Consumer Gradual Learning and Firm Non-stationary Pricing

Wee Chaimanowong

(Chinese University of Hong Kong)

Qiaoyi Chen

(City University of Hong Kong)

Yunfei (Jesse) Yao*

(Chinese University of Hong Kong)

March 2024

* We thank J. Miguel Villas-Boas and Song Lin for constructive comments. All errors are our own.
E-mail address: Wee Chaimanowong: chaimanowongw@gmail.com, Qiaoyi Chen: chenqiaoyi97@gmail.com, Yunfei (Jesse) Yao: jesseyao@cuhk.edu.hk.

Abstract

Consumers gather information gradually to help themselves make the purchasing decision. The increasingly popular privacy regulations have made it harder for firms to track individuals in real time. Even if a firm can track consumers' browsing behavior, it is difficult for the firm to infer whether consumers like the information they see. Without the ability to track consumer's valuation evolution about the product, one may think that firms can only offer a constant price. The major innovation of our paper is to allow the price to be a function of time rather than the consumer's current valuation of the product. We find that constant price is not always optimal for the firm. It can benefit from using non-stationary pricing strategies. When the search cost is zero, the optimal price is arbitrarily close to a constant price if the firm is perfectly patient. In contrast, the slope of the optimal price is bounded from zero if the firm discounts the future. When there is friction in search, the optimal price is non-stationary, even if the firm is perfectly patient. In particular, the firm always increases the price over time if the information is too noisy or the search cost is too high. In other cases where consumers have a stronger incentive to search, the firm charges an increasing price for consumers with high or low initial valuation, whereas charges a decreasing price for medium–value consumers.

1 Introduction

It has been widely documented that consumers gather information gradually to help themselves make the purchasing decision. They can visit the seller's website to see the product description, check reviews on the retailer's storefront, or search review articles through search engines. All these search activities help them reduce their uncertainty about the product's value, but only partially. Since the seminal paper by Weitzman (1979), many papers have studied the optimal search strategy when there are multiple alternatives or multiple attributes of a product (Weitzman, 1979; Wolinsky, 1986; Moscarini and Smith, 2001; Branco et al., 2012, 2016; Ke et al., 2016; Liu and Dukes, 2016; Ke and Villas-Boas, 2019; Guo, 2021; Yao, 2023c; Chaimanowong et al., 2023). Recent papers have started to look at the marketing implications of consumer's gradual learning activities, including information provision policies (Branco et al., 2016; Jerath and Ren, 2021; Ke et al., 2023; Yao, 2023a), search costs manipulation (Bar-Isaac et al., 2010; Dukes and Liu, 2016), product line design (Villas-Boas, 2009; Kuksov and Villas-Boas, 2010; Guo and Zhang, 2012; Liu and Dukes, 2013), consumers' repeat buying and drop–out decision Chaimanowong and Ke (2022), and advertising (Mayzlin and Shin, 2011). Among the possible marketing mixes, pricing is one of the most salient and important marketing strategies. Existing studies mainly consider either constant price (Branco et al., 2012) or price that depends on the current valuation of the consumers (Ning, 2021). In recent years, the increasingly popular privacy regulations such as GDPR and CCPA have made it harder for firms to track individuals in real time. Even if a firm can track consumers' browsing behavior, it is unclear from its perspective how consumers will interpret the information they see. For example, Tesla may be able to observe that a consumer clicks on an image of the interior design of the car but may not know whether the consumer likes the large screen on Tesla or the traditional dashboard. This calls into question whether the firm can track the consumer's belief evolution process when the consumer is searching for information. Though empirical papers have estimated consumer valuation with consumer search (Kim et al., 2010; Seiler, 2013; Koulayev, 2014; Bronnenberg et al., 2016; Kim et al., 2017), they also use the choice data for the estimation. In order to adjust the price during consumer search, firms cannot rely on the eventual choice decision. Furthermore, the empirical literature estimates a single value for a product per consumer. However, the consumer's valuation evolves during gradual learning. There is not enough variation to estimate such dynamic valuation evolution.

Given the difficulty mentioned above, one may think that firms can only offer a constant price when they cannot track consumers. The last hope for firms is one tool they are equipped with that can never be banned by regulations - time. The major innovation of our paper is to allow the price to be a function of time rather than the consumer's current valuation of the product. In other words, we explore non-stationary pricing strategies and ask two questions. First, is constant price always optimal for the firm when it cannot track the consumer's belief evolution process? Second, what should the firm do if the constant price is not optimal?

We find that constant price is not always optimal for the firm. It can benefit from using non-stationary pricing strategies. We prove that a consumer can do almost as well by approximating any bounded price by linear price if she is sufficiently myopic, which can be a building block for future research to simplify the strategy space of non-Markov problems. Given this result, by assuming that the consumer is sufficiently myopic and the price is linear and varies slowly, we show that, when the search cost is zero, the optimal price is arbitrarily close to a constant price if the firm is perfectly patient. In contrast, the slope of the optimal price is bounded from zero if the firm discounts the future. When there is friction in search (positive search costs), the optimal price is non-stationary, even if the firm is perfectly patient. In particular, the firm always increases the price over time if the information is too noisy or the search cost is too high. In other cases where consumers have a stronger incentive to search, the firm charges an increasing price for consumers with high or low initial valuation, whereas charges a decreasing price for medium–value consumers.

Our contribution is twofold. On the one hand, our paper provides new managerial insights for the firm by considering non-stationary pricing. The primary goal of marketing is to reduce the cost and increase the return. Using time as the information source to guide pricing decisions is essentially free. Firms do not need to invest heavily in the tracking technology. Hence, all the increased revenue due to non-stationary pricing becomes profit. It is also immune to privacy regulations. Apple's iPhone privacy upgrades cost publishers like Facebook, YouTube, Twitter, and Snap nearly 10 billion in ad revenue in 2021 alone because the increased privacy restriction limited advertisers' ability to target consumers.[1] Privacy regulations can prevent firms from tracking consumers' demographic information, browsing behavior, and other characteristics, but cannot ban the time, which everyone has access to. Extant research mainly focuses on the economic impact of privacy regulations (Goldfarb and Tucker, 2011; Conitzer et al., 2012; Campbell et al., 2015; Athey et al., 2017; Goldberg et al., 2019; Montes et al., 2019; Choi et al., 2020, 2023; Rafieian and Yoganarasimhan, 2021; Choi et al., 2022; Baik and Larson, 2023; Ke and Sudhir, 2023; Ning et al., 2023; Yao, 2023b). We contribute to this stream of literature by studying what firms can do to respond to such privacy regulations. Not much attention has been paid to this direction. A notable exception is Bondi et al. (2023), where they study in a different context how media firms can use content design to aid their inference of consumer type. The underlying mechanism in that paper is consumer self-selection, whereas our mechanism relies on consumers' forward-looking behavior.

On the other hand, our non-stationary framework and solution method contribute theoretically to optimal control. The vast majority of papers in marketing and economics restrict attention to Markov strategies. The most common reason is tractability rather than managerial justifications. Therefore, this restriction may not be without loss of generality and may cost firms "free dollars" as shown in this paper. Marinovic et al. (2018) contrasts Markov equilibria and non-Markov equilibria in a reputation model. The comparison emphasizes that restricting attention to Markov equilibria may lead to qualitatively different and unre-

[1] Source: https://www.businessinsider.com/apple-iphone-privacy-facebook-youtube-twitter-snap-lose-10-billion-2021-11.

alistic predictions, which highlights the importance of considering non-Markov strategies. To the best of our knowledge, no paper has studied the non-stationary pricing problem under consumer gradual learning. We view this paper as an important first step in understanding firms' non-Markov interventions in the presence of consumer search.

2 The Model

Our model builds on the seminal paper on consumer search, Branco et al. (2012). A firm offers a product with a marginal cost of g and chooses the price. A consumer decides whether to purchase it or not. The consumer's initial valuation is v_0. Before making a decision, she can gradually learn about various product attributes to update her belief about the product's value.

The total utility the consumer gets from consuming the product is the sum of the value of M attributes the product has. Before searching, the consumer's initial valuation for the product, v_0, is her expected utility. The initial valuation represents the consumer's knowledge about the product based on past experiences, word of mouth, or advertising. The consumer can incur a search cost c to learn more about one of the product attributes. After learning about each attribute, the consumer updates the valuation of the product by incorporating the difference between the realized utility of the searched attribute and the expected utility. Denote this difference for attribute i by x_i. Then, the consumer's valuation after searching for m attributes is $v_m := v_0 + \sum_{i=1}^{m} x_i$. Suppose that the value of the difference is binary, $x_i = \pm z$ with equal probability. When there are infinitely many attributes, each with a very small weight in value, v_t becomes a Brownian motion.

$$dv_t = \sigma dW_t,$$

where $\sigma = z/\sqrt{dt}$, the flow search cost is cdt per dt time, and $\{W_t\}_{t\in\mathbb{R}_{\geq 0}}$ is the standard Brownian motion adapted to some filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in\mathbb{R}_{\geq 0}}, \mathbb{P})$. Figure 1 illustrates the evolution of the consumer's valuation as the consumer checks more and more attributes.

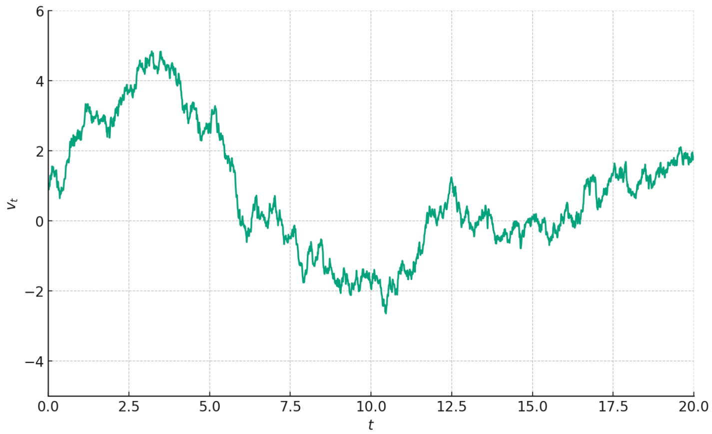

Figure 1: A sample path of the consumer's valuation evolving processes during search

Previous works that study the firm/seller's marketing strategy in the presence of consumer gradual learning either assume that the firm perfectly observes the consumer's valuation evolution processes (v_t is common knowledge between the consumer and the firm) and can condition the price on the consumer's current valuation (Ning, 2021), or that the firm charges a constant price over time (Branco et al., 2012). Suppose we view the consumer's valuation as the state variable, as is the standard and natural way of defining the state variable in the literature. The firm's strategy in the first scenario is allowed to be a function of the state variable v_t. In this case, the firm's problem is to choose the optimal *Markov strategy*. This setup does not fit all real-world examples. Due to the increasingly common privacy regulations such as GDPR and CCPA, it is harder for firms to track consumers' search behavior online. Even if some firms can track consumer's search path, it is hard to know whether or not consumers like the information they find. Moreover, in many offline settings, individual-level tracking is not feasible.

When the firm cannot observe the consumer's valuation evolution and therefore cannot choose the price based on v_t, is offering a constant price the best it can do? The major innovation of this paper is to allow the price to be a function of time t rather than the state variable v_t. In other words, the firm can strategically explore *non-stationary* pricing strategies. Formally, the firm can commit to a pricing scheme $\{p_t\}_{t\geq 0} \in \mathcal{P}$, where $\mathcal{P}$ is a subset of continuously differentiable functions on $[0,\infty)$, $\mathcal{P} \subset C^1[0,\infty)$. This pricing strategy is a *non-Markov strategy* because p_t depends on history (time t) other than the current state v_t. It is widely known in optimal control that it is much harder to characterize *non-Markov strategies* than *Markov strategies*.

The consumer search strategy consists of choosing an appropriate stopping time and we denote by $\mathcal{T}$ the set of all stopping times adapted to $\{\mathscr{F}_t\}_{t\in\mathbb{R}_{\geq 0}}$. We formalize the setup as a game with two players, a consumer and a firm, playing in the following sequence:

1. At $t = 0$, the firm knows the consumer's initial valuation is given by some distribution ϕ over $\mathbb{R}$. It commits to a pricing strategy $p \in \mathcal{P} \subset C^1[0,\infty)$.
2. The firm announces the strategy p.
3. At $t > 0$, the consumer decides whether to purchase the product, exit, or search for more information.
4. The game ends at any time $t \geq 0$ when the consumer makes the purchase or exits.

Importantly, when the consumer decides whether to purchase the product, exit, or keep searching at any given time, she takes into account both the current price and the future price trajectory. The only knowledge the seller has about the buyer is their initial valuation, v_0, which may be derived from a survey conducted over a large population. For any $p \in \mathcal{P}$ and $\tau \in \mathcal{T}$, we define [2]

$$\mathcal{V}^B(t,x;\tau,p) := \mathbb{E}\left[e^{-r(\tau-t)}\max\{v_\tau - p_\tau, 0\} - \int_t^\tau ce^{-r(s-t)}ds \mid v_t = x\right]$$

[2] For simplicity, we use p to denote $\{p_t\}_{t\geq 0}$ whenever this does not cause confusion.

and

$$\mathcal{V}^S(x;\tau,p) := \mathbb{E}\left[e^{-m\tau}(p_\tau - g)\cdot 1_{v_\tau > p_\tau} \mid v_0 = x\right]$$

Motivated by the almost optimality of approximating any bounded price by a linear price from the consumer's perspective, which we will prove in Lemma 2, we use the following equilibrium concept.

