Multi-attribute Search

Yunfei (Jesse) Yao*

(Chinese University of Hong Kong)

Jan. 2024

* I thank J. Miguel Villas-Boas, Yuichiro Kamada, Xinyu Cao, Tony Ke, Chenxi Liao, Zihao Zhou, and seminar participants at the Bay Area Marketing Symposium, the Berkeley/Columbia/Duke/MIT/Northwestern IO Theory Conference and Berkeley Econ Theory Lunch for constructive comments. All errors are my own. E-mail address: jesseyao@cuhk.edu.hk.

Abstract

When considering whether or not to buy a product, the consumer can often evaluate different attributes of it. Due to limited attention, the consumer often can only search for information about one attribute at a time. Assuming that a product has two attributes, we study the optimal search strategy of the consumer by endogenizing the optimal attribute to search, when to keep searching for information, and when to stop searching and make a decision. We find that it is always optimal for the consumer to search for the attribute about which she has the higher uncertainty due to the faster speed of learning. The consumer only searches for the more uncertain attribute if she holds a strong prior belief about one of the attributes, and may search for both attributes otherwise. We also characterize the marginal rate of substitution between the values of two attributes and how the knowledge about one attribute affects the benefits of learning the other attribute. Lastly, we find that reducing search costs benefits the firm only if the consumer's prior belief is intermediate. In that case, a lower search cost prevents the consumer from not searching or quitting searching quickly after receiving negative signals. This gives the consumer more chances to receive enough positive signals and eventually purchase the product.

1 Introduction

When considering whether or not to buy a product, the consumer can often evaluate different attributes of it. An incoming college student finding a laptop can learn about the operating system, weight, exterior design, warranty, and other attributes before making the final decision. Learning costs both time and effort, while the consumer often has limited attention. So, she needs to decide which attribute to learn. Sometimes, she makes the decision based on exogenous reasons, such as an attribute is more prominent (Bordolo et al. 2013, Zhu and Dukes 2017), or her options in an attribute generate a greater range of consumption utility (Kőszegi and Szeidl 2013). If one attribute is much more important than the other, she will search for information about the most important attribute. However, there may not be such exogenous differences among different attributes. What attribute should the consumer search for in that case?

Consider a consumer deciding whether to buy a used car. She can gather information about many different attributes. For example, the consumer can check details about the car's add-on packages through some review articles. She can also purchase a car report to find out the car's accident history. Both options help the consumer learn more about the car and improve the decision. However, it takes time and effort to search for such information. The consumer needs to decide which attribute to pay attention to.

Even if the consumer decides which attribute to search for, she will not learn everything about it immediately. Instead, she gradually gathers information about the attribute. For instance, Even if the consumer spends half an hour searching for information about the car's safety features and finds out that the car has airbags in each of the seats, she still does not know everything about the car's safety. She can continue searching for information about whether the car has an automatic braking system. But, the consumer may not want to stick to one attribute. The relative importance of attributes may change as she learns more. How does the value of learning more about one attribute depends on the value of other attributes? For example, does the consumer's knowledge about the safety of the car affects the benefits from searching for information about the styling of the car? After obtaining enough positive information about the car's safety, she may find it a better use of her time to switch to other attributes. She may feel confident that the car is safe but uncertain whether she will enjoy driving in it. At some point, the consumer may switch to learning more about the car's design. When will she switch to another attribute because the relative importance of attributes changes as she gathers more information?

To answer the above questions, this paper considers a consumer deciding whether to purchase a good or not. The good has two attributes, whose values are independent. The payoff of purchasing the good is the total value of the attributes net of the price. The consumer does not know the value of either attribute. She has a prior belief about the value of each attribute, and can incur a cost to search for information about the attributes before making a decision. By receiving a noisy signal about an attribute from searching, she can update her belief about the value of that attribute and thus about the value of the product. By assuming that the search cost and the informativeness of the signal are the same for each attribute, we ensure that the attributes are symmetric. So, the consumer will not prefer searching for information about one attribute to the other for exogenous reasons. Which attribute to search at any given time is determined endogenously by the expected gain from an extra piece of information about each attribute.

The consumer will stop searching and buy the good if she becomes optimistic enough about its value (the total value of both attributes), and will stop searching without purchasing if she becomes pessimistic enough about its value. When the consumer's belief about the value of the good is in between, she will search for more information. We characterize the search region by a set of ordinary differential equations for intermediate beliefs and by a system of equations for extreme beliefs. We find that it is always optimal for the consumer to search the attribute about which the consumer has the higher uncertainty due to the faster speed of learning. The consumer only searches for the more uncertain attribute if she holds a strong prior belief about one of the attributes and may search for both attributes otherwise. In the car purchasing example, a consumer may not bother to search for the safety features of a Volvo car because Volvo has a good reputation for safety. So, she may instead focus on other aspects of the car. In contrast, Faraday Future has not produced any cars yet. If a consumer considers pre-ordering a car, she probably has a lot of uncertainty about everything. So, she may search for information about every attribute.

The gradual learning model generates valuable insights on cross-attribute dependence. In our paper, the value of learning more about one attribute depends on the value of the other attribute even though the values of different attributes are independent. This cross-attribute dependence is endogenously driven by optimal learning. We characterize the marginal rate of substitution between the values of two attributes and how the knowledge about one attribute affects the benefits of learning the other attribute. To the best of our knowledge, this learning-based endogenous cross-attribute dependence is new to the literature.

We study the comparative statics of the optimal search strategy. An increase in the price shifts the entire search region upwards because the consumer needs to gain a higher value from the good to compensate for the higher price. An increase in either the search cost or the noise of the signal makes searching less attractive for the consumer and shrinks the search region.

We also investigate how the consumer's purchasing likelihood depends on the prior belief. When the consumer is optimistic enough about both attributes, she will purchase the product for sure. When she is pessimistic enough about both attributes, she will never purchase the product. When her belief is in between, she will purchase the product with some probability.

In reality, firms can intervene in the consumer search and purchase processes by changing the search environment. They can affect the difficulty of search by website design or advertising. A consumer will buy the product for sure or with a high probability if she is optimistic about the product prior to searching. Even if the consumer searches for information, she will quickly stop searching and buy the product after receiving positive signals, and is unlikely to have an chance to learn enough negative information and quit. So, the firm does not want to change the search environment. A consumer will neither search nor purchase the product if she is very pessimistic about the product, regardless of firm intervention. Reducing search costs benefits the firm only if the consumer's prior belief is intermediate. In that case, the consumer will either not buy for sure or with a low probability. By reducing search costs and thereby enlarging the search region, the firm prevents the consumer from not searching or quickly quitting searching after receiving negative signals. This gives the consumer more chances to receive enough positive signals and eventually purchase the product.

This paper makes two main contributions. First, we endogenize the search order of different attributes of a product based on the consumer's optimal Bayesian learning. It also generates valuable insights on cross-attribute dependence in search. Second, we provide guidance to firms on how to intervene in the consumer search and purchase processes by changing the search environment.

Related Literature

This paper is related to the literature on how consumers with limited attention allocate their attention to different attributes or options. Existing literature mainly looks at the case in which the attributes or options are asymmetric (Arbatskaya 2007, Armstrong et al. 2009, Xu et al. 2010, Armstrong and Zhou 2011, Bar-Isaac et al. 2012, Bordolo et al. 2013, Kőszegi and Szeidl 2013, Branco et al. 2016, Zhu and Dukes 2017, Jeziorski and Moorthy 2018). In those papers, consumers know that they face attributes with different prominence/importance ex-ante. For example, the search order is exogenous in Arbatskaya (2007). Armstrong et al. (2009) extend the symmetric search model of Wolinsky (1986) by assuming that there is a prominent firm for which all the consumers will search first. In their model, the prominent firm is exogenous. They do not model why consumers want to search for that firm first. In Bordolo et al. (2013), the salient attribute of a good is the attribute furthest away from the average value of the same attribute in the choice set. In Zhu and Dukes (2017), each competing firm can promote one or both attributes of a product. Though the prominence of the product is endogenously determined by competition, it is exogenously given from the consumer's perspective. Jeziorski and Moorthy (2018) examine the effect of prominence in search advertising. There are two types of prominence in their setting, the position of the ads and the prominence of the advertiser. They find that the ad position prominence and the advertiser prominence are substitutes in consumers' clicking behavior. One of the main contributions of our paper is to endogenize the optimal attribute to search from the consumer's perspective. Instead of assuming that the consumer knows the value of each attribute or learns it at once, as is common in this literature, the Bayesian decision-maker in our model gradually learns the value from noisy signals. So, the relative importance of the attributes may change as the consumer gathers more information. In contrast, the prominence attribute/option in the existing literature does not change over time because they impose exogenous differences on the attributes.

This paper also fits into the literature on optimal information acquisition, particularly consumer search. Since the seminal papers by Stigler (1961) and Weitzman (1979), many papers have studied the optimal search problem under simultaneous or sequential search (e.g., Moscarini and Smith

2001, Branco et al. 2012, Ke et al. 2016, and Jerath and Ren 2023). In those papers, the relative importance of different alternatives is exogenous. Consumers observe the distribution of the rewards before making the search decision. Like our paper, the attributes are symmetric in those papers. However, the consumer randomly searches for an attribute. In our model, the consumer decides when to search and which attribute to search. Ke and Villas-Boas (2019) are closely related to our paper. They study the gradual learning of information about multiple alternatives. The decision-maker endogenously determines which alternative to search. There are three main differences between their paper and this one. First, the expected payoff of choosing one of the alternatives depends only on the information gathered from that alternative. So, the objective of searching is to differentiate different alternatives. In our paper, the expected payoff of adopting the product jointly depends on the information gathered from all the alternatives. So, the objective of searching is to learn about the overall distribution of all the attributes. Second, our model allows us to examine the cross-attribute dependence, endogenously driven by the optimal learning. The value of learning more about one attribute depends on the value of the other attribute. Third, they focus on the decision maker's optimal search strategy, whereas we also study the firm's search design problem. We show how the firm can change the consumer's search behavior and increase its profits by changing the seach costs, given the optimal search strategy of the consumer.

Lastly, we study the firm's search design problem in response to the consumer's optimal search strategy. This is related to the literature in the design of search environment. Various papers have studied how a firm can optimally affect consumer's search environment via information provision policy (Branco et al. 2016, Jerath and Ren 2021, Ke et al. 2023, Yao 2023, Gardete and Hunter 2024), product line design (Villas-Boas 2009, Kuksov and Vilas-Boas 2010, Guo and Zhang 2012, Liu and Dukes 2013), pricing (Wee et al. 2024), and advertising (Mayzlin and Shin 2011). Our paper is most closely related to the works on the design of search environment by changing search costs. Empirical studies (Seiler 2013, Ngwe, Ferreira, and Teixeira 2019, and Ursu et al. 2020, 2023) mainly use field experiments and counterfactuals to study the impact of search costs on firm profits under specific contexts. Dukes and Liu (2016) theoretically characterizes the intermediary's strategic choice of search costs in equilibrium. They consider simultanous search and multiple firms, whereas we consider sequential search and multiple attributes. Because we focus on monopoly, we also do not have an intermediary - the seller designs the search environment directly. Bar-Isaac et al. (2010) show that the firm prefers zero search costs if the marginal cost of production is high, infinite search costs if the marginal cost is low, and may prefer intermediate search costs in the presence of heterogeneous consumers. Stivers and Tremblay (2005) show that advertising may increase social welfare by reducing search costs even if it leads to higher prices. There is no gradual learning in both papers - the consumer learns everything in one search.

The remainder of the paper is organized as follows. Section 2 presents the main model. Section 3 presents a benchmark two-period model. Section 4 solves the optimal search strategy of the main model. Section 5 characterizes the purchasing likelihood. Section 6 studies firms' search design problem. Section 7 concludes.

2 Model

A consumer considers whether to purchase a product or not. The product has two attributes whose values are independent. The product's value for the consumer is the sum of the values of the attributes, $U = U_1 + U_2$. In this paper we consider the horizontal match between the attributes and consumer tastes/needs. The value of each attribute is one if it is good a good match and zero if it is a bad match. The consumer's prior belief that attribute i is good is $\mu_i(0)$. Given that we focus on the horizontal preference rather than the vertical one, we assume that the firm does not have private information about the value of the attribute. The price p is exogenously given. We assume that the marginal cost of producing the product is high enough, and thus the price is high enough ($p \geq 3/2$) such that the consumer will quit without purchasing the product for any (μ_1, μ_2), if $\mu_1 + \mu_2 \leq 1$. Hence, we restrict our attention to the case in which $\mu_1 + \mu_2 > 1$. The consumer can learn more about the attributes via costly learning before making a decision. At time t, the consumer can make a purchasing decision or search for information. Because of limited attention, she can only search for information about one attribute at a time. So, if the consumer chooses to search for information, she also needs to decide which attribute to search for information about. The game ends when the consumer makes a decision.

