Non-stationary Pricing and Search

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September 2025

Abstract

Buyers often gather information gradually before making a purchasing decision. This paper introduces a novel framework where sellers adopt *non-stationary pricing strategies* – prices that evolve over time without being contingent on a buyer's current valuation. Such pricing strategies endogenously induce non-stationarity in the buyer's search problem. We show that non-stationary pricing strategies can outperform stationary ones. Given a sufficiently high initial valuation by the buyer, it is optimal for the seller to induce an immediate purchase by increasing the price as sharply as possible. When such fast-rising pricing strategies are not feasible, we consider slow-moving pricing strategies. Under zero search costs, a perfectly patient seller's optimal price is arbitrarily close to constant. In contrast, with discounting, the seller may benefit from charging non-stationary prices. When search costs are positive, the optimal price is non-stationary even if the seller is perfectly patient. The price increases over time if the information is too noisy or the search cost is too high. The direction of price trajectories is more nuanced in other cases where buyers have a stronger incentive to search, with increasing prices for buyers with high or low initial valuation and decreasing prices for medium-valuation buyers.

* We thank J. Miguel Villas-Boas, Tat Chan, Francisco Cisternas, Anthony Dukes, Yuichiro Kamada, Tony Ke, Song Lin, Eddie Ning, Jiwoong Shin, Quitzé Valenzuela-Stookey, and Jungju Yu for constructive comments. We also thank seminar participants at the Berkeley Econ theory lunch. All errors are our own. E-mail address: Wee Chaimanowong: chaimanowongw@gmail.com, Yunfei (Jesse) Yao: jesseyao@cuhk.edu.hk.

1 Introduction

Buyers often gather information gradually to reduce uncertainty about a product's value before making a purchasing decision. They might visit a seller's website, read reviews on the retailer's storefront, or search for review articles through search engines. However, these activities only partially resolve their uncertainty. Building on the seminal work of Weitzman (1979) and Wolinsky (1986), a rich literature has explored optimal search strategies across multiple alternatives or product attributes (Moscarini and Smith, 2001; Armstrong et al., 2009; Branco et al., 2012; Ke and Villas-Boas, 2019; Zhong, 2022) and their implications for sellers' strategies in information provision, pricing, product design, and advertising (Anderson and Renault, 2006; Villas-Boas, 2009; Bar-Isaac et al., 2010; Mayzlin and Shin, 2011; Chaimanowong and Ke, 2022). A critical dimension of these studies is pricing, with existing work often assuming exogenous prices, endogenous constant prices, or endogenous prices contingent on the buyers' current valuation. However, recent privacy regulations have disrupted sellers' ability to track individual buyers in real time, making it challenging to tailor prices to evolving buyer beliefs. Even if a seller can track buyers' browsing behavior, it may be hard for the seller to know how buyers will interpret the information they see. For example, Tesla may be able to observe that a buyer clicks on an image of the interior design of the car, but may not know whether the buyer prefers the large screen on Tesla or the traditional dashboard. This calls into question whether the seller can track the evolution of the buyer's valuation when the buyer is searching for information, and raises an important question: can sellers benefit from dynamic pricing when the evolution of the buyer's valuation is unobservable?

Without the ability to track the evolution of the buyer's valuation, the only stationary pricing strategy is a constant price. This paper introduces a novel framework where sellers adopt *non-stationary pricing strategies* - prices that evolve over time without being contingent on the buyer's current valuation. Such pricing strategies endogenously induce non-stationarity in the buyer's search problem. We address two key questions: (1) Is a stationary pricing strategy always optimal for a seller that cannot observe the evolution of a buyer's valuation? (2) If not, what are the characteristics of the optimal non-stationary pricing strategy?

Our findings challenge the conventional reliance on stationary pricing by showing that non-stationary pricing strategies can outperform stationary ones. Given a sufficiently high initial valuation by the buyer, it is optimal for the seller to induce an immediate purchase by increasing the price as sharply as possible. When such fast-rising pricing strategies are not feasible, we consider slow-moving pricing strategies. We prove that a buyer can do almost as well by approximating any sufficiently slow-moving price with a linear price if she is sufficiently myopic. Given this result, we show that, under zero search costs, a perfectly patient seller's optimal price is arbitrarily close to constant. By contrast, with discounting, the seller may benefit from charging non-stationary prices. When search costs are positive, the optimal price is non-stationary even if the seller is perfectly patient. The price increases over time if the information is too noisy or the search cost is too high. The direction of price trajectories is more nuanced in other cases where buyers have a stronger incentive to search, with increasing prices for buyers with high or low initial valuation and decreasing prices for buyers with a medium level of initial valuation.

By incorporating non-stationary strategies into a search framework, we provide a theoretical advance in optimal control. Unlike most economic models, which impose stationarity for tractability, our results highlight that such restrictions may lead to sub-optimal outcomes. While a few earlier papers have explored non-stationarity in search problems driven by exogenous environments, such as the finite horizon and the evolving distribution of rewards (Gilbert and Mosteller, 1966; Sakaguchi, 1978; Van den Berg, 1990; Smith, 1999; Kamada and Muto, 2015), we model endogenous non-stationarity arising from sellers' strategic pricing in response to buyer search. To our knowledge, this is the first paper to study endogenous non-stationary pricing under buyer gradual learning, providing a foundational step toward understanding sellers' non-stationary interventions in this context.

We also offer practical insights into how sellers can adapt to privacy regulations. Non-stationary pricing leverages time – a freely available and regulation-resistant resource – as an information source for pricing decisions, reducing reliance on costly tracking technologies. Our work provides guidance on how sellers can proactively adjust their pricing strategies to thrive in a privacy-conscious environment.

2 The Model

A seller offers a product with marginal cost g and sets its price. A buyer then decides whether to purchase the product. The buyer's initial valuation is v_0, which is common knowledge. Before making a purchase decision, the buyer can gradually learn about various product attributes and update her belief about the product's value. The buyer's discount rate is r and the seller's discount rate is m. We focus on the learning processes that arise within the general non-linear optimal filtering framework (Liptser and Shiryaev, 2013, Chapter 8).

Assume that the buyer's total utility from consuming the product is given by an unobservable process $\{\pi_t\}_{t\geq 0}$. To learn about π_t, the buyer pays a flow search cost of c per unit of time and observes a process $\{S_t\}_{t\geq 0}$, which generates a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ such that

$v_t := \mathbb{E}[\pi_t|\mathscr{F}_t]$ is a continuous martingale.[1] We denote the valuation process as $\{v_s^{t,x}\}_{s\geq t}$ when emphasizing the initial condition $v_t^{t,x} = x$, or as $\{v_t^x\}_{t\geq 0}$ when the initial condition $v_0^x = x$ is specified at $t = 0$. When the initial value is not of central importance, we simply write v_t. We assume that $\{v_s^{t,x}\}_{s\geq t}$ is the unique strong solution to:

$$dv_s^{t,x} = \mu(v_s^{t,x}, \pi_s)ds + \sigma(v_s^{t,x})dW_s, \qquad v_t^{t,x} = x, \tag{1}$$

where $\{W_t\}_{t\in\mathbb{R}_{\geq 0}}$ is the standard Brownian motion adapted to the filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in\mathbb{R}_{\geq 0}}, \mathbb{P})$. In particular, we have $\mathbb{E}[dv_t|\mathscr{F}_t] = 0$ and $\mathbb{E}[(dv_t)^2|\mathscr{F}_t] = \sigma(v_t)^2dt$. We impose the following assumptions on $\mu(.)$ and $\sigma(.)$:

Assumption 1. *Let $\underline{\pi}, \bar{\pi} \in \mathbb{R}\cup\{\pm\infty\}$ be such that $\underline{\pi} \leq v_t \leq \bar{\pi}$ a.e., for all $t \in \mathbb{R}$. We assume that:*

- $\mu(., \pi), \sigma(.) \in C^\infty(\underline{\pi}, \bar{\pi})$.
- $\sigma(x) > 0$ *for all* $x \in (\underline{\pi}, \bar{\pi})$.
- *The global Lipschitz condition holds for some constant $L \geq 0$, for all $t \in \mathbb{R}$ and $x, y \in [\underline{\pi}, \bar{\pi}]$:*
$$|\mu(x, \pi_t) - \mu(y, \pi_t)| + |\sigma(x) - \sigma(y)| \leq L|x - y|. \tag{2}$$

The constants $\underline{\pi}$ and $\bar{\pi}$ represent the highest and lowest possible values of the product. We allow for the possibility that $\bar{\pi} = +\infty$, $\underline{\pi} = -\infty$. Given $\mathbf{x} = (t, x) \in \mathbb{R}\times[\underline{\pi}, \bar{\pi}]$, we will sometimes write $\mu(\mathbf{x})$ and $\sigma(\mathbf{x})$ instead of $\mu(x)$ and $\sigma(x)$ when it is more convenient to use vector notation, even though $\mu(.)$ and $\sigma(.)$ do not explicitly depend on t.

The assumption that $\{v_t\}_{t\geq 0}$ is a strong solution to the SDE (1) implies that $\{v_t\}_{t\geq 0}$ is a square-integrable martingale, i.e., $\mathbb{E}[v_t^2] < \infty$ for all $t \geq 0$. The global Lipschitz condition plays a role of controlling the growth rate of the square integral. In particular, $\mathbb{E}[v_t^2] = O(e^{L^2t})$. The assumption serves various technical purposes, such as to ensure that the buyer's expected payoff is well-defined.

Previous work studying the seller's endogenous pricing strategy in the presence of buyer gradual learning assumes either that the seller perfectly observes the evolution of the buyer's valuation (v_t is common knowledge between the buyer and the seller) and can condition the price on the buyer's current valuation, or that the seller charges a constant price over time. Suppose we define the state variable by the buyer's current valuation, as is standard in the literature. The seller's problem in the first scenario is to choose the optimal *stationary*

[1] A sufficient condition for $\{v_t\}_{t\geq 0}$ to be martingale with respect to $\{\mathscr{F}_t\}_{t\geq 0}$ is that $\{\pi_t\}_{t\geq 0}$ is martingale with respect to the filtration generated by $\{(S_t, \pi_t)\}_{t\geq 0}$.

strategy because the strategy does not explicitly depend on the time, but only on the current state. This setup does not always fit real-world examples. Recent privacy regulations have disrupted sellers' ability to track individual buyers in real time. Even if a seller can track buyers' browsing behavior, it may be hard for the seller to know how buyers will interpret the information they see. Moreover, in many offline settings, individual-level tracking is not feasible.

Without the ability to observe the evolution of the buyer's valuation and thereby to tailor prices based on v_t, the only stationary pricing strategy is a constant price. Is a stationary pricing strategy always optimal for a seller in such cases? The major innovation of this paper is to consider *non-stationary pricing strategies* – prices that evolve over time without being contingent on a buyer's current valuation. Such pricing strategies endogenously induce non-stationarity in the buyer's search problem. Formally, the seller can commit to a pricing scheme $p := \{p_t\}_{t\geq 0} \in \mathcal{P}$, where $\mathcal{P}$ is the set of *admissible* pricing strategies, which is a subset of smooth functions on $[0,\infty)$, $\mathcal{P} \subset C^{\infty}[0,\infty)$.[2] This pricing strategy is a *non-stationary strategy* because p_t depends explicitly on time. It is widely known in optimal control that it is much harder to characterize *non-stationary strategies* than *stationary strategies*.

The buyer's search strategy consists of choosing an appropriate stopping time. Denote by $\mathcal{T}$ the set of all stopping times adapted to $\{\mathscr{F}_t\}_{t\in\mathbb{R}_{\geq 0}}$. The timing of the game is as follows.

1. At $t = 0$, the seller commits to a pricing strategy $p \in \mathcal{P} \subset C^{\infty}[0,\infty)$.
2. At any $t > 0$, the buyer decides whether to purchase the product, exit, or search for more information.
3. The game ends when the buyer makes a purchase or exits.

The only knowledge the seller has about the buyer is their initial valuation, v_0. Importantly, when the buyer decides whether to purchase the product, exit, or keep searching at any given time, she takes into account both the current price and the future price trajectory. For any $p \in \mathcal{P}$ and $\tau \in \mathcal{T}$, we define the buyer's and seller's expected payoffs as:

$$\mathcal{V}^B(t,x;\tau,p) := \mathbb{E}\left[e^{-r(\tau-t)}\max\{v_\tau^{t,x} - p_\tau, 0\} - \int_t^\tau ce^{-r(s-t)}ds \mid \mathscr{F}_t\right] \tag{3}$$

and

$$\mathcal{V}^S(x;\tau,p) := \mathbb{E}\left[e^{-m\tau}(p_\tau - g)\cdot 1_{v_\tau^x \geq p_\tau}\right]. \tag{4}$$

A sufficient condition for the buyer's expected payoff (3) to be well-defined is $\sup_{\tau\in\mathcal{T}}\mathbb{E}\left[e^{-2r\tau}v_\tau^2\right] < \infty$. This condition holds as long as the global Lipschitz constant L

[2] For simplicity, we use p to denote $\{p_t\}_{t\geq 0}$ whenever this does not cause confusion.

in (2) is less than $\sqrt{r}$. We will assume $L < \sqrt{r}$ for the remainder of this work. This assumption is satisfied in many applications that we consider, including when v_t is the standard Brownian motion ($L = 0$), when v_t is bounded (finite $\underline{\pi}, \bar{\pi}$), or when the buyer has a high discount rate.

Commitment Assumption

We assume that the seller has dynamic commitment power. Due to the hold-up problem, this would be a relatively strong assumption if the seller could track the evolution of the buyer's valuation. When new information (buyer's current valuation) arrives, the seller has an incentive to deviate from the pricing scheme announced at the beginning, in order to extract more surplus from the buyer. Anticipating the seller's incentive to deviate, the buyer will not start searching without a commitment device. In such cases, whether the seller has commitment power will lead to qualitatively different results.

In our setting, the seller does not receive new information about buyer valuation v_t over time. In addition, the only new information the seller can learn at time t is whether the buyer has made a purchasing decision, which will not affect the seller's strategy because the game ends whenever the buyer purchases the product or exits. The seller does not have any new information during the game. Therefore, there is no hold-up problem and the seller does not have an incentive to deviate from the announced pricing strategy. Thus, the commitment assumption does not qualitatively affect the equilibrium outcome in this case. We make the assumption mainly for a cleaner analysis and presentation.

Solution Concept

We consider the following equilibrium concept.

Definition 1. *A subgame perfect ε-equilibrium (ε-SPE) consists of* $(\{\tau^*[p] \in \mathcal{T}\}_{p\in\mathcal{P}}, p^* \in \mathcal{P})$ *such that, for all* $p \in \mathcal{P}$,

$$\mathcal{V}^B(t, x; \tau^*[p], p) \geq \mathcal{V}^B(t, x; \tau, p) - \varepsilon, \quad \forall \tau \in \mathcal{T},$$
$$\textit{and} \;\; \mathcal{V}^S(x; \tau^*[p^*], p^*) \geq \mathcal{V}^S(x; \tau^*[p], p) - \varepsilon, \quad \forall p \in \mathcal{P}.$$

The buyer's value function given the seller's pricing strategy p is:

$$V^B(t, x; p) := \sup_{\tau\in\mathcal{T}} \mathcal{V}^B(t, x; \tau, p). \tag{5}$$

When there is no ambiguity, we will compactly write $V^B(t, x) = V^B(t, x; p)$. Analogously, we define the seller's value function as:

$$V^S(x) := \sup_{p\in\mathcal{P}} \mathcal{V}^S(x;\tau^*[p],p) \tag{6}$$

We work with subgame perfect ε-equilibrium rather than the usual subgame perfect equilibrium for tractability reasons. Specifically, it is challenging to characterize the buyer's search strategy given any non-stationary pricing strategies by the seller. To deal with this technical difficulty, a key idea of this paper is that, if a pricing strategy p is a small perturbation from a pricing strategy with a known solution, then the solution to p could also be a small perturbation from the known solution. The use of sub-game perfect ε-equilibrium allows us to formalize this idea using perturbation theory to the order of ε. The choice of ε can be very close to zero and is not the economic force behind the results.

3 Buyer's Strategy

The buyer faces an optimal stopping problem. She needs to determine the purchasing and quitting boundaries at any time. When the price is non-stationary, the buyer's purchasing and quitting boundaries are also time-contingent. This time-varying property makes her optimal stopping problem challenging even if we fix a pricing scheme. To illustrate the impact of non-stationary pricing on the buyer's problem, we first review the benchmark with constant price.

3.1 Benchmark: Stationary Pricing

When the price is constant, $p_t = p_0 \in \mathbb{R}$, the buyer's search strategy does not depend on time. In particular, we have a time-independent value function $V^B(t,x;p_0) = V_0^B(x;p_0)$, purchasing threshold $\bar{V}_t = \bar{V}[p_0]$, and quitting threshold $\underline{V}_t = \underline{V}[p_0]$. The value function of the buyer satisfies the Hamilton-Jacobi-Bellman (HJB) equation:

$$\frac{\sigma(x)^2}{2}\partial_x^2 V_0^B - rV_0^B - c = 0, \tag{7}$$

subject to the value-matching condition and smooth pasting conditions:

$$\begin{aligned} V_0^B(\bar{V}[p_0];p_0) = \bar{V}[p_0] - p_0, \quad \partial_x V_0^B(\bar{V}[p_0];p_0) = 1, \\ V_0^B(\underline{V}[p_0];p_0) = 0, \quad \partial_x V_0^B(\underline{V}[p_0];p_0) = 0. \end{aligned}$$

We refer to Strulovici and Szydlowski (2015) for the derivation of the free-boundary ODE problem from the optimal stopping problem along with the results, which guarantees the existence and uniqueness of the solution in our setting.[3] In particular, V_0^B is continuously differentiable for all $x \in \mathbb{R}$ and twice continuously differentiable for all $x \in \mathbb{R} \setminus \{\underline{V}[p_0], \bar{V}[p_0]\}$.

We now consider two specific learning structures commonly used in the literature.

