Invitation to Search or Purchase?

Optimal Multi-attribute Advertising

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Abstract

When considering whether or not to buy a product, consumers often evaluate different attributes of it. This learning process requires both time and effort, which incurs costs for the consumer. To encourage consumers to either purchase the product or continue seeking information instead of abandoning their search, firms may provide information through advertising. This paper studies the firm's optimal choice of advertising content and the role of advertising for a two-attribute product. If the firm advertises one attribute, the consumer faces a one-dimensional search problem because she is uncertain about only the other attribute. If the firm does not advertise, the consumer faces a two-dimensional search problem because she is uncertain about both attributes. Given the consumer's search strategy under different advertising strategies, we characterize the optimal advertising strategy. The firm does not advertise if the consumer's prior beliefs about both attributes are extreme. No advertising serves as an invitation to search when the belief is high and as an invitation to purchase when the belief is very high. Otherwise, it advertises the better attribute if the consumer is optimistic enough about the worse attribute, and advertises the worse attribute if the consumer is less optimistic about it. In such cases, the role of advertising is non-monotonic in the belief due to different binding incentives of the firm in different belief regions.

1 Introduction

When considering whether or not to buy a product, consumers often evaluate different attributes of it. For example, an incoming college student choosing a laptop can learn about the operating system, weight, exterior design, warranty, and other attributes before making a final decision. This learning process requires both time and effort, which incurs costs for the consumer. To encourage consumers to either purchase the product or continue seeking information instead of abandoning their search, firms may provide information through advertising. This leads to several managerial decisions for the firm.

Advertising can raise consumers' interest in the product if their preference aligns with the advertised product information. For instance, a car buyer who values electronic entertainment systems might be more inclined toward Tesla if the company highlights its interior design featuring a large screen. Conversely, advertising may also create a negative impression, potentially deterring a sale. For example, a different buyer who prefers the mechanical feel of a car might lose interest in Tesla after seeing the same advertisement, due to the absence of traditional mechanical components. Therefore, the firm must first determine whether to engage in advertising at all. If the firm opts to advertise, it must carefully select the content of the advertisement. Given the limited bandwidth of advertising, the firm cannot communicate all available information to the consumer, necessitating a decision on which attributes to emphasize. For example, Tesla can focus on its interior design or its auto-pilot system in a 20-second TV ad, but not both. Finally, advertising can serve different purposes: it might drive an immediate purchase or prompt further search. Ultimately, this paper seeks to address the following questions:

1. Whether the firm wants to advertise or not?
2. What is the optimal advertising content - which attribute should the firm advertise?
3. What is the role of advertising? Does the firm want to invite the consumer to purchase the product directly or to search for additional information?
4. Should the KPIs for advertising campaigns be bump in immediate purchase or increase in traffic/search about the product, which leads to more sales?

To answer the above questions, this paper considers a consumer deciding whether or not to purchase a good. The good has two attributes, each with independent values. The payoff of purchasing the good is the total value of these attributes minus the price. We focus on the horizontal match between the attributes and the consumer's tastes/needs. The value of each attribute is one if it is a good match and zero if it is a bad match. The consumer initially does not know the actual value of either attribute but holds a prior belief about the value of each. The firm can reveal the value of the attribute by informative advertising. Due to the limited bandwidth of ads, the firm can disclose the value of at most one attribute (Shapiro 2006, Bhardwaj et al. 2008, Mayzlin and Shin 2011). After seeing the ad, the consumer knows the value of the advertised attribute but still has uncertainty about the unadvertised attribute(s). She can incur a cost to search for additional information before making a decision. If the firm advertises one attribute, then the consumer may only search for information about the other attribute. If the firm chooses not to advertise, then the consumer may search for information about either attribute.

We first characterize the consumer's optimal search strategy for a given advertising strategy. If the firm advertises one attribute, the consumer faces a one-dimensional search problem because she is uncertain about only the other attribute. The consumer will stop searching and buy the good if she becomes sufficiently optimistic about the unadvertised attribute. Conversely, she will stop searching without purchasing if her assessment becomes too pessimistic. When her belief about the unadvertised attribute is in between, she will search for more information. If the firm does not advertise, the consumer faces a two-dimensional search problem because she is uncertain about both attributes. We characterize the search region by a set of ordinary differential equations for intermediate beliefs and by a system of equations for extreme beliefs. It is always optimal for the consumer to search for the attribute about which the consumer has greater uncertainty, due to the faster speed of learning. The consumer only searches for one attribute if she holds a strong prior belief about one of the attributes, and may search for both attributes otherwise. In the car purchasing example, a consumer might not investigate the design of a Tesla due to its well-known styling and focus instead on other attributes.[1] In contrast, a consumer considering a pre-order

[1] For example, the consumer may know how Tesla looks like by observing her friend's Tesla before searching for any information about it.

from Faraday Future, a new electric car manufacturer, likely faces significant uncertainty about all aspects and might investigate every attribute. We also find that it may be optimal for the consumer to revisit a previously searched attribute after searching for the other attribute.

Given the consumer's search strategy, we then study the optimal advertising strategy. The firm faces two trade-offs. The first trade-off is whether or not to advertise. The firm benefits from a higher likelihood of purchase if the advertised attribute turns out to be good and moves up the consumer's overall evaluation of the product. However, the consumer will quit if the advertised attribute is bad. In such cases, the firm suffers from no chance of selling the product. So, it is not obvious whether the firm should advertise. The second trade-off is whether to advertise the better attribute (the attribute with a higher prior belief) or the worse attribute (the attribute with a lower prior belief). Because the consumer will not be interested in the product if the advertised attribute is bad, the firm has a higher chance of keeping the consumer interested by advertising the better attribute. The downside is that, conditional on the advertised attribute being good, the consumer's overall evaluation of the product is lower if the firm advertises the better attribute than if the firm advertises the worse attribute. The conditional purchasing probability is lower if the firm advertises the better attribute. It is also not obvious whether the firm should advertise the better or the worse attribute.

We find that the firm will not advertise if the consumer's prior beliefs about both attributes are extreme. If the consumer is very optimistic about both attributes, she will purchase the product for sure or with a very high likelihood. So, the firm does not have an incentive to advertise. No advertising serves as an invitation to search when the belief is high and as an invitation to purchase when the belief is very high. If the consumer is very pessimistic about both attributes, she will never purchase the product even if she knows that one attribute is good. So, the firm does not advertise as well.

If the consumer's prior belief is milder, the firm benefits from advertising. It will advertise the better attribute if the consumer is optimistic enough about the worse attribute, and will advertise the worse attribute if the consumer is less optimistic about it. Suppose the consumer is more optimistic about attribute one than about attribute two. When the prior belief about attribute one is moderate, the firm advertises attribute two to invite the consumer to search for more information.

When the prior belief about attribute one is high, the role of advertising is *non-monotonic* in the prior belief about attribute two. When the prior belief is low, the firm advertises attribute two to invite the consumer to directly purchase the product. When the prior belief is higher, the firm advertises attribute one to invite the consumer to search for more information. When the prior belief is even higher, the firm advertises attribute one to invite the consumer to directly purchase the product. The non-monotonicity is driven by different incentives of the firm in different belief regions. When the consumer is very pessimistic about an attribute, the firm's binding incentive is to convince the consumer to consider the product after seeing the ad. In contrast, when the consumer has a less extreme belief about that attribute, the firm's binding incentive is to avoid or reduce the amount of wasted beliefs upon conversion.

Related Literature

This paper investigates a firm's strategy for information disclosure through advertising. Researchers have begun to consider the informational role of advertising since Nelson (1974). One stream of literature on informative advertising explores scenarios where advertising is the consumer's sole source of information (Butters 1977, Grossman and Shapiro 1984, Villas-Boas 1993, 1994, Lewis and Sappington 1994, Meurer and Stahl 1994, Stahl 1994, Roy 2000, Shaffer and Zettelmeyer 2004, 2009, Soberman 2004, Anderson and Renault 2009, Amaldoss and He 2010, 2016, Kuksov and Lin 2010, Kuksov et al. 2013, 2017, Shen and Villas-Boas 2018, Chatterjee and Zhou 2021, Lauga et al. 2022). In contrast to most studies, where firms provide signals about the overall product value, Sun (2011) examines a seller's disclosure incentives for products with multiple attributes. Assuming that the firm discloses all the information or nothing, it shows that the unraveling result obtained by Grossman (1981) and Milgrom (1981) no longer holds if the product has a vertical attribute and a horizontal one. The Consumer's only source of product information comes from the firm in the aforementioned literature. However, in reality, consumers can seek additional information after viewing advertisements. Recognizing this, another line of research on informative advertising studies the case where consumers can acquire further information post-advertisement (Shin 2005, Anderson and Renault 2006, Mayzlin and Shin 2011, Wang 2017, Berman et al. 2023, Despotakis and Yu 2023).

Despite the extensive literature on informative advertising, there is still considerable scope for exploring the specific content of advertisements. In early work, advertising raises consumer awareness of a product, with the information either pertaining to the existence of the product or providing complete details about it. However, the assumption of full information disclosure may be impractical due to advertising's limited bandwidth or sub-optimal due to strategic considerations. The seminal paper by Anderson and Renault (2006) makes one of the first attempts to consider partial information disclosure through advertising. The authors demonstrate that it can be optimal for the firm to provide a binary signal indicating whether the product match value exceeds a certain threshold. However, they also note the challenges of feasibly and credibly conveying such partial information due to constraints in information transmission. Recent papers using the Bayesian persuasion framework to study the abstract design of advertising content (Shin and Wang 2024, Yao 2024) also abstract away from the issue of how to communicate partial information in practice.

This paper contributes to the literature by examining the choice of specific and feasible advertising content while allowing for the possibility of consumer search. To achieve such goals, we explicitly model multiple attributes of a product in the advertising and search problems. The optimal advertising strategy - specifically, which attribute to advertise - is both managerially relevant and straightforward to implement in practice. Our research closely relates to Mayzlin and Shin (2011). They consider a setting where, by searching for information after seeing the ad, the consumer can obtain an additional and exogenously given signal about product quality. Our paper differs from theirs in several ways. First, while their paper focuses on vertical quality and the advertising strategy is driven by signaling the firm's private information, we concentrate on horizontal quality, with the advertising strategy influenced by differences between one-dimensional search with advertising and two-dimensional search without it. Second, the consumer can only search once and observe an aggregate signal about the firm's quality in Mayzlin and Shin (2011). This is because their paper focuses on whether the firm should advertise attribute information or not rather than which specific attribute should be advertised.[2] We model the search process in detail so that the consumer chooses what attribute and how long to search. This allows us to study which specific

[2] More specifically, the firm can advertise either attribute when both attributes are high-quality and will only advertise the high-quality attribute if only one of the attributes is high-quality. In either case, which attribute to advertise is not a strategic choice.

attribute the firm should advertise. Third, by modeling each attribute in detail, we can study the optimal advertising content within a broader range of product characteristics, accounting for any prior beliefs about each attribute. This leads to new insights about the choice of advertising content and the role of advertising.

This paper is related to the literature on how consumers with limited attention allocate their attention to different attributes or options. Existing literature mainly looks at cases where the search prominence or search order is exogenously given (Arbatskaya 2007, Armstrong et al. 2009, Xu et al. 2010, Armstrong and Zhou 2011, Bar-Isaac et al. 2012, Bordolo et al. 2013, Kőszegi and Szeidl 2013, Branco et al. 2016, Zhu and Dukes 2017, Jeziorski and Moorthy 2018, Ke and Lin 2020). In this paper, we endogenize the optimal attribute to search from the consumer's perspective. Instead of assuming that the consumer knows the value of each attribute or learns it instantly, as is common in this literature, the Bayesian decision-maker in our model gradually learns the value from noisy signals. So, the relative importance of the attributes may change as the consumer gathers more information.

This paper also fits into the literature on optimal information acquisition, particularly consumer search. Following seminal papers by Stigler (1961) and Weitzman (1979), numerous papers have studied the optimal search problem under either simultaneous or sequential search (e.g., Moscarini and Smith 2001, Kuksov and Vilas-Boas 2010, Branco et al. 2012, Liu and Dukes 2013, Dukes and Liu 2016, Ke et al. 2016, Ke and Villas-Boas 2019, Greminger 2022, Pease 2023, Chaimanowong et al. 2025, Jerath and Ren 2025, Ning and Zhou 2025). In these studies, the relative importance of different alternatives is typically exogenous. Consumers randomly choose an attribute to search. In our model, the consumer strategically decides when to search and which attribute to focus on. More importantly, both the attention allocation and the information acquisition literature mainly study the decision-maker's optimal strategy, whereas we focus on the firm's advertising strategy.

The remainder of the paper is organized as follows. Section 2 presents the main model. Section 3 solves the consumer's search strategy. Section 4 characterizes the firm's equilibrium advertising strategy. Section 5 examines three extensions that relax some assumptions in the main model. Section 6 concludes.

2 Model

There is a firm and a consumer. Both are risk-neutral. The consumer considers whether to purchase a product or not. The product has two attributes, each with independent values. The product's value to the consumer is the sum of the values of these attributes, $U = U_1 + U_2$. This paper focuses on the horizontal match between the attributes and the consumer's tastes/needs. The value of each attribute is one if it is a good match and zero if it is a bad match. Given our focus on horizontal preferences rather than vertical ones, we assume the firm does not have private information about the value of the attribute, and thereby the consumer and the firm share a common prior belief about the likelihood that attribute i is a good match, denoted by $\mu_i(0)$. This setup implies that advertising does not have a signaling role.[3] The price p is given exogenously because we want to focus on the role of information in this paper. This modeling choice of abstracting away from endogenous pricing has also been adopted by other papers in the advertising literature (e.g., Wernerfelt 1990, Kuksov et al. 2013). We assume that the marginal cost m of producing the product is sufficiently high, and thus the price is high enough ($p \geq 3/2$), so that the consumer will decide not to purchase the product for any pair of beliefs (μ_1, μ_2) if $\mu_1 + \mu_2 \leq 1$.

The firm can reveal the value of the attribute by informative advertising. Due to the limited bandwidth of ads, we assume that the firm can disclose the value of at most one attribute (Shapiro 2006, Bhardwaj et al. 2008, Mayzlin and Shin 2011).[4] We consider the case in which the consumer knows exactly whether an attribute is a good match or not if the firm advertises that attribute. The firm can provide a noisy signal which does not fully resolve the uncertainty. However, the full disclosure setup gives us the sharpest results and has been widely adopted in the literature.[5]

The consumer has the option to learn more about the attributes via costly learning after seeing the ads. If the firm advertises one attribute, then the consumer may only search for information about the other attribute. If the firm chooses not to advertise, then the consumer may search for

[3] There are two common setups in the advertising literature. Firms usually do not have private information if the focus is on the horizontal match between the product's characteristics and the consumers' taste (a few papers assume instead that both the firm and the consumer have private information). Firms usually have private information if the focus is on the vertical quality of the product/attributes. In such cases, advertising also has a signaling role. Renault (2015) is an excellent survey of both strands of literature.

[4] We relax this assumption by allowing for the firm to advertise both attributes in an extension in section 5.1, and show that the main insights remain.

[5] In section 5.2, we analyze an extension where advertising provides noisy signals about the attribute value, and show that the main results and underlying mechanisms are robust.

information about either attribute. Due to limited attention, we assume that the consumer can only search for information about one attribute at a time. If the consumer opts to search, she also needs to decide which specific attribute to investigate at a given time. The decision-making process ends when the consumer makes a purchasing decision.

Each attribute of the product consists of numerous sub-attributes. So, the consumer cannot learn everything about an attribute with a single evaluation. For instance, if a consumer wishes to learn about the design of a car (an attribute), she might start by looking at an image online to determine the car's exterior color (a sub-attribute). However, she will need to invest additional effort to learn about other sub-attributes, such as the wheel size or the seat material. Given the complexity of modern products, each attribute often includes so many sub-attributes that it becomes impossible for the consumer to fully learn everything.[6] To model this gradual learning process, we assume that the consumer receives noisy signals about an attribute by incurring a flow cost of c. Let $T_i(t)$ denote the cumulative time the consumer spent searching for attribute i up to time t. We represent the signal, S_i, as a Brownian motion, where W_i are independent Wiener processes:

$$dS_i(t) = U_i dT_i(t) + \sigma dW_i(T_i(t))$$

In the above expression, the first term is driven by the true value, U_i, while the second term represents the noise. The parameter σ is a measure of the level of signal noise - a larger σ indicates a noisier signal due to the higher relative weight of the noise term. The consumer is more likely to observe a larger signal realization if the attribute is good, as the first term continuously increases over time when $U_i = 1$. This continuous-time model of Bayesian learning about a binary state has been widely used to study information acquisition (Ke and Villas-Boas 2019, Morris and Strack 2019, Liao 2021), experimentation (Bolton and Harris 1999, Moscarini and Smith 2001), and decision times (Fudenberg et al. 2018). It effectively captures the gradual learning feature and offers tractable analysis. The gradual learning set up in this paper is standard, except that the consumer endogenously chooses which attribute to search for at any given time rather than randomly searching for an attribute.

[6] Gardete and Hunter (2024) provide empirical evidence that consumers often acquire partial instead of complete information and that they are deliberate in choosing which attribute to search for rather than randomly sampling.

Based on the received signals, the consumer continuously updates her belief about the value of each attribute according to Bayes' rule. This belief evolution can be characterized by the following ordinary differential equation:

$$d\mu_i(t) = \frac{1}{\sigma^2}\mu_i(t)[1-\mu_i(t)]\{dS_i(t) - \mu_i dT_i(t)\}, \tag{1}$$

where μ_i is the expected value of attribute i based on the observed information up to time t. A higher σ results in slower belief updating due to noisier signals. If μ_i is closer to $1/2$, there is more uncertainty about attribute i, leading to faster updates in belief. If $dS_i(t) - \mu_i dT_i(t) > 0$, the signal's increasing speed is higher than the current belief about attribute i, and therefore the true value is more likely to be good ($U_i = 1$). So, the consumer will increase her belief about this attribute. Conversely, if the signal increases more slowly, the consumer will decrease her belief. This belief updating process also implies that the consumer's belief about an attribute remains unchanged when she searches for information about the other attribute.

