.$$

Since $\varepsilon, \varepsilon', \varepsilon'' > 0$ are arbitrarily small, we conclude that $V^B(0, x; \tilde{p}) \leq 0$, a contradiction. Therefore, for any $x \in [\underline{\pi}, \bar{\pi}]$, for all sufficiently large $K \geq 0$, either it is optimal to purchase immediately ($x > \tilde{p}_0$), or exit immediately ($x < \tilde{p}_0$). It must be the case that: $\bar{V}_0[\tilde{p}] \searrow \max\{\tilde{p}_0, \underline{\pi}\}, \underline{V}_0[\tilde{p}] \nearrow \min\{\tilde{p}_0, \bar{\pi}\}$ as $K \to +\infty$.

Suppose that $K < 0$, consider any $x \in [\underline{\pi}, \bar{\pi}]$, we note that

$$V^B(0, x; \tilde{p}) \geq \mathcal{V}^B(0, x; \delta, \tilde{p}) \geq e^{-r\delta} \mathbb{E}\left[\max\left\{v_\delta^x - p_\delta - Kh_\delta, 0\right\}\right] - c\delta \geq -e^{-r\delta}(Kh_\delta + p_\delta) - c\delta$$

where δ denotes the simple policy of stopping exactly at some time $\delta > 0$ regardless of the valuation, and the first inequality followed from the sub–optimality of δ. Therefore, for all sufficiently negative $K << 0$, we have $V^B(0, x; \tilde{p}) > 0$, thus it is optimal to continue searching: i.e. $\underline{V}_0[\tilde{p}] < x < \bar{V}_0[\tilde{p}]$ for any $x \in [\underline{\pi}, \bar{\pi}]$, proving $\bar{V}_0[\tilde{p}] \nearrow \bar{\pi}$ and $\underline{V}_0[\tilde{p}] \searrow \underline{\pi}$ as $K \to -\infty$. $\square$

Proof of Lemma 2. For convenience, in the following we will use $A_1(\mathbf{x}, V, \nabla V, \Delta V) := c + rV - \partial_t V - \frac{\sigma(\mathbf{x})^2}{2} \partial_x^2 V$, $A_2(\mathbf{x}, V, \nabla V, \Delta V) := V - x + p_t$, $A_3(\mathbf{x}, V, \nabla V, \Delta V) := V$, so that $H := \min_{i=1,2,3} A_i$. We divide the proof in to two parts, first we show that the viscosity solution to (15) subject to the specified boundary conditions is unique, then we show that the value function is a viscosity solution.

Part 1 (viscosity solution is unique):

We show that the viscosity solution to (15) subjects to the specified condition is unique. Although, this is mostly an application of the comparison principle (Crandall et al., 1992, Theorem 3.3), in our context the domain is unbounded, hence we layout the detail for completeness. Let $u : \mathbb{R} \times [\underline{\pi}, \bar{\pi}] \to \mathbb{R}$ and $v : \mathbb{R} \times [\underline{\pi}, \bar{\pi}] \to \mathbb{R}$ be viscosity subsolution and supersolution to (15), respectively, and suppose that $\lim_{t \to \pm\infty}(u - v) \leq 0$, $\lim_{x \to \underline{\pi}}(u - v) \leq 0$, and $\lim_{x \to \bar{\pi}}(u - v) \leq 0$. We claim that $u \leq v$ everywhere on $\mathbb{R} \times [\underline{\pi}, \bar{\pi}]$. To show this let us assume the contrary that there exists $\hat{\mathbf{x}} \in \mathbb{R} \times (\underline{\pi}, \bar{\pi})$ such that $u(\hat{\mathbf{x}}) - v(\hat{\mathbf{x}}) = \max_{\mathbf{x} \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}]} (u(\mathbf{x}) - v(\mathbf{x})) > 0$. Consider the function: $w_\alpha(\mathbf{x}, \mathbf{y}) := u(\mathbf{x}) - v(\mathbf{y}) - (\alpha/2)\|\mathbf{x} -$

$\mathbf{y}\|_2^2$ for some constant $\alpha \geq 0$. The assumption on the boundary conditions of u and v implies that for any $\alpha \geq 0$ there exists a local maximum $(\mathbf{x}_\alpha, \mathbf{y}_\alpha) \in (\mathbb{R} \times [\underline{\pi}, \bar{\pi}])^2$ of w_α, and by (Crandall et al., 1992, Lemma 3.1):

$$\lim_{\alpha\to\infty} \alpha\|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2^2 = 0, \quad \lim_{\alpha\to\infty} \left(u(\mathbf{x}_\alpha) - v(\mathbf{x}_\alpha) - \frac{\alpha}{2}\|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2^2\right) = u(\hat{\mathbf{x}}) - v(\hat{\mathbf{x}}).$$

By our assumption, we can find $\delta > 0$ such that $u(\mathbf{x}_\alpha) - v(\mathbf{y}_\alpha) \geq \delta$ for all $\alpha \geq 0$. We can apply (Crandall et al., 1992, Theorem 3.2) since $\mathbb{R} \times [\underline{\pi}, \bar{\pi}]$ is locally compact, and we find $X, Y \in \mathcal{S}_2(\mathbb{R})$ such that

$$-3\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \tag{44}$$

with $\mathbf{x}_\alpha$ a local maximum of $u(\mathbf{x}) - \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha)^\intercal(\mathbf{x} - \mathbf{x}_\alpha) - \frac{1}{2}(\mathbf{x} - \mathbf{x}_\alpha)^\intercal X(\mathbf{x} - \mathbf{x}_\alpha)$ and $\mathbf{y}_\alpha$ a local minimum of $v(\mathbf{y}) - \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha)^\intercal(\mathbf{y} - \mathbf{y}_\alpha) - \frac{1}{2}(\mathbf{y} - \mathbf{y}_\alpha)^\intercal Y(\mathbf{y} - \mathbf{y}_\alpha)$. Since u and v are subsolution and supersolution, respectively, we have:

$$H(\mathbf{x}_\alpha, u(\mathbf{x}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X) \leq 0 \leq H(\mathbf{y}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), Y). \tag{45}$$

From (44) we have:

$$\begin{aligned}
&A_1(\mathbf{y}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), Y) - A_1(\mathbf{x}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X) \\
&\qquad = \frac{\sigma(\mathbf{x}_\alpha)^2}{2} X_{xx} - \frac{\sigma(\mathbf{y}_\alpha)^2}{2} Y_{xx} = \begin{pmatrix} \sigma(\mathbf{x}_\alpha) & \sigma(\mathbf{y}_\alpha) \end{pmatrix} \begin{pmatrix} X_{xx} & 0 \\ 0 & -Y_{xx} \end{pmatrix} \begin{pmatrix} \sigma(\mathbf{x}_\alpha) \\ \sigma(\mathbf{y}_\alpha) \end{pmatrix} \\
&\quad \leq 3\alpha \begin{pmatrix} \sigma(\mathbf{x}_\alpha) & \sigma(\mathbf{y}_\alpha) \end{pmatrix} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \begin{pmatrix} \sigma(\mathbf{x}_\alpha) \\ \sigma(\mathbf{y}_\alpha) \end{pmatrix} = 3\alpha(\sigma(\mathbf{x}_\alpha) - \sigma(\mathbf{y}_\alpha))^2 \leq 3\alpha L^2 \|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2^2
\end{aligned}$$

where we used the condition (2) for σ in the last inequality. Similarly, we can check that

$$\begin{aligned}
A_2(\mathbf{y}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), Y) - A_2(\mathbf{x}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X) &\leq \|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2 + |p_{t_x} - p_{t_y}| \\
&\leq \left(1 + \max_{t\in[0,T]} |p'_t|\right) \|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2,
\end{aligned}$$

and $A_3(\mathbf{y}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), Y) - A_3(\mathbf{x}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X) = 0$. Let us define $\omega(r) := \max\left\{3L^2, 1 + \max_{t\in[0,T]} |p'_t|\right\} \cdot r$ and $i^* := \operatorname{argmin}_{i=1,2,3} A_i(\mathbf{x}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X)$, then

$$H(\mathbf{y}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), Y) - H(\mathbf{x}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X)$$

$$
\begin{aligned}
&\leq A_{i^*}(\mathbf{y}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), Y) - A_{i^*}(\mathbf{x}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X) \\
&\qquad\qquad\qquad\qquad \leq \omega\left(\alpha\|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2^2 + \|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
0 < \min\{1,r\}\delta &\leq \min\{1,r\}(u(\mathbf{x}_\alpha) - v(\mathbf{y}_\alpha)) \\
&\leq H(\mathbf{x}_\alpha, u(\mathbf{x}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X) - H(\mathbf{x}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X) \\
&= H(\mathbf{x}_\alpha, u(\mathbf{x}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X) - H(\mathbf{y}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), Y) \\
&+ H(\mathbf{y}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), Y) - H(\mathbf{x}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X) \\
&\qquad\qquad\qquad \leq \omega\left(\alpha\|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2^2 + \|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2\right) \qquad (46)
\end{aligned}
$$

for all $\alpha \geq 0$, where we used (45) to replace the first two terms to zero in the last inequality. But by taking the $\alpha \to \infty$ limit, $\omega\left(\alpha\|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2^2 + \|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2\right) \to 0$, while the inequality above specifies that it is bounded away from zero by $\min\{1,r\}\delta$, which is a contradiction. In other words, we have $u \leq v$ over the entire $\mathbb{R} \times [\underline{\pi}, \bar{\pi}]$. Therefore, if $u : \mathbb{R} \times [\underline{\pi}, \bar{\pi}] \to \mathbb{R}$ and $v : \mathbb{R} \times [\underline{\pi}, \bar{\pi}] \to \mathbb{R}$ are both viscosity solution to (15) with the specified boundary conditions: $\lim_{t\to\pm\infty}(u - v) = 0$, $\lim_{x\to\underline{\pi}}(u - v) = 0$, and $\lim_{x\to\bar{\pi}}(u - v) = 0$, then $u = v$ over the entire $\mathbb{R} \times [\underline{\pi}, \bar{\pi}]$.