Definition 1. *An ε–Subgame perfect Nash's equilibrium (ε–SPNE) consists of:*

$$(\{\tau^*[p] \in \mathcal{T}\}_{p\in\mathcal{P}}, p^* \in \mathcal{P})$$

such that: for all $p \in \mathcal{P}$,

$$\begin{aligned} \mathcal{V}^B(t,x;\tau^*[p],p) &\geq \mathcal{V}^B(t,x;\tau,p) - \varepsilon, \quad \forall \tau \in \mathcal{T}, \\ \textit{and} \;\; \mathcal{V}^S(x;\tau^*[p^*],p^*) &\geq \mathcal{V}^S(x;\tau^*[p],p) - \varepsilon, \quad \forall p \in \mathcal{P}. \end{aligned}$$

The buyer's value function given the seller's pricing strategy p is:

$$V^B(t,x;p) := \sup_{\tau\in\mathcal{T}} \mathcal{V}^B(t,x;\tau,p). \tag{1}$$

When there is no ambiguity, we will compactly write $V^B(t,x) = V^B(t,x;p)$. Analogously, we define the seller's value function:

$$V^S(x) := \sup_{p\in\mathcal{P}} \mathcal{V}^S(x;\tau^*[p],p)$$

3 Consumer's Strategy

The consumer faces an optimal stopping problem. She needs to determine the purchasing and quitting boundaries at any time. The value function of the consumer satisfies the following:

$$\frac{\sigma^2}{2}\partial_x^2 V^B + \partial_t V^B - rV^B - c = 0 \tag{2}$$

Because the consumer needs to make a decision by time T, she will purchase the product at that time if and only if her expected valuation at that time is positive. So, $V^B(T,x) = \max\{x - p_T, 0\}$. The value function also needs to satisfy the value matching and smooth pasting conditions at $t \in (0,T)$:

$$V^B(t,\bar{V}_t) = \bar{V}_t - p_t, \quad \partial_x V^B(t,\bar{V}_t) = 1,$$
$$V^B(t,\underline{V}_t) = 0, \quad \partial_x V^B(t,\underline{V}_t) = 0.$$

When the price is non-stationary, the consumer's purchasing and quitting boundaries are also time-contingent. This time-varying property makes her optimal stopping problem challenging even if we fix a pricing scheme. To illustrate the impact of non-stationary pricing on the consumer's problem, we show a benchmark with constant price and no deadline, which has been characterized in Branco et al. (2012).

3.1 A Constant Price Benchmark

If the price is constant, $p_t = p$, and there is no deadline, $T = \infty$, then intuitively, the consumer's strategy and value function do not depend on time, $V^B(t,x) = V^B(x), \bar{V}_t = \bar{V}$, and $\underline{V}_t = \underline{V}$. The value function of the consumer satisfies:

$$\frac{\sigma^2}{2}\partial_x^2 V^B - rV^B - c = 0$$

Instead of having a value matching and a smooth pasting condition at any time t, we now only have a value matching condition and a smooth pasting condition for the entire problem:

$$V^B(\bar{V}) = \bar{V} - p, \quad \partial_x V^B(t, \bar{V}) = 1,$$
$$V^B(t, \underline{V}) = 0, \quad \partial_x V^B(t, \underline{V}) = 0.$$

The stationary structure leads to closed-form solutions.

$$\begin{aligned}
&V^B(x) = \frac{c}{r}\left[\cosh \frac{\sqrt{2r}}{\sigma}(x - \underline{V} - p) - 1\right], \\
&\bar{V} = p + \sqrt{\frac{c^2}{r^2} + \frac{\sigma^2}{2r}} - \frac{c}{r}, \\
&\underline{V} = p + \left(\sqrt{\frac{c^2}{r^2} + \frac{\sigma^2}{2r}} - \frac{c}{r}\right) - \frac{\sigma}{\sqrt{2r}} \log\left(\sqrt{\frac{r\sigma^2}{2c^2}} + \sqrt{1 + \frac{r\sigma^2}{2c^2}}\right).
\end{aligned}$$

Comparing this benchmark and our problem, we can see that stationarity simplifies the problem significantly. In the benchmark model, the consumer's entire optimal stopping strategy can be summarized by **two unknowns**: the purchasing threshold $\bar{V}$ and the quitting threshold $\underline{V}$, which does not depend on time t. The consumer will purchase the product at any time during the search if her valuation reaches the purchasing threshold and will quit searching at any time if her valuation reaches the quitting threshold. In contrast, the consumer's entire optimal stopping strategy consists of **an infinite number of unknowns**. Knowing that the price changes over time, the consumer's purchasing and quitting thresholds also evolve. She has different purchasing and quitting thresholds at different times. So, instead of pinning down a one-dimensional purchasing/quitting threshold, we need to determine a two-dimensional purchasing/quitting boundary. These time-dependent thresholds significantly complicate our problem.

3.2 Solution to Our Problem

To solve the consumer's problem, we first need to establish a result for the existence and uniqueness of the solution to (2). In stochastic control problems, the strong solution (solution in the usual sense) to the PDE does not always exist. The standard approach is to work with a relaxed notion, the weak solution (Bressan, 2012; Evans, 2022).

Definition 2. *(Weak derivative) Let $G \subset \mathbb{R}^n$ be open. For any $f \in L^p(G)$ and $\alpha \in \mathbb{Z}^n_{\geq 0}$ the α-weak derivative of f (if exists) is the function $v \in L^p(G)$ such that for all $\phi \in C_c^\infty(G)$ we have*

$$\int_G f(x)\partial^\alpha \phi(x)dx = (-1)^{|\alpha|} \int_G g(x)\phi(x)dx,$$

where $\partial^\alpha := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$. If such v exists, it is a.e. unique, and we write $\partial^\alpha f := v$. We denote by $W^{q,p}(G) \subset L^p(G)$ the set of all f such that all α-weak derivatives with $|\alpha| \leq q$ exists.

(Weak solution) A weak solution to the PDE initial boundary value problem (2) on $\Omega := \{(t,x) \in [0,T] \times \mathbb{R} | \underline{V}_t < x < \bar{V}_t\}$ is any $V \in L^2(0,T; W^{2,2}_{loc}(\mathbb{R}))$ with a weak derivative $\partial_t V \in L^2(0,T; W^{1,2}_{loc}(\mathbb{R})^*)$ such that

$$\int_\Omega \left(\frac{\sigma^2}{2} \partial_x^2 V(t,x) + \partial_t V(t,x) - rV(t,x) - c \right) \phi(t,x)dx = 0$$

for all $\phi \in C_c^\infty(\Omega)$.

Now, we can state the existence and uniqueness result.

Lemma 1. *Given a pricing policy $p : [0,\infty) \to \mathbb{R}$ that is continuously differentiable in time t and constant for $t \geq T$, we have the following.*

1. *(Existence) The value function V^B is monotonically increasing, convex in x, and gives a weak solution to the free boundary problem (2) with $V^B \in L^\infty(0,T; W^{2,\infty}_{loc}(\mathbb{R}))$, and weak derivative $\partial_t V^B \in L^\infty(0,T; L^\infty_{loc}(\mathbb{R}))$.*

2. (Uniqueness) If V is a weak solution to the free boundary problem (2) with $V \in L^\infty(0,T;W^{2,\infty}_{loc}(\mathbb{R}))$ and weak derivative $\partial_t V \in L^\infty(0,T;L^\infty_{loc}(\mathbb{R}))$ then $V = V^B|_\Omega$.

The lemma above implies that we can work with (2) to solve for the consumer's value function. The following result shows that a myopic enough consumer can do almost as well if they approximate the price by linear price.

Lemma 2. *(Almost optimality of linear price approximation) Let $p_t = p + h(t)$ be a pricing policy, which is bounded: $|p_t| < C$ for all $t \in \mathbb{R}_{\geq 0}$, for some constant $C > 0$, and $h(t)$ is some differentiable function in t. Let $l_t := p + h'(0)t$ be the linear approximation pricing policy. By assumption, for any $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$|p_t - l_t| < \delta\varepsilon/2, \quad \forall t \in [0, \delta).$$

Let's denote by $\tau^*[p_t] \in \mathcal{T}$ the optimal stopping time given the policy p_t of problem (1). If r is sufficiently large, for instance, such that:

$$e^{-r\delta} < \frac{(1-\delta)\varepsilon/2}{C + \bar{V} + \max\{C, p + |h'(0)|\delta\} + c/r} \quad \text{and} \quad r > \frac{|h'(0)|}{|h'(0)|\delta + p},$$

then $\tau^*[l_t]$ is a consumer's ε–optimal stopping time:

$$\mathbb{E}\left[\mathcal{V}^B(t, v; \tau^*[l_t]; p_t)|v_t = x\right] \geq V^B(t, x) - \varepsilon.$$

Lemma 2 is an important result. It implies that we can restrict our attention to linear price for the consumer's optimal stopping problem if the consumer has a high enough discount factor and is only concerned with almost optimal strategies. Because the space of linear pricing is much smaller than the general pricing space, this can greatly simplify the problem.

Simplifying Assumptions

Solving the full problem is beyond the scope of this research. The lemma above motivates us to make the following simplifying assumptions for analytic traceability.

Assumption 1. *The consumer is sufficiently myopic (the discount factor satisfies the condition in Lemma 2)and only concerned with ε–optimal strategy.*

Assumption 2. *The price is linear and varies slowly, $p_t = p + \sqrt{\varepsilon}Kt$, where $K \in [-1, 1]$.*

Discussion of the Simplifying Assumptions

We impose the two simplifying assumptions to obtain analytic results. They are by no means minimal. However, they do not substantially limit our contribution. We provide several justifications in this section.

The first assumption ensures that any given bounded price function can be approximated by a linear function based on Lemma 2. We do not need the consumer to be completely myopic (discount factor equals 1). The consumer is still forward-looking and rationally anticipates the price evolution in the future. So, the consumer still has time-varying purchasing and quitting boundaries, and we still capture the consumer's equilibrium response to the firm's non-stationary pricing. This assumption means that the consumer cares about the near future more than the far future and simplifies the determination of the optimal stopping time. Since ε can be arbitrarily small, focusing on the consumer's ε–optimal strategy rather than the optimal strategy also does not lose much.

Considering linear pricing from the consumer's perspective can be justified by Lemma 2. However, considering linear pricing from the firm's perspective is not without loss of generality. Nevertheless, linear pricing suffices to answer our first main question fully. By showing that the firm can do better by increasing or decreasing the price linearly, we know that constant price is not generally optimal for the firm when it cannot track the consumer's belief evolution process. In addition, the consumer has time-varying purchasing and quitting boundaries under linear pricing. So, we still capture the consumer's equilibrium response to the firm's non-stationary pricing. Moreover, the most important qualitative property of the firm's optimal pricing strategy is the direction of price evolution. It is managerially relevant to firms and can help guide their pricing decision. Linear pricing is more restricted than the general pricing space but is still general enough for us to see whether the price should stay constant, increase, or decrease over time. In addition, our non-stationary linear pricing strategy can already improve firms' profits.

To summarize, our assumption still captures the main driving force of non-stationary pricing and the consumer's rational response. To the best of our knowledge, no paper has studied the non-stationary pricing problem under consumer gradual learning, even under our simplifying assumptions. Hence, we view our paper as an important first step in pushing forward our understanding of this problem.

Solution

If the consumer simply follows the optimal learning strategy under a constant price by shifting boundaries up and down according to the varying price, then we would have the following value function:

$$V_0^B(x-\sqrt{\varepsilon}Kt)=\frac{c}{r}\left[\cosh\frac{\sqrt{2r}}{\sigma}(x-\underline{V}-p-\sqrt{\varepsilon}Kt)-1\right]. \tag{3}$$

Because the consumer also considers future price evolution in decision-making, we need a correction term for the consumer's value function. The next result shows that the solution of the PDE initial-boundary value problem (2) will not change much given any small changes in the input pricing function p.

Lemma 3. *Let $p, q : [0,\infty) \to \mathbb{R}$ be two differentiable pricing strategies such that $p_t = p_T$, $q_t = q_T$ for all $t \geq T$, and let $V(.,.;p)$ and $V(.,.;q)$ be the their corresponding solution to the PDE IVBP (2), then $|V(t,x;p) - V(t,x;q)| \leq \max_{t\in[t,T]} e^{-r(s-t)}|p_t - q_t|$ for all $(t,x) \in$*

$[0, T] \times \mathbb{R}$.

This lemma implies that we can add a $\sqrt{\varepsilon}$-order correction to the consumer's value function and a $\sqrt{\varepsilon}$-order correction to the purchasing and quitting boundaries. By collecting the $\sqrt{\varepsilon}$-order terms in the PDE of V^B and the boundary conditions, we can obtain a system of equations to solve for the value function and boundary conditions.