3 Consumer's Search Strategy

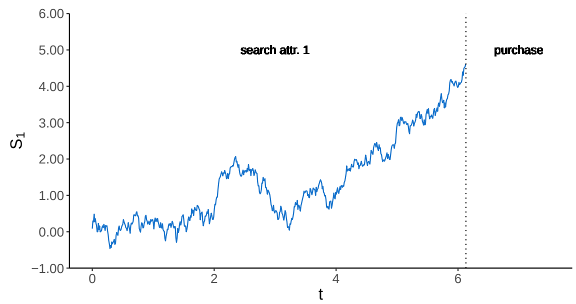
We solve the model by backward induction. In this section, we first look at the consumer's search strategy for a given advertising strategy.

3.1 The Firm Advertises One Attribute

Suppose the firm advertises attribute $i \in \{1, 2\}$. The value of attribute i, U_i, becomes 1 with probability μ_i and 0 with probability $1-\mu_i$.[7] The consumer may only search for information about the other attribute $j \neq i$. She faces a one-dimensional search problem. The consumer can make a decision right away or search for information about the other attribute. From the consumer's perspective, her problem is equivalent to considering a single-attribute product whose value is U_j and whose price is $p' := p - U_i$. One can see that the consumer will quit if $U_i = 0$. So, we only need to investigate the case in which $U_i = 1$ (p' becomes $p - 1$). The optimal search strategy has been shown by Ke and Villas-Boas (2019). There exists $0 < \underline{\mu}_j < \bar{\mu}_j < 1$ such that the consumer

[7] We denote $\mu_i(0)$ by μ_i to simplify the notation in this section.

searches for more information if $\mu_j \in (\underline{\mu}_j, \bar{\mu}_j)$, purchases the product if $\mu_j \geq \bar{\mu}_j$, and quits if $\mu_j \leq \underline{\mu}_j$. Figure 1 illustrates a sample path of the signals and belief evolution when both attributes are good and the firm advertises the second attribute. The consumer knows from advertising that attribute two is good and has uncertainty about only attribute one. She thus searches for information about attribute one, continuously receiving signals about it. Her belief goes up and down initially because she receives both positive and negative signals. After a period of time, the consumer has received sufficient positive signals and becomes confident enough that attribute one is also good. Therefore, she stops searching and purchases the product.



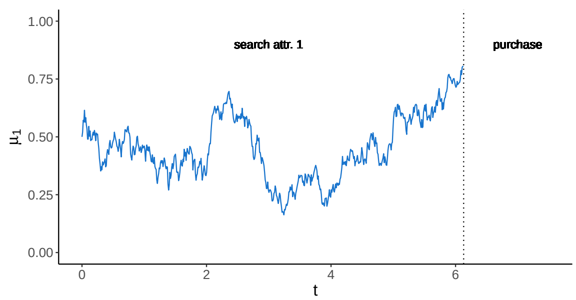


Figure 1: Sample Paths of the Signals and Beliefs When the Firm Advertises Attribute Two for $U_1 = 1, U_2 = 1, \mu_1(0) = 0.5, \mu_2(0) = 0.5$, and $\sigma = 1$.

In the search region ($\underline{\mu}_j < \mu_j < \bar{\mu}_j$), the value function is determined by:

$$\frac{\mu_j^2(1-\mu_j)^2}{2\sigma^2} W''(\mu_j) - c = 0 \;\Rightarrow\; W(\mu_j) = 2\sigma^2 c(1-2\mu_j) \ln \frac{1-\mu_j}{\mu_j} + K_1\mu_j + K_2,$$

where K_1 and K_2 are two unknown constants to be determined. Because $W(\underline{\mu}_j) = W'(\underline{\mu}_j) = 0$, $W(\bar{\mu}_j) = \bar{\mu}_j - p'$, and $W'(\bar{\mu}_j) = 1$, value matching and smooth pasting at $\underline{\mu}_j$ and $\bar{\mu}_j$ determine K_1,

K_2, and the cutoff beliefs:

$$\begin{cases} \phi(\underline{\mu}_j) - \phi(\bar{\mu}_j) = \frac{1}{2\sigma^2 c} \\ \psi(\underline{\mu}_j) - \psi(\bar{\mu}_j) = \frac{p-1}{2\sigma^2 c} \end{cases}, \tag{2}$$

where $\phi(x) = 2\ln\frac{1-x}{x} + \frac{1}{x} - \frac{1}{1-x}$ and $\psi(x) = \ln\frac{1-x}{x} + \frac{1-2x}{1-x}$.

3.2 The Firm Does Not Advertise

If the firm does not advertise, the consumer faces a two-dimensional search problem. At any given time, the consumer can search for information about either attribute. Figure 2 illustrates a sample path of the signals and belief evolution when the first attribute is good, the second one is bad, and the firm does not advertise. Initially, the consumer's belief about each attribute is 1/2. She begins by searching for information about attribute one, continuously receiving signals about it. Although these signals are predominantly positive, the consumer's belief about the first attribute gradually declines because the signal's rate of increase is slower than her belief. As attribute two is not initially searched, no new signals are received for it, and consequently, the consumer's belief regarding this attribute remains unchanged. After a period of time, the consumer shifts her focus to attribute two. Her belief about this attribute first decreases and then increases, while her belief about attribute one stays the same. On receiving positive signals about attribute two, she returns to investigating attribute one. During this phase, the signal for attribute one increases rapidly, causing her belief to rise towards 1. Once the consumer is relatively certain that the first attribute is good, she resumes her search for information about attribute two. She eventually stops searching and decides not to purchase the product after receiving sufficient negative signals, leading her to strongly believe that attribute two is bad.

At any given time, the consumer chooses among four actions: searching for attribute one, searching for attribute two, purchasing the product, or quitting without purchasing. The consumer's search strategy, denoted as α, maps the observed history (the signal realization) up to time t to one of these four actions, for all t. We define the stopping time τ as the first instance when the consumer makes a purchasing decision (either purchasing or quitting). The entire process ends at the stopping

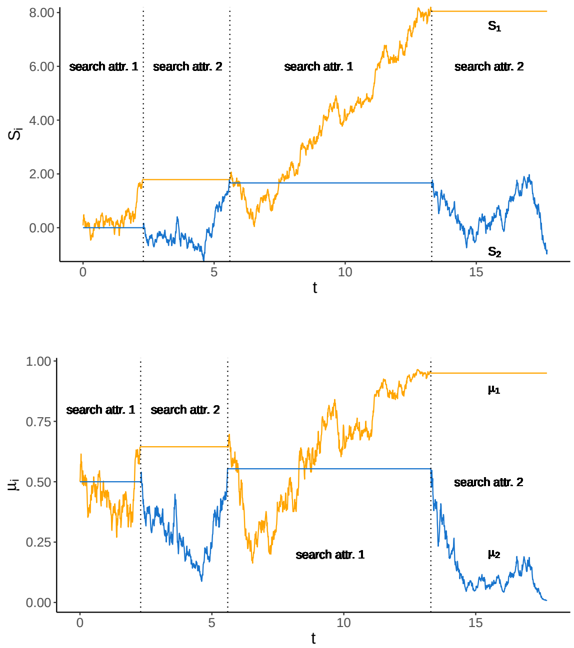


Figure 2: Sample Paths of the Signals and Beliefs When the Firm Does not Advertise for $U_1 = 1, U_2 = 0, \mu_1(0) = 0.5, \mu_2(0) = 0.5$, and $\sigma = 1$.
Note: this figure illustrates one of the possible search strategies and the associated belief evolution. As the subsequent results will show, this sample strategy is not optimal.

time. The consumer's expected payoff for a given initial belief (μ_1, μ_2) and search strategy α is $J(\mu_1, \mu_2, \alpha) := \mathbb{E}\left\{\max\left[\mu_1(\tau) + \mu_2(\tau) - p, 0\right] - \tau c | (\mu_1(0), \mu_2(0)) = (\mu_1, \mu_2)\right\}$. The value function of the consumer's problem is given by:

$$V(\mu_1, \mu_2) := \sup_{\alpha} J(\mu_1, \mu_2, \alpha)$$

Because the search strategy should not depend on future information, the decision at time t must be based only on the information observed up to t. It is well established that the current belief $(\mu_1(t), \mu_2(t))$ is a sufficient statistic for the information available up to time t in this binary-valued setting. Therefore, the search strategy will depend solely on $(\mu_1(t), \mu_2(t))$. If a search strategy α^* achieves $V(\mu_1, \mu_2)$ for any given belief, $V(\mu_1, \mu_2) = J(\mu_1, \mu_2, \alpha^*)$, it will be deemed the optimal search strategy. When the consumer searches for information about attribute one, the value function satisfies (ignoring the time index t for simplicity) $V(\mu_1, \mu_2) = -cdt + \mathbb{E}[V(\mu_1 + d\mu_1, \mu_2)]$. By Taylor's expansion and Ito's lemma, we get:

$$\frac{\mu_1^2(1-\mu_1)^2}{2\sigma^2}V_{\mu_1\mu_1}(\mu_1,\mu_2)-c=0 \tag{3}$$

Similarly, when the consumer searches for information about attribute two, we have:

$$\frac{\mu_2^2(1-\mu_2)^2}{2\sigma^2}V_{\mu_2\mu_2}(\mu_1,\mu_2)-c=0 \tag{4}$$

The Hamilton-Jacobi-Bellman (HJB) equation of the entire problem is:

$$\max\left\{\max_{i=1,2}\left[\frac{\mu_i^2(1-\mu_i)^2}{2\sigma^2}V_{\mu_i\mu_i}(\mu_1,\mu_2)-c\right], \max\left[\mu_1+\mu_2-p,0\right]-V(\mu_1,\mu_2)\right\}=0 \tag{$\star$}$$

We will show that the value function is a viscosity solution of the HJB equation. We will then prove that the viscosity solution is unique. Therefore, if we can find a viscosity solution, it must be the value function. To achieve this, we will construct a search strategy, which we will use to characterize the search region and the expected payoff. Finally, we will verify that the conjectured strategy generates a viscosity solution of the HJB equation, thereby implying that the conjectured search strategy is optimal. Due to symmetry, it is sufficient to consider only the case where $\mu_1 \geq \mu_2$. Analytically, we can fully characterize the optimal search strategy when the search cost is low. We do not believe that focusing on the low search cost case imposes a significant limitation, as our primary interest lies in examining the firm's optimal advertising strategy. When the search cost or the price is high, the consumer's search activity diminishes without advertising. In such cases, the firm faces a sub-problem of the problem in the full model because not advertising is no longer meaningful and the firm is choosing between advertising either attribute (except for the trivial case where the prior belief is so high that the consumer immediately purchases the product).

Proposition 1. *Suppose the firm does not advertise and the search cost is low,* $c \leq \frac{1}{2\sigma^2[\phi(1/2)-\phi(\frac{2}{3}p-\frac{1}{6})]}$. *Conditional on searching, it is optimal for the consumer to search for information about attribute two if* $\mu_1 > \mu_2$ *and to search for information about attribute one if* $\mu_1 < \mu_2$.

Figure 3 illustrates the optimal search strategy if the firm does not advertise. The dashed orange line represents the quitting boundary, while the solid blue line depicts the purchasing boundary. The arrow indicates which attribute the consumer searches for information about, given her cur-

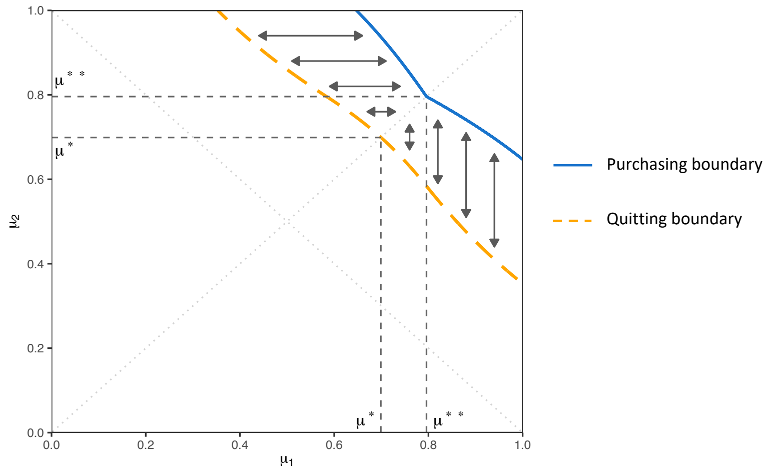


Figure 3: Optimal Search Strategy

rent belief. When the overall beliefs of the attributes are sufficiently low, the likelihood of receiving enough positive signals to warrant a purchase is too low. In this case, the consumer stops searching and quits to save on search costs. Conversely, when the overall beliefs of the attributes are high enough, purchasing the product results in a sufficiently high expected payoff, prompting the consumer to make the purchase. In other cases, the consumer searches for more information to make a better-informed decision.

Denote the point where the quitting boundary intersects the main diagonal as (μ^*, μ^*), and the point where the purchasing boundary intersects the main diagonal as (μ^{**}, μ^{**}). The quitting boundary for $\mu_1 \geq \mu_2$ is represented by $\underline{\mu}(\cdot)$, with a domain of $[\mu^*, 1]$ (the other half of the quitting boundary can be determined by symmetry). The purchasing boundary for $\mu_1 \geq \mu_2$ is represented by $\bar{\mu}(\cdot)$, with a domain of $[\mu^{**}, 1]$ (the other half of the purchasing boundary can also be determined by symmetry).

Intuitively, conditional on searching, the consumer prefers to search for the attribute with a higher rate of learning, because learning costs are identical. From equation (1), it is evident that the more uncertain the belief is, the faster the consumer learns about an attribute. Consequently, she always focuses on learning about the attribute with a belief closer to 1/2. The optimal search strategy implies that the consumer only searches for one attribute if she holds a strong prior belief on one of the attributes, and may search for both attributes otherwise. It also suggests that it may be optimal for the consumer to revisit a previously searched attribute after searching for the other attribute. Gardete and Hunter (2024) find empirical support for such revisiting behavior.

Testable Implications

The optimal search strategy characterized by Proposition 1 leads to the following testable implications. First, if one can elicit a consumer's prior belief through survey or experiments, then our result predicts that the consumer will search for information about fewer attributes when the prior belief about the better attribute is higher. This prediction can be tested by using click-stream or eye-tracking data. Moreover, one can directly assign any prior belief to subjects in a lab experiment by disclosing at the beginning the probability of each attribute being good and giving them incentive compatible rewards. Second, our result not only rationalizes consumers' revisiting behavior, but also provides conditions under which such behavior may occur (intermediate prior beliefs about both attributes). Therefore, the existence of revisiting behavior, and more importantly, the conditions for such behavior, can be empirically tested.

Additional Properties of the Optimal Search Strategy

The following proposition characterizes the slope of the purchasing/quitting boundary and the shape of the search region.

Proposition 2. *For $\mu \in (\mu^*, \mu^{**}]$, we have:*

$$\underline{\mu}'(\mu) = \frac{\phi(\underline{\mu}(\mu)) - \phi(\mu)}{\phi'(\underline{\mu}(\mu))[\mu - \underline{\mu}(\mu)]} \tag{D_1}$$

For $\mu \in [\mu^{**}, 1]$, we have:

$$\bar{\mu}'(\mu) = \frac{\phi(\underline{\mu}(\mu)) - \phi(\bar{\mu}(\mu))}{\phi'(\bar{\mu}(\mu))[\bar{\mu}(\mu) - \underline{\mu}(\mu)]} \tag{$\overline{D_2}$}$$

$$\underline{\mu}'(\mu) = \frac{\phi(\underline{\mu}(\mu)) - \phi(\bar{\mu}(\mu))}{\phi'(\underline{\mu}(\mu))[\bar{\mu}(\mu) - \underline{\mu}(\mu)]} \tag{$\underline{D_2}$}$$

Both $\underline{\mu}(\mu)$ and $\bar{\mu}(\mu)$ strictly decrease in μ, whereas the width of the search region, $\bar{\mu}(\mu) - \underline{\mu}(\mu)$, strictly increases in μ. In addition, if $\underline{\mu}(\mu) \geq 1/2$, then the slope of the quitting boundary is less than -1, and the slope of the purchasing boundary is greater than -1.

The optimal search region exhibits a butterfly shape - the consumer searches for information in a wider region when she is more confident that the more favorable attribute is good. The underlying intuition is as follows: the product's expected value is higher when the consumer is more certain about one attribute being good, prompting her to seek information about the other attribute, even if it is associated with greater uncertainty. This is because the speed of learning is higher for a more uncertain attribute, enhancing the search's benefits while the cost of searching remains unchanged. Consequently, the consumer is motivated to engage in more extensive search activities.

The slope of the search region, representing the marginal rate of substitution between the values of the first and second attributes, is also interesting. It sheds light on the learning process's cross-attribute dependence. If the slope is -1, then the two attributes are perfect substitutes. One may expect this to be the case in general because the product's value is the sum of the values of two attributes, each with independent values. However, both the slope of the quitting boundary and the slope of the purchasing boundary are not -1 in general due to the asymmetry in learning. If the quitting boundary is above $1/2$, the slope of the quitting boundary is less than -1. In such cases, a one-unit increase in the belief about attribute one can compensate for more than a one-unit decrease in the belief about attribute two near the quitting boundary. The consumer will continue her search for attribute two rather than quitting, even if μ_2 declines by slightly more than one unit. This is because the consumer has more uncertainty, and hence a higher rate of learning, about attribute two. So, the benefit of search increases while the search cost remains the same. Similarly, near the purchasing boundary, a one-unit increase in the belief about attribute one compensates for less than a one-unit decrease in the belief about attribute two. This encourages the consumer to keep searching for information about attribute two rather than making an immediate purchase, even if μ_2 decreases by slightly less than a unit.

4 Firm's Advertising Strategy

After characterizing the consumer search strategy for a given advertising strategy, we now study the firm's optimal advertising strategy.