Part 2 (the value function is a viscosity solution):

First, we show that V^B is continuous. Suppose that $\mathbf{x}_0 = (t, x_0), \mathbf{x}_1 = (t, x_1) \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}]$ are given. For any $\varepsilon > 0$, we can find $\tau_a \in \mathcal{T}$ for $a = 0, 1$ such that: $V^B(\mathbf{x}_a) \leq \mathcal{V}^B(\mathbf{x}_a; \tau_a, p) + \varepsilon$, while $V^B(\mathbf{x}_a) \geq \mathcal{V}^B(\mathbf{x}_a; \tau_{1-a}, p)$ by definition. It follows that:

$$
\begin{aligned}
&\left|V^B(\mathbf{x}_0) - V^B(\mathbf{x}_1)\right| \leq \\
&\qquad \max_{a=0,1} \mathbb{E}\left[e^{-r(\tau_a - t)}\left(\max\{v_{\tau_a}^{t,x_a} - p_{\tau_a}, 0\} - \max\{v_{\tau_a}^{t,x_{1-a}} - p_{\tau_a}, 0\}\right)|\mathscr{F}_t\right] + \varepsilon.
\end{aligned}
$$

Using the Lipschitz condition (2) with Gronwall's inequality to upper bounds the growth of the SDE solutions $\{v_s^{t,x_a}\}_{s\geq t}$, it is possible to show that the RHS approaches zero as $x_1 \to x_0$. Suppose that $\mathbf{x}_0 = (t_0, x), \mathbf{x}_1 = (t_1, x) \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}]$ are given, for some $t_1 > t_0$. Then by the optimality principle, we can find $\tau \in \mathcal{T}$ such that

$$
\begin{aligned}
&\mathbb{E}\left[e^{-r(t_1\wedge\tau - t)}V^B(t_1 \wedge \tau, v_{t_1\wedge\tau}^{t_0,x}) - V^B(\mathbf{x}_1) - \int_{t_0}^{t_1\wedge\tau} ce^{-r(s-t_0)}ds|\mathscr{F}_{t_0}\right] + \varepsilon(t_1 - t_0) \\
&\qquad \geq V^B(\mathbf{x}_0) - V^B(\mathbf{x}_1) \geq \mathbb{E}\left[e^{-r(t_1 - t)}V^B(t_1, v_{t_1}^{t_0,x}) - V^B(\mathbf{x}_1) - \int_{t_0}^{t_1} ce^{-r(s-t_0)}ds|\mathscr{F}_{t_0}\right].
\end{aligned}
$$

Using the continuity of V^B in x we have previously proven, we have a pointwise convergence to zero for both integrands as $t_1 \searrow t_0$. We can conclude using the Dominated Convergence Theorem that both expected values converge to zero as $t_1 \searrow t_0$. The convergence to zero as $t_1 \nearrow t_0$ can be obtained similarly. Putting both results together, we find that V^B is continuous in both t and x. Next, we will show that V^B is both a viscosity subsolution and supersolution using the standard argument (e.g. see Yong and Zhou (2012)).

Given $\mathbf{x} = (t, x) \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}]$ and a twice continuously differentiable function ϕ such that $\mathbf{x}$ is a local maximum of $V^B - \phi$. Let us assume that $V^B(t, x) > \max\{x - p_t, 0\}$, i.e. $\mathbf{x}$ is in the learning region, otherwise we have $\min_{i=2,3} A_i(t, x, V^B(t, x), \nabla\phi, \Delta\phi) \le 0$ then it is trivial that $H(t, x, V^B(t, x), \nabla\phi, \Delta\phi) \le 0$. Let $t' > t \ge 0$, we can find $\tau \in \mathcal{T}$ such that $V^B(t, x) \le \mathbb{E}\left[e^{-r(t'\wedge\tau - t)} V^B(t' \wedge \tau, v^{t,x}_{t'\wedge\tau}) - \int_t^{t'\wedge\tau} ce^{-r(s-t)} ds | \mathscr{F}_t\right] + \varepsilon(t' - t)$. Then for all t' sufficiently close to t, we have

$$
\begin{aligned}
0 &\le \mathbb{E}\left[V^B(t, x) - \phi(t, x) - V^B(t' \wedge \tau, v^{t,x}_{t'\wedge\tau}) + \phi(t' \wedge \tau, v^{t,x}_{t'\wedge\tau}) \mid \mathscr{F}_t\right] \\
&\le \mathbb{E}\left[\phi(t' \wedge \tau, v^{t,x}_{t'\wedge\tau}) - \phi(t, x) - \int_t^{t'\wedge\tau} ce^{-r(s-t)} ds - \left(1 - e^{-r(t'\wedge\tau - t)}\right) V^B(t' \wedge \tau, v^{t,x}_{t'\wedge\tau}) \mid \mathscr{F}_t\right] \\
&\quad + \varepsilon(t' - t) = \mathbb{E}\Bigg[\int_t^{t'\wedge\tau} \left(\partial_t\phi(s, v^{t,x}_s) + \frac{\sigma(v_s)^2}{2}\partial_x^2\phi(s, v^{t,x}_s)\right) ds + \int_t^{t'\wedge\tau} \partial_x\phi(s, v^{t,x}_s) dv^{t,x}_s \\
&\qquad - \int_t^{t'\wedge\tau} ce^{-r(s-t)} ds - \left(1 - e^{-r(t'\wedge\tau - t)}\right) V^B(t' \wedge \tau, v^{t,x}_{t'\wedge\tau}) \mid \mathscr{F}_t\Bigg] + \varepsilon(t' - t).
\end{aligned}
$$

Since $\{v^{t,x}_t\}_{t\ge 0}$ is a square–integrable martingale, we know that $\int_t^{t'\wedge\tau} \partial_x\phi(s, v^{t,x}_s) dv^{t,x}_s$ is also a continuous square–integrable martingale (see Karatzas and Shreve (2012)[13]), hence the second term vanishes by the Martingale Stopping Theorem. Dividing both–sides by $t' - t$, taking the limit $t' \to t$ and apply the Dominated Convergence Theorem, we get: $A_1(t, x, V^B(t, x), \nabla\phi, \Delta\phi) \le \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we conclude that $A_1(t, x, V^B(t, x), \nabla\phi, \Delta\phi) \le 0$. Thus, we have $H(t, x, V^B(t, x), \nabla\phi, \Delta\phi) \le 0$, so V^B is a viscosity subsolution of (15).

Given $\mathbf{x} = (t, x) \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}]$ and a twice continuously differentiable function ϕ such that $\mathbf{x}$ is a local minimum of $V^B - \phi$. Then for all $t' > t \ge 0$ sufficiently close, we have

$$
0 \ge \frac{1}{t' - t}\mathbb{E}\left[V^B(t, x) - \phi(t, x) - V^B(t', v^{t,x}_{t'}) + \phi(t', v^{t,x}_{t'}) \mid \mathscr{F}_t\right]
$$

[13] We also need a square–integrability condition: $\mathbb{E}\left[\int_t^{t'\wedge\tau} \left(\sigma(v^{t,x}_s)\partial_x\phi(s, v^{t,x}_s)\right)^2 ds\right] < \infty$, but we can always ensure this by choosing a more appropriate ϕ with the same $\nabla\phi$ and $\Delta\phi$ at (t, x).

$$\geq \frac{1}{t'-t}\mathbb{E}\left[\phi(t', v_{t'}^{t,x}) - \phi(t,x) - \int_t^{t'} ce^{-r(s-t)}ds - \left(1 - e^{-r(t'-t)}\right) V^B(t', v_{t'}^{t,x}) \mid \mathscr{F}_t\right]$$

$$= \frac{1}{t'-t}\mathbb{E}\Bigg[\int_t^{t'} \left(\partial_t\phi(s, v_s^{t,x}) + \frac{\sigma(v_s^{t,x})^2}{2}\partial_x^2\phi(s, v_s^{t,x})\right) ds + \int_t^{t'} \partial_x\phi(s, v_s^{t,x})dv_s^{t,x}$$

$$- \int_t^{t'} ce^{-r(s-t)}ds - \left(1 - e^{-r(t'-t)}\right) V^B(t', v_{t'}^{t,x}) \mid \mathscr{F}_t\Bigg].$$

The second inequality followed from the optimality principle: $V^B(t,x) \geq \mathbb{E}\left[e^{-r(t'-t)}V^B(t', v_{t'}^{t,x}) - \int_t^{t'} ce^{-r(s-t)}ds \mid \mathscr{F}_t\right]$. The second term is a continuous square–integrable martingale, hence vanishes as explained. Taking limit $t' \to t$ and apply the Dominated Convergence Theorem, we get: $A_1(t, x, V^B(t,x), \nabla\phi, \Delta\phi) \geq 0$. The conditions $A_{i=2,3}(t, x, V^B(t,x), \nabla\phi, \Delta\phi) \geq 0$ only depends on $V^B(t,x) \geq \max\{x - p_t, 0\}$, hence they are trivial. Thus, we have shown that $H(t, x, V^B(t,x), \nabla\phi, \Delta\phi) \geq 0$, so V^B is a viscosity supersolution of (15). □

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Online Appendix for Consumer Gradual Learning and Firm Non-stationary Pricing

Proof of Lemma 3. Part 1:

Let such a solution V to (17) be given. Since we have assumed $V(t,x) \geq \max\{x - p_t, 0\}$ and $p_t' + r(\bar{V}_t[p] - p_t) + c \geq 0$, then $V - \max\{x - p_t, 0\} \geq 0$ for all $\mathbf{x} \in \Omega$, and $c + rV - \partial_t V - \frac{\sigma(\mathbf{x})^2}{2}\partial_x^2 V \geq 0$ for all $\mathbf{x} \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}] \setminus \Omega$. By the value–matching, the smooth pasting conditions, and the assumption that $p \in \mathcal{P}_T$ is smooth, we have that V is continuously differentiable[1]. Moreover, V is twice continuously differentiable in x on $\mathbb{R} \times [\underline{\pi}, \bar{\pi}] \setminus \partial\Omega$, as it is a (classical) solution to the PDE on Ω, and $\max\{x - p_t, 0\}$ is twice continuously differentiable in x on $\mathbb{R} \times [\underline{\pi}, \bar{\pi}] \setminus \Omega$. Therefore, we have $H(\mathbf{x}, V, \nabla V, \Delta V) = 0$ classically on $\mathbb{R} \times [\underline{\pi}, \bar{\pi}] \setminus \partial\Omega$. Thus, for any twice continuously differentiable ϕ and any $\mathbf{x}_0 \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}]$, we have $\nabla\phi(\mathbf{x}_0) = \nabla V(\mathbf{x}_0)$, and we can find $\{\mathbf{x}_i\}_{i=0}^{\infty} \subset \mathbb{R} \times [\underline{\pi}, \bar{\pi}] \setminus \partial\Omega$ converging to $\mathbf{x}_0$. If $\mathbf{x}_0$ is a local maximum of $V - \phi$ then $\partial_x^2\phi(\mathbf{x}_0) \geq \lim_{i\to\infty} \partial_x^2 V(\mathbf{x}_i)$ which implies $H(\mathbf{x}_0, V(\mathbf{x}_0), \nabla\phi(\mathbf{x}_0), \Delta\phi(\mathbf{x}_0)) \leq \lim_{i\to\infty} H(\mathbf{x}_i, V(\mathbf{x}_i), \nabla V(\mathbf{x}_i), \Delta V(\mathbf{x}_i)) = 0$. Similarly, if $\mathbf{x}_0$ is a local minimum of $V - \phi$ then $\partial_x^2\phi(\mathbf{x}_0) \leq \lim_{i\to\infty} \partial_x^2 V(\mathbf{x}_i)$ which implies $H(\mathbf{x}_0, V(\mathbf{x}_0), \nabla\phi(\mathbf{x}_0), \Delta\phi(\mathbf{x}_0)) \geq \lim_{i\to\infty} H(\mathbf{x}_i, V(\mathbf{x}_i), \nabla V(\mathbf{x}_i), \Delta V(\mathbf{x}_i)) = 0$.

