

Dynamic Persuasion and Strategic Search

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Abstract

We consider a dynamic model in which a sender sequentially persuades a receiver to take the desired action regardless of the unknown state. In contrast, the receiver only wishes to take that action if the state is good. The receiver could incur costs to search for more information. The updated belief about the state helps the receiver make decisions. The sender designs the information structure. Given the endogenous information environment, the receiver trades off the benefit and cost of information acquisition and decides whether to search or not. The sender could incur higher costs to provide more information so that the receiver would be more likely to take the desired action. Given the information acquisition strategy of the receiver, the sender trades off the benefit and cost of information provision and decides how much information to provide. The sender smoothens information over multiple periods if and only if the prior of the state is good is high. The optimal information structure fully reveals the state being good when the search cost is high and partially when the search cost is low. We also compare the information structures between the profit-maximizing one and the efficient one. The profit-maximizing strategy is efficient if and only if the search cost is high.

1 Introduction

With the rapid development of technologies, it is more and more common for an individual to acquire some information before making a decision. More information could lead to less uncertainty and help the decision-making. As the learning outcome depends on the information environment, other related parties have strong incentives to influence it. Thus, the information environment of the decision-maker will be endogenous. For example, a consumer considering purchasing a product may search on Google to find out whether the product matches his need or not. The seller can control the information available to the consumer by search advertising. She¹ can design the content of the ads. She may also provide information more than once through different channels by retargeting. By providing different kinds of information, the seller affects the search and purchasing decisions of the consumer. The costs of the seller depend on the information structure. If she wants the potential buyer to see the information about the product more often on Google, she needs to spend more money to bid for the search advertising spots. At the same time, it is costly for the buyer to search for information. So, he would only search if he anticipates enough gain from the extra information. For another example, the FDA needs to decide whether to approve or disapprove many new drugs every year. The pharmaceutical company may design tests to convince the regulator that the drug is safe. Even if a test fails to convince the regulator, the firm could offer another test, hoping that the regulator would approve the drug after seeing the new test. Different tests cost differently for the company. In the meantime, monitoring and examining the test is costly for the regulator. So, the regulator would only examine it if the likelihood that the result is positive and the precision of the test is high enough.

To better understand the above scenarios, we consider a dynamic model in which a sender sequentially persuades a receiver to take the desired action (denote it by action G) regardless of the unknown state. In contrast, the receiver only wishes to take that action if the state is good. The receiver could incur costs to search for more information. The updated belief about the state helps him make decisions. If the receiver observes bad news, he knows that the state is less likely to be good and will not take action G without the arrival of new information. If the receiver observes good news, he knows that the state is more likely to be good, and the expected payoff of taking action G increases. The sender designs the information structure. Given the endogenous information environment, the receiver trades off the benefit and cost of information acquisition and decides whether to search or not. The receiver is forward-looking and forms rational expectations of the sender's strategy. The

¹ We refer to the information provider as “she” and the decision-maker as “he” throughout the paper.

sender could incur higher costs to provide more information so that the receiver would be more likely to take action G . Given the information acquisition strategy of the receiver, the sender trades off the benefit and cost of information provision and decides how much information to provide. Therefore, the receiver and the sender are simultaneously trading off the benefit and cost of information acquisition/provision. At each period, the sender chooses the information structure, and the receiver chooses whether to search for more information or make a decision. This paper characterizes the optimal information provision strategy of the sender and the optimal information acquisition strategy of the receiver. In equilibrium, the sender uses one-shot experiments, which induce the receiver to take action G immediately upon observing good news. This way, the sender saves the expected search time and does not need to compensate the receiver for a higher expected search cost. The sender smoothens information over multiple periods if and only if the prior of the state is good is high. The sender extracts all the surplus from the receiver when she provides information in both periods while may leave some rent to the receiver when she provides information in only one period. Due to the convexity of the information provision cost, the sender has an incentive to smoothen the information provision over multiple periods. However, the longer expected search time of the receiver would discourage him from searching if the likelihood of getting a strictly positive ex-post payoff (taking action G at state g) is low. Hence, the sender only smoothens information over multiple periods if the prior is high. In addition, the optimal information structure fully reveals the state being good when the search cost is high and partially when the search cost is low due to the sender's tradeoff between the frequency and precision of the signal. We also compare the information structures between the payoff-maximizing one and the efficient one. When the search cost is high, there is no information distortion, and the optimal strategy of the sender is efficient. There could be upward or downward information distortion when the search cost is lower.

In the main model, the sender can commit to the current-period information structure but cannot promise to the receiver the information structure in the future. In the extension, we study the implications of dynamic commitment power. We find that the ability to commit to future strategies strictly benefits the sender when the search cost is high, while the benefit vanishes as the search cost approaches zero. We also consider the implications of discounting. When the search cost is high, the participation constraints of the receiver are hard to satisfy. Providing enough information to persuade the receiver to search dominates the force of discounting. So, the optimal strategy of the sender does not depend on the discount factor. When the search cost is low and the players become less patient, the present value of the second-period profit decreases and the first-period participation constraint becomes tighter. The sender has a stronger incentive to convert the receiver early. So, she provides less

information in the second period and more information in the first period. Moreover, we consider asymmetric discount factors in a model where the sender is patient while the myopic receiver only cares about the current-period payoff. We find that both players are worse off if the receiver is myopic rather than forward-looking. It implies that the sender should try to better inform the receiver of the possibility of multi-period information revelation and gradual learning. Lastly, we extend the two-period model to an infinite-period one and show that the main insights extend to the richer model. When the search cost approaches zero, the ability to smoothen the information over more extended periods is precious for the sender. She can obtain the equilibrium payoff as if the persuasion cost is zero. We also find that the receiver will decide within a bounded number of periods.

1.1 Related Literature

There is a large stream of literature on optimal information acquisition. In particular, consumer search has raised a growing interest both theoretically (Stigler 1961, Weitzman 1979, Moscarini and Smith 2001, Branco et al. 2012, Ke et al. 2016, and Ke and Villas-Boas 2019) and empirically (Hong and Shum 2006, Kim et al. 2010, 2017, Seiler 2013, Honka 2014, Ma 2016, Chen and Yao 2017, Honka and Chintagunta 2017, Seiler and Pinna 2017, Ursu et al. 2020, Moraga-González and Wildenbeest 2021). In the above papers, the information environment is exogenous. As the strategy and the outcome depend on the information structure, other payoff-relevant parties (e.g., the seller) have strong incentives to influence it. We take it into account by having the firm endogenously determine the information environment of the consumer.

A few papers study the design of the search environment. In Dukes and Liu (2016), consumers choose the breadth and depth of the search. The platform designs the search environment by selecting the search cost. Villas-Boas and Yao (2021) consider the optimal retargeting strategy of the firm which advertises to consumers with a high likelihood of considering the firm’s product. By doing advertising, the firm increases the frequency of consumers learning information and the ability to track consumers. In Zhong (2022), the platform recommends relevant sellers based on match values and prices to consumers. It designs the search algorithm by picking the match precision and the relative importance between prices and match values. Comparing the welfares among information structures emphasizing different vehicle characteristics under the counterfactuals of their structural model, Gardete and Hunter (2020) find that emphasizing the vehicle’s histories and obfuscating price information improves both consumer and firm welfare. In related literature on choice overload, the seller determines the amount of information provided to the consumer

(Kuksov and Villas-Boas 2010, Branco et al. 2016). The decision of the seller affects the consumer’s search cost. While the above papers are related to our paper, there are substantial differences. Instead of looking at specific parameters of the search environment, we study the design of the information environment under a general signal space. We characterize the optimal information structure among any feasible information policy. In the existing literature, the consumer either fully observes the value of a product/attribute or gets a noisy signal from an exogenously specified distribution (e.g., normal distribution). In contrast, the firm determines the distribution of the signal in our paper. The optimal information structure can be asymmetric and may not correspond to standard distributions.

We use a belief-based method of modeling information provision, first introduced by Aumann and Maschler (1995) and Kamenica and Gentzkow (2011) in the Bayesian persuasion and information design literature.² The sender picks a mean-preserving spread of the prior as the posterior belief, which simplifies the analyses. Some papers have studied the persuasion problem when either the receiver or the sender incurs costs. Wei (2021) considers a static persuasion problem in which a rationally inattentive receiver incurs information processing costs. Gentzkow and Kamenica (2014) extend the concavification approach of Kamenica and Gentzkow (2011) to the setting where the sender’s cost is posterior-separable. In our paper, the sender’s cost does not satisfy that assumption, and the receiver searches strategically. So, we cannot use the widely-used concavification method for our problem. Instead, we develop an optimal control method to solve the sender’s information design problem. The constraints of the non-linear program of the sender consist of multiple variables, making the optimization problem challenging. To make the problem tractable, we transform the program into a set of constrained programs, each with one constrained variable. We then select the global solution by comparing the local solutions of each program. Some recent papers extend the static setting of Bayesian persuasion to the dynamic one, but it is usually costless for the sender to provide information (Ely 2017, Ely and Szydlowski 2020, Iyer and Zhong 2021, Orlov et al. 2020). One exception is Che et al. (2020) in which the sender performs costly experimentation to persuade the receiver to take a particular action. The flow cost of providing the information is constant, and the type of experimentation available to the sender is limited. In our paper, the sender could speed up the receiver’s learning by incurring a higher cost. So, we could investigate the optimal tradeoff of the benefit and cost of information provision by the sender.

The remainder of the paper is organized as follows. Section 2 presents the main model. Section 3 characterizes the optimal information provision strategy and the equilibrium outcomes. Section 4 characterizes the efficient information provision strategy and summarizes

² See Bergemann and Morris (2019) for a survey.

the information distortion when the sender rather than the social planner designs the information structure. Section 5 includes several model extensions. Section 6 concludes.

2 The Model

2.1 States, Actions, and Payoffs

There are two players, a sender and a receiver, and two states, good (g) and bad (b). The receiver ultimately makes a binary decision between G and B . The sender wishes to persuade the receiver to take action G regardless of the state while the receiver wishes to match the decision with the state (taking action $G(B)$ when the state is $g(b)$). The payoffs of the decision for the players are the following:

(sender payoff, receiver payoff)	action G	action B
state g	(p, v_g)	$(0, 0)$
state b	(p, v_b)	$(0, 0)$

We assume that the sender's payoff when the receiver takes the desired action is positive, $p > 0$. The receiver's payoff is positive if he takes action G when the state is g , $v_g > 0$, and negative if he takes action G when the state is b , $v_b < 0$. Both players get zero payoff if the receiver takes the action B (which could be thought of as a quitting option). We further assume without loss of generality that $v_g = 1 + v_b$.³ Both the sender and the receiver do not know the state initially but have a common prior belief on it, $\mu_0 := \mathbb{P}(\text{the state is } g) \in (0, 1)$. Each period $t \in \{0, 1\}$, the sender determines and commits to the information structure of the experiment but cannot commit to the experiment in the future. The receiver could search for information (action S) before deciding. The information acquisition is costly but helps the receiver to make better decisions. If the receiver chooses to search, he would observe an experiment that reveals some information about the state. The sender does not observe the realization of the signal but can infer it from the receiver's action.⁴ The players update the belief after the realization of the experiment. The game ends whenever the receiver makes a decision (G or B). Figure 1 illustrates the timing of the game.

³ To see that assuming $v_g = 1 + v_b$ is WLOG, consider the following normalization. Let $v'_g = \frac{v_g}{v_g - v_b}$, $v'_b = \frac{v_b}{v_g - v_b}$, $p' = \frac{p}{v_g - v_b}$. Then, $v'_g - v'_b = 1$. We make this assumption to simplify the analyses and the presentation.

⁴ From Proposition 1, one can see that the sender can perfectly infer the realization of the signal from the receiver's action under the optimal signal structure. So, the receiver does not have private information.

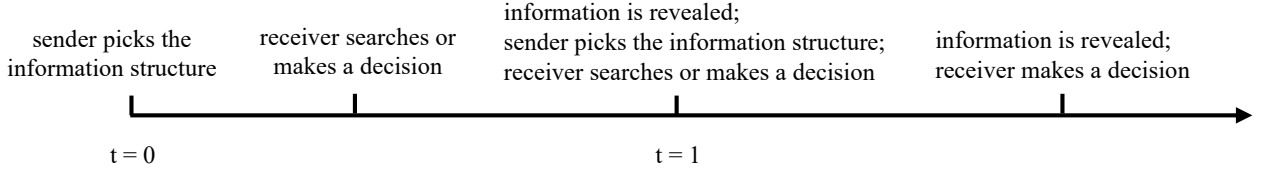


Figure 1: Timing of the Game

We model the experiment by a binary signal $s \in \{0, 1\}$.⁵ $\mathbb{P}[s = 1|g]$ and $\mathbb{P}[s = 1|b]$ uniquely determines the signal. We order the value of the signal such that $\mathbb{P}[s = 1|g] > \mathbb{P}[s = 1|b]$. Hence, $s = 1$ corresponds to good news and $s = 0$ corresponds to bad news. Analogous to Proposition 1 of Kamenica and Gentzkow (2011), we could work with mean-preserving posterior beliefs rather than the specific signal structure to simplify the analyses. Specifically, the existence of a binary signal s such that $\mathbb{P}[s = 1|g] > \mathbb{P}[s = 1|b]$ is equivalent to the existence of a binary-valued posterior belief whose expectation is equal to the prior.⁶

Denote the belief at the beginning of each period by μ_t . At each period, with probability λ_t , the receiver observes good news and the belief increases to $\bar{\mu}_t$. We refer to λ_t as the arrival rate of good news and $\bar{\mu}_t$ as the belief after observing good news. With probability $1 - \lambda_t$, the receiver observes bad news, and the belief decreases to $\underline{\mu}_t$. We refer to $\underline{\mu}_t$ as the belief after observing bad news.

We assume there is no discounting in the main model, and discuss the implication of discounting in the extension. The receiver incurs a flow cost of c per period of search. The sender's cost of information provision depends on the arrival rate of good news, $K = K(\lambda)$. The sender incurs a higher cost to provide information structures with more frequent good news. Throughout this paper, we refer to the arrival rate of good news as the amount of information. More information means more frequent good news. We make the following assumptions on the cost of the sender.

Assumption 1. $K(\cdot) \in \mathcal{C}^2(\mathbb{R}_+)$, $K'(\lambda) > 0$, $K''(\lambda) > 0$, $K(0) = 0$, $\lim_{\lambda \rightarrow 1^-} K'(\lambda) = +\infty$, $\lim_{\lambda \rightarrow 0^+} K'(\lambda) = 0$.

This assumption means that it is relatively cheap for the sender to provide a little information. The marginal cost of the sender increases at an increasing rate when the amount of information provided increases.

The total payoff for each player is the payoff of the decision net of the information provision/acquisition costs. The receiver is forward-looking and forms rational expectations about the sender's strategy in the future. We assume that $\mu_0 v_g + (1 - \mu_0) v_b < 0 \Leftrightarrow \mu_0 < -v_b$.

⁵ Since there are only two states, more than two signal values will not benefit the sender.

⁶ In the appendix, we state this result formally and provide proof.

So, the receiver would never take action G without searching. Otherwise, the sender would provide no information, and the receiver always takes the desired action. We also assume that the search cost is not too high, $c < v_g$. Otherwise, the receiver would never search.

2.2 Applications

2.2.1 Product Sales

A seller offers a product with unit demand. The price is p , and the marginal cost is m . The product may be a good match or a bad match with the buyer. Both the buyer and the seller do not know the state initially but have a common prior belief. The product is of zero value to the buyer if the state is bad and is of value v to the buyer if the state is good. The buyer decides to purchase the product or not. The payoffs of the decision for the players are the following:

(sender payoff, receiver payoff)	purchasing	not purchasing
state good match	$(p - m, v - p)$	$(0, 0)$
state bad match	$(p - m, -p)$	$(0, 0)$

The buyer could incur costs to search for more information about the product. The updated belief helps him make better purchasing decisions. If the buyer observes bad news, he knows that the product is less likely to be a good match and avoids wasting money on it without the arrival of new information. If the buyer observes positive news, he knows that the product is more likely to be a good match and the expected valuation increases. The buyer gets a positive surplus upon purchasing the good if the expected value is higher than the price.

2.2.2 New Drug Launches

A pharmaceutical company develops a new drug and tries to get approval from the regulator (e.g., the FDA) by designing and conducting tests to convince the regulator that the drug is safe. If the regulator approves it, the firm gains an expected profit of p . The regulator gains a positive utility if the drug is safe and a negative utility if it is unsafe. The payoffs of the decision for the players are the following:

(sender payoff, receiver payoff)	approval	disapproval
state safe	$(p, v_g > 0)$	$(0, 0)$
state unsafe	$(p, v_b < 0)$	$(0, 0)$

Monitoring and examining the test is costly for the regulator. So, the regulator would only investigate the test if both the likelihood that the result is positive and the precision of the test are high enough. Even if a single test fails to convince the regulator, the firm could offer another test, hoping that the result would finally convince the regulator to approve the drug.

2.3 Strategies and Equilibrium Concepts

Since the belief is common knowledge and there is no private information, we consider the sender-preferred subgame perfect equilibrium, as in Kamenica and Gentzkow (2011). If multiple actions (B , G , and S) give the receiver the same expected payoff, we assume that the receiver chooses an action that maximizes the sender's expected payoff. We also assume that the sender prefers to give the receiver more surplus in the first period if more than one equilibria lead to the same expected sender's payoff. A perturbation of very little discounting justifies this assumption.

2.3.1 Equilibrium of the Subgame

In the second period, the sender and the receiver play the subgame. Since it is costly for the sender to provide information, the sender would either give no information or provide enough information such that the receiver searches and would take action G if good news arrives. We illustrate the belief evolution of the subgame in Figure 2. Also, the distribution of the belief induced by the experiment should be a mean-preserving spread of the initial belief: $\mathbb{E}[\Delta\mu] = 0$. In sum, the sender either does not provide information and obtains zero payoffs or takes into account the following constraints when designing the information structure:

(1) participation constraint:

$$\lambda_1[\bar{\mu}_1 v_g + (1 - \bar{\mu}_1)v_b] = \lambda_1(\bar{\mu}_1 + v_b) \geq c \quad (IR_1)$$

(2) feasibility constraint:

$$\lambda_1 \bar{\mu}_1 + (1 - \lambda_1) \underline{\mu}_1 = \mu_1 \quad (F_1)$$

If the sender provides information, the constrained program of the sender is:

$$\begin{aligned} & \max_{\lambda_1, \bar{\mu}_1} -K(\lambda_1) + p\lambda_1 \\ \text{s.t. } & (IR_1), (F_1), \lambda_1 \in [0, 1], \bar{\mu}_1 \in [0, \mu_1] \end{aligned} \quad (P_1)$$

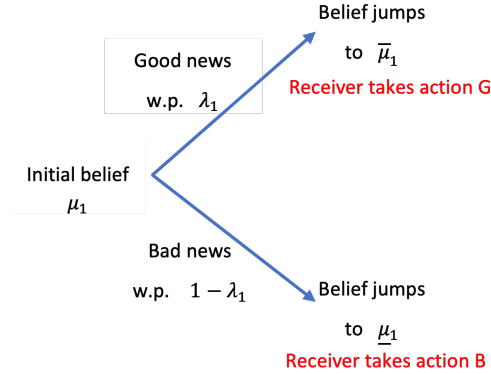


Figure 2: Belief Evolution of the Subgame

We analyze the solution to this problem in the next section. Though the information structure consists of $(\lambda_1, \bar{\mu}_1, \underline{\mu}_1)$, any two of them fully characterize the strategy as the third variable is then uniquely determined by (F_1) . Therefore, we use $(\lambda_1, \bar{\mu}_1)$, the arrival rate of good news and the belief after observing good news, to represent the sender's strategy of the subgame.

2.3.2 Equilibrium of the Full Game

The sender has three options. Firstly, she could provide no information and obtain zero payoffs. Secondly, she could provide information in only one period. If the receiver decides to search, he would observe a one-shot experiment designed by the sender. The receiver takes action G if good news arrives and takes action B if bad news comes. The sender would not provide extra information regardless of the experimental outcome. Her problem is exactly (P_1) (except that the subscript 1 shall be replaced by 0). Lastly, the sender could provide information in both periods. She could give a pair of one-shot experiments, which means that the receiver would take action G upon observing good news at either period. If bad news arrives in the first period, the sender will provide another experiment, hoping that good news will come in the second period. The sender could also offer a pair of iterative experiments. The receiver takes action G if and only if he observes good news at

both periods.⁷ We illustrate the belief evolution of the one-shot experiments and iterative experiments in Figure 3 and 4.

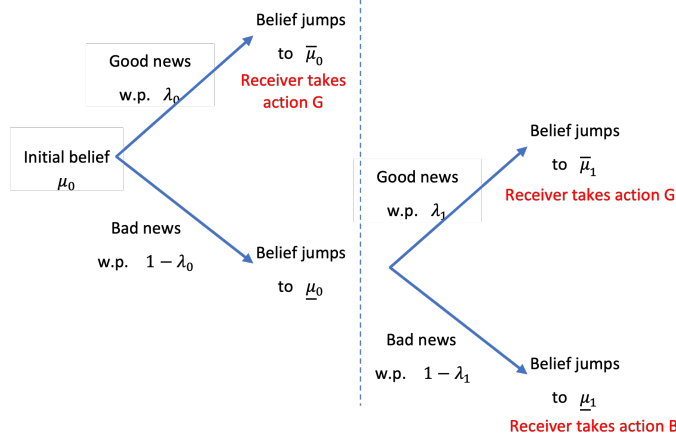


Figure 3: Belief Evolution of the One-shot Experiments

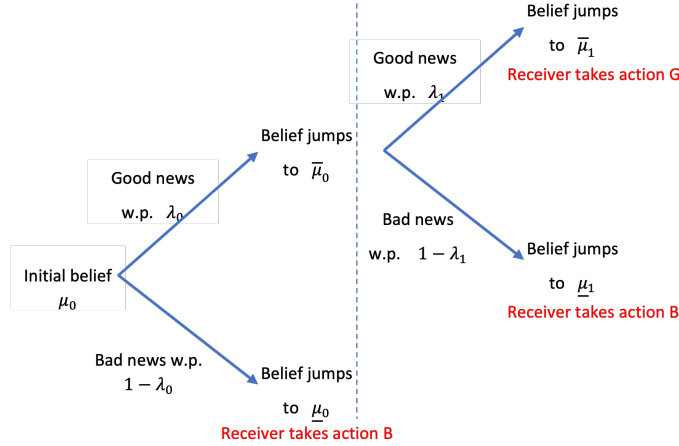


Figure 4: Belief Evolution of the Iterative Experiments

The receiver must search in both periods before taking action G under iterative experiments. Hence, compared to one-shot experiments, iterative experiments requires a longer search time. To compensate the receiver for the higher expected search costs, the sender needs to provide information more favorable to the receiver, which hurts the sender's payoff. The following result shows that the sender always prefers one-shot experiments in equilibrium. So, we limit our attention to the optimal one-shot experiments.