3.1.1 Product attributes learning

We first consider the learning process studied in Branco et al. (2012), where a buyer gradually learns about various product attributes to update her belief about the product's value before making a purchase decision. Each attribute i has a ground-truth utility of x_i. The product's total expected utility relative to the outside option (which we normalized to zero), given t searched attributes, is $\pi_t := \sum_{i=0}^{t} x_i$. When there are an infinite number of attributes, each with a very small weight in the valuation, π_t becomes a Brownian motion: $d\pi_t = \sigma dW_t^\pi$, for some constant σ. An alternative Bayesian learning interpretation (see Ke et al. (2022)) is that the ground-truth value π_t evolves over time according to $d\pi_t = \sigma dW_t^\pi$, and the buyer learns about $\{\pi_t\}_{t\geq 0}$ by observing the signal $\{S_t\}_{t\geq 0}$, where $dS_t := \pi_t dt + \sigma_S dW_t$. The constant σ_S represents the information quality of the signal or the amount of attention the buyer pays. This case differs from typical Bayesian learning settings, where one expects the variance to decrease over time; here, the ground-truth volatility σ and the observation noise σ_S together give the variance an asymptote of $\sigma\sigma_S > 0$. Assuming the normal prior belief $\pi_0 \sim \mathcal{N}(v_0, \sigma\sigma_S)$, then the variance is constant over time and we have a constant volatility process $dv_t = \frac{\sigma}{\sigma_S}(\pi_t - v_t)dt + \sigma dW_t$, or simply:

$$dv_t = \sigma dW_t^v \tag{8}$$

by Lévy characterization, where $\{W_t^v\}_{t\geq 0}$ is a standard Brownian motion adapted to $\{\mathscr{F}_t\}_{t\geq 0}$. We write W_t^v as W_t hereinafter for convenience. In this example, $\underline{\pi} = -\infty$ and $\bar{\pi} = +\infty$. Branco et al. (2012) have characterized closed-form expressions for the value function:

$$V_0^B(x; p_0) = \frac{c}{r}\left[\cosh \frac{\sqrt{2r}}{\sigma}\left(x - \underline{V} - p_0\right) - 1\right], \tag{9}$$

[3] Unlike in Strulovici and Szydlowski (2015) our $\mu(.)$ also depends on another unobservable process $\{\pi_t\}_{t\geq 0}$. One work-around is to consider the process $X_t := (v_t, \pi_t)$ that generates the full information filtration $\{\mathscr{F}_t^X\}_{t\geq 0}$, following Strulovici and Szydlowski (2015), then take another expectation condition on $\{\mathscr{F}_t\}_{t\geq 0}$ at the end. Such detail is not critical in this section because we will merely consider a few benchmark examples for motivation; later, we will study the general version of this problem rigorously via the viscosity solutions framework.

and for the purchasing and quitting boundaries $\bar{V}[p_0] := p_0 + \bar{V}$, $\underline{V}[p_0] := p_0 + \underline{V}$, where:

$$\bar{V} := \sqrt{\frac{c^2}{r^2} + \frac{\sigma^2}{2r}} - \frac{c}{r}, \ \underline{V} := \left(\sqrt{\frac{c^2}{r^2} + \frac{\sigma^2}{2r}} - \frac{c}{r} \right) - \frac{\sigma}{\sqrt{2r}} \log \left(\sqrt{\frac{r\sigma^2}{2c^2}} + \sqrt{1 + \frac{r\sigma^2}{2c^2}} \right). \quad (10)$$

3.1.2 Binary classification

We then consider the learning process of classifying the product's ground-truth value, which is a time-independent binary random variable $\pi_t = \pi \in \{0, 1\}$. A Bayesian decision-maker makes a purchase decision by learning whether the product has a *high* value ($\pi = \bar{\pi} = 1$) or a *low* value ($\pi = \underline{\pi} = 0$). Given the initial expectation $v_0 = \mathbb{E}[\pi|\mathscr{F}_0] \in [0, 1]$, the buyer can further learn the value of π by observing the signal $\{S_t\}_{t\geq 0}$, where $dS_t := \pi dt + \sigma_S dW_t$. Then, the valuation $v_t = \mathbb{E}[\pi|\mathscr{F}_t]$ is updated according to:

$$dv_t = \frac{v_t(1-v_t)}{\sigma_S^2} \left[(\pi - v_t)dt + \sigma_S dW_t \right]. \quad (11)$$

The resulting free-boundary ODE problem has been considered in Ke and Villas-Boas (2019) in the non-discounting case: $r = 0$. In the context of our work, the solution with positive discounting $r > 0$ is more relevant, and is given as follows:

$$V_0^B(x; p_0) = A_+ x^{m_+}(1-x)^{m_-} + A_- x^{m_-}(1-x)^{m_+} - \frac{c}{r} \quad (12)$$

where $m_\pm := \frac{1 \pm \sqrt{1+8r\sigma_S^2}}{2}$, $A_\pm := \frac{\bar{V}[p_0](1-\bar{V}[p_0]) + (\bar{V}[p_0] - p_0 + c/r)(\bar{V}[p_0] - m_\mp)}{(m_\pm - m_\mp)\bar{V}[p_0]^{m_\pm}(1-\bar{V}[p_0])^{m_\mp}}$, and $\bar{V}[p_0]$, $\underline{V}[p_0]$ can be solved from:

$$\frac{\bar{V}[p_0](1-\bar{V}[p_0]) + (\bar{V}[p_0] - p_0 + c/r)(\bar{V}[p_0] - m_\mp)}{\bar{V}[p_0]^{m_\pm}(1-\bar{V}[p_0])^{m_\mp}} = \frac{(c/r)(\underline{V}[p_0] - m_\mp)}{\underline{V}[p_0]^{m_\pm}(1-\underline{V}[p_0])^{m_\mp}} \quad (13)$$

3.1.3 Comparison Between Our Problem and the Benchmarks

Comparing the benchmarks with our problem, we can see that stationarity simplifies the problem significantly. In the benchmark model, the buyer's entire optimal stopping strategy can be summarized by **two unknowns**: $\bar{V}[p_0]$ and $\underline{V}[p_0]$. The buyer will purchase the product at any time during the search if her valuation reaches the purchasing threshold and will quit searching at any time if her valuation reaches the quitting threshold. In contrast, the buyer's entire optimal stopping strategy consists of **an infinite number of unknowns**. The buyer's decision at any time depends on the current price and the future trajectory of the prices. Knowing that prices change over time, the buyer's purchasing and quitting thresholds also evolve. These time-dependent thresholds significantly complicate our problem.

3.2 Buyer's Strategy under Non-Stationary Pricing

The set of admissible pricing strategies that we consider is:

$$\mathcal{P} := \left\{ p \in C^{\infty}[0,\infty) \mid p_t' + r(\bar{\pi} - p_t) + c > 0, p_t > \underline{\pi}, \text{ for all } t \in [0,\infty) \right\}.$$

The conditions on p_t and p_t' ensure that it is optimal for a buyer with $v_t = \bar{\pi}$ or $v_t = \underline{\pi}$ either to immediately make a purchase or to quit. Note that any constant pricing policy $p_0 \in [\underline{\pi}, \bar{\pi}]$ is contained in $\mathcal{P}$, and that the condition on p_t' also controls the price growth rate, i.e., $\lim_{t\to\infty} e^{-rt} p_t = 0$. For a learning process such that $\bar{\pi} = +\infty$, $\underline{\pi} = -\infty$, all the conditions are trivial; therefore, we can take $\mathcal{P} := C^{\infty}[0,\infty)$. We will also work with the subset $\mathcal{P}_T \subset \mathcal{P}$ of strategies that are constant after some amount of time $T > 0$:

$$\mathcal{P}_T := \left\{ p \in \mathcal{P} \mid p_t = p_T, \forall t \geq T \right\},$$

which is helpful in some existence and uniqueness arguments. The idea is to first establish technical results for a finite $T > 0$, and then to take the limit $T \to \infty$ to establish the result for strategies in $\mathcal{P}$. We start with the following intuitive characterization of $V^B(t, x; p)$:

Lemma 1. *Let $p \in \mathcal{P}_T$ be a pricing strategy.*

1. *$V^B(t, x; p)$ is monotonically increasing in x for any fixed t. Moreover, if $V^B(t, x; p) > 0$, then we have a strict inequality: $V^B(t, x'; p) > V^B(t, x; p)$ for any $x' > x$.*

2. *Let $q \in \mathcal{P}_T$ be another pricing strategy such that $q_t \leq p_t$ for all $t \in \mathbb{R}$; then, $V^B(t, x; q) \leq V^B(t, x; p)$ for all fixed $(t, x) \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}]$. Moreover, if $q_t < p_t$ for all $t > 0$, and $V^B(t, x; q) > 0$ for any fixed $(t, x) \in \mathbb{R}_{\geq 0} \times [\underline{\pi}, \bar{\pi}]$, then we have the strict inequality: $V^B(t, x; q) < V^B(t, x; p)$.*

Instead of directly finding the optimal $\tau^*[p] \in \mathcal{T}$ to the optimization problem (5), it is often more convenient to characterize the learning strategy in terms of the *moving* purchasing and quitting thresholds given by a pair of continuously differentiable functions $\bar{V}[p], \underline{V}[p] : \mathbb{R} \to [\underline{\pi}, \bar{\pi}]$ satisfying $\bar{V}_t[p] \geq \underline{V}_t[p]$. The following results characterize some properties of the purchasing and quitting boundaries.

Proposition 1. *Let $p \in \mathcal{P}_T$ be a pricing strategy and let $h : \mathbb{R} \to \mathbb{R}_{\geq 0} \in \mathcal{P}_T$ be strictly increasing over $[0, T)$.*

1. *Suppose that $h_0 = 0$; then, at $t = 0$, the purchasing and quitting boundaries under the pricing strategy $\tilde{p} := p + h$ satisfy $\bar{V}_0[\tilde{p}] < \bar{V}_0[p]$, and $\underline{V}_0[\tilde{p}] \geq \underline{V}_0[p]$.*

2. *Let $K \in \mathbb{R}$ be a constant; then, under the pricing strategy $\tilde{p} := p + Kh$, we have:*

$$\begin{aligned} \bar{V}_t[\tilde{p}] \searrow \max\{\tilde{p}_t, \underline{\pi}\}, \underline{V}_t[\tilde{p}] \nearrow \min\{\tilde{p}_t, \bar{\pi}\} \quad & \text{as } K \to +\infty \\ \bar{V}_t[\tilde{p}] \nearrow \bar{\pi}, \underline{V}_t[\tilde{p}] \searrow \underline{\pi} \quad & \text{as } K \to -\infty \end{aligned}$$

   for any given $t \in [0, T)$.

The first part of Proposition 1 implies that, if $\tilde{p} := p + h$ and $h : \mathbb{R} \to \mathbb{R}_{\leq 0} \in \mathcal{P}_T$ is strictly monotonically decreasing, then $\bar{V}_0[\tilde{p}] \geq \bar{V}_0[p]$, and $\underline{V}_0[\tilde{p}] < \underline{V}_0[p]$. Note that the inequality for the quitting boundary may not be strict; for example, when the search cost is zero, we can have $\underline{V}_t[p] = \underline{\pi}$ for all $p \in \mathcal{P}_T$. However, it is not the case that $\bar{V}_t[\tilde{p}] \leq \bar{V}_t[p]$ and $\underline{V}_t[\tilde{p}] \geq \underline{V}_t[p]$ for all $t \in \mathbb{R}$. Intuitively, although an increasing price has an immediate effect in inducing a high-valuation buyer to purchase instead of searching, such an effect is short-term. If the price continues to increase, then the buyer will eventually demand a higher valuation to trigger the purchasing decision.

For a given $p \in \mathcal{P}_T$, the thresholds $\bar{V}[p], \underline{V}[p] : \mathbb{R} \to [\underline{\pi}, \bar{\pi}]$, along with the value function function $V^B(.,.;p)$, can be determined by solving the corresponding free-boundary backward parabolic PDE initial-value problem: Find $V : \Omega \to \mathbb{R}$, and continuously differentiable functions $\bar{V}[p], \underline{V}[p] : \mathbb{R} \to [\underline{\pi}, \bar{\pi}]$ satisfying $\bar{V}_t[p] \geq \underline{V}_t[p]$, such that

$$\begin{cases} \frac{\sigma(x)^2}{2}\partial_x^2 V(t,x) + \partial_t V(t,x) - rV(t,x) - c = 0, & (t,x) \in \Omega \\ V(t, \bar{V}_t[p]) = \bar{V}_t[p] - p_t, & V(t, \underline{V}_t[p]) = 0, \\ \partial_x V(t, \bar{V}_t[p]) = 1, & \partial_x V(t, \underline{V}_t[p]) = 0, \\ V(T, x) = V_0^B(x; p_T), \end{cases} \tag{14}$$

where

$$\Omega := \{(t,x) \in [0,T] \times [\underline{\pi}, \bar{\pi}] \mid \underline{V}_t[p] < x < \bar{V}_t[p]\}.$$

This PDE connects us back to the constant price benchmark in §3.1, except that now we have the *moving* purchasing and quitting boundaries $\bar{V}[p]$ and $\underline{V}[p]$ instead of the fixed counterparts that we saw in §3.1. The second and third lines of (14) amount to the value-matching and the smooth-pasting conditions at the purchasing and quitting boundaries, respectively. It can be shown that, if V satisfies the free-boundary backward parabolic PDE initial-value problem (14) with the pricing policy $p \in \mathcal{P}_T$, such that $V(t,x) \geq \max\{x - p_t, 0\}$, and $p'_t + r(\bar{V}_t[p] - p_t) + c \geq 0$ for all $(t,x) \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}]$, then V coincides with the buyer's value function: $V^B = V$ (see Lemma 6 in the Online Appendix).

Solving (14) in full generality is beyond the scope of this research. For an arbitrary given pricing policy $p \in \mathcal{P}_T$, there may not exist an analytical closed-form solution. However, if p is a small perturbation from a policy with a known solution, then we expect the solution corresponding to p to be a small perturbation from the known solution. The PDE formulation of the problem enables us to employ the perturbation theory.

Suppose we know that the value function $V^B(.,.;p)$ for a given $p \in \mathcal{P}_T$ is a solution to (14), and we would like to compute $V^B(.,.;p+\sqrt{\varepsilon}h)$ for some $h \in \mathcal{P}_T$ and a small $\varepsilon > 0$. The idea of perturbation theory is to proceed by writing $V^B(.,.;p+\sqrt{\varepsilon}h) = V_0(.,.) + V_1(.,.)\sqrt{\varepsilon} + V_2(.,.)\varepsilon + \cdots$, where $V_0(.,.) := V^B(.,.;p)$, and $\bar{V}_t[p+\sqrt{\varepsilon}h] = \bar{V}_{0,t} + \bar{V}_{1,t}\sqrt{\varepsilon} + \bar{V}_{2,t}\varepsilon + \cdots$, $\underline{V}_t[p+\sqrt{\varepsilon}h] = \underline{V}_{0,t} + \underline{V}_{1,t}\sqrt{\varepsilon} + \underline{V}_{2,t}\varepsilon + \cdots$, where $\bar{V}_{0,t} := \bar{V}_t[p], \underline{V}_{0,t} := \underline{V}_t[p]$. By substituting these expansions into (14) and comparing the $\varepsilon^{k/2}$ terms for $k = 1, 2, \cdots$, we can solve for $V_k, \bar{V}_k, \underline{V}_k$ because we know the value of $V_{k'}, \bar{V}_{k'}, \underline{V}_{k'}$ for $k' = 0, \cdots, k-1$. In the Online Appendix (Lemma 6), we argue that such a technique is valid for all sufficiently small $\varepsilon > 0$. We shall assume this to be the case for the remainder of the work.

We refer readers to the Online Appendix for a more thorough technical discussion of the connection between the problems (5) and (14), as well as further discussion of the validity of the perturbation technique. We will apply the perturbation technique to solve (14) only up to the ε-order, to be consistent with the ε-equilibrium concept. In other words, we can stop at $k = 1$ in the process described, and we will have

$$V^B(t,x;p+\sqrt{\varepsilon}h) = V^B(t,x;p) + V_1(t,x)\sqrt{\varepsilon} + O(\varepsilon).$$

Then, we can take the buyer's learning policy to be given by the corresponding boundaries $\bar{V}^{\varepsilon}_{\leq 1}[p+\sqrt{\varepsilon}h] = \bar{V}[p] + \bar{V}_1\sqrt{\varepsilon}$, and $\underline{V}^{\varepsilon}_{\leq 1}[p+\sqrt{\varepsilon}h] = \underline{V}[p] + \underline{V}_1\sqrt{\varepsilon}$. The following proposition provides a characterization of the $\sqrt{\varepsilon}$-order perturbed boundaries in terms of the zero-th order solution.

Proposition 2. *Let $p \in \mathcal{P}_T$ be a given pricing strategy such that the buyer's value function $V^B(.,.;p)$ is a solution to the PDE (14), which is C^∞-smooth on $\Omega := \{(t,x) \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}] \mid \underline{V}_t[p] < x < \bar{V}_t[p]\}$, with C^∞-smooth corresponding purchasing and quitting boundaries $\bar{V}[p], \underline{V}[p] : \mathbb{R} \to (\underline{\pi}, \bar{\pi})$.*[4] *Let $h \in \mathcal{P}_T$ be arbitrary; then, under the pricing*

[4] We impose the C^∞-smoothness assumptions for simplicity, though they are not necessary conditions. The results hold as long as the classical solution to the $\sqrt{\varepsilon}$-order PDE boundary-value problem $V_1(t,x) \in C^{1,2}(\Omega)$ exists.

strategy $\tilde{p} := p + \sqrt{\varepsilon} h$, we can find an ε-optimal value function taking the form:

$$V^B(t, x; \tilde{p}) = V^B(t, x - \sqrt{\varepsilon} h_t; p) + \sqrt{\varepsilon} V_1^B(t, x) + O(\varepsilon), \tag{15}$$

where $V_1^B(.,.) : \Omega \to \mathbb{R}$ is given by:

$$\begin{aligned} V_1^B(t, x) = -\mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}} h'_s e^{-r(s-t)} \partial_x V^B(s, v_s^{t,x}; p) ds | \mathscr{F}_t\right] \\ + \mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}} h_s e^{-r(s-t)} \sigma(v_s^{t,x}) \sigma'(v_s^{t,x}) \partial_x^2 V^B(s, v_s^{t,x}; p) ds | \mathscr{F}_t\right], \end{aligned} \tag{16}$$

where $\tau_\Omega^{t,x} := \inf\{t' \geq t \mid (t', v_{t'}^{t,x}) \notin \Omega\}$ is the exit time. We can find the ε-optimal purchasing and quitting boundaries taking the form:

$$\begin{aligned} \bar{V}[\tilde{p}] &= (\bar{V}[p] + \sqrt{\varepsilon} h) + \sqrt{\varepsilon} \bar{R} + O(\varepsilon) \\ \underline{V}[\tilde{p}] &= (\underline{V}[p] + \sqrt{\varepsilon} h) + \sqrt{\varepsilon} \underline{R} + O(\varepsilon) \end{aligned} \tag{17}$$

for functions $\bar{R} : \mathbb{R} \to \mathbb{R}$, and $\underline{R} : \mathbb{R} \to \mathbb{R}$ given by:

$$\bar{R}_t = -\frac{\partial_x V_1^B(t, \bar{V}_t[p])}{\partial_x^2 V^B(t, \bar{V}_t[p]; p)}, \qquad \underline{R}_t = -\frac{\partial_x V_1^B(t, \underline{V}_t[p])}{\partial_x^2 V^B(t, \underline{V}_t[p]; p)} \tag{18}$$

Corollary 1. *Under the setting of Proposition 2, if $\sigma'(.) = O(\varepsilon)$ (stable volatility), and if $h := K\tilde{h}$ for some monotonically increasing $\tilde{h} \in \mathcal{P}_T$ and a constant $K \in \mathbb{R} \setminus \{0\}$, then $\bar{S}_t := \bar{R}_t / K \leq 0$ and $\underline{S}_t := \underline{R}_t / K \geq 0$ for all $t \in \mathbb{R}$.*

Lemma 6 in the Online Appendix shows that an $\sqrt{\varepsilon}$-order change in p will result in an $\sqrt{\varepsilon}$-order change in the value of V, and in the boundaries $\bar{V}[p], \underline{V}[p]$. The following result gives a more concrete upper bound:

Lemma 2. *Let $p, q \in \mathcal{P}$; then, $|V^B(t, x; p) - V^B(t, x; q)| \leq \max_{s \geq t} e^{-r(s-t)} |p_s - q_s|$ for all $(t, x) \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}]$.*

Lemma 2 shows that, for buyers who are not perfectly patient, any changes in price in the far future do not have much effect in the present. This enables us to extend our buyer response results for $p \in \mathcal{P}_T$ to an arbitrary $p \in \mathcal{P}$. Note that, by definition, any $p \in \mathcal{P}$ satisfies the asymptotic condition, $\lim_{t \to \infty} e^{-rt} p_t = 0$. Let p^T be given by p over $[0, T - \varepsilon]$, constant for all $t \geq T$, and some in-between smooth transition for $t \in (T - \varepsilon, T)$. We find the solution $V(.,.; p^T)$ of the free-boundary PDE initial value problem (14) corresponding to

$p^T \in \mathcal{P}_T$, which coincides with the value function $V^B(.,.;p^T)$ according to Lemma 6. Then, for all sufficiently large T, we have

$$\left|V\left(t,x;p^T\right) - V^B\left(t,x;p\right)\right| = \left|V^B\left(t,x;p^T\right) - V^B\left(t,x;p\right)\right| < \varepsilon \tag{19}$$

for an arbitrarily given $\varepsilon > 0$. This proves that the sequence of the solutions $\left\{V(.,.;p^T)\right\}_{T\geq 0}$ uniformly converges to the value function $V^B(.,.;p)$ of an infinite horizon pricing strategy p on any compact subset of $\mathbb{R} \times [\underline{\pi}, \bar{\pi}]$.