Part 2:

Repeat the argument from the previous part with the perturbed pricing policy $p + \sqrt{\varepsilon}h \in \mathcal{P}_T$, we have that $H(\mathbf{x}, V_{\leq k}^{\varepsilon}, \nabla V_{\leq k}^{\varepsilon}, \Delta V_{\leq k}^{\varepsilon}) = O(\varepsilon^{(k+1)/2})$ classically on $\mathbb{R} \times [\underline{\pi}, \bar{\pi}] \setminus \partial\Omega_{\leq k}^{\varepsilon}$. Moreover, for any twice continuously differentiable ϕ and any $\mathbf{x}_0 \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}]$, if $\mathbf{x}_0$ is a local maximum of $V_{\leq k}^{\varepsilon} - \phi$ then $H(\mathbf{x}_0, V_{\leq k}^{\varepsilon}, \nabla\phi, \Delta\phi) \leq O(\varepsilon^{(k+1)/2})$, and if $\mathbf{x}_0$ is a local minimum of $V_{\leq k}^{\varepsilon} - \phi$ then $H(\mathbf{x}_0, V_{\leq k}^{\varepsilon}, \nabla\phi, \Delta\phi) \geq O(\varepsilon^{(k+1)/2})$. Since $V_{\leq k}^{\varepsilon}$ satisfies the same asymptotic boundary conditions as the value function V^B, we can repeat the comparison principle argument in the proof of Lemma 2. In particular, setting $u := V_{\leq k}^{\varepsilon}, v := V^B$ we have (45) becomes $H(\mathbf{x}_\alpha, V_{\leq k}^{\varepsilon}, \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X) + O(\varepsilon^{(k+1)/2}) \leq 0 \leq H(\mathbf{y}_\alpha, V^B, \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), Y)$, which means (46) becomes $\min\{1, r\}(V_{\leq k}^{\varepsilon}(\mathbf{x}_\alpha) - V^B(\mathbf{y}_\alpha)) \leq \omega(\alpha\|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2^2 + \|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2) + O(\varepsilon^{(k+1)/2})$. Taking the limit $\alpha \to \infty$, we find that $\sup_{\mathbf{x}\in\mathbb{R}\times[\underline{\pi},\bar{\pi}]} \left(V_{\leq k}^{\varepsilon}(\mathbf{x}) - V^B(\mathbf{x})\right) \leq O(\varepsilon^{(k+1)/2})$, in other words: $V_{\leq k}^{\varepsilon}(\mathbf{x}) \leq V^B(\mathbf{x}) + O(\varepsilon^{(k+1)/2})$. On the other hand, setting $u := V^B, v := V_{\leq k}^{\varepsilon}$ yields $V^B(\mathbf{x}) \leq V_{\leq k}^{\varepsilon}(\mathbf{x}) + O(\varepsilon^{(k+1)/2})$, thus we have $V^B(\mathbf{x}) = V_{\leq k}^{\varepsilon}(\mathbf{x}) + O(\varepsilon^{(k+1)/2})$ as claimed. $\square$

Proof of Proposition 2. We note that $V^B(t, x - \sqrt{\varepsilon}h_t; p)$ is simply the solution $V^B(t, x; p)$ shifted according to $\sqrt{\varepsilon}Kh$ which satisfies the value–matching and smooth–pasting conditions at $\bar{V}[p] + \sqrt{\varepsilon}h$ and $\underline{V}[p] + \sqrt{\varepsilon}h$, but does not satisfies the PDE, hence the $\sqrt{\varepsilon}V_1^B$

[1] To get the continuity of t derivative across the boundary, consider the defining equation: $V^B(t, \bar{V}_t[p]; p) = \bar{V}_t[p] - p_t$. Differentiating with respect to t gives: $\bar{V}_t'[p] \cdot \partial_x V^B(t, \bar{V}_t[p]; p) + \partial_t V^B(t, \bar{V}_t[p]; p) = \bar{V}_t'[p] - p_t'$, or $\partial_t V^B(t, \bar{V}_t[p]; p) = -p_t'$. Similarly, we have $\partial_t V^B(t, \underline{V}_t[p]; p) = 0$

correction is needed. By adding $\sqrt{\varepsilon}V_1^B$ correction, we further need a $\sqrt{\varepsilon}$–order correction to the purchase and exit boundaries $\bar{V}[p] + \sqrt{\varepsilon}h$ and $\underline{V}[p] + \sqrt{\varepsilon}h$ which take the form (20). We find the equation for V_1^B by substituting the ansatz (18) into the PDE for $V^B(.,.;\tilde{p})$ and collecting the $\sqrt{\varepsilon}$–order terms:

$$\frac{\sigma(x)^2}{2}\partial_x^2 V_1^B(t,x) + \partial_t V_1^B(t,x) - rV_1^B(t,x) \\ - h_t'\partial_x V^B(t,x;p) + h_t\sigma(x)\sigma'(x)\partial_x^2 V^B(t,x;p) = 0. \quad (47)$$

To study $\bar{R}$ and $\underline{R}$ we analyze the boundary conditions of $V^B(.,.;\tilde{p})$ to the first–order in $\sqrt{\varepsilon}$. Note that $V^B(t, x-\sqrt{\varepsilon}h_t; p)$ automatically satisfies the value–matching conditions at $\bar{V}[\tilde{p}]$ and $\underline{V}[\tilde{p}]$, as we will confirm below, because $\partial_x V^B(t, \bar{V}_t[p]; p) = 1$ and $\partial_x V^B(t, \underline{V}_t[p]; p) = 0$. We have by substituting the ansatz (18) and (20) into the boundary conditions and comparing the $\sqrt{\varepsilon}$–order terms:

$$V^B(t, \bar{V}_t[\tilde{p}]; \tilde{p}) = \bar{V}_t[\tilde{p}] - \tilde{p}_t \\ \implies V^B(t, \bar{V}_t[p] + \sqrt{\varepsilon}\bar{R}_t; p) + \sqrt{\varepsilon}V_1^B(t, \bar{V}_t[p]) = \bar{V}_t[p] - p_t + \sqrt{\varepsilon}\bar{R}_t \\ V_1^B(t, \bar{V}_t[p]) = -\bar{R}_t\partial_x V^B(t, \bar{V}_t[p]; p) + \bar{R}_t \implies V_1^B(t, \bar{V}_t[p]) = 0. \quad (48)$$

$$\partial_x V^B(t, \bar{V}_t[\tilde{p}]; \tilde{p}) = 1 \implies \partial_x V^B(t, \bar{V}_t[p] + \sqrt{\varepsilon}\bar{R}_t; p) + \sqrt{\varepsilon}\partial_x V_1^B(t, \bar{V}_t[p]) = 1 \\ \partial_x V_1^B(t, \bar{V}_t[p]) = -\bar{R}_t\partial_x^2 V^B(t, \bar{V}_t[p]; p) \implies \bar{R}_t = -\frac{\partial_x V_1^B(t, \bar{V}_t[p])}{\partial_x^2 V^B(t, \bar{V}_t[p]; p)}. \quad (49)$$

$$V^B(t, \underline{V}_t[\tilde{p}]; \tilde{p}) = 0 \implies V^B(t, \underline{V}_t[p] + \sqrt{\varepsilon}\underline{R}_t; p) + \sqrt{\varepsilon}V_1^B(t, \underline{V}_t[p]) = 0 \\ \implies V_1^B(t, \underline{V}_t[p]) = 0. \quad (50)$$

$$\partial_x V^B(t, \underline{V}_t[\tilde{p}]; \tilde{p}) = 0 \implies \partial_x V^B(t, \underline{V}_t[p] + \sqrt{\varepsilon}\underline{R}_t; p) + \sqrt{\varepsilon}\partial_x V_1^B(t, \underline{V}_t[p]) = 0 \\ \partial_x V_1^B(t, \underline{V}_t[p]) = -\underline{R}_t\partial_x^2 V^B(t, \underline{V}_t[p]; p) \implies \underline{R}_t = -\frac{\partial_x V_1^B(t, \underline{V}_t[p])}{\partial_x^2 V^B(t, \underline{V}_t[p]; p)}. \quad (51)$$

Since $p, h \in \mathcal{P}_T$, they are constant for all $t \geq T$, therefore we have the terminal condition at any $T' \geq T$: $V^B(T', x; \tilde{p}) = V_0^B(x; \tilde{p}_T)$ and $V^B(T', x; p) = V_0^B(x; p_T)$, giving the terminal condition for V_1^B:

$$V_1^B(T',x) = V_1^B(T,x) = \frac{1}{\sqrt{\varepsilon}}\left(V_0^B(x;p_T+\sqrt{\varepsilon}h_T) - V_0^B(x-\sqrt{\varepsilon}h_T;p_T)\right) + O(\sqrt{\varepsilon}). \tag{52}$$

We recognize the PDE (47) with (48). (50), and (52) as a backward parabolic (fixed) boundary–value problem. We may transform the problem into the more standard parabolic form for: $\widetilde{V}_1^B(t',x') := V_1^B\left(T-t', \underline{V}_{T-t'}[p] + (\bar{V}_{T-t'}[p] - \underline{V}_{T-t'}[p])x'\right)$ on $\widetilde{\Omega} := [0,\infty)\times[0,1]$ with smooth coefficients $(a_{ij}(.), b_i(.), c(.))$, according to our smoothness assumptions on $\bar{V}[p]$, $\underline{V}[p]$, and $\sigma(.)$. Since $-(h)_t'\partial_x V^B(.,.;p) + h_t\sigma(.)\sigma'(.)\partial_x^2 V^B(.,.;p)$ is assumed smooth on $\widetilde{\Omega}$, and $V_1^B(T,.)$ is smooth on $\{0\}\times[0,1]$, we can apply (Evans, 2022, Chapter 7.1, Theorem 7) or (Friedman, 2008, Chapter 3, §5, Corollary 2) to conclude the existence of the smooth solution $\widetilde{V}_1^B$ to the parabolic initial boundary–value problem. Transform back to the original problem, we get the smooth solution $V_1^B(.,.)$. The solution is unique, and admits a probabilistic expression via the semi–elliptic version of Feynman–Kac formula (Øksendal, 2003, Theorem 9.1.1):

$$\begin{aligned} V_1^B(t,x) = \mathbb{E}\left[e^{-r(T'-t)}V_1^B(T', v_{T'}^{t,x}) \cdot 1\left\{\tau_\Omega^{t,x} \geq T'\right\} | \mathscr{F}_t\right] \\ - \mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}\wedge T'} h_s' e^{-r(s-t)}\partial_x V^B(s, v_s^{t,x};p)ds | \mathscr{F}_t\right] \\ + \mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}\wedge T'} h_s e^{-r(s-t)}\sigma(v_s^{t,x})\sigma'(v_s^{t,x})\partial_x^2 V^B(s, v_s^{t,x};p)ds | \mathscr{F}_t\right], \end{aligned} \tag{53}$$

The first term is upper–bounded by $\sup_{x\in[\underline{V}_T[p],\bar{V}_T[p]]} V_1^B(T,x)e^{-r(T'-t)} \to 0$ as $T' \to \infty$. Since $p,h \in \mathcal{P}_T$, we have that h_t, $V^B(t,x;p)$, and the boundaries $\bar{V}_t[p], \underline{V}_t[p]$ are constant in t for $t \geq T$. Meanwhile, $v_s^{t,x}$ is bounded inside $[\inf_{s\in[t,T]} \underline{V}_s[p], \sup_{s\in[t,T]} \bar{V}_s[p]]$. Therefore, the third and fourth terms are upper–bounded by some constant (which can be determined by the supremum of the absolute value of the integrand over the compact set $[t,T]\times[\inf_{s\in[t,T]} \underline{V}_s[p], \sup_{s\in[t,T]} \bar{V}_s[p]]$) multiple of $\int_t^\infty e^{-r(s-t)}ds = 1/r < \infty$. Taking the limit $T' \to \infty$ of (53) using the Dominated Convergence Theorem for the right–hand–side while noting that the left–hand–side is independent of T', we obtain the expression (19) for $V_1^B(t,x)$. □

Proof of Corollary 1. Suppose that $h := K\tilde{h}$, where $\tilde{h} \in \mathcal{P}_T$ is monotonically increasing in t, and that $\sigma'(.) = O(\varepsilon)$. We define $\bar{S} := \bar{R}/K : \mathbb{R} \to \mathbb{R}$, and $\underline{S} := \underline{R}/K : \mathbb{R} \to \mathbb{R}$. It remains to show that $\bar{S}_t \leq 0$ and $\underline{S}_t \geq 0$. From our assumption that $\sigma'(.) = O(\varepsilon)$, we may ignore the third term in the $\sqrt{\varepsilon}$–order equation (19). Moreover, $\sigma'(.) = O(\varepsilon)$ implies

$V_0^B(x; p_T + \varepsilon h_T) = V_0^B(x - \varepsilon h_T; p_T) + O(\varepsilon)$, hence we can also ignore the first term in (19). Since $V^B(t, .; p)$ is monotonically increasing in x from Lemma 1, it follows from the second term of (19) that $V_1^B(t,x)/K \leq 0$ for any $(t,x) \in \Omega$. In particular, $\partial_x V_1^B(t, \bar{V}_t[p])/K \geq 0$ and $\partial_x V_1^B(t, \underline{V}_t[p])/K \leq 0$.