Proposition 1. *There is an ε-optimal consumer learning strategy that satisfies*

$$\begin{aligned}
&\textit{The value function } V^B(t,x) = V_0^B(x - \sqrt{\varepsilon}Kt) + \sqrt{\varepsilon}V_1^B(t,x) + O(\varepsilon),\\
&\textit{the purchasing boundary } \bar{V}_t = \left(p + \sqrt{\varepsilon}Kt + \bar{V}\right) + \sqrt{\varepsilon}K\bar{S} + O(\varepsilon),\\
&\textit{and the quitting boundary } \underline{V}_t = \left(p + \sqrt{\varepsilon}Kt + \underline{V}\right) + \sqrt{\varepsilon}K\underline{S} + O(\varepsilon),\\
&\textit{where } V_0^B(x - \sqrt{\varepsilon}Kt) = \frac{c}{r}\left[\cosh\frac{\sqrt{2r}}{\sigma}(x - \underline{V} - p - \sqrt{\varepsilon}Kt) - 1\right],\\
&V_1^B(t,x) = A_1(\sqrt{\varepsilon}t)e^{+\frac{\sqrt{2r}}{\sigma}x} + A_2(\sqrt{\varepsilon}t)e^{-\frac{\sqrt{2r}}{\sigma}x} + \frac{cK}{\sigma^2 r}x\cosh\frac{\sqrt{2r}}{\sigma}(x - \underline{V} - p),
\end{aligned}$$

$A_1, A_2, \underline{S}$, and $\bar{S}$ are unknowns determined by the boundary conditions.[3] *Moreover, $\underline{S} > 0$ and $\bar{S} < 0$.*

Compared to the constant price benchmark, an increasing pricing scheme ($K > 0$) with the same initial price has two impacts on the purchasing threshold. On the one hand, the benefit of learning becomes lower because the consumer needs to pay more in the future if she receives positive information and likes the product more. Rationally anticipating this, the consumer has a lower incentive to search and is more inclined to purchase now, reducing the purchasing threshold (captured by the negative $\sqrt{\varepsilon}K\bar{S}$ term). On the other hand, a higher price makes the consumer less willing to purchase, raising the purchasing threshold (captured by the positive $\sqrt{\varepsilon}Kt$ term). Since the first effect remains stable while the second effect increases over time, the purchasing threshold is lower than the benchmark threshold at

[3] The details are in the appendix.

the beginning but eventually exceeds the benchmark threshold as the price keeps increasing.

An increasing pricing scheme also has two impacts on the quitting threshold. Both a lower benefit of searching and a higher price make it more likely for the consumer to quit. So, the quitting threshold is always higher than the benchmark threshold. We also find that the consumer searches in a narrower region (smaller $\bar{V}_t - \underline{V}_t$) if the price increases rather than staying constant because of the lower benefit of searching.

A decreasing pricing scheme ($K < 0$) has the opposite impact on the purchasing and quitting thresholds. The purchasing threshold is higher than the benchmark threshold at the beginning because the consumer has a stronger incentive to search and is less inclined to purchase immediately. It eventually falls below the benchmark threshold as the price keeps decreasing. The quitting threshold is always lower than the benchmark threshold because the benefit of both searching and purchasing is higher. Also, the consumer searches in a broader region.

4 Firm's Strategy

4.1 Firm's Expected Payoff

The firm's expected payoff is the value function V^S. The firm chooses the price scheme to maximize it. It can be computed in the following way.

First, we solve the heat equation:

$$\frac{\sigma^2}{2}\partial_v^2 U(s,v) - \partial_s U(s,v) = 0, \tag{4}$$

over $\{(t,v) \in [0,\infty) \times \mathbb{R} \mid \underline{V}_t < v < \bar{V}_t\}$ with boundary conditions:

$$U(s=t,v) = \delta(v-x), \quad U(s,\bar{V}_t) = 0, \quad U(s,\underline{V}_t) = 0.$$

The solution $U(t,v)$ is the probability density at time t of the consumer valuation being at $v_t = v$. The probability flux of consumer hitting the moving purchase boundary at time s is given by

$$-\frac{\sigma^2}{2}\partial_v U(s,\bar{V}_s) - \bar{V}_s' \cdot U(s,\bar{V}_s) = -\frac{\sigma^2}{2}\partial_v U(s,\bar{V}_s)$$

where $\bar{V}_s'$ is the time weak derivative of the purchase boundary, but the term nevertheless vanishes by construction since $U(s,\bar{V}_s) = 0$. Hence, if $x \in [\underline{V}_t, \bar{V}_t]$ then we have that

$$V^S(t,x) = -\frac{\sigma^2}{2}\int_t^\infty e^{-m(s-t)}(p_s - g)\partial_v U(s,\bar{V}_s)ds, \tag{5}$$

otherwise, we simply have $V^S(t,x) = (p_t - g)1_{x>\bar{V}_t}$, i.e. the consumer purchases immediately and the game ends. More generally, if the distribution of the consumer over $\mathbb{R}$ is known at time t to be $U(s=t,.) = \phi(.)$, then we denote the firm's value function by $V^S(t,\phi)$ and we have:

$$V^S(t,\phi) := \int_{-\infty}^{+\infty} V(t,v)\phi(v)dv + (p_t - g)\int_{\bar{V}_t}^\infty \phi(v)dv. \tag{6}$$

The second term corresponds to the probability that the consumer is outside the search region $[\underline{V}_t, \bar{V}_t]$ who would purchase immediately at time t.

4.2 Direction of Price Evolution

The most important property of the firm's optimal pricing strategy is the direction of price evolution. Whether the price should stay constant, increase, or decrease over time? This is equivalent to determining the sign of parameter K, since the price is $p_t = p + \sqrt{\varepsilon}Kt$. For regularization purposes, we impose a technical assumption that the consumer can only make the purchasing decision after time δ for some small constant $\delta > 0$.

We will discuss two cases. In the first case, the consumer has zero search costs (but still discounts the future). This leads to the sharpest results. In the second case, the consumer has a positive search cost. Using analytical analyses and numerical simulations, we find that search costs play an important role in the problem, and lead to qualitatively new insights.

Zero Search Costs

When the consumer has zero search costs, the continuation value of searching is positive, whereas the payoff from quitting is zero. Therefore, she would never quit searching without purchasing the product. Equivalently, the consumer's quitting boundary is $-\infty$. The consumer's optimal search strategy only has a single boundary $\bar{V}_t$.

If the firm is perfectly patient, it will not have a direct incentive to speed up the consumer's decision-making process. A purchase at any time gives the firm the same payoff. Hence, it does not have a strong incentive to increase the price over time to push the consumer to make an early decision. In addition, the firm will charge a sufficiently high price such that the consumer's payoff from purchasing the product is negative initially. The consumer receives a non-positive expected surplus by purchasing or exiting at the beginning. Therefore, discounting does not reduce her payoff if she delays making a decision by searching for more information. Even if the price does not decrease over time, the consumer will keep searching for information because she has nothing to lose. From the discussion, we can see that the firm has little incentive to charge non-stationary prices in this case. The following proposition shows that the optimal price is arbitrarily close to a constant price in this case. On the contrary, the firm charges non-stationary prices if it discounts the future.

Proposition 2. *Suppose the search cost is zero.*

When the firm is perfectly patient, $m = 0$, for any initial price p and slope K, it can obtain a higher profit by charging a constant price $p_t = p$ if $K < 0$, and by reducing the slope K if $K > 0$.

When the firm's discount factor is sufficiently small or sufficiently large, the slope K of the optimal price $p + \sqrt{\epsilon}Kt$ is bounded from zero.

Positive Search Costs

The previous section shows that $K \to 0$ if both the search cost and the firm's discount factor are 0. In order for the slope of the optimal price to be bounded from zero, the firm must discount the future if there is no search cost. In this section, we consider the case with a positive search cost. In the presence of a positive search cost, the continuation value of searching may be negative. Therefore, both the quitting boundary and the purchasing boundary are non-trivial. At any time t, there are two unknowns, $\bar{V}_t$ and $\underline{V}_t$, in the optimal search problem.

Since the optimal price is non-stationary with K bounded from zero in the presence of search friction, even if the firm is perfectly patient, we will focus on the no-discounting case in this section.[4] To be consistent with the assumption that linear price approximation is almost optimal and the consumer is concerned with $\epsilon-$optimal strategy, we consider K in the vicinity of 0. The main question of interest is whether the firm would benefit from slightly increasing ($K \gtrsim 0$) or decreasing ($K \lesssim 0$) the price over time, compared with the optimal static price.

Proposition 3. *Suppose the search cost is positive and the firm is perfectly patient. The firm's expected profit from a consumer whose initial valuation is x is the following.*

$$\begin{aligned} \mathcal{V}^S(x;p,K) = & \frac{p-g+(\bar{V}_0+x-2\underline{V}_0)}{1-\exp\left(+\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0-\underline{V}_0)\right)} - \frac{2(\bar{V}_0-\underline{V}_0)}{\left(1-\exp\left(+\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0-\underline{V}_0)\right)\right)^2} \\ & - \frac{(p-g+(\bar{V}_0-x))\exp\left(+\frac{2\sqrt{\varepsilon}K}{\sigma^2}(x-\underline{V}_0)\right)}{1-\exp\left(+\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0-\underline{V}_0)\right)} + \frac{2(\bar{V}_0-\underline{V}_0)\exp\left(+\frac{2\sqrt{\varepsilon}K}{\sigma^2}(x-\underline{V}_0)\right)}{\left(1-\exp\left(+\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0-\underline{V}_0)\right)\right)^2} \end{aligned} \quad (7)$$

if $x \in (\bar{V}_0, \underline{V}_0)$ and $K \neq 0$, and $\mathcal{V}^S(x;p,K) = (p-g)\left(\frac{x-\underline{V}_0}{\bar{V}_0-\underline{V}_0}\right)$ if $x \in (\bar{V}_0, \underline{V}_0)$ and $K=0$. $\mathcal{V}^S(x;p,K)=0$ if $x \leq \underline{V}_0$. $\mathcal{V}^S(x;p,K)=p-g$ if $x \geq \bar{V}_0$.

[4] As we can see from the zero search cost case, the firm is more inclined to charge non-stationary prices if it discounts the future.

The firm's value function in the above proposition allows us to characterize under what conditions the firm wants to increase the price over time and under what conditions the firm wants to decrease the price over time. Define $\hat{p}$ as the optimal static price. Define $q := \frac{x-\underline{V}-g}{2(\bar{V}-\underline{V})}$ as the initial relative position of the consumer between the purchasing and quitting boundaries under $\hat{p}$. It turns out that the condition depends only on σ^2/r, c/r and q. Therefore, we summarize the result with respect to these parameters in Figure 2.[5] We divide the figure into four regions.

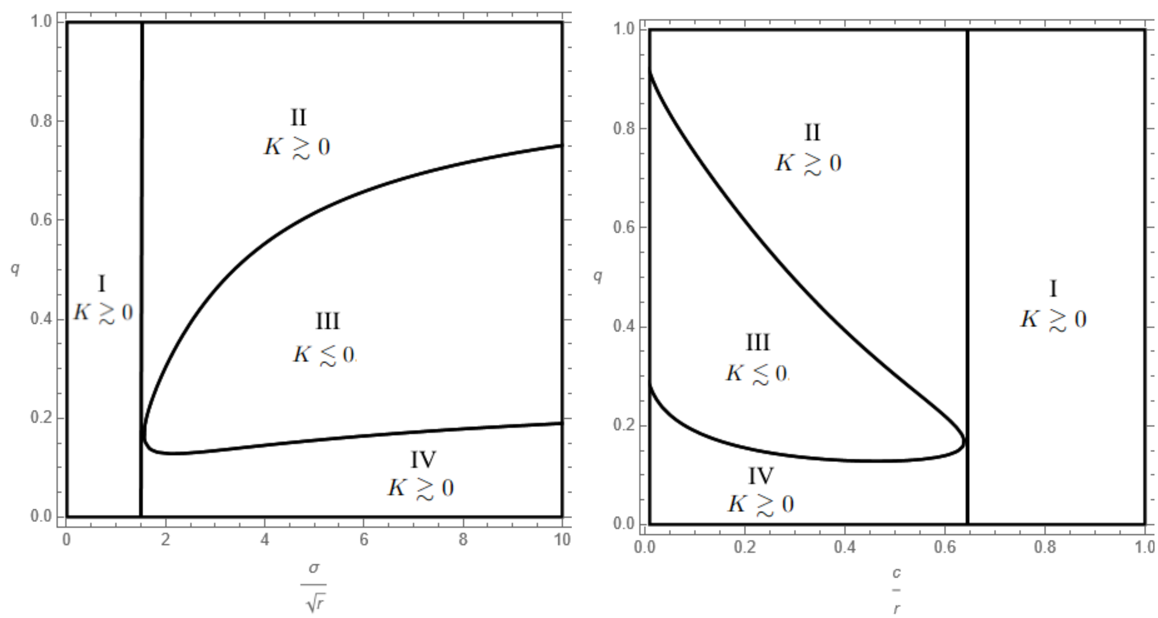

Figure 2: Direction of price evolution

I (Low incentive to search) When the information is too noisy (low σ^2), the search cost is too high (high c), or the consumer values little about the future (high r), the consumer has a low incentive to search for information. In such cases, the firm needs to give the consumer a high surplus to encourage her to search, which hurts its profit. So, it becomes more attractive for the firm to convince the consumer to purchase the product at the beginning, based on the initial valuation and the expected price trajectory. For any given initial price, by charging an increasing price over time, the firm lowers the purchasing

[5] The derivation of the pricing direction is in the appendix.

threshold at the beginning by making it more desirable for the consumer to make an immediate decision. Compared with the stationary pricing strategy of charging a lower constant price, this non-stationary pricing strategy moves the purchasing threshold in the same direction (downwards) without sacrificing the profit conditional on purchase.