⁷ One can see that the receiver will take action B after observing bad news in the first period under iterative experiments. If instead, the receiver searches regardless of the realization of the signal in the first period, the sender can save the persuasion cost by providing the second-period experiment in the first period.

Proposition 1. *For any pair of feasible iterative experiments, there exists a one-shot experiment that gives the sender a strictly higher payoff.*

The sender takes into account the following constraints when designing the optimal one-shot experiment of the first period:

(1) participation constraint:

$$\begin{aligned} & \lambda_0[\bar{\mu}_0 v_g + (1 - \bar{\mu}_0)v_b] + (1 - \lambda_0)\mathbb{E}[\text{receiver surplus at } t = 1 | \text{search at } t = 1] \\ & = \lambda_0(\bar{\mu}_0 + v_b) + (1 - \lambda_0)[\lambda_1(\bar{\mu}_1 + v_b) - c] \geq c \end{aligned} \quad (IR_0)$$

(2) feasibility constraint:

$$\lambda_0 \bar{\mu}_0 + (1 - \lambda_0) \underline{\mu}_0 = \mu_0 \quad (F_0)$$

If the sender provides information in both periods, her problem is:⁸

$$\begin{aligned} & \max_{\lambda_0, \bar{\mu}_0, \mu_1, \lambda_1, \bar{\mu}_1} -K(\lambda_0) + p\lambda_0 + (1 - \lambda_0)[-K(\lambda_1) + p\lambda_1] \quad (P_2) \\ & \text{s.t. } (IR_0), (F_0), (\lambda_1, \bar{\mu}_1) \text{ solves } (P_1) \end{aligned}$$

We analyze the solution to this problem in the next section. We use $(\lambda_0, \bar{\mu}_0, \mu_1, \lambda_1, \bar{\mu}_1)$, the arrival rate of good news at each period, the belief after observing good news at each period, and the initial belief in the second period, to represent the sender's strategy of the entire game.

3 Optimal Strategies

From the previous discussion, the receiver would search (take action S) whenever the sender provides information. The receiver would take action G immediately upon receiving good news and B if the sender does not provide information. Therefore, we only need to characterize the sender's strategy, which implies the receiver's strategy.

⁸ We impose the following implicit assumptions on all of the programs in the remaining of the paper: $\bar{\mu}_t \in [-v_b, 1]$, $\mu_t \in [0, \mu_t)$, $\lambda_t \in [0, 1]$, $\underline{\mu}_0 = \mu_1$. The last equality comes from the fact that the belief at the beginning of the second period, μ_1 , is the belief after observing bad news in the first period, $\underline{\mu}_0$, under one-shot experiments.

3.1 A Benchmark

We first consider a benchmark problem in which the receiver always participates, and the sender could generate an arbitrary amount of information at each period. The sender chooses the information structure to maximize the expected payoff. We will use the solution throughout the subsequent analyses.

$$\max_{\lambda_0, \lambda_1} -K(\lambda_0) + p\lambda_0 + (1 - \lambda_0) [-K(\lambda_1) + p\lambda_1] \quad (P_b)$$

Lemma 1. *The solution to the benchmark problem (P_b) , $(\lambda_0^{**}, \lambda_1^{**})$, does not depend on the search cost c and $\lambda_0^{**} < \lambda_1^{**}$. The benchmark sender's payoff is strictly positive.*

The benchmark sender's payoff is the highest possible payoff the sender could obtain in equilibrium. When the prior is high enough, good news arrives at the rate of λ_t^{**} and the receiver searches. The sender obtains the benchmark payoff. When the prior is lower and good news arrives at the benchmark arrival rate, the receiver may be quite uncertain about the state even after observing good news (low $\bar{\mu}_t$) due to feasibility constraints. So, the receiver chooses not to search. This friction restricts the type of communication the sender and the receiver could have and distorts the optimal strategy. For the problem to be non-degenerate, we concentrate on the case in which the prior is not too high such that the optimal strategy is different from the benchmark solution throughout the paper.

3.2 Optimal Strategy of the Subgame

When the belief at the beginning of the second period is too low, the receiver will not search given any feasible experiments. Thus, the sender does not provide information to minimize the cost. When the belief at the beginning of the second period is higher, and the search cost is not too high, the sender provides information and obtains a positive payoff. The following proposition summarizes the optimal information structure.

Proposition 2. *At the second period, the sender does not provide information when $\mu_1 < \mu_{0,1} := c/v_g$. When $\mu_1 \geq \mu_{0,1}$, the optimal arrival rate of good news and the optimal belief after observing good news, $(\lambda_1^*, \bar{\mu}_1^*)$, depend on the search cost c :*

1. *If $c \geq v_g \lambda_1^{**}$, there exists a unique $\hat{c} \in (v_g \lambda_1^{**}, v_g \mu_1]$ such that the sender does not provide information if $c > \hat{c}$ and $(\lambda_1^*, \bar{\mu}_1^*) = (c/v_g, 1)$ if $c < \hat{c}$. The receiver gets zero surplus.*
2. *If $c \in [\mu_1 + v_b \lambda_1^{**}, v_g \lambda_1^{**})$, $(\lambda_1^*, \bar{\mu}_1^*) = (\frac{\mu_1 - c}{-v_b}, \frac{-v_b \mu_1}{\mu_1 - c})$. The receiver gets zero surplus.*

3. If $c < \mu_1 + v_b \lambda_1^{**} \wedge v_g \lambda_1^{**}$, $(\lambda_1^*, \bar{\mu}_1^*) = (\lambda_1^{**}, \frac{\mu_1}{\lambda_1^{**}} \wedge 1)$. The receiver gets strictly positive surplus.

When the search cost is too high, the sender has to provide a lot of information to persuade the receiver to search. Even if it is feasible for the sender to provide enough information that the receiver would search, it is so costly that the expected sender's payoff is negative. So, the sender chooses not to provide information, and the receiver does not search.

When the search cost is high but not too high, the sender will provide just enough information such that the receiver searches. Since the receiver's participation constraint is hard to satisfy, in equilibrium, the receiver becomes certain ($\bar{\mu}_1 = 1$) that the state is g after observing good news. Suppose, instead, a good signal does not fully reveal the state ($\bar{\mu}_1 < 1$), its arrival rate needs to be higher to persuade the receiver to search. Since the marginal cost of increasing the arrival rate of good news exceeds the marginal benefit, the sender's payoff decreases. The sender trades off the frequency of good news for precision.

When the search cost is intermediate, the receiver's participation constraint is easier to satisfy. Since the marginal benefit of increasing the arrival rate of good news exceeds the marginal cost, in equilibrium, the sender trades off the precision of good news for the frequency. The receiver is still uncertain about the state after observing good news. But the belief would be high enough such that the receiver searches.

When the search cost is low, the information friction does not distort the information structure. The sender provides the benchmark amount of information, and the receiver gets a strictly positive surplus.

3.3 Optimal Strategy of the Entire Game

When the prior is too low, any feasible signal the sender could generate is not attractive enough for the receiver to search. Thus, it is impossible to communicate between the sender and the receiver. When the prior is higher, and the search cost is not too high, the sender provides information and obtains a positive payoff.

Proposition 3. *Suppose the search cost is not too high, $c < \hat{c}$. There exists $\mu_{1,2} \geq c(2v_g - c)/(v_g)^2$ such that the sender does not provide information if the prior is low, $\mu_0 < \mu_{0,1}$, provides information in one period if $\mu_0 \in [\mu_{0,1}, \mu_{1,2})$, and provides information in both periods if $\mu_0 > \mu_{1,2}$. Suppose the sender provides information in both periods. Good news fully reveals the state (being g) when the search cost is high and partially reveals the state when the search cost is low. The receiver gets zero total surplus.*

Since the information provision cost is non-linear in the arrival rate of good news, the sender has an incentive to smoothen the information provision over two periods. When the prior is low, it is not feasible for the sender to provide enough information in both periods so that the receiver would search whenever good news has not arrived. As the prior increases, it becomes feasible for the sender to smoothen the information provision. If the sender finds it optimal to provide information in both periods at a given prior, she also prefers smoothening information for any higher prior.

When it is highly costly for the receiver to search, the optimal information structure fully convinces the receiver that the state is g when good news arrives. In equilibrium, the receiver obtains the highest possible surplus conditional on observing good news and making the purchase. Without providing this type of information favorable to the receiver, the sender cannot persuade the receiver to search. In contrast, when it is less costly for the receiver to search, the optimal information structure adds some noise to good news. In equilibrium, the receiver is not sure that the state is g after observing good news. The state may be b after the receiver takes action G . However, the likelihood of state g after good news is high enough to persuade the receiver to search. By adding some noise to good news, the sender could provide more frequent good news and increase her payoff without violating the feasibility constraint.

When the sender provides information in both periods, she can always extract surplus from the receiver and increases her payoff if the receiver gets a strictly positive surplus. If the sender faces an information under-provision issue, she can increase the payoff by increasing the arrival rate of good news and decreasing the belief after observing good news. If the sender faces an information over-provision issue, she can increase the payoff by reducing the arrival rate of good news and increasing the belief after observing good news. It implies that the receiver gets zero surpluses in equilibrium.

The optimal strategy is consistent with real-world examples. Consider the application of product sales in section 2.2.1 when a buyer decides whether to purchase a pair of shoes and searches on Google to gather some information. The search cost is relatively low, and the buyer is less likely to be sure that it is a good match even after obtaining good news and making the purchase. On the contrary, when a buyer visits a dealership and acquires information on the car of interest through talking to the dealer and test driving it, the search cost is relatively high as it is costly to travel to the physical store and spend a significant amount of time there. When the buyer decides to buy the car, he is usually confident that it is precisely a good match.

3.4 Comparative Statics

When the sender provides information in only one period, the optimal strategy has a closed-form solution and is easy to analyze. Here, we discuss the comparative statics when the sender provides information in both periods. The specific form of the optimal strategy depends on the relative size of the search cost. According to Proposition 2, the sender's strategy of the subgame is constant when the search cost is high (but not too high). When the search cost is lower, it depends on the belief at the beginning of the second period, μ_1 . When $\mu_1 > c - v_b \lambda_1^{**}$, the receiver expects to get a strictly positive surplus in the second period, and thus the first-period participation constraint is relaxed (denote the corresponding strategy by S_+ strategy). When $\mu_1 \in [c/v_g, c - v_b \lambda_1^{**}]$, the receiver expects to get zero surpluses in the second period (denote the corresponding strategy by S_0 strategy). In equilibrium, the sender endogenously determines whether to use the S_0 or S_+ strategy.

3.4.1 Comparative Statics about the Prior

When the search cost is high, the prior determines whether the sender smoothens information but does not affect the information structure conditional on the sender providing information. When the search cost is low, the prior affects the information structure monotonically. When the search cost is intermediate, the sender may switch from the S_0 strategy to the S_+ strategy as the prior increases. There could be a discrete jump in the optimal information structure. We leave the analysis of this case in the appendix.

Proposition 4. *Suppose the sender provides information in both periods. When the search cost is high, $v_g \lambda_1^{**} \leq c < \hat{c}$, good news fully reveals the state. The arrival rate of good news, λ_t^* , and the sender's payoff does not depend on the prior, $(\lambda_t^*, \bar{\mu}_t^*) = (c/v_g, 1)$. When the search cost is low, $c \leq \tilde{c} := v_g K'^{-1} [K(\lambda_1^{**})/\lambda_1^{**}]$, the arrival rate of good news, λ_t^* , is continuous and increases in the prior. The belief after observing good news, $\bar{\mu}_t^*$, is continuous and decreases in the prior. The sender's payoff strictly increases in the prior.*

The optimal information is perfectly smooth when the search cost is high, and the sender provides information in both periods. Since the participation constraint of the receiver is strong, good news fully reveals the state is g . As it is very costly for the receiver to acquire information, the minimal amount of information to persuade the receiver to search is high. The marginal cost of providing more information exceeds the marginal benefit. Even if the prior increases and it is feasible for the sender to provide more information, she would prefer not to do so. Hence, conditional on the sender providing information, the information structure does not depend on the prior.

When the search cost is low, and the sender provides information in both periods, she chooses between S_+ and S_0 strategies. Under the S_+ strategy, the receiver observes less frequent good news in the first period and more frequent good news in the second period. On average, he spends a longer time searching. Consequently, the sender has to provide information more favorable to the receiver to compensate for the higher expected total search cost, which hurts the sender's surplus. Therefore, the sender always chooses the S_0 strategy in equilibrium, and the optimal strategy is continuous in the prior. The sender faces an information under-provision issue in both periods. More frequent good news is feasible when the prior is higher. Even if the receiver becomes less sure about the state being good after observing good news, he would still search as long as the likelihood of receiving good news and earning a strictly positive surplus increases. In equilibrium, the sender trades off the precision of good news for the frequency as the prior increases.

3.4.2 Comparative Statics about Sender's Costs

Providing the same amount of information may cost the sender differently. To study the impact of sender's information provision costs on the optimal strategy, we rewritten the sender's cost function by $K(\lambda) = \eta \tilde{K}(\lambda)$, where $\tilde{K}(1/2) = 1$ for identification. It is more costly for the sender to provide information when η is larger. The following proposition summarizes the comparative statics of the optimal strategy about η .

Proposition 5. *Suppose the sender provides information in both periods. Her payoff strictly decreases in the relative cost of information provision, η . When the search cost is low, $c \leq \tilde{c}$, the sender provides (weakly) less information in the first period and (weakly) more information in the second period, as η increases. When the search cost is high, $c \geq v_g \lambda_1^{**}$, the optimal strategy of the sender does not depend on η .*

When the search cost is high, it is very costly for the receiver to search. The sender needs to provide a lot of information to persuade the receiver to search. As a result, the marginal cost of providing more information exceeds the marginal benefit. So, the sender provides the minimum amount of information for the receiver to search, which does not depend on the sender's cost. Hence, the optimal information structure does not depend on the relative cost of information provision.

When the search cost is low, and the relative cost of information provision increases, the marginal cost of providing information increases while the marginal benefit remains the same. Since the sender incurs the information provision cost in the first period for sure, she provides less information in the first period. This allows her to provide more information in

the second period when the information provision cost is not always incurred, and she faces information under-provision issues.

3.5 A Numerical Example

We illustrate the optimal strategy of the sender by a numerical example. The sender's cost function has the truncated quadratic form, $K(\lambda) = k\lambda^2/(1 - \lambda)$, which satisfies Assumption 1. In each figure below, we present the optimal strategy and the sender's payoffs from the optimal one-period, S_0 , and S_+ strategies⁹. The search cost is low in Figure 5. The sender always prefers S_0 strategy when she provides information in both periods. As illustrated, the arrival rates of good news at both periods, λ_0^* and λ_1^* , are continuous and increase in μ_0 . The beliefs after observing good news at each period, $\bar{\mu}_0^*$ and $\bar{\mu}_1^*$, are continuous and decrease in μ_0 . When the prior is lower than the intercept of the brown line, the sender prefers providing information in only one period to smoothening the information provision. When the search cost increases to the intermediate level (Figure 6), the sender always provides information in both periods as long as it is feasible. So, we do not plot the sender's payoff of providing information in only one period. The sender prefers S_0 strategy for small prior and switches to S_+ strategy as the prior increases. The optimal strategy is non-monotonic and discontinuous in μ_0 due to the switch. When the search cost further increases (Figure 7), the optimal information structure is perfectly smooth, and good news always fully reveals the state.

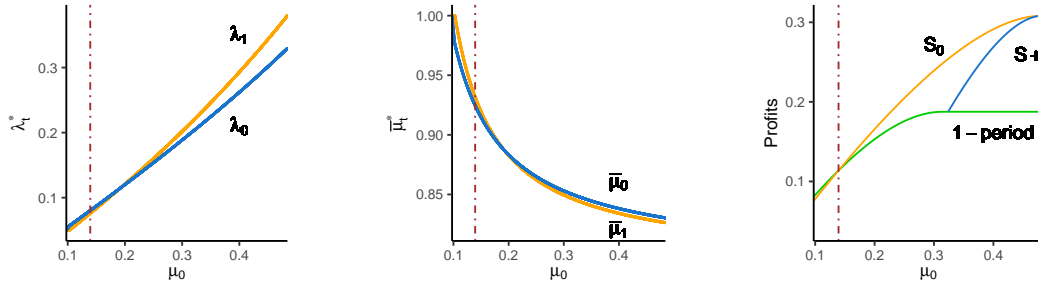


Figure 5: The optimal strategy when $c = 0.01, p = 0.8, v_g = 0.2, v_b = -0.8, k = 0.5$

⁹ The domain of the prior is $[c(2v_g - c)/(v_g)^2, \hat{\mu}_0 \wedge p]$. When $\mu_0 < c(2v_g - c)/(v_g)^2$, the sender provides information in at most one period. When $\mu_0 \geq \hat{\mu}_0 := 2c - v_b\lambda_1^{**} - [c + (1 - \lambda_1^{**})v_b]\lambda_0^{**}$, the sender always chooses the benchmark solution.

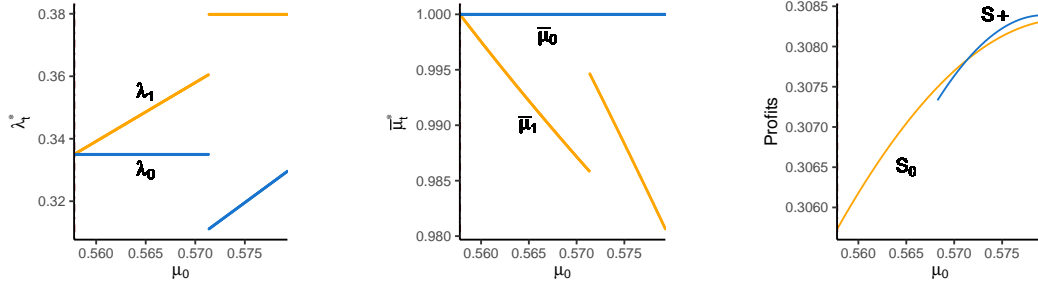


Figure 6: The optimal strategy when $c = 0.067, p = 0.8, v_g = 0.2, v_b = -0.8, k = 0.5$

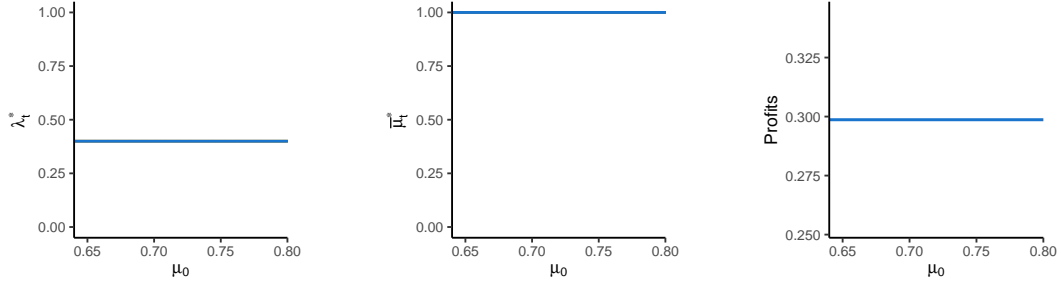


Figure 7: The optimal strategy when $c = 0.08, p = 0.8, v_g = 0.2, v_b = -0.8, k = 0.5$

4 The Efficient Information Structure

In all the previous sections, the sender designs the information structure to maximize the expected payoff. This section characterizes the efficient strategy when a social planner designs the information structure to maximize the total welfare. We then investigate the information distortion caused by not taking into account receiver surplus. For tractability reasons, we use a special form of the payoff function, specified in section 2.2.1. So, in this section, $v_g = 1 - p$ and $v_b = -p$.

4.1 Efficient Strategy of the Subgame

As discussed in the previous section, the sender does not provide information if the belief at the beginning of the second period is too low, $\mu_1 < \mu_{0,1}$. So, we concentrate on the case

in which $\mu_1 \geq \mu_{0,1}$. In the second period, the social planner's problem is:

$$\begin{aligned} & \max_{\lambda_1, \bar{\mu}_1} -K(\lambda_1) + p\lambda_1 + \lambda_1(\bar{\mu}_1 - p) - c \\ & \text{s.t. } (IR_1), (F_1) \end{aligned} \tag{E_1}$$

We discuss below the efficient information structure of the subgame intuitively. The formal characterization of the efficient strategy of the subgame is in the appendix. When the search cost is high, it is very costly for the receiver to search. The sender needs to provide a lot of information to persuade the receiver to search. As a result, the marginal cost of providing more information exceeds the marginal benefit. So, the sender provides the minimum amount of information for the receiver to search, not depending on whether the sender maximizes the sender's surplus or total welfare. Hence, there is no information distortion. When the search cost is lower, it is easier to persuade the receiver to search. The sender provides more than the minimum amount of information for the receiver to search. The marginal costs of providing more information are the same for both the sender and the social planner, while the marginal benefit of providing more information is smaller for the sender. Therefore, the sender provides less information than the social planner does when the search cost is high, or the initial belief is high. One exception is that when both the search cost and the initial belief are low, the sender provides more information than the social planner does because the social planner could only generate low-frequent signals.