Now, let us show that $\partial_x^2 V^B(t, \bar{V}_t[p]; p) \geq 0$. Let $\mathbf{x}_0 = (t, \bar{V}_t[p]) \in \partial\Omega$ be a point on the purchase boundary, then we can find sequences $\{\mathbf{x}_i^+ = (t_i, x_i^+)\}_{i=0}^\infty$ and $\{\mathbf{x}_i^- = (t_i, x_i^-)\}_{i=0}^\infty \subset \Omega$ converging to $\mathbf{x}_0$ such that $x_i^- \leq \bar{V}_{t_i}[p] \leq x_i^+$ for all $i \geq 0$. Since $V^B(.,.;p)$ is the viscosity solution, we have $c + rV^B(\mathbf{x}_i^+; p) - \partial_t V^B(\mathbf{x}_i^+; p) - \frac{\sigma(\mathbf{x}_i^+)^2}{2} \partial_x^2 V^B(\mathbf{x}_i^+; p) \geq 0$, while $c + rV^B(\mathbf{x}_i^-; p) - \partial_t V^B(\mathbf{x}_i^-; p) - \frac{\sigma(\mathbf{x}_i^-)^2}{2} \partial_x^2 V^B(\mathbf{x}_i^-; p) = 0$ for all $i \geq 0$. But $V^B(\mathbf{x}_i^+; p) = x_i^+ - p_{t_i}$, so $\frac{\sigma(\mathbf{x}_i^+)^2}{2} \partial_x^2 V^B(\mathbf{x}_i^+; p) = 0$, hence it follows from the continuous differentiability of $V^B(.,.;p)$ across the boundary $\partial\Omega$ that $\partial_x^2 V^B(t, \bar{V}_t[p]; p) \geq 0$. Similarly, we have that $\partial_x^2 V^B(t, \underline{V}_t[p]; p) \geq 0$.

It follows from (49) and (51) that the sign of $\bar{S}_t$ and $\underline{S}_t$ are the opposite as the sign of $\partial_x V_1^B(t, \bar{V}_t[p])/K$ and $\partial_x V_1^B(t, \underline{V}_t[p])/K$, respectively, therefore, we have $\bar{S}_t \leq 0$ and $\underline{S}_t \geq 0$ for $t \in \mathbb{R}$ as claimed. □

Proof of Lemma 4. Consider a fixed $(t,x) \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}]$, and suppose that $V^B(t,x;q) \leq V^B(t,x;p)$. For an arbitrary $\varepsilon > 0$, let $\tau_{t,x,\varepsilon}[p] \in \mathcal{T}$ be such that $\mathcal{V}^B(t,x;\tau_{t,x,\varepsilon}[p], p) \geq V^B(t,x;p) - \varepsilon$, then

$$V^B(t,x;q) \geq \mathcal{V}^B(t,x;\tau_{t,x,\varepsilon}[p], q) > \mathcal{V}^B(t,x;\tau_{t,x,\varepsilon}[p]; p) - \max_{s \geq t} e^{-r(s-t)} |p_s - q_s|$$
$$\geq V^B(t,x;p) - \max_{s \geq t} e^{-r(s-t)} |p_s - q_s| - \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, it must be the case that:

$$V^B(t,x;p) \geq V^B(t,x;q) \geq V^B(t,x;p) - \max_{s \geq t} e^{-r(s-t)} |p_s - q_s|.$$

If $V^B(t,x;q) \geq V^B(t,x;p)$, then we simply switch the role of p, q and follow through with the above argument, hence we get that

$$|V^B(t,x;p) - V^B(t,x;q)| \leq \max_{s \geq t} e^{-r(s-t)} |p_s - q_s|,$$

which proves the result. □

Proof of Proposition 3. Let $p^T, l_{\mathbf{x}}^T \in \mathcal{P}_T$ be given by some pricing strategies which coincide with $p, l_{\mathbf{x}}$ over $[0, T - \varepsilon]$ and constant for all $t \geq T$. By Lemma 4, we have $|V^B(t,x;p^T) - V^B(t,x;l_{\mathbf{x}}^T)| \leq \max_{s \in [t,T]} e^{-r(s-t)} |p_s - l_{\mathbf{x},s}| \leq \max_{s \geq t} e^{-r(s-t)} |p_s - l_{\mathbf{x},s}|$. Since this inequality holds for all T, we conclude that $|V^B(t,x;p) - V^B(t,x;l_{\mathbf{x}})| \leq \max_{s\geq t} e^{-r(s-t)}|p_s - l_{\mathbf{x},s}|$. But from Taylor's Theorem, we have $|p_s - l_{\mathbf{x},s}| \leq \frac{M}{2}(s-t)^2$ for all $s \geq t$. It follows that $\max_{s\geq t} e^{-r(s-t)}|p_s - l_{\mathbf{x},s}| \leq \frac{M}{2}\max_{s\geq t}(s-t)^2 e^{-r(s-t)} = \frac{2M}{r^2}e^{-2}$. Therefore, $|V^B(t,x;p) - V^B(t,x;l_{\mathbf{x}})| < \varepsilon$ if $r > e^{-1}\sqrt{2M/\varepsilon}$. $\square$

Proof of Corollary 2. The existence and uniqueness of the ODE boundary value problems (23) and (24) follows from the standard theory (Agarwal et al., 2008, Lecture 40). In order to make use of Proposition 2, let us first fix a large $T \geq 0$ and consider $p^T = p_0 + \sqrt{\varepsilon}h^T \in \mathcal{P}_T$ where $h^T \in \mathcal{P}_T$ is given by $h^T_t = Kt$ for $t \in [0, T-\varepsilon]$, constant $h^T_t = KT$ for $t \geq T$, and some in–between smooth transition for $t \in (T-\varepsilon, T)$. We shall assume that $|(h^T)'_t| \leq 1$ for $t \in (T-\varepsilon, T)$. From Proposition 2, we have the following probabilistic expression:

$$V_1^B(t,x;p^T) = -\mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}} (h^T)'_s e^{-r(s-t)}\partial_x V_0^B(v_s^{t,x};p_0)ds|\mathscr{F}_t\right] \\ + \mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}} h^T_s e^{-r(s-t)}\sigma(v_s^{t,x})\sigma'(v_s^{t,x})\partial_x^2 V_0^B(v_s^{t,x};p_0)ds|\mathscr{F}_t\right], \quad (54)$$

where $\tau_\Omega^{t,x} := \inf\{t' \geq t \mid (t', v_{t'}^{t,x}) \notin \Omega\}$ is the exit time. Further, we know that V_1^B satisfies the value–matching boundary conditions $V_1^B(t, \bar{V}[p_0]; p^T) = V_1^B(t, \underline{V}[p_0]; p^T) = 0$, while the smooth–pasting boundary conditions determine $\bar{R}$ and $\underline{R}$. We note that $v_s^{t,x}$ is bounded inside $[\underline{V}[p_0], \bar{V}[p_0]]$, while $\int_0^\infty |h_s^T e^{-r(s-t)}|ds \leq Ke^{rt}/r^2$, and $\int_0^\infty |(h^T)'_s e^{-r(s-t)}|ds \leq Ke^{rt}/r$ by construction for all $T \geq 0$. Therefore, by taking the limit $T \to \infty$ of (54), we have by Lemma 4 and inquality(22) that $V_1^B(t,x;p^T) \to V_1^B(t,x)$, and by applying the Dominated Convergence Theorem to the right–hand–side with $h_s^T \to Ks, (h^T)'_s \to K$, we obtain:

$$V_1^B(t,x) = -K\cdot\mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}} e^{-r(s-t)}\partial_x V_0^B(v_s^{t,x};p_0)ds|\mathscr{F}_t\right] \\ + K\cdot\mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}} se^{-r(s-t)}\sigma(v_s^{t,x})\sigma'(v_s^{t,x})\partial_x^2 V_0^B(v_s^{t,x};p_0)ds|\mathscr{F}_t\right] \quad (55)$$

We rewrite this further as follows:

$$V_1^B(t,x) = -K\cdot\mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}} e^{-r(s-t)}\partial_x V_0^B(v_s^{t,x};p_0)ds|\mathscr{F}_t\right] \\ + K\cdot\mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}} (s-t)e^{-r(s-t)}\sigma(v_s^{t,x})\sigma'(v_s^{t,x})\partial_x^2 V_0^B(v_s^{t,x};p_0)ds|\mathscr{F}_t\right]$$

$$
+ Kt \cdot \mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}} e^{-r(s-t)}\sigma(v_s^{t,x})\sigma'(v_s^{t,x})\partial_x^2 V_0^B(v_s^{t,x};p_0)ds|\mathscr{F}_t\right]
$$

$$
= K\cdot\mathbb{E}\left[\int_0^{\tau_\Omega^x} se^{-rs}\sigma(v_s^x)\sigma'(v_s^x)\partial_x^2 V_0^B(v_s^x;p_0)ds|\mathscr{F}_0\right] - K\cdot\mathbb{E}\left[\int_0^{\tau_\Omega^x} e^{-rs}\partial_x V_0^B(v_s^x;p_0)ds|\mathscr{F}_0\right]
$$

$$
+ Kt\cdot\mathbb{E}\left[\int_0^{\tau_\Omega^x} e^{-rs}\sigma(v_s^x)\sigma'(v_s^x)\partial_x^2 V_0^B(v_s^x;p_0)ds|\mathscr{F}_0\right] =: V_1^B(0,x) + t\widetilde{V}_{1,1}^B(x). \quad (56)
$$