II (High–value consumer) When the consumer has a high initial valuation, she is too valuable to lose from the firm's perspective. Therefore, the firm wants to increase the purchasing probability in this case. Moreover, a high–value consumer can earn a positive payoff from purchasing immediately, which decreases over time because of discounting. Thus, the firm also wants the consumer to purchase quickly. An increasing pricing strategy reduces the benefits of searching and encourages the consumer to make a purchase quickly and more likely.

III (Medium–value consumer) When the consumer has a moderate interest in the product, an increase in price does not suffice to convince the consumer to purchase quickly without learning much additional information. Instead, it reduces the benefit of searching because the consumer knows that she has to pay a higher price if she learns positive things. Therefore, an increasing price will lead to a quick exit rather than a quick purchase.

The firm can benefit from reducing the price gradually in this case. A decreasing price helps the firm keep the consumer engaged in the search process even if she receives some negative information early on. It increases the purchasing likelihood. Because of the moderate initial valuation, the firm can still obtain a decent profit at a lower price. This pricing strategy protects the firm from missing potentially valuable consumers.

IV (Low–value consumer) By charging a decreasing price over time, the firm can keep the consumer engaged in the search process even if she receives some negative information early on. However, it is not worth it for the firm to reduce the price over time for two reasons. First, the profit from an immediate purchase is already low when the consumer has a low initial valuation. The firm will obtain an even lower profit from purchasing if the consumer searches for a while and eventually buys at a lower price. Second, the buyer needs to accumulate a lot of positive information before making a purchase due to the low initial valuation. The purchasing probability will still be low even if the price slightly reduces over time, and cannot offset the cost of a lower profit per purchase.

By implementing an increasing pricing strategy instead, the firm quickly filters out many consumers. On the one hand, the loss from not converting these people is limited due to the low profit per purchase and the low purchasing probability. On the other hand, the benefits of charging a higher price to the remaining consumers are high. Any consumers not quitting despite the increasing price must have learned positive information and are more valuable to the firm.

5 Conclusion

Consumers gather information gradually to help themselves make the purchasing decision. The increasingly popular privacy regulations have made it harder for firms to track individuals in real-time. Even if a firm can track consumers' browsing behavior, it is difficult for the firm to infer whether consumers like the information they see. Without the ability to track consumer's valuation evolution about the product, it seems that firms can only offer a constant price. The major innovation of our paper is to allow the price to be a function of time rather than the consumer's current valuation of the product. We find that constant price is not always optimal for the firm. It can benefit from using non-stationary pricing strategies. By assuming that the consumer is sufficiently myopic and uses $\varepsilon-$ optimal strategies, and that the firm uses linear prices that vary slowly, we show that, when the search cost is zero, the optimal price is arbitrarily close to a constant price if the firm is perfectly patient, whereas the slope of the optimal price is bounded from zero if the firm discounts the future. When the search cost is positive, the optimal price is non-stationary even if the firm is perfectly patient. In particular, the firm always increases the price over time if the information is too noisy or the search cost is too high. In other cases where consumers have a stronger incentive to search, the firm charges an increasing price for consumers with high or low initial valuation, whereas charges a decreasing price for medium–value consumers.

This paper makes two main contributions. On the one hand, it provides new managerial insights for the firm by considering non-stationary pricing. The major goal of marketing is to reduce the cost and increase the return. Using time as the information source to guide pricing decisions is essentially free. Firms do not need to invest heavily in the tracking technology. Hence, all the increased revenue due to non-stationary pricing becomes profit. It is also immune to privacy regulations, which can prevent firms from tracking consumers' demographic information, browsing behavior, and other characteristics, but cannot ban the time, which everyone has access to. Extant research mainly focuses on the economic impact of privacy regulations. We contribute to this stream of literature by studying what firms can do to respond to such privacy regulations.

On the other hand, our non-stationary pricing framework and solution method contribute theoretically to optimal control. The vast majority of papers in marketing and economics restrict attention to Markov strategies. The most common reason is tractability rather than managerial justifications. Therefore, this restriction may not be without loss of generality and may cost firms "free dollars", as shown in this paper. We view this paper as the first step in understanding firms' non-Markov interventions in the presence of consumer search.

Appendix

Proof of Lemma 1.

Part 1: For any fixed t we argue $V^B(t,.)$ is monotonically increasing and convex. At x and for an arbitrary small $\varepsilon > 0$ we can find $\tau_{t,x,\varepsilon} \in \mathcal{T}$ such that $\mathbb{E}\left[\mathcal{V}^B(t,v;\tau_{t,x,\varepsilon},p)|v_t = x\right] \geq V^B(t,x) - \varepsilon$. Fixing $\tau_{t,x,\varepsilon}$, then we note that $\mathbb{E}\left[\mathcal{V}^B(t,v;\tau_{t,x,\varepsilon},p)|v_t = x+\Delta\right] \geq V^B(t,x) - \varepsilon + m_{t,x,\varepsilon}\Delta$ where $m_{t,x,\varepsilon} := \mathbb{E}\left[e^{-r(\tau_{t,x,\varepsilon}-t)}|v_t = x\right] \geq 0$. By the non–optimality of $\tau_{t,x,\varepsilon}$ at $x+\Delta$ we have that $V^B(t,x+\Delta) \geq V^B(t,x) - \varepsilon + m_{x,\varepsilon}\Delta$. Since ε is arbitrary small, we conclude that $V^B(t,x)$ is convex in x, and since $m_{t,x,\varepsilon} \geq 0$ we have that $V^B(t,x)$ is monotonically increasing in x. It then follows that $V^B(t,.)$ is continuous and differentiable a.e. with $|\partial_x V^B| \leq 1$ (hence Lipschitz continuous in x) and with weak derivative $\partial_x V^B(t,.) \in L^\infty_{loc}(\mathbb{R})$ (see §5.8.2 of Evans (2022)).

For any fixed x we show that $V^B(.,x)$ is differentiable a.e.. Given $t,t' \in [0,T]$, by appropriately relabeling $t > t'$ or $t < t'$, we can assume that $V^B(t',x) \leq V^B(t,x)$. Let $\tau_{t,x,\varepsilon} \in \mathcal{T}$ be defined as above, the expected payoff from keep using $\tau_{t,x,\varepsilon}$ at t' (which is the same as shifting p back by $\Delta := t'-t$, keeping t and the stopping time fixed):

$$\mathbb{E}\mathcal{V}^B(\Delta) := \mathbb{E}\left[\mathcal{V}^B(t,v;\tau_{t,x,\varepsilon},p_{\cdot-\Delta})|v_t = x\right]$$

is continuously differentiable in Δ. Clearly, $|(\mathbb{E}V^B)'(\Delta)| \leq K := \max_{s\in[0,T]}|p'_s|$ and so $V^B(t',x) \geq V^B(t,x) - \varepsilon - K|\delta t|$. Since $\varepsilon > 0$ is arbitrary small, we have that $|V^B(t',x) - V^B(t,x)| \leq K|t'-t|$. This means $V^B(.,x)$ is Lipschitz continuous and hence differentiable a.e. with bounded weak derivative $\partial_t V^B(.,.) \in L^\infty([0,T]\times\mathbb{R})$, $|\partial_t V^B| \leq K$ (see §5.8.2 of Evans (2022)).

It follows from the argument above that for any fixed t, the set $U_t := \{x \in \mathbb{R}|V^B(t,x) > \max\{x-p_t,0\}\}$ is an open interval which can be written in the form $(\underline{V}_t,\bar{V}_t)$ for some continuous functions $\bar{V}_t,\underline{V}_t : [0,T] \to \mathbb{R}, \bar{V}_t > \underline{V}_t$.

We show that $\partial_x V^B(t,.)$ is continuous. We already knew that $\partial_x V^B(t,.)$ is monotonically increasing, therefore for any $x^* \in \mathbb{R}$ we have

$$\partial_x V^B(t, x^{*+}) := \lim_{x' \to x^{*+}} \partial_x V^B(t, x') \geq \lim_{x' \to x^{*-}} \partial_x V^B(t, x') =: \partial_x V^B(t, x^{*-}),$$

and at $(t-\delta t, x^*)$ the consumer is forced to continues learning for δt, before continue optimally under τ^*, then the expected payoff is:

$$\begin{aligned}
&e^{-r\delta t}\mathbb{E}\left[V^B(t, v_t)|v_t \geq x^*\right]\mathbb{P}\left[v_t \geq x^*|v_{t-\delta t} = x^*\right] \\
&\qquad + e^{-r\delta t}\mathbb{E}\left[V^B(t, v_t)|v_t < x^*\right]\mathbb{P}\left[v_t < x^*|v_{t-\delta t} = x^*\right] - \frac{c}{r}\left(1 - e^{-r\delta t}\right) \\
&\qquad\qquad \geq \frac{1}{2}\left[V^B(t-\delta t, x^*) + \partial_x V^B(t, x^{*+}) \cdot \frac{\sigma\sqrt{2\delta t}}{\sqrt{\pi}}\right] \\
&\qquad\qquad + \frac{1}{2}\left[V^B(t-\delta t, x^*) - \partial_x V^B(t, x^{*-}) \cdot \frac{\sigma\sqrt{2\delta t}}{\sqrt{\pi}}\right] + O(\delta t) \\
&\qquad\qquad = V^B(t-\delta t, x^*) + \left(\partial_x V^B(t, x^{*+}) - \partial_x V^B(t, x^{*-})\right) \cdot \frac{\sigma\sqrt{\delta t}}{\sqrt{2\pi}} + O(\delta t).
\end{aligned}$$

The inequality followed from the convexity and $V^B(t-\delta t, x^*) = V^B(t, x^*) + O(\delta t)$ by the continuity in t. We can see that the payoff is $> V^B(t-\delta t, x^*)$ for sufficiently small δt unless $\partial_x V^B(t, x^{*+}) = \partial_x V^B(t, x^{*-})$. Therefore, $V^B(t, .)$ is in fact continuously differentiable for each fixed t.

Now, consider any $(t, x) \in \Omega$. From the principle of optimality, we have for any $t' \geq t$:

$$V^B(t, x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[e^{-r(\tau \wedge t' - t)} V^B(\tau \wedge t', v_{\tau \wedge t'}) - \int_t^{\tau \wedge t'} c e^{-r(s-t)} ds | v_t = x\right].$$

For any $\varepsilon' > 0$ and $\delta t > 0$ it is possible to find $\tau_{t,x,\varepsilon'} \in \mathcal{T}$ and because V^B is Lipschitz continuous it is also possible to find $\varepsilon > 0$ independent of (t, x) such that

$$\begin{aligned}
&\mathbb{E}\left[e^{-r\tau_{t,x,\varepsilon'} \wedge \delta t} V^B(t' + \tau_{t,x,\varepsilon'} \wedge \delta t, x' + \sigma W_{\tau_{t,x,\varepsilon'} \wedge \delta t}) - \frac{c}{r}\left(1 - e^{-r\tau_{t,x,\varepsilon'} \wedge \delta t}\right)\right] + \varepsilon' \delta t \\
&\qquad\qquad \geq V^B(t', x') \geq \mathbb{E}\left[e^{-r\delta t} V^B(t' + \delta t, x' + \sigma W_{\delta t}) - \frac{c}{r}\left(1 - e^{-r\delta t}\right)\right]
\end{aligned}$$

for all $(t', x') \in (t, x) + [-\varepsilon, +\varepsilon]^2$. The second inequality followed simply from the non–optimality. The usual way to proceed is by applying Ito's Lemma. Although V^B is not known to be $C^{1,2}(\Omega)$, more general versions of Ito's Lemma are available such as the one for convex functions (see §3.6 Karatzas and Shreve (2012)), or the weak derivative version Aebi (1992). In any case, the standard treatment is by mollifying the problematic function which we shall provide detail for our simple case for completeness.