Notice that each attribute of the product can have many sub-attributes. So, the consumer cannot learn everything about an attribute with a single evaluation. For example, suppose the consumer is interested in learning about the design of the car (an attribute). By checking an image online, she knows the exterior color (a sub-attribute) of the car. However, the consumer still needs to incur additional efforts to find out the size of wheels or the material of seats. In today's world with sophisticated products, every attribute contains so many sub-attributes that the consumer can never fully learn everything. To capture the gradual learning feature, we assume that the consumer can obtain noisy signals about an attribute by incurring a flow cost of c. Define $T_i(t)$ as the cumulative time that attribute i has been searched until time t. We model the signal, S_i, by a Brownian motion (W_i are independent Wiener processes):

$$dS_i(t) = U_i dT_i(t) + \sigma dW_i(T_i(t))$$

In the above expression, the first term is driven by the true value, U_i, whereas the second term is the noise. The parameter σ is a measure of the level of signal noise - a larger σ means the signal is noisier because the relative weight of the noise term is higher. The consumer will be more likely to observe a larger signal realization if the attribute is good, because the first term keeps increasing in time if $U_i = 1$. Given the received signal, the consumer continuously updates her belief on the value of each attribute according to Bayes' rule. The belief evolution can be characterized by the following ODE:

$$d\mu_i(t) = \frac{1}{\sigma^2}\mu_i(t)[1-\mu_i(t)]\{dS_i(t) - \mu_i dT_i(t)\} \tag{1}$$

, where μ_i is the expected value of attribute i based on the observed information up to time t. If σ is higher, the consumer will update her belief more slowly because the signal is noisier. If μ_i is closer to 1/2, the consumer will have more uncertainty about attribute i and update her belief about attribute i faster. If $dS_i(t) - \mu_i dT_i(t) > 0$, the signal increases faster than the current belief about attribute i and is more likely to be good. So, the consumer will increase her belief about attribute i. If the signal increases more slowly than the current belief about attribute i and is more likely to be bad, the consumer will decrease her belief about attribute i. This belief updating process also implies that the consumer's belief about an attribute will remain the same when she searches for information about the other attribute. This continuous-time model of Bayesian learning about a binary state has been widely adopted to study information acquisition (Ke and Villas-Boas 2019, Morris and Strack 2019), experimentation (Bolton and Harris 1999, Moscarini and Smith 2001), decision times (Fudenberg et al. 2018), and Bayesian persuasion (Liao 2021). It captures the gradual learning feature and offers nice tractability.

Figure 1 illustrates a sample path of the signals and belief evolution when the first attribute is good and the second one is bad. The consumer's prior belief for each attribute is 1/2. The consumer starts by searching attribute 1 and continuously receives signals about it. Though the signal remains mainly positive, the consumer's belief about the first attribute gradually declines because the signal increases at a lower rate than her belief. Because the consumer does not search for attribute 2 initially, she does not receive new signals on it, and her belief about it remains the same. After searching for a while, the consumer switches to searching attribute 2. Her belief about it first decreases and then increases while her belief about attribute 1 stays the same. When she receives some positive signals about attribute 2, she returns to attribute 1. This time, the signal keeps increasing at a high speed, and the belief about attribute 1 rises toward 1. When the consumer becomes fairly certain that the first attribute is good, she searches for attribute 2 again. She stops searching and quits after receiving enough negative signals and strongly believes attribute 2 is bad.

At any given time, the consumer needs to choose among four actions: searching attribute 1, searching attribute 2, purchasing the product, and quitting without purchasing. The consumer's search strategy, α, is a mapping from the observed history (the signal realization) until time t to one of the four actions, for all t. Define the stopping time τ to be the first time the consumer makes a purchasing decision (purchasing or quitting). The entire process ends at the stopping time. The consumer's expected payoff for a given initial belief (μ_1, μ_2) and search strategy α is:

$$J(\mu_1, \mu_2, \alpha) = \mathbb{E}\left\{\max\left[\mu_1(\tau) + \mu_2(\tau) - p, 0\right] - \tau c | (\mu_1(0), \mu_2(0)) = (\mu_1, \mu_2)\right\}$$

The value function of the consumer's problem is:

$$V(\mu_1, \mu_2) := \sup_{\alpha} J(\mu_1, \mu_2, \alpha)$$

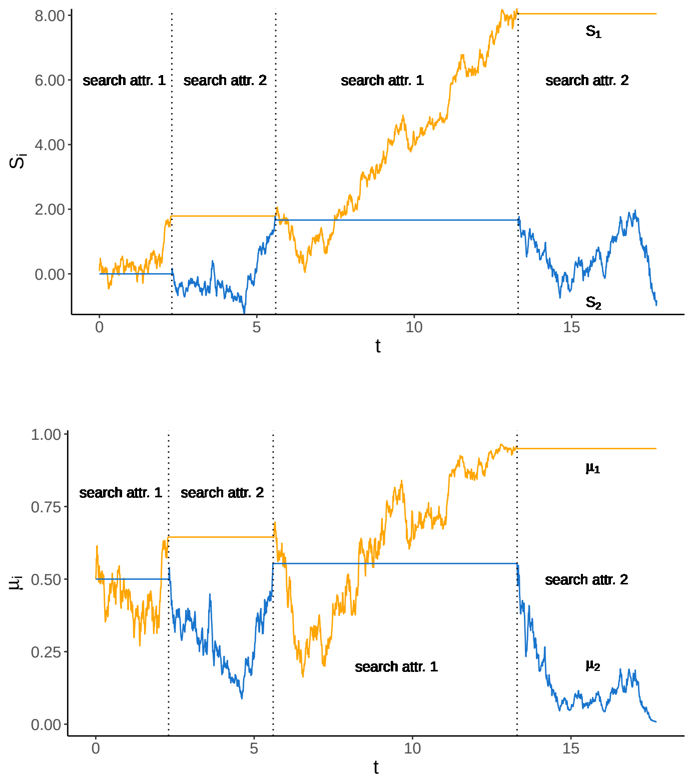

Figure 1: Sample Paths of the Signals and Beliefs for $U_1 = 1, U_2 = 0, \mu_1(0) = 0.5, \mu_2(0) = 0.5$, and $\sigma = 1$.

Since the search strategy should not depend on any future information, the decision at time t should only be based on the observed information up to time t. It is well known that the current belief $(\mu_1(t), \mu_2(t))$ is a sufficient statistic for the information up to time t in this binary-valued setting. So, the search strategy will depend only on $(\mu_1(t), \mu_2(t))$. If a search strategy α^* achieves $V(\mu_1, \mu_2)$ for any given belief, it will be the optimal search strategy.

$$V(\mu_1, \mu_2) = J(\mu_1, \mu_2, \alpha^*)$$

We will first present a simply two-period model to provide some intuition in the next section. Then, section 4 characterizes the consumer's value function and optimal search strategy of the main model.

3 A Two-period One-shot Learning Model

The main model looks complicated and is difficult to solve. A natural question is whether a simpler model can provide the same insight. In this section, we present and solve a two-period benchmark model, where there is no gradual learning for each attribute. The consumer can learn everything about an attribute by searching once. In the next section, we will compare the results of this simpler one-shot learning model and the results of the main model, and show that they lead to qualitatively different insights about the consumer's search behavior. The difference highlights the importance of modeling gradual learning explicitly and using a infinite-period model so that the result is driven by the learning problem rather than the deadline effect.

The consumer still decides whether to purchase a product or not. There are two periods, $t = 1, 2$. In each period, the consumer can incur a search cost c and search for information about one attribute. The consumer learns everything about an attribute if she searches. So, her belief about that attribute will become 1 with probability μ_i and 0 with probability $1 - \mu_i$ after searching. The consumer can make the purchasing decision without searching, after searching once, or after searching twice. To be consistent with the main model, we assume that $p \geq 3/2$, $\mu_1 + \mu_2 > 1$, and $\mu_1 \geq \mu_2$.[1] Similar to the main model, we assume that the search cost is small, $c < (p-1)(2-p)$.

Let us first consider the optimal strategy of the consumer in the second period.

Lemma 1 (Subgame)**.** *The consumer quits if the attribute she searches in the first period is bad.*

Now suppose the consumer searches attribute 1(2) and finds out it is good in the first period. The consumer purchases directly if $\mu_2(\mu_1) \geq 1 - c/(p-1)$*, searches attribute 2(1) and buy if only if it is also good if* $\mu_2(\mu_1) \in [c/(2-p), 1 - c/(p-1))$*, and quits if* $\mu_2(\mu_1) < c/(2-p)$*.*

The consumer's strategy in this subgame is intuitive. The consumer only obtains a positive payoff if both attributes are good. So, she will quit if the first attribute turns out to be bad. If the first attribute is good, her strategy depends on her prior belief about the other attribute. If she is very optimistic about the other attribute, then she can earn a positive payoff with a high probability by buying immediately. This way, she can save the search cost without bearing too much risk. As the search cost increases, the incentive to save the search cost is stronger. So, the belief range of

[1] The case of $\mu_1 < \mu_2$ is symmetric to the case of $\mu_1 \geq \mu_2$. So, we only need to study one of them.

this case, $\mu_2(\mu_1) \geq 1 - c/(p-1)$, becomes larger. If the consumer has a modest belief about the other attribute, the benefit of learning is larger because of the higher uncertainty. Therefore, she prefers to search again and make a fully informed purchasing decision. If she is pessimistic about the other attribute, the likelihood of purchasing after searching it is too low. So, the consumer will quit. Given the consumer's search strategy in the second period, we can characterize her strategy in the first period.

Proposition 1. *In the first period, the consumer purchases without searching if* $\begin{cases} \mu_1 + \mu_2 \geq p \\ \mu_2 \geq \max\{1 - \frac{c}{p-\mu_1}, \frac{p-\mu_1-c}{1+c-\mu_1(2-p)}\} \end{cases}$*, quits without searching if* $\begin{cases} \mu_1 + \mu_2 < p \\ \mu_1 < \min\{p - 1 + \frac{c}{\mu_2}, \frac{(1+\mu_2)c}{\mu_2(2-p)}\} \end{cases}$*, and search attribute 2 in other cases.*

The most salient feature of the optimal search strategy is that the consumer always searches for the more uncertain attribute first if she searches at all. We will compare this benchmark model with our main model in the next section.

4 Optimal Strategy

When the consumer searches for information about attribute one, the value function satisfies (ignoring the time index t for simplicity):

$$V(\mu_1, \mu_2) = -cdt + \mathbb{E}[V(\mu_1 + d\mu_1, \mu_2)]$$

By Taylor's expansion and Ito's lemma, we get:

$$\frac{\mu_1^2(1-\mu_1)^2}{2\sigma^2} V_{\mu_1\mu_1}(\mu_1, \mu_2) - c = 0 \tag{2}$$

Similarly, when the consumer searches for information about attribute two, we have:

$$\frac{\mu_2^2(1-\mu_2)^2}{2\sigma^2} V_{\mu_2\mu_2}(\mu_1, \mu_2) - c = 0 \tag{3}$$

The HJB equation of the entire problem is:

$$\max\left\{\max_{i=1,2}\left[\frac{\mu_i^2(1-\mu_i)^2}{2\sigma^2}V_{\mu_i\mu_i}(\mu_1,\mu_2)-c\right],\max\left[\mu_1+\mu_2-p,0\right]-V(\mu_1,\mu_2)\right\}=0 \qquad (\star)$$

A standard method of solving this kind of stochastic control problem is the"guess and verify" approach. We will show that the value function is a viscosity solution of the HJB equation. Then, we will prove that the viscosity solution of the HJB equation is unique. Therefore, if we can find a viscosity solution, it must be the value function. To do so, we will construct a learning rule and stopping time, and use it to characterize the search region and the expected payoff. Lastly, we will verify that the conjectured strategy generates a viscosity solution of the HJB equation, which implies that the conjectured learning rule and stopping time are optimal. Because of symmetry, we only need to consider the case in which $\mu_1 \geq \mu_2$. Analytically, we can fully characterize the optimal search strategy when the search cost is low. We do not think the result for the low search cost case is a strong restriction, as we are interested in the consumer's search behavior and how the firm can influence it through informative advertising. Naturally, the more interesting case is when the consumer searches more given the low search cost. When the search cost or the price is very high, the consumer searches little, and the problem is less interesting and relevant.

Theorem 1. *Suppose the search cost is low, $c \leq \frac{1}{2\sigma^2[\phi(1/2)-\phi(\frac{2}{3}p-\frac{1}{6})]}$. Conditional on searching, it is optimal for the consumer to search for information about attribute two (one) if $\mu_1 \geq \mu_2$ ($\mu_1 < \mu_2$).*

Figure 2 illustrates the optimal search strategy. The dashed orange line is the quitting boundary, and the solid blue line is the purchasing boundary. The grey arrow represents which attribute the consumer searches for information about, given the current belief. When the overall beliefs of the attributes are low enough, the likelihood of obtaining lots of positive signals and purchasing the good is too low. The consumer stops searching and quits to save the search cost. When the overall beliefs of the attributes are high enough, purchasing the good gives the consumer a higher enough expected surplus. So, she makes the purchase. In other cases, the consumer searches for more information to make a better decision. Denote the intersection of the quitting boundary and the main diagonal by (μ^*, μ^*), the intersection of the purchasing boundary and the main diagonal by (μ^{**}, μ^{**}). Represent the quitting boundary when $\mu_1 \geq \mu_2$ by $\underline{\mu}(\cdot)$, whose domain is $[\mu^*, 1]$

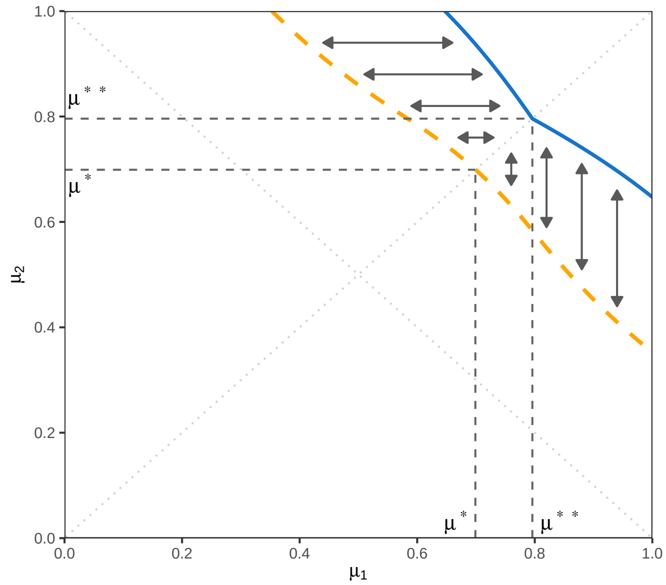

Figure 2: Optimal Search Strategy

(the other half of the quitting boundary is determined by symmetry). Represent the purchasing boundary when $\mu_1 \geq \mu_2$ by $\bar{\mu}(\cdot)$, whose domain is $[\mu^{**}, 1]$ (the other half of the purchasing boundary is determined by symmetry).