4.1 Advertising One Attribute

We first consider the purchasing probability if the firm advertises one attribute. We know from the last section that the consumer will quit if she knows that one attribute is bad. Therefore, the consumer's purchase probability is the multiplication of two terms: the likelihood that the advertised attribute is good and the conditional purchasing probability given that the advertised attribute is good.

We first look at the case where the firm advertises attribute one. In this case, the consumer will know that the first attribute is bad with probability $1 - \mu_1$, and will quit after receiving such information. With the complementary probability μ_1, the consumer will learn that attribute one is good, and depending on her belief about attribute two, may purchase the product directly, search for more information about the second attribute, or quit. The following proposition characterizes the overall purchasing probability if the firm advertises attribute one.

Proposition 3. *Suppose the firm advertises attribute one. The probability that the consumer purchases the product is:*

$$
\begin{aligned}
P_1(\mu_1, \mu_2) :=\ & \mathbb{P}[\textit{purchasing}|\textit{firm advertises attribute one, prior belief } (\mu_1, \mu_2)] \\
=\ & \begin{cases} \mu_1, & \textit{if } \mu_2 \geq \bar{\mu}(1) \\ \mu_1 \cdot \frac{\mu_2 - \underline{\mu}(1)}{\bar{\mu}(1) - \underline{\mu}(1)}, & \textit{if } \mu_2 \in [\underline{\mu}(1), \bar{\mu}(1)] \\ 0, & \textit{if } \mu_2 \leq \underline{\mu}(1) \end{cases}
\end{aligned}
$$

The case of advertising attribute two is symmetric to the case of advertising attribute one. One can obtain a similar result about the consumer's purchasing likelihood given the advertising strategy.

Corollary 1. *Suppose the firm advertises attribute two. The probability that the consumer purchases the product is:*

$$
P_2(\mu_1, \mu_2) := \mathbb{P}[\textit{purchasing}|\textit{firm advertises attribute two, prior belief } (\mu_1, \mu_2)]
$$

$$
= \begin{cases} \mu_2, & \textit{if } \mu_1 \geq \bar{\mu}(1) \\ \mu_2 \cdot \frac{\mu_1 - \underline{\mu}(1)}{\bar{\mu}(1) - \underline{\mu}(1)}, & \textit{if } \mu_1 \in [\underline{\mu}(1), \bar{\mu}(1)] \\ 0, & \textit{if } \mu_1 \leq \underline{\mu}(1) \end{cases}
$$

4.2 Not Advertising

The consumer may search for information about either attribute if the firm does not advertise. If the consumer is highly confident about one of the attributes, she will not seek further information about it but will focus on learning about the other attribute. A decision to purchase is made once enough positive information is acquired to reach the purchasing boundary. Conversely, if sufficient negative information leads her belief to the quitting boundary, she will decide not to make a purchase. For instance, a consumer considering a Tesla might skip researching its design because she has seen the body styling of her friend's Tesla, choosing instead to investigate other aspects of the vehicle.

In cases where the consumer has moderate beliefs about both attributes, she must gather information on both attributes before purchasing the product.[8] Moreover, she will be equally certain about the value of each attribute upon deciding to buy. For example, a consumer considering a pre-order from Faraday Future, a new manufacturer, likely faces significant uncertainty about all aspects and might investigate every attribute. Given the consumer's optimal search strategy, we can calculate the purchasing likelihood if the firm advertises neither attribute.

Proposition 4. *Suppose the firm does not advertise. If $\mu_1 \geq \mu_2$, the probability that the consumer purchases the product is:*

$$
P(\mu_1, \mu_2) := \mathbb{P}[\textit{purchasing} | \textit{firm does not advertise, prior belief } (\mu_1, \mu_2)]
$$

[8] The consumer may have searched for only one attribute if she decides not to purchase the product after receiving enough negative information about that attribute.

$$
= \begin{cases} 1, & \textit{if } \mu_1 \in [\mu^{**}, 1] \textit{ and } \mu_2 \in [\bar{\mu}(\mu_1), \mu_1] \\ \frac{\mu_2 - \underline{\mu}(\mu_1)}{\bar{\mu}(\mu_1) - \underline{\mu}(\mu_1)}, & \textit{if } \mu_1 \in [\mu^{**}, 1] \textit{ and } \mu_2 \in [\underline{\mu}(\mu_1), \bar{\mu}(\mu_1)] \\ h(\mu_1, \mu_2)\tilde{P}(\mu_1), & \textit{if } \mu_1 \in [\mu^{*}, \mu^{**}] \textit{ and } \mu_2 \in [\underline{\mu}(\mu_1), \mu_1] \\ 0, & \textit{if } \mu_1 \leq \mu^{*} \textit{ or } \mu_2 \leq \bar{\mu}(\mu_1) \end{cases}
$$

, where $h(\mu_1, \mu_2) = \frac{\mu_2 - \underline{\mu}(\mu_1)}{\mu_1 - \underline{\mu}(\mu_1)}$ *and* $\tilde{P}(\mu) = e^{-\int_{\mu}^{\mu^{**}} \frac{2}{x - \underline{\mu}(x)} dx}$*. By symmetry,* $P(\mu_1, \mu_2) = P(\mu_2, \mu_1)$ *if* $\mu_1 < \mu_2$*.*

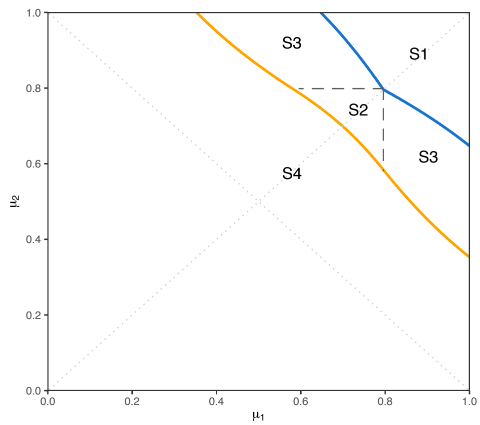


Figure 4: Four Regions for Purchase

Figure 4 delineates four regions that outline the consumer's path to purchase based on her initial beliefs. Different purchasing paths lead to different forms of purchasing probability. In region $S1$, the consumer purchases immediately, while in region $S4$, she quits without searching. Beliefs that fall within the intermediate regions, S_2 and S_3, indicate a high value of information, prompting further search before a decision is made.

Specifically, in region $S3$, the consumer strongly believes that the first attribute is good and focuses her search on the second, more uncertain attribute. Sufficient positive information about the second attribute leads to a purchase, while enough negative information results in quitting. Due to the low information value about the first attribute, the consumer never switches her focus back to it regardless of the outcomes of her search regarding the second attribute. In this scenario, the second attribute is pivotal in determining the consumer's purchasing decision.

In region $S2$, the consumer is quite uncertain about the values of both attributes. She will initially focus her search on attribute two, given her greater uncertainty about it compared to attribute one. However, she is not confident about the value of attribute one either. Consequently, if she receives sufficient positive signals about attribute two, she will then shift her focus to searching for information about attribute one. If subsequent positive signals about attribute one are received, she may toggle her search focus back to attribute two, and this pattern of switching and revisiting may continue until she gains enough confidence in both attributes to make a purchasing decision. Conversely, if she encounters sufficient negative information about either attribute, she will discontinue her search and opt not to purchase the product.

4.3 Optimal Advertising Strategy

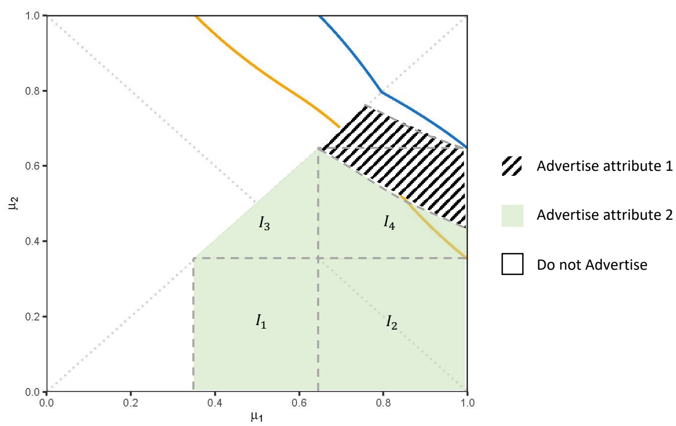
When choosing the advertising strategy, the firm faces two trade-offs. The first trade-off is whether or not to advertise. The firm benefits from a higher likelihood of purchase if the advertised attribute turns out to be good and moves up the consumer's overall evaluation of the product. However, the consumer will quit if the advertised attribute is bad. In such cases, the firm suffers from no chance of selling the product. So, it is not obvious whether the firm should advertise.

The second trade-off is whether to advertise the better attribute (the attribute with a higher prior belief) or the worse attribute (the attribute with a lower prior belief). Because the consumer will not be interested in the product if the advertised attribute is bad, the firm has a higher chance of keeping the consumer interested by advertising the better attribute. The downside is that, conditional on the advertised attribute being good, the consumer's overall evaluation of the product is lower if the firm advertises the better attribute than if the firm advertises the worse attribute. The conditional purchasing probability is lower if the firm advertises the better attribute. It is also not obvious whether the firm should advertise the better or the worse attribute.

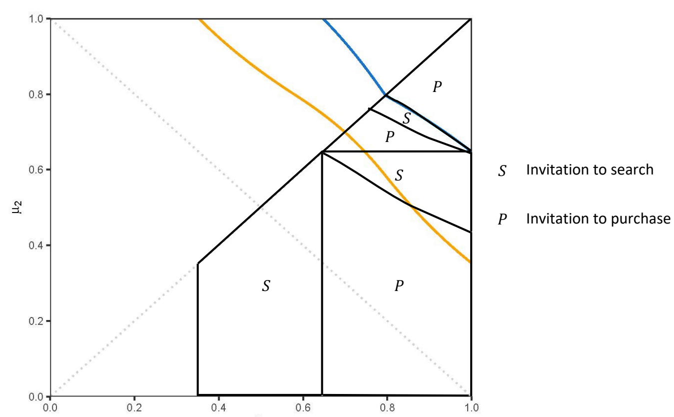
It turns out that any of the options, advertising the better attribute, advertising the worse attribute, and not advertising, may be optimal. By symmetry, we only need to consider the case when $\mu_1 \geq \mu_2$. The following proposition summarizes the main result of this paper.

Proposition 5. *Suppose $\mu_1 \geq \mu_2$. There exists $\tilde{\mu}(\mu_1)$ and $\widehat{\mu}(\mu_1)$ such that $\underline{\mu}(1) < \tilde{\mu}(\mu_1) \leq \widehat{\mu}(\mu_1) < \bar{\mu}(\mu_1)$ and $\tilde{\mu}(\mu_1)$ decreases in μ_1. The firm does not advertise if $\mu_1 \leq \underline{\mu}(1)$ or $\mu_2 \geq \widehat{\mu}(\mu_1)$, advertises* *attribute two if* $\mu_1 \in (\underline{\mu}(1), \bar{\mu}(1)]$*, or* $\mu_1 > \bar{\mu}(1)$ *and* $\mu_2 \leq \tilde{\mu}(\mu_1)$*, advertises attribute one if* $\mu_1 > \bar{\mu}(1)$ *and* $\mu_2 \in (\tilde{\mu}(\mu_1), \widehat{\mu}(\mu_1))$*.*

Advertising serves as an invitation to search if $\mu_1 \in (\underline{\mu}(1), \bar{\mu}(1)]$ *or if* $\mu_1 > \bar{\mu}(1)$ *and* $\mu_2 \in (\tilde{\mu}(\mu_1), \bar{\mu}(\mu_1))$*, and serves as an invitation to purchase if* $\mu_1 > \bar{\mu}(1)$ *and* $\mu_2 < \tilde{\mu}(\mu_1)$ *or if* $\mu_1 > \bar{\mu}(1)$ *and* $\mu_2 \in (\bar{\mu}(1), \widehat{\mu}(\mu_1))$*. No advertising serves as an invitation to search if* $\mu_1 > \bar{\mu}(1)$ *and* $\mu_2 \in (\widehat{\mu}(\mu_1), \bar{\mu}(\mu_1))$*, and serves as an invitation to purchase if* $\mu_1 > \bar{\mu}(1)$ *and* $\mu_2 > \bar{\mu}(\mu_1)$*.*



(a) Optimal Advertising Strategy



(b) Consumer Behavior Induced by the Advertising Strategy

Figure 5: Optimal Advertising Strategy and the Induced Consumer Behavior for $\mu_1 \geq \mu_2$

Figure 5a illustrates the optimal advertising strategy. The firm advertises attribute one in the diagonally striped black region, attribute two in the solid green region, and does not advertise in the blank region. In other words, the firm will not advertise if the consumer's prior beliefs about both attributes are extreme and will advertise if the consumer's prior belief is milder. In the latter case, the firm will advertise the better attribute if the consumer is optimistic enough about the worse attribute, and will advertise the worse attribute if the consumer is less optimistic about it.

Figure 5b presents the role of advertising. When the prior belief about attribute one is moderate, the firm advertises the other attribute to invite the consumer to search for more information about attribute one. When the prior belief about attribute one is high, the role of advertising is *non-monotonic* in the prior belief about attribute two. When μ_2 is low, the firm advertises attribute two to invite the consumer to directly purchase the product. When μ_2 is higher, the firm advertises attribute one to invite the consumer to search for more information. When μ_2 is even higher, the firm advertises attribute one to invite the consumer to directly purchase the product. We will discuss the mechanism in detail in the following paragraphs. Roughly speaking, the non-monotonicity is driven by different incentives of the firm in different belief regions. When the consumer is very pessimistic about an attribute, the firm's binding incentive is to convince the consumer to consider the product after seeing the ad. In contrast, when the consumer has a less extreme belief about that attribute, the firm's binding incentive is to avoid or reduce the amount of wasted beliefs upon conversion.[9]

We will now discuss the mechanism and intuition of the optimal advertising strategy. If the consumer's prior beliefs about both attributes are too low, the product will not be attractive to the consumer even if she knows that one attribute is good. The consumer will neither search for information nor purchase the product even if the firm advertises. So, the firm does not advertise. If the consumer has high enough prior beliefs about both attributes, she will purchase the product without additional information. The firm also has no incentive to advertise. Even if the consumer's belief is within the search region, she will purchase the product after receiving a little positive information as long as her belief is close to the purchasing boundary. The purchasing probability is close to 1. In contrast, if the firm advertises, the consumer will quit for sure if she finds out that one attribute is bad. So, the purchasing probability is lower. The firm is better off by not

[9] "Wasted belief" means that the consumer's belief about the product value is higher than what is needed for the consumer to buy the product immediately. Such excessive belief is wasteful from the firm's standpoint.

advertising, which serves as an invitation to search. The intuition is the following. If the consumer finds out that one attribute is good from advertising, her belief about the product value will be higher than what is needed for her to purchase the product immediately. Such excessive belief is wasteful from the firm's standpoint. If the firm does not advertise, the consumer will be just indifferent between searching for more information and purchasing the product after receiving a little positive information. The firm does not waste any belief, and the consumer will be more likely to purchase the product. Therefore, the firm does not advertise in the blank region.

Now let's consider the solid green region and the diagonally striped black region. We divide the solid green region into four sub-regions. If the belief lies in region I_1 or I_2, the consumer is very pessimistic about the second attribute. Even if she knows for sure that the first attribute is good, she needs to receive a lot of positive signals about attribute two to purchase the product. The search cost outweighs the benefit of the search. So, she will not search for information. The only way of convincing the consumer to consider the product is to advertise attribute two. With a high probability, the consumer will find out that attribute two is bad and quit. However, her overall valuation of the product increases by a lot if she finds out that attribute two is good. In that case, if the consumer's belief about attribute one is high (region I_2), she will purchase the product directly. Even if the consumer's belief about attribute one is low (region I_1), she needs fewer positive signals to purchase the product by searching for attribute one rather than attribute two. The benefit of search outweighs the search cost. So, the consumer will search for information about attribute one and purchase the product with a positive probability. Therefore, the firm advertises attribute two in both regions.

If the belief lies in region I_3, the consumer will never purchase the product without advertising but may purchase the product if the firm advertises either attribute. So, the firm advertises. Advertising either attribute invites the consumer to search for additional information. On one hand, the consumer is more optimistic about attribute one and will be more likely to search for information if the firm advertises attribute one rather than two. On the other hand, she needs more positive signals to purchase the product if the firm advertises attribute one. So, the conversion rate conditional on searching is lower. Because it is costless for the consumer to receive ads but costly for her to search for information, the firm can extract more surplus from the consumer by reducing the expected search time. Therefore, the firm prefers to have a lower search probability but a higher purchasing probability conditional on searching by advertising attribute two.

Lastly, we consider the case where the belief lies in region I_4 or the diagonally striped black region. If the consumer has a sufficiently high belief about attribute two, she will purchase the product immediately if she knows for sure that either attribute is good. One can see that the firm always prefers advertising attribute one to advertising attribute two because of the higher purchasing probability. In this case, advertising invites direct purchase. If the consumer's belief about attribute two is lower, the decision between advertising attribute one or two becomes more complex. If the firm advertises attribute one and the consumer finds out it is good, the consumer will always search for information about attribute two before making a decision. In contrast, the consumer will be very positive about the product value if the firm advertises attribute two and the consumer knows that attribute two is good. In that case, she will purchase the product immediately. So, some beliefs are "wasted" - the consumer will purchase the product immediately even if her belief is lower. The more optimistic she is about the first attribute, the more beliefs are wasted. So, the firm will be more likely to advertise attribute one, which invites further search, rather than advertising attribute two, which invites direct purchase.