For any $\varepsilon > 0$ the mollification of any function $f \in L^1_{loc}([0,T] \times \mathbb{R})$ is the smooth function $f_\varepsilon := \eta_\varepsilon * f$, where $\eta_\varepsilon(t,x) \propto e^{-\frac{1}{1-\|(t,x)\|_2/\varepsilon^2}} 1_{(t,x)\in[-\varepsilon,+\varepsilon]^2}$ is a standard compactly supported bump–function. Applying the mollification in (t', x') centered at (t, x) to the above inequality before applying Ito's Lemma we obtain:

$$
\begin{aligned}
&\mathbb{E}\Bigg[\int_t^{t+\tau_{t,x,\varepsilon'}\wedge\delta t} e^{-r(s-t)} \\
&\qquad \times \left(\frac{\sigma^2}{2}\partial_x^2 V_\varepsilon^B(s, v_s) + \partial_t V_\varepsilon^B(s, v_s) - rV_\varepsilon^B(s, v_s) - c\right) ds | v_t = x\Bigg] + \varepsilon'\delta t \\
&\quad \geq 0 \geq \mathbb{E}\left[\int_t^{t+\delta t} e^{-r(s-t)} \left(\frac{\sigma^2}{2}\partial_x^2 V_\varepsilon^B(s, v_s) + \partial_t V_\varepsilon^B(s, v_s) - rV_\varepsilon^B(s, v_s) - c\right) ds | v_t = x\right].
\end{aligned}
$$

Taking the limit $\delta t \to 0$ we get

$$
\left|\frac{\sigma^2}{2}\partial_x^2 V_\varepsilon^B(t, x) + \partial_t V_\varepsilon^B(t, x) - rV_\varepsilon^B(t, x) - c\right| \leq \varepsilon'.
$$

Recall that $\varepsilon' > 0$ arbitrary and $\varepsilon > 0$ is independent of (t, x), this establishes a uniform convergence $\frac{\sigma^2}{2}\partial_x^2 V_\varepsilon^B + \partial_t V_\varepsilon^B - rV_\varepsilon^B - c \to 0$ on Ω as $\varepsilon \to 0$. Let $g := \partial_t V^B - rV^B - c$. Since $|V^B| \leq L := \max_{s\in[0,T]}(\bar{V}_s - p_s)$ and $|\partial_t V^B| \leq K := \max_{s\in[0,T]} |p'_s|$, therefore g is bounded: $|g(t, x)| \leq K + rL + c$. It follows that $g(t, .) \in L^\infty_{loc}(\mathbb{R}) \subset L^1_{loc}(\mathbb{R})$. From the properties of mollification (see §5.3.1 of Evans (2022)) we already know the $L^1_{loc}(\mathbb{R})$ convergence $\partial_t V_\varepsilon^B(t, .) - rV_\varepsilon^B(t, .) - c \to g(t, .)$, it follows from the uniform convergence above that $\partial_x^2 V_\varepsilon^B(t, .) \to g(t, .)$ in $L^1_{loc}(\mathbb{R})$ as $\varepsilon \to 0$. Therefore the weak derivative $\partial_x^2 V^B(t, .)$

exists, coincides with $g(t,.)$ a.e.,

$$\frac{\sigma^2}{2}\partial_x^2 V^B + \partial_t V^B - rV^B - c = 0 \quad \text{a.e. on } \Omega$$

and $\partial_x^2 V^B(t,.) \in L^\infty(\mathbb{R})$ for each t.

Part 2: Let V be the weak solution as given in Lemma's statement but for convenience, let's also extend the definition by $V(t,x) := \max\{x - p_t, 0\}$ for $(t,x) \notin \Omega$, and $V(t \geq T, x) := V_0^B(x; p_T)$. As before, we consider the mollification $V_\varepsilon := \eta_\varepsilon * V$ before proceeding with Ito's Lemma. For any given stopping time $\tau \in \mathcal{T}$ we have that

$$\begin{aligned}\mathbb{E}&\left[e^{-r\tau}V_\varepsilon(\tau, v_\tau)|v_t = x\right] - e^{-rt}V_\varepsilon(t,x)\\ &= \mathbb{E}\left[\int_t^\tau e^{-rs}\left(-rV_\varepsilon(s,v_s) + \partial_t V_\varepsilon(s,v_s) + \frac{\sigma^2}{2}\partial_x^2 V_\varepsilon(s,v_s)\right) ds|v_t = x\right]. \quad (8)\end{aligned}$$

Since we have assumed $p_t = p_T$ constant for $t \geq T$, we may restrict our attention only to $\tau \in \mathcal{T}$ which coincides with the constant price stopping rule (characterized by the purchase and exit boundaries $p_T + \bar{V}, p_T + \underline{V}$) whenever $\tau \geq T$. In other words, we only consider any arbitrarily stopping time τ which is only potentially non–optimal for $0 \leq \tau < T$.

Recall the standard result (see §5.3.1. of Evans (2022) and Bressan (2012)) that $\partial^k V_\varepsilon(t,x) = \eta_\varepsilon * \partial^k V(t,x)$ for $k \geq 0, 1, 2$, ∂ denotes derivative w.r.t. t or x (and ∂V must be interpreted in weak sense) and that for any function $f \in L^1_{loc}(\mathbb{R})$ we have a pointwise convergence: $(\eta_\varepsilon * f)(x) \to f(x)$ as $\varepsilon \to 0$, for all $x \in \mathbb{R}$. We proceed by taking the limit as $\varepsilon \to 0$ on both–sides of (8). In the following, let's assume that $\varepsilon < \bar{\varepsilon}$ for some $\bar{\varepsilon} > 0$. For the LHS, we have a pointwise convergence: $V_\varepsilon \to V$ over $[0,T] \times \mathbb{R}$. Let $L := \max_{s \in [0,T]}(\bar{V}_s - p_s)$ then $|V_\varepsilon(t,x)| \leq \max\{x - p_t, L\}$. Further,

$$\begin{aligned}\mathbb{E}\left[e^{-r\tau}\max\{v_\tau - p_\tau, L\}|v_t = x\right] &\leq V_0^B(x; p = \min_{t\geq 0} p_t, c = 0) + L\\ &\leq \max\left\{x - \min_{t\geq 0} p_t, \frac{\sigma}{\sqrt{2r}}\right\} + L < \infty\end{aligned}$$

and so $\mathbb{E}\left[e^{-r\tau}V_\varepsilon(\tau,v_\tau)|v_t=x\right] \to \mathbb{E}\left[e^{-r\tau}V(\tau,v_\tau)|v_t=x\right]$ by the Dominated Convergence Theorem. We now turn to the RHS. By the convexity of V, and $V(t,.) \in W^{2,\infty}_{loc}(\mathbb{R})$ we have that the weak derivative $\partial_x^2 V(t,.)$ coincides a.e. with the classical second derivative, hence $\partial_x^2 V(t,.) \geq 0$ a.e. It follows that $\partial_x^2 V(t,x) \geq \partial_x^2 V_\varepsilon(t,x) \geq 0$ a.e., where the first inequality is an equality if $x \in (\underline{V}_t + \varepsilon, \bar{V}_t - \varepsilon V_t)$. Next, we consider the behavior of $\partial_t V(t,.)$ across the boundary of Ω at any fixed $t \in [0,T]$. We focus on the $x = \bar{V}_t$ boundary, the $x = \underline{V}_t$ boundary is similar. Since $V \in L^\infty(0,T;W^{2,\infty}(\mathbb{R}))$ implies the uniform essential boundedness of $\partial_x^2 V(t,.)$ for $t \in [0,T]$, for any given $\bar{\delta} > 0$ let $\bar{\varepsilon} > 0$ be such that $1 \geq \partial_x V_\varepsilon(t,x) > 1 - \bar{\delta}$ for all $(t,x) \in \{\bar{V}_t - \bar{\varepsilon} < x < \bar{V}_t + \bar{\varepsilon}, t \in [0,T]\}$ and all $\varepsilon < \bar{\varepsilon}$. For $\delta > 0$ let's define $\bar{V}_{t,\delta,\varepsilon} := \min\{x|V_\varepsilon(t,x) = x - p_t + \delta\}$ which is smooth in t. For any $t \in [0,T]$ and $\bar{V}_t > x > \bar{V}_t - \bar{\varepsilon}$ we can choose $\delta > 0$ such that $\bar{V}_{t,\delta,\varepsilon} = x$ and it follows from differentiating the defining equation $V_\varepsilon(t,\bar{V}_{t,\delta,\varepsilon}) = \bar{V}_{t,\delta,\varepsilon} - p_t + \delta$ that

$$\begin{aligned}
&\partial_t V_\varepsilon(t,\bar{V}_{t,\delta,\varepsilon}) + \bar{V}'_{t,\delta,\varepsilon}\partial_x V_\varepsilon(t,\bar{V}_{t,\delta,\varepsilon}) = \bar{V}'_{t,\delta,\varepsilon} - p'_t \\
&\qquad\qquad\qquad\qquad \implies |\partial_t V_\varepsilon(t,x) + p'_t| < |\bar{V}'_{t,\delta,\varepsilon}| \cdot |1 - \partial_x V_\varepsilon(t,x)| < M\bar{\delta},
\end{aligned}$$

where $M := \max_{s\in[0,T]}|p'_s| + \operatorname{ess\,sup}_{[0,T]\times\mathbb{R}}|\partial_t V|$. Since V is a weak solution, we have $\frac{\sigma^2}{2}\partial_x^2 V_\varepsilon + \partial_t V_\varepsilon - rV_\varepsilon - c = 0$ for all $(t,x) \in \{\underline{V}_t + \varepsilon < x < \bar{V}_t - \varepsilon, t \in [0,T]\}$. The argument we have been through above shows that for (t,x) in the $\bar{\varepsilon}$ vicinity of the boundary $\partial\bar{\Omega}$ we have $\partial_t V_\varepsilon$ varies no more than $M\bar{\delta}$, whereas $\partial_x^2 V_\varepsilon$ decreases rapidly to zero from Ω to the outside. Therefore, $\frac{\sigma^2}{2}\partial_x^2 V_\varepsilon + \partial_t V_\varepsilon - rV_\varepsilon - c - M\bar{\delta} < 0$ for $(t,x) \notin \{\underline{V}_t + \varepsilon < x < \bar{V}_t - \varepsilon, t \in [0,T]\}$. It follows that for all $\varepsilon > 0$ with $\varepsilon < \bar{\varepsilon}$ the RHS of (8) is always $\leq \mathbb{E}\left[\int_t^\tau ce^{-rs}ds|v_t = x\right] + M\bar{\delta}$. Since $\bar{\delta} > 0$ is arbitrary small, under the limit $\varepsilon \to 0$ (8) becomes

$$\mathbb{E}\left[e^{-r\tau}V(\tau,v_\tau)|v_t=x\right] - e^{-rt}V(t,x) \leq \mathbb{E}\left[\int_t^\tau ce^{-rs}ds|v_t=x\right],$$

Using $V(t, v_t) \geq \max\{v_t - p_t, 0\}$ and rearranging the inequality above, we obtain:

$$V(t,x) \geq \mathbb{E}\left[e^{-r(\tau-t)}\max\{v_\tau - p_\tau, 0\} - \int_t^\tau ce^{-r(s-t)}ds|v_t = x\right]. \tag{9}$$

Since τ is arbitrary, we have by the definition of supremum that $V(t,x) \geq V^B(t,x)$.

For any $(t,x) \in \Omega$, consider the hitting time $\tau^* \in \mathcal{T}$ of the valuation process starting at x to the boundary $\partial\bar{\Omega}$. Any sample path of v_t from x to $v_{\tau^*} \in \partial\bar{\Omega}$ is contained entirely inside Ω where $\frac{\sigma^2}{2}\partial_x^2 V + \partial_t V - rV - c = 0$ a.e.. Take the $\varepsilon \to 0$ limit of (8), use the boundedness of each term on the RHS's integrand over $\bar{\Omega}$ to obtain the convergence of RHS's integral via Dominated Convergence Theorem, and the fact $V(\tau^*, v_{\tau^*}) = \max\{v_{\tau^*} - p_{\tau^*}, 0\}$ to conclude that we get (9) with equality. Therefore, the supremum can be reached with τ^*, hence $V(t,x) = V^B(t,x)$. □

Proof of Lemma 2. Without the loss of generality, let $t = 0$ and fix an $x \in \mathbb{R}$. For convenience, let $\tilde{V}^B(0,x) := \mathbb{E}[\mathcal{V}^B(0, v; \tau^*[l_t], p_t)|v_0 = x]$ be the expected payoff for a consumer following the $\tau^*[l_t]$ as opposed to the value function $V^B(0,x) = \mathbb{E}[\mathcal{V}^B(0, v; \tau^*[p_t], p_t)|v_0 = x]$ which is the payoff for a consumer following the optimal stopping policy $\tau^*[p_t]$. We have

$$\begin{aligned} V^B(0,x) = {} & \mathbb{E}\left[\mathcal{V}^B(0, v; \tau^*[p_t], p_t)|v_0 = x, \tau^*[p_t] < \delta\right]\mathbb{P}[\tau^*[p_t] < \delta] \\ & + \mathbb{E}\left[\mathcal{V}^B(0, v; \tau^*[p_t], p_t)|v_0 = x, \tau^*[p_t] \geq \delta\right]\mathbb{P}[\tau^*[p_t] \geq \delta]. \end{aligned}$$

Now, we note that for $\tau < \delta$ we have $\max\{v_\tau - p_\tau, 0\} \leq \max\{v_\tau - l_\tau + \delta\varepsilon/2, 0\} \leq \max\{v_\tau - l_\tau, 0\} + \delta\varepsilon/2$, which implies

$$\begin{aligned} & \mathbb{E}\left[\mathcal{V}^B(0, v; \tau^*[p_t], p_t)|v_0 = x, \tau^*[p_t] < \delta\right] \\ & \qquad \leq \mathbb{E}\left[\mathcal{V}^B(0, v; \tau^*[p_t], l_t)|v_0 = x, \tau^*[p_t] < \delta\right] + \delta\varepsilon/2. \end{aligned}$$

Similarly, using $\max\{v_\tau - l_\tau, 0\} \leq \max\{v_\tau - p_\tau, 0\} + \delta\varepsilon/2$ for $\tau < \delta$, we have:

$$\mathbb{E}\left[\mathcal{V}^B(0, v; \tau^*[l_t], l_t)|v_0 = x, \tau^*[l_t] < \delta\right]$$
$$\leq \mathbb{E}\left[\mathcal{V}^B(0, v; \tau^*[l_t], p_t)|v_0 = x, \tau^*[l_t] < \delta\right] + \delta\varepsilon/2.$$