Note that the above expression is linear in t, in particular, the first two terms $V_1^B(0,x)$ and the factor $\widetilde{V}_{1,1}^B(x)$ of t in the last term are functions of x only. The boundary conditions of $V_1^B(t,\bar{V}[p_0]) = V_1^B(t,\underline{V}[p_0]) = 0$ holds for all t, and therefore we also have $V_1^B(0,\bar{V}[p_0]) = V_1^B(0,\underline{V}[p_0]) = 0$ and $\widetilde{V}_{1,1}^B(\bar{V}[p_0]) = \widetilde{V}_{1,1}^B(\underline{V}[p_0]) = 0$. We recognize the probabilistic expression for $\widetilde{V}_{1,1}^B$ as that of the solution to the boundary–value problem (23), thus, we have $\widetilde{V}_{1,1}^B = V_{1,1}^B$. Let us define:

$$
V_{1,1}^B(x;\beta) := \mathbb{E}\left[\int_0^{\tau_\Omega^x} e^{-\beta s}\sigma(v_s^x)\sigma'(v_s^x)\partial_x^2 V_0^B(v_s^x;p_0)ds\right],
$$

which satisfies the ODE:

$$
\frac{\sigma(x)^2}{2}\partial_x^2 V_{1,1}^B(x;\beta) - \beta V_{1,1}^B(x;\beta) + K\sigma(x)\sigma'(x)\partial_x^2 V_0^B(x;p_0) = 0. \quad (57)
$$

We can see that $V_{1,1}^B(x) := V_{1,1}^B(x;\beta = r)$, and that the first term of (56) is given by $-\partial_\beta V_{1,1}^B(x;\beta)|_{\beta=r}$, therefore:

$$
V_1^B(0,x) + \partial_\beta V_{1,1}^B(x;\beta)|_{\beta=r} = -K\cdot\mathbb{E}\left[\int_0^{\tau_\Omega^x} e^{-rs}\partial_x V_0^B(v_s^x;p_0)ds\right].
$$

Substituting this into the corresponding ODE of the right–hand–side of the above, we get:

$$
\frac{\sigma(x)^2}{2}\partial_x^2\left(V_1^B(0,x) + \partial_\beta V_{1,1}^B(x;\beta)|_{\beta=r}\right) - r\left(V_1^B(0,x) + \partial_\beta V_{1,1}^B(x;\beta)|_{\beta=r}\right) - K\partial_x V_0^B(x;p_0) = 0
$$

$$
\implies \frac{\sigma(x)^2}{2}\partial_x^2 V_1^B(0,x) - rV_1^B(0,x) + V_{1,1}^B(x) - K\partial_x V_0^B(x;p_0) = 0,
$$

where we used $V_{1,1}^B(x) = \frac{\sigma(x)^2}{2}\partial_x^2\partial_\beta V_{1,1}^B(x;\beta)|_{\beta=r} - \beta\partial_\beta V_{1,1}^B(x;\beta)|_{\beta=r}$ which is obtained by differentiating (57) at $\beta = r$, note that although $V_0^B(x;p_0)$ depends on r, it does not depends on β, hence its β derivative vanishes. Therefore, $V_1^B(0,.)$ satisfies the ODE (24) with the specified boundary conditions, therefore it must coincides with $V_{1,0}$. □

Proof of Proposition 4. In the special case of linear pricing $t \mapsto p_t := p_0 + \sqrt{\varepsilon}Kt$ the value function takes the form (25) over Ω as we can directly check that it satisfies the PDE of (17). Let's define $K_\pm := \frac{\sqrt{\varepsilon}K \pm \sqrt{\varepsilon K^2 + 2r\sigma^2}}{\sigma^2}$ for convenience. The purchase and exit boundaries ansatz take the form (26). We determine the unknown $A_1, A_2, \bar{V}[\sqrt{\varepsilon}K]$, and $\underline{V}[\sqrt{\varepsilon}K]$ from the boundary conditions

$$V^B(t, \bar{V}_t) = \bar{V}_t - p_t \implies A_1 e^{K_- \bar{V}[\sqrt{\varepsilon}K]} + A_2 e^{K_+ \bar{V}[\sqrt{\varepsilon}K]} - \frac{c}{r} = \bar{V}[\sqrt{\varepsilon}K] \tag{58}$$

$$\partial_x V^B(t, \bar{V}_t) = 1 \implies A_1 K_- e^{K_- \bar{V}[\sqrt{\varepsilon}K]} + A_2 K_+ e^{K_+ \bar{V}[\sqrt{\varepsilon}K]} = 1 \tag{59}$$

$$V^B(t, \underline{V}_t) = 0 \implies A_1 e^{K_- \underline{V}[\sqrt{\varepsilon}K]} + A_2 e^{K_+ \underline{V}[\sqrt{\varepsilon}K]} - \frac{c}{r} = 0 \tag{60}$$

$$\partial_x V^B(t, \bar{V}_t) = 0 \implies A_1 K_- e^{K_- \underline{V}[\sqrt{\varepsilon}K]} + A_2 K_+ e^{K_+ \underline{V}[\sqrt{\varepsilon}K]} = 0 \tag{61}$$

From (60) and (61) we find that

$$A_1 = \frac{c}{r}\left(\frac{K_+}{K_+ - K_-}\right) e^{-K_- \underline{V}[\sqrt{\varepsilon}K]}, \quad A_2 = \frac{c}{r}\left(\frac{K_-}{K_- - K_+}\right) e^{-K_+ \underline{V}[\sqrt{\varepsilon}K]}. \tag{62}$$

Substituting (62) back into (59), we obtain the equation to be solved for $(\bar{V}[\sqrt{\varepsilon}K] - \underline{V}[\sqrt{\varepsilon}K])$:

$$e^{K_+(\bar{V}[\sqrt{\varepsilon}K] - \underline{V}[\sqrt{\varepsilon}K])} - e^{K_-(\bar{V}[\sqrt{\varepsilon}K] - \underline{V}[\sqrt{\varepsilon}K])} = \frac{r}{c} \cdot \frac{K_- - K_+}{K_- K_+}, \tag{63}$$

we note that the LHS is an increasing function, hence the solution always exists. Finally, we find $\bar{V}[\sqrt{\varepsilon}K]$ by substituting (62) back into (58) and simplify:

$$\bar{V}[\sqrt{\varepsilon}K] = \frac{1}{K_-} + \frac{c}{r}\left(e^{K_+(\bar{V}[\sqrt{\varepsilon}K] - \underline{V}[\sqrt{\varepsilon}K])} - 1\right) \tag{64}$$

from this it is simple to find $\underline{V}[\sqrt{\varepsilon}K]$. Equation (63) and (64) is equivalent to the following non–linear system of equations:

$$\begin{cases} e^{\frac{\sqrt{\varepsilon}K + \sqrt{\varepsilon K^2 + 2r\sigma^2}}{\sigma^2}(\bar{V}[\sqrt{\varepsilon}K] - \underline{V}[\sqrt{\varepsilon}K])} - e^{\frac{\sqrt{\varepsilon}K - \sqrt{\varepsilon K^2 + 2r\sigma^2}}{\sigma^2}(\bar{V}[\sqrt{\varepsilon}K] - \underline{V}[\sqrt{\varepsilon}K])} & = \dfrac{\sqrt{\varepsilon K^2 + 2r\sigma^2}}{c} \\ \dfrac{c}{r}\left(e^{\frac{\sqrt{\varepsilon}K + \sqrt{\varepsilon K^2 + 2r\sigma^2}}{\sigma^2}(\bar{V}[\sqrt{\varepsilon}K] - \underline{V}[\sqrt{\varepsilon}K])} - 1\right) - \bar{V}[\sqrt{\varepsilon}K] & = \dfrac{\sqrt{\varepsilon}K + \sqrt{\varepsilon K^2 + 2r\sigma^2}}{2r} \end{cases} \tag{65}$$

When $\sqrt{\varepsilon}K \sim 0$, we may obtain a simple expression for $\bar{V}[\sqrt{\varepsilon}K]$ and $\underline{V}[\sqrt{\varepsilon}K]$ to the ε–order. We substituting the ansatz (27) into (63), (64), and comparing the zeroth–order and $\sqrt{\varepsilon}$–order terms we get the claimed expression for $\bar{S} := \bar{R}/K, \underline{S} := \underline{R}/K$. The signs of $\bar{S}$ and $\underline{S}$ followed from the Proposition 2, but one can also verify explicitly. □

Proof of Proposition 5. The solution (28) and (29) to (24) and (23) can be obtained using a standard ODE solving technique such as the "variation of parameters". The rest of the results are easily taken care of by Corollary 2. $\square$

Proof of Lemma 5. Part 1:

It follows from (31) that $\mathbb{P}[\tau^*[p_0] < \infty] = 1$. Therefore, $\mathbb{P}[v^x_{\tau^*[p_0]} \geq \bar{V}[p_0]] + \mathbb{P}[v^x_{\tau^*[p_0]} \leq \bar{V}[p_0]] = 1$, and since $\{v^x_{t\wedge\tau^*[p_0]}\}_{t\geq 0}$ is a uniformly integrable martingale, we have by the Martingale Stopping Theorem that: $x = v^x_0 = \mathbb{E}[v^x_{\tau^*[p_0]}] = \bar{V}[p_0]\mathbb{P}[v^x_{\tau^*[p_0]} \geq \bar{V}[p_0]] + \underline{V}[p_0]\mathbb{P}[v^x_{\tau^*[p_0]} \leq \bar{V}[p_0]]$. If $m = 0$ then $\mathcal{V}^S(x;p_0) = (p_0 - g)\mathbb{P}[v^x_{\tau^*[p_0]} \geq \bar{V}[p_0]]$ hence the first part is proven by solving the system of linear equations for $\mathbb{P}[v^x_{\tau^*[p_0]} \geq \bar{V}[p_0]]$.

Part 2:

We can consider $v_t = \sigma W_t$, where $\{W_t\}_{t\geq 0}$ denotes the standard Brownian motion. Then $\mathcal{V}^S(x;p_0) = (p_0 - g)\mathbb{E}[e^{-m\tau^*[p_0]} \cdot 1\{v^x_{\tau^*[p_0]} = \bar{V}[p_0]\}]$ which can also be evaluated using the standard technique involving Martingale Stopping Theorem (see Karatzas and Shreve (2012)). $\square$

Proof of Proposition 6. We would like to solve the PDE initial boundary value problem (35) up to the ε–order. The idea is similar to the proof of Proposition 2, except it is easier here since the boundaries $\bar{V}[p], \underline{V}[p]$ are already fixed for us by Proposition 2. The claim is that if $\mathcal{V}^S(.,.;p)$ solves (35) exactly for the given $\bar{V}[p], \underline{V}[p]$, and $p \in \mathcal{P}_T$, and if $\mathcal{V}^\varepsilon_{\leq k}(.,.;p)$ solves (35) up to the $\varepsilon^{(k+1)/2}$–order with the same given $\bar{V}[p], \underline{V}[p]$, and $p \in \mathcal{P}_T$, then by comparing their corresponding Feynman–Kac expressions, we have $\mathcal{V}^S = \mathcal{V}^\varepsilon_{\leq k} + O(\varepsilon^{(k+1)/2})$. We omit further detail, and proceed with the $k = 1$ for the seller's expected payoff up to the ε–order.