Advertising Costs

In the previous discussion, we did not consider the advertising costs to focus on the role of advertising content. The zero advertising cost assumption has been made in other papers on informative advertising, such as in Mayzlin and Shin (2011). In reality, the firm needs to incur a cost to advertise. Our framework can incorporate this cost, but the analysis will be more tedious. So, we abstract away the advertising costs in the previous analysis. We now briefly discuss what happens if we take into account the advertising costs. Suppose the firm needs to incur a cost c_A to advertise attribute i. The comparison between advertising attributes one and two will not change because both require an extra cost, c_A. However, whether the firm prefers to advertise or not may change. If the prior belief of the consumer without advertising is close to the purchasing boundary, then the firm will not advertise. Even without advertising, the consumer will purchase the product with a high probability. By not advertising, the firm saves advertising costs. The firm will also not advertise if the belief about one of the attributes is too low. Even if the firm can raise the purchasing probability above zero by advertising, the purchasing likelihood is very low. The profit will be negative because of the advertising costs. So, the firm will not advertise, and the consumer will neither search nor purchase. For all other beliefs, the firm's advertising strategy is the same as the case without advertising costs.

5 Extensions

5.1 Relaxing Limited Bandwidth

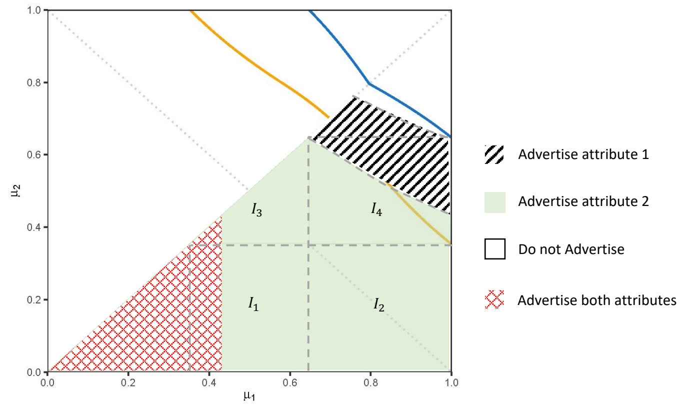
In the main model, we assume that a firm can disclose the value of at most one attribute because of the limited bandwidth of ads. Though the assumption of limited bandwidth is a common one in the advertising literature, there are definitely scenarios in which the firm can reveal information about both attributes. In this extension, we relax the assumption of limited bandwidth and allow the firm to choose from not advertising, advertising one of the attributes, and advertising both attributes.

Proposition 6. *Suppose $\mu_1 \geq \mu_2$. There exists $\mu^b \in (\underline{\mu}(1), \bar{\mu}(1))$, $\tilde{\mu}(\mu_1)$, and $\widehat{\mu}(\mu_1)$ such that $\underline{\mu}(1) < \tilde{\mu}(\mu_1) \leq \widehat{\mu}(\mu_1) < \bar{\mu}(\mu_1)$ and $\tilde{\mu}(\mu_1)$ decreases in μ_1. The firm does not advertise if $\mu_2 \geq \widehat{\mu}(\mu_1)$, advertises both attribute if $\mu_1 < \mu^b$, advertises attribute two if $\mu_1 \in (\mu^b, \bar{\mu}(1)]$, or $\mu_1 > \bar{\mu}(1)$ and $\mu_2 \leq \tilde{\mu}(\mu_1)$, advertises attribute one if $\mu_1 > \bar{\mu}(1)$ and $\mu_2 \in (\tilde{\mu}(\mu_1), \widehat{\mu}(\mu_1))$.*

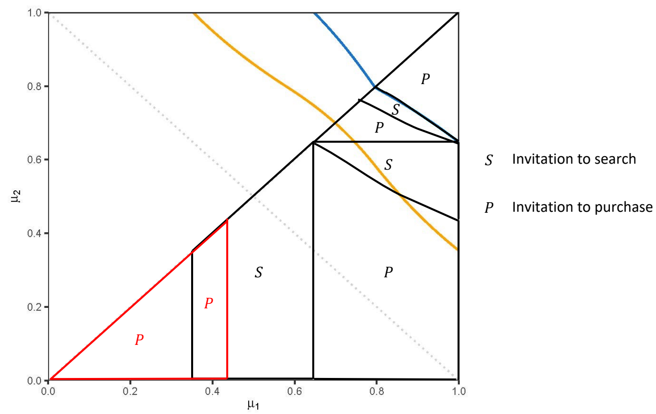
Advertising serves as an invitation to search if $\mu_1 \in (\mu^b, \bar{\mu}(1)]$ or if $\mu_1 > \bar{\mu}(1)$ and $\mu_2 \in (\tilde{\mu}(\mu_1), \bar{\mu}(\mu_1))$, and serves as an invitation to purchase if $\mu_1 < \mu^b$, if $\mu_1 > \bar{\mu}(1)$ and $\mu_2 < \tilde{\mu}(\mu_1)$, or if $\mu_1 > \bar{\mu}(1)$ and $\mu_2 \in (\bar{\mu}(1), \widehat{\mu}(\mu_1))$. No advertising serves as an invitation to search if $\mu_1 > \bar{\mu}(1)$ and $\mu_2 \in (\widehat{\mu}(\mu_1), \bar{\mu}(\mu_1))$, and serves as an invitation to purchase if $\mu_1 > \bar{\mu}(1)$ and $\mu_2 > \bar{\mu}(\mu_1)$.

Figure 6 illustrates the results. The firm advertises both attributes in the crosshatched red region, attribute one in the diagonally striped black region, attribute two in the solid green region, and does not advertise in the blank region. Compared with Proposition 5 and Figure 5, allowing ads for both attributes does little to alter the firm's optimal strategy or the role of advertising.

A difference arises only when the consumer's prior beliefs about both attributes are low. In this case, the firm cannot generate sufficient consumer interest even if it advertises one attribute



(a) Optimal Advertising Strategy Without Limited Bandwidth



(b) Consumer Behavior Induced by the Advertising Strategy

Figure 6: Optimal Advertising Strategy and the Induced Consumer Behavior for $\mu_1 \geq \mu_2$ Without Limited Bandwidth

and that attribute turns out to be good. Consequently, the consumer never buys the product when advertising faces limited bandwidth. Without bandwidth limits, however, the firm can induce a positive purchase probability by advertising both attributes: with a small but positive chance, the consumer learns that both attributes are good and purchases after seeing the ads. This "Hail Mary" strategy is optimal only when the firm has nothing to lose.

In all other cases, advertising both attributes is never optimal because either the consumer refrains from buying if at least one attribute is bad, or the strategy wastes too much belief when both are good. The firm's trade-off among advertising attribute one, advertising attribute two, and not advertising remains the same as in the main model. Thus, the optimal advertising strategy, the role of advertising, and the underlying mechanism remain consistent with the main model.

5.2 Noisy Signal from Advertising

To obtain the sharpest result, the main model assumes that the firm fully reveals the value of an attribute by informative advertising. One interpretation is that learning is easy if the firm reveals information, but finding information from third-party sources could be a lengthy process if the firm conceals information.[10] Realistically, an ad may not be able to resolve all the consumer's uncertainty about one attribute. To capture the limitation on information transmission through informative ads, this extension consider the possibility that the firm may only provide a noisy signal about one attribute. Specifically, the firm can provide a noisy signal $s_i \in \{g, b\}$ about attribute i via informative advertising. The signal realization matches the true state with probability $\gamma > 1/2$. Formally, $\mathbb{P}(s_i = g|\text{attribute } i \text{ is good}) = \mathbb{P}(s_i = b|\text{attribute } i \text{ is bad}) = \gamma$. If $\gamma = 1$, then advertising fully reveals the value of the advertised attribute and the problem reduces to that in the main model. If $\gamma < 1$, then the consumer still has some uncertainty about the value of the advertised attribute after seeing the ad. Suppose the firm advertises attribute i. Denote the consumer's belief after seeing the ad by μ_i^g if the signal realization is good, $s_i = g$, and by μ_i^b if $s_i = b$. By Bayes' rule, $\mu_i^g = \mathbb{P}(\text{attribute } i \text{ is good}|s_i = g) = \gamma\mu_i/[\gamma\mu_i + (1-\gamma)(1-\mu_i)]$ and $\mu_i^b = \mathbb{P}(\text{attribute } i \text{ is good}|s_i = b) = (1-\gamma)\mu_i/[(1-\gamma)\mu_i + \gamma(1-\mu_i)]$. The following analytical result characterizes some properties of the optimal advertising strategy.

Proposition 7. *Suppose $\mu_1 \geq \mu_2$ and advertising is noisy, $\gamma < 1$. There exists $\widehat{\mu}'(\mu_1) < \bar{\mu}(\mu_1)$ such that the firm does not advertise if $\mu_1 \leq \underline{\mu}(1)$ or $\mu_2 \geq \widehat{\mu}'(\mu_1)$. In region I_1 and I_2, there exists $\hat{\gamma}(\mu_1, \mu_2) \in (1/2, 1)$, which is decreasing in μ_1 and μ_2, such that the firm advertises attribute two if $\gamma > \hat{\gamma}(\mu_1, \mu_2)$ and does not advertise if $\gamma < \hat{\gamma}(\mu_1, \mu_2)$.*[11]

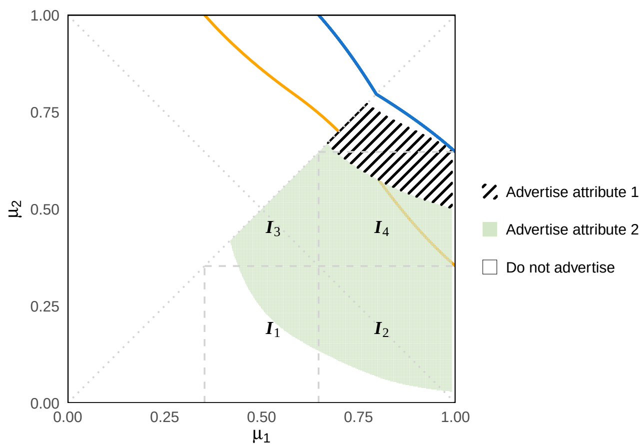
Compared with the full disclosure case illustrated by Figure 5a, the firm still refrains from advertising in the blank region above the purchasing boundary ($\mu_2 \geq \bar{\mu}(\mu_1)$) and to the left of region I_1. The logic mirrors that in the main model. In the first case, the consumer always

[10] We thank the DE for providing this interpretation.

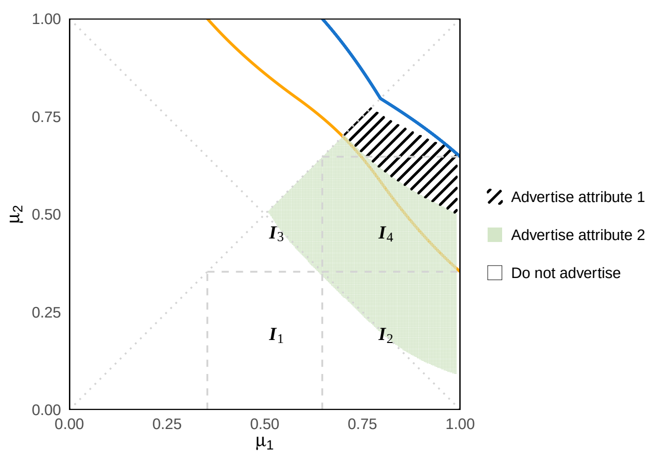
[11] The last result in this Proposition is equivalent to the following statement: in region I_1 and I_2, there exists $\check{\mu}(\mu_2, \gamma) \in (0, \underline{\mu}(1)]$, which is decreasing in μ_1 and γ, such that the firm advertises attribute two if $\mu_2 \in (\check{\mu}(\mu_2, \gamma), \underline{\mu}(1))$ and does not advertise if $\mu_2 < \check{\mu}(\mu_2, \gamma)$.

purchases without an ad. In the second case, the consumer will never search or purchase even if the firm advertises. Moreover, the firm does not advertise when the belief is below but close to the purchasing boundary: the consumer is already highly likely to buy without an ad, and advertising would "waste" some beliefs and reduce the purchasing likelihood.

The main difference between this extension and the full-disclosure model is that, when the consumer has a low belief about the worse attribute and a moderate or high belief about the better attribute (region I_1 and I_2), a noisier signal (smaller γ) leads the firm to refrain from advertising over a wider range. This is because a noisier signal can induce a pessimistic consumer to search



(a) More Precise Signal, $\gamma = .95$.



(b) Noisier Signal, $\gamma = .85$.

Figure 7: Optimal Advertising Strategy When the Ad is Noisy

only when the prior belief is higher.

We supplement these analytical results with numerical simulations in Figure 7. Figure 7a illustrates the optimal advertising strategy when the signal from advertising is more accurate ($\gamma = 0.95$), whereas Figure 7a presents the optimal advertising strategy when the signal from advertising is noisier ($\gamma = 0.85$). As shown there, the overall optimal advertising strategy closely parallels that in the main model, except for a smaller advertising region in I_1 and I_2. In these regions, as the advertising content becomes noisier, the firm is less inclined to advertise because it is increasingly difficult to induce consumers to search even after a positive advertising signal.

5.3 Manipulation of Initial Beliefs

The main model abstracts away from potentially screening consumer segments through targeting. In practice, firms can employ various screening tools, such as data analytics, product placement, website design, and influencer marketing, to attract certain consumer segments. Although these efforts do not alter the beliefs of the overall population, which adheres to the statistical property of Bayes plausibility, they can influence the beliefs of the consumers the firm attracts through selection. Although such screening activities are costly, they can help build a customer base more likely to align well with the firm's product (i.e., a higher initial belief).[12]

Formally, we assume that if a firm engages in broad, untargeted marketing, the consumers in its customer base initially have low prior beliefs (μ_1, μ_2), located in region I_1 of Figure 5a. By investing in targeting activities, the firm can pre-screen consumers to improve the likelihood of a match. Subsequently, it can select advertising content to suit consumers within its customer base. We assume that the screening outcome improves as the firm incurs a higher targeting cost. Specifically, by incurring cost $C(\Delta\mu)$, the firm can increase the matching probability on one attribute from μ_i to $\mu_i + \Delta\mu$ among consumers in the customer base. Because it is more straightforward to screen consumers along one dimension rather than two, we focus on the simpler scenario where the firm screens for one attribute only. To capture the notion that increasing targeting accuracy becomes progressively harder as it becomes more precise, we assume $C(0) = 0$, $C'(\Delta\mu) > 0$, and

[12] This extension is related to the consumer inference literature where being targeted by an ad may affect what a consumer can infer about their product valuation (e.g., Shin and Yu 2021). The main difference is that the firm does not have private information in this paper. Consequently, advertising does not have a signaling role.

$C''(\Delta\mu) > 0$. The game proceeds as follows. In the first stage, the firm determines whether and to what extent it will invest in targeting activities to pre-screen consumers, increasing the matching probability on one attribute from μ_i to $\mu_i + \Delta\mu$ for consumers in the customer base. The game then continues as described in the main model.

Proposition 8. *Suppose the initial belief (μ_1, μ_2) is located in region I_1. If the screening cost is high, $C'(\bar{\mu}(1) - \mu_1) > (p - m)\mu_2/[\bar{\mu}(1) - \underline{\mu}(1)]$ and $C'(\bar{\mu}(1) - \mu_2) > (p - m)\mu_1/[\bar{\mu}(1) - \underline{\mu}(1)]$, then the manipulated belief given the optimal targeting activity will be located in region I_1 or I_3.*[13] *The role of optimal advertising is to invite consumers to search. If the screening cost is low, $C'(\bar{\mu}(1) - \mu_1) \leq (p - m)\mu_2/[\bar{\mu}(1) - \underline{\mu}(1)]$ and $C'(\bar{\mu}(1) - \mu_2) \leq (p - m)\mu_1/[\bar{\mu}(1) - \underline{\mu}(1)] \leq C'(\tilde{\mu}^{-1}(\max\{\mu_1, \tilde{\mu}(1)\}) - \mu_2)$, then the manipulated belief given the optimal targeting activity will be either $(\bar{\mu}(1), \mu_2)$ or $(\mu_1, \bar{\mu}(1))$ (i.e., located in the boundary of region I_2 or I_4). The role of optimal advertising is to invite consumers to purchase.*

The proposition reveals that the firm's ability to manipulate initial beliefs may or may not serve as a strategic substitute for consumer search. When the cost of targeting is high (e.g., for a mass-market product or a new brand with limited data), the firm cannot easily identify its "true believers." It is optimal to run broad targeting campaigns, leaving the consumer's priors in region I_1 or I_3. Subsequently, the firm relies on an "Invitation to Search" advertising strategy, outsourcing the final verification of product matching to consumers because screening a pre-convinced customer is too expensive. In this case, manipulation of initial beliefs cannot replace consumer search.

Conversely, when targeting costs are low (e.g., for niche products or firms with rich customer data), it is profitable for the firm to invest heavily in targeting campaigns so that the prior belief reaches the boundary of region I_2 or I_4. By ensuring that consumers in its customer base are already sufficiently optimistic about one attribute, the firm can use its subsequent ad to resolve uncertainty about the other attribute and trigger a direct sale. In this case, the ability to manipulate initial beliefs serves as a strategic substitute for consumer search.

[13] Because the manipulated belief may lie above the main diagonal, we extend the domains of regions I_1 through I_4 and the diagonally striped region to include their symmetric counterparts across the main diagonal.

6 Discussion and Concluding Remarks

Choosing the optimal advertising content is an important managerial decision faced by all firms. This paper studies the firm's optimal choice of advertising content and the role of advertising for a two-attribute product. If the firm advertises one attribute, the consumer faces a one-dimensional search problem because she is uncertain about only the other attribute. If the firm does not advertise, the consumer faces a two-dimensional search problem because she is uncertain about both attributes. In that case, the consumer only searches for one attribute if she holds a strong prior belief about one of the attributes, and may search for both attributes otherwise.

Given the consumer's search strategy under different advertising strategies, we characterize the optimal advertising strategy. The firm does not advertise if the consumer's prior beliefs about both attributes are extreme. No advertising serves as an invitation to search when the belief is high and as an invitation to purchase when the belief is very high. Otherwise, the firm advertises the better attribute if the consumer is optimistic enough about the worse attribute, and advertises the worse attribute if the consumer is less optimistic about it. When the prior belief about the better attribute is moderate, the firm advertises the other attribute to invite the consumer to search. When the prior belief about the better attribute is high, the role of advertising is *non-monotonic* in the prior belief about attribute two. The non-monotonicity is driven by different incentives of the firm in different belief regions. When the consumer is very pessimistic about an attribute, the firm's binding incentive is to convince the consumer to consider the product after seeing the ad. In contrast, when the consumer has a less extreme belief about that attribute, the firm's binding incentive is to avoid or reduce the amount of wasted beliefs upon conversion.