4.2 Efficient Strategy of the Full Game

Like the previous section, the social planner does not provide information if the prior is too low or the search cost is too high. When she provides information in only one period, the previous subsection (and Proposition 15 in the appendix, where the subscript 1 needs to be replaced by 0) characterized the efficient strategy. When she provides information in both periods, her problem is:

$$\begin{aligned} & \max_{\lambda_0, \bar{\mu}_0, \mu_1, \lambda_1, \bar{\mu}_1} -K(\lambda_0) + \lambda_0\bar{\mu}_0 - c + (1 - \lambda_0)[-K(\lambda_1) + \lambda_1 - c] \\ & \text{s.t. } (IR_0), (F_0), (\lambda_1, \bar{\mu}_1) \text{ solves } (E_1) \end{aligned} \tag{E_2}$$

The following proposition compares the payoff-maximizing strategy and the efficient one when the sender provides information in both periods.

Proposition 6. *Suppose the sender provides information in both periods. When $c \geq v_g \lambda_1^{**}$, the payoff-maximizing strategy is efficient. When $c < v_g \lambda_1^{**}$, the sender, who maximizes*

the payoff, provides less information in the first period and more information in the second period than the social planner does, who maximizes the total welfare.

When the search cost is high, similar to the argument in the previous subsection, the sender provides the minimum amount of information for the receiver to search, which does not depend on the sender's objective (maximizing the sender surplus or total welfare). Hence, there is no information distortion. When the search cost is lower, it is easier to persuade the receiver to search. The sender provides more than the minimum amount of information for the receiver to search. The marginal costs of providing more information are the same for both the sender and the social planner, while the marginal benefit of providing more information is smaller for the sender. Therefore, the sender provides less information than the social planner does when the search cost is high, or the initial belief is high. One exception is that when both the search cost and the initial belief are low, the sender provides more information than the social planner does because the social planner could only generate low-frequent signals.

5 Model Extensions

5.1 Dynamic Commitment

Under many circumstances, the assumption that the sender could generate credible signals within each period but does not have dynamic commitment power is reasonable. It is hard for the sender to commit to the entire information structure across all periods ex-ante and convince the receiver that the sender would stick to the information structure when it is profitable to deviate to a different information structure during intermediate periods. However, factors such as reputation could give the sender a stronger commitment power. Here, we consider the case in which the sender has dynamic commitment power and study its implications. She chooses and commits to the entire information structure to maximize the ex-ante expected surplus.

$$\begin{aligned} & \max_{\lambda_0, \bar{\mu}_0, \mu_1, \lambda_1, \bar{\mu}_1} -K(\lambda_0) + p\lambda_0 + (1 - \lambda_0) [-K(\lambda_1) + p\lambda_1] & (P_{dc}) \\ \text{s.t. } & (IR_0), (F_0), (IR_1), (F_1) \end{aligned}$$

Proposition 7. *Suppose the search cost is high, $v_g \lambda_1^{**} < c < \hat{c}$, and $\mu_0 > c(2v_g - c)/(v_g)^2$. The sender provides information in both periods regardless of the dynamic commitment power. If the sender has dynamic commitment power, her payoff is strictly higher, and*

the receiver gets a strictly positive surplus in the second period. The benefit of dynamic commitment power for the seller vanishes as the search cost approaches zero.

When the search cost is high, we have shown that the sender would perfectly smoothen the information if she doesn't have dynamic commitment power. If the sender instead has dynamic commitment power, she would commit to providing information more favorable to the receiver in the second period. As a result, the receiver will search even if the sender provides less information in the first period, relaxing the information over-provision issue. Though it hurts the sender's payoff in the second period, it increases the sender's payoff in the first period by reducing the information provision cost. The overall effect is strictly positive. So, the optimal information provision would not be perfectly smooth. This finding is related to the result on durable good pricing. Without dynamic commitment power, the monopolist tends to reduce the price as time goes on, hurting the profit as a rational receiver would strategically wait. Here, dynamic commitment power also benefits the sender, but the underlying mechanisms differ. This paper focuses on persuasion (information provision) rather than incentive (pricing). In addition, in the durable good pricing example, the ability to commit not to provide a more favorable price in the future benefits the sender. While in this paper, the ability to commit to more favorable information in the future benefits the sender.

However, when the search cost approaches zero, the difference between the sender surplus with and without dynamic commitment power approaches zero. The intuition is that the benefit of dynamic commitment power comes from relaxing the participation constraint in the first period by committing to providing more favorable information in the second period. However, the participation constraint is already very loose when the search cost is low. Thus, The benefit of dynamic commitment power approaches zero.

5.2 Discounting

In the main model, we assume that there is no discounting. In reality, information acquisition and provision usually happen in a short period, and that assumption is reasonable. However, some communications between the sender and the receiver could take longer. Here, we study the information provision strategy when the sender and the receiver have the same

discount factor, $\delta \in (0, 1)$. With discounting, the sender's problem becomes:

$$\begin{aligned} \max_{\lambda_0, \bar{\mu}_0, \mu_1, \lambda_1, \bar{\mu}_1} & -K(\lambda_0) + p\lambda_0 + \delta(1 - \lambda_0)[-K(\lambda_1) + p\lambda_1] & (P_{2,\delta}) \\ \text{s.t. } & \lambda_0(\bar{\mu}_0 + v_b) + \delta(1 - \lambda_0)[\lambda_1(\bar{\mu}_1 + v_b) - c] \geq c & (IR_{0,\delta}) \\ & (F_0), (\lambda_1, \bar{\mu}_1) \text{ solves } (P_1) \end{aligned}$$

One can see that both the objective function and the first-period participation constraint change. When the players become less patient (the discount factor δ decreases), the present value of the second-period sender surplus decreases, and the first-period participation constraint becomes tighter. Thus, it is less attractive for the sender to sell the goods in the second period.

Proposition 8. *When the search cost is high, $c \geq v_g \lambda_1^{**}$, the optimal strategy of the sender does not depend on the discount factor, δ . When the search cost is low, $c \leq \tilde{c}$, the sender provides (weakly) more information in the first period and (weakly) less information in the second period, as the players become less patient.*

When the search cost is high, the participation constraints of the receiver are hard to satisfy. Providing enough information to persuade the receiver to search dominates the force of discounting. Therefore, the sender's strategy remains the same as the no-discounting case, and the sender perfectly smoothens information provision. When the search cost is low, the sender provides (weakly) less information in the second period when the players are less patient because of discounting. The sender provides more information in the first period as she becomes more tempted to convert the receiver early.

We assume in the main model that the receiver is forward-looking and takes into account the potential payoff of the second period when he chooses the action of the first period. It is reasonable if the information environment is transparent and the receiver knows that gradual learning is possible. Here, we consider the possibility of asymmetric discount factors. In particular, the sender is perfectly patient while the receiver is perfectly impatient (myopic). If the receiver is myopic, then he only trades off the current-period benefit and cost in deciding whether to search or not. The information structure in the second period cannot relax the first-period participation constraint. Therefore, when the receiver is myopic, the feasible information structure is a subset of when the receiver is forward-looking. If the receiver surplus in the second period is strictly positive when the receiver is forward-looking, the optimal information structure may not be feasible when the receiver is myopic. Hence, the sender is (weakly) worse off if the receiver is myopic rather than forward-looking. This result has managerial implications as it suggests that the sender should try to better inform

the receiver of the possibility of gradual learning. Common knowledge of gradual information revelation improves the sender surplus.

5.3 Infinite Number of Periods

The two-period model can capture information smoothing and gradual learning. We extend the main model to a model with an infinite number of periods and show that the main insights extend to this richer model. We also gain some additional insights.

Time is discrete, $t = 0, 1, 2, \dots$. Each period, the sender determines and commits to the information structure of the experiment but cannot commit to the experiment in the future. The receiver could search for information before deciding. Different from the two-period model, there is no deadline. The receiver can search for however long. Since the sender's payoff is bounded by p , the payoff function is well-defined even without the discount factor. So, we do not consider discounting for consistency with the main model. We can analyze the problem similarly if we include discounting.

Proposition 9. *The receiver searches for at most $\lfloor \frac{v_g \mu_0}{c} \rfloor$ periods. When the search cost is high, $v_g \lambda_1^{**} \leq c < \hat{c}$, the sender provides perfectly smooth information for $k := \lfloor \frac{\ln(1-\mu_0)}{\ln(1-c/v_g)} \rfloor$ periods. Good news fully reveals the state, $(\lambda_t^*, \bar{\mu}_t^*) = (c/v_g, 1)$, for $t = 0, 1, \dots, k-1$. When the search cost is low, good news partially reveals the state. The sender's equilibrium payoff when it is costless for her to provide information is μ_0 . For any persuasion costs satisfying assumption 1, the sender can achieve that payoff as the search cost approaches zero.*

This proposition shows that the main insights are robust to the specification of the length of time. The two-period model corresponds to the case when there is a deadline in the information acquisition. When time is infinite, there is no limit on how long the receiver can search. Under high search costs, the optimal information structure fully convinces the receiver that the state is g when good news arrives. Under low search costs, the optimal information structure adds some noise to good news. The same intuition for proposition 3 applies. Because the receiver can keep searching for a longer period, the sender can better smoothen the information. When the search cost is high, the sender may provide information for more than two periods if she believes that the state is likely to be g and the receiver is willing to spend more time searching. When the search cost approaches zero, the ability to smoothen the information is precious for the sender. She can obtain the equilibrium payoff as if the persuasion cost is zero. Because of the low search cost, the sender can convince the receiver to search with very little information in each period. The sender's cost becomes very low by smoothening the information over many periods.

The model predicts that the receiver will decide within a bounded number of periods under endogenous information structures. On the contrary, consumers may be stuck in the continuation region and keep searching when the information environment is exogenous (e.g., Moscarini and Smith 2001, Branco et al. 2012). Too much search hurts not only the consumer but also the firm. The consumer incurs a high search cost when he spends a long time searching. To persuade the consumer to keep searching, the firm must incur higher persuasion costs. In equilibrium, the firm will design the information structure such that the consumer searches for a limited period.

6 Concluding Remarks

This paper examines the optimal information provision strategy of a sender and the optimal information acquisition strategy of a receiver when the sender sequentially persuades a receiver to take the desired action. The sender prefers that action regardless of the unknown state, while the receiver only wishes to take that action if the state is good. In our model, the sender incurs a cost to provide information, and the receiver incurs a cost to search. The receiver trades off the cost of searching and the benefit of obtaining more accurate information to make better decisions. The sender trades off the cost of information provision and the benefit of persuading the receiver to search and take the desired action. We allow for sequential information provision and acquisition, so the communication between the receiver and the sender is gradual. Consequently, the sender also makes the intertemporal tradeoff of smoothening the information to reduce the total cost. In equilibrium, she uses one-shot experiments that induce the receiver to take the desired action upon observing good news immediately. The sender smoothenes information over multiple periods if and only if the prior on the state being good is high. The sender extracts all the surplus from the receiver when she provides information in both periods while may leave some rent to the receiver when she provides information in only one period. When the search cost of the receiver is high, the receiver is sure that the state is good when he takes the desired action. When the search cost is low, the optimal information structure does not fully reveal the state, which may be bad despite the receiver taking the desired action. We compare the payoff-maximizing information structure with the efficient information structure and find no information distortion when the search cost is high. There could be upward or downward information distortion when the search cost is lower.

This paper endogenizes the consumer's information environment from the firm's perspective. Instead of looking at specific parameters of the search environment, we study the design of the information environment under general signal space. We characterize the optimal in-

formation structure among any feasible information policy. The implementation of a given signal depends on the institutional details of the specific problem. Further empirical works can complement the current paper by putting the theoretical results into practice. In addition, it would be interesting to study the optimal persuasion strategy when there is more than one sender. The competition may lead the sender to provide more information and improve equilibrium efficiency. Moreover, the sender has complete control of the information structure in this paper. It would be interesting to consider the case when the sender could only partially control the information environment.

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Appendix

The following proposition formalizes the claim that we could work with mean-preserving posterior beliefs rather than the specific signal structure in section 2.1.

Proposition 10. (*Equivalent Representations of the Signal*) *The following are equivalent:*

1. *There exists a binary signal $s \in \Delta(\{0, 1\})$ such that $\mathbb{P}[s = 1|g] > \mathbb{P}[s = 1|b]$.*
2. *There exists a binary-valued posterior belief whose expectation is equal to the prior.*

Proof of Proposition 10. $1 \Rightarrow 2$: Given a binary signal such that $\mathbb{P}[s = 1|g] > \mathbb{P}[s = 1|b]$, law of iterated expectation implies that $\mathbb{E}[\mathbb{P}[g|s]] = \mathbb{E}[\mathbb{E}[\mathbf{1}_{[g]}|s]] = \mathbb{E}[\mathbf{1}_{[g]}] = \mathbb{P}[g]$. So, the expectation of the posterior belief is equal to the prior. Note that $\mathbb{P}[s = 0|g] < \mathbb{P}[s = 0|b]$. By Bayes' rule, $\mathbb{P}[g|s = 1] = \frac{\mathbb{P}[s=1|g]\mathbb{P}[g]}{\mathbb{P}[s=1|g]\mathbb{P}[g] + \mathbb{P}[s=1|b]\mathbb{P}[b]} > \frac{\mathbb{P}[s=1|g]\mathbb{P}[g]}{\mathbb{P}[s=1|g]\mathbb{P}[g] + \mathbb{P}[s=1|g]\mathbb{P}[b]} = \mathbb{P}[g] = \frac{\mathbb{P}[s=0|g]\mathbb{P}[g]}{\mathbb{P}[s=0|g]\mathbb{P}[g] + \mathbb{P}[s=0|b]\mathbb{P}[b]} > \frac{\mathbb{P}[s=0|g]\mathbb{P}[g]}{\mathbb{P}[s=0|g]\mathbb{P}[g] + \mathbb{P}[s=0|b]\mathbb{P}[b]} = \mathbb{P}[g|s = 0]$. So, the posterior belief is binary-valued.

$2 \Rightarrow 1$: Given a binary-valued posterior belief whose expectation is equal to the prior, μ_0 . Denote the distribution of the belief by $\mu = \begin{cases} \bar{\mu}_1 > \mu_0 & w.p. \lambda_1 \\ \underline{\mu}_1 < \mu_0 & w.p. 1 - \lambda_1 \end{cases}$. We now construct a binary signal $s \in \Delta(\{0, 1\})$. Define $\mathbb{P}[s = 1|g] = \frac{\bar{\mu}_1 \lambda_1}{\mu_0}$ and $\mathbb{P}[s = 1|b] = \frac{(1 - \bar{\mu}_1) \lambda_1}{(1 - \mu_0)}$. One can verify by Bayes' rule that this signal s induces exactly the same posterior belief, using the assumption that $\mu_0 = \lambda_1 \bar{\mu}_1 + (1 - \lambda_1) \underline{\mu}_1$. We just need to show that $\mathbb{P}[s = 1|g] > \mathbb{P}[s = 1|b]$, which follows from the fact that $\bar{\mu}_1 > \mu_0$. \square

Proof of Proposition 1. We first characterize the optimal one-period strategy (providing an one-shot experiment) of the sender. Analogous to section 2.3.1, the sender's problem is:

$$\begin{aligned}
 & \max_{\lambda_0, \bar{\mu}_0} -K(\lambda_0) + p\lambda_0 & (P_0) \\
 \text{s.t. } & \lambda_0(\bar{\mu}_0 + v_b) \geq c & (IR'_0) \\
 & \lambda_0 \bar{\mu}_0 + (1 - \lambda_0) \underline{\mu}_0 = \mu_0, & (F_0) \\
 & \lambda_0 \in [0, 1], \underline{\mu}_0 \in [0, \mu_0)
 \end{aligned}$$

We transform (P_0) into an equivalent constrained program that is easier to analyze.

Lemma 2. If $\mu_0 < c/v_g$, the sender does not provide information in the second period. If $\mu_0 \geq c/v_g$, (P_0) is equivalent to:

$$\begin{aligned} \Pi_1(\mu_0) &:= \max -K(\lambda_0) + p\lambda_0 \\ \text{s.t. } \lambda_0 &\in \left[\frac{c}{v_g}, \frac{\mu_0 - c}{-v_b} \right] \end{aligned} \quad (P'_0)$$

Proof. We first show that any (λ_0, μ_0) satisfying the constraints in (P_0) , (IR'_0) , (F_0) , also satisfy the constraints in (P'_0) :

$$(IR'_0) \Rightarrow \lambda_0 \geq \frac{c}{\mu_0 + v_b} \geq \frac{c}{v_g}. \quad (IR'_0) \ \& \ (F_0) \Rightarrow \lambda_0 \leq \frac{\mu_0 - c}{-v_b - \mu_0} \leq \frac{\mu_0 - c}{-v_b}. \quad \text{Thus, } \lambda_0 \in \left[\frac{c}{v_g}, \frac{\mu_0 - c}{-v_b} \right].$$

It is feasible for the sender to provide information in the second period iff $\left[\frac{c}{v_g}, \frac{\mu_0 - c}{-v_b} \right]$ is non-empty: $\frac{c}{v_g} \leq \frac{\mu_0 - c}{-v_b} \Leftrightarrow \mu_0 \geq \frac{c}{v_g}$. So, If $\mu_0 < \frac{c}{v_g}$, the sender would not provide information in the second period.

We then show that for any (λ_0, μ_0) satisfying the constraints in (P'_0) and $\mu_0 \geq \frac{c}{v_g}$, we could find $\bar{\mu}_0, \underline{\mu}_0$ such that $(\lambda_0, \mu_0, \bar{\mu}_0, \underline{\mu}_0)$ satisfies the constraints in (P_0) . The conclusion then follows.

Suppose (λ_0, μ_0) satisfies the constraints in (P'_0) : $\lambda_0 \in \left[\frac{c}{v_g}, \frac{\mu_0 - c}{-v_b} \right]$, $\mu_0 \geq \frac{c}{v_g}$. Consider $\bar{\mu}_0 = \frac{c}{\lambda_0} - v_b$ and $\underline{\mu}_0 = \frac{\mu_0 - c + v_b \lambda_0}{1 - \lambda_0}$. One can verify that $(\lambda_0, \mu_0, \bar{\mu}_0, \underline{\mu}_0)$ satisfies (IR'_0) & (F_0) . So, we just need to show that $-v_b \leq \bar{\mu}_0 \leq 1$ and $\underline{\mu}_0 \geq 0$. $\bar{\mu}_0 = \frac{c}{\lambda_0} - v_b \geq -v_b$. $\lambda_0 \geq \frac{c}{v_g} \Rightarrow \bar{\mu}_0 = \frac{c}{\lambda_0} - v_b \leq 1$. $\lambda_0 \leq \frac{\mu_0 - c}{-v_b} \Rightarrow \mu_0 \geq c - v_b \lambda_0 \Rightarrow \underline{\mu}_0 = \frac{\mu_0 - c + v_b \lambda_0}{1 - \lambda_0} \geq 0$. \square

Now consider the transformed program (P'_0) when $\mu_0 \geq \frac{c}{v_g}$.

1. If $c \geq v_g \lambda_1^{**}$ (i.e. $\lambda_1^{**} \leq \frac{c}{v_g}$) and the sender provides information, then $\lambda_0^* = \frac{c}{v_g}$ due to strict concavity of the objective function. One can show that $(\lambda_0, \bar{\mu}_0, \underline{\mu}_0) = (\frac{c}{v_g}, 1, \frac{\mu_0 v_g - c}{v_g - c})$ is the only feasible information structure that satisfies (IR'_0) and (F_0) . Thus, the sender would provide information with $(\lambda_0, \bar{\mu}_0, \underline{\mu}_0) = (\frac{c}{v_g}, 1, \frac{\mu_0 v_g - c}{v_g - c})$ iff the sender surplus, $-K(\frac{c}{v_g}) + p \cdot \frac{c}{v_g}$, is positive (when it is 0, the sender is indifferent between providing information or not). Let $f(\tilde{c}) = -K(\frac{\tilde{c}}{v_g}) + p \cdot \frac{\tilde{c}}{v_g}$. We have $f(0) = 0$, f is strictly concave and obtains the maximum at $\tilde{c}^* = v_g \lambda_1^{**} < c < 1$. In addition, $f(v_g) < 0$ because $\lim_{\lambda \rightarrow 1} K'(\lambda) = +\infty$. Therefore, there exists a unique

$\hat{c} \in (v_g \lambda_1^{**}, v_g)$ s.t. $f(c) \begin{cases} \geq 0, & \text{if } 0 \leq c \leq \hat{c} \\ < 0, & \text{if } c > \hat{c} \end{cases}$. Moreover, when the sender provides information, $\mu_0 \geq \frac{c}{v_g} \Rightarrow \hat{c} \leq \mu_0 v_g$. So, the sender does not provide information if $c > \hat{c}$ and provides information with $(\lambda_0^*, \bar{\mu}_0^*) = (\frac{c}{v_g}, 1)$ if $c < \hat{c}$. The receiver surplus is 0.

2. If $c \in [\mu_0 + v_b \lambda_1^{**}, v_g \lambda_1^{**})$ (i.e. $\lambda_1^{**} \geq \frac{\mu_0 - c}{-v_b} > \frac{c}{v_g}$) and the sender provides information, then $\lambda_0^* = \frac{\mu_0 - c}{-v_b}$ due to strict concavity of the objective function. One can show

that $(\lambda_0, \bar{\mu}_0, \underline{\mu}_0) = (\frac{\mu_0 - c}{-v_b}, \frac{-\mu_0 v_b}{\mu_0 - c}, 0)$ is the only feasible information structure that satisfies (IR'_0) and (F_0) . Thus, the sender would provide information with $(\lambda_0, \bar{\mu}_0, \underline{\mu}_0) = (\frac{\mu_0 - c}{-v_b}, \frac{-\mu_0 v_b}{\mu_0 - c}, 0)$ iff the sender surplus, $-K(\frac{\mu_0 - c}{-v_b}) + p \cdot \frac{\mu_0 - c}{-v_b}$, is positive. Since $-K(0) + p \cdot 0 = 0$, $\frac{\mu_0 - c}{-v_b} < \lambda_1^{**}$, and the objective function is strictly concave, the sender surplus is always strictly positive. So, the sender would always provide information. The receiver surplus is 0.