Intuitively, the consumer will stop searching, not buy the product if the belief becomes too low, and will purchase the product if the belief becomes high enough. When the belief is in between, she keeps searching for information. Conditional on searching, the intuition for the optimal learning rule is that the consumer prefers to search for the attribute with a higher rate of learning, as the learning costs are identical. From equation (1), one can see that the more uncertain the belief is, the faster the consumer learns about an attribute. Therefore, she always learns the attribute with a belief closer to 1/2. The optimal search strategy implies that the decision-maker only searches the more uncertain attribute if she holds a strong prior belief on one of the attributes and may search both attributes otherwise.

The following proposition characterizes the slope of the purchasing/quitting boundary and the shape of the search region.

Proposition 2. *For $\mu \in (\mu^*, \mu^{**}]$, we have:*

$$\underline{\mu}'(\mu) = \frac{\phi(\underline{\mu}(\mu)) - \phi(\mu)}{\phi'(\underline{\mu}(\mu))[\mu - \underline{\mu}(\mu)]} \tag{D_1}$$

For $\mu \in [\mu^{**}, 1]$, we have:

$$\bar{\mu}'(\mu) = \frac{\phi(\underline{\mu}(\mu)) - \phi(\bar{\mu}(\mu))}{\phi'(\bar{\mu}(\mu))[\bar{\mu}(\mu) - \underline{\mu}(\mu)]} \tag{$\overline{D_2}$}$$

$$\underline{\mu}'(\mu) = \frac{\phi(\underline{\mu}(\mu)) - \phi(\bar{\mu}(\mu))}{\phi'(\underline{\mu}(\mu))[\bar{\mu}(\mu) - \underline{\mu}(\mu)]} \tag{$\underline{D_2}$}$$

Both $\underline{\mu}(\mu)$ and $\bar{\mu}(\mu)$ strictly decrease in μ, while the width of the search region, $\bar{\mu}(\mu) - \underline{\mu}(\mu)$, strictly increases in μ. In addition, if $\underline{\mu}(\mu) \geq 1/2$, then the slope of the quitting boundary is less than -1 and the slope of the purchasing boundary is greater than -1.

We find that the optimal search region has a butterfly shape - the consumer searches for information in a broader region when the consumer is more certain that the more favorable attribute is good. The intuition is the following. The product has a higher expected value if the consumer is more confident about one attribute being good. So, the consumer will search for information about the other attribute even if she has more uncertainty about it. Because the speed of learning is higher when searching a more uncertain attribute, the benefit of search increases while the search cost remains the same. Therefore, the consumer will search more.

If the consumer likes an attribute more, she will purchase the product even if she has a higher uncertainty about the other attribute. She will also be less likely to stop searching and quit. Therefore, the search region shifts downwards as the belief about one attribute, μ, increases. The value of the slope of the search region is also interesting. It is the marginal rate of substitution between the values of attributes one and two, and helps us understand the cross-attribute dependence in learning. If the slope equals -1, then the two attributes are perfect substitutes. One may expect this to be the case in general because the product's value is the sum of the values of two attributes. However, both the slope of the quitting boundary and the slope of the purchasing boundary are not -1 in general because of the asymmetry of learning. If the quitting boundary is above $1/2$, a unit increase of the belief about attribute one can substitute for more than a unit of the belief about attribute two near the quitting boundary, $\underline{\mu}'(\mu^*) < -1$. The consumer will keep searching for information about attribute two instead of quitting even if μ_2 decreases by slightly more than a unit. This is because the consumer has more uncertainty about attribute 2. The speed of learning is higher when the consumer searches a more uncertain attribute. So, the benefit of search increases while the search cost remains the same. Similarly, a unit increase of the belief about attribute one can substitute for less than a unit of the belief about attribute two near the purchasing boundary, $\bar{\mu}'(\mu^*) > -1$. The consumer will keep searching for information about attribute two instead of purchasing the product even if μ_2 only decreases by slightly less than a unit.

4.1 Comparison with the Two-period Model

There is no gradual learning in the benchmark model we examined in the previous section. The consumer resolves all the uncertainty about an attribute in one period and thus will only search for information in at most two periods. In contrast, the main model in continuous-time features gradual learning. The consumer only resolves partial uncertainty about an attribute each time she searches, and there is no deadline to stop searching and making a decision. The robust finding is that the consumer searches for the attribute about which she has higher uncertainty. We now outline reasons in support of the main model.

First, the two-period model is not stationary due to the deadline. So, it is harder to distinguish the main economic forces and interpret the findings. For example, we cannot tell if the search order is driven by the speed of learning or by the deadline effect. Second, the discrete model becomes complicated quickly as the periods increase because the number of cases to consider grows fast. Consequently, it is challenging to extend the two-period model to the N-period model to allow for gradual learning. Third, partial evaluation is ubiquitous in the real world (Häubl and Trifts 2000, Hauser 2011). A consumer will not revisit the same attribute during search if she can learn everything in one shot. However, empirical evidences have shown that return visits account for a significant portion of all search activity (Bronnenberg et al. 2016). A gradual learning model not only generates return visits, but also allows us to know conditions for attribute revisits. The insights can guide managers in allocating their marketing resources. For instance, online car sellers will have a better sense whether they should focus on engine powers if they observe that a consumer has clicked on images about the looking of the car, depending on whether the consumer will revisit the car design attribute. Lastly, the gradual learning model generates valuable insights on cross-attribute dependence. Few papers in consumer search consider this issue, partly due to the technical difficulty. Ke and Lin (2020) allow for an exogenous correlation between attributes of different products and find an information complementarity effect where a lower price of one product can increase the demand for the others. In their paper, different products have some common attributes. So, the values of the product attributes are correlated. In our paper, the values of different attributes are independent, whereas the value of learning more about one attribute depends on the value of the other attribute. Moreover, this cross-attribute dependence is endogenously driven by optimal learning. Our main model with gradual learning makes it possible to know the marginal rate of substitution between the values of two attributes and how the knowledge about one attribute affects the benefits of learning the other attribute. To the best of our knowledge, this learning-based endogenous cross-attribute dependence is new to the literature. It provides valuable managerial implications.

4.2 Comparative Statics

If the firm wants to use the above results, it needs to understand how the model primitives affect the consumer's search behavior. The following proposition summarizes the comparative statics of the search region with regard to the price, search cost, and noise of the signal.

Proposition 3. *Suppose $\mu_1 \geq \mu_2$. The purchasing threshold $\bar{\mu}(\mu)$ increases in the price p, and decreases in the search cost c and the noise of the signal σ^2. The quitting threshold $\underline{\mu}(\mu)$ increases in the price p, the search cost c, and the noise of the signal σ^2.*

An increase in the price shifts the entire search region upwards because the consumer needs to gain a higher value from the good to compensate for the higher price. For example, as Figure 3 illustrates, the consumer may be willing to pay 1.5 when she believes that each attribute has an 80% probability of being good. She will obtain a positive expected surplus from purchasing the product. However, if the price of the good increases to 1.75, she will not buy the good given the same belief because of the negative expected utility. She may not even keep searching for information because the likelihood that the belief becomes high enough to compensate for the high price is low. She will be better off stopping searching, saving the search cost. Similarly, the consumer may be willing to search for more information when she believes that each attribute has a 70% probability of being good if the price is 1.5. Though she will obtain a negative utility from purchasing the product right away, she may like the product more after some search and gain a positive surplus by purchasing it. In contrast, if the price of the good increases to 1.75, she will stop searching because the likelihood of receiving a lot of positive information and raising the valuation for the product above the high price is very low.

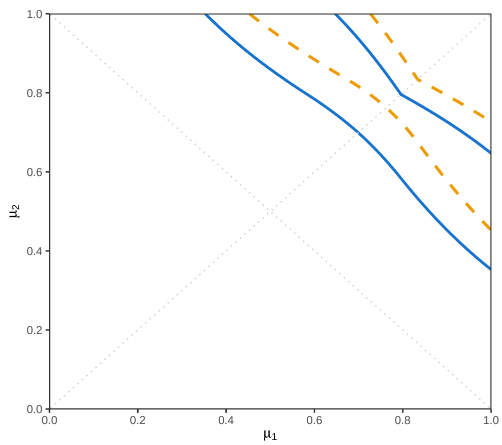

Figure 3: Optimal Search Region for $p = 1.5$ (solid blue) or 1.6 (dashed orange), $c = 0.1, \sigma^2 = 1$.

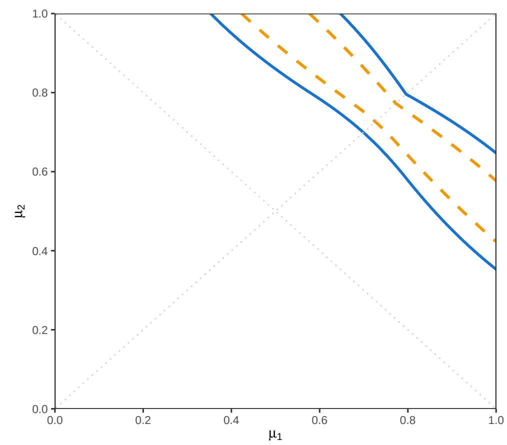

Figure 4: Optimal Search Region for $p = 1.5$, $c\sigma^2 = 0.1$ (solid blue) or 0.2 (dashed orange).

Given a prior belief (μ_1, μ_2), increasing the price has two opposite effects on the firm. A higher price raises the profit conditional on purchasing but reduces the purchasing likelihood. The next section discusses in detail how the consumer's purchasing likelihood depends on the prior belief.

The change in the search cost or the signal's noise has the same effect on the consumer's search behavior because they always appear together in the value function as $c\sigma^2$. An increase in either the search cost or the signal noise makes searching less attractive for the consumer and shrinks the search region. The consumer will only search for information in a narrower range of beliefs. Figure 4 illustrates how the search region depends on the search cost and the signal noise. For example, for a product whose price is 1.5, the consumer may want to keep searching if she believes that each attribute has a 78% probability of being good and $c\sigma^2 = 0.1$. She can obtain a positive surplus by purchasing immediately. However, she may receive some negative information about the product and avoid purchasing a bad product by mistake. So, she may prefer to make a decision when she becomes more certain about the value of the product. However, if it takes more time or effort to search for information or the information is not very accurate, $c\sigma^2 = 0.2$, the benefit from search will be lower and the consumer may instead purchase the good immediately.

5 Purchasing Likelihood

We now look at the consumer's belief path to purchase. If the consumer strongly believes that one of the attributes is good, she will never search for information on that attribute. The consumer will keep searching for information about the other attribute. She will purchase the product if she obtains enough positive information and the belief reaches the purchasing boundary $\bar{\mu}$. If she receives enough negative information and the belief reaches the quitting boundary $\underline{\mu}$, she will quit searching without buying the good. For example, when deciding whether to buy a Volvo, a consumer may not bother to search for its safety features because Volvo has a good reputation for safety. She gains more from searching for other attributes of the car.

In contrast, the consumer must search for information on both attributes before purchasing the good if she has mild beliefs about both attributes. Moreover, she will be equally certain about the value of each attribute if she decides to buy the good. For example, Faraday Future has not produced any cars yet. If a consumer considers pre-ordering a car, she probably has a lot of uncertainty about everything. So, she may search for information about every attribute. Given the consumer's optimal search strategy, we can calculate the purchasing likelihood given a prior belief (μ_1, μ_2).

Proposition 4. *Suppose $\mu_1 \geq \mu_2$. The probability that the consumer purchases the product is:*

$$P(\mu_1, \mu_2) := \mathbb{P}[\textit{purchasing}|\textit{starting at }(\mu_1, \mu_2)]$$
$$= \begin{cases} 1, \ \textit{if } \mu_1 \in [\mu^{**}, 1] \textit{ and } \mu_2 \in [\bar{\mu}(\mu_1), \mu_1] \\ \frac{\mu_2 - \underline{\mu}(\mu_1)}{\bar{\mu}(\mu_1) - \underline{\mu}(\mu_1)}, \ \textit{if } \mu_1 \in [\mu^{**}, 1] \textit{ and } \mu_2 \in [\underline{\mu}(\mu_1), \bar{\mu}(\mu_1)] \\ h(\mu_1, \mu_2)\tilde{P}(\mu_1), \ \textit{if } \mu_1 \in [\mu^{*}, \mu^{**}] \textit{ and } \mu_2 \in [\underline{\mu}(\mu_1), \mu_1] \\ 0, \ \textit{if } \mu_1 \leq \mu^{*} \textit{ or } \mu_2 \leq \bar{\mu}(\mu_1) \end{cases}$$

, where $h(\mu_1, \mu_2) = \frac{\mu_2 - \underline{\mu}(\mu_1)}{\mu_1 - \underline{\mu}(\mu_1)}$ and $\tilde{P}(\mu) = e^{-\int_{\mu}^{\mu^{**}} \frac{2}{x - \underline{\mu}(x)} dx}$. By symmetry, $P(\mu_1, \mu_2) = P(\mu_2, \mu_1)$ if $\mu_1 < \mu_2$.

We can see that there are four regions, as Figure 5 illustrates. The consumer makes the purchase immediately if the belief lies in the region $S1$ and quits without purchasing immediately if the belief lies in the region $S4$. For beliefs in between, the value of information is the highest. The consumer will search for more information before making a decision. If the belief lies in the region $S3$ on the right-hand side of the figure, the consumer strongly believes that the first attribute is good. So, instead of spending more time confirming it, she searches for information about the more uncertain attribute, attribute two. If she receives enough positive information about the second attribute, she will be very optimistic about the product's value and will make the purchase. If she receives enough negative information about the second attribute, she will be pessimistic about the product's value and will stop searching. Because the consumer has had a pretty good sense of the first attribute's value, she will not switch back to searching for information about it regardless of what she learns about the second attribute. Therefore, the second attribute is the pivotal attribute in this case.

If the belief lies on the right-hand side of the region $S2$, the consumer is quite uncertain about the value of both attributes. She will search for information about attribute two because she is more uncertain about attribute two than attribute one. However, the consumer also does not have a strong belief about the value of attribute one. So, the consumer will switch to search for information about attribute one if she receives enough positive signals about attribute two. She may switch back to attribute two if she gets enough positive signals about attribute one and may switch back and forth before being confident about both attributes and purchasing the product. As shown in Figure 5, the belief must reach (μ^{**}, μ^{**}) for the consumer to make the purchase decision. So, she will be equally confident about the value of both attributes when she stops searching and buying the good. She will stop searching and quit if she receives enough negative signals about either attribute.