We recall that the buyer's ε–optimal response given by Proposition 2, characterized by the purchase and exit boundaries: $\bar{V}[p] = (\bar{V}[p_0] + \sqrt{\varepsilon}h) + \sqrt{\varepsilon}\bar{R}$, and $\underline{V}[p] = (\underline{V}[p_0] + \sqrt{\varepsilon}h) + \sqrt{\varepsilon}\underline{R}$, respectively. We propose the perturbation ansatz:

$$
\begin{aligned}
\mathcal{V}^S(t,x;p) &= \mathcal{V}^S_0\left(\frac{\bar{V}[p_0] - \underline{V}[p_0]}{\bar{V}_t[p] - \underline{V}_t[p]}(x - \underline{V}_t[p]) + \underline{V}[p_0]; p_0\right) + \sqrt{\varepsilon}\mathcal{V}^S_1(t,x) + O(\varepsilon) \\
&= \mathcal{V}^S_0\left((1 - \sqrt{\varepsilon}r_{1,t})x - \sqrt{\varepsilon}r_{0,t}; p_0\right) + \sqrt{\varepsilon}\mathcal{V}^S_1(t,x) + O(\varepsilon),
\end{aligned}
$$

where, in the second equality, we expanded the argument of $\mathcal{V}^S_0(.;p_0)$ to the first order in $\sqrt{\varepsilon}$. The first term represents a naive rescaling of the constant price solution according to the buyer's response moving boundaries. Substituting the ansatz into the PDE (35) and collect the $\sqrt{\varepsilon}$–terms, we obtain the PDE for $\mathcal{V}^S_1$:

$$
\begin{aligned}
&\frac{\sigma(x)^2}{2}\partial^2_x\mathcal{V}^S_1(t,x) + \partial_t\mathcal{V}^S_1(t,x) - m\mathcal{V}^S_1(t,x) \\
&\quad + \left(\sigma(x)\sigma'(x)(r_{1,t}x + r_{0,t}) - \sigma(x)^2 r_{1,t}\right)\partial^2_x\mathcal{V}^S_0(x;p_0) - (r'_{1,t}x + r'_{0,t})\partial_x\mathcal{V}^S_0(x;p_0) = 0,
\end{aligned}
$$

along with the boundary conditions up to the ε–order:

$$\mathcal{V}^S(t,\bar{V}_t[p];p) = p_t - g \implies \mathcal{V}_0^S(\bar{V}[p_0];p_0) + \sqrt{\varepsilon}\mathcal{V}_1^S(t,\bar{V}[p_0]) + O(\varepsilon) = p_0 + \sqrt{\varepsilon}h_t - g$$
$$\implies \mathcal{V}_1^S(t,\bar{V}[p_0]) = h_t$$

$$\mathcal{V}^S(t,\underline{V}_t[p];p) = 0 \implies \mathcal{V}_0^S(\underline{V}[p_0];p_0) + \sqrt{\varepsilon}\mathcal{V}_1^S(t,\underline{V}[p_0]) + O(\varepsilon) = 0$$
$$\implies \mathcal{V}_1^S(t,\underline{V}[p_0]) = 0,$$

and finally we have the terminal condition at any $T' \geq T$: $\mathcal{V}^S(T',x;p) = \mathcal{V}_0^S(x;p_T)$ which gives

$$\mathcal{V}_1^S(T',x) = \frac{1}{\sqrt{\varepsilon}}\left(\mathcal{V}_0^S(x;p_0+\sqrt{\varepsilon}h_T) - \mathcal{V}_0^S((1-\sqrt{\varepsilon}r_{1,T})x - \sqrt{\varepsilon}r_{0,T};p_0)\right) + O(\sqrt{\varepsilon}).$$

Note that $\mathcal{V}_1^S(T',x)$ is determined by the constant price expected profits $\mathcal{V}_0^S$ which is not difficult to find, and it is in fact independent of $T' \geq T$. We can reverse the time–axis, then apply (Evans, 2022, Chapter 7, Theorem 7) or (Friedman, 2008, Chapter 3, §5, Corollary 2) to conclude the existence of the smooth solution $\mathcal{V}_1^S$ to the parabolic initial boundary–value problem. The solution is unique and admits the following probabilistic expression via the semi–elliptic version of Feynman–Kac Formula (Øksendal, 2003, Theorem 9.1.1):

$$\begin{aligned}
\mathcal{V}_1^S(t,x) = \mathbb{E}&\left[e^{-m(T'-t)}\mathcal{V}_1^S(T',v_{T'}^{t,x})\cdot 1\left\{\tau_\Omega^{t,x} \geq T'\right\}|\mathscr{F}_t\right] \\
&+ \mathbb{E}\left[h_{\tau_\Omega^{t,x}}e^{-m(\tau_\Omega^{t,x}-t)}\cdot 1\left\{v_{\tau_\Omega^{t,x}}^{t,x} \geq \bar{V}[p_0], \tau_\Omega^{t,x} < T'\right\}|\mathscr{F}_t\right] \\
&- \mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}\wedge T'} e^{-m(s-t)}(r'_{1,s}v_s^{t,x} + r'_{0,s})\partial_x\mathcal{V}_0^S(v_s^{t,x};p_0)ds|\mathscr{F}_t\right] \\
+ \mathbb{E}&\left[\int_t^{\tau_\Omega^{t,x}\wedge T'} e^{-m(s-t)}\left(\sigma(v_s^{t,x})\sigma'(v_s^{t,x})(r_{1,s}v_s^{t,x}+r_{0,s}) - \sigma(v_s^{t,x})^2 r_{1,s}\right)\partial_x^2\mathcal{V}_0^S(v_s^{t,x};p_0)ds|\mathscr{F}_t\right].
\end{aligned} \tag{66}$$

The first term is upper–bounded by $\sup_{x\in[\underline{V}[p_0],\bar{V}[p_0]]}\mathcal{V}_1^S(T,x)\mathbb{P}\left[\tau_\Omega^{t,x} \geq T'|\mathscr{F}_t\right] \to 0$ as $T' \to \infty$ by (31). Since $h \in \mathcal{P}_T$, we have that the second term is upper–bounded by $\sup_{s\in[t,T]} h_s$. As a consequence of $h \in \mathcal{P}_T$, it also follows that $r_{0,t}$, $r_{1,t}$ are constant for all $t \geq T$. Meanwhile, $v_s^{t,x}$ is bounded inside $[\underline{V}[p_0],\bar{V}[p_0]]$. Therefore, the third and fourth terms are upper–bounded by some constant (which can be determined by the supremum of the absolute value of the integrand over the compact set $[t,T]\times[\underline{V}[p_0],\bar{V}[p_0]]$) multiple of $\mathbb{E}[\tau_\Omega^{t,x}|\mathscr{F}_t] < \infty$. Taking the limit $T' \to \infty$ using the Dominated Convergence Theorem, then set $t = 0$, we obtain the expression (37) for $\mathcal{V}_1^S(0, x)$. □

Proof of Proposition 7. Equation (38) in the proposition follows from an application of Proposition 1 to the pricing strategy $p := p_0 + Kh$. It follows that as $K \to \infty$, the corresponding purchase and exit boundaries $\bar{V}_t[p_0 + Kh]$ and $\underline{V}_t[p_0 + Kh]$ will monotonically decrease and increase toward $p_0 + Kh_t$, respectively. If $x > p_0$ then only the purchase boundary $\bar{V}_0[p_0 + Kh]$ will reach x as $K \to \infty$, giving the seller the payoff $p_0 - g$. Likewise, for $x \leq p_0$ only $\underline{V}_0[p_0 + Kh]$ will reach x as $K \to \infty$ giving the seller the payoff 0.

Further, suppose that x is sufficiently high such that we can find a seller's ε–optimal pricing strategy $\tilde{p} \in \mathcal{P}_T$ satisfying $\underline{V}_t[\tilde{p}] > g$ for all $t \in [0, \infty)$. Let $\tau^*[\tilde{p}] \in \mathcal{T}$ denotes the corresponding ε–optimal buyer's stopping time to the pricing strategy $\tilde{p}$. It follows that

$$\begin{aligned}\mathcal{V}^S(x; \tau^*[\tilde{p}], \tilde{p}) &= \mathbb{E}\left[e^{-m\tau^*[\tilde{p}]}(p_{\tau^*[\tilde{p}]} - g) \cdot 1_{v_{\tau^*[\tilde{p}]} \geq \tilde{p}_{\tau^*[\tilde{p}]}} \mid v_0 = x\right] \\ &\leq \mathbb{E}\left[\left(\tilde{p}_{\tau^*[\tilde{p}]} - g\right) \cdot 1_{v_{\tau^*[\tilde{p}]} \geq \tilde{p}_{\tau^*[\tilde{p}]}} \mid v_0 = x\right] \leq \mathbb{E}\left[v_{\tau^*[\tilde{p}]} - g \mid v_0 = x\right] = x - g. \quad (67)\end{aligned}$$

The first inequality follows from removing the discounting factor. The second inequality follows by noting that if v_t hits the purchase boundary $\bar{V}_t[\tilde{p}]$ first we would have $v_{\tau^*[\tilde{p}]} - g \geq \tilde{p}_{\tau^*[p]} - g$, and if v_t hits the exit boundary first we would have $v_{\tau^*[\tilde{p}]} < \tilde{p}_{\tau^*[\tilde{p}]}$, so $v_{\tau^*[\tilde{p}]} - g = \underline{V}_{\tau^*[\tilde{p}]}[\tilde{p}] - g \geq 0 = (p_{\tau^*[\tilde{p}]} - g) \cdot 1_{v_{\tau^*[\tilde{p}]} \geq \tilde{p}_{\tau^*[\tilde{p}]}}$. The final equality followed from the Martingale stopping theorem since $\left|v_{t \wedge \tau^*[\tilde{p}]}\right|$ is bounded by $\max_{s \in [0,\infty)}\{|\underline{V}_s[\tilde{p}]|, |\bar{V}_s[\tilde{p}]|\} = \max_{s \in [0,T]}\{|\underline{V}_s[\tilde{p}]|, |\bar{V}_s[\tilde{p}]|\}$ where the latter is finite because both boundaries are continuous over $[0, T]$ and are constant over $[T, \infty)$ by the definition of $\mathcal{P}_T$. So $x - g \geq V^S(x) - \varepsilon$ for any arbitrary $\varepsilon > 0$, hence we conclude that $V^S(x) = x - g$. The claim that this supremum can be approached by (39) follows from (38). □

Proof of Theorem 1. We obtain the expression of $\mathcal{V}^S(x; p_0, K) = \mathcal{V}^S(x; p)$ from Proposition 6 with $h := Kt$, although strictly speaking, some justification is needed as $p \in \mathcal{P}$ instead of $p \in \mathcal{P}_T$. We will now discuss the detail. Following the proof of Proposition 6, we obtain the expression (66) for $\mathcal{V}_1^B(t, x)$ for any arbitrary large $T' \geq t$[2]. Since $m = 0$, we have from Lemma 5 that $\mathcal{V}_0^S(x; p_0) = (p_0 - g)\frac{x - \underline{V}[p_0]}{\bar{V}[p_0] - \underline{V}[p_0]}$. This means that the fourth term of (66) vanishes and (66) simplified to:

$$\mathcal{V}_1^S(t, x) = \mathbb{E}\left[\mathcal{V}_1^S(T', v_{T'}^{t,x}) \cdot 1\left\{\tau_\Omega^{t,x} \geq T'\right\} | \mathscr{F}_t\right] + \mathbb{E}\left[\tau_\Omega^{t,x} \cdot 1\left\{v_{\tau_\Omega^{t,x}}^{t,x} \geq \bar{V}[p_0], \tau_\Omega^{t,x} < T'\right\} | \mathscr{F}_t\right]$$

[2] Except now $\mathcal{V}_1^S(T', x)$ is not constant in T'. Instead, since p is linear, $\mathcal{V}_1^S(T', x)$ is at most linear in T', and this will be sufficient for us to argue that the first term of (66) converges to zero as $T' \to 0$.