3. If $c < \mu_0 + v_b \lambda_1^{**} \wedge v_g \lambda_1^{**}$ (i.e. $\lambda_1^{**} \in (\frac{c}{v_g}, \frac{\mu_0 - c}{-v_b})$), then the sender could obtain the maximum possible payoff by setting $(\lambda_0, \bar{\mu}_0) = (\lambda_1^{**}, \frac{\mu_0}{\lambda_1^{**}} \wedge 1)$. Let $\underline{\mu}_0 = \begin{cases} 0, & \text{if } \mu_0 \leq \lambda_1^{**} \\ \frac{\mu_0 - \lambda_1^{**}}{1 - \lambda_1^{**}}, & \text{if } \mu_0 > \lambda_1^{**} \end{cases}$.

One can verify that $(\lambda_0, \bar{\mu}_0, \underline{\mu}_0)$ is feasible and satisfies (IR'_0) and (F_0) . We have shown in the proof of Lemma 1 that the sender surplus is strictly positive. So, the sender would provide information and $(\lambda_0^*, \bar{\mu}_0^*) = (\lambda_1^{**}, \frac{\mu_0}{\lambda_1^{**}} \wedge 1)$. The receiver surplus is

$$\begin{cases} \mu_0 + v_b \lambda_1^{**} - c, & \text{if } \mu_0 \leq \lambda_1^{**} \\ \lambda_1^{**} v_g - c, & \text{if } \mu_0 > \lambda_1^{**} \end{cases} > 0.$$

If $c \geq v_g \lambda_1^{**}$, or $c < v_g \lambda_1^{**}$ and $\mu_1 \leq c - v_b \lambda_1^{**}$, the expected receiver surplus in the second period is 0. The receiver incurs search cost without any immediate benefit in the first period under iterative signals. The expected receiver surplus in the first period is strictly negative if he searches. Therefore, he would not search, and iterative signals are not feasible. Now we consider the case in which $c < v_g \lambda_1^{**}$ and $\mu_1 > c - v_b \lambda_1^{**}$.

The sender's problem when she uses iterative signals is:

$$\begin{aligned} \Pi_{iter}(\mu_0) &:= \max -K(\lambda_0) + \lambda_0 [-K(\lambda_1^{**}) + p \lambda_1^{**}] & (P_{iter}) \\ \text{s.t. } \lambda_1(\bar{\mu}_1 + v_b) &\geq \frac{1 + \lambda_0}{\lambda_0} c & (IR_{0,iter}) \\ \lambda_1(\bar{\mu}_1 + v_b) &\geq c & (IR_{1,iter}) \\ \lambda_0 \bar{\mu}_0 + (1 - \lambda_0) \underline{\mu}_0 &= \mu_0 & (F_0) \\ \lambda_1 \bar{\mu}_1 + (1 - \lambda_1) \underline{\mu}_1 &= \mu_1 & (F_1) \\ \mu_1 &= \bar{\mu}_0, \lambda_1 = \lambda_1^{**} \end{aligned}$$

Note that $(IR_{0,iter})$ implies (IR_1^{iter}) and $(\lambda_1^*, \bar{\mu}_1^*) = (\lambda_1^{**}, \frac{\mu_1}{\lambda_1^{**}})$ satisfies (F_1) . $\forall \mu_1 \geq \lambda_1^{**}$, the optimal second-period strategy is always $(\lambda_1^*, \bar{\mu}_1^*) = (\lambda_1^{**}, 1)$. Therefore, choosing μ_1 above λ_1^{**} does not increase the second-period sender's payoff or relax the first-period constraints. So, we could restrict μ_1 to be less than or equal to λ_1^{**} .

- i) $\mu_0 \geq c - v_b \lambda_1^{**}$

$$\Pi_1(\mu_0) = -K(\lambda_1^{**}) + p\lambda_1^{**} > \Pi_{iter}(\mu_0).$$

ii) $\mu_0 < c - v_b\lambda_1^{**}$

$$\Pi_1(\mu_0) = -K\left(\frac{\mu_0 - c}{-v_b}\right) + \frac{(\mu_0 - c)p}{-v_b}$$

$$\begin{aligned} (IR_{0,iter}) \Rightarrow \lambda_0 &\geq \frac{c}{\mu_1 - c + v_b\lambda_1^{**}} (\Rightarrow \mu_1 \geq \frac{1 + \lambda_0}{\lambda_0}c - v_b\lambda_1^{**}) \\ &\geq \frac{c}{v_g\lambda_1^{**} - c} \\ (F_0) \Rightarrow \lambda_0 &= \frac{\mu_0 - \underline{\mu}_0}{\mu_1 - \underline{\mu}_0} \leq \frac{\mu_0}{\mu_1} \stackrel{(1)}{\leq} \frac{\mu_0}{\frac{1 + \lambda_0}{\lambda_0}c - v_b\lambda_1^{**}} \Rightarrow \lambda_0 \leq \frac{\mu_0 - c}{c - v_b\lambda_1^{**}} \end{aligned} \quad (1)$$

A necessary condition for λ_0 to be well-defined is:

$$\frac{c}{v_g\lambda_1^{**} - c} \leq \frac{\mu_0 - c}{c - v_b\lambda_1^{**}} \Leftrightarrow \mu_0 \geq \frac{\lambda_1^{**}c}{v_g\lambda_1^{**} - c} (> \frac{c}{v_g})$$

Therefore, it is feasible for the sender to provide a one-period signal whenever it is feasible to provide iterative signals.

Define $\bar{\Pi}_{iter}(\mu_0) := \max_{0 \leq \lambda_0 \leq \frac{\mu_0 - c}{c - v_b\lambda_1^{**}}} -K(\lambda_0) + \lambda_0 [-K(\lambda_1^{**}) + p\lambda_1^{**}]$. One can see that $\bar{\Pi}_{iter}(\mu_0) \geq \Pi_{iter}(\mu_0)$. Let $\lambda_0^*(\mu_0) := \arg\max_{0 \leq \lambda_0 \leq \frac{\mu_0 - c}{c - v_b\lambda_1^{**}}} -K(\lambda_0) + \lambda_0 [-K(\lambda_1^{**}) + p\lambda_1^{**}]$.

We want to show that:

$$\begin{aligned} \bar{\Pi}_{iter}(\mu_0) &< \Pi_1(\mu_0) \\ \Leftrightarrow -K(\lambda_0^*(\mu_0)) + \lambda_0^*(\mu_0) [-K(\lambda_1^{**}) + p\lambda_1^{**}] &< -K\left(\frac{\mu_0 - c}{-v_b}\right) + \frac{\mu_0 - c}{-v_b}p \\ \frac{d}{d\lambda_0} \{-K(\lambda_0) + \lambda_0 [-K(\lambda_1^{**}) + p\lambda_1^{**}]\} &= -K'(\lambda_0) - K(\lambda_1^{**}) + p\lambda_1^{**} \\ \Rightarrow \lambda_0^*(\mu_0) &= \begin{cases} \frac{\mu_0 - c}{c - v_b\lambda_1^{**}} & \text{if } \mu_0 \leq \mu_0^t \\ \frac{\mu_0^t - c}{c - v_b\lambda_1^{**}} & \text{if } \mu_0 > \mu_0^t \end{cases} \end{aligned} \quad (2)$$

, where $\lambda_0^t > 0$ is defined by $-K'(\lambda_0^t) - K(\lambda_1^{**}) + p\lambda_1^{**} = 0$, $\mu_0^t = \lambda_0^t(c - v_b\lambda_1^{**}) + c$.

$$\text{When } \mu_0 \leq \mu_0^t, \begin{cases} \lambda_0^*(\mu_0) = \frac{\mu_0 - c}{c - v_b\lambda_1^{**}} > \frac{\mu_0 - c}{-v_b} \Rightarrow -K(\lambda_0^*(\mu_0)) < -K\left(\frac{\mu_0 - c}{-v_b}\right) \\ \lambda_0^*(\mu_0)p\lambda_1^{**} = \frac{\mu_0 - c}{c - v_b\lambda_1^{**}}p\lambda_1^{**} < \mu_0 - c \\ -\lambda_0^*(\mu_0)K(\lambda_1^{**}) < 0 \end{cases} \Rightarrow (2) \text{ holds,}$$

where the first inequality holds because $-v_b > \mu_1 > c - v_b\lambda_1^{**}$.

When $\mu_0 > \mu_0^t$, $\bar{\Pi}_{iter}(\mu_0) = \bar{\Pi}_{iter}(\mu_0^t) < \Pi_1(\mu_0^t) < \Pi_1(\mu_0)$. So, (2) holds.

Thus, we have shown that $\Pi_1(\mu_0) > \bar{\Pi}_{iter}(\mu_0)$ for any μ_0 such that iterative signals are feasible. \square

Proof of Lemma 1. λ_0^{**} and λ_1^{**} are determined by the first order conditions: $-K'(\lambda_1^{**}) + p = 0$ and $-K'(\lambda_0^{**}) + p + K(\lambda_1^{**}) - p\lambda_1^{**} = 0$. $-K(0) + p \cdot 0 = 0$ & $-K'(\lambda) + p > 0$ for small $\lambda \Rightarrow -K(\lambda_1^{**}) + p\lambda_1^{**} > 0$. Therefore, $-K'(\lambda_0^{**}) + p + K(\lambda_1^{**}) - p\lambda_1^{**} = 0$ implies that $-K'(\lambda_0^{**}) + p > 0 = -K'(\lambda_1^{**}) + p, \Rightarrow K'(\lambda_0^{**}) < K'(\lambda_1^{**}) \Rightarrow \lambda_0^{**} < \lambda_1^{**}$. $-K(0) + p \cdot 0 + (1 - 0)[-K(\lambda_1^{**}) + p\lambda_1^{**}] > 0$ and strict concavity (w.r.t. λ_0) of the objective function imply that $-K(\lambda_0^{**}) + p\lambda_0^{**} + (1 - \lambda_0^{**})[-K(\lambda_1^{**}) + p\lambda_1^{**}] > 0$. \square

Proof of Proposition 2. It can be proved in the same way as in the proof of Proposition 1 when we derive the optimal one-period strategy of the sender, except that we need to replace the subscript 0 by 1 here. \square

Proof of Proposition 3.

$$(1) v_g \lambda_1^{**} \leq c < \hat{c}$$

If the sender provides information in both periods, the constrained program of the sender is:

$$\begin{aligned} \max & -K(\lambda_0) + p\lambda_0 + (1 - \lambda_0) \left[-K\left(\frac{c}{v_g}\right) + \frac{cp}{v_g} \right] \\ \text{s.t. } & (IR_0), (F_0), \mu_1 \geq \frac{c}{v_g} \end{aligned} \quad (P_{2H})$$

We first transform (P_{2H}) into an equivalent constrained program that is easier to analyze.

Lemma 3. Suppose $v_g \lambda_1^{**} \leq c < \hat{c}$. If $\mu_{0,1} \leq \mu_0 < \frac{2v_g - c}{(v_g)^2}c$, the sender provides information in one period. If $\mu_0 \geq \frac{2v_g - c}{(v_g)^2}c$, (P_{2H}) is equivalent to:

$$\begin{aligned} \max & -K(\lambda_0) + p\lambda_0 + (1 - \lambda_0) \left[-K\left(\frac{c}{v_g}\right) + \frac{cp}{v_g} \right] \\ \text{s.t. } & \lambda_0 \in \left[\frac{c}{v_g}, \frac{v_g \mu_0 - (1 + v_g)c}{-v_b v_g - c} \right] \end{aligned} \quad (P'_{2H})$$

Proof. The proof of the equivalence between (P_{2H}) and (P'_{2H}) is similar to that of Lemma 2.

It is feasible for the sender to provide information at both periods if and only if the domain of λ_1 is non-empty: $\frac{c}{v_g} \leq \frac{v_g \mu_0 - (2-p)c}{pv_g - c} \Leftrightarrow \mu_0 \geq \frac{2v_g - c}{(v_g)^2}c$. \square

Denote the optimal λ_0 without constraints by $\lambda_{0,H}^{**}$. $\lambda_{0,H}^{**} = \arg\max_{\lambda_0} -K(\lambda_0) + p\lambda_0 + (1 - \lambda_0) \left[-K\left(\frac{c}{v_g}\right) + \frac{cp}{v_g} \right]$. The following lemma summarizes the relative size of $\lambda_{0,H}^{**}$, λ_0^{**} , and $\frac{c}{v_g}$.

Lemma 4. $0 < \lambda_{0,H}^{**} < \lambda_1^{**} \leq \frac{c}{v_g}$.

Proof. $\lambda_1^{**} < \frac{c}{v_g}$ is the assumption. F.O.C $\Rightarrow K'(\lambda_{0,H}^{**}) = p + K(\frac{c}{v_g}) - \frac{cp}{v_g}$. From Lemma 1, $K'(\lambda_1^{**}) = p$. $-K(\frac{c}{v_g}) + \frac{cp}{v_g} > 0$ when $c < \hat{c}$. So, $K'(\lambda_{0,H}^{**}) < K'(\lambda_1^{**}) \Rightarrow \lambda_{0,H}^{**} < \lambda_1^{**}$. $-K'(\lambda_{0,H}^{**}) + p + K(\frac{c}{v_g}) - \frac{cp}{v_g} = 0 \Rightarrow K'(\lambda_{0,H}^{**}) = p + K(\frac{c}{v_g}) - \frac{cp}{v_g} > p - \frac{cp}{v_g} = (1 - \frac{c}{v_g})p > 0$, where the last inequality follows from the assumption that $c < v_g$. Thus, $\lambda_{0,H}^{**} > 0$. $\lambda_1^{**} \leq \frac{c}{v_g}$ is the assumption. \square

When it is feasible for the sender to provide information in both periods, $\mu_0 \geq \frac{2v_g - c}{(v_g)^2}c$, Lemma 4 and strict concavity of the objective function imply that the optimal two-period strategy of the sender is $(\lambda_t^*, \bar{\mu}_t^*) = (\frac{c}{v_g}, 1)$, $t = 0, 1$. The sender surplus is $(2 - \frac{c}{v_g}) \left[-K(\frac{c}{v_g}) + \frac{cp}{v_g} \right] > -K(\frac{c}{v_g}) + \frac{cp}{v_g}$, the sender surplus of the optimal one-period strategy. Therefore, the sender would always provide information in both periods as long as it is feasible.

(2) $c < v_g \lambda_1^{**}$

If the sender provides information in both periods, we first show that we could restrict the domain of μ_1 to be $\leq \lambda_1^{**}$ given that $c < v_g \lambda_1^{**}$. The intuition is that $\forall \mu_1 \geq \lambda_1^{**}$, the optimal second-period strategy is always $(\lambda_1^*, \bar{\mu}_1^*) = (\lambda_1^{**}, 1)$. Therefore, choosing μ_1 above λ_1^{**} does not increase the second-period sender's payoff or relax the first-period constraints. So, we could restrict μ_1 to be less than or equal to λ_1^{**} . Formally, when $\lambda_1^{**} \leq \mu_1 < \mu_0$, the constrained program of the sender is:

$$\begin{aligned} & \max -K(\lambda_0) + p\lambda_0 + (1 - \lambda_0) [-K(\lambda_1^{**}) + p\lambda_1^{**}] \\ & \text{s.t. } \lambda_0(\bar{\mu}_0 + v_b) + (1 - \lambda_0)[\lambda_1^{**}v_g - c] \geq c \quad (\tilde{I}\tilde{R}'_0) \\ & \lambda_0\bar{\mu}_0 + (1 - \lambda_0)\mu_1 = \mu_0 \quad (F_0) \\ & \mu_1 \in [\lambda_1^{**}, \mu_0] \end{aligned}$$

$(\tilde{I}\tilde{R}'_0) \ \& \ (F_0) \Rightarrow \lambda_0 \leq \frac{-2c + \mu_0 - \mu_1 + v_g \lambda_1^{**}}{v_g \lambda_1^{**} - c - v_b - \mu_1} \leq \frac{\mu_0 - 2c + v_b \lambda_1^{**}}{-v_b(1 - \lambda_1^{**}) - c}$ (“=” when $\mu_1 = \lambda_1^{**}$). $(F_0) \Rightarrow \lambda_0 \geq \frac{\mu_0 - \mu_1}{1 - \mu_1}$. The domain of λ_0 is non-empty iff $\frac{\mu_0 - \mu_1}{1 - \mu_1} \leq \frac{-2c + \mu_0 - \mu_1 + v_g \lambda_1^{**}}{v_g \lambda_1^{**} - c - v_b - \mu_1} \Leftrightarrow \mu_1 \leq \frac{-2c + v_g \lambda_1^{**} + \mu_0[1 - v_g \lambda_1^{**} + c + v_b]}{v_g - c}$.

Therefore, smaller μ_1 means it is more likely for the domain of λ_0 to be non-empty and larger upper bound of λ_0 . So, the optimal μ_1 would never $\in (\lambda_1^{**}, \mu_0)$. Hence, the optimal strategy

of the subgame is $(\lambda_1^*, \bar{\mu}_1^*) = \begin{cases} (\lambda_1^{**}, \frac{\mu_1}{\lambda_1^{**}}) & , \text{ if } \mu_1 \in (c - v_b \lambda_1^{**}, \lambda_1^{**}] \\ (\frac{\mu_1 - c}{-v_b}, \frac{-\mu_1 v_b}{\mu_1 - c}) & , \text{ if } \mu_1 \in [\frac{c}{v_g}, c - v_b \lambda_1^{**}] \end{cases}$ and the constrained

program of the full game is either¹⁰:

$$\begin{aligned} & \max -K(\lambda_0) + p\lambda_0 + (1 - \lambda_0) [-K(\lambda_1^{**}) + p\lambda_1^{**}] & (P_{2S_+}) \\ \text{s.t. } & (IR_0), (F_0), \mu_1 \in [c - v_b\lambda_1^{**}, \lambda_1^{**}] \end{aligned}$$

or:

$$\begin{aligned} & \max -K(\lambda_0) + p\lambda_0 + (1 - \lambda_0) \left[-K\left(\frac{\mu_1 - c}{-v_b}\right) + \frac{(\mu_1 - c)p}{-v_b} \right] & (P_{2S_0}) \\ \text{s.t. } & (IR_0), (F_0), \mu_1 \in \left[\frac{c}{v_g}, c - v_b\lambda_1^{**}\right] \end{aligned}$$

1. S_+ strategy (solution to (P_{2S_+}))

Proposition 11. Suppose $c < v_g\lambda_1^{**}$ and $\mu_0 < \hat{\mu}_0 = 2c - v_b\lambda_1^{**} - [c + (1 - \lambda_1^{**})v_b]\lambda_0^{**}$. If $\mu_0 > 2c - v_b\lambda_1^{**}$ and $\mu_0 \geq \frac{(2 - \lambda_1^{**})c}{v_g(1 - \lambda_1^{**}) + c}$, (P_{2S_+}) is feasible with the following solution. $\lambda_0^* = \frac{\mu_0 - 2c + v_b\lambda_1^{**}}{-v_b(1 - \lambda_1^{**}) - c}$; $\bar{\mu}_0^* = \begin{cases} \frac{(v_b - c)(c - v_b\lambda_1^{**}) - v_b\mu_0}{\mu_0 - 2c + v_b\lambda_1^{**}} \in (-v_b, 1) & , \text{ if } \hat{\mu}_1(\mu_0) < c - v_b\lambda_1^{**} \\ 1 & , \text{ if } \hat{\mu}_1(\mu_0) \geq c - v_b\lambda_1^{**} \end{cases}$; $(\lambda_1^*, \bar{\mu}_1^*) = (\lambda_1^{**}, \frac{\mu_1}{\lambda_1^{**}})$; $\mu_1^* = \hat{\mu}_1(\mu_0) \vee c - v_b\lambda_1^{**}$, where $\hat{\mu}_1(\mu_0) = \frac{2c - v_b\lambda_1^{**} - (1 + c - v_b\lambda_1^{**} + v_b)\mu_0}{c - v_b - \mu_0}$. The receiver gets zero surplus.

Proof. We first transform (P_{2S_+}) into an equivalent constrained program that is easier to analyze.

Lemma 5. Suppose $c < v_g\lambda_1^{**}$ and $\mu_0 \geq \mu_{0,1}$. If $\mu_0 \leq 2c - v_b\lambda_1^{**}$ or $\mu_0 < \frac{(2 - \lambda_1^{**})c}{v_g(1 - \lambda_1^{**}) + c}$, the sender provides information in one period. If $\mu_0 > 2c - v_b\lambda_1^{**}$ and $\mu_0 \geq \frac{(2 - \lambda_1^{**})c}{v_g(1 - \lambda_1^{**}) + c}$, (P_{2S_+}) is equivalent to:

$$\begin{aligned} & \max -K(\lambda_0) + p\lambda_0 + (1 - \lambda_0) [-K(\lambda_1^{**}) + p\lambda_1^{**}] & (P_{2S_+}'') \\ \text{s.t. } & \lambda_0 \in \left(0, \frac{\mu_0 - 2c + v_b\lambda_1^{**}}{-v_b(1 - \lambda_1^{**}) - c}\right] \end{aligned}$$

Proof. We first show that, if $\mu_0 > 2c - v_b\lambda_1^{**}$ and $\mu_0 \geq \frac{(2 - \lambda_1^{**})c}{v_g(1 - \lambda_1^{**}) + c}$, (P_{2S_+}) is equivalent

¹⁰ We include $\mu_1 = c - v_b\lambda_1^{**}$ in (P_{2S_+}) as well to simplify the exposition.

to:

$$\begin{aligned}
& \max -K(\lambda_0) + p\lambda_0 + (1 - \lambda_0) [-K(\lambda_1^{**}) + p\lambda_1^{**}] & (P'_{2S_+}) \\
\text{s.t. } & \lambda_0 \in \left[\frac{\mu_0 - \mu_1}{1 - \mu_1}, \frac{\mu_0 - 2c + v_b\lambda_1^{**}}{-v_b(1 - \lambda_1^{**}) - c} \wedge \frac{\mu_0 - \mu_1}{-v_b - \mu_1} \right] \\
& \mu_1 \in [c - v_b\lambda_1^{**} \vee \hat{\mu}_1(\mu_0), \lambda_1^{**}] \\
& , \text{ where } \hat{\mu}_1(\mu_0) = \frac{2c - v_b\lambda_1^{**} - (1 + c - v_b\lambda_1^{**} + v_b)\mu_0}{c - v_b - \mu_0}
\end{aligned}$$

$(F_0) \Rightarrow \lambda_0 = \frac{\mu_0 - \mu_1}{\bar{\mu}_0 - \mu_1} \in \left[\frac{\mu_0 - \mu_1}{1 - \mu_1}, \frac{\mu_0 - \mu_1}{-v_b - \mu_1} \right]$. $(IR_0) \& (F_0) \Rightarrow \lambda_0 \leq \frac{\mu_0 - 2c + v_b\lambda_1^{**}}{-v_b(1 - \lambda_1^{**}) - c}$. For λ_0 to be positive, we need $\mu_0 > 2c - v_b\lambda_1^{**}$. The domain of λ_0 is non-empty iff $\frac{\mu_0 - \mu_1}{1 - \mu_1} \leq \frac{\mu_0 - 2c + v_b\lambda_1^{**}}{-v_b(1 - \lambda_1^{**}) - c} \Leftrightarrow \mu_1 \geq \hat{\mu}_1(\mu_0)$. For $\mu_1 \leq \lambda_1^{**}$, we need $\hat{\mu}_1(\mu_0) \leq \lambda_1^{**} \Leftrightarrow \mu_0 \geq \frac{(2 - \lambda_1^{**})c}{v_b(1 - \lambda_1^{**}) + c}$. We also have that $\mu_1 = \underline{\mu}_0 < \mu_0$. Thus, the constraints in (P_{2S_+}) imply the constraints in (P'_{2S_+}) .