In our work, we pay special attention to pricing policies that are linear in time. Such linear pricing p does not belong to $\mathcal{P}_T$ for any $T > 0$. However, this is not a problem according to Lemma 2. By choosing a sufficiently large T, an ε-optimal buyer will not differentiate between p and $p^T \in \mathcal{P}_T$. This enables us to utilize the theory we have developed so far for $\mathcal{P}_T$ on linear pricing.

As it turns out, when p is linear in t, the buyer's value function admits an analytic closed-form under some simple learning settings, such as when $\{v_t\}_{t\geq 0}$ is a vanilla Brownian motion. It is also simpler to analyze the seller's strategies when they are restricted to the space of linear pricing. The fact that the linear pricing space is much smaller than the general pricing space also simplifies the problem, especially when searching for the seller's optimal pricing strategy, which we will do in §4.

Consideration of linear pricing may seem restrictive. However, the following proposition, which is an application of Lemma 2, shows that, for myopic enough ε-optimal buyers, any pricing strategy that is sufficiently slow-moving can be approximated by linear pricing. Intuitively, unless the price changes very drastically in the far future, such as growing super-exponentially, myopic buyers do not look too far into the future, and any differentiable functions *look like* a linear function over any sufficiently short time interval.

Proposition 3. *(Near-optimality of linear price approximation) Let $p \in \mathcal{P}$ be an admissible pricing policy with $\sup_{t\in\mathbb{R}} |p''_t| \leq M$. At any $\mathbf{x} = (t,x) \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}]$, we consider the linear approximation pricing policy $l_{\mathbf{x}} \in \mathcal{P} : s \mapsto l_{\mathbf{x},s} := p_t + p'_t \cdot (s-t)$.[5] Let the buyer's optimal learning strategy given the linear pricing $l_{\mathbf{x}}$ be $\tau^*[l_{\mathbf{x}}] \in \mathcal{T}$. If the buyer is sufficiently myopic: $r > e^{-1}\sqrt{2M/\varepsilon}$, then $\tau^*[l_{\mathbf{x}}]$ is also the buyer's ε-optimal stopping time under the p pricing strategy:*

$$\mathcal{V}^B(t,x;\tau^*[l_{\mathbf{x}}];p) \geq V^B(t,x;p) - \varepsilon.$$

[5] It is also possible to apply Lemma 2 to the constant price approximation, i.e., we assume $\sup_{t\in R} |p'_t| \leq M$ and consider $p_0 \in \mathcal{P} : t \mapsto p_0$ for all $t \in \mathbb{R}$. We have $\mathcal{V}^B(t,x;\tau^*[p_0];p) \geq V^B(t,x;p) - \varepsilon$ if $r > e^{-1}M/\varepsilon$. In other words, if the buyer is very myopic, $r = O(1/\varepsilon)$, then every pricing $p \in \mathcal{P}$ can be treated as constant, which is a trivial result.

The following corollary shows that linear perturbation is particularly simple.

Corollary 2. *Consider a linear pricing strategy $p : t \mapsto p_0 + \sqrt{\varepsilon}Kt \in \mathcal{P}$, where p_0 is a constant. Suppose that the constant price buyer's value function $V_0^B(.;p_0)$ is a solution to the PDE (14), which is C^∞-smooth on $\Omega := \{(t,x) \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}] \mid \underline{V}[p_0] < x < \bar{V}[p_0]\}$, where $\bar{V}[p_0], \underline{V}[p_0] \in (\underline{\pi}, \bar{\pi})$ are the corresponding constant purchasing and quitting boundaries. Then, we can find an ε-optimal value function given by:*

$$V^B(t,x;p) = V_0^B(x - \sqrt{\varepsilon}Kt; p_0) + \sqrt{\varepsilon}V_1^B(t,x) + O(\varepsilon)$$

where $V_1^B(t,x) = V_{1,0}^B(x) + tV_{1,1}^B(x)$ is linear in t, with $V_{1,1}^B$ being the unique solution to the ODE boundary-value problem:

$$\frac{\sigma(x)^2}{2}\partial_x^2 V_{1,1}^B(x) - rV_{1,1}^B(x) + K\sigma(x)\sigma'(x)\partial_x^2 V_0^B(x;p_0) = 0, \quad V_{1,1}^B(\bar{V}[p_0]) = V_{1,1}^B(\underline{V}[p_0]) = 0, \tag{20}$$

and $V_{1,0}^B$ being the unique solution to the ODE boundary-value problem:

$$\frac{\sigma(x)^2}{2}\partial_x^2 V_{1,0}^B(x) - rV_{1,0}^B(x) + V_{1,1}^B(x) - K\partial_x V_0^B(x;p_0) = 0, \quad V_{1,0}^B(\bar{V}[p_0]) = V_{1,0}^B(\underline{V}[p_0]) = 0. \tag{21}$$

We can find that the ε-optimal purchasing and quitting boundaries: $\bar{V}_t[p] = \bar{V}[p_0] + \sqrt{\varepsilon}Kt + \sqrt{\varepsilon}\bar{R}_t + O(\varepsilon)$, and $\underline{V}_t[p] = \underline{V}[p_0] + \sqrt{\varepsilon}Kt + \sqrt{\varepsilon}\underline{R}_t + O(\varepsilon)$, where

$$\bar{R}_t = -\frac{\partial_x V_1^B(t, \bar{V}[p_0])}{\partial_x^2 V_0^B(\bar{V}[p_0]; p_0)} =: K\bar{S}_{0,0} + K\bar{S}_{0,1}t, \quad \underline{R}_t = -\frac{\partial_x V_1^B(t, \underline{V}[p_0])}{\partial_x^2 V_0^B(\underline{V}[p_0]; p_0)} =: K\underline{S}_{0,0} + K\underline{S}_{0,1}t$$

are linear in t, for some constants $\bar{S}_{0,0}, \bar{S}_{0,1}$, $\underline{S}_{0,0}, \underline{S}_{0,1}$.

By Corollary 1, we know that $\bar{S}_{0,0} \le 0$ and $\underline{S}_{0,0} \ge 0$. We now revisit the two buyer learning processes considered in §3.1.1 and §3.1.2 under linear pricing. We can obtain the closed-form expression for the value function given the first learning process and can obtain the perturbative solution up to the ε-order given the second learning process.

Solution: Product attributes learning

The learning process in §3.1.1 is a rare example where the free-boundary PDE (14) can be solved exactly. This leads to the exact value function according to the first part of Lemma 6. The main reason is that $\sigma(.)$ is a constant in this case; thus, the probability measure of $\{v_s^{t,x}\}_{s \ge t}$ is x-translation-invariant. Therefore, we can transform the original problem to a simpler problem where the price is fixed at p_0, while the buyer valuation process is a drifted

Brownian motion $v_t = -\sqrt{\varepsilon}Kt + \sigma W_t$. The transformed problem is stationary in time, with the corresponding HJB: $\frac{\sigma^2}{2}\partial_x^2 V(x) - \sqrt{\varepsilon}K\partial_x V(x) - rV(x) - c = 0$.

Therefore, the free-boundary problem (14) can be solved in this case by first solving the HJB above, and then making an inverse transformation back to the original problem.

Proposition 4. *Consider the buyer's learning process as in §3.1.1. Under a linear pricing strategy $p : t \mapsto p_0 + \sqrt{\varepsilon}Kt \in \mathcal{P}$, the buyer's value function is given by*

$$V^B(t,x) = A_1 e^{\frac{\sqrt{\varepsilon}K - \sqrt{\varepsilon K^2 + 2r\sigma^2}}{\sigma^2}(x - p_0 - \sqrt{\varepsilon}Kt)} + A_2 e^{\frac{\sqrt{\varepsilon}K + \sqrt{\varepsilon K^2 + 2r\sigma^2}}{\sigma^2}(x - p_0 - \sqrt{\varepsilon}Kt)} - \frac{c}{r} \tag{22}$$

with purchasing and quitting boundaries given by

$$\bar{V}_t = p_0 + \bar{V}[\sqrt{\varepsilon}K] + \sqrt{\varepsilon}Kt, \qquad \underline{V}_t = p_0 + \underline{V}[\sqrt{\varepsilon}K] + \sqrt{\varepsilon}Kt \tag{23}$$

where the constants $\bar{V}[\sqrt{\varepsilon}K]$, $\underline{V}[\sqrt{\varepsilon}K], A_1$, and A_2 are determined by boundary conditions in the appendix. To the $\sqrt{\varepsilon}$-order, $\bar{V}[\sqrt{\varepsilon}K]$ and $\underline{V}[\sqrt{\varepsilon}K]$ take the following analytical form,

$$\bar{V}[\sqrt{\varepsilon}K] = \bar{V} + \sqrt{\varepsilon}\bar{R} + O(\varepsilon), \qquad \underline{V}[\sqrt{\varepsilon}K] = \underline{V} + \sqrt{\varepsilon}\underline{R} + O(\varepsilon), \tag{24}$$

where

$$\underline{S} := \frac{\underline{R}}{K} = \left(\frac{\bar{V} - \underline{V}}{\sigma^2}\right)\left(\bar{V} + \frac{c}{r}\right) - \frac{1}{2r} = \frac{\sqrt{\frac{c^2}{r^2} + \frac{\sigma^2}{2r}}}{\sigma\sqrt{2r}} \log\left(\sqrt{\frac{r\sigma^2}{2c^2}} + \sqrt{1 + \frac{r\sigma^2}{2c^2}}\right) - \frac{1}{2r} > 0$$

$$\bar{S} := \frac{\bar{R}}{K} = \underline{S} - \frac{1}{2r} \cdot \frac{\bar{V} - \underline{V}}{\bar{V} + c/r} = \frac{1/(\sigma\sqrt{2r})}{\sqrt{\frac{c^2}{r^2} + \frac{\sigma^2}{2r}}} \cdot \frac{c^2}{r^2} \log\left(\sqrt{\frac{r\sigma^2}{2c^2}} + \sqrt{1 + \frac{r\sigma^2}{2c^2}}\right) - \frac{1}{2r} < 0.$$

and $\bar{V}, \underline{V}$ are given by (10).

Compared to the result of Proposition 2, $\bar{R}$ and $\underline{R}$ are constant in this case. Compared to the constant price benchmark, an increasing pricing scheme ($K > 0$) with the same initial price has two impacts on the purchasing threshold. On the one hand, the benefit of learning becomes lower because the buyer will have to pay more in the future if she receives positive information that causes her to like the product more. Rationally anticipating this, the buyer has a lower incentive to search and is more inclined to purchase now, which reduces the purchasing threshold (captured by the negative $\sqrt{\varepsilon}K\bar{S}$ term). On the other hand, a higher price makes the buyer less willing to purchase, which raises the purchasing threshold (captured by the positive $\sqrt{\varepsilon}Kt$ term). Because the first effect remains stable while the second effect increases over time, the purchasing threshold is lower than the benchmark threshold at the beginning but eventually exceeds the benchmark threshold as the price keeps increasing.

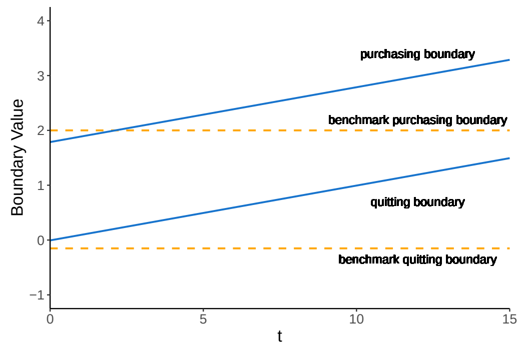

Figure 1: Purchasing and quitting boundaries when $c = .2, p = 1, r = .1, \sigma = 1, \epsilon = 0.01$, and $K = 1$.

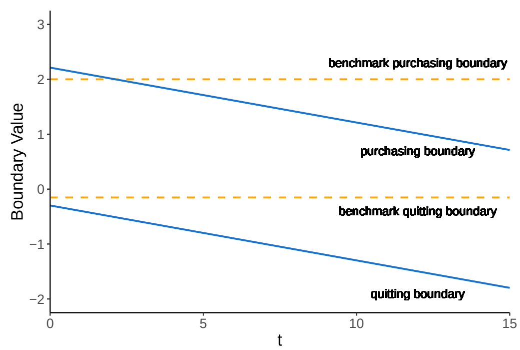

Figure 2: Purchasing and quitting boundaries when $c = .2, p = 1, r = .1, \sigma = 1, \epsilon = 0.01$, and $K = -1$.

An increasing pricing scheme also has two impacts on the quitting threshold. Both a lower benefit of searching and a higher price make it more likely for the buyer to quit. So, the quitting threshold is always higher than the benchmark threshold. We also find that the buyer searches in a narrower region (smaller $\bar{V}_t - \underline{V}_t$) if the price increases rather than staying constant because of the lower benefit of searching. Figure 1 illustrates the purchasing and quitting boundaries in this case, under both non-stationary pricing and constant price.

A decreasing pricing scheme ($K < 0$) has the opposite impact on the purchasing and quitting thresholds. The purchasing threshold is higher than the benchmark threshold at the beginning because the buyer has a stronger incentive to search and is less inclined to purchase immediately. It eventually falls below the benchmark threshold as the price keeps decreasing. The quitting threshold is always lower than the benchmark threshold because the benefit of both searching and purchasing is higher. Also, the buyer searches in a broader region. Figure 2 illustrates the purchasing and quitting boundaries in this case, under both non-stationary pricing and constant price.

Solution: Binary classification

We revisit the learning process in §3.1.2 under a linear pricing strategy. This problem cannot be solved exactly. However, with the help of Corollary 2, it is possible to obtain an analytically closed form of the buyer's value function, as well as the purchasing and quitting boundaries, in terms of the constant price parameters $\bar{V}[p_0]$, $\underline{V}[p_0]$, up to the ε-order. Such an analytical closed form is lengthy and will be omitted; nevertheless, it can be obtained by evaluating the elementary integrals and solving the system of linear equations associated with the boundary conditions, which is summarized in the following proposition.

Proposition 5. *Consider the buyer's binary classification process, as in §3.1.2, under a linear pricing strategy $p : t \mapsto p_0 + \sqrt{\varepsilon}Kt \in \mathcal{P}$. Let $\bar{V}[p_0], \underline{V}[p_0] \in (0,1)$ be the constant price p_0 purchasing and quitting boundaries as specified by the solution to (13). For convenience, let us define: $u_\pm(x) := x^{m_\pm}(1-x)^{m_\mp}$. According to Corollary 2 there is an ε-optimal buyer learning strategy with the value function, purchase, and quitting boundaries taking the form:*

$$V^B(t,x) = V_0^B(x - \sqrt{\varepsilon}Kt; p_0) + \sqrt{\varepsilon}V_1^B(t,x) + O(\varepsilon),$$

$\bar{V}_t[p] = \bar{V}[p_0] + \sqrt{\varepsilon}Kt + \sqrt{\varepsilon}\bar{R}_t + O(\varepsilon)$, and $\underline{V}_t[p] = \underline{V}[p_0] + \sqrt{\varepsilon}Kt + \sqrt{\varepsilon}\underline{R}_t + O(\varepsilon)$, respectively, where $V_0^B(x;p_0) = A_+u_+(x) + A_-u_-(x) - \frac{c}{r}$ is given by (12); $V_1^B(t,x) := V_{1,0}^B(x) + tV_{1,1}^B(x)$ where

$$V_{1,0}^B(x) := \left(B_+ + \frac{2\sigma_S^2}{\sqrt{1+8r\sigma_S^2}}\int \frac{K\partial_x V_0^B(x;p_0) - V_{1,1}^B(x)}{x^{2-m_-}(1-x)^{2-m_+}}dx\right)u_+(x) \\ + \left(B_- - \frac{2\sigma_S^2}{\sqrt{1+8r\sigma_S^2}}\int \frac{K\partial_x V_0^B(x;p_0) - V_{1,1}^B(x)}{x^{2-m_+}(1-x)^{2-m_-}}dx\right)u_-(x), \quad (25)$$

$$V_{1,1}^B(x) := \left(C_+ - \frac{4r\sigma_S^2 K}{\sqrt{1+8r\sigma_S^2}}\int \frac{1-2x}{x^{3-m_-}(1-x)^{3-m_+}}\left(V_0(x) + \frac{c}{r}\right)dx\right)u_+(x) \\ + \left(C_- + \frac{4r\sigma_S^2 K}{\sqrt{1+8r\sigma_S^2}}\int \frac{1-2x}{x^{3-m_+}(1-x)^{3-m_-}}\left(V_0(x) + \frac{c}{r}\right)dx\right)u_-(x), \quad (26)$$

and $\bar{R}_t = K\bar{S}_{0,0} + K\bar{S}_{0,1}t$, $\underline{R}_t = K\underline{S}_{0,0} + K\underline{S}_{0,1}t$ *are given in terms of* $V_0^B(.,.;p_0)$, $V_{1,0}^B(.)$, *and* $V_{1,1}^B(.)$, *as in Corollary 2. The constants* $B_\pm$ *and* $C_\pm$ *are determined by the boundary conditions:* $V_{1,0}^B(\bar{V}[p_0]) = V_{1,0}^B(\underline{V}[p_0]) = 0$ *and* $V_{1,1}^B(\bar{V}[p_0]) = V_{1,1}^B(\underline{V}[p_0]) = 0$, *respectively.*

3.3 Generalizability of the Results

We have shown in the previous section that we can obtain many analytical results under a linear pricing strategy with a small slope of $\sqrt{\varepsilon}$-order. Proposition 3 indicates that ε-optimal buyers will respond to broader classes of non-linear pricing p as if they were linear under the assumptions below:

Assumption 2. *For a given* $\varepsilon > 0$, *we assume that:*

- *The buyer is* ε*-optimal,*
- *The buyer is sufficiently myopic (sufficiently large* $r >> 0$*),*
- *The seller adjusts the price slowly over time:* $|p_t'| = O(\sqrt{\varepsilon})$,

such that the conditions for Proposition 3 are satisfied.

Assumption 2 ensures the validity of our perturbation technique and the consistency with the ε-equilibrium concept. It clarifies when we can apply our linear pricing results to more general non-linear pricing strategies. For a given price function $p \in \mathcal{P}_T$ that satisfies Assumption 2, the buyer will derive the learning strategy from the linear pricing approximation based on Proposition 3:

$$t \mapsto p_0 + \sqrt{\varepsilon}Kt, \qquad \sqrt{\varepsilon}K := p_0' = O(\sqrt{\varepsilon}). \tag{27}$$

4 Seller's Strategy

4.1 Seller's Expected Payoff

The expected payoff for a seller implementing the pricing strategy $p \in \mathcal{P}$ is given by $\mathcal{V}^S(x;\tau^*[p],p)$, where $\tau^*[p] \in \mathcal{T}$ denotes the buyer's ε-optimal response to p. We will denote $\mathcal{V}^S(x;\tau^*[p],p)$ by $\mathcal{V}^S(x;p)$ hereinafter for simplicity. To characterize the optimal pricing, we need to compute $\mathcal{V}^S(x;p)$ for a given pricing strategy p.

4.1.1 Constant Price

In the simplest cases of a constant price, the seller's expected payoff can be derived from the properties of martingales.