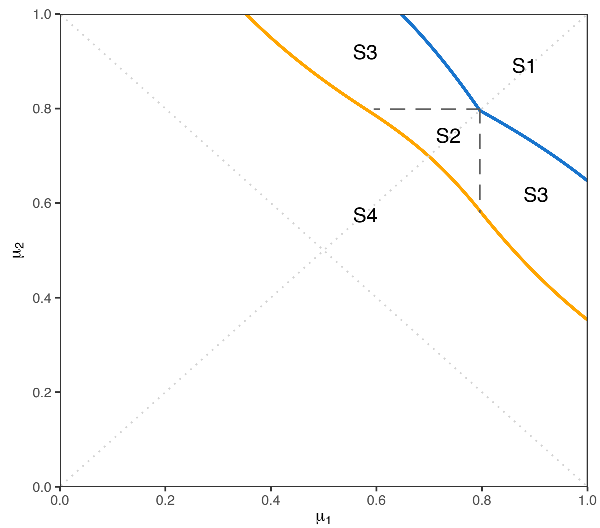

Figure 5: Four Regions for Purchase

6 Search Design

The previous section determines the purchasing probability given the prior belief. The consumer's search environment is exogeneously given. In reality, the firm can affect the difficulty of search by website design or advertising. For example, many firms allow consumers to filter the search results, but differs in the filtering precision. Some websites give consumers a lot of control over the information they want to see, whereas other websites only allow for limited customization in search. Different designs are correlated with different search costs or noise of search. Firms can also make it easier for consumers to find relevant information through advertising. Empirically, De Corniere (2016) find that targeted advertising reduces search costs.

Since only the multiplication of search cost and signal noise, $c\sigma^2$, are identifiable in our model, we view the search design as the decision to change the search cost without loss of generality.

6.1 Symmetric Case

The firm is endowed with the default search cost c, and decides whether to reduce it to $\tilde{c} < c$ by redesigning the website.[2] Denote the purchasing and quitting boundaries under the reduced search cost by $\tilde{\bar{\mu}}$ and $\underline{\tilde{\mu}}$. Proposition 3 and Figure 4 have shown that the consumer will search in a wider range if the search cost is lower. Does it benefit the firm? The following proposition shows that it does only if the consumer is relatively pessimistic about the product given the default search environment.

Proposition 5. *Suppose $\mu_1 \geq \mu_2$. There exists $\tilde{\mu}(\mu_1)$ such that $\underline{\mu}(\mu_1) < \tilde{\mu}(\mu_1) < \bar{\mu}(\mu_1)$ if $\mu_1 \geq \mu^{**}$ and $\underline{\mu}(\mu_1) < \tilde{\mu}(\mu_1) \leq \mu_1$ if $\mu_1 < \mu^{**}$. The firm reduces the search cost if and only if $\mu_2 \in (\underline{\tilde{\mu}}(\mu_1), \tilde{\mu}(\mu_1))$.*

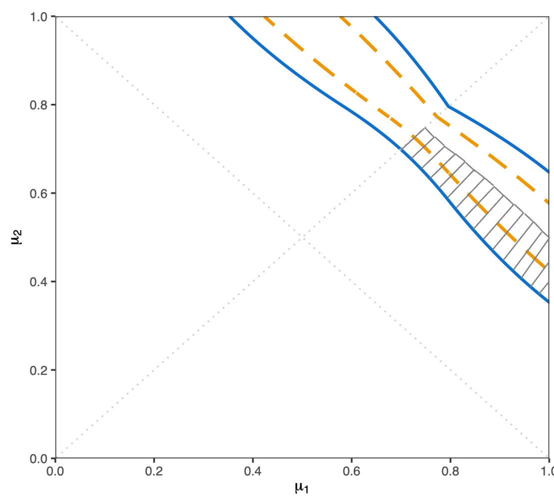

Figure 6: Illustration of Search Design for the Symmetric Case. Dashed line: stopping boundary under the default search cost. Solid line: stopping boundary under the reduced search cost. Striped shaded region: the firm prefers reducing search costs.

A consumer will buy the product without searching if her prior belief is sufficiently high. The firm does not have an incentive to reduce search costs in this case. Even if the consumer does not make a purchase directly, the firm does not want to reduce the search cost if she is optimistic about the product and will buy with a high probability. By keeping the search region tight, the firm makes sure that the consumer will quickly stop searching and buy the product after receiving

[2] The results of increasing the search cost are symmetric to the ones of decreasing the search cost. So, we only discuss one case. Also, by symmetry, we only need to consider the firm's advertising strategy when $\mu_1 \geq \mu_2$.

positive signals, and is unlikely to have a chance to learn enough negative information and quit. A consumer will neither search nor purchase the product if she is very pessimistic about the product, regardless of firm intervention.

Reducing search costs benefits the firm only if the consumer's prior belief is intermediate. The firm will reduce search costs if the consumer never buys under the default search cost but may buy under the reduced search cost. Even if the consumer buys with a low probability without firm intervention, the firm can enlarge the search region and prevent the consumer from quickly quitting searching after receiving negative signals by reducing search costs. This gives the consumer more chances to receive enough positive signals and eventually purchase the product. So, the firm is better off reducing search costs.

Figure 6 illustrates the firm's strategy. The dashed line is the stopping boundary under the default search cost. The firm can expand the search region to the area within the solid lines by reducing search costs. It does so when the belief lies in the striped shaded region.

6.2 Asymmetric Case

It is also interesting to consider the case where the firm can reduce the search cost from c to $\tilde{c}$ for one attribute. A complete characterization of the optimal strategy is beyond the scope of this paper. Instead, we discuss below what insights from the symmetric case will extend to this asymmetric case. Suppose the firm reduces the search cost for attribute 2. Searching for information about attribute 2 becomes more attractive to the consumer because the benefit of searching for either attribute remains unchanged. Therefore, the consumer may search attribute 2 for some beliefs such that $\mu_2 > \mu_1$. The consumer only searches for attribute 2 if $\mu_1 \geq \underline{\mu^{**}}(< \mu^{**})$, and only searches for attribute 1 if $\mu_2 \geq \overline{\mu^{**}}(> \mu^{**})$. Suppose instead the firm reduces the search cost for attribute 1. By symmetry, one can see that the consumer only searches for attribute 1 if $\mu_2 \geq \underline{\mu^{**}}(< \mu^{**})$, and only searches for attribute 2 if $\mu_1 \geq \overline{\mu^{**}}(> \mu^{**})$.

Consider the case where $\mu_1 \geq \overline{\mu^{**}}$. The consumer will only search for information about attribute 2 regardless of firm intervention. The firm will not reduce the search cost for attribute 1 because it does not change the consumer's search behavior. Therefore, it either reduces the search cost for attribute 2 or does not intervene in the search environment. Since the consumer only searches for attribute 2, reducing the search cost for attribute 2 has the same effect on the consumer's search behavior as reducing the search cost for both attributes. The result in Proposition 5 for this case stays the same, both qualitatively and quantitatively. The firm reduces the search cost for attribute 2 if and only if $\mu_2 \in (\underline{\tilde{\mu}}(\mu_1), \tilde{\mu}(\mu_1))$ for the same $\underline{\tilde{\mu}}$ and $\tilde{\mu}$ as in the symmetric case. When $\mu_2 \geq \overline{\mu^{**}}$, by symmetry, the firm reduces the search cost for attribute 1 if and only if $\mu_1 \in (\underline{\tilde{\mu}}(\mu_2), \tilde{\mu}(\mu_2))$.

Now consider the case where $\mu_2 \leq \mu_1 < \overline{\mu^{**}}$. By similar arguments as the proof of Proposition 5, one can see that the firm has an incentive to reduce the search cost for attribute 2 for intermediate μ_2 such that the consumer does not purchase or purchases the product with a low probability under the default search cost, whereas search and buy with a positive probability under the reduced search cost. This finding is also consistent with the insight from the symmetric case. What is not known in this case is whether the firm is better off reducing the search cost for attribute 1. We leave it for future research.

7 Conclusion

Understanding how consumers decide which attribute to pay more attention to has important managerial implications. It helps the firm decide how to design the product and which attributes to emphasize. In this paper, we study the optimal search strategy of a Bayesian decision-maker by endogenizing the optimal attribute to search for, when to keep searching, and when to stop and make a decision. We characterize the search region by a set of ordinary differential equations for moderate beliefs and by a system of equations for extreme beliefs. We find that it is always optimal to search the attribute the consumer has the highest uncertainty due to the fastest learning speed. The decision-maker only searches the more uncertain attribute if she holds a strong prior belief about one of the attributes, and may search both attributes otherwise. We also study the firm's search design problem. In reality, it can affect the difficulty of search by website design or advertising. We find that reducing search costs benefits it only if the consumer's prior belief is intermediate. In that case, a lower search cost prevents the consumer from not searching or quitting searching quickly after receiving negative signals. This gives the consumer more chances to receive enough positive signals and eventually purchase the product.

There are some limitations to this paper. The consumer only considers one product in our model. If there are multiple products, the consumer needs to make two decisions - which product to search for and which attribute of the product to search for. Studying this richer problem can lead to interesting findings. It will also be interesting to extend the number of attributes beyond two and see whether the consumer still searches for the attribute with the highest uncertainty due to the fastest learning speed. Lastly, we consider an exogenous price throughout the paper to focus on the role of information. Future research can study the optimal pricing of the product given the consumer's optimal search strategy.

Appendix

Proof of Proposition 2. We have derived (D_1) in the main text. It implies immediately that $\underline{\mu}'(\mu) < 0$ for $\mu \in (\mu^*, \mu^{**}]$. For $\mu \in [\mu^{**}, 1]$, by the implicit function theorem, we have:

$$\begin{bmatrix} \bar{\mu}'(\mu) \\ \underline{\mu}'(\mu) \end{bmatrix} = - \begin{bmatrix} -\phi'(\bar{\mu}(\mu)) & \phi'(\underline{\mu}(\mu)) \\ -\psi'(\bar{\mu}(\mu)) & \psi'(\underline{\mu}(\mu)) \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 \\ \frac{1}{2\sigma^2 c} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{2\sigma^2 c} \frac{1}{\phi'(\bar{\mu}(\mu))[\bar{\mu}(\mu)-\underline{\mu}(\mu)]} \\ \frac{1}{2\sigma^2 c} \frac{1}{\phi'(\underline{\mu}(\mu))[\bar{\mu}(\mu)-\underline{\mu}(\mu)]} \end{bmatrix} = \begin{bmatrix} \frac{\phi(\underline{\mu}(\mu))-\phi(\bar{\mu}(\mu))}{\phi'(\bar{\mu}(\mu))[\bar{\mu}(\mu)-\underline{\mu}(\mu)]} & < 0 \\ \frac{\phi(\underline{\mu}(\mu))-\phi(\bar{\mu}(\mu))}{\phi'(\underline{\mu}(\mu))[\bar{\mu}(\mu)-\underline{\mu}(\mu)]} & < 0 \end{bmatrix}$$

This gives us the expression for $(\overline{D_2})$ and $(\underline{D_2})$. One can see from the negative sign of the derivative that both $\underline{\mu}(\mu)$ and $\bar{\mu}(\mu)$ strictly decrease in μ.

We now look at the width of the search region.

$$\begin{aligned}
&[\bar{\mu}(\mu) - \underline{\mu}(\mu)]' \\
=&\frac{\phi(\underline{\mu}(\mu)) - \phi(\bar{\mu}(\mu))}{\phi'(\bar{\mu}(\mu))[\bar{\mu}(\mu) - \underline{\mu}(\mu)]} - \frac{\phi(\underline{\mu}(\mu)) - \phi(\bar{\mu}(\mu))}{\phi'(\underline{\mu}(\mu))[\bar{\mu}(\mu) - \underline{\mu}(\mu)]} \\
=&\frac{\phi(\underline{\mu}(\mu)) - \phi(\bar{\mu}(\mu))}{\bar{\mu}(\mu) - \underline{\mu}(\mu)} \left[1/\phi'(\bar{\mu}(\mu)) - 1/\phi'(\underline{\mu}(\mu))\right] \\
=&\frac{\phi(\underline{\mu}(\mu)) - \phi(\bar{\mu}(\mu))}{\bar{\mu}(\mu) - \underline{\mu}(\mu)} \left[\underline{\mu}(\mu)^2(1 - \underline{\mu}(\mu))^2 - \bar{\mu}(\mu)^2(1 - \bar{\mu}(\mu))^2\right]
\end{aligned}$$

One can see that $\frac{\phi(\underline{\mu}(\mu))-\phi(\bar{\mu}(\mu))}{\bar{\mu}(\mu)-\underline{\mu}(\mu)} > 0$. So, $[\bar{\mu}(\mu) - \underline{\mu}(\mu)]' > 0 \Leftrightarrow \underline{\mu}(\mu)^2(1 - \underline{\mu}(\mu))^2 > \bar{\mu}(\mu)^2(1 - \bar{\mu}(\mu))^2 \Leftrightarrow \underline{\mu}(\mu)(1 - \underline{\mu}(\mu)) > \bar{\mu}(\mu)(1 - \bar{\mu}(\mu)) \Leftrightarrow |\underline{\mu}(\mu) - 1/2| < |\bar{\mu}(\mu) - 1/2|$. Thus, the width of the search region, $\bar{\mu}(\mu) - \underline{\mu}(\mu)$, increases in the belief, μ, if and only if the quitting boundary is closer to 1/2 than the purchasing boundary. We know that $\forall \mu \geq \mu^{**},\ p = \mu + \frac{\bar{\mu}(\mu)+\underline{\mu}(\mu)}{2}$ due to the symmetry of the one-dimensional learning problem.[3] Therefore,

[3] More specifically, the sum of the purchasing and quitting thresholds is zero when the price is zero in the one-dimensional optimal search strategy, as shown by Branco et al. (2012). It implies that the price equals to the average of the two boundaries. In our two-dimensional problem, the consumer only searches the more uncertain attribute when $\mu \geq \mu^{**}$. So, it can be translated to a one-dimensional search problem with the price p normalized to $p - \mu$.

$$
\begin{aligned}
&\frac{\bar{\mu}(\mu)+\underline{\mu}(\mu)}{2}=p-\mu \geq 3/2-1=1/2\\
\Rightarrow\ &\frac{\bar{\mu}(\mu)+\underline{\mu}(\mu)}{2} \geq 1/2\\
\Leftrightarrow\ &\bar{\mu}(\mu)+\underline{\mu}(\mu)>1\\
\Leftrightarrow\ &|\underline{\mu}(\mu)-1/2|<|\bar{\mu}(\mu)-1/2|,\ \forall \mu \geq \mu^{**}
\end{aligned}
$$

Thus, the width of search region, $\bar{\mu}(\mu)-\underline{\mu}(\mu)$, always increases in the belief μ.