For any (λ_0, μ_1) satisfying the constraints in (P'_{2S_+}) , consider $(\lambda_0, \mu_1, \bar{\mu}_0 = \frac{\mu_0 - (1 - \lambda_0)\mu_1}{\lambda_0}, \bar{\mu}_1 = \frac{\mu_1}{\lambda_1^{**}} \wedge 1, \underline{\mu}_1 = \frac{\mu_1 - \lambda_1^{**}\bar{\mu}_1}{1 - \lambda_1^{**}})$. $(IR_0) \& (F_0)$ are satisfied by construction. $\bar{\mu}_0 = \frac{\mu_0 - (1 - \lambda_0)\mu_1}{\lambda_0} > \frac{\mu_0 - (1 - \lambda_0)\mu_0}{\lambda_0} = \mu_0$. $\bar{\mu}_0 = \frac{\mu_0 - (1 - \lambda_0)\mu_1}{\lambda_0} \leq \frac{\mu_0 - (1 - \frac{\mu_0 - \mu_1}{1 - \mu_1})\mu_1}{\frac{\mu_0 - \mu_1}{1 - \mu_1}} = 1$. One can verify that $\bar{\mu}_1 \in (-v_b, 1]$, $\underline{\mu}_1 \in [0, -v_b)$. Therefore, the $(\lambda_0, \mu_1, \bar{\mu}_0, \bar{\mu}_1, \underline{\mu}_1)$ we constructed satisfies all the constraints in (P_{2S_+}) and is feasible.

Therefore, the two programs are equivalent.

We then show that (P'_{2S_+}) is equivalent to (P''_{2S_+}) . It is clear that the constraints in (P'_{2S_+}) imply the constraints in (P''_{2S_+}) . We now show that for any $\lambda_0 \in \left(0, \frac{\mu_0 - 2c + v_b\lambda_1^{**}}{-v_b(1 - \lambda_1^{**}) - c}\right]$, we could find (λ_0, μ_1) that satisfies the constraints in (P'_{2S_+}) and is feasible. Since $\lambda_0 = \frac{\mu_0 - 2c + v_b\lambda_1^{**}}{-v_b(1 - \lambda_1^{**}) - c}$ maximizes the objective function among $\lambda_0 \in \left(0, \frac{\mu_0 - 2c + v_b\lambda_1^{**}}{-v_b(1 - \lambda_1^{**}) - c}\right]$ when $\mu_0 < \hat{\mu}_0$, we only need to verify (by construction) that $\lambda_0 = \frac{\mu_0 - 2c + v_b\lambda_1^{**}}{-v_b(1 - \lambda_1^{**}) - c}$ can be obtained.

i) $\hat{\mu}_1(\mu_0) < c - v_b\lambda_1^{**}$

Consider $\mu_1 = c - v_b\lambda_1^{**}$, $\lambda_0 = \frac{\mu_0 - 2c + v_b\lambda_1^{**}}{-v_b(1 - \lambda_1^{**}) - c}$, $\bar{\mu}_0 = \frac{\mu_0 - (1 - \lambda_0)\mu_1}{\lambda_0} = \frac{(v_b - c)(c - v_b\lambda_1^{**}) - v_b\mu_0}{\mu_0 - 2c + v_b\lambda_1^{**}}$. By construction, (IR_0) and (F_0) are satisfied; μ_1 's constraints are also satisfied. So, we just need to verify that $\bar{\mu}_0 \in (p, 1)$. $\bar{\mu}_0 < 1 \Leftrightarrow (v_b - c)(c - v_b\lambda_1^{**}) - v_b\mu_0 < \mu_0 - 2c + v_b\lambda_1^{**} \Leftrightarrow \hat{\mu}_1(\mu_0) < c - v_b\lambda_1^{**}$, which is the assumption. $\bar{\mu}_0 > -v_b \Leftrightarrow c < -v_b(1 - \lambda_1^{**})$, which holds because $\mu_0 > 2c - v_b\lambda_1^{**} \Rightarrow c < \frac{1}{2}(\mu_0 + v_b\lambda_1^{**}) \leq \mu_0 + v_b\lambda_1^{**} < -v_b + v_b\lambda_1^{**} = -v_b(1 - \lambda_1^{**})$.

ii) $\hat{\mu}_1(\mu_0) \geq c - v_b\lambda_1^{**}$

Consider $\lambda_0 = \frac{\mu_0 - 2c + v_b \lambda_1^{**}}{-v_b(1 - \lambda_1^{**}) - c}$, $\bar{\mu}_0 = 1$, $\mu_1 = \frac{\mu_0 - \lambda_0 \bar{\mu}_0}{1 - \lambda_0} = \hat{\mu}_1(\mu_0)$. By construction, (IR_0) and (F_0) are satisfied; μ_1 's constraints are also satisfied. So, we just need to verify that $\mu_1 = \hat{\mu}_1(\mu_0) \in [c - v_b \lambda_1^{**}, \lambda_1^{**}]$. $\hat{\mu}_1(\mu_0) \geq c - v_b \lambda_1^{**}$ is the assumption. $\mu_0 \geq \frac{(2 - \lambda_1^{**})c}{v_g(1 - \lambda_1^{**}) + c} \Rightarrow \hat{\mu}_1(\mu_0) \leq \lambda_1^{**}$.

□

When $\frac{\mu_0 - 2c + v_b \lambda_1^{**}}{-v_b(1 - \lambda_1^{**}) - c} \geq \lambda_0^{**} (\Leftrightarrow \mu_0 \geq \hat{\mu}_0)$, the optimal λ_0 is λ_0^{**} . When $\frac{\mu_0 - 2c + v_b \lambda_1^{**}}{-v_b(1 - \lambda_1^{**}) - c} < \lambda_0^{**}$, the optimal λ_0 is $\frac{\mu_0 - 2c + v_b \lambda_1^{**}}{-v_b(1 - \lambda_1^{**}) - c}$ due to strict concavity of the objective function (denote it by $J(\lambda_0)$). Since the one-period optimal sender surplus is $-K(\lambda_1^{**}) + p\lambda_1^{**} = J(0)$, $J(\cdot)$ is strictly concave and obtains the unique maximum value at $\lambda_0^{**} > \frac{\mu_0 - 2c + v_b \lambda_1^{**}}{-v_b(1 - \lambda_1^{**}) - c}$, we have $J(\frac{\mu_0 - 2c + v_b \lambda_1^{**}}{-v_b(1 - \lambda_1^{**}) - c}) > J(0)$. So, the sender always provides information in both periods when it is feasible ($\mu_0 > 2c - v_b \lambda_1^{**}$ and $\mu_0 \geq \frac{(2 - \lambda_1^{**})c}{v_g(1 - \lambda_1^{**}) + c}$). We will use this observation in the later proofs. According to the proof of Lemma 5, the receiver always gets zero surpluses when $\mu_0 < \hat{\mu}_0$. $\mu_1 = \hat{\mu}_1(\mu_0) \vee c - v_b \lambda_1^{**}$ is the smallest μ_1 that supports $\lambda_0^* = \frac{\mu_0 - 2c + v_b \lambda_1^{**}}{-v_b(1 - \lambda_1^{**}) - c}$, which gives the receiver the largest surplus in the first period. So, $\mu_1^* = \hat{\mu}_1(\mu_0) \vee c - v_b \lambda_1^{**}$. $(F_0) \Rightarrow \bar{\mu}_0^* = \frac{\mu_0 - (1 - \lambda_0)\mu_1}{\lambda_0} = \begin{cases} \frac{(v_b - c)(c - v_b \lambda_1^{**}) - v_b \mu_0}{\mu_0 - 2c + v_b \lambda_1^{**}} \in (p, 1) & , \text{if } \hat{\mu}_1(\mu_0) < c - v_b \lambda_1^{**} \\ 1 & , \text{if } \hat{\mu}_1(\mu_0) \geq c - v_b \lambda_1^{**} \end{cases}$ □

2. S_0 strategy (solution to (P_{2S_0}))

Proposition 12. Suppose $c \leq v_g \lambda_0^{**}$. When $\mu_0 \geq \frac{2v_g - c}{(v_g)^2}c$, (P'_{2S_0}) is feasible. λ_0^* , λ_1^* , and μ_1^* are continuous and increase in μ_0 , while $\bar{\mu}_0^*$ and $\bar{\mu}_1^*$ are continuous and decrease in μ_0 , in the solution to (P'_{2S_0}) . The receiver gets zero surplus at each period.

Proof. We first transform (P_{2S_0}) into an equivalent constrained program that is easier to analyze.

Lemma 6. Suppose $c < v_g \lambda_1^{**}$. If $\mu_{0,1} \leq \mu_0 < \frac{2v_g - c}{(v_g)^2}c$, the sender provides information in one period. If $\mu_0 \geq \frac{2v_g - c}{(v_g)^2}c$, (P_{2S_0}) is equivalent to:

$$\begin{aligned} \max \quad & -K(\lambda_0) + p\lambda_0 + (1 - \lambda_0) \left[-K\left(\frac{\mu_1 - c}{-v_b}\right) + \frac{(\mu_1 - c)p}{-v_b} \right] & (P'_{2S_0}) \\ \text{s.t.} \quad & \lambda_0 \in \left[\frac{\mu_0 - \mu_1}{1 - \mu_1}, \frac{\mu_0 - \mu_1 - c}{-v_b - \mu_1} \right] \\ & \mu_1 \in \left[\frac{c}{v_g}, \frac{v_g \mu_0 - c}{v_g - c} \right] \end{aligned}$$

Proof. Using the same argument as the proof of Lemma 5, one can show that (P_{2S_0}) is equivalent to:

$$\begin{aligned} & \max -K(\lambda_0) + p\lambda_0 + (1 - \lambda_0) \left[-K\left(\frac{\mu_1 - c}{-v_b}\right) + \frac{(\mu_1 - c)p}{-v_b} \right] & (P''_{2S_0}) \\ \text{s.t. } & \lambda_0 \in \left[\frac{\mu_0 - \mu_1}{1 - \mu_1}, \frac{\mu_0 - \mu_1 - c}{-v_b - \mu_1} \right] \\ & \mu_1 \in \left[\frac{c}{v_g}, \frac{v_g\mu_0 - c}{v_g - c} \wedge c - v_b\lambda_1^{**} \right] \end{aligned}$$

We just need to show that (P''_{2S_0}) is equivalent to (P'_{2S_0}) .

If $\frac{v_g\mu_0 - c}{v_g - c} \leq c - v_b\lambda_1^{**}$, μ_1 's constraint becomes $\mu_1 \in \left[\frac{c}{v_g}, \frac{v_g\mu_0 - c}{v_g - c} \right]$. So, (P''_{2S_0}) is equivalent to (P'_{2S_0}) .

If $\frac{v_g\mu_0 - c}{v_g - c} > c - v_b\lambda_1^{**}$, denote the solution to (P'_{2S_0}) by (λ_0, μ_1) .

(a) $(\lambda_0^{**}, \lambda_1^{**})$ could be obtained $(\lambda_1 = \lambda_1^{**} \Leftrightarrow \mu_1 = c - v_b\lambda_1^{**})$ in (P''_{2S_0})
 $\mu_1 \in \left[\frac{c}{v_g}, \frac{v_g\mu_0 - c}{v_g - c} \wedge c - v_b\lambda_1^{**} \right]$ is equivalent to $\mu_1 \in \left[\frac{c}{v_g}, \frac{v_g\mu_0 - c}{v_g - c} \right]$, as the optimal μ_1 under the latter (relaxed) constraint would be $c - v_b\lambda_1^{**}$. So, (P''_{2S_0}) is equivalent to (P'_{2S_0}) .

(b) $(\lambda_0^{**}, \lambda_1^{**})$ could not be obtained in (P''_{2S_0})

Suppose $\mu_1 > c - v_b\lambda_1^{**}$. If $\lambda_0 > \frac{\mu_0 - \mu_1}{1 - \mu_1}$, consider $(\lambda'_0 = \lambda_0, \mu'_1 = \mu_1 - \varepsilon)$. For small enough ε , it is feasible and gives the sender a strictly higher payoff. A contradiction! If $\lambda_0 = \frac{\mu_0 - \mu_1}{1 - \mu_1}$ instead, we have $\frac{\mu_0 - \mu_1}{1 - \mu_1} \geq \lambda_0^{**}$. A contradiction!

Therefore, $\mu_1 \leq c - v_b\lambda_1^{**}$ and thus (P''_{2S_0}) is equivalent to (P'_{2S_0}) .

In sum, (P''_{2S_0}) is equivalent to (P'_{2S_0}) . □

To solve (P'_{2S_0}) , we introduce another lemma:

Lemma 7. Suppose $c < v_g\lambda_1^{**}$. For $\mu_0 < \hat{\mu}_0$, λ_0 is binding at the upper bound in the solution to (P'_{2S_0}) .

Proof. To solve (P'_{2S_0}) , we consider several cases.

i) $\lambda_0 \leq \frac{\mu_0 - \mu_1 - c}{-v_b - \mu_1}$ is binding and μ_1 's constraints are not binding.

The Lagrangian is $\mathcal{L} = -K(\lambda_0) + p\lambda_0 + (1 - \lambda_0) \left[-K\left(\frac{\mu_1 - c}{-v_b}\right) + \frac{(\mu_1 - c)p}{-v_b} \right] + \eta \left(\frac{\mu_0 - \mu_1 - c}{-v_b - \mu_1} - \lambda_0 \right)$

s.t. $\eta \geq 0, \eta \left(\frac{\mu_0 - \mu_1 - c}{-v_b - \mu_1} - \lambda_0 \right) = 0$.

$$\text{F.O.C.} \Rightarrow \begin{cases} -K'(\lambda_0) + p + K\left(\frac{\mu_1 - c}{-v_b}\right) + \frac{(\mu_1 - c)p}{v_b} - \eta = 0 \\ (1 - \lambda_0) \left[K'\left(\frac{\mu_1 - c}{-v_b}\right) \cdot \frac{1}{v_b} - \frac{p}{v_b} \right] + \eta \cdot \frac{\mu_0 + v_b - c}{(v_b + \mu_1)^2} = 0 \end{cases}$$

Plug in $\lambda_0 = \frac{\mu_0 - \mu_1 - c}{-v_b - \mu_1}$. Dividing the second equality by $\frac{\mu_0 + v_b - c}{(v_b + \mu_1)^2}$ and comparing with the first equality, we obtain:

$$\begin{aligned} \eta &= -(v_b + \mu_1) \left[K'\left(\frac{\mu_1 - c}{-v_b}\right) \cdot \frac{1}{v_b} - \frac{p}{v_b} \right] \\ &= -K'(\lambda_0) + p + K\left(\frac{\mu_1 - c}{-v_b}\right) + \frac{(\mu_1 - c)p}{v_b} \\ \Rightarrow K\left(\frac{\mu_1 - c}{-v_b}\right) + \frac{v_b + \mu_1}{v_b} K'\left(\frac{\mu_1 - c}{-v_b}\right) - K'\left(\frac{\mu_0 - \mu_1 - c}{-v_b - \mu_1}\right) - \frac{cp}{v_b} &= 0 \quad (*) \end{aligned}$$

$\frac{\partial}{\partial \mu_1} \left[K\left(\frac{\mu_1 - c}{-v_b}\right) + \frac{v_b + \mu_1}{v_b} K'\left(\frac{\mu_1 - c}{-v_b}\right) \right] = -\frac{v_b + \mu_1}{(v_b)^2} K''\left(\frac{\mu_1 - c}{-v_b}\right) > 0$. So, the sum of the first two terms of the LHS of (*) strictly increases in μ_1 . $\frac{\mu_0 - \mu_1 - c}{-v_b - \mu_1}$ strictly decreases in μ_1 , $K'(\cdot)$ strictly increases in μ_1 . So, $-K'\left(\frac{\mu_0 - \mu_1 - c}{-v_b - \mu_1}\right)$ strictly increases in μ_1 . Thus, the LHS of (*) strictly increases in μ_1 . When μ_0 increases, the LHS of (*) is strictly negative if μ_1 is unchanged. Therefore, μ_1 also has to increase. So, the sum of the first two terms of the LHS of (*) increases. As a result, the third term, $-K'\left(\frac{\mu_0 - \mu_1 - c}{-v_b - \mu_1}\right) = -K'(\lambda_0)$ has to decrease strictly. So, λ_0 has to increase strictly. In sum, the optimal λ_0 and μ_1 are strictly increasing in μ_0 .

ii) $\mu_1 \leq \frac{v_g \mu_0 - c}{v_g - c}$ is binding.

When $\mu_1 = \frac{v_g \mu_0 - c}{v_g - c}$, $\lambda_0 \in \left\{ \frac{c}{v_g} \right\}$. So, λ_0 is binding at the upper bound.

iii) $\mu_1 \geq \frac{c}{v_g}$ is binding and λ_0 is not binding.

The Lagrangian is $\mathcal{L} = -K(\lambda_0) + p\lambda_0 + (1 - \lambda_0) \left[-K\left(\frac{\mu_1 - c}{-v_b}\right) + \frac{(\mu_1 - c)p}{-v_b} \right] + \eta \left(\mu_1 - \frac{c}{v_g} \right)$
s.t. $\eta \geq 0, \eta \left(\mu_1 - \frac{c}{v_g} \right) = 0$.

$$\text{F.O.C.} \Rightarrow \begin{cases} -K'(\lambda_0) + p + K\left(\frac{\mu_1 - c}{-v_b}\right) + \frac{(\mu_1 - c)p}{v_b} = 0 \\ (1 - \lambda_0) \left[K'\left(\frac{\mu_1 - c}{-v_b}\right) \cdot \frac{1}{v_b} - \frac{p}{v_b} \right] + \eta = 0 \end{cases}$$

The second equality $\Rightarrow \eta = -\frac{1 - \lambda_0}{v_b} \left[K'\left(\frac{\mu_1 - c}{-v_b}\right) - p \right] \stackrel{c < v_g \lambda_1^{**}}{<} -\frac{1 - \lambda_0}{v_b} [K'(\lambda_1^{**}) - p] = 0$.

But $\eta \geq 0$. A contradiction! So, this case cannot happen.

iv) $\mu_1 \geq \frac{c}{v_g}$ is binding and $\lambda_0 \geq \frac{\mu_0 - \mu_1}{1 - \mu_1}$ is binding.

The Lagrangian is $\mathcal{L} = -K(\lambda_0) + p\lambda_0 + (1 - \lambda_0) \left[-K\left(\frac{\mu_1 - c}{-v_b}\right) + \frac{(\mu_1 - c)p}{-v_b} \right] + \eta \left(\mu_1 - \frac{c}{v_g} \right) + \xi \left(\lambda_0 - \frac{\mu_0 - \mu_1}{1 - \mu_1} \right)$ *s.t.* $\eta \geq 0, \eta \left(\mu_1 - \frac{c}{v_g} \right) = 0, \xi \geq 0, \xi \left(\lambda_0 - \frac{\mu_0 - \mu_1}{1 - \mu_1} \right) = 0$.

$$\text{F.O.C.} \Rightarrow \begin{cases} -K'(\lambda_0) + p + K\left(\frac{\mu_1 - c}{-v_b}\right) + \frac{(\mu_1 - c)p}{v_b} + \xi = 0 \\ (1 - \lambda_0) \left[K'\left(\frac{\mu_1 - c}{-v_b}\right) \cdot \frac{1}{v_b} - \frac{p}{v_b} \right] + \eta + \xi \frac{1 - \mu_0}{(1 - \mu_1)^2} = 0 \end{cases}$$

Similar to the previous case, the LHS of the second equality > 0 . A contradiction!

So, this case cannot happen.

v) $\lambda_0 \geq \frac{\mu_0 - \mu_1}{1 - \mu_1}$ is binding and μ_1 is not binding.

The Lagrangian is $\mathcal{L} = -K(\lambda_0) + p\lambda_0 + (1 - \lambda_0) \left[-K\left(\frac{\mu_1 - c}{-v_b}\right) + \frac{(\mu_1 - c)p}{-v_b} \right] + \xi \left(\lambda_0 - \frac{\mu_0 - \mu_1}{1 - \mu_1} \right)$

s.t. $\xi \geq 0, \xi \left(\lambda_0 - \frac{\mu_0 - \mu_1}{1 - \mu_1} \right) = 0$.

$$\text{F.O.C.} \Rightarrow \begin{cases} -K'(\lambda_0) + p + K\left(\frac{\mu_1 - c}{-v_b}\right) + \frac{(\mu_1 - c)p}{-v_b} + \xi = 0 \\ (1 - \lambda_0) \left[K'\left(\frac{\mu_1 - c}{-v_b}\right) \cdot \frac{1}{v_b} - \frac{p}{v_b} \right] + \xi \frac{1 - \mu_0}{(1 - \mu_1)^2} = 0 \end{cases}$$

Similar to the previous case, the LHS of the second equality > 0 . A contradiction!