On the other hand, we note that $|p_t| < C$ also implies $v_{\tau^*[p_t]} \leq C + \bar{V}$, so we have

$$\mathbb{E}\left[\mathcal{V}^B(0, v; \tau^*[p_t], p_t)|v_0 = x, \tau^*[p_t] \geq \delta\right] \leq -\frac{c}{r} + e^{-r\delta}\left(2C + \bar{V} + c/r\right)$$
$$< -\frac{c}{r} + (1-\delta)\varepsilon/2 \leq \mathbb{E}\left[\mathcal{V}^B(0, v; \tau^*[p_t], l_t)|v_0 = x, \tau^*[p_t] \geq \delta\right] + (1-\delta)\varepsilon/2.$$

We can obtain a similar inequality for $\mathbb{E}[\mathcal{V}^B(0, v; \tau^*[l_t], l_t)|\tau^*[l_t] \geq \delta]$. If $h'(0) < 0$ then the purchase boundary should be decreasing with time, which means $v_{\tau^*[l_t]} < C + \bar{V}$. Also we have that $l_{\tau^*[l_t]}e^{-r\tau^*[l_t]} \leq \max_{t\geq\delta}(p + |h'(0)|t)e^{-rt} = e^{-r\delta}(p + |h'(0)|\delta)$, where the last equality follows if $r > \frac{|h'(0)|}{|h'(0)|\delta+p}$. Therefore, we have

$$\mathbb{E}\left[\mathcal{V}^B(0, v; \tau^*[l_t], l_t)|v_0 = x, \tau^*[l_t] \geq \delta\right] \leq -\frac{c}{r} + e^{-r\delta}(C + \bar{V} + p + |h'(0)|\delta + c/r)$$
$$< -\frac{c}{r} + (1-\delta)\varepsilon/2 \leq \mathbb{E}\left[\mathcal{V}^B(0, v; \tau^*[l_t], p_t)|v_0 = x, \tau^*[l_t] \geq \delta\right] + (1-\delta)\varepsilon/2.$$

For the case $h'(0) \geq 0$, we must have that the payoff cannot be better than if the price was to stay constant at p, and in the constant price case we have $v_{\tau^*[p]} < C + \bar{V}$, therefore:

$$\mathbb{E}\left[\mathcal{V}^B(0, v; \tau^*[l_t], l_t)|v_0 = x, \tau^*[l_t] \geq \delta\right] \leq -\frac{c}{r} + e^{-r\delta}(C + \bar{V} + p + c/r)$$
$$< -\frac{c}{r} + (1-\delta)\varepsilon/2 \leq \mathbb{E}\left[\mathcal{V}^B(0, v; \tau^*[l_t], p_t)|v_0 = x, \tau^*[l_t] \geq \delta\right] + (1-\delta)\varepsilon/2.$$

Putting everything back into the expression for $V^B(0,x)$ we had earlier yields:

$$\begin{aligned}
V^B(0,x) \leq \mathbb{E}\left[\mathcal{V}^B(0,v;\tau^*[p_t],l_t)|v_0 = x, \tau^*[p_t] < \delta\right] \mathbb{P}[\tau^*[p_t] < \delta] \\
+ \mathbb{E}\left[\mathcal{V}^B(0,v;\tau^*[p_t],l_t)|v_0 = x, \tau^*[p_t] \geq \delta\right] \mathbb{P}[\tau^*[p_t] \geq \delta] \\
+ \delta(\varepsilon/2)\mathbb{P}[\tau^*[p_t] < \delta] + (1-\delta)(\varepsilon/2)\mathbb{P}[\tau^*[p_t] \geq \delta] \\
\leq \mathbb{E}\left[\mathcal{V}^B(0,v;\tau^*[p_t],l_t)|v_0 = x\right] + \varepsilon/2 \\
\leq \mathbb{E}\left[\mathcal{V}^B(0,v;\tau^*[l_t],l_t)|v_0 = x\right] + \varepsilon/2 \\
= \mathbb{E}\left[\mathcal{V}^B(0,v;\tau^*[l_t],l_t)|v_0 = x, \tau^*[l_t] < \delta\right] \mathbb{P}[\tau^*[l_t] < \delta] \\
+ \mathbb{E}\left[\mathcal{V}^B(0,v;\tau^*[l_t],l_t)|v_0 = x, \tau^*[l_t] \geq \delta\right] \mathbb{P}[\tau^*[l_t] \geq \delta] + \varepsilon/2 \\
\leq \mathbb{E}\left[\mathcal{V}^B(0,v;\tau^*[l_t],p_t)|v_0 = x, \tau^*[l_t] < \delta\right] \mathbb{P}[\tau^*[l_t] < \delta] \\
+ \mathbb{E}\left[\mathcal{V}^B(0,v;\tau^*[l_t],p_t)|v_0 = x, \tau^*[l_t] \geq \delta\right] \mathbb{P}[\tau^*[l_t] \geq \delta] \\
+ \delta(\varepsilon/2)\mathbb{P}[\tau^*[l_t] < \delta] + (1-\delta)(\varepsilon/2)\mathbb{P}[\tau^*[l_t] \geq \delta] + \varepsilon/2 \\
\leq \mathbb{E}\left[\mathcal{V}^B(0,v;\tau^*[l_t],p_t)|v_0 = x\right] + \varepsilon = \tilde{V}^B(0,t) + \varepsilon
\end{aligned}$$

where the third inequality followed from the optimality of $\tau^*[l_t]$ under the pricing policy l_t. □

Proof of Lemma 3. According to Lemma 1, the solution V must coincide with the value function V^B on $[0,T] \times \mathbb{R}$, therefore, we may take $V(.,.;p)$ and $V(.,.;q)$ to be given by (1) with p and q, respectively. Let's consider a fixed $(t,x) \in [0,T] \times \mathbb{R}$, and let's suppose that $V(t,x;q) \leq V(t,x;p)$. For an arbitrary $\varepsilon > 0$, let $\tau_{t,x,\varepsilon}[p] \in \mathcal{T}$ be such that $\mathbb{E}[\mathcal{V}^B(t,x;\tau_{t,x,\varepsilon}[p],p)|v_t = x] \geq V(t,x;p) - \varepsilon$, then

$$\begin{aligned}
V(t,x;q) \geq \mathbb{E}[\mathcal{V}^B(t,x;\tau_{t,x,\varepsilon}[p],q)|v_t = x] \\
> \mathbb{E}[\mathcal{V}^B(t,x;\tau_{t,x,\varepsilon}[p];p)|v_t = x] - \max_{s\in[t,T]} e^{-r(s-t)}|p_s - q_s| \\
\geq V(t,x;p) - \max_{s\in[t,T]} e^{-r(s-t)}|p_s - q_s| - \varepsilon
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it must be the case that:

$$V(t,x;p) \geq V(t,x;q) \geq V(t,x;p) - \max_{s\in[t,T]} e^{-r(s-t)}|p_s - q_s|.$$

If $V(t,x;q) \geq V(t,x;p)$, then we simply switch the role of p,q and follow through with the above argument, hence we get that

$$|V(t,x;p) - V(t,x;q)| \leq \max_{s\in[t,T]} e^{-r(s-t)}|p_s - q_s|,$$

which proves the result. □

Proof of Proposition 1. Define $\underline{R} = \underline{S}K$ and $\bar{R} = \bar{R}K$. Then, the purchasing and quitting boundaries take the following form:

$$\begin{aligned}\bar{V}_t &= \left(p + \sqrt{\varepsilon}Kt + \bar{V}\right) + \sqrt{\varepsilon}\bar{R} + O(\varepsilon)\\ \underline{V}_t &= \left(p + \sqrt{\varepsilon}Kt + \underline{V}\right) + \sqrt{\varepsilon}\underline{R} + O(\varepsilon).\end{aligned}$$

The equation for V_1^B can be found by collecting the $\sqrt{\varepsilon}$–order terms in the PDE of V^B:

$$\begin{aligned}&\frac{\sigma^2}{2}\sqrt{\varepsilon}\partial_x^2 V_1^B(t,x) + \partial_t V_0^B(x - \sqrt{\varepsilon}Kt) - \sqrt{\varepsilon}rV_1^B(t,x) = 0\\ &\qquad\qquad\qquad \Longrightarrow \frac{\sigma^2}{2}\partial_x^2 V_1^B(t,x) - rV_1^B(t,x) = \frac{cK\sqrt{2}}{\sigma\sqrt{r}}\sinh\frac{\sqrt{2r}}{\sigma}(x - \underline{V} - p).\end{aligned}$$

From this, the general solution takes the form:

$$V_1^B(t,x) = A_1(\sqrt{\varepsilon}t)e^{+\frac{\sqrt{2r}}{\sigma}x} + A_2(\sqrt{\varepsilon}t)e^{-\frac{\sqrt{2r}}{\sigma}x} + \frac{cK}{\sigma^2 r}x\cosh\frac{\sqrt{2r}}{\sigma}(x - \underline{V} - p). \qquad (10)$$

Similarly, the boundary conditions are obtained by writing the boundary conditions of V^B

to the first–order in $\sqrt{\varepsilon}$:

$$V^B(t,\bar{V}_t) = \bar{V}_t - p_t \implies V_1^B(t, p+\bar{V}) = 0. \tag{11}$$

$$\partial_x V^B(t,\bar{V}_t) = 1 \implies \partial_x V_1^B(t, p+\bar{V}) = -\frac{2r}{\sigma^2}\left(\bar{V}+\frac{c}{r}\right)\bar{R} \tag{12}$$

$$V^B(t,\underline{V}_t) = 0 \implies V_1^B(t, p+\underline{V}) = 0 \tag{13}$$

$$\partial_x V^B(t,\underline{V}_t) = 0 \implies \partial_x V_1^B(t, p+\underline{V}) = -\frac{2c}{\sigma^2}\underline{R} \tag{14}$$

Substituting (10) into (11), (12), (13),(14), we get the following system of equations to solve for $A_1, A_2, \bar{R}, \underline{R}$, and R:

$$\begin{aligned}
A_1 e^{+\frac{\sqrt{2r}}{\sigma}(p+\bar{V})} + A_2 e^{-\frac{\sqrt{2r}}{\sigma}(p+\bar{V})} &= -\frac{K}{\sigma^2}(p+\bar{V})\left(\bar{V}+\frac{c}{r}\right)\\
\frac{\sqrt{2r}}{\sigma}A_1 e^{+\frac{\sqrt{2r}}{\sigma}(p+\bar{V})} - \frac{\sqrt{2r}}{\sigma}A_2 e^{-\frac{\sqrt{2r}}{\sigma}(p+\bar{V})} &= -\frac{K}{\sigma^2}\left(2\bar{V}+p+\frac{c}{r}\right) - \frac{2r}{\sigma^2}\left(\bar{V}+\frac{c}{r}\right)\bar{R}\\
A_1 e^{+\frac{\sqrt{2r}}{\sigma}(p+\underline{V})} + A_2 e^{-\frac{\sqrt{2r}}{\sigma}(p+\underline{V})} &= -\frac{cK}{\sigma^2 r}(p+\underline{V})\\
\frac{\sqrt{2r}}{\sigma}A_1 e^{+\frac{\sqrt{2r}}{\sigma}(p+\underline{V})} - \frac{\sqrt{2r}}{\sigma}A_2 e^{-\frac{\sqrt{2r}}{\sigma}(p+\underline{V})} &= -\frac{cK}{\sigma^2 r} - \frac{2c}{\sigma^2}\underline{R}.
\end{aligned}$$

We find that:

$$\underline{S} = \frac{\underline{R}}{K} = \left(\frac{\bar{V}-\underline{V}}{\sigma^2}\right)\left(\bar{V}+\frac{c}{r}\right) - \frac{1}{2r} = \frac{\sqrt{\frac{c^2}{r^2}+\frac{\sigma^2}{2r}}}{\sigma\sqrt{2r}}\log\left(\sqrt{\frac{r\sigma^2}{2c^2}}+\sqrt{1+\frac{r\sigma^2}{2c^2}}\right) - \frac{1}{2r}$$

$$\bar{S} = \frac{\bar{R}}{K} = \underline{S} - \frac{1}{2r}\cdot\frac{\bar{V}-\underline{V}}{\bar{V}+c/r} = \frac{1/(\sigma\sqrt{2r})}{\sqrt{\frac{c^2}{r^2}+\frac{\sigma^2}{2r}}}\cdot\frac{c^2}{r^2}\log\left(\sqrt{\frac{r\sigma^2}{2c^2}}+\sqrt{1+\frac{r\sigma^2}{2c^2}}\right) - \frac{1}{2r}$$

It only remains to show the last claims about the sign of $\underline{S}$ and $\bar{S}$. For large r, we have $\underline{S} \sim \frac{1}{r}\log r - \frac{1}{r} > 0$ and $\bar{S} \sim \frac{1}{r^2}\log r - \frac{1}{r} < 0$. □

Proof of Proposition 2. By taking the $c \to 0$ limit in Proposition 1, we have:

$$\bar{V}_t = p + \sqrt{\varepsilon} K t + \frac{\sigma}{\sqrt{2r}} - \sqrt{\varepsilon}\frac{K}{2r}.$$

We assume throughout that $p \geq g$.