Despite extensive research on informative advertising, relatively little attention has been paid to advertising content in a way that yields implementable guidance for managers. This paper addresses this gap by examining firms' choices of specific, feasible advertising content in the presence of consumer search. In particular, the findings shed light on firms' trade-off between highlighting their strengths and remedying their weaknesses. In addition, by examining the role of advertising, this paper informs the design of KPIs for advertising campaigns - for example, prioritizing an immediate increase in purchases versus an increase in traffic or search activity related to the product.

In terms of policy implications, the firm's profit-maximizing strategy may or may not coincide with the consumer welfare-maximizing strategy. Conditional on advertising, consumer surplus is higher when consumers face less uncertainty at the time of purchase and when they spend less time searching for additional information. Consequently, the firm's and the consumer's incentives are aligned in region $I_1 - I_4$ of Figure 5a. In contrast, in the diagonally striped region, the firm is better off advertising the better attribute, whereas consumers are better off when the worse attribute is advertised. This incentive misalignment suggests that, to enhance consumer welfare, regulators may wish to impose mandatory disclosure requirements for attributes that are more uncertain from the consumer's perspective, especially for products with a relatively high match likelihood across attributes (the prior belief lies within the diagonally striped region of Figure 5a).

There are some limitations to this paper. We consider a monopoly in this paper. Introducing competition can lead to interesting findings. For example, integrating our model's endogenous attribute selection with the competitive search framework of Ke and Villas-Boas (2019) suggests a nested switching behavior. Consumers would likely go back and forth between the attributes of a single product to resolve uncertainties about the absolute valuation, while also switching across competing products to evaluate the relative valuation. In such a setting, in addition to managing a consumer's propensity to search versus purchase, advertising would also serve as a strategic tool to hijack attention toward a given product.

It will also be interesting to extend the number of attributes beyond two. We expect the core learning mechanism to remain intact: the consumer would still locally prefer to search the attribute with the highest uncertainty because of the fastest learning speed. However, characterizing the exact multidimensional search boundaries remains a complex avenue for future research.

We consider an exogenous price throughout the paper to focus on the role of information. Consequently, the uncertain attributes in our model are purely horizontal. Future research can study the optimal pricing of the product or expand the model to include vertical, price-related search frictions where consumers must expend effort to uncover hidden fees, shipping costs, or applicable coupons. In such cases, the firm would possess private information about its own pricing structure. This would introduce an additional signaling mechanism.

Lastly, our analysis focuses on a parameter range characterized by high marginal costs and, consequently, high prices. This restriction ensures that the consumer needs to consider the value of the unadvertised attribute even if the advertised one is known to be a good match. The necessity of joint evaluation generates the rich trade-offs observed in both consumer search behavior and firm decision-making. Conversely, if the price were low enough that a single good attribute justified a purchase, consumers would always buy immediately upon learning that an advertised attribute is a good match, regardless of their beliefs about the remaining attribute. Under such conditions, the firm would have a strong incentive to default to advertising the better attribute to minimize wasted beliefs.

Appendix

Proof of Proposition 1. By symmetry, we only need to prove the case of $\mu_1 \geq \mu_2$. We first show that the viscosity solution of the HJB equation $(\star)$ exists and is unique. Because the value function is a viscosity solution of $(\star)$, the viscosity solution of $(\star)$ must be the value function by uniqueness. We then conjecture an optimal search strategy and characterize its properties. Lastly, we verify that the conjectured strategy indeed generates a viscosity solution to $(\star)$. So, the conjectured strategy is optimal.

Lemma 1. *The viscosity solution of the HJB equation* $(\star)$ *exists and is unique.*

Proof. Because the consumer can guarantee a payoff of zero by quitting immediately and cannot achieve a payoff higher than $\sup\{\mu_1 + \mu_2 - p\} = 1 + 1 - p \leq 2$, the value function is bounded and thus exists. This implies the existence of the viscosity solution because the value function is a viscosity solution to $(\star)$.

The proof of the uniqueness uses a modification of a comparison principle in Crandall et al. (1992). Given that it very much resembles the proof of Lemma 1 in Ke and Villas-Boas (2019), we refer the reader to their proof. □

<u>**Conjecture:**</u> Conditional on searching, it is optimal for the consumer to search for information about attribute two (one) if $\mu_1 \geq \mu_2$ ($\mu_1 < \mu_2$).

Given this conjecture, we now characterize the search region (illustrated in Figure 3).

The PDE when the consumer searches attribute two, equation (4), has the following general solution (B_1 and B_2 are unknown functions to be determined):

$$V(\mu_1,\mu_2) = 2\sigma^2 c(1-2\mu_2)\ln\frac{1-\mu_2}{\mu_2} + B_1(\mu_1)\mu_2 + B_2(\mu_1), \mu_1 \in [\mu^*, 1].$$

Also, $V(\mu_1,\mu_2) = 0$ at the quitting boundary $\mu_2 = \underline{\mu}(\mu_1)$. For the value function in the search region, value matching and smooth pasting (wrt μ_2) at the quitting boundary $(\mu_1, \underline{\mu}(\mu_1))$ imply:[14]

$$\frac{V(\mu_1,\mu_2)}{2\sigma^2 c} = (1-2\mu_2)\ln\frac{1-\mu_2}{\mu_2} + \phi(\underline{\mu}(\mu_1))\mu_2 - \psi(\underline{\mu}(\mu_1)), \tag{5}$$

where $\phi(x) = 2\ln\frac{1-x}{x} + \frac{1}{x} - \frac{1}{1-x}$ and $\psi(x) = \ln\frac{1-x}{x} + \frac{1-2x}{1-x}$.

By symmetry, for $\mu_1 < \mu_2$, the value function in the search region satisfies:

$$\frac{V(\mu_1,\mu_2)}{2\sigma^2 c} = (1-2\mu_1)\ln\frac{1-\mu_1}{\mu_1} + \phi(\underline{\mu}(\mu_2))\mu_1 - \psi(\underline{\mu}(\mu_2)) \tag{6}$$

Equation (5) characterizes the value function for beliefs $\mu_1 \geq \mu_2$. Equation (6) characterizes the value function for beliefs $\mu_1 < \mu_2$. The two regions are separated by the main diagonal $\{(\mu_1,\mu_2) : \mu_1 = \mu_2\}$. Continuity of $V_{\mu_1}(\mu_1,\mu_2)$ at this boundary implies that:

$$\underline{\mu}'(\mu) = \frac{\phi(\underline{\mu}(\mu)) - \phi(\mu)}{\phi'(\underline{\mu}(\mu))[\mu - \underline{\mu}(\mu)]}, \text{ for } \mu \in (\mu^*, \mu^{**}]. \tag{D_1}$$

For $\mu_1 \in [\mu^{**}, 1]$, $V(\mu_1,\mu_2) = \mu_1 + \mu_2 - p$ at the purchasing boundary $\mu_2 = \bar{\mu}(\mu_1)$. Value matching and smooth pasting (w.r.t. μ_2) at the purchasing boundary $(\mu_1, \underline{\mu}(\mu_1))$ imply (in the search region):

$$\frac{V(\mu_1,\mu_2)}{2\sigma^2 c} = (1-2\mu_2)\ln\frac{1-\mu_2}{\mu_2} + \phi(\bar{\mu}(\mu_1))\mu_2 - \psi(\bar{\mu}(\mu_1)) + \frac{\mu_1 - \mu_2 - p}{2\sigma^2 c}. \tag{7}$$

Equations (5) and (7) use the quitting and purchasing boundaries to pin down the value function, respectively. The resulting expression should be equivalent in the common domain $\mu_1 \in [\mu^{**}, 1]$.

[14] For technical details, please refer to Dixit (1993).

By equalizing V and V_{μ_2} of equations (5) and (7), we obtain the following system of equations:

$$\begin{cases} \phi(\underline{\mu}(\mu)) - \phi(\bar{\mu}(\mu)) = \frac{1}{2\sigma^2 c} \\ \psi(\underline{\mu}(\mu)) - \psi(\bar{\mu}(\mu)) = \frac{p-\mu}{2\sigma^2 c} \end{cases}, \text{ for } \mu \in [\mu^{**}, 1]. \tag{8}$$

For each belief, μ, the system of equations above consists of two unknowns ($\bar{\mu}(\mu)$ and $\underline{\mu}(\mu)$) and two equations. They uniquely determine the function for the purchasing boundary $\bar{\mu}(\mu)$ and the function for the quitting boundary $\underline{\mu}(\mu)$, for $\mu \in [\mu^{**}, 1]$, given a cutoff belief μ^{**}.

Instead of determining $\bar{\mu}(\mu)$ and $\underline{\mu}(\mu)$ by a system of equations (8), we can also implicitly determine $\bar{\mu}(\mu)$ and $\underline{\mu}(\mu)$ in two separate equations. Representing $\bar{\mu}(\mu)$ by $\underline{\mu}(\mu)$ from the first equation of (8), we have:

$$\bar{\mu}(\mu) = \phi^{-1}\left[\phi(\underline{\mu}(\mu)) - \frac{1}{2\sigma^2 c}\right].$$

Plugging it into the second equation of (8), we have:

$$\underline{\mu}(\mu) = \psi^{-1}\left\{\psi\left(\phi^{-1}\left[\phi(\underline{\mu}(\mu)) - \frac{1}{2\sigma^2 c}\right]\right) + \frac{p-\mu}{2\sigma^2 c}\right\}.$$

The equation above implicitly determines $\underline{\mu}(\mu)$, for $\mu \in [\mu^{**}, 1]$. Similarly, we can implicitly determine $\bar{\mu}(\mu)$ by the following equation:

$$\bar{\mu}(\mu) = \psi^{-1}\left\{\psi\left(\phi^{-1}\left[\phi(\bar{\mu}(\mu)) + \frac{1}{2\sigma^2 c}\right]\right) - \frac{p-\mu}{2\sigma^2 c}\right\}.$$

We now solve for the cutoff belief at the intersection of the purchasing boundary and the main diagonal, μ^{**}. Because (μ^{**}, μ^{**}) is on the purchasing boundary, we have $\mu^{**} = \bar{\mu}(\mu^{**})$. Hence,

$$\begin{cases} \phi(\underline{\mu}(\mu^{**})) - \phi(\mu^{**}) = \frac{1}{2\sigma^2 c} \\ \psi(\underline{\mu}(\mu^{**})) - \psi(\mu^{**}) = \frac{p-\mu^{**}}{2\sigma^2 c} \end{cases}. \tag{9}$$

The system of equations above consists of two unknowns (μ^{**} and $\underline{\mu}(\mu^{**})$ and two equations.

They uniquely determine the cutoff belief μ^{**} via the following equations:

$$\phi^{-1}\left[\phi(\mu^{**})+\frac{1}{2\sigma^2 c}\right]=\psi^{-1}\left[\psi(\mu^{**})+\frac{p-\mu^{**}}{2\sigma^2 c}\right]. \tag{I^{**}}$$

We have pinned down the cutoff belief μ^{**}. Given this cutoff beliefs, we have determined the purchasing boundary $(\mu, \bar{\mu}(\mu))$ and the quitting boundary $(\mu, \underline{\mu}(\mu))$, for $\mu \in [\mu^{**}, 1]$. Equation (D_1) and the initial condition (I^{**}) implicitly determine the function for the quitting boundary $\underline{\mu}(\mu)$, for $\mu \in (\mu^*, \mu^{**}]$, given a cutoff belief μ^*.

We now solve for the cutoff belief at the intersection of the quitting boundary and the main diagonal, μ^*. Because (μ^*, μ^*) is on the quitting boundary, we have $\mu^* = \underline{\mu}(\mu^*)$. This initial condition determines μ^*.

In sum, we have pinned down the cutoff belief μ^* and the quitting boundary $(\mu, \underline{\mu}(\mu))$, for $\mu \in [\mu^*, \mu^{**}]$. We have fully characterized the purchasing boundary $(\mu, \bar{\mu}(\mu))$ and the quitting boundary $(\mu, \underline{\mu}(\mu))$, for $\mu_1 \geq \mu_2$. The other case in which $\mu_1 < \mu_2$ is determined by symmetry.

<u>**Verification:**</u> To verify that the conjectured strategy indeed generates a viscosity solution to the HJB equation $(\star)$:

$$\max\left\{\max_{i=1,2}\left[\frac{\mu_i^2(1-\mu_i)^2}{2\sigma^2}V_{\mu_i\mu_i}(\mu_1,\mu_2)-c\right], \max\left[\mu_1+\mu_2-p,0\right]-V(\mu_1,\mu_2)\right\}=0,$$

We just need to show that (everything else holds by our construction):

$$\frac{\mu_1^2(1-\mu_1)^2}{2\sigma^2}V_{\mu_1\mu_1}(\mu_1,\mu_2)-c\leq 0 \;\Leftrightarrow\; \mu_1^2(1-\mu_1)^2V_{\mu_1\mu_1}(\mu_1,\mu_2)/2\sigma^2 c\leq 1, \tag{10}$$

if $\mu_1 + \mu_2 > 1$, $\mu_1 \geq \mu_2$, and $\underline{\mu}(\mu_1) < \mu_2 < \bar{\mu}(\mu_1)$.

For $\mu_1 \in (\mu^*, \mu^{**}]$, we have:

$$\begin{aligned}
&V_{\mu_1}(\mu_1,\mu_2)/2\sigma^2 c = \phi'(\underline{\mu}(\mu_1))\underline{\mu}'(\mu_1)[\mu_2-\underline{\mu}(\mu_1)]\\
&\overset{(D_1)}{=}\frac{\phi(\underline{\mu}(\mu_1))-\phi(\mu_1)}{\mu_1-\underline{\mu}(\mu_1)}[\mu_2-\underline{\mu}(\mu_1)]\\
&\Rightarrow\; V_{\mu_1\mu_1}(\mu_1,\mu_2)/2\sigma^2 c = \phi'(\underline{\mu}(\mu_1))\underline{\mu}'(\mu_1)[\mu_2-\underline{\mu}(\mu_1)]
\end{aligned}$$

$$\overset{(D_1)}{=}\frac{\phi'(\underline{\mu}(\mu_1))\underline{\mu}'(\mu_1)-\phi'(\mu_1)}{\mu_1-\underline{\mu}(\mu_1)}[\mu_2-\underline{\mu}(\mu_1)]+[\phi(\underline{\mu}(\mu_1))-\phi(\mu_1)]\frac{(\mu_2-\mu_1)\underline{\mu}'(\mu_1)+\underline{\mu}(\mu_1)-\mu_2}{[\mu_1-\underline{\mu}(\mu_1)]^2}$$
$$=-\frac{\phi'(\mu_1)[\mu_2-\underline{\mu}(\mu_1)]}{\mu_1-\underline{\mu}(\mu_1)}+(\mu_2-\mu_1)\frac{[\phi(\underline{\mu}(\mu_1))-\phi(\mu_1)]^2}{\phi'(\underline{\mu}(\mu_1))[\mu_1-\underline{\mu}(\mu_1)]^3}$$
$$\Rightarrow\ \mu_1^2(1-\mu_1)^2V_{\mu_1\mu_1}(\mu_1,\mu_2)/2\sigma^2c$$
$$=\frac{\mu_2-\underline{\mu}(\mu_1)}{\mu_1-\underline{\mu}(\mu_1)}+(\mu_1-\mu_2)\mu_1^2(1-\mu_1)^2\frac{\underline{\mu}(\mu_1)^2[1-\underline{\mu}(\mu_1)]^2}{[\mu_1-\underline{\mu}(\mu_1)]^3}[\phi(\underline{\mu}(\mu_1))-\phi(\mu_1)]^2$$

So,

$$\begin{aligned}
&\mu_1^2(1-\mu_1)^2V_{\mu_1\mu_1}(\mu_1,\mu_2)/2\sigma^2c\le 1\\
\Leftrightarrow&\mu_1^2(1-\mu_1)^2\frac{\underline{\mu}(\mu_1)^2[1-\underline{\mu}(\mu_1)]^2}{[\mu_1-\underline{\mu}(\mu_1)]^2}[\phi(\underline{\mu}(\mu_1))-\phi(\mu_1)]^2\le 1\\
\Leftrightarrow&\mu_1(1-\mu_1)\frac{\underline{\mu}(\mu_1)^2[1-\underline{\mu}(\mu_1)]}{[\mu_1-\underline{\mu}(\mu_1)]}[\phi(\underline{\mu}(\mu_1))-\phi(\mu_1)]\le 1\\
\Leftrightarrow&H(\mu_1):=\mu_1(1-\mu_1)[\phi(\underline{\mu}(\mu_1))-\phi(\mu_1)]-\frac{\mu_1-\underline{\mu}(\mu_1)}{\underline{\mu}(\mu_1)[1-\underline{\mu}(\mu_1)]}\le 0
\end{aligned}\tag{11}$$

Observe that $H(\mu^*)=0$. Ignoring the subscript 1 for notational ease, we have:

$$\begin{aligned}
H'(\mu)=&(1-2\mu)[\phi(\underline{\mu}(\mu))-\phi(\mu)]+\frac{\mu(1-\mu)}{\mu-\underline{\mu}(\mu)}[\phi(\underline{\mu}(\mu))-\phi(\mu)]+\frac{1}{\mu(1-\mu)}\\
&-\frac{1}{\underline{\mu}(\mu)(1-\underline{\mu}(\mu))}+\frac{\phi(\underline{\mu}(\mu))-\phi(\mu)}{\mu-\underline{\mu}(\mu)}[-\mu+2\mu\underline{\mu}(\mu)-\underline{\mu}(\mu)^2]\\
=&[1-3\mu+\underline{\mu}(\mu)][\phi(\underline{\mu}(\mu))-\phi(\mu)]+\frac{1}{\mu(1-\mu)}-\frac{1}{\underline{\mu}(\mu)(1-\underline{\mu}(\mu))}
\end{aligned}$$

Suppose (11) does not hold. There would exist $\widehat{\mu}$ such that $H(\widehat{\mu})=0$ and $H'(\widehat{\mu})>0$.

$$(11)\Rightarrow\phi(\underline{\mu}(\widehat{\mu}))-\phi(\widehat{\mu})=\frac{\widehat{\mu}-\underline{\mu}(\widehat{\mu})}{\widehat{\mu}(1-\widehat{\mu})\underline{\mu}(\widehat{\mu})[1-\underline{\mu}(\widehat{\mu})]}$$

Hence, we get an expression for $1/[\widehat{\mu}(1-\widehat{\mu})]$ and $1/\{\underline{\mu}(\mu)[1-\underline{\mu}(\mu)]\}$. Plugging these expressions into the previous expression for $H'(\mu)$, we have $H'(\widehat{\mu})=-2[\phi(\underline{\mu}(\mu))-\phi(\mu)][\mu-\underline{\mu}(\mu)]\le 0$. A contradiction! So, (11) and thus (10) hold, $\forall\mu_1\in[\mu^*,\mu^{**}]$.