So, this case cannot happen.

vi) both λ_0 and μ_1 are not binding.

The solution is the unconstrained optimal solution $(\lambda_0^{**}, \lambda_1^{**})$. But we have assumed that it is not feasible.

i) to vi) finish the proof of Lemma 7. □

According to Lemma 7, if $\mu_0 \geq \frac{2v_g - c}{(v_g)^2}c$, (P'_{2S_0}) is equivalent to:

$$\max -K\left(\frac{\mu_0 - \mu_1 - c}{-v_b - \mu_1}\right) + p \frac{\mu_0 - \mu_1 - c}{-v_b - \mu_1} + \left(1 - \frac{\mu_0 - \mu_1 - c}{-v_b - \mu_1}\right) \left[-K\left(\frac{\mu_1 - c}{-v_b}\right) + \frac{(\mu_1 - c)p}{-v_b} \right] \\ (P'''_{2S_0})$$

$$\text{s.t. } \mu_1 \in \left[\frac{c}{v_g}, \frac{v_g \mu_0 - c}{v_g - c} \right]$$

The first order derivative of the objective function w.r.t. μ_1 is:

$$D(\mu_0, \mu_1) \\ := \frac{\partial}{\partial \mu_1} \left\{ -K\left(\frac{\mu_0 - \mu_1 - c}{-v_b - \mu_1}\right) + p \frac{\mu_0 - \mu_1 - c}{-v_b - \mu_1} + \left(1 - \frac{\mu_0 - \mu_1 - c}{-v_b - \mu_1}\right) \left[-K\left(\frac{\mu_1 - c}{-v_b}\right) + \frac{(\mu_1 - c)p}{-v_b} \right] \right\} \\ = \frac{\mu_0 + v_b - c}{(\mu_1 + v_b)^2} \left[K\left(\frac{\mu_1 - c}{-v_b}\right) + \frac{v_b + \mu_1}{v_b} K'\left(\frac{\mu_1 - c}{-v_b}\right) - K'\left(\frac{\mu_0 - \mu_1 - c}{-v_b - \mu_1}\right) - \frac{cp}{v_b} \right]$$

The first term of $D(\mu_0, \mu_1)$, $\frac{\mu_0 + v_b - c}{(\mu_1 + v_b)^2}$, is always strictly negative. The second term, $K\left(\frac{\mu_1 - c}{-v_b}\right) + \frac{v_b + \mu_1}{v_b} K'\left(\frac{\mu_1 - c}{-v_b}\right) - K'\left(\frac{\mu_0 - \mu_1 - c}{-v_b - \mu_1}\right) - \frac{cp}{v_b}$, is the LHS of (*), which has been shown

to be strictly increasing in μ_1 in the proof of Lemma 7. One can see that $D(\mu_0, \mu_1)$ is strictly negative when μ_1 is large. Thus, $D(\mu_0, \mu_1)$ is always negative or positive for μ_1 small and negative for μ_1 large. Let $\mu_1^{**}(\mu_0)$ be the cutoff value such that $D(\mu_0, \mu_1) \geq 0$ for $\mu_1 \leq \mu_1^{**}(\mu_0)$ and $D(\mu_0, \mu_1) \leq 0$ for $\mu_1 \geq \mu_1^{**}(\mu_0)$ ($\mu_1^{**}(\mu_0) := -\infty$ if $D(\mu_0, \mu_1)$ is always negative). Since $\mu_1 \in \left[\frac{c}{v_g}, \frac{v_g\mu_0 - c}{v_g - c}\right]$, the optimal $\mu_1^*(\mu_0) = \left[\frac{c}{v_g} \vee \mu_1^{**}(\mu_0)\right] \wedge \frac{v_g\mu_0 - c}{v_g - c}$.

One can see that we could define $\tilde{\mu}_1^{**}(\mu_0) := \begin{cases} \frac{c}{v_g}, & \text{if } D(\mu_0, \mu_1) \text{ is always negative} \\ \mu_1^{**}(\mu_0), & \text{otherwise} \end{cases}$. $\tilde{\mu}_1^{**}(\mu_0) \in (-\infty, +\infty)$ and $\mu_1^*(\mu_0) = \left[\frac{c}{v_g} \vee \tilde{\mu}_1^{**}(\mu_0)\right] \wedge \frac{v_g\mu_0 - c}{v_g - c}$. Since $\tilde{\mu}_1^{**}(\mu_0)$ is continuous in μ_0 , $\mu_1^*(\mu_0)$ is also continuous in μ_0 . It then implies that $\lambda_0^*(\mu_0) = \frac{\mu_0 - \mu_1^*(\mu_0) - c}{-v_b - \mu_1^*(\mu_0)}$, $\lambda_1^*(\mu_0) = \frac{\mu_1^*(\mu_0) - c}{-v_b}$, and $\bar{\mu}_1^*(\mu_0) = \frac{-\mu_1^*(\mu_0)v_b}{\mu_1^*(\mu_0) - c}$ are continuous in μ_0 .

We have shown in the proof of Lemma 7 that λ_0^* and μ_1^* strictly increase in μ_0 when μ_1^* is the interior solution. Now we consider the case when μ_1 is binding. When $\mu_1^* = \frac{c}{v_g}$, $\lambda_0^* = \frac{\mu_0 - \mu_1^* - c}{-v_b - \mu_1^*} = \frac{\mu_0 - \frac{c}{v_g} - c}{-v_b - \frac{c}{v_g}}$ strictly increases in μ_0 . When $\mu_1^* = \frac{v_g\mu_0 - c}{v_g - c}$, it is strictly increasing in μ_0 and $\lambda_0^* = \frac{\mu_0 - \mu_1^* - c}{-v_b - \mu_1^*} = \frac{c}{v_g}$ is constant. Together with the continuity property we just established, we have shown that λ_0^* and μ_1^* are (weakly) increasing in μ_0 . Thus, $\lambda_1^* = \frac{\mu_1^* - c}{-v_b}$ is (weakly) increasing in μ_0 and $\bar{\mu}_1^* = \frac{-\mu_1^*v_b}{\mu_1^* - c}$ is (weakly) decreasing in μ_0 . \square

According to the proof of Proposition 1 and Lemma 6, the sender does not provide information iff $\mu_0 < \mu_{0,1}$ and provide information in one period if $\mu_{0,1} \leq \mu_0 < \frac{2v_g - c}{(v_g)^2}c$. Thus, we just need to determine whether she provides information in one period or in both periods when $\mu_0 \geq \frac{2v_g - c}{(v_g)^2}c$ by comparing the sender surplus of the optimal one-period strategy and the optimal sender surplus of the S_0 strategy.

(a) $c \leq v_g\lambda_0^{**}$

Define $\mu_{1,2} := \inf\{\mu_0 \geq \frac{2v_g - c}{(v_g)^2}c : \Pi_{S_0}(\mu_0) \geq \Pi_1(\mu_0)\}$. One can see that $\mu_{1,2} \in \left[\frac{2v_g - c}{(v_g)^2}c, \hat{\mu}_0\right)$ and $\Pi_{S_0}(\mu_{1,2}) \geq \Pi_1(\mu_{1,2})$. According to Lemma 7,

$$\begin{aligned} \Pi_1(\mu_0) &= -K\left(\frac{\mu_0 - c}{-v_b} \wedge \lambda_1^{**}\right) + p\left(\frac{\mu_0 - c}{-v_b} \wedge \lambda_1^{**}\right) \\ \Pi_{S_0}(\mu_0) &= \max_{\mu_1} -K\left(\frac{\mu_0 - \mu_1 - c}{-v_b - \mu_1}\right) + p\frac{\mu_0 - \mu_1 - c}{-v_b - \mu_1} + \left(1 - \frac{\mu_0 - \mu_1 - c}{-v_b - \mu_1}\right) \left[-K\left(\frac{\mu_1 - c}{-v_b}\right) + \frac{(\mu_1 - c)p}{-v_b}\right] \\ s.t. \mu_1 &\in \left[\frac{c}{v_g}, \frac{v_g\mu_0 - c}{v_g - c}\right] \end{aligned}$$

i) $\mu_{1,2} \geq c - v_b \lambda_1^{**}$

$$\forall \mu_0 \in (\mu_{1,2}, \widehat{\mu}_0], \Pi_{S_0}(\mu_0) > \Pi_{S_0}(\mu_{1,2}) \geq \Pi_1(\mu_{1,2}) = \Pi_1(\mu_0).$$

ii) $\mu_{1,2} < c - v_b \lambda_1^{**}$

$$\forall \mu_0 \in [\mu_{1,2}, c - v_b \lambda_1^{**}), \frac{d\Pi_1(\mu_0)}{d\mu_0} = K'(\frac{\mu_0 - c}{-v_b}) \frac{1}{v_b} - \frac{p}{v_b}.$$

$$A. \mu_1(\mu_0) = \mu_1^u(\mu_0) = \frac{v_g \mu_0 - c}{v_g - c}$$

$$\Pi_{S_0}(\mu_0) = -K(\frac{c}{v_g}) + \frac{cp}{v_g} + (1 - \frac{c}{v_g})[-K(\frac{\mu_1^u(\mu_0) - c}{-v_b}) + \frac{(\mu_1^u(\mu_0) - c)p}{-v_b}].$$

For $\Delta > 0$ small enough, we have $\mu_0 + \delta < c - v_b \lambda_1^{**}$, $\forall \delta \in (0, \Delta)$. Consider $\mu_{0,\delta} = \mu_0 + \delta \in (\mu_0, \mu_0 + \Delta)$, we have $\Pi_{S_0}(\mu_{0,\delta}) \geq \underline{\Pi_{S_0}}(\mu_{0,\delta}) := -K(\frac{c}{v_g}) + \frac{cp}{v_g} + (1 - \frac{c}{v_g})[-K(\frac{\mu_1^u(\mu_{0,\delta}) - c}{-v_b}) + \frac{(\mu_1^u(\mu_{0,\delta}) - c)p}{-v_b}]$. Noticing that $\Pi_{S_0}(\mu_0) = \underline{\Pi_{S_0}}(\mu_0)$, we have

$$\frac{d\Pi_{S_0}(\mu_0)}{d\mu_0} \geq \frac{d\underline{\Pi_{S_0}}(\mu_0)}{d\mu_0} = K'(\frac{\mu_1^u(\mu_0) - c}{-v_b}) \frac{1}{v_b} - \frac{p}{v_b} > \frac{d\Pi_1(\mu_0)}{d\mu_0}$$

So, $\Pi_{S_0}(\mu_0) \geq \Pi_1(\mu_0)$, $\forall \mu_0 \in [\mu_{1,2}, c - v_b \lambda_1^{**})$, and the inequality is strict when $\mu_0 > \mu_{1,2}$.

B. $\mu_1(\mu_0) < \mu_1^u(\mu_0)$

Let $\lambda_0 = \frac{\mu_0 - \mu_1(\mu_0) - c}{-v_b - \mu_1(\mu_0)}$. For $\Delta > 0$ small enough, we have $\mu_0 + \delta < c - v_b \lambda_1^{**}$ and $\mu_1(\mu_0) + \frac{\delta}{1 - \lambda_0} < \mu_1^u(\mu_0) < \mu_1^u(\mu_0 + \delta)$, $\forall \delta \in (0, \Delta)$. Consider $\mu_{0,\delta} = \mu_0 + \delta \in (\mu_0, \mu_0 + \Delta)$, noticing that $\frac{\mu_0 + \delta - (\mu_1 + \frac{\delta}{1 - \lambda_0}) - c}{-v_b - (\mu_1 + \frac{\delta}{1 - \lambda_0})} = \lambda_0$, we have

$$\Pi_{S_0}(\mu_{0,\delta}) \geq \widetilde{\Pi_{S_0}}(\mu_{0,\delta}) := -K(\lambda_0) + p\lambda_0 + (1 - \lambda_0)[-K(\frac{\mu_1(\mu_0) + \frac{\delta}{1 - \lambda_0} - c}{p}) + \frac{(\mu_1(\mu_0) + \frac{\delta}{1 - \lambda_0} - c)p}{-v_b}].$$

$$\begin{aligned} \frac{d\Pi_{S_0}(\mu_0)}{d\mu_0} &\geq \frac{d\widetilde{\Pi_{S_0}}(\mu_0)}{d\mu_0} \\ &= (1 - \lambda_0) \left[-\frac{1}{p} K'(\frac{\mu_1(\mu_0) - c}{p}) \frac{1}{1 - \lambda_0} + \frac{1}{1 - \lambda_0} \right] \\ &= 1 - \frac{1}{p} K'(\frac{\mu_1(\mu_0) - c}{p}) \\ &\geq 1 - \frac{1}{p} K'(\frac{\mu_1^u(\mu_0) - c}{p}) \\ &> \frac{d\Pi_1(\mu_0)}{d\mu_0} \end{aligned}$$

So, $\Pi_{S_0}(\mu_0) \geq \Pi_1(\mu_0)$ and the inequality is strict when $\mu_0 > \mu_{1,2}$.

In sum, $\Pi_{S_0}(\mu_0) > \Pi_1(\mu_0)$, $\forall \mu_0 \in (\mu_{1,2}, \widehat{\mu}_0]$.

(b) $v_g \lambda_0^{**} < c < v_g \lambda_1^{**}$

The following lemma provides a closed-form solution to program (P_{2S_0}) when the search cost is intermediate:

Lemma 8. *Suppose $v_g\lambda_0^{**} < c < v_g\lambda_1^{**}$. $\lambda_0^* = \frac{c}{v_g}, \mu_1^* = \frac{v_g\mu_0 - c}{v_g - c}$ in the solution to (P_{2S_0}) .*

Proof. Fixing a μ_1 , consider the unconstrained program:

$$\lambda'_0 \in \operatorname{argmax} -K(\lambda_0) + p\lambda_0 + (1 - \lambda_0) \left[-K\left(\frac{\mu_1 - c}{-v_b}\right) + \frac{(\mu_1 - c)p}{-v_b} \right]$$

Since $-K(\frac{\mu_1 - c}{-v_b}) + \frac{(\mu_1 - c)p}{-v_b} \leq -K(\lambda_1^{**}) + p\lambda_1^{**}$, $\lambda'_0 \leq \lambda_0^{**} < \frac{c}{v_g}$. Since $\lambda_0^* = \frac{\mu_0 - \mu_1 - c}{-v_b - \mu_1}$ decreases in μ_1 and $-K(\frac{\mu_1 - c}{-v_b}) + \frac{(\mu_1 - c)p}{-v_b}$ increases in μ_1 for $\mu_1 < c - v_b\lambda_1^{**}$, it is always better for the sender to choose the largest possible μ_1 ($\lambda_0 = \frac{c}{v_g}$ when μ_1 is the upper bound) when it is less than $c - v_b\lambda_1^{**}$, which is equivalent to $\frac{v_g\mu_0 - c}{v_g - c} < c - v_b\lambda_1^{**} \Leftrightarrow \mu_0 < \hat{\mu}_0 := \frac{c + (v_g - c)(c - v_b\lambda_1^{**})}{v_g}$. $\hat{\mu}_0 = 2c - v_b\lambda_1^{**} + p\lambda_0^{**}(p - c - p\lambda_1^{**}) < 2c - v_b\lambda_1^{**} + p \cdot \frac{c}{v_g}(p - c - p\lambda_1^{**}) = \hat{\mu}_0$. Thus, one can see that for $\mu_0 \leq \hat{\mu}_0$, $\lambda_0^* = \frac{c}{v_g}, \mu_1^* = \frac{v_g\mu_0 - c}{v_g - c}$ in the solution to (P'_{2S_0}) . Lemma 6 shows that (P'_{2S_0}) is equivalent to (P_{2S_0}) . So, $\lambda_0^* = \frac{c}{v_g}, \mu_1^* = \frac{v_g\mu_0 - c}{v_g - c}$ are also the solutions to (P_{2S_0}) . \square

Define $\mu_{1,2} := \inf\{\mu_0 \geq \frac{2v_g - c}{(1-p)^2}c : \Pi_{S_0}(\mu_0) \geq \Pi_1(\mu_0) \text{ or } S_+ \text{ strategy is feasible}\}$. Note that $\mu_{1,2} \leq \mu_{2,+}$. If S_+ strategy is feasible $\forall \mu_0 > \mu_{1,2}$, the 1-period sender surplus would always be dominated by the 2-period sender surplus $\forall \mu_0 > \mu_{1,2}$, as the optimal S_+ strategy generates a strictly higher sender surplus than the optimal 1-period strategy. We now consider the case in which S_+ strategy is not feasible for some $\mu_0 > \mu_{1,2}$, which implies that $\Pi_{S_0}(\mu_{1,2}) \geq \Pi_1(\mu_{1,2})$.

$$\begin{aligned} \Pi_1(\mu_0) &= -K\left(\frac{\mu_0 - c}{-v_b} \wedge \lambda_1^{**}\right) + p\left(\frac{\mu_0 - c}{-v_b} \wedge \lambda_1^{**}\right) \\ \Pi_{S_0}(\mu_0) &= -K\left(\frac{c}{v_g}\right) + \frac{cp}{v_g} + \left(1 - \frac{c}{v_g}\right) \left[-K\left(\frac{\mu_1^u(\mu_0) - c}{-v_b}\right) + \frac{(\mu_1^u(\mu_0) - c)p}{-v_b} \right] \end{aligned}$$

- i) $\mu_{1,2} \geq c - v_b\lambda_1^{**}$
 $\forall \mu_0 \in (\mu_{1,2}, \mu_{2,+}]$, $\Pi_{S_0}(\mu_0) > \Pi_{S_0}(\mu_{1,2}) \geq \Pi_1(\mu_{1,2}) = \Pi_1(\mu_0)$.
- ii) $\mu_{1,2} < c - v_b\lambda_1^{**}$

$$\forall \mu_0 \in [\mu_{1,2}, c - v_b \lambda_1^{**}],$$

$$\begin{aligned} \frac{d\Pi_1(\mu_0)}{d\mu_0} &= K' \left(\frac{\mu_0 - c}{-v_b} \right) \frac{1}{v_b} - \frac{p}{v_b} \\ \frac{d\Pi_{S_0}(\mu_0)}{d\mu_0} &= K' \left(\frac{\mu_1^u(\mu_0) - c}{-v_b} \right) \frac{1}{v_b} - \frac{p}{v_b} > \frac{d\Pi_1(\mu_0)}{d\mu_0} \end{aligned}$$

So, $\Pi_{S_0}(\mu_0) \geq \Pi_1(\mu_0)$ and the inequality is strict when $\mu_0 > \mu_{1,2}$. $\Pi_{S_0}(c - v_b \lambda_1^{**}) > \Pi_1(c - v_b \lambda_1^{**})$. $\forall \mu_0 \in [c - v_b \lambda_1^{**}, \mu_{2,+}]$, $\Pi_{S_0}(\mu_0) \geq \Pi_{S_0}(c - v_b \lambda_1^{**}) > \Pi_1(c - v_b \lambda_1^{**}) = \Pi_1(\mu_0)$.

In sum, $\forall \mu_0 \in (\mu_{1,2}, \mu_{2,+}]$, $\Pi_{S_0}(\mu_0) > \Pi_1(\mu_0)$.

One can see that the optimal S_+ strategy always generates a strictly higher (and strictly positive) sender surplus than the optimal 1-period strategy. Therefore, the sender always provides information in both periods when S_+ strategy is feasible.¹¹ By Lemma 5, S_+ strategy is feasible iff $\mu_0 > 2c - v_b \lambda_1^{**}$ and $\mu_0 \geq \frac{(2-\lambda_1^{**})c}{v_g(1-\lambda_1^{**})+c}$ when $c < v_g \lambda_1^{**}$. Hence, together with the above results on S_0 strategy, there exists $\mu_{1,2} \in [\frac{2v_g-c}{(v_g)^2}c, 2c - v_b \lambda_1^{**} \vee \frac{(2-\lambda_1^{**})c}{v_g(1-\lambda_1^{**})+c}]$ such that the sender does not provide information if $\mu_0 < \mu_{0,1}$, provides information in one period if $\mu_0 \in [\mu_{0,1}, \mu_{1,2})$, and provides information in both periods if $\mu_0 > \mu_{1,2}$. \square

Proof of Proposition 4.

(1) High Search Cost ($v_g \lambda_1^{**} \leq c < \hat{c}$)

It has been shown in the proof of Proposition 3. Since the optimal strategy does not depend on the prior, the sender's payoff does not depend on the prior either.

(2) Low Search Cost ($c \leq \tilde{c} = v_g K'^{-1} \left[\frac{K(\lambda_1^{**})}{\lambda_1^{**}} \right]$)

We first prove the following claim:

Lemma 9. $\tilde{c} < v_g \lambda_0^{**} < v_g \lambda_1^{**}$.

¹¹ But the optimal 2-period strategy may be either S_+ or S_0 strategy.

Proof. We have $K'(\frac{\tilde{c}}{v_g}) = \frac{K(\lambda_1^{**})}{\lambda_1^{**}}$.

$$\begin{aligned}
\text{FOC of } (P_b) &\Rightarrow K'(\lambda_0^{**}) = K(\lambda_1^{**}) + p(1 - \lambda_1^{**}) \\
&\Rightarrow \lambda_1^{**} K'(\lambda_0^{**}) - K(\lambda_1^{**}) = (1 - \lambda_1^{**}) [-K(\lambda_1^{**}) + p\lambda_1^{**}] > 0 \\
&\Rightarrow K'(\lambda_0^{**}) > \frac{K(\lambda_1^{**})}{\lambda_1^{**}} = K'(\frac{\tilde{c}}{v_g}) \\
&\Rightarrow \lambda_0^{**} > \frac{\tilde{c}}{v_g} \\
&\Rightarrow \tilde{c} < v_g \lambda_0^{**} < v_g \lambda_1^{**}
\end{aligned}$$

□

We then compare the optimal sender surplus between the solution to (P_{2S_+}) and the solution to (P_{2S_0}) , and show that the optimal S_0 strategy is always preferred to the optimal S_+ strategy when both types of strategy are feasible.