The solution $U(t,v)$ to the heat equation with the single linearly moving absorbing boundary with initial condition $U(t=0,v) = \delta(v-x), x \leq \bar{V}_0$, is well–known:

$$U(t,v) = \frac{\exp\left(-\frac{\sqrt{\varepsilon}K}{\sigma^2}(v - x - \sqrt{\varepsilon}Kt) - \frac{\varepsilon K^2}{2\sigma^2}t\right)}{\sigma\sqrt{2\pi t}} \left(e^{-\frac{(v-\sqrt{\varepsilon}Kt-x)^2}{2t\sigma^2}} - e^{-\frac{(v-\sqrt{\varepsilon}Kt+x-2\bar{V}_0)^2}{2t\sigma^2}}\right).$$

Therefore, the purchase probability flux is:

$$-\frac{\sigma^2}{2}\partial_v U(t,\bar{V}_t) = \frac{\bar{V}_0 - x}{\sigma\sqrt{2\pi t^3}} \exp\left(-\frac{\left(\bar{V}_t - x\right)^2}{2t\sigma^2}\right).$$

It is now straightforward to compute the expected firm's payoff at $t = 0$:

$$\begin{aligned}
\mathcal{V}^S(x;p,K) &:= -\frac{\sigma^2}{2}\int_0^\infty e^{-ms}(p_s - g)\partial_v U(s,\bar{V}_s)ds \\
&= \left(p - g + \frac{\sqrt{\varepsilon}K}{\sqrt{2m\sigma^2 + \varepsilon K^2}}\left(p + \frac{\sigma}{\sqrt{2r}} - x - \sqrt{\varepsilon}\frac{K}{2r}\right)\right) \\
&\quad \times \exp\left(-\left(\frac{\sqrt{\varepsilon}K + \sqrt{2m\sigma^2 + \varepsilon K^2}}{\sigma^2}\right)\left(p + \frac{\sigma}{\sqrt{2r}} - x - \sqrt{\varepsilon}\frac{K}{2r}\right)\right). \quad (15)
\end{aligned}$$

This formula subjects to the condition that $x \leq \bar{V}_0$, which means $\sqrt{\varepsilon}K \leq 2r(p-x) + \sqrt{2r}\sigma$, otherwise if $p < x$ then we have $\mathcal{V}^S(x;p,K) = p - g$ for $2r(p-x) + \sqrt{2r}\sigma < K < \sqrt{2r}\sigma$.

In the special case where $m = 0$, we have

$$\mathcal{V}^S(x;p,K) = \begin{cases} p-g, & \sqrt{2r}\sigma > \sqrt{\varepsilon}K > 2r(p-x) + \sqrt{2r}\sigma \\ \left(2p + \frac{\sigma}{\sqrt{2r}} - g - x - \sqrt{\varepsilon}\frac{K}{2r}\right) & \\ \quad \times \exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^2}\left(p + \frac{\sigma}{\sqrt{2r}} - x - \sqrt{\varepsilon}\frac{K}{2r}\right)\right), & 0 < \sqrt{\varepsilon}K \leq \min\{2r(p-x) + \sqrt{2r}\sigma, \sqrt{2r}\sigma\} \\ p-g, & \sqrt{\varepsilon}K = 0 \\ x - g - \frac{\sigma}{\sqrt{2r}} + \sqrt{\varepsilon}\frac{K}{2r}, & \sqrt{\varepsilon}K < 0 \end{cases}$$

For any fixed p, we can approach the supremum $2p + \frac{\sigma}{\sqrt{2r}} - g - x \geq p - g$ of $\mathcal{V}^S$ by choosing $K > 0$ as close to 0 as possible. The firm could also earn an extra of $\left(2p + \frac{\sigma}{\sqrt{2r}} - g - x\right) - (p-g) = p - x + \frac{\sigma}{\sqrt{2r}}$ by setting K infinitesimally higher than 0.

For $m > 0$, the optimal K is now bounded from 0. This can also be seen for a general p by computing:

$$\frac{\partial}{\partial K}\mathcal{V}^S(x;p,K)\Big|_{K=0} = e^{-\frac{\sqrt{2m}}{\sigma}\left(p-x+\frac{\sigma}{\sqrt{2r}}\right)} \times \left(\frac{p-x+\sigma/\sqrt{2r}}{\sigma\sqrt{2m}} - (p-g)\left(\frac{p-x+\sigma/\sqrt{2r} - (\sigma/r)\sqrt{m/2}}{\sigma^2}\right)\right),$$

we can see that this is always > 0 for sufficiently small and sufficiently large $m > 0$. $\square$

Proof of Proposition 3. The standard solution U_0 to the heat equation (4) with 2 absorbing non–moving boundaries at $\bar{V}_0 := p + \bar{V} + \sqrt{\varepsilon}\bar{S}K$ and $\underline{V}_0 := p + \underline{V} + \sqrt{\varepsilon}\underline{S}K$ and the initial condition $U_0(0,v) = \delta(v-x)$ is given by Karatzas and Shreve (2012):

$$U_0(t,v) = \frac{1}{\sigma\sqrt{2\pi t}} \sum_{k=-\infty}^{+\infty} \left[e^{-\frac{(v-x+2k(\bar{V}_0 - \underline{V}_0))^2}{2t\sigma^2}} - e^{-\frac{(v+x-2\underline{V}_0+2k(\bar{V}_0-\underline{V}_0))^2}{2t\sigma^2}}\right]. \quad (16)$$

Equivalently:

$$U_0(t,v)dv = \mathbb{P}\left[x + \sigma W_t \in dv, \underline{V}_0 < x + \sigma W_s < \bar{V}_0, s \in [0,t]\right].$$

Instead of moving the boundary according to $\sqrt{\varepsilon}Kt$ we may consider the consumer valuation process to be the Brownian process with drift starting at x: $\tilde{v}_t = x - \sqrt{\varepsilon}Kt + \sigma W_t$ with fixed absorbing boundaries at $\bar{V}_0, \underline{V}_0$. If $\{W_t\}$ is the standard Brownian process on $(\Omega, \mathscr{F}, \Sigma, \mathbb{P})$ then $\{x + \sigma W_t\}$ is the Brownian process with drift starting at x, i.e. $\{\tilde{v}_t\}$ on $(\Omega, \mathscr{F}, \Sigma, \mathbb{Q})$ where

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathscr{F}_t} = \exp\left(-\frac{\sqrt{\varepsilon}K}{\sigma}W_t - \frac{\varepsilon K^2}{2\sigma^2}t\right).$$

Consequently, we have that the solution U to the heat equation (4) with moving boundaries $\bar{V}_t, \underline{V}_t$ is given by

$$\begin{aligned} U(t,v)dv &= \mathbb{P}\left[\tilde{v}_t \in v - \sqrt{\varepsilon}Kt + dv, \underline{V}_0 < \tilde{v}_s < \bar{V}_0, s \in [0,t]\right] \\ &= \mathbb{Q}\left[x + \sigma W_t \in v - \sqrt{\varepsilon}Kt + dv, \underline{V}_0 < x + \sigma W_s < \bar{V}_0, s \in [0,t]\right] \\ &= \exp\left(-\frac{\sqrt{\varepsilon}K}{\sigma^2}(v - x - \sqrt{\varepsilon}Kt) - \frac{\varepsilon K^2}{2\sigma^2}t\right) U_0(t, v - \sqrt{\varepsilon}Kt)dv \end{aligned}$$

Therefore, the purchase probability flux is:

$$\begin{aligned} -\frac{\sigma^2}{2}\partial_v U(t,\bar{V}_t) = \sum_{k=-\infty}^{+\infty} &\frac{(2k+1)(\bar{V}_0 - \underline{V}_0) - (x - \underline{V}_0)}{\sigma\sqrt{2\pi t^3}} \\ &\times e^{\frac{2\sqrt{\varepsilon}Kk}{\sigma^2}(\bar{V}_0 - \underline{V}_0)} e^{-\frac{((2k+1)(\bar{V}_0 - \underline{V}_0) - (x - \underline{V}_0) + \sqrt{\varepsilon}Kt)^2}{2t\sigma^2}}. \end{aligned} \tag{17}$$

The term–by–term differentiation is justified at $v = \bar{V}_t$ for any fixed $x \in (\bar{V}_0, \underline{V}_0)$ because $0 < |\bar{V}_0 - x| < |\bar{V}_0 - \underline{V}_0|$, hence the series representation of $U_0(t, v - \sqrt{\varepsilon}Kt)$, and the derivative series both converge absolutely and uniformly for all v in some neighborhoods of $\bar{V}_t$ and $t \in [0,\infty)$. We now compute the seller's expected profit:

Claim 1. *The seller's expected profit from the buyer initially at* $x \in (\bar{V}_0, \underline{V}_0)$ *is:*

$$\mathcal{V}^S(x;p,K) = \frac{\left(p - g - \frac{\sqrt{\varepsilon}K}{\sqrt{2m\sigma^2+\varepsilon K^2}}(\bar{V}_0 + x - 2\underline{V}_0)\right) e^{-\frac{\sqrt{\varepsilon}K+\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(\bar{V}_0 - x)}}{1 - e^{-\frac{2\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(\bar{V}_0-\underline{V}_0)}}$$
$$+ \frac{\frac{2\sqrt{\varepsilon}K}{\sqrt{2m\sigma^2+\varepsilon K^2}}(\bar{V}_0 - \underline{V}_0)e^{-\frac{\sqrt{\varepsilon}K+\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(\bar{V}_0 - x)}}{\left(1 - e^{-\frac{2\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(\bar{V}_0-\underline{V}_0)}\right)^2}$$
$$- \frac{\left(p - g - \frac{\sqrt{\varepsilon}K}{\sqrt{2m\sigma^2+\varepsilon K^2}}(\bar{V}_0 - x)\right) e^{-\frac{\sqrt{\varepsilon}K+\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(\bar{V}_0 - \underline{V}_0)}e^{+\frac{\sqrt{\varepsilon}K-\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(x-\underline{V}_0)}}{1 - e^{-\frac{2\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(\bar{V}_0-\underline{V}_0)}}$$
$$- \frac{\frac{2\sqrt{\varepsilon}K}{\sqrt{2m\sigma^2+\varepsilon K^2}}(\bar{V}_0 - \underline{V}_0)e^{-\frac{\sqrt{\varepsilon}K+\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(\bar{V}_0 - \underline{V}_0)}e^{+\frac{\sqrt{\varepsilon}K-\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(x-\underline{V}_0)}}{\left(1 - e^{-\frac{2\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(\bar{V}_0-\underline{V}_0)}\right)^2}, \quad (18)$$

if $m > 0$ *or* $K \neq 0$, *and* $\mathcal{V}^S(x;p,K) = (p-g)\left(\frac{x-\underline{V}_0}{\bar{V}_0-\underline{V}_0}\right)$ *if* $m = 0, K = 0$. *On the other hand, if* $x \leq \underline{V}_0$ *then* $\mathcal{V}^S(x;p,K) = 0$, *and if* $x \geq \bar{V}_0$ *then* $\mathcal{V}^s(x;p,K) = p - g$.

Proof. We shall only cover the non–trivial case where $x \in (\bar{V}_0, \underline{V}_0)$. First, let's assume that either $m > 0$ or $K \neq 0$. We compute $\mathcal{V}^S(x;p,K)$ by substituting (17) into (5):

$$\mathcal{V}^S(x;p,K) = -\frac{\sigma^2}{2}\int_0^\infty e^{-ms}(p_s - g)\partial_v U(s, \bar{V}_s)ds$$
$$= \sum_{k=-\infty}^{+\infty} \left((2k+1)(\bar{V}_0 - \underline{V}_0) - (x - \underline{V}_0)\right) e^{\frac{2\sqrt{\varepsilon}Kk}{\sigma^2}(\bar{V}_0-\underline{V}_0)}$$
$$\times \int_0^{+\infty} \frac{(p + \sqrt{\varepsilon}Ks - g)}{\sigma\sqrt{2\pi s^3}} e^{-ms-\frac{\left((2k+1)(\bar{V}_0-\underline{V}_0)-(x-\underline{V}_0)+\sqrt{\varepsilon}Ks\right)^2}{2s\sigma^2}} ds$$
$$= \sum_{k=0}^{+\infty} \left(p - g + \frac{\sqrt{\varepsilon}K}{\sqrt{2m\sigma^2+\varepsilon K^2}}\left((2k+1)(\bar{V}_0 - \underline{V}_0) - (x - \underline{V}_0)\right)\right)$$
$$\times \exp\left(+\frac{\sqrt{\varepsilon}K}{\sigma^2} \cdot 2k(\bar{V}_0 - \underline{V}_0) - \frac{\sqrt{\varepsilon}K + \sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}\left((2k+1)(\bar{V}_0 - \underline{V}_0) - (x - \underline{V}_0)\right)\right)$$
$$- \sum_{k=1}^{+\infty} \left(p - g + \frac{\sqrt{\varepsilon}K}{\sqrt{2m\sigma^2+\varepsilon K^2}}\left((2k-1)(\bar{V}_0 - \underline{V}_0) + (x - \underline{V}_0)\right)\right)$$

$$\times \exp\left(-\frac{\sqrt{\varepsilon}K}{\sigma^2}\cdot 2k(\bar{V}_0-\underline{V}_0)+\frac{\sqrt{\varepsilon}K-\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}\left((2k-1)(\bar{V}_0-\underline{V}_0)+(x-\underline{V}_0)\right)\right)$$

In the second equality, we switched the order of summation and integration, which can be justified by Fubini's theorem for $m > 0$ or $K \neq 0$. The resulting infinite series can be evaluated using standard geometric series results to yield (18). If $m = 0$ and $K = 0$, then it is known (see Branco et al. (2012)) that the seller's expected profit is $(p-g)\left(\frac{x-\underline{V}_0}{\bar{V}_0-\underline{V}_0}\right)$. $\square$

In the limit $\underline{V}_0 \to -\infty$ (i.e. the limit $c \to 0$) (18) reduces to (15) we previously studied. Unlike in the single boundary case, in the presence of the exit boundary, the expected seller's profit is not only continuous at $K = 0$, but also differentiable, even when $m = 0$, as we will show below. We now focus on the $m = 0$ case.