Lemma 3. *Consider a constant pricing $p = p_0 \in \mathbb{R}$. Suppose that the constant purchasing and quitting boundaries $\bar{V}[p_0], \underline{V}[p_0] \in (\underline{\pi}, \bar{\pi})$ are finite. For any given $x \in [\underline{V}[p_0], \bar{V}[p_0]]$:*

1. *If $m = 0$, then $\mathcal{V}^S(x; p_0) = (p_0 - g)\frac{x - \underline{V}[p_0]}{\bar{V}[p_0] - \underline{V}[p_0]}$.*

2. *If the volatility is constant: $\sigma(x)^2 = \sigma^2$, then $\mathcal{V}^S(x; p_0) = (p_0 - g)\frac{\sinh \frac{\sqrt{2m}}{\sigma}(x - \underline{V}[p_0])}{\sinh \frac{\sqrt{2m}}{\sigma}(\bar{V}[p_0] - \underline{V}[p_0])}$.*

Assumption 1 implies that $\sigma(x) \geq \underline{\sigma}, \forall x \in [\underline{V}[p_0], \bar{V}[p_0]]$, for some constant $\underline{\sigma} > 0$. It ensures that the buyer will almost surely purchase or quit within a finite amount of time.[6]

4.1.2 General Price

For a buyer with initial valuation x, let $\bar{V}[p], \underline{V}[p] : [0, \infty) \to \mathbb{R}$ denote the buyer's stopping boundaries corresponding to the $\tau^*[p]$ learning strategy and let $\Omega := \{(t, v) \in [0, \infty) \times \mathbb{R} \mid \underline{V}_t[p] < v < \bar{V}_t[p]\}$. We consider $U(s, v; t, x)$, the transition probability density of a particle starting from x at time t to some point v at a later time $s \geq t$, as described by the process $\{v_s^{t,x}\}_{s \geq t}$, without leaving the domain Ω. For any fixed $(s, v) \in \Omega$, $U(s, v; t, x)$ satisfies the Kolmogorov backward equation with absorbing boundary condition:

$$\begin{cases} \dfrac{\sigma(x)^2}{2}\partial_v^2 U(s, v; t, x) + \partial_t U(s, v; t, x) = 0, & (t, x) \in \Omega \\ U(s, v; t, \bar{V}_t[p]) = 0, & U(s, v : t, \underline{V}_t[p]) = 0 \\ U(s, v; t = s, x) = \delta(v - x) \end{cases}, \tag{28}$$

where $\delta(v - x)$ denotes the Dirac-Delta distribution concentrated at v. When it is clear from the context, we denote $U(t, v; t_0 = 0, x)$ simply as $U(t, v)$.

Remark 1. *Alternatively, $U(s, v; t, x)$ for any fixed $(t, x) \in \Omega$ satisfies the Kolmogorov* forward *equation (a.k.a. the Fokker-Planck equation) in (s, v) with absorbing boundary con-*

[6] Specifically, using the Dubins–Schwarz theorem, by computing the survival probability of the standard Brownian motion from the Heat equation series solution, we can derive the following probability tail bound $\mathbb{P}[\tau^*[p_0] > T] \leq C \cdot \exp\left(-\frac{\pi^2 \underline{\sigma}^2}{2(\bar{V}[p_0] - \underline{V}[p_0])^2} \cdot T\right)$, for some constant C, for any $T > 0$. This implies that $\mathbb{E}[\tau^*[p_0]] < \infty$.

dition:

$$
\begin{cases}
\frac{1}{2}\partial_v^2\left[\sigma(v)^2 U(s,v;t,x)\right] - \partial_t U(s,v;t,x) = 0, & (s,v)\in\Omega \\
U(s,\bar{V}_s[p];t,x) = 0, & U(s,\underline{V}_s[p];t,x) = 0\cdot \\
U(s=t,v;t,x) = \delta(v-x)
\end{cases}
\tag{29}
$$

The existence and properties of the solution $U(s,v;t,x)$ depend on the smoothness conditions of $\bar{V}[p]$, $\underline{V}[p]$ (Friedman, 2008, Chapter 3). We assume all necessary conditions are satisfied so that the solution $U(s,v;t,x)\in C^{1,2}(\Omega)$ exists. The probability flux of the buyer hitting the moving purchasing boundary, and thus getting absorbed, at time s is:

$$
-\frac{1}{2}\partial_v\left[\sigma(v)^2 U(s,v;t,x)\right]|_{v=\bar{V}_s[p]} - \bar{V}_s'[p]\cdot U(s,\bar{V}_s;t,x) = -\frac{1}{2}\partial_v\left[\sigma(v)^2 U(s,v;t,x)\right]|_{v=\bar{V}_s[p]}.
$$

In the above equation, the term $\bar{V}_s'[p]$ is needed to take into account the boundary movement, which nevertheless vanishes because of the boundary condition: $U(s,\bar{V}_s;t,x)=0$. Hence, for a pricing policy $p\in\mathcal{P}$, the seller's expected payoff from a buyer starting at time t with valuation x is:

$$
\mathcal{V}^S(t,x;p) = -\frac{1}{2}\int_t^\infty e^{-m(s-t)}(p_s-g)\partial_v\left[\sigma(v)^2 U(s,v;t,x)\right]|_{v=\bar{V}_s[p]}ds,\ \text{ if } x\in(\underline{V}_t[p],\bar{V}_t[p]),
\tag{30}
$$

and $\mathcal{V}^S(t,x;p)=(p_t-g)1_{x\geq\bar{V}_t[p]}$ otherwise.

If $p\in\mathcal{P}_T$, then (30) and (28) imply that $\mathcal{V}^S$ satisfies the following backward parabolic PDE initial boundary value problem:

$$
\begin{cases}
\frac{\sigma(x)^2}{2}\partial_x^2\mathcal{V}^S(t,x;p) + \partial_t\mathcal{V}^S(t,x;p) - m\mathcal{V}^S(t,x;p) = 0, & (t,x)\in\Omega \\
\mathcal{V}^S(t,\bar{V}_t[p];p) = p_t - g, & \mathcal{V}_0^S(t,\underline{V}_t[p];p) = 0\,, \\
\mathcal{V}^S(T,x;p) = \mathcal{V}_0^S(x;p_T)
\end{cases}
\tag{31}
$$

where $\mathcal{V}_0^S(x;p_T)$ denotes the seller's payoff under the constant price policy: $p_t=p_T$ for all $t\geq T$. $\mathcal{V}_0^S$ is time-independent and is determined by the ODE: $\frac{\sigma^2(x)}{2}\partial_x^2\mathcal{V}_0^S - m\mathcal{V}_0^S = 0$. An important case is $t=0$, in which we write $\mathcal{V}^S(x;p)=\mathcal{V}^S(t=0,x;p)$. When the price is close to being constant, we can solve perturbatively for the seller's payoff as follows.

Proposition 6. *Consider a pricing strategy $p := p_0+\sqrt{\varepsilon}h\in\mathcal{P}_T$, where $p_0\in\mathbb{R}$ is a constant, and $h\in\mathcal{P}_T$. Suppose that $\bar{V}[p_0],\underline{V}[p_0]\in(\underline{\pi},\bar{\pi})$ are the purchasing and quitting boundaries corresponding to the constant price p_0 strategy, and $\bar{R},\underline{R}:\mathbb{R}\to\mathbb{R}$ are the buyer's $\sqrt{\varepsilon}$-order responses to p, as given in Proposition 2. Let $\varepsilon>0$ be sufficiently small such* *that* $\bar{V}_t[p] = (\bar{V}[p_0] + \sqrt{\varepsilon}h_t) + \sqrt{\varepsilon}\bar{R}_t \in (\underline{\pi}, \bar{\pi})$, *and* $\underline{V}_t[p] = (\underline{V}[p_0] + \sqrt{\varepsilon}h_t) + \sqrt{\varepsilon}\underline{R}_t \in (\underline{\pi}, \bar{\pi})$ *for all* $t \in [0,T]$. *Then the seller's expected payoff from the buyer with initial valuation* $x \in (\underline{V}[p_0], \bar{V}[p_0])$ *under the pricing strategy* p *up to the* ε*-order is given by:*

$$\mathcal{V}^S(x;p) = \mathcal{V}_0^S\left((1-\sqrt{\varepsilon}r_{1,0})x - \sqrt{\varepsilon}r_{0,0};p_0\right) + \sqrt{\varepsilon}\mathcal{V}_1^S(0,x) + O(\varepsilon), \tag{32}$$

where $r_{1,t} := \frac{\bar{R}_t - \underline{R}_t}{\bar{V}[p_0]-\underline{V}[p_0]}$ *and* $r_{0,t} := h_t + \underline{R}_t - r_{1,t}\underline{V}[p_0]$, *and* $\mathcal{V}_1^S(.,.) : \Omega := \{(t,x) \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}] \mid \underline{V}[p_0] < x < \bar{V}[p_0]\} \to \mathbb{R}$ *is given by:*

$$\begin{aligned}\mathcal{V}_1^S(0,x) = \mathbb{E}\left[h_{\tau_\Omega^x} e^{-m\tau_\Omega^x} \cdot 1\left\{v^x_{\tau_\Omega^x} \geq \bar{V}[p_0]\right\}\right] - \mathbb{E}\left[\int_0^{\tau_\Omega^x} e^{-ms}(r'_{1,s}v_s^x + r'_{0,s})\partial_x\mathcal{V}_0^S(v_s^x;p_0)ds\right] \\ + \mathbb{E}\left[\int_0^{\tau_\Omega^x} e^{-ms}\left(\sigma(v_s^x)\sigma'(v_s^x)(r_{1,s}v_s^x + r_{0,s}) - \sigma(v_s^x)^2 r_{1,s}\right)\partial_x^2\mathcal{V}_0^S(v_s^x;p_0)ds\right]\end{aligned} \tag{33}$$

where $\tau_\Omega^x := \inf\{t \geq 0 \mid (t, v_t^x) \notin \Omega\}$ *is the stopping time.*

4.2 Fast-rising Price

In this section, we consider a non-stationary strategy of rapidly increasing the price. Specifically, when the buyer's initial valuation is sufficiently high, it is optimal for the seller to induce an immediate purchase by increasing the price as sharply as possible, which can be interpreted as a take-it-or-leave-it offer. In this case, we can characterize the supremum of the seller's payoff over a very general set of admissible pricing strategies under minimal assumptions.[7]

Proposition 7. *Let* $h \in \mathcal{P}_T$ *be an arbitrary pricing strategy strictly increasing over* $[0,T)$ *with* $h_0 = 0$, *and let* $p_0 \in \mathbb{R}$ *be a constant. Consider the pricing strategy* $p = p_0 + Kh$ *for* $K \in \mathbb{R}$. *Then,*

$$\lim_{K\to\infty} \mathcal{V}^S(x;p) = \begin{cases} p_0 - g, & \text{if } x > p_0 \\ 0, & \text{if } x \leq p_0 \end{cases}. \tag{34}$$

Further, suppose that, for all sufficiently high x *and any given* $\varepsilon > 0$, *there exists an* ε*-optimal pricing strategy* $\tilde{p} \in \mathcal{P}_T$ *such that* $\underline{V}_t[\tilde{p}] \geq g$ *for all* $t \in [0,\infty)$. *Then,*

$$V^S(x) = \sup_{p\in\mathcal{P}_T} \mathcal{V}^S(x;\tau^*[p],p) = x - g,$$

which can be approached by the sequence $\{p_n := p_{0,n} + K_n h \in \mathcal{P}_T\}_{n\in\mathbb{Z}_{\geq 0}}$, *where* $p_{0,n} \nearrow x$ *and*

[7] In this section, unlike the case under Assumption 2, we neither require the buyer to be myopic nor require the price to be slow-moving.

$K_n \to +\infty$.

The condition $\underline{V}_t[\tilde{p}] \geq g$ ensures that selling at the buyer's current valuation v_t is always profitable for the seller if it can hypothetically observe v_t at any given time. If the buyer's willingness-to-pay is greater than the production cost, then selling the product at the willingness-to-pay is the best possible outcome for the seller. Proposition 7 shows how the seller can achieve such a best-case scenario via non-stationary pricing. One can see that this condition is always satisfied if $g \leq \underline{\pi}$. Under a positive search cost $c > 0$, and a constant volatility σ, Branco et al. (2012) shows that, for all sufficiently high x ($\geq 2\bar{V} - \underline{V} + g$), the optimal static pricing strategy is $\hat{p}_0 = x - \bar{V}$. In this case, $\mathcal{V}^S(x; \hat{p}_0, K = 0) = x - \bar{V} - g$, which is lower than the upper-bound $x - g$ approachable by non-stationary pricing. While a static price seller risks losing the customer by raising the price above $\hat{p}_0 = x - \bar{V}$, a non-stationary price seller uses the threat of a rapid price increase to induce a high-value customer to immediately accept the price $p_0 = x$.

However, in some buyer search models, it is possible that the condition $\underline{V}_t[\tilde{p}] \geq g$ cannot be satisfied for any x. For example, in §4.3, we consider the model with $c = 0$, a constant volatility σ, and $m = 0$. Then $\underline{V}_t[p_0] = -\infty$, and the buyer valuation process hits $\bar{V}[p_0]$ with probability 1. Therefore, it is optimal to use static pricing strategies where we can set an arbitrarily high p_0, in which case $V^S(x) = +\infty$. Even if we impose an artificial constraint $p_0 < x$, we still find that the optimal linear pricing strategy is to set the slope $K \gtrsim 0$ as close as possible to 0 and achieve $\lim_{p_0 \nearrow x, K \searrow 0} \mathcal{V}^S(x; p_0, K) = x - g + \frac{\sigma}{\sqrt{2r}} > x - g$. This example demonstrates how Proposition 7 breaks down when $\underline{V}_t[p] = -\infty < g$. Intuitively, a higher profit than $x - g$ can be obtained because the buyer will never exit, given $c = 0$.

As we can see, fast-rising pricing strategies can be restrictive. They are only useful for sufficiently high-valuation buyers, and may not work for any buyers in some cases. Moreover, it may not be practical for the seller to increase the price drastically due to regulatory or reputational considerations. Therefore, in the following sections, we consider another non-stationary pricing strategy, which can be applied to a broader range of scenarios.

4.3 Slow-moving Price

The fast-rising pricing strategy in §4.2 is not always feasible. In this section, we focus on the implementation of slow-moving linear pricing by the seller. Specifically, the set of admissible pricing is:

$$\mathcal{P}_{lin}^{\varepsilon} := \left\{ t \mapsto p_0 + \sqrt{\varepsilon} K t \mid p_0 \in \mathbb{R}, K \in [-1, +1] \right\} \subset C^{\infty}[0, \infty).$$

Within $\mathcal{P}^{\varepsilon}_{lin}$, we denote the expected payoff by $\mathcal{V}^S(x;p_0,K)$. The seller only needs to determine the optimal $(p_0,K)=(p_0^*,K^*)$. Considering linear pricing from the seller's perspective is not without loss of generality, but it is sufficient to answer the economically relevant questions of whether a constant price is always optimal when the seller cannot track the evolution of the buyer's valuation, and what would be the profitable direction of a slow-moving price otherwise. In particular, denote by $\hat{p}_0 := \hat{p}_0(x)$ the optimal constant price given the buyer's initial valuation x. By computing $\frac{\partial \mathcal{V}^S}{\partial K}(x;p_0=\hat{p}_0,K=0)$, we can determine whether $K^*>0$, $K^*<0$, or $K^*=0$, which characterizes (p_0^*,K^*), the optimal policy in some vicinity of $K=0$. The above analysis provides normative guidance to a seller initially using optimal constant pricing $\hat{p}_0$ on how it can improve its profit with non-stationary pricing.

The discussion of linear pricing also serves as a template for understanding pricing strategies in more general settings, where $\mathcal{P}$ could include non-linear pricing strategies as long as Assumption 2 holds. With these assumptions, the buyer's ε-optimal learning decision in response to any $p\in\mathcal{P}$ is entirely determined by the value of p_t and its slope p'_t at any given time t according to Proposition 3. Because $|p'_t|=O(\sqrt{\varepsilon})$, we are able to utilize our linear perturbation framework. In practice, such assumptions hold if the seller's buyers are impatient and impulsive in their purchasing decisions, and if, due to regulation, the seller is restricted on how quickly it can change the price over time. We will elaborate on this in §4.5.

We focus on the case where the seller is perfectly patient ($m=0$). We start with the following result on linear perturbation from an arbitrary constant price p_0.

Theorem 1. *Consider a linear pricing strategy $p : t\mapsto p_0+\sqrt{\varepsilon}Kt\in\mathcal{P}$ of a perfectly patient seller. Suppose that $\bar{V}[p_0],\underline{V}[p_0]\in(\underline{\pi},\bar{\pi})$ are the purchasing and quitting boundaries corresponding to the constant price p_0 strategy. For all sufficiently small $\varepsilon>0$, the seller's expected payoff from the buyer with initial valuation $x\in(\underline{V}[p_0],\bar{V}[p_0])$ under the pricing strategy p up to the ε-order is given by:*

$$\begin{aligned}\mathcal{V}^S(x;p_0,K) &= \frac{(p_0-g)(x-\underline{V}[p_0])}{\bar{V}[p_0]-\underline{V}[p_0]} + \sqrt{\varepsilon}K\mathbb{E}\left[\tau^x_\Omega\cdot 1\left\{v^x_{\tau^x_\Omega}\geq \bar{V}[p_0]\right\}\right] \\ &\quad - \sqrt{\varepsilon}K\cdot\frac{p_0-g}{\bar{V}[p_0]-\underline{V}[p_0]}\left((1+\underline{S}_{0,1})\mathbb{E}\left[\tau^x_\Omega\right]+(\bar{S}_{0,1}-\underline{S}_{0,1})\mathbb{E}\left[\tau^x_\Omega\cdot 1\left\{v^x_{\tau^x_\Omega}\geq\bar{V}[p_0]\right\}\right]\right) \\ &\quad - \sqrt{\varepsilon}K\cdot\frac{p_0-g}{\bar{V}[p_0]-\underline{V}[p_0]}\left(\underline{S}_{0,0}+(\bar{S}_{0,0}-\underline{S}_{0,0})\mathbb{P}\left[v^x_{\tau^x_\Omega}\geq\bar{V}[p_0]\right]\right)+O(\varepsilon) \qquad (35)\end{aligned}$$

where $\tau^x_\Omega := \inf\{t\geq 0 \mid (t,v^x_t)\notin\Omega\}$ is the stopping time, $\Omega := \{(t,x)\in\mathbb{R}\times[\underline{\pi},\bar{\pi}] \mid \underline{V}[p_0]<x<\bar{V}[p_0]\}$, and the constants $\bar{S}_{0,0},\bar{S}_{0,1}$, $\underline{S}_{0,0},\underline{S}_{0,1}$ determine the $\sqrt{\varepsilon}$-order buyer's response to p as given in Corollary 2.

The first term of the expression in (35) represents the seller's payoff from the constant price policy p_0. Below, we consider each of the following terms, where the second term affects the profit per purchase and the last two terms affect the probability of purchase.

Change to the profit per purchase:

$$+\sqrt{\varepsilon}K\mathbb{E}\left[\tau_\Omega^x \cdot 1\left\{v_{\tau_\Omega^x}^x \geq \bar{V}[p_0]\right\}\right]$$

This term is related to the expected change in price (relative to the initial price) at the time of purchase. If the price is increasing over time, the buyer will pay a price higher than the initial price p_0 if she ends up buying the product, reflected by the non-negative value of this term. If the price is decreasing over time, the seller can only extract a lower profit if the buyer buys after searching, reflected by the non-positive value of this term. Note that $\mathbb{E}\left[\tau_\Omega^x \cdot 1\left\{v_{\tau_\Omega^x}^x \geq \bar{V}[p_0]\right\}\right]$ satisfies the ODE $\frac{1}{2}\sigma(x)^2 w''(x) = -\frac{x-\underline{V}[p_0]}{\bar{V}[p_0]-\underline{V}[p_0]}$ with $w(\bar{V}[p_0]) = w(\underline{V}[p_0]) = 0$. Solving the ODE with the boundary condition gives us:

$$\begin{aligned}\mathbb{E}\left[\tau_\Omega^x \cdot 1\left\{v_{\tau_\Omega^x}^x \geq \bar{V}[p_0]\right\}\right] = \frac{2(x-\underline{V}[p_0])}{(\bar{V}[p_0]-\underline{V}[p_0])^2}\int_{\underline{V}[p_0]}^{\bar{V}[p_0]} \frac{(\bar{V}[p_0]-z)(z-\underline{V}[p_0])}{\sigma(z)^2}dz \\ - \frac{2}{\bar{V}[p_0]-\underline{V}[p_0]}\int_{\underline{V}[p_0]}^{x} \frac{(x-z)(z-\underline{V}[p_0])}{\sigma(z)^2}dz.\end{aligned}$$

Change to purchase probability due to rescaling of the search interval:

$$-\sqrt{\varepsilon}K \cdot \frac{p_0 - g}{\bar{V}[p_0]-\underline{V}[p_0]}\left(\underline{S}_{0,0} + (\bar{S}_{0,0} - \underline{S}_{0,0})\mathbb{P}\left[v_{\tau_\Omega^x}^x \geq \bar{V}[p_0]\right]\right).$$

Expecting the price to change over time rather than stay constant, the buyer will adjust the search region. An increasing price trajectory shrinks the search region, whereas a decreasing price trajectory enlarges the search region. This economic force affects the search interval even at time 0 when the prices are identical, for the cases of time-varying price and constant price. Consider an increasing price. Observing that $\mathbb{P}\left[v_{\tau_\Omega^x}^x \geq \bar{V}[p_0]\right]$ increases as the buyer's initial valuation becomes higher, and that $\bar{S}_{0,0} - \underline{S}_{0,0} \leq 0$ according to Corollary 2, one can see that an increasing price increases the buyer's probability of purchase if she has a high initial valuation (near the purchasing boundary), and reduces the buyer's probability of purchase if she has a low initial valuation (near the quitting boundary). This term can be evaluated using the fact that $\mathbb{P}\left[v_{\tau_\Omega^x}^x \geq \bar{V}[p_0]\right] = \frac{x-\underline{V}[p_0]}{\bar{V}[p_0]-\underline{V}[p_0]}$, as shown in the proof of Lemma 3.