Proposition 13. *Suppose $c \leq \tilde{c}$ and $\mu_0 < \hat{\mu}_0$. The sender uses S_0 strategy when she provides information in both periods.*

Proof. $\forall \mu_0 < \hat{\mu}_0$ such that S_0 (S_+) strategy is feasible, denote the optimal sender surplus by $\Pi_{S_0}(\mu_0)$ ($\Pi_{S_+}(\mu_0)$).

i) $\mu'_0 := \frac{(v_g - c)(c - v_b \lambda_1^{**}) + c}{v_g} \leq \mu_0 < \hat{\mu}_0$
 $\frac{(v_g - c)(c - v_b \lambda_1^{**}) + c}{v_g} \leq \mu_0 \Leftrightarrow \hat{\mu}_1(\mu_0) \leq c - v_b \lambda_1^{**}$. According to Proposition 11, $(\lambda_{0,S_+}, \mu_{1,S_+}) = (\frac{\mu_0 - 2c + v_b \lambda_1^{**}}{-v_b(1 - \lambda_1^{**}) - c}, c - v_b \lambda_1^{**})$ gives $\Pi_{S_+}(\mu_0)$. $\mu_1 \in [\frac{c}{v_g}, \frac{v_g \mu_0 - c}{v_g - c} \wedge c - v_b \lambda_1^{**}]$ in (P_{2S_0}'') and $\frac{(v_g - c)(c - v_b \lambda_1^{**}) + c}{v_g} \leq \mu_0 \Leftrightarrow \frac{v_g \mu_0 - c}{v_g - c} \geq c - v_b \lambda_1^{**}$. Consider $(\lambda_0, \mu_1) = (\frac{\mu_0 - \mu_1 - c}{-v_b - \mu_1}, c - v_b \lambda_1^{**})$, which satisfies the constraints in (P_{2S_0}'') and is identical to $(\lambda_{0,S_+}, \mu_{1,S_+})$. So, $\Pi_{S_0}(\mu_0) \geq \Pi_{S_+}(\mu_0)$.

ii) $\mu_0 < \mu'_0$
 $\mu_0 < \mu'_0 \Leftrightarrow \hat{\mu}_1(\mu_0) > c - v_b \lambda_1^{**}$. According to Proposition 11, $(\lambda_{0,S_+}, \mu_{1,S_+}) = (\frac{\mu_0 - 2c + v_b \lambda_1^{**}}{-v_b(1 - \lambda_1^{**}) - c}, \hat{\mu}_1(\mu_0))$ gives $\Pi_{S_+}(\mu_0)$. One can verify that $\frac{v_g \mu'_0 - c}{v_g - c} = c - v_b \lambda_1^{**}$, $\frac{\mu'_0 - 2c + v_b \lambda_1^{**}}{-v_b(1 - \lambda_1^{**}) - c} = \frac{c}{v_g} \leq \lambda_0^{**}$. So, $\mu'_0 \leq \hat{\mu}_0$, which implies that $\lambda_{0,S_+}(\mu'_0) \leq \lambda_0^{**}$.

First, consider μ_0 such that both S_0 and S_+ strategies are feasible. Let $\mu_1^u(\mu_0) := \frac{v_g \mu_0 - c}{v_g - c}$.

$$\begin{aligned}
\Pi_{S_+}(\mu_0) &= -K(\lambda_{0,S_+}(\mu_0)) + p\lambda_{0,S_+}(\mu_0) + (1 - \lambda_{0,S_+}(\mu_0))[-K(\lambda_1^{**}) + p\lambda_1^{**}] \\
\Pi_{S_0}(\mu_0) &\geq \Pi_{S_0}(\mu_0) := -K(\frac{c}{v_g}) + \frac{cp}{v_g} + (1 - \frac{c}{v_g}) \left[-K(\frac{\mu_1^u(\mu_0) - c}{-v_b}) + \frac{(\mu_1^u(\mu_0) - c)p}{-v_b} \right]
\end{aligned}$$

Lemma 10. $\frac{\mu_1^u(\mu_0)-c}{-v_b} \geq \lambda_{0,S_+}(\mu_0), \forall \mu_0 < \mu'_0$.

Proof. $\frac{d}{d\mu_0}[\frac{\mu_1^u(\mu_0)-c}{-v_b}] = \frac{v_g}{-v_b(v_g-c)}, \frac{d}{d\mu_0}[\lambda_{0,S_+}(\mu_0)] = \frac{1}{-v_b(1-\lambda_1^{**})-c}.$

$$\frac{d}{d\mu_0}[\frac{\mu_1^u(\mu_0)-c}{-v_b}] \leq \frac{d}{d\mu_0}[\lambda_{0,S_+}(\mu_0)] \Leftrightarrow c(-2v_b-1) \leq -v_b v_g \lambda_1^{**} \quad (*)$$

If $v_b \geq -1/2$, $(*)$ always holds. If $(-1 <) v_b < -1/2$, we have that $\frac{-v_b}{-2v_b-1} \geq 1 \Rightarrow c \leq v_g \lambda_1^{**} \leq \frac{-v_b v_g}{-2v_b-1} \lambda_1^{**} \Rightarrow (*)$ also holds. So, $\frac{d}{d\mu_0}[\frac{\mu_1^u(\mu_0)-c}{-v_b}] \leq \frac{d}{d\mu_0}[\lambda_{0,S_+}(\mu_0)], \forall \mu_0 < \mu'_0$.

Note that $\frac{\mu_1^u(\mu'_0)-c}{-v_b} = \lambda_1^{**} > \lambda_0^{**} \geq \lambda_{0,S_+}(\mu'_0)$. This concludes the proof. \square

Now we calculate the increasing rate of the sender surplus as a function of μ_0 :

$$\begin{aligned} \frac{d\Pi_{S_+}(\mu_0)}{d\mu_0} &= \frac{K'(\lambda_{0,S_+}(\mu_0)) - K(\lambda_1^{**}) + \frac{cp}{v_b}}{v_b(1-\lambda_1^{**})+c} - \frac{p}{v_b} \\ \frac{d\underline{\Pi}_{S_0}(\mu_0)}{d\mu_0} &:= \frac{K'(\frac{\mu_1^u(\mu_0)-c}{-v_b})}{v_b} - \frac{p}{v_b} \\ \frac{d\Pi_{S_+}(\mu_0)}{d\mu_0} &\geq \frac{d\underline{\Pi}_{S_0}(\mu_0)}{d\mu_0} \Leftrightarrow \frac{K'(\frac{\mu_1^u(\mu_0)-c}{-v_b})}{-v_b} + \frac{K(\lambda_1^{**}) - \frac{cp}{v_b}}{-v_b(1-\lambda_1^{**})-c} \geq \frac{K'(\lambda_{0,S_+}(\mu_0))}{-v_b(1-\lambda_1^{**})-c} \quad (*) \end{aligned}$$

$$\begin{aligned} c &\leq \tilde{c} = v_g K'^{-1} \left[\frac{K(\lambda_1^{**})}{\lambda_1^{**}} \right] \\ &\Leftrightarrow K(\lambda_1^{**}) \geq \lambda_1^{**} K'(\frac{c}{v_g}) \\ &\Rightarrow -v_b \left[K(\lambda_1^{**}) - \frac{cp}{v_b} \right] \geq (c - v_b \lambda_1^{**}) K'(\frac{c}{v_g}) \left(c \leq v_g \lambda_1^{**} \Rightarrow K'(\frac{c}{v_g}) \leq K'(\lambda_1^{**}) = p \right) \\ &\Rightarrow -v_b \left[K(\lambda_1^{**}) - \frac{cp}{v_b} \right] \geq (c - v_b \lambda_1^{**}) K'(\lambda_{0,S_+}(\mu_0)) \left(\lambda_{0,S_+}(\mu_0) < \lambda_{0,S_+}(\mu'_0) = \frac{c}{v_g} \right) \\ &\Leftrightarrow \frac{K'(\lambda_{0,S_+}(\mu_0))}{-v_b} + \frac{K(\lambda_1^{**}) - \frac{cp}{v_b}}{-v_b(1-\lambda_1^{**})-c} \geq \frac{K'(\lambda_{0,S_+}(\mu_0))}{-v_b(1-\lambda_1^{**})-c} \\ &\stackrel{\text{Lemma 10}}{\Rightarrow} \frac{K'(\frac{\mu_1^u(\mu_0)-c}{-v_b})}{-v_b} + \frac{K(\lambda_1^{**}) - \frac{cp}{v_b}}{-v_b(1-\lambda_1^{**})-c} \geq \frac{K'(\lambda_{0,S_+}(\mu_0))}{-v_b(1-\lambda_1^{**})-c} \\ &\stackrel{(*)}{\Leftrightarrow} \frac{d\Pi_{S_+}(\mu_0)}{d\mu_0} \geq \frac{d\underline{\Pi}_{S_0}(\mu_0)}{d\mu_0} \end{aligned}$$

One can verify that $\lambda_{0,S_+}(\mu'_0) = \frac{c}{v_g}$, $\frac{\mu_1^u(\mu'_0)-c}{-v_b} = \lambda_1^{**}$. So, $\Pi_{S_+}(\mu'_0) = \underline{\Pi}_{S_0}(\mu'_0)$. Therefore, $\underline{\Pi}_{S_0} \geq \Pi_{S_+}(\mu_0)$. $\Pi_{S_0}(\mu_0) \geq \underline{\Pi}_{S_0} \Rightarrow \Pi_{S_0}(\mu_0) \geq \Pi_{S_+}(\mu_0)$.

Now we show that the S_0 strategy is feasible whenever the S_+ strategy is feasible, which concludes the proof of Proposition 13.

Lemma 11. *Suppose $c < v_g \lambda_1^{**}$. For any μ_0 such that S_+ strategy is feasible, S_0 strategy is also feasible.*

Proof. Suppose there exists μ_0 such that S_+ strategy is feasible while S_0 strategy is not feasible. Then, $2c - v_b \lambda_1^{**} < \frac{2v_g - c}{(v_g)^2}$ and $\frac{(2 - \lambda_1^{**})c}{v_g(1 - \lambda_1^{**}) + c} < \frac{2v_g - c}{(v_g)^2}$, which is equivalent to $\lambda_1^{**} < \frac{c^2}{v_b(v_g)^2} + \frac{2c}{v_g}$ and $c > v_g \lambda_1^{**}$, which is not possible as we assumed that $c < v_g \lambda_1^{**}$. \square

\square

Proposition 13 tells us that we could limit our attention to S_0 strategy when $c \leq \tilde{c}$. Proposition 12 has characterized the optimal S_0 strategy. One can see from the characterization of the optimal strategy that the sender's payoff strictly increases in the prior.

\square

Proposition 14. (*comparative statics wrt the prior when the search cost is intermediate*) *When the search cost is intermediate, $v_g \lambda_0^{**} < c < v_g \lambda_1^{**}$, and the sender provides information in both periods. There exists $\mu_{2,+} \in [\frac{2v_g - c}{(v_g)^2} c, \hat{\mu}_0)$ and $\mu_{2,0} \in [\frac{2v_g - c}{(v_g)^2} c, \mu_{2,+}]$. The arrival rate of good news in the first period, λ_0^* , remains the same when $\mu_0 < \mu_{2,0}$ and strictly increases in the prior when $\mu_0 > \mu_{2,+}$. The arrival rate of good news in the second period, λ_1^* , strictly increases in the prior when $\mu_0 < \mu_{2,0}$ and remains the same when $\mu_0 > \mu_{2,+}$. The belief after observing good news in the second period, $\bar{\mu}_1^*$, strictly decreases in the prior when $\mu_0 < \mu_{2,0}$ or $\mu_0 > \mu_{2,+}$. Good news always fully reveals the state in the first period, $\bar{\mu}_0^* \equiv 1$.*

When the expected receiver surplus in the second period is zero (S_0 strategy), the minimum amount of information for the receiver to search in the first period is already too high. Under the S_+ strategy, the receiver anticipates the sender to provide favorable information in the second period, which relaxes the first-period participation constraint. Therefore, the receiver is willing to search even if the sender provides less information in the first period. This benefits the sender. However, the S_+ strategy has the disadvantage of inducing higher expected receiver search costs.

When the prior is low, the disadvantage of the S_+ strategy dominates the advantage. The sender prefers the S_0 strategy to the S_+ strategy. She faces an information over-provision issue in the first period and an under-provision issue in the second period. In the first period, she provides the minimum amount of information for the receiver to search. So, the arrival

rate of good news in the first period, λ_0^* , does not depend on the prior. In the second period, she provides the maximum amount of information. More frequent good signals are feasible when the prior is higher. Even if the receiver becomes less certain about the state being good after observing good news, he would still search as long as the likelihood of receiving good news and earning a strictly positive surplus increases. In equilibrium, the sender trades off the precision of good news for the frequency as the prior increases.

When the prior is high, the advantage of the S_+ strategy dominates the disadvantage. The sender prefers the S_+ strategy to the S_0 strategy. She faces an information underprovision issue in the first period and no distortion in the second period. In the first period, she provides the maximum amount of feasible information, which strictly increases in the prior. In the second period, she provides the optimal amount of information, which does not depend on the prior. When the prior increases, the participation constraint in the first period is relaxed. To persuade the receiver to search in the first period, the sender could provide information less favorable to the receiver in the second period. She trades off the precision of good news in the second period for the frequency of good news in the first period. So, the belief after observing good news, $\bar{\mu}_1^*$, strictly decreases in the prior. The optimal strategy is discontinuous when the sender switches the types of strategy (as illustrated in Figure 6).

Proof of Proposition 14. We compare the sender surplus between the solution to (P_{2S_+}) and the solution to (P_{2S_0}) . The following result shows that the optimal S_+ strategy generates a strictly higher sender surplus than the optimal S_0 strategy when the prior is high.

Lemma 12. *If $v_g\lambda_0^{**} < c < v_g\lambda_1^{**}$, the receiver gets strictly positive surplus in the second period when μ_0 is close to $\hat{\mu}_0$.*

Proof. According to Proposition 11, the sender achieves the benchmark payoff $-K(\lambda_0^{**}) + p\lambda_0^{**} + (1 - \lambda_0^{**})[-K(\lambda_1^{**}) + p\lambda_1^{**}]$ when $\mu_0 \rightarrow \hat{\mu}_0$ by S_+ strategy. According to Lemma 8, the sender surplus of S_0 strategy is $\leq -K(\frac{c}{v_g}) + \frac{cp}{v_g} + (1 - \frac{c}{v_g})[-K(\lambda_1^{**}) + p\lambda_1^{**}] < -K(\lambda_0^{**}) + p\lambda_0^{**} + (1 - \lambda_0^{**})[-K(\lambda_1^{**}) + p\lambda_1^{**}]$ as $\lambda_0^{**} < \frac{c}{v_g} = \lambda_0$. The difference of the benchmark sender surplus and the sender surplus of S_0 strategy is larger than a strictly positive constant. So, when $\mu_0 \rightarrow \hat{\mu}_0$, S_+ strategy gives the sender strictly higher payoff. $v_g\lambda_0^{**} < c \Leftrightarrow \hat{\mu}_1(\hat{\mu}_0) > c - v_b\lambda_1^{**} \Rightarrow$ the receiver gets strictly positive surplus from S_+ strategy when $\mu_0 \rightarrow \hat{\mu}_0$. Thus, the receiver gets strictly positive surplus in the second period in equilibrium when $\mu_0 \rightarrow \hat{\mu}_0$. Continuity of the optimal strategy and sender surplus then implies that there exists a neighborhood $[\hat{\mu}_0 - \delta, \hat{\mu}_0]$ for some $\delta > 0$ such that the receiver gets strictly positive surplus in the second period in equilibrium when $\mu_0 \in [\hat{\mu}_0 - \delta, \hat{\mu}_0]$. \square

When $\frac{c}{v_g} \leq \mu_0 < \mu_{1,2}$, the sender provides information in one period. When $\mu_{1,2} \leq \mu_0 < \hat{\mu}_0$, the sender provides information in both periods. Proposition 11, and Lemma 8 imply the explicit form of the optimal strategy. When $\mu_{1,2} \leq \mu_0 < \mu_{2,+}$, $(\lambda_0^*, \bar{\mu}_0^*) = (\frac{c}{v_g}, 1)$, $(\lambda_1^*, \bar{\mu}_1^*) = (\frac{v_g(\mu_0 - c) - c(1 - c)}{p(v_g - c)}, \frac{[v_g\mu_0 - c]p}{v_g(\mu_0 - c) - c(1 - c)})$, $\mu_1^* = \frac{v_g\mu_0 - c}{v_g - c} < c - v_b\lambda_1^{**}$. The receiver gets zero surplus at each period. When $\mu_{2,+} \leq \mu_0 < \hat{\mu}_0$, $(\lambda_0^*, \bar{\mu}_0^*) = (\frac{\mu_0 - 2c + v_b\lambda_1^{**}}{-v_b(1 - \lambda_1^{**}) - c}, 1)$, $(\lambda_1^*, \bar{\mu}_1^*) = (\lambda_1^{**}, \frac{\mu_1}{\lambda_1^{**}})$, $\mu_1^* = \hat{\mu}_1(\mu_0) > c - v_b\lambda_1^{**}$, where $\hat{\mu}_1(\mu_0) = \frac{2c - v_b\lambda_1^{**} - (1 + c - v_b\lambda_1^{**} + v_b)\mu_0}{c - v_b - \mu_0}$. The receiver gets strictly positive surplus in the second period and zero total surplus. \square

Proof of Propostion 5. When the search cost is high, $v_g\lambda_1^{**} \leq c < \hat{c}$, the optimal strategy of the sender is $(\lambda_t^*, \bar{\mu}_t^*) = (\frac{c}{v_g}, 1)$, $t = 0, 1$ according to Proposition 4, which does not depend on η .

When the search cost is low, $c < \tilde{c} < v_g\lambda_0^{**}$. The boundary solution does not depend on η . Consider the interior solution to (P_{2S_0}) , we have: $\eta\tilde{K}(\frac{\mu_1 - c}{-v_b}) + \frac{p - \mu_1}{p}\eta\tilde{K}'(\frac{\mu_1 - c}{-v_b}) - \eta\tilde{K}'(\frac{\mu_0 - \mu_1 - c}{-v_b - \mu_1}) + c = 0$. The LHS strictly increases in μ_1 and strictly decreases in η . So, $\eta \uparrow \Rightarrow \mu_1^* \uparrow \Rightarrow \lambda_1^* = \frac{\mu_1^* - c}{p} \uparrow, \lambda_0^* = \frac{\mu_0 - \mu_1^* - c}{p - \mu_1^*} \downarrow$. \square

Proposition 15. (efficient strategy of the subgame) *At the second period, the social planner does not provide information when $\mu_1 < \mu_{0,1}$. When $\mu_1 \geq \mu_{0,1}$, the efficient signal fully reveals the state when good news arrives, $\bar{\mu}_{1,e} = 1$; the arrival rate of good news, $\lambda_{1,e}$, depends on the search cost:*

1. if $c \geq v_g\lambda_1^{**}$, then there exists a unique $\hat{c} \in (v_g\lambda_1^{**}, \mu_1 v_g]$ such that the social planner does not provide information if $c > \hat{c}$ and $\lambda_{1,e} = \frac{c}{v_g} \vee (\tilde{\lambda}_1 \wedge \mu_1)$ if $c \leq \hat{c}$.
2. if $c \in [\mu_1 + v_b\lambda_1^{**}, v_g\lambda_1^{**})$, then $\lambda_{1,e} = \mu_1$.
3. if $c < \mu_1 + v_b\lambda_1^{**} \wedge v_g\lambda_1^{**}$, then $\lambda_{1,e} = \tilde{\lambda}_1 \wedge \mu_1$.

Proof of Proposition 15. We first introduce a benchmark problem, in which the receiver is forced to participate and the social planner could generate any signal that fully reveals the state when good news arrives. The social planner chooses the information structure to maximize total welfare. We will use the solution throughout the remaining section.

$$\max_{\lambda_1} -K(\lambda_1) + \lambda_1 \quad (E_b)$$

Lemma 13. *The optimal solution to (E_b) exists and is unique. Denote it by $\tilde{\lambda}_1$. The objective function under $\tilde{\lambda}_1$ is strictly positive. $\tilde{\lambda}_1$ does not depend on the search cost c and $\tilde{\lambda}_1 > \lambda_1^{**}$, the solution to the payoff-maximizing benchmark problem.*

Proof. All the results follow from the same argument as the proof of Lemma 1 except $\tilde{\lambda}_1 > \lambda_1^{**}$. The F.O.C.'s imply $K'(\tilde{\lambda}_1) = 1 > p = K'(\lambda_1^{**}) \Rightarrow \tilde{\lambda}_1 > \lambda_1^{**}$. \square

As the social planner wants to maximize the total welfare, she always wants to make the precision of the signal, $\bar{\mu}_1$, as high as possible (subject to the feasibility constraint) given μ_1 and λ_1 , to increase receiver surplus while holding sender surplus fixed. So, if the social planner provides information, then $\bar{\mu}_1 = \frac{\mu_1}{\lambda_1} \wedge 1$.