Now suppose that $\underline{\mu}(\mu) \geq 1/2$, then $\forall \mu \in (\mu^{*}, \mu^{**}]$, we have

$$
\begin{aligned}
\underline{\mu}'(\mu) &\overset{(D_1)}{=} \frac{\phi(\underline{\mu}(\mu))-\phi(\mu)}{\phi'(\underline{\mu}(\mu))[\mu-\underline{\mu}(\mu)]}\\
&=\frac{-\phi'(\xi_1(\mu))[\mu-\underline{\mu}(\mu)]}{\phi'(\underline{\mu}(\mu))[\mu-\underline{\mu}(\mu)]}\ (\xi_1(\mu)\in(\underline{\mu}(\mu),\mu))\\
&=-\frac{\phi'(\xi_1(\mu)}{\phi'(\underline{\mu}(\mu))}\\
&<-1
\end{aligned}
$$

, where the last inequality comes from the fact that the absolute value of $\phi'(x)=-\frac{1}{x^2(1-x)^2}$ is strictly increasing in x for $x \geq 1/2$. Similarly, $\forall \mu \in [\mu^{**}, 1]$, we have

$$
\begin{aligned}
\underline{\mu}'(\mu) &\overset{(\underline{D_2})}{=} \frac{\phi(\underline{\mu}(\mu))-\phi(\bar{\mu}(\mu))}{\phi'(\underline{\mu}(\mu))[\bar{\mu}(\mu)-\underline{\mu}(\mu)]}\\
&=\frac{-\phi'(\xi_2(\mu))[\bar{\mu}(\mu)-\underline{\mu}(\mu)]}{\phi'(\underline{\mu}(\mu))[\bar{\mu}(\mu)-\underline{\mu}(\mu)]}\ (\xi_2(\mu)\in(\underline{\mu}(\mu),\bar{\mu}(\mu)))\\
&=-\frac{\phi'(\xi_2(\mu)}{\phi'(\underline{\mu}(\mu))}<-1\\
\bar{\mu}'(\mu) &\overset{(\overline{D_2})}{=} \frac{\phi(\underline{\mu}(\mu))-\phi(\bar{\mu}(\mu))}{\phi'(\bar{\mu}(\mu))[\bar{\mu}(\mu)-\underline{\mu}(\mu)]}\\
&=\frac{-\phi'(\xi_3(\mu))[\bar{\mu}(\mu)-\underline{\mu}(\mu)]}{\phi'(\bar{\mu}(\mu))[\bar{\mu}(\mu)-\underline{\mu}(\mu)]}\ (\xi_3(\mu)\in(\underline{\mu}(\mu),\bar{\mu}(\mu)))\\
&=-\frac{\phi'(\xi_3(\mu))}{\phi'(\bar{\mu}(\mu))}>-1
\end{aligned}
$$

□

Proof of Theorem 1. By symmetry, we only need to prove the case of $\mu_1 \geq \mu_2$. We first show that the viscosity solution of the HJB equation ($\star$) exists and is unique. Since the value function is a viscosity solution of ($\star$), the viscosity solution of ($\star$) must be the value function by uniqueness. We then conjecture an optimal search strategy and characterize its properties. Lastly, we verify that the conjectured strategy indeed generates a viscosity solution to ($\star$). So, the conjectured strategy is optimal.

Lemma 2. *The viscosity solution of the HJB equation ($\star$) exists and is unique.*

Proof. Since the consumer can guarantee a payoff of zero by quitting immediately and cannot achieve a payoff higher than $\sup\{\mu_1 + \mu_2 - p\} = 1 + 1 - p \leq 2$, the value function is bounded and thus exists. This implies the existence of the viscosity solution because the value function is a viscosity solution to ($\star$).

The proof of the uniqueness uses a modification of a comparison principle in Crandall et al. (1992). Given that it very much resembles the proof of Lemma 1 in Ke and Villas-Boas (2019), we refer the reader to their proof. □

Conjecture:

Conditional on searching, it is optimal for the consumer to search for information about attribute two (one) if $\mu_1 \geq \mu_2$ ($\mu_1 < \mu_2$).

Given this conjecture, we now characterize the search region (illustrated in Figure 2).

The PDE when the consumer searches attribute two, equation (3), has the following general solution:

$$V(\mu_1, \mu_2) = 2\sigma^2 c(1 - 2\mu_2) \ln \frac{1 - \mu_2}{\mu_2} + B_1(\mu_1)\mu_2 + B_2(\mu_1), \mu_1 \in [\mu^*, 1]$$

We also have $V(\mu_1, \mu_2) = 0$ at the quitting boundary $\mu_2 = \underline{\mu}(\mu_1)$. For the value function in the search region, value matching and smooth pasting (wrt μ_2) at the quitting boundary $(\mu_1, \underline{\mu}(\mu_1))$ imply:[4]

$$\frac{V(\mu_1, \mu_2)}{2\sigma^2 c} = (1 - 2\mu_2) \ln \frac{1 - \mu_2}{\mu_2} + \phi(\underline{\mu}(\mu_1))\mu_2 - \psi(\underline{\mu}(\mu_1)) \tag{4}$$

[4] For technical details, please refer to Dixit (1993).

, where $\phi(x) = 2\ln\frac{1-x}{x} + \frac{1}{x} - \frac{1}{1-x}$ and $\psi(x) = \ln\frac{1-x}{x} + \frac{1-2x}{1-x}$.

By symmetry, for $\mu_1 < \mu_2$, the value function in the search region satisfies:

$$\frac{V(\mu_1,\mu_2)}{2\sigma^2 c} = (1-2\mu_1)\ln\frac{1-\mu_1}{\mu_1} + \phi(\underline{\mu}(\mu_2))\mu_1 - \psi(\underline{\mu}(\mu_2)) \tag{5}$$

Equation (4) characterizes the value function for beliefs $\mu_1 \geq \mu_2$. Equation (5) characterizes the value function for beliefs $\mu_1 < \mu_2$. The two regions are separated by the main diagonal $\{(\mu_1,\mu_2) : \mu_1 = \mu_2\}$. Continuity of $V_{\mu_1}(\mu_1,\mu_2)$ at this boundary implies that:

$$\underline{\mu}'(\mu) = \frac{\phi(\underline{\mu}(\mu)) - \phi(\mu)}{\phi'(\underline{\mu}(\mu))[\mu - \underline{\mu}(\mu)]}, \text{ for } \mu \in (\mu^*, \mu^{**}] \tag{D_1}$$

For $\mu_1 \in [\mu^{**}, 1]$, $V(\mu_1,\mu_2) = \mu_1 + \mu_2 - p$ at the purchasing boundary $\mu_2 = \bar{\mu}(\mu_1)$. Value matching and smooth pasting (w.r.t. μ_2) at the purchasing boundary $(\mu_1, \underline{\mu}(\mu_1))$ imply (in the search region):

$$\frac{V(\mu_1,\mu_2)}{2\sigma^2 c} = (1-2\mu_2)\ln\frac{1-\mu_2}{\mu_2} + \phi(\bar{\mu}(\mu_1))\mu_2 - \psi(\bar{\mu}(\mu_1)) + \frac{\mu_1 - \mu_2 - p}{2\sigma^2 c} \tag{6}$$

Equation (4) and (6) use the quitting boundary and the purchasing boundary to pin down the value function, respectively. The resulting expression should be equivalent in the common domain $\mu_1 \in [\mu^{**}, 1]$. By equalizing V and V_{μ_2} of equation (4) and (6), we obtain the following system of equations:

$$\begin{cases} \phi(\underline{\mu}(\mu)) - \phi(\bar{\mu}(\mu)) = \frac{1}{2\sigma^2 c} \\ \psi(\underline{\mu}(\mu)) - \psi(\bar{\mu}(\mu)) = \frac{p-\mu}{2\sigma^2 c} \end{cases}, \text{ for } \mu \in [\mu^{**}, 1] \tag{7}$$

For each belief, μ, the system of equations above consists of two unknowns ($\bar{\mu}(\mu)$ and $\underline{\mu}(\mu)$) and two equations. They uniquely determine the function for the purchasing boundary $\bar{\mu}(\mu)$ and the function for the quitting boundary $\underline{\mu}(\mu)$, for $\mu \in [\mu^{**}, 1]$, given a cutoff belief μ^{**}.

Instead of determining $\bar{\mu}(\mu)$ and $\underline{\mu}(\mu)$ by a system of equations (7), we can also implicitly determine $\bar{\mu}(\mu)$ and $\underline{\mu}(\mu)$ in two separate equations. Representing $\bar{\mu}(\mu)$ by $\underline{\mu}(\mu)$ from the first equation of (7), we have:

$$\bar{\mu}(\mu) = \phi^{-1}\left[\phi(\underline{\mu}(\mu)) - \frac{1}{2\sigma^2 c}\right]$$

Plugging it into the second equation of (7), we have:

$$\underline{\mu}(\mu) = \psi^{-1}\left\{\psi\left(\phi^{-1}\left[\phi(\underline{\mu}(\mu)) - \frac{1}{2\sigma^2 c}\right]\right) + \frac{p-\mu}{2\sigma^2 c}\right\}$$

The equation above implicitly determines $\underline{\mu}(\mu)$, for $\mu \in [\mu^{**}, 1]$. Similarly, we can implicitly determine $\bar{\mu}(\mu)$ by the following equation:

$$\bar{\mu}(\mu) = \psi^{-1}\left\{\psi\left(\phi^{-1}\left[\phi(\bar{\mu}(\mu)) + \frac{1}{2\sigma^2 c}\right]\right) - \frac{p-\mu}{2\sigma^2 c}\right\}$$

We now solve for the cutoff belief at the intersection of the purchasing boundary and the main diagonal, μ^{**}. Since (μ^{**}, μ^{**}) is on the purchasing boundary, we have $\mu^{**} = \bar{\mu}(\mu^{**})$, μ^{**} is determined by:

$$\begin{cases} \phi(\underline{\mu}(\mu^{**})) - \phi(\mu^{**}) = \frac{1}{2\sigma^2 c} \\ \psi(\underline{\mu}(\mu^{**})) - \psi(\mu^{**}) = \frac{p-\mu^{**}}{2\sigma^2 c} \end{cases} \tag{8}$$

The system of equations above consists of two unknowns (μ^{**} and $\underline{\mu}(\mu^{**})$ and two equations. They uniquely determine the cutoff belief μ^{**} via the following equations:

$$\phi^{-1}\left[\phi(\mu^{**}) + \frac{1}{2\sigma^2 c}\right] = \psi^{-1}\left[\psi(\mu^{**}) + \frac{p-\mu^{**}}{2\sigma^2 c}\right] \tag{I^{**}}$$

We have pinned down the cutoff belief μ^{**}. Given this cutoff beliefs, we have determined the purchasing boundary $(\mu, \bar{\mu}(\mu))$ and the quitting boundary $(\mu, \underline{\mu}(\mu))$, for $\mu \in [\mu^{**}, 1]$.

The ODE (D_1) and the initial condition (I^{**}) implicitly determine the function for the quitting boundary $\underline{\mu}(\mu)$, for $\mu \in (\mu^*, \mu^{**}]$, given a cutoff belief μ^*.

We now solve for the cutoff belief at the intersection of the quitting boundary and the main diagonal, μ^*. Since (μ^*, μ^*) is on the quitting boundary, we have $\mu^* = \underline{\mu}(\mu^*)$. This initial condition determines μ^*.

In sum, we have pinned down the cutoff belief μ^* and the quitting boundary $(\mu, \underline{\mu}(\mu))$, for $\mu \in [\mu^*, \mu^{**}]$.

We have fully characterized the purchasing boundary $(\mu, \bar{\mu}(\mu))$ and the quitting boundary $(\mu, \underline{\mu}(\mu))$, for $\mu_1 \geq \mu_2$. The other case in which $\mu_1 < \mu_2$ is readily determined by symmetry.

<u>Verification:</u>

To verify that the conjectured strategy indeed generates a viscosity solution to the HJB equation $(\star)$:

$$\max\left\{\max_{i=1,2}\left[\frac{\mu_i^2(1-\mu_i)^2}{2\sigma^2}V_{\mu_i\mu_i}(\mu_1,\mu_2)-c\right], \max\left[\mu_1+\mu_2-p,0\right]-V(\mu_1,\mu_2)\right\}=0$$

We just need to show that (everything else holds by our construction):

$$\begin{aligned}&\frac{\mu_1^2(1-\mu_1)^2}{2\sigma^2}V_{\mu_1\mu_1}(\mu_1,\mu_2)-c\leq 0\\ \Leftrightarrow\ &\mu_1^2(1-\mu_1)^2V_{\mu_1\mu_1}(\mu_1,\mu_2)/2\sigma^2c\leq 1\end{aligned} \tag{9}$$

if $\mu_1+\mu_2>1$, $\mu_1\geq\mu_2$, and $\underline{\mu}(\mu_1)<\mu_2<\bar{\mu}(\mu_1)$.