Change to purchase probability due to moving boundaries and price:

$$
\begin{aligned}
&-\sqrt{\varepsilon}K\cdot\frac{p_0-g}{\bar{V}[p_0]-\underline{V}[p_0]}\left((1+\underline{S}_{0,1})\mathbb{E}\left[\tau_\Omega^x\right]+(\bar{S}_{0,1}-\underline{S}_{0,1})\mathbb{E}\left[\tau_\Omega^x\cdot 1\left\{v^x_{\tau^x_\Omega}\geq\bar{V}[p_0]\right\}\right]\right)\\
=&-\sqrt{\varepsilon}K\cdot\frac{p_0-g}{\bar{V}[p_0]-\underline{V}[p_0]}\left(\mathbb{E}\left[\tau_\Omega^x\right]+\bar{S}_{0,1}\mathbb{E}\left[\tau_\Omega^x\cdot 1\left\{v^x_{\tau^x_\Omega}\geq\bar{V}[p_0]\right\}\right]+\underline{S}_{0,1}\mathbb{E}\left[\tau_\Omega^x\cdot 1\left\{v^x_{\tau^x_\Omega}\leq\underline{V}[p_0]\right\}\right]\right).
\end{aligned}
$$

The first term in the bracket, $\mathbb{E}[\tau_\Omega^x]$, reflects that the probability of purchase is affected by the expected amount of price change over the entire search process. The buyer is less likely to make a purchase if the price increases over time, and is more likely to make a purchase if the price decreases over time. The second term in the bracket, $\bar{S}_{0,1}\mathbb{E}\left[\tau_\Omega^x\cdot 1\left\{v^x_{\tau^x_\Omega}\geq\bar{V}[p_0]\right\}\right]$, accounts for the fact that the purchasing boundary has moved a certain distance by the time a buyer reaches the original purchasing boundary. Analogously, the third term in the bracket, $\underline{S}_{0,1}\mathbb{E}\left[\tau_\Omega^x\cdot 1\left\{v^x_{\tau^x_\Omega}\leq\underline{V}[p_0]\right\}\right]$, accounts for the fact that the quitting boundary has moved a certain distance by the time a buyer reaches the original quitting boundary.

To evaluate the above formula, it remains to compute $\mathbb{E}[\tau_\Omega^x]$, which satisfies the ODE $\frac{1}{2}\sigma(x)^2w''(x)=-1$ with $w(\bar{V}[p_0])=w(\underline{V}[p_0])=0$. Solving the ODE with the boundary condition gives us:

$$
\mathbb{E}[\tau_\Omega^x]=\frac{2(x-\underline{V}[p_0])}{\bar{V}[p_0]-\underline{V}[p_0]}\int_{\underline{V}[p_0]}^{\bar{V}[p_0]}\frac{\bar{V}[p_0]-z}{\sigma(z)^2}dz-2\int_{\underline{V}[p_0]}^{x}\frac{x-z}{\sigma(z)^2}dz.
$$

The derivative $\frac{\partial\mathcal{V}^S}{\partial K}(x;p_0,K=0)$

Theorem 1 allows us to compute the exact value of $\frac{\partial\mathcal{V}^S}{\partial K}(x;p_0,K=0)$ at any arbitrary $x\in[\underline{V}[p_0],\bar{V}[p_0]]$ and p_0, in terms of the buyer's $\sqrt{\varepsilon}$-order response characterized by the parameters: $\underline{V}[p_0],\bar{V}[p_0],\underline{S}_{0,0},\bar{S}_{0,0},\underline{S}_{0,1},\bar{S}_{0,1}$. For convenience, we use $q:=\frac{x-\underline{V}[\hat{p}_0]}{\bar{V}[\hat{p}_0]-\underline{V}[\hat{p}_0]}$ to denote the buyer's initial valuation relative to the purchasing and exiting boundaries under the optimal static price $\hat{p}_0$. Because there is a bijection between x and q, we can equivalently consider $\frac{\partial\mathcal{V}^S}{\partial K}(x;p_0,K=0)$ or $\frac{\partial\mathcal{V}^S}{\partial K}(q;p_0,K=0)$.

Importantly, if $\frac{\partial\mathcal{V}^S}{\partial K}(q;p_0=\hat{p}_0,K=0)$ is non-zero for a given $q\in[0,1]$, then the optimal K^* will be bounded away from 0. The seller can improve its expected profit by setting $K\gtrsim 0$ if $\frac{\partial\mathcal{V}^S}{\partial K}(q;p_0=\hat{p},K=0)>0$, and by setting $K\lesssim 0$ if $\frac{\partial\mathcal{V}^S}{\partial K}(q;p_0=\hat{p},K=0)<0$.

For the product attributes learning process discussed in §3.1.1, with constant volatility σ, the optimal static price is:

$$
\hat{p}_0=\hat{p}_0(x)=\begin{cases}\frac{x+g-\underline{V}}{2}, & \underline{V}+g<x<2\bar{V}-\underline{V}+g\\ x-\bar{V}, & x\geq 2\bar{V}-\underline{V}+g\end{cases},
$$

where $\bar{V}, \underline{V}$ are given by (10). In this case, $q = \frac{x - \underline{V} - g}{2(\bar{V} - \underline{V})}$. Substituting the buyer's linear perturbation solution from Proposition 4 into Theorem 1 leads to:

$$\frac{1}{\sqrt{\varepsilon}} \frac{\partial \mathcal{V}^S}{\partial K}(q; p_0 = \hat{p}_0, K = 0) = \frac{(\bar{V} - \underline{V})^2}{3\sigma^2}(1 - 2q)q(1 - q) - (\bar{S}q + \underline{S}(1 - q))q. \tag{36}$$

Because the sign of the above expression depends only on q, c/r, and σ^2/r, in Figure 3, we illustrate the direction of a slow-moving price that improves the seller's expected profit over the optimal constant price strategy as a function of q, c/r, and σ^2/r.

For the learning process of binary classification, discussed in §3.1.2, we can substitute the buyer's linear perturbation solution given by Proposition 5 into Theorem 1. Although it is possible to obtain the closed-form expression for $\frac{\partial \mathcal{V}^S}{\partial K}(x; p_0 = \hat{p}_0, K = 0)$ in terms of $\underline{V}[\hat{p}_0]$ and $\underline{V}[\hat{p}_0]$, the expression is very tedious. On the other hand, the static price p_0 decision boundaries $\bar{V}[p_0]$ and $\underline{V}[p_0]$ are only implicitly specified through a system of non-linear algebraic equations, as discussed in §3.1.2. Consequently, we can only evaluate $\hat{p}_0$ and $\bar{V}[\hat{p}_0]$, $\underline{V}[\hat{p}_0]$ numerically. In Figure 4, we present the direction of a slow-moving price that improves the seller's expected profit over the optimal constant price strategy.

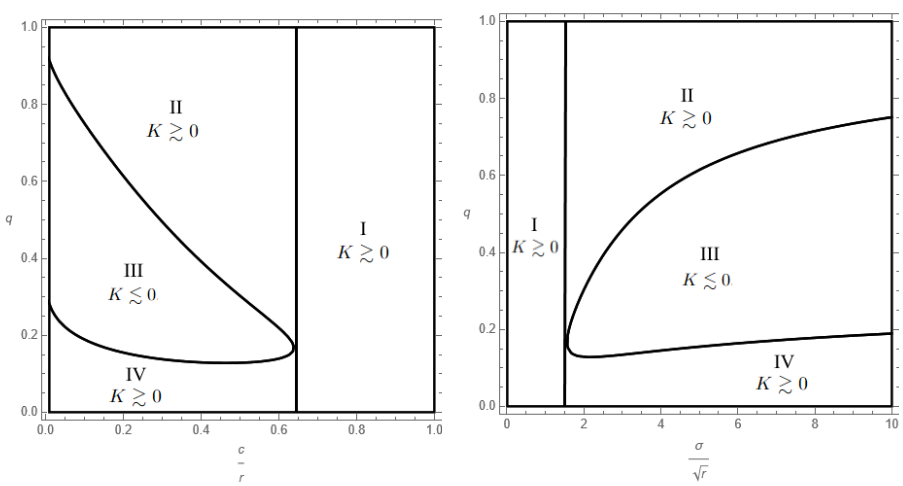

Figure 3: Direction of slow-moving price for a perfectly patient seller where the buyer follows the product attributes learning process, $\sigma^2/r = 1$ in the left plot, and $c/r = 1$ in the right plot.

Figures 3 and 4 are qualitatively similar. We can classify each plot into four regions.

I (Low incentive to search) When the search cost c is too high, the buyer has a low incentive to search for information. The seller needs to give the buyer a high surplus to encourage her to search, which hurts its profit. So, it becomes more attractive for the seller to convince the buyer to purchase the product at the beginning, based on

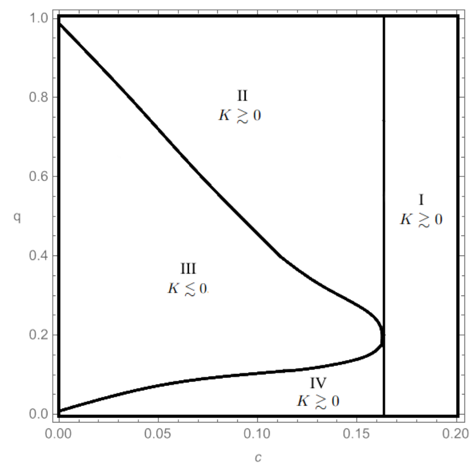

Figure 4: Direction of slow-moving price for a perfectly patient seller where the buyer follows the binary classification process, $g = 0.3$, $r = 1$, and $\sigma_S = 1$.

the initial valuation and the expected price trajectory. For any given initial price, by charging an increasing price over time, the seller lowers the purchasing threshold at the beginning by making it more desirable for the buyer to make an immediate decision. Compared with the stationary pricing strategy of charging a lower constant price, this non-stationary pricing strategy moves the purchasing threshold in the same direction (downward) without sacrificing the profit conditional on purchase. In other words, it increases the probability of purchase without reducing the profit per purchase.

II (High-valuation buyer) When the buyer has a high initial valuation, she is too valuable to lose from the seller's perspective. Therefore, the seller wants to increase the purchasing probability in this case. Moreover, a high-valuation buyer can earn a positive payoff from purchasing immediately, which decreases over time because of discounting. Thus, the seller also wants the buyer to make a quick purchase. An increasing pricing strategy reduces the benefits of searching and encourages the buyer to buy quickly and with a higher likelihood.

III (Medium-valuation buyer) When the buyer has a moderate interest in the product, an increase in price does not suffice to convince the buyer to purchase quickly without learning much additional information. Instead, it reduces the benefit of searching because the buyer knows she will have to pay a higher price if she learns positive things. Therefore, an increasing price will lead to a quick exit rather than a quick purchase.

The seller can benefit from reducing the price gradually in this case. A decreasing price helps the seller keep the buyer engaged in the search process, even if she receives some negative information early on. As a result, it increases the purchasing likelihood.

Because of the moderate initial valuation, the seller can still obtain a decent profit at a lower price. This pricing strategy protects the seller from missing potentially valuable buyers.

IV (Low-valuation buyer) By charging a decreasing price over time, the seller can keep the buyer engaged in the search process even if she receives some negative information early on. However, it is not worthwhile for the seller to reduce the price over time, for two reasons. First, the profit from an immediate purchase is already low when the buyer has a low initial valuation. The seller will obtain an even lower profit from an eventual purchase if the buyer searches for a while and eventually buys from the seller at a lower price. Second, due to the low initial valuation, the buyer must accumulate a lot of positive information before purchasing. The purchasing probability will still be low even if the price is slightly reduced over time, and cannot offset the cost of a lower profit per purchase.

In this case, the seller quickly filters out many buyers by implementing an increasing pricing strategy. On the one hand, the loss from not making a deal with these people is limited due to the low profit per purchase and the low purchasing probability. On the other hand, the benefits of charging a higher price to the remaining buyers are high. Any buyers who do not quit despite the increasing price must have learned positive information and therefore are more valuable to the seller.

4.4 Constant Volatility with Linear Pricing

When the buyer follows the product attributes learning process in §3.1.1 with constant volatility $\sigma > 0$, one can derive the seller's payoff $\mathcal{V}^S(x; p_0, K)$ exactly under any linear pricing strategy $p : t \mapsto p_0 + \sqrt{\varepsilon}Kt \in \mathcal{P}^{\varepsilon}_{lin}$, for any $\varepsilon > 0$ and $p_0 \in \mathbb{R}$, $K \in [-1, +1]$. The key reason is that, in this case, Proposition 4 has shown that the resulting purchasing and quitting boundaries are linear in t. The series solution to the heat equation $U(s, v; t, x)$ with two absorbing fixed boundaries has been well-studied, and we can use the Girsanov Theorem to transform such a solution to the solution to the equation with two linearly moving absorbing boundaries. By substituting this solution into (30), we can obtain a closed form expression for $\mathcal{V}^S(x; p_0, K)$.

We first consider the special case with zero search costs. Then, we study the more interesting case with positive search costs.

Zero Search Costs

When the buyer has zero search costs, the continuation value of searching is positive, whereas the payoff from quitting is zero. Therefore, she would never quit searching without purchasing the product. This implies that her quitting boundary is $-\infty$. Therefore, her optimal search strategy is characterized by a single boundary, the purchasing boundary $\bar{V}[p]$.

If the seller is perfectly patient, it will not have a direct incentive to speed up the buyer's decision-making process. A purchase at any time yields the same payoff for the seller. Hence, it does not have a strong incentive to increase the price over time to push the buyer to make an early decision. In addition, the seller will charge a sufficiently high price such that the buyer's payoff from purchasing the product is negative initially. Therefore, the buyer's discounting of the future does not reduce her surplus if she delays the decision by searching for more information. Even if the price does not decrease over time, the buyer will keep searching for information because she has nothing to lose. Therefore, the seller has no incentive to reduce the price over time to prevent the buyer from quitting. In sum, in this case, the seller has little incentive to charge non-stationary prices. The following proposition shows that the optimal price is arbitrarily close to constant when the seller is perfectly patient. In contrast, the seller may benefit from charging non-stationary prices if it discounts the future.

Proposition 8. *Suppose the search cost is zero, $c = 0$.*

1. *When the seller is perfectly patient ($m = 0$), for any fixed initial price p_0, the seller can approach the profit supremum by choosing $K > 0$ as close to zero as possible, i.e.:*

$$V^S(x) = \sup_{K \searrow 0} \mathcal{V}^S(x; p_0, K) = 2p_0 + \frac{\sigma}{\sqrt{2r}} - g - x.$$

2. *When the seller discounts the future ($m > 0$) – in particular, when $m >> 0$ or $m \sim 0$ – then the slope K of the optimal linear pricing is bounded away from zero.*

Positive Search Costs

We now consider the case with a positive search cost. In this case, the continuation value of searching may be negative. Hence, both the purchasing and quitting boundaries are finite. We focus on the case of a perfectly patient seller because we will show that the seller will charge non-stationary prices even in this case. Also – as we can see from the zero-search-cost case – the seller is more inclined to charge non-stationary prices if it discounts the future.[8]

Proposition 9. *Suppose the search cost is positive $c > 0$, and the seller is perfectly patient $m = 0$. The seller's expected profit from a buyer whose initial valuation is x is:*

$$\begin{aligned}\mathcal{V}^S(x; p_0, K) &= \frac{p_0 - g + (\bar{V}_0 + x - 2\underline{V}_0)}{1 - \exp\left(+\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0 - \underline{V}_0)\right)} - \frac{2(\bar{V}_0 - \underline{V}_0)}{\left(1 - \exp\left(+\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0 - \underline{V}_0)\right)\right)^2} \\ &- \frac{(p_0 - g + (\bar{V}_0 - x))\exp\left(+\frac{2\sqrt{\varepsilon}K}{\sigma^2}(x - \underline{V}_0)\right)}{1 - \exp\left(+\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0 - \underline{V}_0)\right)} + \frac{2(\bar{V}_0 - \underline{V}_0)\exp\left(+\frac{2\sqrt{\varepsilon}K}{\sigma^2}(x - \underline{V}_0)\right)}{\left(1 - \exp\left(+\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0 - \underline{V}_0)\right)\right)^2} \end{aligned} \tag{37}$$

if $x \in (\underline{V}_0, \bar{V}_0)$ and $K \neq 0$, and $\mathcal{V}^S(x; p_0, K) = (p_0 - g)\left(\frac{x - \underline{V}_0}{\bar{V}_0 - \underline{V}_0}\right)$ if $x \in (\underline{V}_0, \bar{V}_0)$ and $K = 0$. $\mathcal{V}^S(x; p_0, K) = 0$ if $x \leq \underline{V}_0$. $\mathcal{V}^S(x; p_0, K) = p_0 - g$ if $x \geq \bar{V}_0$.

By taking the derivative of (37) at $K = 0$, one obtains equation (36), derived in the previous perturbative analysis. Unlike the no-search-cost case, here the slope K^* of the optimal pricing strategy can be bounded away from zero even if $m = 0$.

So far, we know that $\hat{p}_0 = \frac{x+g-\underline{V}}{2}$ maximizes $\mathcal{V}^S(x; ,., K = 0)$. The closed-form expression (37) enables us to find the optimal initial price $p_0^* = p_0^*(x, K)$ that maximizes $\mathcal{V}^S(x; ., K)$ for any $K \neq 0$ by solving $\frac{\partial \mathcal{V}^S}{\partial p_0}(x; p_0^*, K) = 0$:

$$\begin{aligned} p_0^*(x, K) := \frac{x + g}{2} + \frac{\sigma^2}{2\sqrt{\varepsilon}K} - \frac{\underline{V}[\sqrt{\varepsilon}K]}{2}\left(1 - \coth\frac{\sqrt{\varepsilon}K}{\sigma^2}\left(\bar{V}[\sqrt{\varepsilon}K] - \underline{V}[\sqrt{\varepsilon}K]\right)\right) \\ - \frac{\bar{V}[\sqrt{\varepsilon}K]}{2}\coth\frac{\sqrt{\varepsilon}K}{\sigma^2}\left(\bar{V}[\sqrt{\varepsilon}K] - \underline{V}[\sqrt{\varepsilon}K]\right) \end{aligned}$$

for $x \in (\underline{V}_0, \bar{V}_0)$. One can verify that $\lim_{K \to 0} p_0^*(x, K) = \hat{p}_0(x)$.