For $\mu_1\in[\mu^{**},1]$, we have:

$$V_{\mu_1}(\mu_1,\mu_2)/2\sigma^2c=\phi'(\underline{\mu}(\mu_1))\underline{\mu}'(\mu_1)[\mu_2-\underline{\mu}(\mu_1)]\overset{(D_2)}{=}\frac{\mu_2-\underline{\mu}(\mu_1)}{\bar{\mu}(\mu_1)-\underline{\mu}(\mu_1)}$$

$$V_{\mu_1\mu_1}(\mu_1,\mu_2)/2\sigma^2 c = \frac{-\underline{\mu}'(\mu_1)[\bar{\mu}(\mu_1)-\underline{\mu}(\mu_1)]-[\bar{\mu}'(\mu_1)-\underline{\mu}'(\mu_1)][\mu_2-\underline{\mu}(\mu_1)]}{[\bar{\mu}(\mu_1)-\underline{\mu}(\mu_1)]^2}$$
$$= \frac{1}{2\sigma^2 c}\frac{1}{[\bar{\mu}(\mu_1)-\underline{\mu}(\mu_1)]^3}[\frac{\mu_2-\bar{\mu}(\mu_1)}{\phi'(\underline{\mu}(\mu_1))}-\frac{\mu_2-\underline{\mu}(\mu_1)}{\phi'(\bar{\mu}(\mu_1))}]$$
$$\Rightarrow V_{\mu_1\mu_1}(\mu_1,\mu_2) = \frac{1}{[\bar{\mu}(\mu_1)-\underline{\mu}(\mu_1)]^3}[\frac{\mu_2-\bar{\mu}(\mu_1)}{\phi'(\underline{\mu}(\mu_1))}-\frac{\mu_2-\underline{\mu}(\mu_1)}{\phi'(\bar{\mu}(\mu_1))}]$$

Because $\frac{\partial V_{\mu_1\mu_1}(\mu_1,\mu_2)}{\partial \mu_2} < 0$, we only need to show that (10) holds for $\mu_2 = \underline{\mu}(\mu_1)$:

$$\mu_1^2(1-\mu_1)^2 V_{\mu_1\mu_1}(\mu_1,\underline{\mu}(\mu_1))/2\sigma^2 c \leq 1 \;\Leftrightarrow\; \frac{\mu_1^2(1-\mu_1)^2}{[\bar{\mu}(\mu_1)-\underline{\mu}(\mu_1)]^2}\frac{-1}{\phi'(\underline{\mu}(\mu_1))} \leq 1 \tag{12}$$

Let's first show that $\underline{\mu}(\mu^{**}) \leq 1/2$ by contradiction. Suppose instead $\underline{\mu}(\mu^{**}) > 1/2$.

$$p-\mu^{**} = \frac{\bar{\mu}(\mu^{**})+\underline{\mu}(\mu^{**})}{2} \;\Leftrightarrow\; p-\mu^{**} = \frac{\mu^{**}+\underline{\mu}(\mu^{**})}{2} \;\Leftrightarrow\; \underline{\mu}(\mu^{**}) = 2p-3\mu^{**}$$

Hence, $2p-3\mu^{**} > 1/2 \Rightarrow \mu^{**} < \frac{2}{3}p-\frac{1}{6}$. Because $\phi(x)$ is strictly decreasing in x, the first equation of (9) implies:

$$\frac{1}{2\sigma^2 c} = \phi(\underline{\mu}(\mu^{**}))-\phi(\mu^{**}) < \phi(1/2)-\phi(\frac{2}{3}p-\frac{1}{6}) \;\Leftrightarrow\; c > \frac{1}{2\sigma^2[\phi(1/2)-\phi(\frac{2}{3}p-\frac{1}{6})]}.$$

A contradiction! Therefore, $\underline{\mu}(\mu^{**}) \leq 1/2$. Because $\underline{\mu}(\mu_1)$ is decreasing in μ_1, we have $\underline{\mu}(\mu_1) \leq 1/2$, $\forall \mu \in [\mu^{**},1]$. One can see that the LHS of (12), $\frac{\mu_1^2(1-\mu_1)^2}{[\bar{\mu}(\mu_1)-\underline{\mu}(\mu_1)]^2}\frac{-1}{\phi'(\underline{\mu}(\mu_1))}$, decreases in $\mu_1 \in [\mu^{**},1]$. And we know that (12) holds for $\mu_1 = \mu^{**}$ (we have shown that (10) and thus (12) hold for $\forall \mu_1 \in [\mu^*,\mu^{**}]$). Therefore, (12) and thus (10) hold for $\forall \mu_1 \in [\mu^{**},1]$.

□

Proof of Proposition 2. We have derived (D_1) in the proof of Proposition 1. It implies immediately that $\underline{\mu}'(\mu) < 0$ for $\mu \in (\mu^*,\mu^{**}]$. For $\mu \in [\mu^{**},1]$, by the implicit function theorem, we have:

$$\begin{bmatrix}\bar{\mu}'(\mu)\\ \underline{\mu}'(\mu)\end{bmatrix} = -\begin{bmatrix}-\phi'(\bar{\mu}(\mu)) & \phi'(\underline{\mu}(\mu))\\ -\psi'(\bar{\mu}(\mu)) & \psi'(\underline{\mu}(\mu))\end{bmatrix}^{-1}\cdot\begin{bmatrix}0\\ \frac{1}{2\sigma^2 c}\end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2\sigma^2 c} \frac{1}{\phi'(\bar{\mu}(\mu))[\bar{\mu}(\mu)-\underline{\mu}(\mu)]} \\ \frac{1}{2\sigma^2 c} \frac{1}{\phi'(\underline{\mu}(\mu))[\bar{\mu}(\mu)-\underline{\mu}(\mu)]} \end{bmatrix} = \begin{bmatrix} \frac{\phi(\underline{\mu}(\mu))-\phi(\bar{\mu}(\mu))}{\phi'(\bar{\mu}(\mu))[\bar{\mu}(\mu)-\underline{\mu}(\mu)]} & < 0 \\ \frac{\phi(\underline{\mu}(\mu))-\phi(\bar{\mu}(\mu))}{\phi'(\underline{\mu}(\mu))[\bar{\mu}(\mu)-\underline{\mu}(\mu)]} & < 0 \end{bmatrix}$$

This gives us the expression for ($\overline{D_2}$) and ($\underline{D_2}$). One can see from the negative sign of the derivative that both $\underline{\mu}(\mu)$ and $\bar{\mu}(\mu)$ strictly decrease in μ.

We now look at the width of the search region.

$$\begin{aligned} [\bar{\mu}(\mu) - \underline{\mu}(\mu)]' =& \frac{\phi(\underline{\mu}(\mu)) - \phi(\bar{\mu}(\mu))}{\phi'(\bar{\mu}(\mu))[\bar{\mu}(\mu) - \underline{\mu}(\mu)]} - \frac{\phi(\underline{\mu}(\mu)) - \phi(\bar{\mu}(\mu))}{\phi'(\underline{\mu}(\mu))[\bar{\mu}(\mu) - \underline{\mu}(\mu)]} \\ =& \frac{\phi(\underline{\mu}(\mu)) - \phi(\bar{\mu}(\mu))}{\bar{\mu}(\mu) - \underline{\mu}(\mu)} \left[1/\phi'(\bar{\mu}(\mu)) - 1/\phi'(\underline{\mu}(\mu))\right] \\ =& \frac{\phi(\underline{\mu}(\mu)) - \phi(\bar{\mu}(\mu))}{\bar{\mu}(\mu) - \underline{\mu}(\mu)} \left[\underline{\mu}(\mu)^2(1 - \underline{\mu}(\mu))^2 - \bar{\mu}(\mu)^2(1 - \bar{\mu}(\mu))^2\right] \end{aligned}$$

One can see that $\frac{\phi(\underline{\mu}(\mu))-\phi(\bar{\mu}(\mu))}{\bar{\mu}(\mu)-\underline{\mu}(\mu)} > 0$. So, $[\bar{\mu}(\mu) - \underline{\mu}(\mu)]' > 0 \Leftrightarrow \underline{\mu}(\mu)^2(1 - \underline{\mu}(\mu))^2 > \bar{\mu}(\mu)^2(1 - \bar{\mu}(\mu))^2 \Leftrightarrow \underline{\mu}(\mu)(1 - \underline{\mu}(\mu)) > \bar{\mu}(\mu)(1 - \bar{\mu}(\mu)) \Leftrightarrow |\underline{\mu}(\mu) - 1/2| < |\bar{\mu}(\mu) - 1/2|$. Thus, the width of the search region, $\bar{\mu}(\mu) - \underline{\mu}(\mu)$, increases in the belief, μ, if and only if the quitting boundary is closer to 1/2 than the purchasing boundary. We know that $\forall \mu \geq \mu^{**},\ p = \mu + \frac{\bar{\mu}(\mu)+\underline{\mu}(\mu)}{2}$ due to the symmetry of the one-dimensional learning problem.[15] Therefore, for any $\mu \in [\mu^{**}, 1)$,

$$\begin{aligned} & \frac{\bar{\mu}(\mu) + \underline{\mu}(\mu)}{2} = p - \mu > 3/2 - 1 = 1/2 \\ \Rightarrow\ & \frac{\bar{\mu}(\mu) + \underline{\mu}(\mu)}{2} > 1/2 \\ \Leftrightarrow\ & \bar{\mu}(\mu) + \underline{\mu}(\mu) > 1 \\ \Leftrightarrow\ & |\underline{\mu}(\mu) - 1/2| < |\bar{\mu}(\mu) - 1/2|. \end{aligned}$$

Thus, the width of the search region, $\bar{\mu}(\mu) - \underline{\mu}(\mu)$, always strictly increases in the belief μ.

Now suppose that $\underline{\mu}(\mu) \geq 1/2$, then $\forall \mu \in (\mu^*, \mu^{**}]$, we have

$$\underline{\mu}'(\mu) \overset{(\underline{D_1})}{=} \frac{\phi(\underline{\mu}(\mu)) - \phi(\mu)}{\phi'(\underline{\mu}(\mu))[\mu - \underline{\mu}(\mu)]} = \frac{-\phi'(\xi_1(\mu))[\mu - \underline{\mu}(\mu)]}{\phi'(\underline{\mu}(\mu))[\mu - \underline{\mu}(\mu)]}\ (\xi_1(\mu) \in (\underline{\mu}(\mu), \mu))$$

[15] More specifically, the sum of the purchasing and quitting thresholds is zero when the price is zero in the one-dimensional optimal search strategy, as shown by Branco et al. (2012). It implies that the price equals to the average of the two boundaries. In our two-dimensional problem, the consumer only searches the more uncertain attribute when $\mu \geq \mu^{**}$. So, it can be translated to a one-dimensional search problem with the price p normalized to $p - \mu$.

$$= - \frac{\phi'(\xi_1(\mu)}{\phi'(\underline{\mu}(\mu))} < -1,$$

where the last inequality comes from the fact that the absolute value of $\phi'(x) = -1/[x^2(1-x)^2]$ is strictly increasing in x for $x \geq 1/2$. Similarly, $\forall \mu \in [\mu^{**}, 1]$, we have

$$\begin{aligned}
\underline{\mu}'(\mu) &\overset{(\underline{D_2})}{=} \frac{\phi(\underline{\mu}(\mu)) - \phi(\bar{\mu}(\mu))}{\phi'(\underline{\mu}(\mu))[\bar{\mu}(\mu) - \underline{\mu}(\mu)]} = \frac{-\phi'(\xi_2(\mu))[\bar{\mu}(\mu) - \underline{\mu}(\mu)]}{\phi'(\underline{\mu}(\mu))[\bar{\mu}(\mu) - \underline{\mu}(\mu)]} \ (\xi_2(\mu) \in (\underline{\mu}(\mu), \bar{\mu}(\mu))) \\
&= - \frac{\phi'(\xi_2(\mu)}{\phi'(\underline{\mu}(\mu))} < -1 \\
\bar{\mu}'(\mu) &\overset{(\overline{D_2})}{=} \frac{\phi(\underline{\mu}(\mu)) - \phi(\bar{\mu}(\mu))}{\phi'(\bar{\mu}(\mu))[\bar{\mu}(\mu) - \underline{\mu}(\mu)]} = \frac{-\phi'(\xi_3(\mu))[\bar{\mu}(\mu) - \underline{\mu}(\mu)]}{\phi'(\bar{\mu}(\mu))[\bar{\mu}(\mu) - \underline{\mu}(\mu)]} \ (\xi_3(\mu) \in (\underline{\mu}(\mu), \bar{\mu}(\mu))) \\
&= - \frac{\phi'(\xi_3(\mu))}{\phi'(\bar{\mu}(\mu))} > -1
\end{aligned}$$

□

Proof of Proposition 3. We have shown that the consumer will quit if the firm advertises attribute one and the attribute is bad. It happens with probability $1 - \mu_1$. With the complementary probability, the consumer purchases the product if $\mu_2 \geq \bar{\mu}_2$, searches for more information about attribute two if $\underline{\mu}_2 < \mu_2 < \bar{\mu}_2$, and quit if $\mu_2 \leq \underline{\mu}_2$, according to section 3.1. When the consumer searches for more information about attribute two, by Dynkin's lemma, one can see that the conditional purchasing probability is $(\mu_2 - \underline{\mu}_2)/(\bar{\mu}_2 - \underline{\mu}_2)$. So, the probability that the consumer purchases the product is:

$$P_1(\mu_1, \mu_2) = \begin{cases} \mu_1, & \text{if } \mu_2 \geq \bar{\mu}_2 \\ \mu_1 \cdot \frac{\mu_2 - \underline{\mu}_2}{\bar{\mu}_2 - \underline{\mu}_2}, & \text{if } \mu_2 \in [\underline{\mu}_2, \bar{\mu}_2] \\ 0, & \text{if } \mu_2 \leq \underline{\mu}_2 \end{cases} .$$

Comparing equation (2) with equation (8), one can see that $\underline{\mu}_2 = \underline{\mu}(1)$ and that $\bar{\mu}_2 = \bar{\mu}(1)$. □

Proof of Proposition 4. We first consider $\mu_1 \in [\mu^{**}, 1]$ and $\mu_1 \geq \mu_2$. Under this circumstance, the consumer only learns about attribute two until μ_2 hits either the purchasing boundary or the quitting boundary. As μ_2 is a martingale, by Dynkin's formula, we get:

$$P(\mu_1,\mu_2) := \mathbb{P}[\text{purchasing}|\text{starting at } (\mu_1,\mu_2)] = \frac{\mu_2 - \underline{\mu}(\mu_1)}{\bar{\mu}(\mu_1) - \underline{\mu}(\mu_1)}.$$

Now we consider $\mu_1 \in [\mu^*, \mu^{**}]$ and $\mu_1 \geq \mu_2$. The belief either hits (μ^{**}, μ^{**}) and the consumer purchases the good or the belief hits $\{(x, \underline{\mu}(x)) : x \in [\mu_1, \mu^{**})\} \cup \{(\underline{\mu}(x), x) : x \in [\mu_1, \mu^{**})\}$ and the consumer quits. To calculate the purchasing likelihood, let's first calculate the likelihood of the belief hitting $(\mu_1, \underline{\mu}(\mu_1))$ before hitting the main diagonal (μ_1, μ_1), $q(\mu_1, \mu_2) = (\mu_1 - \mu_2)/[\mu_1 - \underline{\mu}(\mu_1)]$.

Now we calculate the probability of purchasing given belief (μ, μ), $\tilde{P}(\mu)$, by considering the infinitesimal learning on attribute two. Noticing that $q(\mu,\mu) = 0$, $\frac{\partial q}{\partial \mu_1}|_{\mu_1=\mu_2=\mu} = \frac{1}{\mu - \underline{\mu}(\mu)}$, $\frac{\partial q}{\partial \mu_2}|_{\mu_1=\mu_2=\mu} = -\frac{1}{\mu-\underline{\mu}(\mu)}$, we have:

$$\begin{aligned}
\tilde{P}(\mu) &= \frac{1}{2}\mathbb{P}[\text{purchasing}|(\mu,\mu), d\mu \geq 0] + \frac{1}{2}\mathbb{P}[\text{purchasing}|(\mu,\mu), d\mu < 0] \\
&= \frac{1}{2}[1 - q(\mu + |d\mu|, \mu)]\tilde{P}(\mu + |d\mu|) + \frac{1}{2}[1 - q(\mu - |d\mu|, \mu)]\tilde{P}(\mu) \\
&= \tilde{P}(\mu) + \frac{|d\mu|}{2}\tilde{P}'(\mu) + |d\mu|\frac{\tilde{P}(\mu)}{\underline{\mu}(\mu) - \mu} + o(d\mu) \\
\Rightarrow 0 &= \frac{|d\mu|}{2}\left[\tilde{P}'(\mu) + 2\frac{\tilde{P}(\mu)}{\underline{\mu}(\mu) - \mu}\right] + o(d\mu) \\
\Rightarrow \frac{\tilde{P}'(\mu)}{\tilde{P}(\mu)} &= -\frac{2}{\underline{\mu}(\mu) - \mu}, \ \forall \mu \in (\mu^*, \mu^{**}),
\end{aligned}$$

where the last equality comes from dividing the previous equation by $|d\mu|$ and take the limit of $d\mu$ to 0. Together with the initial condition $\tilde{P}(\mu^{**}) = 1$, we obtain $\tilde{P}(\mu) = e^{-\int_{\mu}^{\mu^{**}} \frac{2}{x - \underline{\mu}(x)} dx}$.