$$- \frac{p_0 - g}{\bar{V}[p_0] - \underline{V}[p_0]} \mathbb{E}\left[\int_t^{\tau_\Omega^{t,x} \wedge T'} (r'_{1,s} v_s^{t,x} + r'_{0,s}) ds | \mathscr{F}_t\right], \quad (68)$$

where

$$\begin{aligned}
r_{1,t} &:= \frac{\bar{R}_t - \underline{R}_t}{\bar{V}[p_0] - \underline{V}[p_0]} = K \cdot \frac{\bar{S}_{0,0} - \underline{S}_{0,0} + (\bar{S}_{0,1} - \underline{S}_{0,1})t}{\bar{V}[p_0] - \underline{V}[p_0]} \\
r_{0,t} &:= h_t + \underline{R}_t - r_{1,t}\underline{V}[p_0] \\
&= K \cdot \left(\underline{S}_{0,0} - \frac{\bar{S}_{0,0} - \underline{S}_{0,0}}{\bar{V}[p_0] - \underline{V}[p_0]}\underline{V}[p_0]\right) + Kt \cdot \left(1 + \underline{S}_{0,1} - \frac{\bar{S}_{0,1} - \underline{S}_{0,1}}{\bar{V}[p_0] - \underline{V}[p_0]}\underline{V}[p_0]\right)
\end{aligned} \quad (69)$$

are as defined in Corollary 2. We can further simplify the third term as follows:

$$\begin{aligned}
&- \frac{p_0 - g}{\bar{V}[p_0] - \underline{V}[p_0]} \mathbb{E}\left[\int_t^{\tau_\Omega^x \wedge T'} (r'_{1,s} v_s^x + r'_{0,s}) ds | \mathscr{F}_t\right] \\
&\quad = - \frac{p_0 - g}{\bar{V}[p_0] - \underline{V}[p_0]} \mathbb{E}\left[\int_t^{\tau_\Omega^x \wedge T'} d(r_{1,s} v_s^x) - \int_t^{\tau_\Omega^x \wedge T'} r_{1,s} dv_s^x + \int_t^{\tau_\Omega^x \wedge T'} r'_{0,s} ds | \mathscr{F}_t\right] \\
&\qquad\qquad = - \frac{p_0 - g}{\bar{V}[p_0] - \underline{V}[p_0]} \left(\mathbb{E}\left[r_{1,\tau_\Omega^x \wedge T'} v^x_{\tau_\Omega^x \wedge T'} + r_{0,\tau_\Omega^x \wedge T'} | \mathscr{F}_t\right] - (r_{1,t} x - r_{0,t})\right).
\end{aligned}$$

We used Ito's Lemma in the first equality. For the second equality, note that $\{v_t^x\}_{t \geq 0}$ is a square–integrable martingale, hence we know that $\int_0^{\tau_\Omega^x} r_{1,s} dv_s^x$ is a continuous square–integrable martingale, therefore its expectation vanishes.

Since h is linear in t, we find that $\mathcal{V}_1^S(T', x)$ is at most linear in T', and we have already seen from (69) that $r_{1,t}$, $r_{0,t}$ are linear in t. Meanwhile, $v_s^{t,x}$ is bounded inside $[\underline{V}[p_0], \bar{V}[p_0]]$. Therefore, the first term is upper–bounded by $\sup_{x \in [\underline{V}[p_0], \bar{V}[p_0]]} \mathcal{V}_1^S(T', x) \mathbb{P}\left[\tau_\Omega^{t,x} \geq T' | \mathscr{F}_t\right] \to 0$ as $T' \to \infty$ since $\mathbb{P}\left[\tau_\Omega^{t,x} \geq T' | \mathscr{F}_t\right] \to 0$ exponentially according to (31). Both the second and third terms are upper–bounded by some constant multiple of $\mathbb{E}[\tau_\Omega^{t,x} | \mathscr{F}_t] < \infty$. Taking the limit $T' \to \infty$ using the Dominated Convergence Thoerem, then set $t = 0$, we obtain the expression for $\mathcal{V}_1^S(0, x)$. Substituting the expression for $\mathcal{V}_1^S(0, x)$ into the perturbative expansion (36) we obtain

$$\begin{aligned}
\mathcal{V}^S(x; p_0, K) &= (p_0 - g) \frac{x - \underline{V}[p_0]}{\bar{V}[p_0] - \underline{V}[p_0]} \\
&\quad + \sqrt{\varepsilon} K \mathbb{E}\left[\tau_\Omega^x \cdot 1\left\{v^x_{\tau_\Omega^x} \geq \bar{V}[p_0]\right\}\right] - \frac{\sqrt{\varepsilon}(p_0 - g)}{\bar{V}[p_0] - \underline{V}[p_0]} \mathbb{E}\left[r_{1,\tau_\Omega^x} v^x_{\tau_\Omega^x} + r_{0,\tau_\Omega^x}\right] + O(\varepsilon)
\end{aligned}$$

which leads to (40) after some simplifications and substitution of (69). □

Proof of Proposition 8. Part 1:

From the first equation of (65), when $c \searrow 0$, the RHS becomes large which means $\bar{V}[\sqrt{\varepsilon}K] - \underline{V}[\sqrt{\varepsilon}K]$ becomes large, and the LHS is $\sim e^{\frac{\sqrt{\varepsilon}K+\sqrt{\varepsilon K^2+2r\sigma^2}}{\sigma^2}(\bar{V}[\sqrt{\varepsilon}K]-\underline{V}[\sqrt{\varepsilon}K])}$. Therefore, the second equation of (65) together with (26) gives:

$$\bar{V}_t = p_0 + \sqrt{\varepsilon}Kt + \frac{\sqrt{\varepsilon K^2 + 2r\sigma^2} - \sqrt{\varepsilon}K}{2r}.$$

and $\underline{V}_t = -\infty$. Therefore, we only have one linearly moving boundary $\bar{V}_t$. Let's assume throughout also that $p_0 \geq g$. The solution $U(t,v)$ to the heat equation with the single linearly moving absorbing boundary with initial condition $U(t=0,v) = \delta(v-x), x \leq \bar{V}_0$, is well–known:

$$U(t,v) = \frac{\exp\left(-\frac{\sqrt{\varepsilon}K}{\sigma^2}(v - x - \sqrt{\varepsilon}Kt) - \frac{\varepsilon K^2}{2\sigma^2}t\right)}{\sigma\sqrt{2\pi t}} \left(e^{-\frac{(v-\sqrt{\varepsilon}Kt-x)^2}{2t\sigma^2}} - e^{-\frac{(v-\sqrt{\varepsilon}Kt+x-2\bar{V}_0)^2}{2t\sigma^2}}\right).$$

Therefore, the purchase probability flux is:

$$-\frac{\sigma^2}{2}\partial_v U(t,\bar{V}_t) = \frac{\bar{V}_0 - x}{\sigma\sqrt{2\pi t^3}} \exp\left(-\frac{\left(\bar{V}_t - x\right)^2}{2t\sigma^2}\right).$$

It is now straightforward to compute the expected firm's payoff at $t=0$:

$$\begin{aligned}
\mathcal{V}^S(x;p_0,K) &:= -\frac{\sigma^2}{2}\int_0^\infty e^{-ms}(p_s - g)\partial_v U(s,\bar{V}_s)ds \\
&= \left(p_0 - g + \frac{\sqrt{\varepsilon}K}{\sqrt{2m\sigma^2 + \varepsilon K^2}}\left(p_0 - x + \frac{\sqrt{\varepsilon K^2 + 2r\sigma^2} - \sqrt{\varepsilon}K}{2r}\right)\right) \\
&\times \exp\left(-\left(\frac{\sqrt{\varepsilon}K + \sqrt{2m\sigma^2 + \varepsilon K^2}}{\sigma^2}\right)\left(p_0 - x + \frac{\sqrt{\varepsilon K^2 + 2r\sigma^2} - \sqrt{\varepsilon}K}{2r}\right)\right), \quad (70)
\end{aligned}$$

for $x \leq \bar{V}_0$, otherwise if $x > \bar{V}_0$ then we have $\mathcal{V}^S(x;p_0,K) = p_0 - g$. In the special case where

$m = 0$, we have

$$\mathcal{V}^S(x; p_0, K) = \begin{cases} \left(2p_0 - g - x + \dfrac{\sqrt{\varepsilon K^2 + 2r\sigma^2} - \sqrt{\varepsilon}K}{2r}\right) \\ \quad \times \exp\left(-\dfrac{2\sqrt{\varepsilon}K}{\sigma^2}\left(p_0 - x + \dfrac{\sqrt{\varepsilon K^2 + 2r\sigma^2} - \sqrt{\varepsilon}K}{2r}\right)\right), & \sqrt{\varepsilon}K > 0 \\ p_0 - g, & \sqrt{\varepsilon}K = 0 \\ x - g - \dfrac{\sqrt{\varepsilon K^2 + 2r\sigma^2} - \sqrt{\varepsilon}K}{2r}, & \sqrt{\varepsilon}K < 0 \end{cases}.$$

For any fixed p_0, we can approach the supremum $2p_0 + \frac{\sigma}{\sqrt{2r}} - g - x \geq p_0 - g$ of $\mathcal{V}^S$ by choosing $\sqrt{\varepsilon}K \gtrsim 0$ as close to 0 as possible, and earning an extra of $\left(2p_0 + \frac{\sigma}{\sqrt{2r}} - g - x\right) - (p_0 - g) = p_0 - x + \frac{\sigma}{\sqrt{2r}}$. If we can also vary the initial price p_0, then it is optimal to set p_0 as large as possible, i.e. the optimal price is unbounded.

Part 2:

For $m > 0$, the optimal K is now bounded from 0. This can also be seen for a general p_0 by computing:

$$\begin{aligned} \frac{\partial \mathcal{V}^S}{\partial K}(x; p_0, K = 0) &= e^{-\frac{\sqrt{2m}}{\sigma}\left(p_0 - x + \frac{\sigma}{\sqrt{2r}}\right)} \\ &\quad \times \left(\frac{p_0 - x + \sigma/\sqrt{2r}}{\sigma\sqrt{2m}} - (p_0 - g)\left(\frac{p_0 - x + \sigma/\sqrt{2r} - (\sigma/r)\sqrt{m/2}}{\sigma^2}\right)\right), \end{aligned}$$

we can see that this is always > 0 for sufficiently small and sufficiently large $m > 0$. $\square$

Proof of Proposition 9. The standard solution U_0 to the heat equation (33) with 2 absorbing non–moving boundaries at $\bar{V}_0 := p_0 + \bar{V}[\sqrt{\varepsilon}K]$, $\underline{V}_0 := p_0 + \underline{V}[\sqrt{\varepsilon}K]$, and the initial condition $U_0(0, v) = \delta(v - x)$ is given by Karatzas and Shreve (2012):

$$U_0(t, v) = \frac{1}{\sigma\sqrt{2\pi t}} \sum_{k=-\infty}^{+\infty} \left[e^{-\frac{(v - x + 2k(\bar{V}_0 - \underline{V}_0))^2}{2t\sigma^2}} - e^{-\frac{(v + x - 2\underline{V}_0 + 2k(\bar{V}_0 - \underline{V}_0))^2}{2t\sigma^2}}\right]. \tag{71}$$

Equivalently:

$$U_0(t, v)dv = \mathbb{P}\left[x + \sigma W_t \in dv, \underline{V}_0 < x + \sigma W_s < \bar{V}_0, s \in [0, t]\right].$$