Lemma 4. *Consider a perfectly patient seller ($m = 0$). Suppose that $r, \sigma, c > 0$; then, there exists $\varepsilon > 0$ sufficiently small such that the seller's profit maximizing strategy (p_0^*, K^*) in $\mathcal{P}_{lin}^{\varepsilon}$ satisfies either $p_0^* < \hat{p}_0, K^* \gtrsim 0$ or $p_0^* > \hat{p}_0, K^* \lesssim 0$.*

Lemma 4 characterizes the profit maximization pricing strategy in the linear perturbative regime $\mathcal{P}_{lin}^{\varepsilon}$ for the buyer under the constant volatility learning process. Specifically, the optimal strategy with increasing price is always coupled with a lower initial price, and the optimal strategy with decreasing price is always coupled with a higher initial price. Examples of both types of strategies can be found in Figure 3. In particular, at the boundary between regions III and VI, $(p_0^*, K^*) = (\hat{p}, 0)$ is the maximum point in $\mathcal{P}_{lin}^{\varepsilon}$ for all sufficiently small

[8] We derive the expected profit when the seller discounts the future in equation (72) in the online appendix.

$\varepsilon > 0$. Due to the continuity of $\mathcal{V}^S(.;.,.)$ and $p_0^*(.,.)$, by choosing a slightly higher q, we obtain an example of a maximum point with $p_0^* > \hat{p}_0, K^* \lesssim 0$; by choosing a slightly lower q, we obtain an example of a maximum point with $p_0^* < \hat{p}_0, K^* \gtrsim 0$. A similar argument can be made at the boundary between regions II and III.

Lastly, we note that the closed-form formula (37) gives us a unified picture of the fast-rising pricing strategy studied in §4.2, together with the perturbative analysis in §4.3, at least for the case of linear pricing under constant volatility σ. Because (37) is valid for all $\sqrt{\varepsilon}K \in \mathbb{R}$, it is relatively simple to find the globally optimal linear pricing strategy $t \mapsto p_0 + Kt$ for each given x by maximizing $\mathcal{V}^S(x; p_0, K)$ over all $(p_0, K) \in \mathbb{R}^2$.[9] We show results for some representative choices of x and other parameters in Figure 5. The optimal fast-rising pricing strategy can be approached asymptotically along the left side of the vertical line in each plot. As discussed in §4.2, the seller can approach the payoff supremum $x - g$ via rapid price increases when, for any given $\varepsilon > 0$, there exists an ε-optimal pricing strategy $\tilde{p}$ satisfying $\underline{V}_t[\tilde{p}] \geq g$ for all $t \in [0, \infty)$. Intuitively, this occurs when x or c is sufficiently high, which is illustrated by the bottom row of Figure 5. On the other hand, the top row of Figure 5 shows that the global maximum lies in the perturbative regime when x is not too high.

4.5 Generalizability of the Results

We have studied two types of non-stationary pricing strategies. One is the fast-rising pricing strategy, where the price rapidly increases to induce a high-valuation buyer to make an immediate purchase. The other is the slow-moving linear perturbation strategy, where the slope is constrained to some neighborhood of zero. Because the first type of strategy requires minimal assumptions, we will elaborate on the generalizability of the second type of strategy in this section.

Firstly, in practice, sellers may not know precisely the buyer's initial valuation x. Instead, they may only know the distribution of the buyer's initial valuation, denoted by ϕ. In such cases, the seller's expected profit under the strategy $p \in \mathcal{P}$ is $\mathcal{V}^S(\phi; p) := \int_{\mathbb{R}} \mathcal{V}^S(x; \tau^{x*}[p], p)\phi(x)dx$, where we denote by $\tau^{x*}[p] \in \mathcal{T}$ the ε-optimal stopping time for the buyer with initial valuation x.[10] Consider a perfectly patient seller. To find out the direction of a slow-moving price that improves the profits relative to charging a constant price p_0, we

[9] In other words, we study the global maximum over $\mathcal{P}_{lin}^{\varepsilon}$ for any arbitrarily large $\varepsilon > 0$. For notational convenience, in the following discussion, we absorb the coefficient $\sqrt{\varepsilon}$ into K and allow K to span the domain $\mathbb{R}$.

[10] Note that the optimal learning strategy of the buyers at any initial valuation x can be characterized by the same purchasing and quitting boundaries $\bar{V}[p], \underline{V}[p] : \mathbb{R} \to [\underline{\pi}, \bar{\pi}]$.

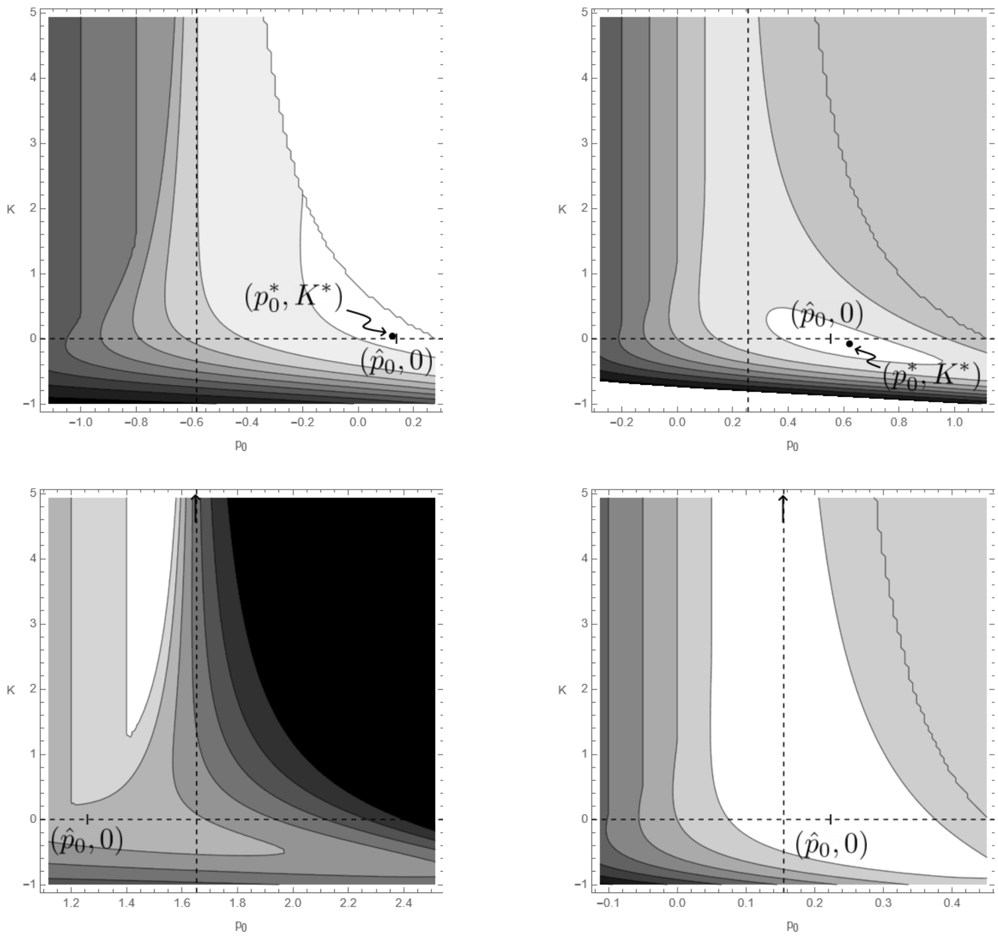

Figure 5: Contour plots of the seller's expected profit from the linear pricing strategy $t \mapsto p_0 + Kt$ when $r = \sigma = \varepsilon = 1$. The dashed vertical line illustrates the initial valuation x. $\hat{p}_0$: the optimal constant pricing. Top-left figure: $q = 0.1$, $c = 0.2$, with the global maximum (p_0^*, K^*) in the perturbative regime where $p_0^* > \hat{p}_0$, $K^* \lesssim 0$. Top-right figure: $q = 0.4$, $c = 0.2$, with the global maximum (p_0^*, K^*) in the perturbative regime where $p_0^* < \hat{p}_0$, $K^* \gtrsim 0$. Bottom-left figure: $q = 0.9$, $c = 0.2$. Bottom-right figure: $q = 0.4$, $c = 0.8$.

can integrate the linear perturbation result (35) in Theorem 1 against the distribution ϕ.

Secondly, let us expand the set of admissible pricing strategies from the linear one $\mathcal{P}^{\varepsilon}_{lin}$ to a more general one $\mathcal{P}^{\varepsilon,M} := \left\{p \in C^\infty[0,\infty) \mid \sup_{t\in\mathbb{R}_{>0}} |p'_t| \le \sqrt{\varepsilon}, \sup_{t\in\mathbb{R}_{>0}} |p''_t| \le M\right\}$ for some $\varepsilon > 0$ and $M > 0$. The expanded set includes non-linear pricing strategies that are not *too* far away from linear strategies. In practice, sellers may be restricted by regulations or reputational concerns in how quickly they can change the price over time, indicating that $\varepsilon > 0$ cannot be too large. According to Proposition 3, a myopic buyer with $r > e^{-1}\sqrt{2M/\varepsilon}$ can make an ε-optimal learning decision at any time t by approximating the pricing strategy $p \in \mathcal{P}^{\varepsilon,M}$ with the linear pricing $l_{\mathbf{x}} : s \mapsto p_t + p'_t \cdot (s - t)$, i.e., the learning decision at any time t is entirely determined by p_t and p'_t.

From the seller's perspective, suppose that the search process is very informative, $\sigma(x)^2 \geq \underline{\sigma}^2$, for all x over the relevant learning region and for some constant $\underline{\sigma}^2 >> 0$. Suppose also that the search cost is sufficiently expensive $c >> 0$ to keep the purchasing and quitting boundaries bounded. In such cases, the buyer updates her valuation and reaches a purchasing decision quickly. We can argue that it is also ε-optimal for the seller to approximate any $p \in \mathcal{P}^{\varepsilon,M}$ with linear pricing.

Let $\tau^*[l] \in \mathcal{T}$ denote the ε-optimal buyer's stopping time, corresponding to $p \in \mathcal{P}^{\varepsilon,M}$, by the linear approximation $l : t \mapsto p_0 + p_0' t$. Then, for any $\delta > 0$, we have:

$$V^S(x) = \sup_{p \in \mathcal{P}^{\varepsilon,\varepsilon'}} \left[\mathbb{E}\left[e^{-m\tau^*[l]}(p_{\tau^*[l]} - g) \cdot 1\left\{ v^x_{\tau^*[l]} \geq p_{\tau^*[l]}, \tau^*[l] < \delta \right\} \right] + e^{-m\delta} \mathcal{V}^S\left(U(\delta, .; 0, x); p_{.+\delta} \right) \right], \tag{38}$$

where $p_{.+\delta} : t \mapsto p_{t+\delta}$ is the corresponding time-shifted pricing strategy, and $U(\delta, .; 0, x)$ is the transition probability density (satisfying equation (28)). Given a finite $M > 0$, we can choose $\delta = O(\sqrt{\varepsilon})$, so that the first term of (38) can be approximated with a linear pricing l up to an order of some $\varepsilon > 0$, i.e.,

$$\mathbb{E}\left[e^{-m\tau^*[l]}(p_{\tau^*[l]} - l_{\tau^*[l]}) \cdot 1\left\{ v^x_{\tau^*[l]} \geq p_{\tau^*[l]}, \tau^*[l] < \delta \right\} \right] < \frac{1}{2} M \delta^2 = O(\varepsilon).$$

The survival probability $\mathbb{P}[\tau^*[l] > \delta]$ can be upper-bounded by the survival probability $O\left(\frac{1}{\underline{\sigma}\sqrt{\delta}}\right)$ of the Brownian motion with volatility $\underline{\sigma}$ and a single absorbing boundary. Suppose that $\underline{\sigma}^2 >> 0$; then, we may argue that the second term of (38) is $O(\varepsilon)$. In conclusion, for all sufficiently large $\underline{\sigma}^2 >> 0$, it is ε-optimal for the seller to plan only for the $\delta = O(\sqrt{\varepsilon})$ time ahead using the linear perturbation theory for $\mathcal{P}^{\varepsilon}_{lin}$, which we studied to specify $l : t \mapsto p_0 + Kt$, i.e., specifying (p_0^*, K^*).

After time δ, the seller can repeat the process to further improve profit by maximizing $\mathcal{V}^S(\phi^1; p_{.+\delta})$, where $\phi^1 := U(\delta, .; 0, x)$, over the next δ-period using a linear approximation. More generally, the seller can specify the ε-optimal pricing strategy over the interval $t \in [k\delta, (k+1)\delta)$ for any $k \geq 1$ using the linear approximation $l^{k+1} : t \mapsto l^k_{k\delta} + K^{k+1} \cdot (t - k\delta) \in \mathcal{P}^{\varepsilon}_{lin}$, which can be specified from Theorem 1 integrated against ϕ^k. By patching together the linear segments, we obtain the piecewise linear pricing strategy which should approximate the optimal $p \in \mathcal{P}^{\varepsilon,M}$[11].

[11] Although the assumption of Proposition 3 is not entirely satisfied for piecewise linear pricing strategies, due to the lack of second derivatives, this is not really an issue as long as we impose the constraint: $(K^{k+1} - K^k)/\delta \leq M$ for all $k \geq 1$. Intuitively, linear approximation is a better approximation to a piecewise linear function than to a non-linear function.

5 Conclusion

This paper introduces a novel framework where sellers adopt non-stationary pricing strategies. Our findings challenge the conventional reliance on stationary pricing by showing that non-stationary pricing strategies can outperform stationary ones. We provide a theoretical advance in optimal control by incorporating non-stationary strategies into a buyer search framework. Unlike previous work, the non-stationarity in the buyer's search problem arises endogenously from sellers' strategic pricing in response to buyers' gradual learning.

Appendix

Proof of Lemma 1. Part 1: Consider any $x, x' \in \mathbb{R}$, and suppose that $x' > x$. Let $\{v_s^{t,x}\}_{s \geq t}$ and $\{v_s^{t,x'}\}_{s \geq t}$ be the two strong solutions of the SDE (1), and we have $v_s^{t,x'} > v_s^{t,x}$ a.e., for all $s \geq t$. This can be seen by using the Lipschitz condition (2) to analyze the difference process $d_s := v_s^{t,x'} - v_s^{t,x}$. It follows that $\mathcal{V}^B(t, x'; \tau, p) \geq \mathcal{V}^B(t, x; \tau, p)$ for all $\tau \in \mathcal{T}$. For any $\varepsilon > 0$, we can find $\tau_{t,x} \in \mathcal{T}$ such that $\mathcal{V}^B(t, x; \tau_{t,x}, p) \geq V(t, x; p) - \varepsilon$. Then, $V^B(t, x'; p) \geq \mathcal{V}^B(t, x'; \tau_{t,x}, p) \geq \mathcal{V}^B(t, x; \tau_{t,x}, p) \geq V(t, x; p) - \varepsilon$. Since $\varepsilon > 0$ is arbitrarily small, hence $V^B(t, x'; p) \geq V^B(t, x; p)$ as claimed. Further, if $V^B(t, x; p) > 0$, then either we can find $\tau_{t,x} \in \mathcal{T}$ such that $\mathbb{P}[\tau_{t,x} > t] > 0$ for any given $\varepsilon > 0$, or $V^B(t, x; p) = x - p_t$. In both cases, we have $\mathcal{V}^B(t, x'; \tau_{t,x}, p) > \mathcal{V}^B(t, x; \tau_{t,x}, p)$, which implies the strict inequality: $V^B(t, x'; p) > V^B(t, x; p)$ for any $x' > x$.

Part 2: It is clear from (3) that $\mathcal{V}^B(t, x; \tau, q) \leq \mathcal{V}^B(t, x; \tau, p)$ for all $\tau \in \mathcal{T}$, hence following the similar argument as in the previous part, we get $V^B(t, x; q) \leq V^B(t, x; p)$. Further, if $V^B(t, x; q) > 0$ then we can find $\tau_{t,x} \in \mathcal{T}$ such that $V^B(t, x; q) - \varepsilon \leq \mathcal{V}^B(t, x; \tau_{t,x}, q)$ for any given $\varepsilon > 0$, and either $\mathbb{P}[\tau_{t,x} > t] > 0$ or $V^B(t, x; q) = x - q_t$. In both cases, we have $\mathcal{V}^B(t, x; \tau_{t,x}, q) < \mathcal{V}^B(t, x; \tau_{t,x}, p)$, proving the strict inequality $V^B(t, x; q) < V^B(t, x; p)$. □

Proof of Proposition 1. Part 1: From Lemma 1, we already know that $V^B(0, x; \tilde{p}) \leq V^B(0, x; p)$. In fact, we have $V^B(0, x; \tilde{p}) < V^B(0, x; p)$ at any x such that $V^B(0, x; \tilde{p}) > 0$. Meanwhile, we have $\max\{x - \tilde{p}_0, 0\} = \max\{x - p_0, 0\}$ from the assumption $h_0 = 0$. It follows that $\bar{V}_0[\tilde{p}] = \sup\{x \in \mathbb{R} \mid V^B(0, x; \tilde{p}) > \max\{x - p_0, 0\}\} < \sup\{x \in \mathbb{R} \mid V^B(0, x; p) > \max\{x - p_0, 0\}\} = \bar{V}_0[p]$, and similarly $\underline{V}_0[\tilde{p}] = \inf\{x \in \mathbb{R} \mid V^B(0, x; \tilde{p}) > \max\{x - p_0, 0\}\} \geq \inf\{x \in \mathbb{R} \mid V^B(0, x; p) > \max\{x - p_0, 0\}\} = \underline{V}_0[p]$, which proves the claim.

Part 2: Without the loss of generality, let's only consider $t = 0$ and h such that $h_0 = 0$, we can always redefine t and shift the x–axis by a constant, otherwise. Let us suppose for a contradiction that there exists $x \in [\underline{\pi}, \bar{\pi}]$ where it is optimal to continue learning for any

$K \geq 0$, i.e. $V^B(0,x;\tilde{p}) > 0$ is bounded away from zero for all $K \geq 0$. In other words, for any $\varepsilon, \varepsilon' > 0$, we can choose $\{\tau[K]\}_{K\geq 0} \subset \mathcal{T}$ and $\delta > 0$ such that $\sup_{K\geq 0} \mathbb{E}[1_{\tau[K]<\delta}] < \varepsilon'^2$ and:

$$V^B(0,x;\tilde{p}) \leq \mathbb{E}\left[e^{-r\delta\wedge\tau[K]}V^B(\delta\wedge\tau[K], v^x_{\delta\wedge\tau[K]};\tilde{p}) - \int_0^{\delta\wedge\tau[K]} ce^{-rs}ds\right] + \varepsilon\delta, \quad \forall K \geq 0.$$

We can separate the expression above further into two terms corresponding to the events $\tau[K] \geq \delta$ and $\tau[K] < \delta$. Then, using Lemma 1: $V^B(t,x;\tilde{p}) \leq V^B(t,x;p)$ for all $(t,x) \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}]$, and $V^B(\delta,x;\tilde{p}) \leq V^B(\delta,x;p_\delta + Kh_\delta)$, where $p_\delta + Kh_\delta$ denotes a constant pricing policy (i.e. the buyer is better–off if $\tilde{p}_t$ stopped increasing after $t = \delta$), we obtain:

$$\begin{aligned} V^B(0,x;\tilde{p}) &\leq \mathbb{E}\left[e^{-r\tau[K]}V^B(\tau[K], v^x_{\tau[K]};\tilde{p}) \cdot 1_{\tau[K]<\delta}\right] + \mathbb{E}\left[V^B(\delta, v^x_\delta;\tilde{p}) \cdot 1_{\tau[K]\geq\delta}\right] + \varepsilon\delta \\ &\leq \mathbb{E}\left[e^{-r\tau[K]}V^B(\tau[K], v^x_{\tau[K]};p) \cdot 1_{\tau[K]<\delta}\right] + \mathbb{E}\left[V^B(\delta, v^x_\delta;p_\delta + Kh_\delta) \cdot 1_{\tau[K]\geq\delta}\right] + \varepsilon\delta. \end{aligned}$$

We bound the first term using the restriction on the growth–rate of the square–integral of the process $v^{t,x}_s$ implied by Assumption 1, and (63) the asymptotically linear condition in x for $V^B(t,x;p)$, we have:

$$\mathbb{E}\left[e^{-r\tau[K]}V^B(\tau[K], v^x_{\tau[K]};p) \cdot 1_{\tau[K]<\delta}\right] \leq \sup_{\tau\in\mathcal{T}} \mathbb{E}\left[e^{-2r\tau}V^B(\tau, v^x_\tau;p)^2\right]^{1/2} \cdot \sup_{K\geq 0} \mathbb{E}\left[1_{\tau[K]<\delta}\right]^{1/2},$$

where the first factor is finite, and the second factor is $< \varepsilon'$ by our choice of δ. For the second term, we know from the result on constant pricing policy value function that $V^B(\delta, v^x_\delta; p_\delta + Kh_\delta) = 0$ for all sufficiently large $K > 0$, giving a pointwise convergence of $V^B(\delta, v^x_\delta; p_\delta + Kh_\delta) \cdot 1_{\tau[K]\geq\delta}$ in the probability space. Thus, given any $\varepsilon'' > 0$, we can find a sufficiently large $K > 0$ such that $\mathbb{E}\left[V^B(\delta, v^x_\delta; p_\delta + Kh_\delta) \cdot 1_{\tau[K]\geq\delta}\right] < \varepsilon''$ by the Dominated Convergence Theorem. Overall, we have

$$V^B(0,x;\tilde{p}) \leq \varepsilon' \cdot \sup_{\tau\in\mathcal{T}} \mathbb{E}\left[e^{-2r\tau}V^B(\tau, v^x_\tau;p)^2\right]^{1/2} + \varepsilon'' + \varepsilon\delta.$$

Since $\varepsilon, \varepsilon', \varepsilon'' > 0$ are arbitrarily small, we conclude that $V^B(0,x;\tilde{p}) \leq 0$, a contradiction. Therefore, for any $x \in [\underline{\pi}, \bar{\pi}]$, for all sufficiently large $K \geq 0$, either it is optimal to purchase immediately ($x > \tilde{p}_0$), or exit immediately ($x < \tilde{p}_0$). It must be the case that: $\bar{V}_0[\tilde{p}] \searrow \max\{\tilde{p}_0, \underline{\pi}\}, \underline{V}_0[\tilde{p}] \nearrow \min\{\tilde{p}_0, \bar{\pi}\}$ as $K \to +\infty$.