In sum, the purchasing likelihood when $\mu_1 \geq \mu_2$ and $\mu_1 \in (\mu^*, \mu^{**})$ is $P(\mu_1, \mu_2) = [1 - q(\mu_1, \mu_2)]\tilde{P}(\mu_1) = h(\mu_1, \mu_2)\tilde{P}(\mu_1)$, where $h(\mu_1, \mu_2) = [\mu_2 - \underline{\mu}(\mu_1)]/[\mu_1 - \underline{\mu}(\mu_1)]$. By symmetry, the purchasing likelihood when $\mu_1 < \mu_2$ and $\mu_2 \in (\mu^*, \mu^{**})$ is $P(\mu_1, \mu_2) = P(\mu_2, \mu_1) = [1 - q(\mu_2, \mu_1)]\tilde{P}(\mu_2) = h(\mu_2, \mu_1)\tilde{P}(\mu_2)$. □

Proof of Proposition 5. One can see that the consumer will not purchase the product if $\mu_1 \leq \underline{\mu}(1)$, even if the firm advertises one attribute which turns out to be good. So, the firm does not advertise if $\mu_1 \leq \underline{\mu}(1)$. Also, the consumer will purchase the product for sure if $\mu_2 \geq \bar{\mu}(\mu_1)$ without advertising.

So, the firm does not advertise if $\mu_2 \geq \bar{\mu}(\mu_1)$. We now look at other cases.

(1) $\mu_1 > \underline{\mu}(1)$ and $\mu_2 \leq \underline{\mu}(1)$ (region I_1 and I_2)

The consumer will never purchase the product if the firm advertises attribute one or does not advertise. In contrast, the consumer may purchase the product if the firm advertises on attribute two. The consumer will not purchase if attribute two is bad. However, if attribute two is good, the consumer will purchase the product immediately in the region I_2, and will search for information about attribute one in the region I_1. In the region I_1, the consumer will purchase the product after receiving enough positive information. So, the purchasing likelihood is strictly positive. Hence, the firm advertises attribute two.

(2) $\mu_1 \in (\underline{\mu}(1), \bar{\mu}(1)]$ and $\mu_2 > \underline{\mu}(1)$ (region I_3)

The purchasing probability is zero if the firm does not advertise, and is positive if the firm advertises either attriutes. Thus, we need to compare the purchasing likelihoods between advertising attribute one and attribute two.

$$
\begin{aligned}
P_1(\mu_1, \mu_2) = {} & \mu_1 \cdot \frac{\mu_2 - \underline{\mu}(1)}{\bar{\mu}(1) - \underline{\mu}(1)} \\
P_2(\mu_1, \mu_2) = {} & \mu_2 \cdot \frac{\mu_1 - \underline{\mu}(1)}{\bar{\mu}(1) - \underline{\mu}(1)} \overset{\mu_1 \geq \mu_2}{\geq} P_1(\mu_1, \mu_2),
\end{aligned}
$$

where the inequality is strict if $\mu_1 > \mu_2$. So, the firm advertises attribute two.

(3) $\mu_1 > \bar{\mu}(1)$ and $\mu_2 \in (\underline{\mu}(1), \bar{\mu}(\mu))$ (region I_4, the diagonally striped region, and the blank search region)

To characterize the advertising strategy, we need to determine two things. First, whether the firm wants to advertise. Second, whether the firm prefers advertising attribute one or two, conditional on advertising.

We first compare advertising attribute one and two.

$$
P_1(\mu_1, \mu_2) = \begin{cases} \mu_1 \cdot \frac{\mu_2 - \underline{\mu}(1)}{\bar{\mu}(1) - \underline{\mu}(1)}, \text{ if } \mu_2 < \bar{\mu}(1) \\ \mu_1, \text{ if } \mu_2 \geq \bar{\mu}(1) \end{cases}, \quad P_2(\mu_1, \mu_2) = \mu_2.
$$

If $\mu_2 \geq \bar{\mu}(1)$, then $P_1(\mu_1, \mu_2) \geq P_2(\mu_1, \mu_2)$. If $\mu_2 < \bar{\mu}(1)$, then

$$P_1(\mu_1, \mu_2) > P_2(\mu_1, \mu_2) \Leftrightarrow \frac{\mu_2 - \underline{\mu}(1)}{\bar{\mu}(1) - \underline{\mu}(1)} > \frac{\mu_2}{\mu_1} \Leftrightarrow \mu_2 > \frac{\underline{\mu}(1)\mu_1}{\mu_1 - \bar{\mu}(1) + \underline{\mu}(1)} := \tilde{\mu}(\mu_1)$$

Because $\tilde{\mu}(\mu_1) \leq \bar{\mu}(1)$, the firm prefers advertising attribute one to advertising attribute two if and only if $\mu_2 > \tilde{\mu}(\mu_1)$. One can see that $\tilde{\mu}(\mu_1)$ decreases in μ_1.

We then determine whether the firm wants to advertise or not. If the belief is below the quitting boundary, the firm always prefers advertising because the consumer will never purchase without advertising. Now suppose the belief is in the search region. According to Proposition 4, the purchasing likelihood without advertising is $P(\mu_1, \mu_2) = [\mu_2 - \underline{\mu}(\mu_1)]/[\bar{\mu}(\mu_1) - \underline{\mu}(\mu_1)]$.

If the firm advertises attribute one, the purchasing likelihood is:

$$P_1(\mu_1, \mu_2) = \begin{cases} \mu_1 \cdot \frac{\mu_2 - \underline{\mu}(1)}{\bar{\mu}(1) - \underline{\mu}(1)}, \text{ if } \mu_2 < \bar{\mu}(1) \\ \mu_1, \text{ if } \mu_2 \geq \bar{\mu}(1) \end{cases}$$

If the firm advertises attribute two, the purchasing likelihood is: $P_2(\mu_1, \mu_2) = \mu_2$.

Observe that $P(\mu_1, \underline{\mu}(\mu_1)) = 0$, $P(\mu_1, \bar{\mu}(\mu_1)) = 1$, $P_1(\mu_1, \underline{\mu}(1)) = 0$, $P_1(\mu_1, \bar{\mu}(1)) = \mu_1$, and $\underline{\mu}(1) \leq \underline{\mu}(\mu)$. By (quasi-) linearity of the purchasing likelihood, one can see that $P(\mu_1, \mu_2)$ crosses $\max\{P_1(\mu_1, \mu_2), P_2(\mu_1, \mu_2)\}$ once as μ_2 increases, fixing a μ_1. Hence, there exists $\widehat{\mu}(\mu_1) \in [\tilde{\mu}(\mu_1), \bar{\mu}(\mu_1))$ such that the firm does not advertise if and only if $\mu_2 \geq \widehat{\mu}(\mu_1)$.

The role of advertising follows from the consumer's strategy characterized in section 3.

□

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ONLINE APPENDIX

Proof of Proposition 6. One can see that the consumer will purchase the product for sure when $\mu_2 \geq \bar{\mu}(\mu_1)$ without advertising. So, the firm does not advertise when $\mu_2 \geq \bar{\mu}(\mu_1)$. Also, the consumer will not purchase the product when $\mu_1 \leq \underline{\mu}(1)$ if the firm does not advertise or advertises only one attribute, but will buy the product if the firm advertises both attributes and both turn out to be good. So, the firm advertises both attributes if $\mu_1 \leq \underline{\mu}(1)$. We now look at other cases.

(1) $\mu_1 > \underline{\mu}(1)$ and $\mu_2 \leq \underline{\mu}(1)$ (region I_1 and I_2)

Proposition 5 has shown that the firm prefers advertising attribute two to advertising attribute one or not advertising. Therefore, we just need to compare the expected profits from advertising attribute two and from advertising both attributes. For a given prior belief (μ_1, μ_2), denote the consumer's probability of purchasing if the firm advertises both attributes by $P_b(\mu_1, \mu_2)$.

$$P_b(\mu_1, \mu_2) = \mu_1 \cdot \mu_2,$$

$$P_2(\mu_1, \mu_2) = \begin{cases} \mu_2 \cdot \frac{\mu_1 - \underline{\mu}(1)}{\bar{\mu}(1) - \underline{\mu}(1)}, \ if \ \underline{\mu}(1) < \mu_1 < \bar{\mu}(1) \text{ (region } I_1) \\ \mu_2, \ if \ \mu_1 \geq \bar{\mu}(1) \text{ (region } I_2) \end{cases}.$$

In region I_1, one can see that there exists $\mu^b \in (\underline{\mu}(1), \bar{\mu}(1))$ such that $P_b(\mu_1, \mu_2) > P_2(\mu_1, \mu_2)$ if $\mu_1 < \mu^b$ and $P_b(\mu_1, \mu_2) < P_2(\mu_1, \mu_2)$ if $\mu_1 > \mu^b$. In region I_2, one can see that advertising attribute two strictly dominates advertising both attributes.

(2) $\mu_1 \in (\underline{\mu}(1), \bar{\mu}(1)]$ and $\mu_2 > \underline{\mu}(1)$ (region I_3)

Similar to the previous case, we only need to compare the expected profits from advertising attribute two and from advertising both attributes. The same argument implies that $P_b(\mu_1, \mu_2) > P_2(\mu_1, \mu_2)$ if $\mu_1 < \mu^b$ and $P_b(\mu_1, \mu_2) < P_2(\mu_1, \mu_2)$ if $\mu_1 > \mu^b$.

(3) $\mu_1 > \bar{\mu}(1)$ and $\mu_2 \in (\underline{\mu}(1), \bar{\mu}(\mu))$ (region I_4, the diagonally striped black region, and the blank search region)

Consider first region I_4. We just need to compare the expected profits from advertising attribute two and from advertising both attributes. Because $P_b(\mu_1, \mu_2) = \mu_1 \cdot \mu_2 < \mu_2 = P_2(\mu_1, \mu_2)$, advertising attribute two is still optimal in this case.

Consider then the diagonally striped black region. We just need to compare the expected profits from advertising attribute one and from advertising both attributes. Note that we have shown in Proposition 5 that advertising attribute one is better than advertising attribute two, $P_1(\mu_1, \mu_2) > P_2(\mu_1, \mu_2)$. Note also that $P_2(\mu_1, \mu_2) = \mu_2 > \mu_1 \cdot \mu_2 = P_b(\mu_1, \mu_2)$. So, $P_1(\mu_1, \mu_2) > P_2(\mu_1, \mu_2) > P_b(\mu_1, \mu_2)$.

Lastly, consider the blank search region. We just need to compare the expected profits from not advertising and from advertising both attributes. Note that we have shown in Proposition 5 that not advertising is better than advertising attribute one, $P(\mu_1, \mu_2) > P_1(\mu_1, \mu_2)$. Note also that $P_1(\mu_1, \mu_2) = \mu_1 > \mu_1 \cdot \mu_2 = P_b(\mu_1, \mu_2)$. So, $P(\mu_1, \mu_2) > P_1(\mu_1, \mu_2) > P_b(\mu_1, \mu_2)$.

The role of advertising follows from the consumer's strategy characterized in section 3.

□

Proof of Proposition 7. If $\mu_2 \leq \mu_1 \leq \underline{\mu}(1)$, then one can see that the consumer's belief after seeing an ad about either attribute will be below the quitting boundary. So, the firm does not advertise. Also, the consumer will purchase the product for sure if $\mu_2 \geq \bar{\mu}(\mu_1)$ without advertising. So, the firm does not advertise if $\mu_2 \geq \bar{\mu}(\mu_1)$. We now look at other cases.

(1) μ_2 less than but close to $\bar{\mu}_1$ (Belief near the purchasing boundary)

(a) We first compare advertising attribute two with not advertising.

Suppose the firm advertises attribute two. With probability $\mu_2\gamma + (1-\mu_2)(1-\gamma)$, the consumer observes a good signal. In this case, the purchasing probability is at most one. With probability $\mu_2(1-\gamma) + (1-\mu_2)\gamma \in (\min\{1-\mu_2, 1/2\}, \max\{1-\mu_2, 1/2\})$, the consumer observes a bad signal. The consumer's belief on attribute two given a bad signal is $\mu_2^b = (1-\gamma)\mu_2/[(1-\gamma)\mu_2 + \gamma(1-\mu_2)]$. The probability of not purchasing in this case is:

$$\mathbb{P}(\text{not purchasing}|s_2 = b) = \frac{\bar{\mu}(\mu_1) - \mu_2^b}{\bar{\mu}(\mu_1) - \underline{\mu}(\mu_1)} \wedge 1.$$

In sum, the probability of not purchasing if the firm advertises attribute two satisfies:

$$
\begin{aligned}
&\mathbb{P}(\text{not purchasing}|\text{advertise attribute 2})\\
=&\mathbb{P}(s_2=g)\mathbb{P}(\text{not purchasing}|s_2=g)+\mathbb{P}(s_2=b)\mathbb{P}(\text{not purchasing}|s_2=b)\\
\geq&\mathbb{P}(s_2=b)\mathbb{P}(\text{not purchasing}|s_2=b)\\
=&[\mu_2(1-\gamma)+(1-\mu_2)\gamma]\cdot\left[\frac{\bar{\mu}(\mu_1)-\mu_2^b}{\bar{\mu}(\mu_1)-\underline{\mu}(\mu_1)}\wedge 1\right].
\end{aligned}
$$

The probability of not purchasing if the firm does not advertise is:

$$
\mathbb{P}(\text{not purchasing}|\text{not advertise})=\frac{\bar{\mu}(\mu_1)-\mu_2}{\bar{\mu}(\mu_1)-\underline{\mu}(\mu_1)}.
$$

Because $\mu_2(1-\gamma)+(1-\mu_2)\gamma\geq\min\{1-\mu_2,1/2\}\geq\min\{1-\mu^{**},1/2\}:=K\in(0,1/2]$, a sufficient condition for $\mathbb{P}(\text{purchasing}|\text{advertise attribute 2})\leq\mathbb{P}(\text{purchasing}|\text{not advertise})\Leftrightarrow$ $\mathbb{P}(\text{not purchasing}|\text{advertise attribute 2})\geq\mathbb{P}(\text{not purchasing}|\text{not advertise})$ is:

$$
K\cdot\left[\frac{\bar{\mu}(\mu_1)-\mu_2^b}{\bar{\mu}(\mu_1)-\underline{\mu}(\mu_1)}\wedge 1\right]\geq\frac{\bar{\mu}(\mu_1)-\mu_2}{\bar{\mu}(\mu_1)-\underline{\mu}(\mu_1)} \tag{13}
$$

If $\frac{\bar{\mu}(\mu_1)-\mu_2^b}{\bar{\mu}(\mu_1)-\underline{\mu}(\mu_1)}\geq 1$, then inequality (13) is equivalent to:

$$
K\geq\frac{\bar{\mu}(\mu_1)-\mu_2}{\bar{\mu}(\mu_1)-\underline{\mu}(\mu_1)}\Leftrightarrow\mu_2\geq\bar{\mu}(\mu_1)-K[\bar{\mu}(\mu_1)-\underline{\mu}(\mu_1)].
$$

Let $\widehat{\mu}_2'(\mu_1)=\bar{\mu}(\mu_1)-K[\bar{\mu}(\mu_1)-\underline{\mu}(\mu_1)]$. One can see that $\widehat{\mu}_2'(\mu_1)<\bar{\mu}(\mu_1)$ and that the firm prefers not advertising to advertising attribute two if $\mu_2\geq\widehat{\mu}_2'(\mu_1)$.

If $\frac{\bar{\mu}(\mu_1)-\mu_2^b}{\bar{\mu}(\mu_1)-\underline{\mu}(\mu_1)}<1$, then inequality (13) is equivalent to:

$$
K\cdot\frac{\bar{\mu}(\mu_1)-\mu_2^b}{\bar{\mu}(\mu_1)-\underline{\mu}(\mu_1)}\geq\frac{\bar{\mu}(\mu_1)-\mu_2}{\bar{\mu}(\mu_1)-\underline{\mu}(\mu_1)}\Leftrightarrow\mu_2\geq\bar{\mu}(\mu_1)-\frac{1}{1+\frac{1-K}{K}\frac{(1-\gamma)\frac{\mu_2}{1-\mu_2}+\gamma}{2\gamma-1}}\bar{\mu}(\mu_1) \tag{14}
$$

Let $M(\mu_1) := [(1-\gamma)\bar{\mu}(\mu_1)/(1-\bar{\mu}(\mu_1)) + \gamma]/(2\gamma-1) > 0$. One can see that:

$$\begin{aligned}
&\bar{\mu}(\mu_1) > \mu_2 \Rightarrow M(\mu_1) > \frac{(1-\gamma)\frac{\mu_2}{1-\mu_2} + \gamma}{2\gamma - 1} \\
\Rightarrow &\bar{\mu}(\mu_1) - \frac{1}{1 + \frac{1-K}{K} M(\mu_1)} \bar{\mu}(\mu_1) > \bar{\mu}(\mu_1) - \frac{1}{1 + \frac{1-K}{K} \frac{(1-\gamma)\frac{\mu_2}{1-\mu_2} + \gamma}{2\gamma-1}} \bar{\mu}(\mu_1)
\end{aligned}$$

Let $\widehat{\mu}_2'(\mu_1) = \bar{\mu}(\mu_1) - \frac{1}{1+\frac{1-K}{K}M(\mu_1)}\bar{\mu}(\mu_1) < \bar{\mu}(\mu_1)$. A sufficient condition for inequality (14) and thus inequality (13) to hold is $\mu_2 \geq \widehat{\mu}_2'(\mu_1)$. So, the firm prefers not advertising to advertising attribute two if $\mu_2 \geq \widehat{\mu}_2'(\mu_1)$.