Instead of moving the boundary according to $\sqrt{\varepsilon}Kt$ we may consider the consumer valuation process to be the Brownian process with drift starting at x: $\tilde{v}_t = x - \sqrt{\varepsilon}Kt + \sigma W_t$ with fixed absorbing boundaries at $\bar{V}_0, \underline{V}_0$. By Girsanov Theorem, if $\{W_t\}$ is the standard Brownian process on $(\Omega, \mathscr{F}, \Sigma, \mathbb{P})$ then $\{x + \sigma W_t\}$ is the Brownian process with drift starting at x, i.e. $\{\tilde{v}_t\}$ on $(\Omega, \mathscr{F}, \Sigma, \mathbb{Q})$ where

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_{\mathscr{F}_t} = \exp\left(-\frac{\sqrt{\varepsilon}K}{\sigma}W_t - \frac{\varepsilon K^2}{2\sigma^2}t\right).$$

Consequently, we have that the solution U to the heat equation (33) with moving boundaries $\bar{V}_t, \underline{V}_t$ is given by

$$\begin{aligned} U(t,v)dv &= \mathbb{P}\left[\tilde{v}_t \in v - \sqrt{\varepsilon}Kt + dv, \underline{V}_0 < \tilde{v}_s < \bar{V}_0, s \in [0,t]\right] \\ &= \mathbb{Q}\left[x + \sigma W_t \in v - \sqrt{\varepsilon}Kt + dv, \underline{V}_0 < x + \sigma W_s < \bar{V}_0, s \in [0,t]\right] \\ &= \exp\left(-\frac{\sqrt{\varepsilon}K}{\sigma^2}(v - x - \sqrt{\varepsilon}Kt) - \frac{\varepsilon K^2}{2\sigma^2}t\right) U_0(t, v - \sqrt{\varepsilon}Kt)dv \end{aligned}$$

Therefore, the purchase probability flux is:

$$-\frac{\sigma^2}{2}\partial_v U(t,\bar{V}_t) = \sum_{k=-\infty}^{+\infty} \frac{(2k+1)(\bar{V}_0 - \underline{V}_0) - (x - \underline{V}_0)}{\sigma\sqrt{2\pi t^3}} e^{\frac{2\sqrt{\varepsilon}Kk}{\sigma^2}(\bar{V}_0 - \underline{V}_0)} e^{-\frac{((2k+1)(\bar{V}_0 - \underline{V}_0) - (x - \underline{V}_0) + \sqrt{\varepsilon}Kt)^2}{2t\sigma^2}}. \tag{72}$$

The term–by–term differentiation is justified at $v = \bar{V}_t$ for any fixed $x \in (\underline{V}_0, \bar{V}_0)$ because $0 < |\bar{V}_0 - x| < |\bar{V}_0 - \underline{V}_0|$, hence the series representation of $U_0(t, v - \sqrt{\varepsilon}Kt)$, and the derivative series both converge absolutely and uniformly for all v in some neighborhoods of $\bar{V}_t$ and $t \in [0, \infty)$. We now compute the seller's expected profit:

Claim 1. *The seller's expected profit from the consumer initially at $x \in (\underline{V}_0, \bar{V}_0)$ is:*

$$\begin{aligned} \mathcal{V}^S(x; p_0, K) = &\frac{\left(p_0 - g - \frac{\sqrt{\varepsilon}K}{\sqrt{2m\sigma^2+\varepsilon K^2}}(\bar{V}_0 + x - 2\underline{V}_0)\right) e^{-\frac{\sqrt{\varepsilon}K + \sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(\bar{V}_0 - x)}}{1 - e^{-\frac{2\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(\bar{V}_0 - \underline{V}_0)}} \\ &+ \frac{\frac{2\sqrt{\varepsilon}K}{\sqrt{2m\sigma^2+\varepsilon K^2}}(\bar{V}_0 - \underline{V}_0)e^{-\frac{\sqrt{\varepsilon}K + \sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(\bar{V}_0 - x)}}{\left(1 - e^{-\frac{2\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(\bar{V}_0 - \underline{V}_0)}\right)^2} \\ &- \frac{\left(p_0 - g - \frac{\sqrt{\varepsilon}K}{\sqrt{2m\sigma^2+\varepsilon K^2}}(\bar{V}_0 - x)\right) e^{-\frac{\sqrt{\varepsilon}K + \sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(\bar{V}_0 - \underline{V}_0)} e^{+\frac{\sqrt{\varepsilon}K - \sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(x - \underline{V}_0)}}{1 - e^{-\frac{2\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(\bar{V}_0 - \underline{V}_0)}} \\ &- \frac{\frac{2\sqrt{\varepsilon}K}{\sqrt{2m\sigma^2+\varepsilon K^2}}(\bar{V}_0 - \underline{V}_0)e^{-\frac{\sqrt{\varepsilon}K + \sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(\bar{V}_0 - \underline{V}_0)} e^{+\frac{\sqrt{\varepsilon}K - \sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(x - \underline{V}_0)}}{\left(1 - e^{-\frac{2\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(\bar{V}_0 - \underline{V}_0)}\right)^2}, \end{aligned} \tag{73}$$

if $m > 0$ or $K \neq 0$, and $\mathcal{V}^S(x;p_0,K) = (p_0 - g)\left(\frac{x-\underline{V}_0}{\bar{V}_0-\underline{V}_0}\right)$ if $m = 0, K = 0$. On the other hand, if $x \leq \underline{V}_0$ then $\mathcal{V}^S(x;p_0,K) = 0$, and if $x \geq \bar{V}_0$ then $\mathcal{V}^s(x;p_0,K) = p_0 - g$.

Proof. We shall only cover the non–trivial case where $x \in (\underline{V}_0, \bar{V}_0)$. First, let's assume that either $m > 0$ or $K \neq 0$. We compute $\mathcal{V}^S(x;p_0,K)$ by substituting (72) into (34):

$$\begin{aligned}
\mathcal{V}^S(x;p_0,K) &= -\frac{\sigma^2}{2}\int_0^\infty e^{-ms}(p_s-g)\partial_v U(s,\bar{V}_s)ds \\
&= \sum_{k=-\infty}^{+\infty}\left((2k+1)(\bar{V}_0-\underline{V}_0)-(x-\underline{V}_0)\right)e^{\frac{2\sqrt{\varepsilon}Kk}{\sigma^2}(\bar{V}_0-\underline{V}_0)} \\
&\quad\times\int_0^{+\infty}\frac{(p_0+\sqrt{\varepsilon}Ks-g)}{\sigma\sqrt{2\pi s^3}}e^{-ms-\frac{\left((2k+1)(\bar{V}_0-\underline{V}_0)-(x-\underline{V}_0)+\sqrt{\varepsilon}Ks\right)^2}{2s\sigma^2}}ds \\
&= \sum_{k=0}^{+\infty}\left(p_0-g+\frac{\sqrt{\varepsilon}K}{\sqrt{2m\sigma^2+\varepsilon K^2}}\left((2k+1)(\bar{V}_0-\underline{V}_0)-(x-\underline{V}_0)\right)\right) \\
&\quad\times\exp\left(+\frac{\sqrt{\varepsilon}K}{\sigma^2}\cdot 2k(\bar{V}_0-\underline{V}_0)-\frac{\sqrt{\varepsilon}K+\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}\left((2k+1)(\bar{V}_0-\underline{V}_0)-(x-\underline{V}_0)\right)\right) \\
&\quad-\sum_{k=1}^{+\infty}\left(p_0-g+\frac{\sqrt{\varepsilon}K}{\sqrt{2m\sigma^2+\varepsilon K^2}}\left((2k-1)(\bar{V}_0-\underline{V}_0)+(x-\underline{V}_0)\right)\right) \\
&\quad\times\exp\left(-\frac{\sqrt{\varepsilon}K}{\sigma^2}\cdot 2k(\bar{V}_0-\underline{V}_0)+\frac{\sqrt{\varepsilon}K-\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}\left((2k-1)(\bar{V}_0-\underline{V}_0)+(x-\underline{V}_0)\right)\right)
\end{aligned}$$

In the second equality, we switched the order of summation and integration, which can be justified by Fubini's theorem for $m > 0$ or $K \neq 0$. The resulting infinite series can be evaluated using standard geometric series results to yield (73). If $m = 0$ and $K = 0$, then it is known (see Branco et al. (2012)) that the seller's expected profit is $(p_0 - g)\left(\frac{x-\underline{V}_0}{\bar{V}_0-\underline{V}_0}\right)$. □

In the limit $\underline{V}_0 \to -\infty$ (i.e. the limit $c \to 0$) (73) reduces to (70) we previously studied. Unlike in the single boundary case, in the presence of the exit boundary, the expected seller's profit is not only continuous at $K = 0$, but also differentiable, even when $m = 0$, as we will show below. We now focus on the $m = 0$ case.

From (73) we have that $\mathcal{V}^S(x;p_0,K<0)|_{m=0}$ is given by (42), and that:

$$\begin{aligned}
&\mathcal{V}^S(x;p_0,K>0)|_{m=0} = \\
&\quad\frac{(p_0-g-(\bar{V}_0+x-2\underline{V}_0))\exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0-x)\right)}{1-\exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0-\underline{V}_0)\right)}+\frac{2(\bar{V}_0-\underline{V}_0)\exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0-x)\right)}{\left(1-\exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0-\underline{V}_0)\right)\right)^2}
\end{aligned}$$

$$
- \frac{(p_0 - g - (\bar{V}_0 - x)) \exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0 - \underline{V}_0)\right)}{1 - \exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0 - \underline{V}_0)\right)} - \frac{2(\bar{V}_0 - \underline{V}_0) \exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0 - \underline{V}_0)\right)}{\left(1 - \exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0 - \underline{V}_0)\right)\right)^2}. \quad (74)
$$

Both (42) and (74) are valid expressions for all $K \neq 0$, and with some works, we can show them to be equal for all $K \neq 0$. This proves $\mathcal{V}^S(x; p_0, K)$ is given by (42) for all $K \neq 0$. □

Proof of Lemma 6. We can compute that

$$
\frac{\partial p_0^*}{\partial K}(x, K = 0) = \frac{1}{12r\sigma} \left(3\sigma - 3\sqrt{\frac{2c^2}{r} + \sigma^2} \sinh^{-1} \sqrt{\frac{r\sigma^2}{2c^2}} - \sigma \left(\sinh^{-1} \sqrt{\frac{r\sigma^2}{2c^2}} \right)^2 \right) \leq 0
$$

where the inequality is strict everywhere except when $r\sigma^2/c^2 = 0$. Given that $r\sigma^2/c^2 > 0$, we can find a sufficiently small $\varepsilon > 0$ such that $p_0^*(x, .)$ is a decreasing function for $K \in [-1, +1]$. Any local maximum point of $\mathcal{V}^S(x; ., .)$ would take the form $(p_0^*(x; K^*), K^*)$ where $K^* := \arg\max_K \mathcal{V}^S(x; p_0^*(x, K), K)$. Hence, for all sufficiently small $\varepsilon > 0$, the profit maximizer (p_0^*, K^*) in $\mathcal{P}_{lin}^{\varepsilon}$ either satisfies $p_0^* < \hat{p}_0, K^* \gtrsim 0$, or $p_0^* > \hat{p}_0, K^* \lesssim 0$. □