Suppose that $K < 0$, consider any $x \in [\underline{\pi}, \bar{\pi}]$, we note that

$$V^B(0,x;\tilde{p}) \geq \mathcal{V}^B(0,x;\delta,\tilde{p}) \geq e^{-r\delta}\mathbb{E}\left[\max\{v^x_\delta - p_\delta - Kh_\delta, 0\}\right] - c\delta \geq -e^{-r\delta}(Kh_\delta + p_\delta) - c\delta$$

where δ denotes the simple policy of stopping exactly at some time $\delta > 0$ regardless of the valuation, and the first inequality followed from the sub–optimality of δ. Therefore, for all sufficiently negative $K << 0$, we have $V^B(0, x; \tilde{p}) > 0$, thus it is optimal to continue searching: i.e. $\underline{V}_0[\tilde{p}] < x < \bar{V}_0[\tilde{p}]$ for any $x \in [\underline{\pi}, \bar{\pi}]$, proving $\bar{V}_0[\tilde{p}] \nearrow \bar{\pi}$ and $\underline{V}_0[\tilde{p}] \searrow \underline{\pi}$ as $K \to -\infty$. $\square$

Proof of Proposition 2. We note that $V^B(t, x - \sqrt{\varepsilon} h_t; p)$ is simply the solution $V^B(t, x; p)$ shifted according to $\sqrt{\varepsilon} K h$ which satisfies the value–matching and smooth–pasting conditions at $\bar{V}[p] + \sqrt{\varepsilon} h$ and $\underline{V}[p] + \sqrt{\varepsilon} h$, but does not satisfies the PDE, hence the $\sqrt{\varepsilon} V_1^B$ correction is needed. By adding $\sqrt{\varepsilon} V_1^B$ correction, we further need a $\sqrt{\varepsilon}$–order correction to the purchase and quitting boundaries $\bar{V}[p] + \sqrt{\varepsilon} h$ and $\underline{V}[p] + \sqrt{\varepsilon} h$ which take the form (17). We find the equation for V_1^B by substituting the ansatz (15) into the PDE for $V^B(., .; \tilde{p})$ and collecting the $\sqrt{\varepsilon}$–order terms:

$$\frac{\sigma(x)^2}{2} \partial_x^2 V_1^B(t, x) + \partial_t V_1^B(t, x) - r V_1^B(t, x) - h_t' \partial_x V^B(t, x; p) + h_t \sigma(x) \sigma'(x) \partial_x^2 V^B(t, x; p) = 0. \quad (39)$$

To study $\bar{R}$ and $\underline{R}$ we analyze the boundary conditions of $V^B(., .; \tilde{p})$ to the first–order in $\sqrt{\varepsilon}$. Note that $V^B(t, x - \sqrt{\varepsilon} h_t; p)$ automatically satisfies the value–matching conditions at $\bar{V}[\tilde{p}]$ and $\underline{V}[\tilde{p}]$, as we will conseller below, because $\partial_x V^B(t, \bar{V}_t[p]; p) = 1$ and $\partial_x V^B(t, \underline{V}_t[p]; p) = 0$. We have by substituting the ansatz (15) and (17) into the boundary conditions and comparing the $\sqrt{\varepsilon}$–order terms:

$$\begin{aligned}
&V^B(t, \bar{V}_t[\tilde{p}]; \tilde{p}) = \bar{V}_t[\tilde{p}] - \tilde{p}_t \\
&\qquad \Longrightarrow V^B(t, \bar{V}_t[p] + \sqrt{\varepsilon} \bar{R}_t; p) + \sqrt{\varepsilon} V_1^B(t, \bar{V}_t[p]) = \bar{V}_t[p] - p_t + \sqrt{\varepsilon} \bar{R}_t \\
&\qquad\qquad V_1^B(t, \bar{V}_t[p]) = -\bar{R}_t \partial_x V^B(t, \bar{V}_t[p]; p) + \bar{R}_t \Longrightarrow V_1^B(t, \bar{V}_t[p]) = 0. \quad (40)
\end{aligned}$$

$$\begin{aligned}
&\partial_x V^B(t, \bar{V}_t[\tilde{p}]; \tilde{p}) = 1 \Longrightarrow \partial_x V^B(t, \bar{V}_t[p] + \sqrt{\varepsilon} \bar{R}_t; p) + \sqrt{\varepsilon} \partial_x V_1^B(t, \bar{V}_t[p]) = 1 \\
&\qquad \partial_x V_1^B(t, \bar{V}_t[p]) = -\bar{R}_t \partial_x^2 V^B(t, \bar{V}_t[p]; p) \Longrightarrow \bar{R}_t = -\frac{\partial_x V_1^B(t, \bar{V}_t[p])}{\partial_x^2 V^B(t, \bar{V}_t[p]; p)}. \quad (41)
\end{aligned}$$

$$\begin{aligned}
&V^B(t, \underline{V}_t[\tilde{p}]; \tilde{p}) = 0 \Longrightarrow V^B(t, \underline{V}_t[p] + \sqrt{\varepsilon} \underline{R}_t; p) + \sqrt{\varepsilon} V_1^B(t, \underline{V}_t[p]) = 0 \\
&\qquad \Longrightarrow V_1^B(t, \underline{V}_t[p]) = 0. \quad (42)
\end{aligned}$$

$$\begin{aligned}
&\partial_x V^B(t, \underline{V}_t[\tilde{p}]; \tilde{p}) = 0 \Longrightarrow \partial_x V^B(t, \underline{V}_t[p] + \sqrt{\varepsilon} \underline{R}_t; p) + \sqrt{\varepsilon} \partial_x V_1^B(t, \underline{V}_t[p]) = 0 \\
&\qquad \partial_x V_1^B(t, \underline{V}_t[p]) = -\underline{R}_t \partial_x^2 V^B(t, \underline{V}_t[p]; p) \Longrightarrow \underline{R}_t = -\frac{\partial_x V_1^B(t, \underline{V}_t[p])}{\partial_x^2 V^B(t, \underline{V}_t[p]; p)}. \quad (43)
\end{aligned}$$

Since $p, h \in \mathcal{P}_T$, they are constant for all $t \geq T$, therefore we have the terminal condition at any $T' \geq T$: $V^B(T', x; \tilde{p}) = V_0^B(x; \tilde{p}_T)$ and $V^B(T', x; p) = V_0^B(x; p_T)$, giving the terminal condition for V_1^B:

$$V_1^B(T',x) = V_1^B(T,x) = \frac{1}{\sqrt{\varepsilon}}\left(V_0^B(x;p_T+\sqrt{\varepsilon}h_T) - V_0^B(x-\sqrt{\varepsilon}h_T;p_T)\right) + O(\sqrt{\varepsilon}). \tag{44}$$

We recognize the PDE (39) with (40). (42), and (44) as a backward parabolic (fixed) boundary–value problem. We may transform the problem into the more standard parabolic form for: $\widetilde{V}_1^B(t',x') := V_1^B\left(T-t', \underline{V}_{T-t'}[p] + (\bar{V}_{T-t'}[p] - \underline{V}_{T-t'}[p])x'\right)$ on $\widetilde{\Omega} := [0,\infty)\times[0,1]$ with smooth coefficients $(a_{ij}(.), b_i(.), c(.))$, according to our smoothness assumptions on $\bar{V}[p]$, $\underline{V}[p]$, and $\sigma(.)$. Since $-(h)'_t\partial_x V^B(.,.;p) + h_t\sigma(.)\sigma'(.)\partial_x^2 V^B(.,.;p)$ is assumed smooth on $\widetilde{\Omega}$, and $V_1^B(T,.)$ is smooth on $\{0\}\times[0,1]$, we can apply (Evans, 2022, Chapter 7.1, Theorem 7) or (Friedman, 2008, Chapter 3, §5, Corollary 2) to conclude the existence of the smooth solution $\widetilde{V}_1^B$ to the parabolic initial boundary–value problem. Transforming back to the original problem, we get the smooth solution $V_1^B(.,.)$. The solution is unique, and admits a probabilistic expression via the semi–elliptic version of Feynman–Kac formula (Øksendal, 2003, Theorem 9.1.1):

$$\begin{aligned} V_1^B(t,x) = \mathbb{E}&\left[e^{-r(T'-t)}V_1^B(T',v_{T'}^{t,x})\cdot 1\left\{\tau_\Omega^{t,x}\geq T'\right\}|\mathscr{F}_t\right] \\ &- \mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}\wedge T'} h'_s e^{-r(s-t)}\partial_x V^B(s,v_s^{t,x};p)ds|\mathscr{F}_t\right] \\ &+ \mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}\wedge T'} h_s e^{-r(s-t)}\sigma(v_s^{t,x})\sigma'(v_s^{t,x})\partial_x^2 V^B(s,v_s^{t,x};p)ds|\mathscr{F}_t\right], \end{aligned} \tag{45}$$

The first term is upper–bounded by $\sup_{x\in[\underline{V}_T[p],\bar{V}_T[p]]} V_1^B(T,x)e^{-r(T'-t)} \to 0$ as $T'\to\infty$. Since $p,h\in\mathcal{P}_T$, we have that h_t, $V^B(t,x;p)$, and the boundaries $\bar{V}_t[p], \underline{V}_t[p]$ are constant in t for $t\geq T$. Meanwhile, $v_s^{t,x}$ is bounded inside $[\inf_{s\in[t,T]}\underline{V}_s[p], \sup_{s\in[t,T]}\bar{V}_s[p]]$. Therefore, the third and fourth terms are upper–bounded by some constant (which can be determined by the supremum of the absolute value of the integrand over the compact set $[t,T]\times[\inf_{s\in[t,T]}\underline{V}_s[p], \sup_{s\in[t,T]}\bar{V}_s[p]]$) multiple of $\int_t^\infty e^{-r(s-t)}ds = 1/r < \infty$. Taking the limit $T'\to\infty$ of (45) using the Dominated Convergence Theorem for the right–hand–side while noting that the left–hand–side is independent of T', we obtain expression (16). □

Proof of Corollary 1. Suppose that $h := K\tilde{h}$, where $\tilde{h}\in\mathcal{P}_T$ is monotonically increasing in t, and that $\sigma'(.) = O(\varepsilon)$. We define $\bar{S} := \bar{R}/K : \mathbb{R}\to\mathbb{R}$, and $\underline{S} := \underline{R}/K : \mathbb{R}\to\mathbb{R}$. It remains to show that $\bar{S}_t\leq 0$ and $\underline{S}_t\geq 0$. From our assumption that $\sigma'(.) = O(\varepsilon)$, we may ignore the third term in the $\sqrt{\varepsilon}$–order equation (16). Moreover, $\sigma'(.) = O(\varepsilon)$ implies $V_0^B(x;p_T+\varepsilon h_T) = V_0^B(x-\varepsilon h_T;p_T) + O(\varepsilon)$, hence we can also ignore the first term in (16).

Since $V^B(t,.;p)$ is monotonically increasing in x from Lemma 1, it follows from the second term of (16) that $V_1^B(t,x)/K \leq 0$ for any $(t,x) \in \Omega$. In particular, $\partial_x V_1^B(t,\bar{V}_t[p])/K \geq 0$ and $\partial_x V_1^B(t,\underline{V}_t[p])/K \leq 0$.

Now, let us show that $\partial_x^2 V^B(t,\bar{V}_t[p];p) \geq 0$. Let $\mathbf{x}_0 = (t,\bar{V}_t[p]) \in \partial\Omega$ be a point on the purchasing boundary, then we can find sequences $\{\mathbf{x}_i^+ = (t_i,x_i^+)\}_{i=0}^{\infty}$ and $\{\mathbf{x}_i^- = (t_i,x_i^-)\}_{i=0}^{\infty} \subset \Omega$ converging to $\mathbf{x}_0$ such that $x_i^- \leq \bar{V}_{t_i}[p] \leq x_i^+$ for all $i \geq 0$. Since $V^B(.,.;p)$ is the viscosity solution, we have $c + rV^B(\mathbf{x}_i^+;p) - \partial_t V^B(\mathbf{x}_i^+;p) - \frac{\sigma(\mathbf{x}_i^+)^2}{2}\partial_x^2 V^B(\mathbf{x}_i^+;p) \geq 0$, while $c + rV^B(\mathbf{x}_i^-;p) - \partial_t V^B(\mathbf{x}_i^-;p) - \frac{\sigma(\mathbf{x}_i^-)^2}{2}\partial_x^2 V^B(\mathbf{x}_i^-;p) = 0$ for all $i \geq 0$. But $V^B(\mathbf{x}_i^+;p) = x_i^+ - p_{t_i}$, so $\frac{\sigma(\mathbf{x}_i^+)^2}{2}\partial_x^2 V^B(\mathbf{x}_i^+;p) = 0$, hence it follows from the continuous differentiability of $V^B(.,.;p)$ across the boundary $\partial\Omega$ that $\partial_x^2 V^B(t,\bar{V}_t[p];p) \geq 0$. Similarly, we have that $\partial_x^2 V^B(t,\underline{V}_t[p];p) \geq 0$.

It follows from (41) and (43) that the sign of $\bar{S}_t$ and $\underline{S}_t$ are opposite to the sign of $\partial_x V_1^B(t,\bar{V}_t[p])/K$ and $\partial_x V_1^B(t,\underline{V}_t[p])/K$, respectively. So, $\bar{S}_t \leq 0$ and $\underline{S}_t \geq 0$ for $t \in \mathbb{R}$. □

Proof of Lemma 2. Consider a fixed $(t,x) \in \mathbb{R} \times [\underline{\pi},\bar{\pi}]$, and suppose that $V^B(t,x;q) \leq V^B(t,x;p)$. For an arbitrary $\varepsilon > 0$, let $\tau_{t,x,\varepsilon}[p] \in \mathcal{T}$ be such that $\mathcal{V}^B(t,x;\tau_{t,x,\varepsilon}[p],p) \geq V^B(t,x;p) - \varepsilon$, then $V^B(t,x;q) \geq \mathcal{V}^B(t,x;\tau_{t,x,\varepsilon}[p],q) > \mathcal{V}^B(t,x;\tau_{t,x,\varepsilon}[p];p) - \max_{s\geq t} e^{-r(s-t)}|p_s - q_s| \geq V^B(t,x;p) - \max_{s\geq t} e^{-r(s-t)}|p_s - q_s| - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, it must be the case that $V^B(t,x;p) \geq V^B(t,x;q) \geq V^B(t,x;p) - \max_{s\geq t} e^{-r(s-t)}|p_s - q_s|$.

If $V^B(t,x;q) \geq V^B(t,x;p)$, then we simply switch the role of p,q and follow through with the above argument, hence we get that $|V^B(t,x;p) - V^B(t,x;q)| \leq \max_{s\geq t} e^{-r(s-t)}|p_s - q_s|$, which proves the result. □

Proof of Proposition 3. Let $p^T, l_{\mathbf{x}}^T \in \mathcal{P}_T$ be given by some pricing strategies which coincide with $p, l_{\mathbf{x}}$ over $[0,T-\varepsilon]$ and constant for all $t \geq T$. By Lemma 2, we have $|V^B(t,x;p^T) - V^B(t,x;l_{\mathbf{x}}^T)| \leq \max_{s\in[t,T]} e^{-r(s-t)}|p_s - l_{\mathbf{x},s}| \leq \max_{s\geq t} e^{-r(s-t)}|p_s - l_{\mathbf{x},s}|$. Since this inequality holds for all T, we conclude that $|V^B(t,x;p) - V^B(t,x;l_{\mathbf{x}})| \leq \max_{s\geq t} e^{-r(s-t)}|p_s - l_{\mathbf{x},s}|$. But from Taylor's Theorem, we have $|p_s - l_{\mathbf{x},s}| \leq \frac{M}{2}(s-t)^2$ for all $s \geq t$. It follows that $\max_{s\geq t} e^{-r(s-t)}|p_s - l_{\mathbf{x},s}| \leq \frac{M}{2}\max_{s\geq t}(s-t)^2 e^{-r(s-t)} = \frac{2M}{r^2}e^{-2}$. Therefore, $|V^B(t,x;p) - V^B(t,x;l_{\mathbf{x}})| < \varepsilon$ if $r > e^{-1}\sqrt{2M/\varepsilon}$. □

Proof of Corollary 2. The existence and uniqueness of the ODE boundary value problems (20) and (21) follows from the standard theory (Agarwal et al., 2008, Lecture 40). In order to make use of Proposition 2, let us first fix a large $T \geq 0$ and consider $p^T = p_0 + \sqrt{\varepsilon}h^T \in \mathcal{P}_T$ where $h^T \in \mathcal{P}_T$ is given by $h_t^T = Kt$ for $t \in [0,T-\varepsilon]$, constant $h_t^T = KT$ for $t \geq T$, and some in–between smooth transition for $t \in (T-\varepsilon,T)$. We shall assume that $|(h^T)_t'| \leq 1$ for $t \in (T-\varepsilon,T)$. From Proposition 2, we have the following probabilistic expression:

$$V_1^B(t,x;p^T) = -\mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}} (h^T)'_s e^{-r(s-t)} \partial_x V_0^B(v_s^{t,x};p_0) ds | \mathscr{F}_t\right] + \mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}} h_s^T e^{-r(s-t)} \sigma(v_s^{t,x})\sigma'(v_s^{t,x}) \partial_x^2 V_0^B(v_s^{t,x};p_0) ds | \mathscr{F}_t\right], \quad (46)$$

where $\tau_\Omega^{t,x} := \inf\{t' \geq t \mid (t', v_{t'}^{t,x}) \notin \Omega\}$ is the stopping time. Further, we know that V_1^B satisfies the value–matching boundary conditions $V_1^B(t, \bar{V}[p_0]; p^T) = V_1^B(t, \underline{V}[p_0]; p^T) = 0$, while the smooth–pasting boundary conditions determine $\bar{R}$ and $\underline{R}$. We note that $v_s^{t,x}$ is bounded inside $[\underline{V}[p_0], \bar{V}[p_0]]$, while $\int_0^\infty |h_s^T e^{-r(s-t)}| ds \leq K e^{rt}/r^2$, and $\int_0^\infty |(h^T)'_s e^{-r(s-t)}| ds \leq K e^{rt}/r$ by construction for all $T \geq 0$. Therefore, by taking the limit $T \to \infty$ of (46), we have by Lemma 2 and inquality(19) that $V_1^B(t,x;p^T) \to V_1^B(t,x)$, and by applying the Dominated Convergence Theorem to the right–hand–side with $h_s^T \to Ks$, $(h^T)'_s \to K$, we obtain:

$$V_1^B(t,x) = -K \cdot \mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}} e^{-r(s-t)} \partial_x V_0^B(v_s^{t,x};p_0) ds | \mathscr{F}_t\right] + K \cdot \mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}} s e^{-r(s-t)} \sigma(v_s^{t,x})\sigma'(v_s^{t,x}) \partial_x^2 V_0^B(v_s^{t,x};p_0) ds | \mathscr{F}_t\right] \quad (47)$$