For $\mu_1 \in (\mu^*, \mu^{**}]$, we have

$$\begin{aligned}
&V_{\mu_1}(\mu_1,\mu_2)/2\sigma^2c\\
=&\phi'(\underline{\mu}(\mu_1))\underline{\mu}'(\mu_1)[\mu_2-\underline{\mu}(\mu_1)]\\
\overset{(D_1)}{=}&\frac{\phi(\underline{\mu}(\mu_1))-\phi(\mu_1)}{\mu_1-\underline{\mu}(\mu_1)}[\mu_2-\underline{\mu}(\mu_1)]\\
\Rightarrow\ &V_{\mu_1\mu_1}(\mu_1,\mu_2)/2\sigma^2c\\
=&\phi'(\underline{\mu}(\mu_1))\underline{\mu}'(\mu_1)[\mu_2-\underline{\mu}(\mu_1)]\\
\overset{(D_1)}{=}&\frac{\phi'(\underline{\mu}(\mu_1))\underline{\mu}'(\mu_1)-\phi'(\mu_1)}{\mu_1-\underline{\mu}(\mu_1)}[\mu_2-\underline{\mu}(\mu_1)]+[\phi(\underline{\mu}(\mu_1))-\phi(\mu_1)]\frac{(\mu_2-\mu_1)\underline{\mu}'(\mu_1)+\underline{\mu}(\mu_1)-\mu_2}{[\mu_1-\underline{\mu}(\mu_1)]^2}\\
=&-\frac{\phi'(\mu_1)[\mu_2-\underline{\mu}(\mu_1)]}{\mu_1-\underline{\mu}(\mu_1)}+(\mu_2-\mu_1)\frac{[\phi(\underline{\mu}(\mu_1))-\phi(\mu_1)]^2}{\phi'(\underline{\mu}(\mu_1))[\mu_1-\underline{\mu}(\mu_1)]^3}\\
\Rightarrow\ &\mu_1^2(1-\mu_1)^2V_{\mu_1\mu_1}(\mu_1,\mu_2)/2\sigma^2c
\end{aligned}$$

$$=\frac{\mu_2-\underline{\mu}(\mu_1)}{\mu_1-\underline{\mu}(\mu_1)}+(\mu_1-\mu_2)\mu_1^2(1-\mu_1)^2\frac{\underline{\mu}(\mu_1)^2[1-\underline{\mu}(\mu_1)]^2}{[\mu_1-\underline{\mu}(\mu_1)]^3}[\phi(\underline{\mu}(\mu_1))-\phi(\mu_1)]^2$$

So,

$$\begin{aligned}
&\mu_1^2(1-\mu_1)^2V_{\mu_1\mu_1}(\mu_1,\mu_2)/2\sigma^2c\leq 1\\
\Leftrightarrow&\mu_1^2(1-\mu_1)^2\frac{\underline{\mu}(\mu_1)^2[1-\underline{\mu}(\mu_1)]^2}{[\mu_1-\underline{\mu}(\mu_1)]^2}[\phi(\underline{\mu}(\mu_1))-\phi(\mu_1)]^2\leq 1\\
\Leftrightarrow&\mu_1(1-\mu_1)\frac{\underline{\mu}(\mu_1)^2[1-\underline{\mu}(\mu_1)]}{[\mu_1-\underline{\mu}(\mu_1)]}[\phi(\underline{\mu}(\mu_1))-\phi(\mu_1)]\leq 1\\
\Leftrightarrow&H(\mu_1):=\mu_1(1-\mu_1)[\phi(\underline{\mu}(\mu_1))-\phi(\mu_1)]-\frac{\mu_1-\underline{\mu}(\mu_1)}{\underline{\mu}(\mu_1)[1-\underline{\mu}(\mu_1)]}\leq 0
\end{aligned}\tag{10}$$

Observe that $H(\mu^*)=0$. Ignoring the subscript 1 for notational ease, we have:

$$\begin{aligned}
H'(\mu)=&(1-2\mu)[\phi(\underline{\mu}(\mu))-\phi(\mu)]+\frac{\mu(1-\mu)}{\mu-\underline{\mu}(\mu)}[\phi(\underline{\mu}(\mu))-\phi(\mu)]+\frac{1}{\mu(1-\mu)}\\
&-\frac{1}{\underline{\mu}(\mu)(1-\underline{\mu}(\mu))}+\frac{\phi(\underline{\mu}(\mu))-\phi(\mu)}{\mu-\underline{\mu}(\mu)}[-\mu+2\mu\underline{\mu}(\mu)-\underline{\mu}(\mu)^2]\\
=&[1-3\mu+\underline{\mu}(\mu)][\phi(\underline{\mu}(\mu))-\phi(\mu)]+\frac{1}{\mu(1-\mu)}-\frac{1}{\underline{\mu}(\mu)(1-\underline{\mu}(\mu))}
\end{aligned}$$

Suppose (10) does not hold. There would exist $\widehat{\mu}$ such that $H(\widehat{\mu})=0$ and $H'(\widehat{\mu})>0$.

$$(10)\Rightarrow\phi(\underline{\mu}(\widehat{\mu}))-\phi(\widehat{\mu})=\frac{\widehat{\mu}-\underline{\mu}(\widehat{\mu})}{\widehat{\mu}(1-\widehat{\mu})\underline{\mu}(\widehat{\mu})[1-\underline{\mu}(\widehat{\mu})]}$$

Hence, we get an expression for $\frac{1}{\widehat{\mu}(1-\widehat{\mu})}$ and $\frac{1}{\underline{\mu}(\mu)[1-\underline{\mu}(\mu)]}$. Plugging these expressions into the previous expression for $H'(\mu)$, we have:

$$H'(\widehat{\mu})=-2[\phi(\underline{\mu}(\mu))-\phi(\mu)][\mu-\underline{\mu}(\mu)]\leq 0$$

A contradiction! So, (10) and thus (9) hold, $\forall\mu_1\in[\mu^*,\mu^{**}]$.

For $\mu_1 \in [\mu^{**}, 1]$, we have

$$
\begin{aligned}
V_{\mu_1}(\mu_1,\mu_2)/2\sigma^2 c =& \phi'(\underline{\mu}(\mu_1))\underline{\mu}'(\mu_1)[\mu_2 - \underline{\mu}(\mu_1)] \\
\overset{(D_2)}{=} & \frac{\mu_2 - \underline{\mu}(\mu_1)}{\bar{\mu}(\mu_1) - \underline{\mu}(\mu_1)} \\
V_{\mu_1\mu_1}(\mu_1,\mu_2)/2\sigma^2 c =& \frac{-\underline{\mu}'(\mu_1)[\bar{\mu}(\mu_1) - \underline{\mu}(\mu_1)] - [\bar{\mu}'(\mu_1) - \underline{\mu}'(\mu_1)][\mu_2 - \underline{\mu}(\mu_1)]}{[\bar{\mu}(\mu_1) - \underline{\mu}(\mu_1)]^2} \\
=& \frac{1}{2\sigma^2 c}\frac{1}{[\bar{\mu}(\mu_1) - \underline{\mu}(\mu_1)]^3}[\frac{\mu_2 - \bar{\mu}(\mu_1)}{\phi'(\underline{\mu}(\mu_1))} - \frac{\mu_2 - \underline{\mu}(\mu_1)}{\phi'(\bar{\mu}(\mu_1))}] \\
\Rightarrow V_{\mu_1\mu_1}(\mu_1,\mu_2) =& \frac{1}{[\bar{\mu}(\mu_1) - \underline{\mu}(\mu_1)]^3}[\frac{\mu_2 - \bar{\mu}(\mu_1)}{\phi'(\underline{\mu}(\mu_1))} - \frac{\mu_2 - \underline{\mu}(\mu_1)}{\phi'(\bar{\mu}(\mu_1))}]
\end{aligned}
$$

Since $\frac{\partial V_{\mu_1\mu_1}(\mu_1,\mu_2)}{\partial \mu_2} < 0$, we only need to show that (9) holds for $\mu_2 = \underline{\mu}(\mu_1)$:

$$
\begin{aligned}
& \mu_1^2(1-\mu_1)^2 V_{\mu_1\mu_1}(\mu_1, \underline{\mu}(\mu_1))/2\sigma^2 c \leq 1 \\
\Leftrightarrow & \frac{\mu_1^2(1-\mu_1)^2}{[\bar{\mu}(\mu_1) - \underline{\mu}(\mu_1)]^2}\frac{-1}{\phi'(\underline{\mu}(\mu_1))} \leq 1
\end{aligned} \tag{11}
$$

Let's first show that $\underline{\mu}(\mu^{**}) \leq 1/2$ by contradiction. Suppose instead $\underline{\mu}(\mu^{**}) > 1/2$.

$$
\begin{aligned}
& p - \mu^{**} = \frac{\bar{\mu}(\mu^{**}) + \underline{\mu}(\mu^{**})}{2} \\
\Leftrightarrow \ & p - \mu^{**} = \frac{\mu^{**} + \underline{\mu}(\mu^{**})}{2} \\
\Leftrightarrow \ & \underline{\mu}(\mu^{**}) = 2p - 3\mu^{**}
\end{aligned}
$$

Hence, $2p - 3\mu^{**} > 1/2 \Rightarrow \mu^{**} < \frac{2}{3}p - \frac{1}{6}$. Since $\phi(x)$ is strictly decreasing in x, the first equation of (8) implies

$$
\begin{aligned}
\frac{1}{2\sigma^2 c} =& \phi(\underline{\mu}(\mu^{**})) - \phi(\mu^{**}) \\
<& \phi(1/2) - \phi(\frac{2}{3}p - \frac{1}{6}) \\
\Leftrightarrow c >& \frac{1}{2\sigma^2[\phi(1/2) - \phi(\frac{2}{3}p - \frac{1}{6})]}
\end{aligned}
$$

A contradiction! Therefore, $\underline{\mu}(\mu^{**}) \leq 1/2$. Since $\underline{\mu}(\mu_1)$ is decreasing in μ_1, we have $\underline{\mu}(\mu_1) \leq$

$1/2$, $\forall \mu \in [\mu^{**}, 1]$. One can see that the LHS of (11), $\frac{\mu_1^2(1-\mu_1)^2}{[\bar{\mu}(\mu_1)-\underline{\mu}(\mu_1)]^2} \frac{-1}{\phi'(\underline{\mu}(\mu_1))}$, decreases in $\mu_1 \in [\mu^{**}, 1]$. And we know that (11) holds for $\mu_1 = \mu^{**}$ (we have shown that (9) and thus (11) hold for $\forall \mu_1 \in [\mu^*, \mu^{**}]$). Therefore, (11) and thus (9) hold for $\forall \mu_1 \in [\mu^{**}, 1]$.

□

Proof of Proposition 4. We first consider $\mu_1 \in [\mu^{**}, 1]$ and $\mu_1 \geq \mu_2$. Under this circumstance, the consumer only learns about attribute two until μ_2 hits either the purchasing boundary or the quitting boundary. As μ_2 is a martingale, by Dynkin's formula, we get:

$$P(\mu_1, \mu_2) := \mathbb{P}[\text{purchasing}|\text{starting at } (\mu_1, \mu_2)] = \frac{\mu_2 - \underline{\mu}(\mu_1)}{\bar{\mu}(\mu_1) - \underline{\mu}(\mu_1)}$$

Now we consider $\mu_1 \in [\mu^*, \mu^{**}]$ and $\mu_1 \geq \mu_2$. The belief either hits (μ^{**}, μ^{**}) and the consumer purchases the good or the belief hits $\{(x, \underline{\mu}(x)) : x \in [\mu_1, \mu^{**})\} \cup \{(\underline{\mu}(x), x) : x \in [\mu_1, \mu^{**})\}$ and the consumer quits. To calculate the purchasing likelihood, let's first calculate the likelihood of the belief hitting $(\mu_1, \underline{\mu}(\mu_1))$ before hitting the main diagonal (μ_1, μ_1), $q(\mu_1, \mu_2)$.

$$q(\mu_1, \mu_2) = \frac{\mu_1 - \mu_2}{\mu_1 - \underline{\mu}(\mu_1)}$$

Now we calculate the probability of purchasing given belief (μ, μ), $\tilde{P}(\mu)$ by consider the infinitesimal learning on attribute two. Noticing that $q(\mu, \mu) = 0$, $\frac{\partial q}{\partial \mu_1}|_{\mu_1=\mu_2=\mu} = \frac{1}{\mu - \underline{\mu}(\mu)}$, $\frac{\partial q}{\partial \mu_2}|_{\mu_1=\mu_2=\mu} = -\frac{1}{\mu - \underline{\mu}(\mu)}$, we have:

$$\begin{aligned}
\tilde{P}(\mu) &= \frac{1}{2}\mathbb{P}[\text{purchasing}|(\mu, \mu), d\mu \geq 0] + \frac{1}{2}\mathbb{P}[\text{purchasing}|(\mu, \mu), d\mu < 0] \\
&= \frac{1}{2}[1 - q(\mu + |d\mu|, \mu)]\tilde{P}(\mu + |d\mu|) + \frac{1}{2}[1 - q(\mu - |d\mu|, \mu)]\tilde{P}(\mu) \\
&= \tilde{P}(\mu) + \frac{|d\mu|}{2}\tilde{P}'(\mu) + |d\mu|\frac{\tilde{P}(\mu)}{\underline{\mu}(\mu) - \mu} + o(d\mu) \\
\Rightarrow 0 &= \frac{|d\mu|}{2}\left[\tilde{P}'(\mu) + 2\frac{\tilde{P}(\mu)}{\underline{\mu}(\mu) - \mu}\right] + o(d\mu) \\
\Rightarrow \frac{\tilde{P}'(\mu)}{\tilde{P}(\mu)} &= -\frac{2}{\underline{\mu}(\mu) - \mu}, \ \forall \mu \in (\mu^*, \mu^{**})
\end{aligned}$$

, where the last equality comes from dividing the previous equation by $|d\mu|$ and take the limit of

$d\mu$ to 0. Together with the initial condition $\tilde{P}(\mu^{**}) = 1$, we obtain:

$$\tilde{P}(\mu) = e^{-\int_{\mu}^{\mu^{**}} \frac{2}{x-\underline{\mu}(x)} dx}$$

In sum, the purchasing likelihood when $\mu_1 \geq \mu_2$ and $\mu_1 \in (\mu^*, \mu^{**})$ is:

$$P(\mu_1, \mu_2) = \mathbb{P}[\text{purchasing}|\text{starting at } (\mu_1, \mu_2)] = [1 - q(\mu_1, \mu_2)]\tilde{P}(\mu_1) = h(\mu_1, \mu_2)\tilde{P}(\mu_1)$$

, where $h(\mu_1, \mu_2) = \frac{\mu_2 - \underline{\mu}(\mu_1)}{\mu_1 - \underline{\mu}(\mu_1)}$.