From (18) we have that $\mathcal{V}^S(x;p,K<0)|_{m=0}$ is given by (7), and that:

$$\begin{aligned}
&\mathcal{V}^S(x;p,K>0)|_{m=0} = \\
&\frac{(p-g-(\bar{V}_0+x-2\underline{V}_0))\exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0-x)\right)}{1-\exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0-\underline{V}_0)\right)}+\frac{2(\bar{V}_0-\underline{V}_0)\exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0-x)\right)}{\left(1-\exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0-\underline{V}_0)\right)\right)^2} \\
&-\frac{(p-g-(\bar{V}_0-x))\exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0-\underline{V}_0)\right)}{1-\exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0-\underline{V}_0)\right)}-\frac{2(\bar{V}_0-\underline{V}_0)\exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0-\underline{V}_0)\right)}{\left(1-\exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0-\underline{V}_0)\right)\right)^2}. \quad (19)
\end{aligned}$$

Both (7) and (19) are valid expressions for all $K \neq 0$, and with some works, we can show them to be equal for all $K \neq 0$. This proves $\mathcal{V}^S(x;p,K)$ is given by (7) for all $K \neq 0$. $\square$

Derivation of the pricing direction:

For $x \in (\bar{V}_0, \underline{V}_0)$, one can see that $\mathcal{V}^S(x;p,.)$ is differentiable at $K = 0$ with:

$$\begin{aligned}
\frac{\partial \mathcal{V}^S}{\partial K}(x;p,K=0) = &\frac{((\bar{S}-\underline{S})p-(x-\underline{V})\bar{S}-(\bar{V}-x)\underline{S})(p-g)}{(\bar{V}-\underline{V})^2} \\
&+\frac{(p+\underline{V}-x)(p+\bar{V}-x)(4p-3g+2\underline{V}-\bar{V}-x)}{3\sigma^2(\bar{V}-\underline{V})}. \quad (20)
\end{aligned}$$

The sign of (20) is all we need to find out how to improve the seller's expected profit in the vicinity of $K = 0$. We will later show that it is optimal to set the initial price p to be the optimal static price $\hat{p}$ when $K \to 0$. According to Branco et al. (2012), $\hat{p} = \hat{p}(x) = \frac{x+g-\underline{V}}{2}, \quad \underline{V} + g < x < 2\bar{V} - \underline{V} + g$. When $p = \hat{p}$, (20) simplifies to:

$$
\begin{aligned}
\frac{\partial \mathcal{V}^S}{\partial K}(q; p = \hat{p}, K = 0) &= \frac{(\bar{V} - \underline{V})^2}{3\sigma^2}(1-2q)q(1-q) - (\bar{S}q + \underline{S}(1-q))q \\
&= \frac{q}{2r}\left(1 - \frac{2(c/r)^2 + (1-q)\sigma^2/r}{(\sigma/\sqrt{r})\sqrt{2(c/r)^2 + \sigma^2/r}} \log\left(\frac{\sigma/\sqrt{r} + \sqrt{2(c/r)^2 + \sigma^2/r}}{\sqrt{2}c/r}\right)\right) \\
&\qquad + \frac{(1-2q)q(1-q)}{6r}\left(\log\left(\frac{\sigma/\sqrt{r} + \sqrt{2(c/r)^2 + \sigma^2/r}}{\sqrt{2}c/r}\right)\right)^2
\end{aligned}
$$

It is clear that $\frac{\partial \mathcal{V}^S}{\partial K}(q; p = \hat{p}, K = 0)$ is not identically zero for all $q \in [0, 1]$, which shows that for a generic $q \in [0, 1]$ the optimal strategy (p^*, K^*) is such that K^* must be bounded away from 0 even for $m = 0$. The seller can immediately improve its expected profit by setting $K \gtrsim 0$ if $\frac{\partial \mathcal{V}^S}{\partial K}(q; p = \hat{p}, K = 0) > 0$, and by setting $K \lesssim 0$ if $\frac{\partial \mathcal{V}^S}{\partial K}(q; p = \hat{p}, K = 0) < 0$. This gives us Figure 2.

So far, we discussed K in the vicinity of 0 by analyzing the derivative of $\mathcal{V}^S(x; p, K)$ at $K = 0$ with $p = \hat{p} = \frac{x+g-\underline{V}}{2}$. For any $K \neq 0$, the optimal initial price p that maximizes $\mathcal{V}^S(x; ., K)$ is:

$$
\begin{aligned}
p^*(x, K) := (\bar{V} - \underline{V})q + \frac{1}{4\sqrt{\varepsilon}K}&\left(\coth\left(\frac{\sqrt{\varepsilon}K\left(\bar{V} - \underline{V} + \sqrt{\varepsilon}K(\bar{S} - \underline{S})\right)}{\sigma^2}\right) - 1\right) \\
\times &\left[e^{\frac{2\sqrt{\varepsilon}K\left(\bar{V} - \underline{V} + \sqrt{\varepsilon}K(\bar{S} - \underline{S})\right)}{\sigma^2}}\left(\sigma^2 - \sqrt{\varepsilon}K\left(\bar{V} + \sqrt{\varepsilon}K\bar{S} - \underline{V} - 2g\right)\right)\right. \\
&\qquad \left. - \sigma^2 - \sqrt{\varepsilon}K\left(\bar{V} - \underline{V} + \sqrt{\varepsilon}K(\bar{S} - \underline{S}) - \underline{S} - 2g\right)\right]
\end{aligned}
$$

for $K \in [-1, 1]$ such that $\underline{V}_0 < x$, $\bar{V}_0 > x$ and we can check that $\lim_{K \to 0} p^*(x, K) = \hat{p}(x)$.

References

Aebi, R. (1992). Itô's formula for non-smooth functions. *Publications of the Research Institute for Mathematical Sciences*, 28(4):595–602.

Athey, S., Catalini, C., and Tucker, C. (2017). The digital privacy paradox: Small money, small costs, small talk. Technical report, National Bureau of Economic Research.

Baik, Simon Anderson, A. and Larson, N. (2023). Price discrimination in the information age: Prices, poaching, and privacy with personalized targeted discounts. *The Review of Economic Studies*, 90(5):2085–2115.

Bar-Isaac, H., Caruana, G., and Cuñat, V. (2010). Information gathering and marketing. *Journal of Economics & Management Strategy*, 19(2):375–401.

Bondi, T., Tech, C., Rafieian, O., and Yao, Y. J. (2023). Privacy and polarization: An inference-based framework.

Branco, F., Sun, M., and Villas-Boas, J. M. (2012). Optimal search for product information. *Management Science*, 58(11):2037–2056.

Branco, F., Sun, M., and Villas-Boas, J. M. (2016). Too much information? information provision and search costs. *Marketing Science*, 35(4):605–618.

Bressan, A. (2012). Lecture notes on functional analysis. *Graduate studies in mathematics*, 143.

Bronnenberg, B. J., Kim, J. B., and Mela, C. F. (2016). Zooming in on choice: How do consumers search for cameras online? *Marketing science*, 35(5):693–712.

Campbell, J., Goldfarb, A., and Tucker, C. (2015). Privacy regulation and market structure. *Journal of Economics & Management Strategy*, 24(1):47–73.

Chaimanowong, W. and Ke, T. T. (2022). A simple micro-founded model of repeat buying based on continuous information tracking. *Available at SSRN 3954939*.

Chaimanowong, W., Ke, T. T., and Villas-Boas, J. M. (2023). Searching for breakthroughs. *Available at SSRN 4425117*.

Choi, W. J., Jerath, K., et al. (2022). Privacy and consumer empowerment in online advertising. *Foundations and Trends® in Marketing*, 15(3):153–212.

Choi, W. J., Jerath, K., and Sarvary, M. (2020). Advertising and price competition under consumer data privacy choices. *Available at SSRN 3708273*.

Choi, W. J., Jerath, K., and Sarvary, M. (2023). Consumer privacy choices and (un) targeted advertising along the purchase journey. *Journal of Marketing Research*, 60(5):889–907.

Conitzer, V., Taylor, C. R., and Wagman, L. (2012). Hide and seek: Costly consumer privacy in a market with repeat purchases. *Marketing Science*, 31(2):277–292.

Dukes, A. and Liu, L. (2016). Online shopping intermediaries: The strategic design of search environments. *Management Science*, 62(4):1064–1077.

Evans, L. C. (2022). *Partial differential equations*, volume 19. American Mathematical Society.

Goldberg, S., Johnson, G., and Shriver, S. (2019). Regulating privacy online: An economic evaluation of the gdpr. *Available at SSRN 3421731*.

Goldfarb, A. and Tucker, C. E. (2011). Privacy regulation and online advertising. *Management science*, 57(1):57–71.

Guo, L. (2021). Endogenous evaluation and sequential search. *Marketing Science*, 40(3):413–427.

Guo, L. and Zhang, J. (2012). Consumer deliberation and product line design. *Marketing Science*, 31(6):995–1007.

Jerath, K. and Ren, Q. (2021). Consumer rational (in) attention to favorable and unfavorable product information, and firm information design. *Journal of Marketing Research*, 58(2):343–362.

Karatzas, I. and Shreve, S. (2012). *Brownian motion and stochastic calculus*, volume 113. Springer Science & Business Media.

Ke, T. T., Lin, S., and Lu, M. Y. (2023). Information design of online platforms. *Available at SSRN*.

Ke, T. T., Shen, Z.-J. M., and Villas-Boas, J. M. (2016). Search for information on multiple products. *Management Science*, 62(12):3576–3603.

Ke, T. T. and Sudhir, K. (2023). Privacy rights and data security: Gdpr and personal data markets. *Management Science*, 69(8):4389–4412.

Ke, T. T. and Villas-Boas, J. M. (2019). Optimal learning before choice. *Journal of Economic Theory*, 180:383–437.

Kim, J. B., Albuquerque, P., and Bronnenberg, B. J. (2010). Online demand under limited consumer search. *Marketing science*, 29(6):1001–1023.

Kim, J. B., Albuquerque, P., and Bronnenberg, B. J. (2017). The probit choice model under sequential search with an application to online retailing. *Management Science*, 63(11):3911–3929.

Koulayev, S. (2014). Search for differentiated products: identification and estimation. *The RAND Journal of Economics*, 45(3):553–575.

Kuksov, D. and Villas-Boas, J. M. (2010). When more alternatives lead to less choice. *Marketing Science*, 29(3):507–524.

Liu, L. and Dukes, A. (2013). Consideration set formation with multiproduct firms: The case of within-firm and across-firm evaluation costs. *Management Science*, 59(8):1871–1886.

Liu, L. and Dukes, A. (2016). Consumer search with limited product evaluation. *Journal of Economics & Management Strategy*, 25(1):32–55.

Marinovic, I., Skrzypacz, A., and Varas, F. (2018). Dynamic certification and reputation for quality. *American Economic Journal: Microeconomics*, 10(2):58–82.

Mayzlin, D. and Shin, J. (2011). Uninformative advertising as an invitation to search. *Marketing science*, 30(4):666–685.

Montes, R., Sand-Zantman, W., and Valletti, T. (2019). The value of personal information in online markets with endogenous privacy. *Management Science*, 65(3):1342–1362.

Moscarini, G. and Smith, L. (2001). The optimal level of experimentation. *Econometrica*, 69(6):1629–1644.

Ning, Z. E. (2021). List price and discount in a stochastic selling process. *Marketing Science*, 40(2):366–387.

Ning, Z. E., Shin, J., and Yu, J. (2023). Targeted advertising: Strategic mistargeting and personal data opt-out. *KAIST College of Business Working Paper Series*.

Rafieian, O. and Yoganarasimhan, H. (2021). Targeting and privacy in mobile advertising. *Marketing Science*, 40(2):193–218.

Seiler, S. (2013). The impact of search costs on consumer behavior: A dynamic approach. *Quantitative Marketing and Economics*, 11:155–203.

Villas-Boas, J. M. (2009). Product variety and endogenous pricing with evaluation costs. *Management Science*, 55(8):1338–1346.

Weitzman, M. L. (1979). Optimal search for the best alternative. *Econometrica: Journal of the Econometric Society*, pages 641–654.

Wolinsky, A. (1986). True monopolistic competition as a result of imperfect information. *The Quarterly Journal of Economics*, 101(3):493–511.

Yao, Y. (2023a). Dynamic persuasion and strategic search. *Management Science*.

Yao, Y. J. (2023b). Failure of reputation for privacy.

Yao, Y. J. (2023c). Multi-attribute search and informative advertising. *Available at SSRN 4512698*.