1. if $c \geq v_g \lambda_1^{**}$ (i.e. $\lambda_1^{**} \leq \frac{c}{v_g}$), then $TS = \begin{cases} -K(\lambda_1) + \mu_1 - c, & \text{if } \lambda_1 \geq \mu_1 \\ -K(\lambda_1) + \lambda_1 - c, & \text{if } \lambda_1 < \mu_1 \end{cases} \Rightarrow \lambda_{1,e} = \frac{c}{v_g} \vee (\tilde{\lambda}_1 \wedge \mu_1)$ when the social planner provides information. The social planner would provide information if the total surplus is non-negative. Similar to the proof of Proposition 2, one can show that there exists a unique $\hat{c} \in (v_g \lambda_1^{**}, \mu_1 v_g]$ such that the social planner does not provide information iff $c > \hat{c}$. Moreover, if $\hat{c} \geq (1-p)\tilde{\lambda}_1$, $\lambda_{1,e} = \frac{c}{1-p} \Rightarrow \bar{\mu}_{1,e} = 1 \Rightarrow TS = -K(\lambda_{1,e}) + p\lambda_{1,e} \Rightarrow \hat{c} = \hat{c}$.
2. if $c \in [\mu_1 - p\lambda_1^{**}, v_g \lambda_1^{**})$ (i.e. $\lambda_1^{**} \geq \frac{\mu_1 - c}{-v_b} > \frac{c}{1-p}$), then by the previous argument, $\lambda_{1,e} = \frac{c}{1-p} \vee (\tilde{\lambda}_1 \wedge \mu_1) = \mu_1$.
3. if $c < \mu_1 - p\lambda_1^{**} \wedge (1-p)\lambda_1^{**}$ (i.e. $\lambda_1^{**} \in \left(\frac{c}{1-p}, \frac{\mu_1 - c}{-v_b}\right)$), then the total welfare $TS = \begin{cases} -K(\lambda_1) + \mu_1 - c, & \text{if } \lambda_1 \geq \mu_1 \\ -K(\lambda_1) + \lambda_1 - c, & \text{if } \lambda_1 < \mu_1 \end{cases} \Rightarrow \lambda_{1,e} = \tilde{\lambda}_1 \wedge \mu_1$.

\square

Proof of Proposition 6. When $(1-p)\tilde{\lambda}_1 \leq c \leq \hat{c}$, the constrained program of the social planner is:

$$\begin{aligned} & \max -K(\lambda_0) + \lambda_0 \bar{\mu}_0 - c + (1 - \lambda_0) \left[-K\left(\frac{c}{1-p}\right) + \frac{cp}{1-p} \right] & (E_{2H}) \\ & \text{s.t. } (IR_0), (F_0), \mu_1 \geq \frac{c}{1-p} \end{aligned}$$

Using similar methods of finding the optimal sender strategy in the main text, one can show that the solution to (E_{2H}) is $(\lambda_{0,e}, \lambda_{1,e}) = (\frac{c}{1-p}, \frac{c}{1-p})$. Therefore, there is no information distortion.

When $v_g \lambda_1^{**} \leq c < (1-p)\tilde{\lambda}_1 \wedge \hat{c}$, the constrained program of the social planner is:

$$\begin{aligned} & \max -K(\lambda_0) + \lambda_0 \bar{\mu}_0 - c + (1 - \lambda_0) \left[-K(\tilde{\lambda}_1 \wedge \mu_1) + \tilde{\lambda}_1 \wedge \mu_1 - c \right] \\ & \text{s.t. } (IR_0), (F_0), \mu_1 \geq \frac{c}{1-p} \end{aligned} \quad (E_{2I})$$

Using similar methods of finding the optimal sender strategy in the main text, one can show that the solution to (E_{2I}) is $(\lambda_{0,e}, \lambda_{1,e}) = (\frac{c}{1-p}, \frac{c}{1-p})$. Therefore, there is no information distortion.

When $c < v_g \lambda_1^{**}$, one can see that we could restrict μ_1 to be less than or equal to $\tilde{\lambda}_1$ without loss of generality. The constrained program of the social planner is:

$$\begin{aligned} & \max -K(\lambda_0) + \lambda_0 \bar{\mu}_0 - c + (1 - \lambda_0) [-K(\mu_1) + \mu_1 - c] \\ & \text{s.t. } (IR_0), (F_0), \mu_1 \geq \frac{c}{1-p} \end{aligned} \quad (E_{2S_0})$$

Using similar methods of finding the optimal sender strategy in the main text, one can show that the solution to (E_{2S_0}) is $(\lambda_{0,e}, \lambda_{1,e}) = (\frac{\mu_0 - \frac{c}{1-p} - c}{p - \frac{c}{1-p}}, \frac{c}{1-p})$. Therefore, the sender provides too much information (relative to the efficient solution) in the second period and too little information in the first period. \square

Proof of Proposition 7. One can see that if the expected surplus of the receiver in the second period is 0 in the optimal solution to (P_{dc}) , then $(\lambda_1^*, \bar{\mu}_1^*)$ solves (P_1) . Otherwise, the sender can strictly increase the payoff by using the same $(\lambda_0^*, \bar{\mu}_0^*)$ and replacing $(\lambda_1^*, \bar{\mu}_1^*)$ by the optimal solution to (P_1) , holding the same μ_1 . Hence, if dynamic commitment power strictly increases the sender surplus, the solution to (P_{dc}) must satisfy: $\mathbb{E}[\text{receiver surplus at } t = 1] = (1 - \lambda_0)[\lambda_1(\bar{\mu}_1 + v_b) - c] > 0$. Denote the optimal sender surplus when the sender does not have dynamic commitment power by Π_{wo} . The corresponding optimal strategy of the sender is $(\lambda_t^*, \bar{\mu}_t^*) = (\frac{c}{v_g}, 1)$ according to Proposition 4. Consider the following strategy: $(\lambda_0, \bar{\mu}_0, \lambda_1, \bar{\mu}_1) = (\frac{c - \delta v_g}{(1-\delta)v_g}, 1, \frac{c}{v_g} + \delta, 1)$. Denote the corresponding sender surplus by $\Pi(\delta)$. One can verify that it is feasible when $\delta > 0$ is small and the sender has dynamic commitment power, and leads to a payoff no larger than the optimal sender surplus with dynamic commitment (denote it by Π_w).

$$\begin{aligned}
\Pi(\delta) &= -K \left[\frac{c - \delta v_g}{(1 - \delta)v_g} \right] + p \frac{c - \delta v_g}{(1 - \delta)v_g} + \left[1 - \frac{c - \delta v_g}{(1 - \delta)v_g} \right] \left[-K \left(\frac{c}{v_g} + \delta \right) + \frac{cp}{v_g} + \delta p \right] \\
\frac{d\Pi(\delta)}{d\delta} &= \frac{v_g - c}{(1 - \delta)^2 v_g} [I_1(\delta) + I_2(\delta)] \\
, \text{ where } I_1(\delta) &= K' \left[\frac{c - \delta v_g}{(1 - \delta)v_g} \right] - (1 - \delta) K' \left(\frac{cp}{v_g} + \delta \right) \\
I_2(\delta) &= \frac{cp}{v_g} - K \left(\frac{c}{v_g} + \delta \right)
\end{aligned}$$

$I_1(\delta) \rightarrow 0$, $I_2(\delta) \rightarrow \frac{cp}{v_g} - K(\frac{c}{v_g}) > 0$ as $\delta \rightarrow 0$. Therefore, $\exists \hat{\delta} > 0$ s.t. $\frac{d\Pi(\delta)}{d\delta} > 0$, $\forall \delta \in (0, \hat{\delta}]$. Since $\Pi(\delta)$ is continuous in δ and $\Pi(0) = \Pi_{wo}$, we have $\Pi_w \geq \Pi(\delta) > \Pi_{wo}$, $\forall \delta \in (0, \hat{\delta}]$.

We now show that the benefit of dynamic commitment power vanishes as the search cost approaches zero in several steps. Denote the optimal sender surplus when the sender has (does not have) dynamic commitment power and the search cost is c by $\Pi_{w,c}$ ($\Pi_{wo,c}$). Note that $\Pi_{w,c} \geq \Pi_{wo,c}$, $\forall c \geq 0$.

Lemma 14. *When the search cost is zero, the optimal sender surpluses with and without dynamic commitment power are the same, $\Pi_{w,0} = \Pi_{wo,0}$.*

Proof. Suppose the sender provides information in both periods. When the search cost is 0 and the sender uses one-shot experiments, one can see that (IR_0) and (IR_1) are equivalent to $\bar{\mu}_0 \geq -v_b$ and $\bar{\mu}_1 \geq -v_b$. Therefore, even if the receiver obtains strictly positive surplus in the second period, the first-period participation constraint is not relaxed. Hence, the second-period strategy of the sender maximizes her payoff of the subgame in the solution to (P_{dc}) , and thus satisfies the constraints of (P_2) . To show that dynamic commitment power does not improve the sender's payoff when $c = 0$, we just need to show that iterative signals are not optimal when the sender has dynamic commitment power.¹² When $c = 0 < v_g \lambda_1^{**}$ and $\mu_1 > c - v_b \lambda_1^{**} = -v_b \lambda_1^{**}$, the optimal strategy with and without dynamic commitment power coincides, as $(\lambda_1^*, \bar{\mu}_1^*) = (\lambda_1^{**}, \frac{\mu_1}{\lambda_1^{**}} \wedge 1)$. In the proof of Proposition 1, we have shown that iterative signals are not optimal. So, we just need to show that iterative signals are not optimal when $\mu_1 \leq -v_b \lambda_1^{**}$.

If $\mu_0 \geq -v_b \lambda_1^{**}$, then $\mu_1 = \bar{\mu}_0 > \mu_0 \geq -v_b \lambda_1^{**}$. So, iterative signals are not optimal.

¹² We have shown in Proposition 1 that iterative signals are not optimal when the sender does not have dynamic commitment power.

If $\mu_0 < -v_b \lambda_1^{**}$, we have:

$$\begin{aligned}\Pi_1(\mu_0) &= -K\left(\frac{\mu_0}{-v_b}\right) + \frac{\mu_0 p}{-v_b} \\ \Pi_{iter}(\mu_0) &= \max_{\lambda_0, \mu_1} -K(\lambda_0) + \lambda_0 \left[-K\left(\frac{\mu_1}{-v_b}\right) + \frac{\mu_1 p}{-v_b} \right] \\ \text{s.t. } &(IR_{0,iter}), (IR_{1,iter}), (F_0), (F_1), \mu_1 = \tilde{\mu}_0\end{aligned}$$

Denote the optimal solution when the sender uses iterative signals by $(\tilde{\lambda}_0, \tilde{\mu}_1)$. $\Pi_{iter}(\mu_0) \leq \Pi_1(\mu_0) \Leftrightarrow -K(\tilde{\lambda}_0) + \tilde{\lambda}_0 \left[-K\left(\frac{\tilde{\mu}_1}{-v_b}\right) + \frac{\tilde{\mu}_1 p}{-v_b} \right] \leq -K\left(\frac{\mu_0}{-v_b}\right) + \frac{\mu_0 p}{-v_b}$. To show that iterative signals are not optimal, it suffices to show that $-\tilde{\lambda}_0 K\left(\frac{\tilde{\mu}_1}{-v_b}\right) + \frac{\tilde{\lambda}_0 \tilde{\mu}_1 p}{-v_b} \leq -K\left(\frac{\mu_0}{-v_b}\right) + \frac{\mu_0 p}{-v_b}$. Strict convexity of $K(\cdot) \Rightarrow \tilde{\lambda}_0 K\left(\frac{\tilde{\mu}_1}{-v_b}\right) = \tilde{\lambda}_0 K\left(\frac{\tilde{\mu}_1}{-v_b}\right) + (1 - \tilde{\lambda}_0)K(0) \geq K\left(\frac{\tilde{\lambda}_0 \tilde{\mu}_1}{-v_b}\right)$. Thus, $-\tilde{\lambda}_0 K\left(\frac{\tilde{\mu}_1}{-v_b}\right) \leq -K\left(\frac{\tilde{\lambda}_0 \tilde{\mu}_1}{-v_b}\right)$. It suffices to show that $-K\left(\frac{\tilde{\lambda}_0 \tilde{\mu}_1}{-v_b}\right) + \frac{\tilde{\lambda}_0 \tilde{\mu}_1 p}{-v_b} \leq -K\left(\frac{\mu_0}{-v_b}\right) + \frac{\mu_0 p}{-v_b}$, which holds because $(F_0) \Rightarrow \tilde{\lambda}_0 \tilde{\mu}_1 \leq \mu_0$ and we know from the F.O.C. that $-K(\lambda) + p\lambda$ strictly increases in λ when $\lambda < \lambda_1^{**}$ (here, $\lambda \leq \frac{\mu_0}{-v_b} < \lambda_1^{**}$). \square

We now make the following observation. Since $\Pi_{w,c} \geq \Pi_{wo,c}, \forall c \geq 0$ and $\Pi_{w,0} = \Pi_{wo,0}$, we must have $\Pi_{wo,c} \rightarrow \Pi_{w,c}$ as $c \rightarrow 0$ if $\Pi_{wo,c} \rightarrow \Pi_{wo,0}$ as $c \rightarrow 0$.¹³ The next result confirms that it is indeed the case and thus finishes the proof.

Lemma 15. $\Pi_{wo,c} \rightarrow \Pi_{wo,0}$ as $c \rightarrow 0$.

Proof. Proposition 13 shows that the optimal strategy is S_0 strategy when the search cost is low. So, according to Lemma 6 and 7, for any c small enough, the sender's problem is:

$$\begin{aligned}\Pi_{wo,c} &= \max_{\mu_1} -K\left(\frac{\mu_0 - \mu_1 - c}{-v_b - \mu_1}\right) + p \frac{\mu_0 - \mu_1 - c}{-v_b - \mu_1} + \left(1 - \frac{\mu_0 - \mu_1 - c}{-v_b - \mu_1}\right) \left[-K\left(\frac{\mu_1 - c}{-v_b}\right) + \frac{(\mu_1 - c)p}{-v_b} \right] \\ &\quad (P_{2S_0}''') \\ \text{s.t. } \mu_1 &\in \left[\frac{c}{v_g}, \frac{v_g \mu_0 - c}{v_g - c} \right]\end{aligned}$$

Denote the solution when the search cost is 0 by $\mu_{1,0}^*$. Define $\mu_{1,c}$ to be the closest value to $\mu_{1,0}^*$ among $\left[\frac{c}{v_g}, \frac{v_g \mu_0 - c}{v_g - c} \right]$ and denote the corresponding sender surplus by $\underline{\Pi}_{wo,c}$. One can see that $\underline{\Pi}_{wo,c} \leq \Pi_{wo,c}$. Since $\mu_{1,0}^* \in \left[0, \frac{v_g \mu_0}{v_g} \right]$, we have $\mu_{1,c} \rightarrow \mu_{1,0}^*$ as $c \rightarrow 0$. Therefore, $\underline{\Pi}_{wo,c} \rightarrow \Pi_{wo,0}$ as $c \rightarrow 0$. Since $\underline{\Pi}_{wo,c} \leq \Pi_{wo,c} \leq \Pi_{wo,0}$, we also have $\Pi_{wo,c} \rightarrow \Pi_{wo,0}$ as $c \rightarrow 0$. \square

¹³ One can easily show this observation formally by the triangle inequality. \square

Proof of Proposition 8. When $c \geq \widehat{c}$, the sender does not provide information for any δ .

(1) $v_g \lambda_1^{**} \leq c < \widehat{c}$

By the same argument as the proof of Proposition 4, one can see that the optimal strategy of the sender does not depend on δ . For low prior, $\mu_0 \in [\frac{c}{v_g}, \frac{2v_g - c}{(v_g)^2}c)$, the sender provides information in one period, $(\lambda^*, \bar{\mu}^*) = (\frac{c}{v_g}, 1)$. For high prior, $\mu_0 \geq \frac{2v_g - c}{(v_g)^2}c$, the sender provides information in both periods, $(\lambda_t^*, \bar{\mu}_t^*) = (\frac{c}{v_g}, 1), t = 0, 1$.

(2) $c \leq \tilde{c}$

The sender's problem can be divided into 2 cases:

$$\begin{aligned} & \max -K(\lambda_0) + p\lambda_0 + \delta(1 - \lambda_0) [-K(\lambda_1^{**}) + p\lambda_1^{**}] & (P_{2S_+}^\delta) \\ \text{s.t. } & (IR_{0,\delta}), (F_0), \mu_1 \in [c - v_b \lambda_1^{**}, \lambda_1^{**}] \end{aligned}$$

$$\begin{aligned} & \max -K(\lambda_0) + p\lambda_0 + \delta(1 - \lambda_0) \left[-K\left(\frac{\mu_1 - c}{-v_b}\right) + \frac{(\mu_1 - c)p}{-v_b} \right] & (P_{2S_0}^\delta) \\ \text{s.t. } & (IR_{0,\delta}), (F_0), \mu_1 \in \left[\frac{c}{v_g}, c - v_b \lambda_1^{**} \right] \end{aligned}$$

Consider the solution to $(P_{2S_0}^\delta)$. By the same argument as the proof of Lemma 6, $(P_{2S_0}^\delta)$ is equivalent to

$$\begin{aligned} & \max -K(\lambda_0) + p\lambda_0 + \delta(1 - \lambda_0) \left[-K\left(\frac{\mu_1 - c}{-v_b}\right) + \frac{(\mu_1 - c)p}{-v_b} \right] & (P_{2S_0}^{\delta'}) \\ \text{s.t. } & \lambda_0 \in \left[\frac{\mu_0 - \mu_1}{1 - \mu_1}, \frac{\mu_0 - \mu_1 - c}{-v_b - \mu_1} \right] \\ & \mu_1 \in \left[\frac{c}{v_g}, \frac{v_g \mu_0 - c}{v_g - c} \right] \end{aligned}$$

By arguments similar to the proof of Lemma 7, $\lambda_0 \leq \frac{\mu_0 - \mu_1 - c}{-v_b - \mu_1}$ is binding for $\mu_0 < \widehat{\mu}_0$. Suppose that μ_1 's constraints are not binding.

The Lagrangian is $\mathcal{L} = -K(\lambda_0) + p\lambda_0 + \delta(1 - \lambda_0) \left[-K\left(\frac{\mu_1 - c}{-v_b}\right) + \frac{(\mu_1 - c)p}{-v_b} \right] + \eta \left(\frac{\mu_0 - \mu_1 - c}{-v_b - \mu_1} - \lambda_0 \right)$ s.t. $\eta \geq$

$$0, \eta \left(\frac{\mu_0 - \mu_1 - c}{-v_b - \mu_1} - \lambda_0 \right) = 0. \text{ F.O.C. } \Rightarrow$$

$$\begin{aligned} \eta &= \delta(p - \mu_1) - \delta \frac{p - \mu_1}{p} K' \left(\frac{\mu_1 - c}{-v_b} \right) \\ &= -K' \left(\frac{\mu_0 - \mu_1 - c}{-v_b - \mu_1} \right) + p + \delta \left[K \left(\frac{\mu_1 - c}{-v_b} \right) - \mu_1 + c \right] \\ &\Rightarrow \delta \left[K \left(\frac{\mu_1 - c}{-v_b} \right) + \frac{p - \mu_1}{p} K' \left(\frac{\mu_1 - c}{-v_b} \right) - p + c \right] - K' \left(\frac{\mu_0 - \mu_1 - c}{-v_b - \mu_1} \right) + p = 0 \quad (*_\delta) \end{aligned}$$

The sum of the first two terms of the LHS of $(*_\delta)$ strictly increases in δ , and the LHS of $(*_\delta)$ strictly increases in μ_1 . So, the optimal λ_0 strictly decreases in δ ; the optimal μ_1 and λ_1 strictly increase in δ . When one of μ_1 's constraints is binding, λ_0 and λ_1 does not depend on δ .

We finish the proof of this case by showing that S_0 strategy dominates S_+ strategy. Denote the sender surplus of the optimal S_0 (S_+) strategy when the discount factor is $\delta \in (0, 1)$ by $\Pi_{S_0}(\delta)$ ($\Pi_{S_+}(\delta)$) and denote the corresponding optimal S_0 (S_+) strategy at time t by $\lambda_{t,S_0}(\delta)$ ($\lambda_{t,S_+}(\delta)$). We have:

$$\begin{aligned} \Pi_{S_0}(\delta) &= -K(\lambda_{0,S_0}(\delta)) + p\lambda_{0,S_0}(\delta) + \delta(1 - \lambda_{0,S_0}(\delta)) [-K(\lambda_{1,S_0}(\delta)) + p\lambda_{1,S_0}(\delta)] \\ &\geq -K(\lambda_{0,S_0}(1)) + p\lambda_{0,S_0}(1) + \delta(1 - \lambda_{0,S_0}(1)) [-K(\lambda_{1,S_0}(1)) + p\lambda_{1,S_0}(1)] \\ &= -K(\lambda_{0,S_0}(1)) + p\lambda_{0,S_0}(1) + (1 - \lambda_{0,S_0}(1)) [-K(\lambda_{1,S_0}(1)) + p\lambda_{1,S_0}(1)] \\ &\quad - (1 - \delta)(1 - \lambda_{0,S_0}(1)) [-K(\lambda_{1,S_0}(1)) + p\lambda_{1,S_0}(1)] \\ &\stackrel{(\dagger)}{\geq} -K(\lambda_{0,S_+}(1)) + p\lambda_{0,S_+}(1) + (1 - \lambda_{0,S_+}(1)) [-K(\lambda_1^{**}) + p\lambda_1^{**}] \\ &\quad - (1 - \delta)(1 - \lambda_{0,S_+}(1)) [-K(\lambda_1^{**}) + p\lambda_1^{**}] \\ &= -K(\lambda_{0,S_+}(1)) + p\lambda_{0,S_+}(1) + \delta(1 - \lambda_{0,S_+}(1)) [-K(\lambda_1^{**}) + p\lambda_1^{**}] \\ &\geq -K(\lambda_{0,S_+}(\delta)) + p\lambda_{0,S_+}(\delta) + \delta(1 - \lambda_{0,S_+}(\delta)) [-K(\lambda_1^{**}) + p\lambda_1^{**}] \\ &= \Pi_{S_+}(\delta) \end{aligned}$$

, where the inequality (\dagger) holds because Proposition 13 implies that $-K(\lambda_{0,S_0}(1)) + p\lambda_{0,S_0}(1) + (1 - \lambda_{0,S_0}(1)) [-K(\lambda_{1,S_0}(1)) + p\lambda_{1,S_0}(1)] \geq -K(\lambda_{0,S_+}(1)) + p\lambda_{0,S_+}(1) + (1 - \lambda_{0,S_+}(1)) [-K(\lambda_1^{**}) + p\lambda_1^{**}]$ and we have $\lambda_{0,S_0}(1) \geq \lambda_{0,S_+}(1)$, $-K(\lambda_{1,S_0}(1)) + p\lambda_{1,S_0}(1) \leq -K(\lambda_1^{**}) + p\lambda_1^{**}$.

□