By symmetry, the purchasing likelihood when $\mu_1 < \mu_2$ and $\mu_2 \in (\mu^*, \mu^{**})$ is:

$$P(\mu_1, \mu_2) = P(\mu_2, \mu_1) = [1 - q(\mu_2, \mu_1)]\tilde{P}(\mu_2) = h(\mu_2, \mu_1)\tilde{P}(\mu_2)$$

$\square$

Proof of Proposition 5. We first define $\tilde{\mu}^*$ by $\underline{\tilde{\mu}}(\tilde{\mu}^*) = \tilde{\mu}^*$ and $\tilde{\mu}^{**}$ by $\tilde{\bar{\mu}}(\tilde{\mu}^{**}) = \tilde{\mu}^{**}$.

If $\mu_2 \leq \underline{\tilde{\mu}}(\mu_1)$ or $\mu_2 \geq \tilde{\bar{\mu}}(\mu_1)$, the purchasing probability is 0 regardless the search costs. Also, the purchasing probability is positive under the reduced search cost $\tilde{c}$ and 0 under the default search cost c if $\underline{\tilde{\mu}}(\mu_1) < \mu_2 \leq \underline{\mu}(\mu_1)$. So, it is always better for the firm to reduce search costs in that region. The purchasing probability is less than 1 under the reduced search cost $\tilde{c}$ and 1 under the default search cost c if $\bar{\mu}(\mu_1) \leq \mu_2 < \tilde{\bar{\mu}}(\mu_1)$. So, it is always worse for the firm to reduce search costs in that region. We restrict our attention to the remaining case where $\underline{\mu}(\mu_1) < \mu_2 < \bar{\mu}(\mu_1)$ in the subsequent analyses. Consider three cases.

1. $\mu_1 \geq \tilde{\mu}^{**}$

   The purchasing probability under the default search cost c is:

$$P(\mu_1, \mu_2) = \frac{\mu_2 - \underline{\mu}(\mu_1)}{\bar{\mu}(\mu_1) - \underline{\mu}(\mu_1)}$$

The purchasing probability under the reduced search cost $\tilde{c}$ is:

$$\tilde{P}(\mu_1, \mu_2) = \frac{\mu_2 - \tilde{\underline{\mu}}(\mu_1)}{\tilde{\bar{\mu}}(\mu_1) - \tilde{\underline{\mu}}(\mu_1)}$$

We also know that $\mu_1 + \frac{\bar{\mu}(\mu_1)+\underline{\mu}(\mu_1)}{2} = \mu_1 + \frac{\tilde{\bar{\mu}}(\mu_1)+\tilde{\underline{\mu}}(\mu_1)}{2} = p \Rightarrow \bar{\mu}(\mu_1)+\underline{\mu}(\mu_1) = \tilde{\bar{\mu}}(\mu_1)+\tilde{\underline{\mu}}(\mu_1) = 2(p-\mu_1)$. Since both $P(\mu_1, \mu_2)$ and $\tilde{P}(\mu_1, \mu_2)$ are linear in μ_2, $P(\mu_1, p-\mu_1) = \tilde{P}(\mu_1, p-\mu_1) = 1/2$, and $P(\mu_1, \mu_2)$ has a lower slope than $\tilde{P}(\mu_1, \mu_2)$ as a linear function of μ_2, one can see that $P(\mu_1, \mu_2) < \tilde{P}(\mu_1, \mu_2)$ if and only if $\mu_2 < p - \mu_1$. Therefore, $\tilde{\mu}(\mu_1) = p - \mu_1 \in (\underline{\mu}(\mu_1), \bar{\mu}(\mu_1))$ in this case.

2. $\mu^{**} < \mu_1 < \tilde{\mu}^{**}$

   The purchasing probability under the default search cost c is still:

$$P(\mu_1, \mu_2) = \frac{\mu_2 - \underline{\mu}(\mu_1)}{\bar{\mu}(\mu_1) - \underline{\mu}(\mu_1)}$$

   The purchasing probability under the reduced search cost $\tilde{c}$ is now:

$$\tilde{P}(\mu_1, \mu_2) = \frac{\mu_2 - \tilde{\underline{\mu}}(\mu_1)}{\mu_1 - \tilde{\underline{\mu}}(\mu_1)} e^{-\int_{\mu_1}^{\tilde{\mu}^{**}} \frac{2}{x - \tilde{\underline{\mu}}(x)} dx}$$

   Both $P(\mu_1, \mu_2)$ and $\tilde{P}(\mu_1, \mu_2)$ are linear in μ_2. Observe that $P(\mu_1, \underline{\mu}(\mu_1)) = 0 < \tilde{P}(\mu_1, \underline{\mu}(\mu_1))$ and $P(\mu_1, \bar{\mu}(\mu_1)) = 1 > \tilde{P}(\mu_1, \bar{\mu}(\mu_1))$, one can see that there exists an unique $\tilde{\mu}(\mu_1) \in (\underline{\mu}(\mu_1), \bar{\mu}(\mu_1))$ such that $P(\mu_1, \mu_2) < \tilde{P}(\mu_1, \mu_2)$ if and only if $\mu_2 < \tilde{\mu}(\mu_1)$.

3. $\mu^{*} < \mu_1 < \mu^{**}$

   The purchasing probability under the default search cost c is:

$$P(\mu_1, \mu_2) = \frac{\mu_2 - \underline{\mu}(\mu_1)}{\mu_1 - \underline{\mu}(\mu_1)} e^{-\int_{\mu_1}^{\mu^{**}} \frac{2}{x - \underline{\mu}(x)} dx}$$

   The purchasing probability under the reduced search cost $\tilde{c}$ is:

$$\tilde{P}(\mu_1, \mu_2) = \frac{\mu_2 - \tilde{\underline{\mu}}(\mu_1)}{\mu_1 - \tilde{\underline{\mu}}(\mu_1)} e^{-\int_{\mu_1}^{\tilde{\mu}^{**}} \frac{2}{x - \tilde{\underline{\mu}}(x)} dx}$$

Hence,

$$
\begin{aligned}
& P(\mu_1, \mu_2) < \tilde{P}(\mu_1, \mu_2) \\
\Leftrightarrow\ & \frac{\mu_2 - \underline{\mu}(\mu_1)}{\mu_2 - \underline{\tilde{\mu}}(\mu_1)} < \frac{\mu_1 - \underline{\mu}(\mu_1)}{\mu_1 - \underline{\tilde{\mu}}(\mu_1)} e^{\int_{\mu_1}^{\mu^{**}} \frac{2}{x - \underline{\mu}(x)} dx - \int_{\mu_1}^{\tilde{\mu}^{**}} \frac{2}{x - \underline{\tilde{\mu}}(x)} dx}
\end{aligned}
$$

The left-hand side strictly increases in μ_2 whereas the right-hand side does not depend on μ_2. Furthermore, observe that the left-hand side $\to 0$ as $\mu_2 \to \underline{\mu}(\mu_1)$ while the right-hand side is positive, one can see that there exists an unique $\tilde{\mu}(\mu_1) \in (\underline{\mu}(\mu_1), \mu_1] \subset (\underline{\mu}(\mu_1), \mu^{**})$ such that $P(\mu_1, \mu_2) < \tilde{P}(\mu_1, \mu_2)$ if and only if $\mu_2 < \tilde{\mu}(\mu_1)$.[5]

In sum, there exists $\tilde{\mu}(\mu_1)$ such that $\underline{\mu}(\mu_1) < \tilde{\mu}(\mu_1) < \bar{\mu}(\mu_1)$ if $\mu_1 \geq \mu^{**}$ and $\underline{\mu}(\mu_1) < \tilde{\mu}(\mu_1) \leq \mu_1$ if $\mu_1 < \mu^{**}$. The firm reduces the search cost if and only if $\mu_2 \in (\underline{\tilde{\mu}}(\mu_1), \tilde{\mu}(\mu_1))$. $\square$

Funding and Competing Interests

All authors certify that they have no affiliations with or involvement in any organization or entity with any financial interest or non-financial interest in the subject matter or materials discussed in this manuscript. The authors have no funding to report.

[5] For μ_1 close to μ^*, $\tilde{\mu}(\mu_1)$ may equal μ_1.

References

Arbatskaya, M. (2007). Ordered search. The RAND Journal of Economics, 38(1), 119-126.

Armstrong, M., Zhou, J. (2011). Paying for prominence. The Economic Journal, 121(556), F368-F395.

Armstrong, M., Vickers, J., Zhou, J. (2009). Prominence and consumer search. The RAND Journal of Economics, 40(2), 209-233.

Bar-Isaac, H., Caruana, G., & Cuñat, V. (2010). Information gathering and marketing. Journal of Economics & Management Strategy, 19(2), 375-401.

Bar-Isaac, H., Caruana, G., & Cuñat, V. (2012). Information gathering externalities for a multi-attribute good. The Journal of Industrial Economics, 60(1), 162-185.

Bolton, P., & Harris, C. (1999). Strategic experimentation. Econometrica, 67(2), 349-374.

Bordalo, P., Gennaioli, N., Shleifer, A. (2013). Salience and consumer choice. Journal of Political Economy, 121(5), 803-843.

Branco, F., Sun, M., Villas-Boas, J. M. (2012). Optimal search for product information. Management Science, 58(11), 2037-2056.

Branco, F., Sun, M., Villas-Boas, J. M. (2016). Too much information? Information provision and search costs. Marketing Science, 35(4), 605-618.

Bronnenberg, B. J., Kim, J. B., & Mela, C. F. (2016). Zooming in on choice: How do consumers search for cameras online?. Marketing science, 35(5), 693-712.

Chaimanowong, W., Chen Q., & Yao, Y. (2024). Consumer Gradual Learning and Firm Non-stationary Pricing. Working Paper

Crandall, M. G., Ishii, H., Lions, P. L. (1992). User's guide to viscosity solutions of second order partial differential equations. Bulletin of the American mathematical society, 27(1), 1-67.

De Corniere, A. (2016). Search advertising. American Economic Journal: Microeconomics, 8(3), 156-188.

Dixit, A. (1993). The art of smooth pasting. Harwood Academic Publishers: Chur, Switzerland.

Dukes, A., & Liu, L. (2016). Online shopping intermediaries: The strategic design of search environments. Management Science, 62(4), 1064-1077.

Fudenberg, D., Strack, P., & Strzalecki, T. (2018). Speed, accuracy, and the optimal timing of choices. American Economic Review, 108(12), 3651-3684.

Gardete, P., & Hunter, M., (2024). Multiattribute Search: Empirical Evidence and Information Design. Available at SSRN: http://dx.doi.org/10.2139/ssrn.3180811

Guo, L., & Zhang, J. (2012). Consumer deliberation and product line design. Marketing Science, 31(6), 995-1007.

Häubl, G., & Trifts, V. (2000). Consumer decision making in online shopping environments: The effects of interactive decision aids. Marketing science, 19(1), 4-21.

Hauser, J. (2011). A marketing science perspective on recognition-based heuristics (and the fast-and-frugal paradigm). Judgment and Decision Making, 6(5), 396-408.

Jerath, K., & Ren, Q. (2021). Consumer rational (in) attention to favorable and unfavorable product information, and firm information design. Journal of Marketing Research, 58(2), 343-362.

Jerath, K., & Ren, Q. (2023). Consumer search and product returns. Working paper.

Jeziorski, P., Moorthy, S. (2018). Advertiser prominence effects in search advertising. Management science, 64(3), 1365-1383.

Ke, T. T., & Lin, S. (2020). Informational complementarity. Management Science, 66(8), 3699-3716.

Ke, T. T., Lin, S., Lu, M. Y. (2023). Information design of online platforms. Working Paper.

Ke, T. T., Shen, Z. J. M., Villas-Boas, J. M. (2016). Search for information on multiple products. Management Science, 62(12), 3576-3603.

Ke, T. T., Villas-Boas, J. M. (2019). Optimal learning before choice. Journal of Economic Theory, 180, 383-437.

Kőszegi, B., Szeidl, A. (2013). A model of focusing in economic choice. The Quarterly journal of economics, 128(1), 53-104.

Kuksov, D., Villas-Boas, J. M. (2010). When more alternatives lead to less choice. Marketing Science, 29(3), 507-524.

Liao, X. (2021). Bayesian persuasion with optimal learning. Journal of Mathematical Economics, 97, 102534.

Liu, L., & Dukes, A. (2013). Consideration set formation with multiproduct firms: The case of within-firm and across-firm evaluation costs. Management Science, 59(8), 1871-1886.

Mayzlin, D., Shin, J. (2011). Uninformative advertising as an invitation to search. Marketing science, 30(4), 666-685.

Morris, S., & Strack, P. (2019). The wald problem and the relation of sequential sampling and ex-ante information costs. Available at SSRN 2991567.

Moscarini, G., Smith, L. (2001). The optimal level of experimentation. Econometrica, 69(6), 1629-1644.

Ngwe, D., Ferreira, K. J., & Teixeira, T. (2019). The impact of increasing search frictions on online shopping behavior: Evidence from a field experiment. Journal of Marketing Research, 56(6), 944-959.

Seiler, S. (2013). The impact of search costs on consumer behavior: A dynamic approach. Quantitative Marketing and Economics, 11, 155-203.

Stigler, G. J. (1961). The economics of information. Journal of political economy, 69(3), 213-225.