(b) We then compare advertising attribute one with not advertising.

Suppose the firm advertises attribute one. With probability $\mu_1\gamma + (1-\mu_1)(1-\gamma)$, the consumer observes a good signal. In this case, the purchasing probability is at most one. With probability $\mu_1(1-\gamma) + (1-\mu_1)\gamma \in (\min\{1-\mu_1, 1/2\}, \max\{1-\mu_1, 1/2\})$, the consumer observes a bad signal. The consumer's belief on attribute one given a bad signal is $\mu_1^b = (1-\gamma)\mu_1/[(1-\gamma)\mu_1 + \gamma(1-\mu_1)]$. We divide the problem into three cases.

i. $\mu_1^b > \mu_2$ and $\mu_1^b \geq \mu^{**}$.

$$\begin{aligned}
\mathbb{P}(\text{purchasing}|s_1 = b) = P(\mu_1^b, \mu_2) = &\frac{\mu_2 - \underline{\mu}(\mu_1^b)}{\bar{\mu}(\mu_1^b) - \underline{\mu}(\mu_1^b)} \vee 0 \\
\Rightarrow &\mathbb{P}(\text{not purchasing}|\text{advertise attribute 1}) \\
\geq &\mathbb{P}(\text{not purchasing}|s_1 = b)\mathbb{P}(s_1 = b) \\
= &[1 - \frac{\mu_2 - \underline{\mu}(\mu_1^b)}{\bar{\mu}(\mu_1^b) - \underline{\mu}(\mu_1^b)} \vee 0] \cdot [\mu_1(1-\gamma) + (1-\mu_1)\gamma] \\
= &[\frac{\bar{\mu}(\mu_1^b) - \mu_2}{\bar{\mu}(\mu_1^b) - \underline{\mu}(\mu_1^b)} \wedge 1] \cdot [\mu_1(1-\gamma) + (1-\mu_1)\gamma], \\
\mathbb{P}(\text{not purchasing}|\text{not advertise}) = &\frac{\bar{\mu}(\mu_1) - \mu_2}{\bar{\mu}(\mu_1) - \underline{\mu}(\mu_1)}.
\end{aligned}$$

A sufficient condition for $\mathbb{P}(\text{purchasing}|\text{advertise attribute 1}) \leq \mathbb{P}(\text{purchasing}|\text{not advertise}) \Leftrightarrow \mathbb{P}(\text{not purchasing}|\text{advertise attribute 1}) \geq \mathbb{P}(\text{not purchasing}|\text{not advertise})$ is:

$$[\frac{\bar{\mu}(\mu_1^b) - \mu_2}{\bar{\mu}(\mu_1^b) - \underline{\mu}(\mu_1^b)} \wedge 1] \cdot [\mu_1(1-\gamma) + (1-\mu_1)\gamma] \geq \frac{\bar{\mu}(\mu_1) - \mu_2}{\bar{\mu}(\mu_1) - \underline{\mu}(\mu_1)} \tag{15}$$

According to Proposition 2, $\mu_1^b < \mu_1$ implies that $\bar{\mu}(\mu_1^b) > \bar{\mu}(\mu_1)$. One can see that the left-hand side of inequality (15) is strictly positive, whereas the right-hand side of inequality (15) is zero when $\mu_2 = \bar{\mu}(\mu_1)$. By continuity of the expressions in (15), there exists $\widehat{\mu}_1'(\mu_1) < bar\mu(\mu_1)]$ such that inequality (15) holds for any $\mu_2 \in [\widehat{\mu}_1'(\mu_1), \bar{\mu}(\mu_1)]$. Hence, the firm prefers not advertising to advertising attribute one, when $\mu_2 \geq \widehat{\mu}_1'(\mu_1)$.

ii. $\mu_1^b > \mu_2$ and $\mu_1^b < \mu^{**}$.

$$\begin{aligned}
&\mathbb{P}(\text{purchasing}|s_1 = b) = P(\mu_1^b, \mu_2) = h(\mu_1^b, \mu_2)\tilde{P}(\mu_1^b) \\
\Rightarrow &\mathbb{P}(\text{not purchasing}|\text{advertise attribute 1}) \\
\geq &\mathbb{P}(\text{not purchasing}|s_1 = b)\mathbb{P}(s_1 = b) \\
= &[1 - h(\mu_1^b, \mu_2)\tilde{P}(\mu_1^b)] \cdot [\mu_1(1-\gamma) + (1-\mu_1)\gamma], \\
&\mathbb{P}(\text{not purchasing}|\text{not advertise}) = \frac{\bar{\mu}(\mu_1) - \mu_2}{\bar{\mu}(\mu_1) - \underline{\mu}(\mu_1)}.
\end{aligned}$$

A sufficient condition for $\mathbb{P}(\text{purchasing}|\text{advertise attribute 1}) \leq \mathbb{P}(\text{purchasing}|\text{not advertise}) \Leftrightarrow \mathbb{P}(\text{not purchasing}|\text{advertise attribute 1}) \geq \mathbb{P}(\text{not purchasing}|\text{not advertise})$ is:

$$\begin{aligned}
&[1 - h(\mu_1^b, \mu_2)\tilde{P}(\mu_1^b)] \cdot [\mu_1(1-\gamma) + (1-\mu_1)\gamma] \geq \frac{\bar{\mu}(\mu_1) - \mu_2}{\bar{\mu}(\mu_1) - \underline{\mu}(\mu_1)} \\
\Leftrightarrow &\mu_2 \geq \bar{\mu}(\mu_1) - [\bar{\mu}(\mu_1) - \underline{\mu}(\mu_1)] \cdot [1 - h(\mu_1^b, \mu_2)\tilde{P}(\mu_1^b)] \cdot [\mu_1(1-\gamma) + (1-\mu_1)\gamma] \quad (16)
\end{aligned}$$

Note that $h(\mu_1^b, \mu_2) \leq 1$, we have:

$$\begin{aligned}
&\bar{\mu}(\mu_1) - [\bar{\mu}(\mu_1) - \underline{\mu}(\mu_1)] \cdot [1 - \tilde{P}(\mu_1^b)] \cdot [\mu_1(1-\gamma) + (1-\mu_1)\gamma] \\
\geq &\bar{\mu}(\mu_1) - [\bar{\mu}(\mu_1) - \underline{\mu}(\mu_1)] \cdot [1 - h(\mu_1^b, \mu_2)\tilde{P}(\mu_1^b)] \cdot [\mu_1(1-\gamma) + (1-\mu_1)\gamma]
\end{aligned}$$

Let $\widehat{\mu}_1'(\mu_1) = \bar{\mu}(\mu_1) - [\bar{\mu}(\mu_1) - \underline{\mu}(\mu_1)] \cdot [1 - \tilde{P}(\mu_1^b)] \cdot [\mu_1(1-\gamma) + (1-\mu_1)\gamma]$. One can see that inequality (16) holds, and thus the firm prefers not advertising to advertising attribute one, when $\mu_2 \geq \widehat{\mu}_1'(\mu_1)$. Because $\tilde{P}(\mu_1^b) < 1$ when $\mu_1^b < \mu^{**}$, we have $\widehat{\mu}_1'(\mu_1) < \bar{\mu}(\mu_1)$.

iii. $\mu_1^b < \mu_2$.

$$\mathbb{P}(\text{purchasing}|s_1 = b) = P(\mu_1^b, \mu_2) \overset{\mu_1^b<\mu_2}{<} P(\mu_2, \mu_2) = h(\mu_2, \mu_2)\tilde{P}(\mu_2) = 1 \cdot e^{-\int_{\mu_2}^{\mu^{**}} \frac{2}{x-\underline{\mu}(x)} dx}$$

$$\Rightarrow \mathbb{P}(\text{not purchasing}|\text{advertise attribute 1})$$

$$\geq \mathbb{P}(\text{not purchasing}|s_1 = b)\mathbb{P}(s_1 = b)$$

$$> [1 - e^{-\int_{\mu_2}^{\mu^{**}} \frac{2}{x-\underline{\mu}(x)} dx}] \cdot [\mu_1(1-\gamma) + (1-\mu_1)\gamma],$$

$$\mathbb{P}(\text{not purchasing}|\text{not advertise}) = \frac{\bar{\mu}(\mu_1) - \mu_2}{\bar{\mu}(\mu_1) - \underline{\mu}(\mu_1)}.$$

A sufficient condition for $\mathbb{P}(\text{purchasing}|\text{advertise attribute 1}) \leq \mathbb{P}(\text{purchasing}|\text{not advertise}) \Leftrightarrow$ $\mathbb{P}(\text{not purchasing}|\text{advertise attribute 1}) \geq \mathbb{P}(\text{not purchasing}|\text{not advertise})$ is:

$$[1 - e^{-\int_{\mu_2}^{\mu^{**}} \frac{2}{x-\underline{\mu}(x)} dx}] \cdot [\mu_1(1-\gamma) + (1-\mu_1)\gamma] \geq \frac{\bar{\mu}(\mu_1) - \mu_2}{\bar{\mu}(\mu_1) - \underline{\mu}(\mu_1)} \tag{17}$$

Because $1 - e^{-\int_{\mu_2}^{\mu^{**}} \frac{2}{x-\underline{\mu}(x)} dx} > 1 - e^{-\int_{\bar{\mu}(\mu_1)}^{\mu^{**}} \frac{2}{x-\underline{\mu}(x)} dx}$, a sufficient condition for inequality (17) to hold is:

$$[1 - e^{-\int_{\bar{\mu}(\mu_1)}^{\mu^{**}} \frac{2}{x-\underline{\mu}(x)} dx}] \cdot [\mu_1(1-\gamma) + (1-\mu_1)\gamma] \geq \frac{\bar{\mu}(\mu_1) - \mu_2}{\bar{\mu}(\mu_1) - \underline{\mu}(\mu_1)}$$

$$\Leftrightarrow \mu_2 \geq \bar{\mu}(\mu_1) - [\bar{\mu}(\mu_1) - \underline{\mu}(\mu_1)] \cdot [1 - e^{-\int_{\bar{\mu}(\mu_1)}^{\mu^{**}} \frac{2}{x-\underline{\mu}(x)} dx}] \cdot [\mu_1(1-\gamma) + (1-\mu_1)\gamma].$$

Let $\widehat{\mu}_1'(\mu_1) = \bar{\mu}(\mu_1) - [\bar{\mu}(\mu_1) - \underline{\mu}(\mu_1)] \cdot [1 - e^{-\int_{\bar{\mu}(\mu_1)}^{\mu^{**}} \frac{2}{x-\underline{\mu}(x)} dx}] \cdot [\mu_1(1-\gamma) + (1-\mu_1)\gamma] <$ $\bar{\mu}(\mu_1)$. One can see that the firm prefers not advertising to advertising attribute one if $\mu_2 \geq \widehat{\mu}_1'(\mu_1)$.

Let $\widehat{\mu}'(\mu_1) := \max\{\widehat{\mu}_1'(\mu_1), \widehat{\mu}_2'(\mu_1)\}$. One can see that $\widehat{\mu}'(\mu_1) < \bar{\mu}(\mu_1)$ and that the firm does not advertise if $\widehat{\mu}'(\mu_1) \leq \mu_2 < \bar{\mu}(\mu_1)$.

(2) $\mu_1 > \underline{\mu}(1)$ and $\mu_2 \leq \underline{\mu}(1)$ (region I_1 and I_2)

The consumer will never purchase the product if the firm advertises attribute one or does not advertise. In contrast, the consumer may purchase the product if the firm advertises on attribute two. The consumer will not purchase if the signal is bad. So, we focus on the case in which the signal is good. Advertising attribute 2 is strictly better than not advertising if and only if the posterior belief is above the quitting boundary when the signal is good:

$$
\begin{aligned}
&\mathbb{P}(\text{attribute 2 is good}|s_2 = g) > \underline{\mu}(\mu_1) \\
\Leftrightarrow &\frac{\gamma\mu_2}{\gamma\mu_2 + (1-\gamma)(1-\mu_2)} > \underline{\mu}(\mu_1)
\end{aligned}
\tag{18}
$$

One can see that the left-hand side of equation (18) increases in γ and μ_2, and that the right-hand side of equation (18) decreases in μ_1. Notice that the inequality holds if $\gamma = 1$ and does not hold if $\gamma = 1/2$. Therefore, there exists $\hat{\gamma}(\mu_1, \mu_2) \in (1/2, 1)$, which is decreasing in μ_1 and μ_2, such that the firm advertises attribute two if $\gamma > \hat{\gamma}(\mu_1, \mu_2)$ and does not advertise if $\gamma < \hat{\gamma}(\mu_1, \mu_2)$.

□

Proof of Proposition 8. Consider first manipulating μ_1. By incurring cost $C(\Delta\mu)$, the firm can increase the initial belief about attribute one from μ_1 to $\mu_1 + \Delta\mu$. According to Proposition 5, it is optimal for the firm to advertise attribute two after belief manipulation. One can see that the purchasing probability does not increase in μ_1 once the belief reaches the boundary of region I_2. Thus, the firm never increases μ_1 above $\bar{\mu}(1)$, $\Delta\mu \in [0, \bar{\mu}(1) - \mu_1]$. In such cases, one can see that the purchasing probability given belief (μ_1, μ_2) is:

$$
P_2(\mu_1, \mu_2) = \mu_2 \cdot \frac{\mu_1 - \underline{\mu}(1)}{\bar{\mu}(1) - \underline{\mu}(1)}.
$$

Therefore, the marginal benefit of increasing a unit of μ_1 is $(p-m)\mu_2/[\bar{\mu}(1) - \underline{\mu}(1)]$. If $C'(\bar{\mu}(1) - \mu_1) > (p-m)\mu_2/[\bar{\mu}(1) - \underline{\mu}(1)]$, then the optimal Δp will be smaller than $\bar{\mu}(1) - \mu_1$ and thus the manipulated belief will be located in region I_1. If $C'(\bar{\mu}(1) - \mu_1) \leq (p-m)\mu_2/[\bar{\mu}(1) - \underline{\mu}(1)]$, then, the optimal Δp will be $\bar{\mu}(1) - \mu_1$ and thus the manipulated belief will be $(\bar{\mu}(1), \mu_2)$.[1]

Now consider manipulating μ_2. By incurring cost $C(\Delta\mu)$, the firm can increase the initial belief about attribute two from μ_2 to $\mu_2 + \Delta\mu$. According to Proposition 5 and by symmetry, it is optimal

[1] We are analyzing the optimal targeting strategy and manipulated belief *conditional on* manipulating μ_1. It may be optimal for the firm to manipulate μ_2 instead.

for the firm to advertise attribute one if the manipulated belief $\mu_2 + \Delta\mu < \tilde{\mu}^{-1}(\max\{\mu_1, \tilde{\mu}(1)\})$, and to advertise attribute two if the manipulated belief $\mu_2 + \Delta\mu > \tilde{\mu}^{-1}(\max\{\mu_1, \tilde{\mu}(1)\})$. Similar to the previous argument, the marginal benefit of increasing a unit of μ_2 is $(p-m)\mu_1/[\bar{\mu}(1) - \underline{\mu}(1)]$ if $\mu_2 + \Delta\mu \leq \bar{\mu}(1)$, 0 if $\mu_2 + \Delta\mu \in (\bar{\mu}(1), \tilde{\mu}^{-1}(\max\{\mu_1, \tilde{\mu}(1)\}))$, and $(p-m)(\mu_1 - \underline{\mu}(1))/[\bar{\mu}(1) - \underline{\mu}(1)]$ if $\mu_2 + \Delta\mu \geq \tilde{\mu}^{-1}(\max\{\mu_1, \tilde{\mu}(1)\})$. As a result, as long as the screening cost is not too low, $C'(\tilde{\mu}^{-1}(\max\{\mu_1, \tilde{\mu}(1)\}) - \mu_2) \geq \mu_1/[\bar{\mu}(1) - \underline{\mu}(1)]$, the firm never increases μ_2 above $\bar{\mu}(1)$.

If $C'(\bar{\mu}(1) - \mu_2) > \mu_1/[\bar{\mu}(1) - \underline{\mu}(1)]$, then the optimal Δp will be smaller than $\bar{\mu}(1) - \mu_2$ and thus the manipulated belief will be located in region I_1 or I_3. If $C'(\bar{\mu}(1) - \mu_2) \leq \mu_1/[\bar{\mu}(1) - \underline{\mu}(1)]$, then, the optimal Δp will be $\bar{\mu}(1) - \mu_2$ and thus the manipulated belief will be $(\mu_1, \bar{\mu}(1))$.[2]

Putting the above together, if the screening cost is high, $C'(\bar{\mu}(1) - \mu_1) > \mu_2/[\bar{\mu}(1) - \underline{\mu}(1)]$ and $C'(\bar{\mu}(1) - \mu_2) > \mu_1/[\bar{\mu}(1) - \underline{\mu}(1)]$ (which implies $C'(\tilde{\mu}^{-1}(\max\{\mu_1, \tilde{\mu}(1)\}) - \mu_2) \geq \mu_1/[\bar{\mu}(1) - \underline{\mu}(1)]$ because of the convexity of $C(\cdot)$), then the manipulated belief given the optimal targeting activity will be located in region I_1 or I_3. The role of optimal advertising is to invite consumers to search. If the screening cost is low, $C'(\bar{\mu}(1) - \mu_1) \leq \mu_2/[\bar{\mu}(1) - \underline{\mu}(1)]$ and $C'(\bar{\mu}(1) - \mu_2) \leq \mu_1/[\bar{\mu}(1) - \underline{\mu}(1)] \leq C'(\tilde{\mu}^{-1}(\max\{\mu_1, \tilde{\mu}(1)\}) - \mu_2)$, then the manipulated belief given the optimal targeting activity will be either $(\bar{\mu}(1), \mu_2)$ or $(\mu_1, \bar{\mu}(1))$. The role of optimal advertising is to invite consumers to purchase.

$\square$

[2] We are analyzing the optimal targeting strategy and manipulated belief *conditional on* manipulating μ_2. It may be optimal for the firm to manipulate μ_1 instead.