Stivers, A., & Tremblay, V. J. (2005). Advertising, search costs, and social welfare. Information Economics and Policy, 17(3), 317-333.

Ursu, R. M., Wang, Q., Chintagunta, P. K. (2020). Search duration. Marketing Science, 39(5), 849-871.

Ursu, R. M., Zhang, Q., & Honka, E. (2023). Search gaps and consumer fatigue. Marketing Science, 42(1), 110-136.

Villas-Boas, J. M. (2009). Product variety and endogenous pricing with evaluation costs. Management Science, 55(8), 1338-1346.

Weitzman, M. L. (1979). Optimal search for the best alternative. Econometrica: Journal of the Econometric Society, 641-654.

Wolinsky, A. (1986). True monopolistic competition as a result of imperfect information. The Quarterly Journal of Economics, 101(3), 493-511.

Xu, L., Chen, J., Whinston, A. (2010). Oligopolistic pricing with online search. Journal of Management information systems, 27(3), 111-142.

Yao, Y. (2023). Dynamic persuasion and strategic search. Management Science.

Zhu, Y., Dukes, A. (2017). Prominent attributes under limited attention. Marketing Science, 36(5), 683-698.

Online Appendix for Multi-attribute Search

Yunfei (Jesse) Yao

Jan. 2024

1 Proofs of the Benchmark Model

Proof of Lemma 1. If one attribute is bad, the consumer's payoff from purchasing is negative regardless of the other attribute's value. So, the consumer will quit directly.

Now suppose the consumer searches attribute 1 and finds out it is good in the first period. Conditional on the first-period action and outcome, the consumer's second-period utility is $1+\mu_2-p$ if she purchases without searching again, $-c+\mu_2(2-p)$ if she searches attribute 2 and buy if only if it is also good, and 0 if she quits.

The consumer prefers searching again to quitting if and only if $-c+\mu_2(2-p) \geq 0 \Leftrightarrow \mu_2 > c/(2-p)$. She prefers purchasing directly to searching again if and only if $1+\mu_2-p \geq -c+\mu_2(2-p) \Leftrightarrow \mu_2 \geq 1-c/(p-1)$. The assumption $c < (p-1)[1-\sqrt{(3-p)(p-1)}]/(2-p)$ implies that $c/(2-p) < 1-c/(p-1)$. Hence, the consumer purchases directly if $\mu_2 \geq 1-c/(p-1)$, searches attribute 2 and buy if only if it is also good if $\mu_2 \in [c/(2-p), 1-c/(p-1))$, and quits if $\mu_2 < c/(2-p)$.

The case of searching attribute 2 in the first period is symmetric to that of searching attribute 1 in the first period. □

Proof of Proposition 1. The consumer's expected utility is $\mu_1+\mu_2-p$ if she buys without searching, $-c+\mu_1(1+\mu_2-p)$ if she searches attribute 1 and makes a decision, $-c+\mu_2(1+\mu_1-p)$ if she searches attribute 2 and makes a decision, $-(1+\mu_1)c+\mu_1\mu_2(2-p)$ if she searches attribute 1 and then attribute 2 when attribute 1 is good, and $-(1+\mu_2)c+\mu_2\mu_1(2-p)$ if she searches attribute 2 and then attribute 1 when attribute 2 is good.

The consumer will buy without searching if $\begin{cases} \mu_1+\mu_2-p \geq 0 \\ \mu_1+\mu_2-p \geq -c+\mu_1(1+\mu_2-p) \\ \mu_1+\mu_2-p \geq -c+\mu_2(1+\mu_1-p) \\ \mu_1+\mu_2-p \geq -(1+\mu_1)c+\mu_1\mu_2(2-p) \\ \mu_1+\mu_2-p \geq -(1+\mu_2)c+\mu_2\mu_1(2-p) \end{cases}$

One can see that the third inequality implies the second inequality, and that the fifth inequality implies the fourth inequality. Therefore, the above conditions are equivalent to

$$\begin{cases} \mu_1+\mu_2-p \geq 0 \\ \mu_1+\mu_2-p \geq -c+\mu_2(1+\mu_1-p) \\ \mu_1+\mu_2-p \geq -(1+\mu_2)c+\mu_2\mu_1(2-p) \end{cases} \Leftrightarrow \begin{cases} \mu_1+\mu_2 \geq p \\ \mu_2 \geq \max\{1-\frac{c}{p-\mu_1}, \frac{p-\mu_1-c}{1+c-\mu_1(2-p)}\} \end{cases}$$

Similarly, The consumer will buy without searching if $\begin{cases} \mu_1+\mu_2-p < 0 \\ -c+\mu_1(1+\mu_2-p) < 0 \\ -c+\mu_2(1+\mu_1-p) < 0 \\ -(1+\mu_1)c+\mu_1\mu_2(2-p) < 0 \\ -(1+\mu_2)c+\mu_2\mu_1(2-p) < 0 \end{cases}$

One can see that the third inequality implies the second inequality, and that the fifth inequality implies the fourth inequality. Therefore, the above conditions are equivalent to

$$\begin{cases} \mu_1+\mu_2-p < 0 \\ -c+\mu_2(1+\mu_1-p) < 0 \\ -(1+\mu_2)c+\mu_2\mu_1(2-p) < 0 \end{cases} \Leftrightarrow \begin{cases} \mu_1+\mu_2 < p \\ \mu_1 < \min\{p-1+\frac{c}{\mu_2}, \frac{(1+\mu_2)c}{\mu_2(2-p)}\} \end{cases}$$

Now consider the case in which the consumer searches first. There are four possibilities before pinning down which attribute the consumer will search first.

1. The consumer will make the purchasing decision without searching again if she searches either attribute first.

2. The consumer will search again if she searches either attribute first and it turns out to be good.

3. The consumer will make the purchasing decision without searching again if she searches attribute 1 first, but will search again if she searches attribute 2 first and it turns out to be good.

4. The consumer will make the purchasing decision without searching again if she searches attribute 2 first, but will search again if she searches attribute 1 first and it turns out to be good.

By comparing the expected utility from searching one of the attributes first, one can see that the consumer will search attribute 2 first in the first two cases. By lemma 1, one can see that the third case is impossible, because it requires that $\mu_2 > 1 - c/(p-1) > \mu_1$, condtradicted with the assumption $\mu_1 \geq \mu_2$. We now show that the consumer searches attribute 2 first in the last case. Therefore, the consumer never searches attribute 1 first. The necessary and sufficient conditions for the last case to hold and for the consumer to prefer searching attribute 1 first are:

$$\begin{cases} \mu_1 > 1 - \frac{c}{p-1} \ (lemma1) \\ \mu_2 \in (\frac{c}{2-p}, 1 - \frac{c}{p-1}) \ (lemma1) \\ -(1+\mu_1)c + \mu_1\mu_2(2-p) > \mu_1 + \mu_2 - p \text{ (searching attribute 1 first is better than buying directly)} \\ -(1+\mu_1)c + \mu_1\mu_2(2-p) > 0 \text{ (searching attribute 1 first is better than quitting directly)} \\ -(1+\mu_1)c + \mu_1\mu_2(2-p) > -c + \mu_2(1+\mu_1-p) \\ \text{(searching attribute 1 first is better than searching attribute 2 first)} \end{cases}$$

However, we have:

$$\begin{aligned} -(1+\mu_1)c + \mu_1\mu_2(2-p) &\overset{\mu_1 \geq \mu_2}{\leq} -(1+\mu_2)c + \mu_1\mu_2(2-p) \\ &\overset{\mu_1 > 1 - \frac{c}{p-1}}{<} -c + \mu_2(1+\mu_1-p) \end{aligned}$$

, which contradicts with the last condition. □

2 Proof of the Comparative Statics

Proof of Proposition 3.

(1) Comparative statics w.r.t. p

We first consider $\bar{\mu}(\mu)$. Fixing an arbitrary $\mu \in (\mu^{**}, 1]$, recall the system of equations (7):

$$\phi(\underline{\mu}(\mu)) - \phi(\bar{\mu}(\mu)) = \frac{1}{2\sigma^2 c}$$
$$\psi(\underline{\mu}(\mu)) - \psi(\bar{\mu}(\mu)) = \frac{p-\mu}{2\sigma^2 c}$$

By the implicit function theorem, we obtain:

$$\begin{bmatrix} \frac{\partial \bar{\mu}(\mu)}{\partial p} \\ \frac{\partial \underline{\mu}(\mu)}{\partial p} \end{bmatrix} = - \begin{bmatrix} -\phi'(\bar{\mu}(\mu)) & \phi'(\underline{\mu}(\mu)) \\ -\psi'(\bar{\mu}(\mu)) & \psi'(\underline{\mu}(\mu)) \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 \\ -\frac{1}{2\sigma^2 c} \end{bmatrix} = \begin{bmatrix} -\frac{\phi(\bar{\mu}(\mu)) - \phi(\bar{\mu}(\mu))}{\phi'(\underline{\mu}(\mu))[\bar{\mu}(\mu) - \underline{\mu}(\mu)]} & > 0 \\ -\frac{\phi(\underline{\mu}(\mu)) - \phi(\bar{\mu}(\mu))}{\phi'(\underline{\mu}(\mu))[\bar{\mu}(\mu) - \underline{\mu}(\mu)]} & > 0 \end{bmatrix}$$

We now consider $\underline{\mu}(\mu)$. Suppose there exists $p_1 > p_2$ with the corresponding quitting boundaries $(\mu_1, \underline{\mu}_{p_1}(\mu_1))$ and $(\mu_1, \underline{\mu}_{p_2}(\mu_1))$, respectively. Denote the cutoff beliefs by $(\mu^*_{p_1}, \mu^{**}_{p_1})$ for price p_1 and by $(\mu^*_{p_2}, \mu^{**}_{p_2})$ for price p_2. Fixing an arbitrary $\mu_1 \in (\mu^*_{p_1}, 1]$, we know that the consumer is indifferent between quitting and searching for information when her belief is $(\mu_1, \underline{\mu}_{p_1}(\mu_1))$ and the price is p_1. Since $p_2 < p_1$, one can see that the value of searching for information when her belief is $(\mu_1, \underline{\mu}_{p_1}(\mu_1))$ and the price is p_2 is strictly higher than zero. So, the consumer will keep searching for information. Thus, $\underline{\mu}_{p_2}(\mu_1) < \underline{\mu}_{p_1}(\mu_1)$.

Therefore, the entire search region shifts upwards as the price increases.

(2) Comparative statics w.r.t. c

We first consider $\bar{\mu}(\mu)$. Fixing an arbitrary $\mu \in (\mu^{**}, 1]$, recall the system of equations (7):

$$\phi(\underline{\mu}(\mu)) - \phi(\bar{\mu}(\mu)) = \frac{1}{2\sigma^2 c}$$
$$\psi(\underline{\mu}(\mu)) - \psi(\bar{\mu}(\mu)) = \frac{p-\mu}{2\sigma^2 c}$$

By the implicit function theorem, we obtain:

$$\begin{bmatrix} \frac{\partial \bar{\mu}(\mu)}{\partial c} \\ \frac{\partial \underline{\mu}(\mu)}{\partial c} \end{bmatrix} = - \begin{bmatrix} -\phi'(\bar{\mu}(\mu)) & \phi'(\underline{\mu}(\mu)) \\ -\psi'(\bar{\mu}(\mu)) & \psi'(\underline{\mu}(\mu)) \end{bmatrix}^{-1} \cdot \begin{bmatrix} \frac{1}{2\sigma^2 c^2} \\ \frac{p-\mu}{2\sigma^2 c^2} \end{bmatrix}$$

$$= \frac{1}{2\sigma^2 c^2 \phi'(\bar{\mu}(\mu))\phi'(\underline{\mu}(\mu))[\bar{\mu}(\mu) - \underline{\mu}(\mu)]} \cdot \begin{bmatrix} \phi'(\underline{\mu}(\mu))(p - \mu - \underline{\mu}(\mu)) \\ \phi'(\bar{\mu}(\mu))(p - \mu - \bar{\mu}(\mu)) \end{bmatrix}$$

The consumer purchases the product when the belief is $(\mu, \bar{\mu}(\mu))$. So, $\mu + \bar{\mu}(\mu) - p > 0$. The consumer stops searching and does not purchase the product when the belief is $(\mu, \underline{\mu}(\mu))$. So, $\mu + \underline{\mu}(\mu) - p < 0$. We also have $\phi'(x) = -\frac{1}{x^2(1-x)^2} \Rightarrow \phi'(x) < 0, \forall x$. Thus, we obtain:

$$\begin{bmatrix} \frac{\partial \bar{\mu}(\mu)}{\partial c} < 0 \\ \frac{\partial \underline{\mu}(\mu)}{\partial c} > 0 \end{bmatrix}$$

We now consider $\underline{\mu}(\mu)$. Suppose there exists $c_1 > c_2$ with the corresponding quitting boundaries $(\mu_1, \underline{\mu}_{c_1}(\mu_1))$ and $(\mu_1, \underline{\mu}_{c_2}(\mu_1))$, respectively. Denote the cutoff beliefs by $(\mu^*_{c_1}, \mu^{**}_{c_1})$ for price c_1 and by $(\mu^*_{c_2}, \mu^{**}_{c_2})$ for price c_2. Fixing an arbitrary $\mu_1 \in (\mu^*_{c_1}, 1]$, we know that the consumer is indifferent between quitting and searching for information when her belief is $(\mu_1, \underline{\mu}_{c_1}(\mu_1))$ and the price is c_1. Since $c_2 < c_1$, one can see that the value of searching for information when her belief is $(\mu_1, \underline{\mu}_{c_1}(\mu_1))$ and the price is c_2 is strictly higher than zero. So, the consumer will keep searching for information. Thus, $\underline{\mu}_{c_2}(\mu_1) < \underline{\mu}_{c_1}(\mu_1)$.

(3) Comparative statics w.r.t. σ^2

c and σ^2 always appear together as $2\sigma^2 c$ in the equations. So, the qualitative result of the comparative statics w.r.t. σ^2 is the same as the comparative statics w.r.t. c.

$\square$