We rewrite this further as follows:

$$\begin{aligned} V_1^B(t,x) = &-K \cdot \mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}} e^{-r(s-t)} \partial_x V_0^B(v_s^{t,x};p_0) ds | \mathscr{F}_t\right] \\ &+ K \cdot \mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}} (s-t) e^{-r(s-t)} \sigma(v_s^{t,x})\sigma'(v_s^{t,x}) \partial_x^2 V_0^B(v_s^{t,x};p_0) ds | \mathscr{F}_t\right] \\ &+ Kt \cdot \mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}} e^{-r(s-t)} \sigma(v_s^{t,x})\sigma'(v_s^{t,x}) \partial_x^2 V_0^B(v_s^{t,x};p_0) ds | \mathscr{F}_t\right] \\ = &K \cdot \mathbb{E}\left[\int_0^{\tau_\Omega^{x}} s e^{-rs} \sigma(v_s^{x})\sigma'(v_s^{x}) \partial_x^2 V_0^B(v_s^{x};p_0) ds | \mathscr{F}_0\right] - K \cdot \mathbb{E}\left[\int_0^{\tau_\Omega^{x}} e^{-rs} \partial_x V_0^B(v_s^{x};p_0) ds | \mathscr{F}_0\right] \\ &+ Kt \cdot \mathbb{E}\left[\int_0^{\tau_\Omega^{x}} e^{-rs} \sigma(v_s^{x})\sigma'(v_s^{x}) \partial_x^2 V_0^B(v_s^{x};p_0) ds | \mathscr{F}_0\right] =: V_1^B(0,x) + t\widetilde{V}_{1,1}^B(x). \quad (48) \end{aligned}$$

Note that the above expression is linear in t, in particular, the first two terms $V_1^B(0,x)$ and the factor $\widetilde{V}_{1,1}^B(x)$ of t in the last term are functions of x only. The boundary conditions of $V_1^B(t, \bar{V}[p_0]) = V_1^B(t, \underline{V}[p_0]) = 0$ holds for all t, and therefore we also have $V_1^B(0, \bar{V}[p_0]) = V_1^B(0, \underline{V}[p_0]) = 0$ and $\widetilde{V}_{1,1}^B(\bar{V}[p_0]) = \widetilde{V}_{1,1}^B(\underline{V}[p_0]) = 0$. We recognize the probabilistic expression for $\widetilde{V}_{1,1}^B$ as that of the solution to the boundary–value problem (20), thus, we have $\widetilde{V}_{1,1}^B = V_{1,1}^B$.

Let us define:

$$V_{1,1}^B(x;\beta) := \mathbb{E}\left[\int_0^{\tau_\Omega^x} e^{-\beta s}\sigma(v_s^x)\sigma'(v_s^x)\partial_x^2 V_0^B(v_s^x;p_0)ds\right],$$

which satisfies the ODE:

$$\frac{\sigma(x)^2}{2}\partial_x^2 V_{1,1}^B(x;\beta) - \beta V_{1,1}^B(x;\beta) + K\sigma(x)\sigma'(x)\partial_x^2 V_0^B(x;p_0) = 0. \tag{49}$$

We can see that $V_{1,1}^B(x) := V_{1,1}^B(x;\beta = r)$, and that the first term of (48) is given by $-\partial_\beta V_{1,1}^B(x;\beta)|_{\beta=r}$. Therefore, $V_1^B(0,x)+\partial_\beta V_{1,1}^B(x;\beta)|_{\beta=r} = -K\cdot\mathbb{E}\left[\int_0^{\tau_\Omega^x} e^{-rs}\partial_x V_0^B(v_s^x;p_0)ds\right]$. Substituting this into the corresponding ODE of the right–hand–side of the above, we get:

$$\begin{aligned}\frac{\sigma(x)^2}{2}\partial_x^2\left(V_1^B(0,x)+\partial_\beta V_{1,1}^B(x;\beta)|_{\beta=r}\right) - r\left(V_1^B(0,x)+\partial_\beta V_{1,1}^B(x;\beta)|_{\beta=r}\right) - K\partial_x V_0^B(x;p_0) = 0\\ \implies \frac{\sigma(x)^2}{2}\partial_x^2 V_1^B(0,x) - rV_1^B(0,x) + V_{1,1}^B(x) - K\partial_x V_0^B(x;p_0) = 0,\end{aligned}$$

where we used $V_{1,1}^B(x) = \frac{\sigma(x)^2}{2}\partial_x^2\partial_\beta V_{1,1}^B(x;\beta)|_{\beta=r} - \beta\partial_\beta V_{1,1}^B(x;\beta)|_{\beta=r}$, which is obtained by differentiating (49) at $\beta = r$. Note that although $V_0^B(x;p_0)$ depends on r, it does not depends on β, hence its β derivative vanishes. Therefore, $V_1^B(0,.)$ satisfies the ODE (21) with the specified boundary conditions, therefore it must coincide with $V_{1,0}$. □

Proof of Proposition 4. In the special case of linear pricing $t \mapsto p_t := p_0 + \sqrt{\varepsilon}Kt$ the value function takes the form (22) over Ω as we can directly check that it satisfies the PDE of (14). Let's define $K_\pm := \frac{\sqrt{\varepsilon}K\pm\sqrt{\varepsilon K^2+2r\sigma^2}}{\sigma^2}$ for convenience. The purchase and quitting boundaries ansatz take the form (23). We determine the unknown $A_1, A_2, \bar{V}[\sqrt{\varepsilon}K]$, and $\underline{V}[\sqrt{\varepsilon}K]$ from the boundary conditions

$$V^B(t,\bar{V}_t) = \bar{V}_t - p_t \implies A_1 e^{K_-\bar{V}[\sqrt{\varepsilon}K]} + A_2 e^{K_+\bar{V}[\sqrt{\varepsilon}K]} - \frac{c}{r} = \bar{V}[\sqrt{\varepsilon}K] \tag{50}$$

$$\partial_x V^B(t,\bar{V}_t) = 1 \implies A_1 K_- e^{K_-\bar{V}[\sqrt{\varepsilon}K]} + A_2 K_+ e^{K_+\bar{V}[\sqrt{\varepsilon}K]} = 1 \tag{51}$$

$$V^B(t,\underline{V}_t) = 0 \implies A_1 e^{K_-\underline{V}[\sqrt{\varepsilon}K]} + A_2 e^{K_+\underline{V}[\sqrt{\varepsilon}K]} - \frac{c}{r} = 0 \tag{52}$$

$$\partial_x V^B(t,\bar{V}_t) = 0 \implies A_1 K_- e^{K_-\underline{V}[\sqrt{\varepsilon}K]} + A_2 K_+ e^{K_+\underline{V}[\sqrt{\varepsilon}K]} = 0 \tag{53}$$

From (52) and (53) we find that

$$A_1 = \frac{c}{r}\left(\frac{K_+}{K_+ - K_-}\right)e^{-K_-\underline{V}[\sqrt{\varepsilon}K]},\quad A_2 = \frac{c}{r}\left(\frac{K_-}{K_- - K_+}\right)e^{-K_+\underline{V}[\sqrt{\varepsilon}K]}. \tag{54}$$

Substituting (54) back into (51), we obtain the equation to be solved for $(\bar{V}[\sqrt{\varepsilon}K]-\underline{V}[\sqrt{\varepsilon}K])$:

$$e^{K_+(\bar{V}[\sqrt{\varepsilon}K]-\underline{V}[\sqrt{\varepsilon}K])}-e^{K_-(\bar{V}[\sqrt{\varepsilon}K]-\underline{V}[\sqrt{\varepsilon}K])}=\frac{r}{c}\cdot\frac{K_- - K_+}{K_-K_+}, \tag{55}$$

we note that the LHS is an increasing function, hence the solution always exists. Finally, we find $\bar{V}[\sqrt{\varepsilon}K]$ by substituting (54) back into (50) and simplify:

$$\bar{V}[\sqrt{\varepsilon}K]=\frac{1}{K_-}+\frac{c}{r}\left(e^{K_+(\bar{V}[\sqrt{\varepsilon}K]-\underline{V}[\sqrt{\varepsilon}K])}-1\right) \tag{56}$$

from this it is simple to find $\underline{V}[\sqrt{\varepsilon}K]$. Equation (55) and (56) is equivalent to the following non–linear system of equations:

$$\begin{cases} e^{\frac{\sqrt{\varepsilon}K+\sqrt{\varepsilon K^2+2r\sigma^2}}{\sigma^2}(\bar{V}[\sqrt{\varepsilon}K]-\underline{V}[\sqrt{\varepsilon}K])}-e^{\frac{\sqrt{\varepsilon}K-\sqrt{\varepsilon K^2+2r\sigma^2}}{\sigma^2}(\bar{V}[\sqrt{\varepsilon}K]-\underline{V}[\sqrt{\varepsilon}K])} & =\dfrac{\sqrt{\varepsilon K^2+2r\sigma^2}}{c} \\ \dfrac{c}{r}\left(e^{\frac{\sqrt{\varepsilon}K+\sqrt{\varepsilon K^2+2r\sigma^2}}{\sigma^2}(\bar{V}[\sqrt{\varepsilon}K]-\underline{V}[\sqrt{\varepsilon}K])}-1\right)-\bar{V}[\sqrt{\varepsilon}K] & =\dfrac{\sqrt{\varepsilon}K+\sqrt{\varepsilon K^2+2r\sigma^2}}{2r} \end{cases} \tag{57}$$

When $\sqrt{\varepsilon}K \sim 0$, we may obtain a simple expression for $\bar{V}[\sqrt{\varepsilon}K]$ and $\underline{V}[\sqrt{\varepsilon}K]$ to the ε–order. We substituting the ansatz (24) into (55), (56), and comparing the zeroth–order and $\sqrt{\varepsilon}$–order terms we get the claimed expression for $\bar{S} := \bar{R}/K, \underline{S} := \underline{R}/K$. The signs of $\bar{S}$ and $\underline{S}$ followed from the Proposition 2, but one can also verify explicitly. □

Proof of Proposition 5. The solution (25) and (26) to (21) and (20) can be obtained using a standard ODE solving technique such as the "variation of parameters". The rest of the results are easily taken care of by Corollary 2. □

Proof of Lemma 3. Part 1: It follows from footnote 6 that $\mathbb{P}[\tau^*[p_0] < \infty] = 1$. Therefore, $\mathbb{P}[v^x_{\tau^*[p_0]} \geq \bar{V}[p_0]] + \mathbb{P}[v^x_{\tau^*[p_0]} \leq \bar{V}[p_0]] = 1$. Since $\{v^x_{t\wedge\tau^*[p_0]}\}_{t\geq 0}$ is a uniformly integrable martingale, by the Martingale Stopping Theorem, $x = v^x_0 = \mathbb{E}[v^x_{\tau^*[p_0]}] = \bar{V}[p_0]\mathbb{P}[v^x_{\tau^*[p_0]} \geq \bar{V}[p_0]] + \underline{V}[p_0]\mathbb{P}[v^x_{\tau^*[p_0]} \leq \bar{V}[p_0]]$. If $m = 0$ then $\mathcal{V}^S(x;p_0) = (p_0-g)\mathbb{P}[v^x_{\tau^*[p_0]} \geq \bar{V}[p_0]]$. One can prove the first part by solving the system of linear equations for $\mathbb{P}[v^x_{\tau^*[p_0]} \geq \bar{V}[p_0]]$.

Part 2: We consider $v_t = \sigma W_t$, where $\{W_t\}_{t\geq 0}$ denotes the standard Brownian motion. Then $\mathcal{V}^S(x;p_0) = (p_0-g)\mathbb{E}[e^{-m\tau^*[p_0]} \cdot 1\{v^x_{\tau^*[p_0]} = \bar{V}[p_0]\}]$, which can be evaluated using the standard technique involving Martingale Stopping Theorem (see Karatzas and Shreve (2012)). □

Proof of Proposition 6. We would like to solve the PDE initial boundary value problem (31) up to the ε–order. The idea is similar to the proof of Proposition 2, except it is easier here since the boundaries $\bar{V}[p], \underline{V}[p]$ are already fixed for us by Proposition 2. The claim is that if $\mathcal{V}^S(.,.;p)$ solves (31) exactly for the given $\bar{V}[p], \underline{V}[p]$, and $p \in \mathcal{P}_T$, and if $\mathcal{V}^\varepsilon_{\leq k}(.,.;p)$ solves (31) up to the $\varepsilon^{(k+1)/2}$–order with the same given $\bar{V}[p], \underline{V}[p]$, and $p \in \mathcal{P}_T$, then by comparing their corresponding Feynman–Kac expressions, we have $\mathcal{V}^S = \mathcal{V}^\varepsilon_{\leq k} + O(\varepsilon^{(k+1)/2})$. We omit further detail, and proceed with the $k = 1$ for the seller's expected payoff up to the ε–order.

We recall that the buyer's ε–optimal response given by Proposition 2, characterized by the purchase and quitting boundaries: $\bar{V}[p] = (\bar{V}[p_0] + \sqrt{\varepsilon}h) + \sqrt{\varepsilon}\bar{R}$, and $\underline{V}[p] = (\underline{V}[p_0] + \sqrt{\varepsilon}h) + \sqrt{\varepsilon}\underline{R}$, respectively. We propose the perturbation ansatz:

$$\begin{aligned}\mathcal{V}^S(t,x;p) &= \mathcal{V}^S_0\left(\frac{\bar{V}[p_0] - \underline{V}[p_0]}{\bar{V}_t[p] - \underline{V}_t[p]}(x - \underline{V}_t[p]) + \underline{V}[p_0]; p_0\right) + \sqrt{\varepsilon}\mathcal{V}^S_1(t,x) + O(\varepsilon) \\ &= \mathcal{V}^S_0\left((1 - \sqrt{\varepsilon}r_{1,t})x - \sqrt{\varepsilon}r_{0,t}; p_0\right) + \sqrt{\varepsilon}\mathcal{V}^S_1(t,x) + O(\varepsilon),\end{aligned}$$

where, in the second equality, we expanded the argument of $\mathcal{V}^S_0(.;p_0)$ to the first order in $\sqrt{\varepsilon}$. The first term represents a naive rescaling of the constant price solution according to the buyer's response moving boundaries. Substituting the ansatz into the PDE (31) and collect the $\sqrt{\varepsilon}$–terms, we obtain the PDE for $\mathcal{V}^S_1$:

$$\begin{aligned}&\frac{\sigma(x)^2}{2}\partial_x^2\mathcal{V}^S_1(t,x) + \partial_t\mathcal{V}^S_1(t,x) - m\mathcal{V}^S_1(t,x) \\ &\qquad + \left(\sigma(x)\sigma'(x)(r_{1,t}x + r_{0,t}) - \sigma(x)^2 r_{1,t}\right)\partial_x^2\mathcal{V}^S_0(x;p_0) - (r'_{1,t}x + r'_{0,t})\partial_x\mathcal{V}^S_0(x;p_0) = 0,\end{aligned}$$

along with the boundary conditions up to the ε–order:

$$\begin{aligned}\mathcal{V}^S(t,\bar{V}_t[p];p) = p_t - g \implies &\mathcal{V}^S_0(\bar{V}[p_0];p_0) + \sqrt{\varepsilon}\mathcal{V}^S_1(t,\bar{V}[p_0]) + O(\varepsilon) = p_0 + \sqrt{\varepsilon}h_t - g \\ &\implies \mathcal{V}^S_1(t,\bar{V}[p_0]) = h_t\end{aligned}$$

$$\mathcal{V}^S(t,\underline{V}_t[p];p) = 0 \implies \mathcal{V}^S_0(\underline{V}[p_0];p_0) + \sqrt{\varepsilon}\mathcal{V}^S_1(t,\underline{V}[p_0]) + O(\varepsilon) = 0 \implies \mathcal{V}^S_1(t,\underline{V}[p_0]) = 0,$$

and finally the terminal condition at any $T' \geq T$: $\mathcal{V}^S(T',x;p) = \mathcal{V}^S_0(x;p_T)$ gives

$$\mathcal{V}^S_1(T',x) = \frac{1}{\sqrt{\varepsilon}}\left(\mathcal{V}^S_0(x;p_0 + \sqrt{\varepsilon}h_T) - \mathcal{V}^S_0((1 - \sqrt{\varepsilon}r_{1,T})x - \sqrt{\varepsilon}r_{0,T};p_0)\right) + O(\sqrt{\varepsilon}).$$

Note that $\mathcal{V}^S_1(T',x)$ is determined by the constant price expected profits $\mathcal{V}^S_0$ which is not difficult to find, and it is in fact independent of $T' \geq T$. We can reverse the time–axis, then apply (Evans, 2022, Chapter 7, Theorem 7) or (Friedman, 2008, Chapter 3, §5, Corollary 2) to conclude the existence of the smooth solution $\mathcal{V}^S_1$ to the parabolic initial boundary–value problem. The solution is unique and admits the following probabilistic expression via the semi–elliptic version of Feynman–Kac Formula (Øksendal, 2003, Theorem 9.1.1):

$$\begin{aligned}\mathcal{V}_1^S(t,x) &= \mathbb{E}\left[e^{-m(T'-t)}\mathcal{V}_1^S(T',v_{T'}^{t,x})\cdot 1\left\{\tau_\Omega^{t,x}\geq T'\right\}|\mathscr{F}_t\right]\\ &\quad+\mathbb{E}\left[h_{\tau_\Omega^{t,x}}e^{-m(\tau_\Omega^{t,x}-t)}\cdot 1\left\{v_{\tau_\Omega^{t,x}}^{t,x}\geq \bar{V}[p_0],\tau_\Omega^{t,x}<T'\right\}|\mathscr{F}_t\right]\\ &\quad-\mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}\wedge T'} e^{-m(s-t)}(r'_{1,s}v_s^{t,x}+r'_{0,s})\partial_x\mathcal{V}_0^S(v_s^{t,x};p_0)ds|\mathscr{F}_t\right]\\ &+\mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}\wedge T'} e^{-m(s-t)}\left(\sigma(v_s^{t,x})\sigma'(v_s^{t,x})(r_{1,s}v_s^{t,x}+r_{0,s})-\sigma(v_s^{t,x})^2r_{1,s}\right)\partial_x^2\mathcal{V}_0^S(v_s^{t,x};p_0)ds|\mathscr{F}_t\right].\end{aligned} \tag{58}$$

The first term is upper–bounded by $\sup_{x\in[\underline{V}[p_0],\bar{V}[p_0]]}\mathcal{V}_1^S(T,x)\mathbb{P}\left[\tau_\Omega^{t,x}\geq T'|\mathscr{F}_t\right]\to 0$ as $T'\to\infty$ by footnote 6. Since $h\in\mathcal{P}_T$, we have that the second term is upper–bounded by $\sup_{s\in[t,T]}h_s$. As a consequence of $h\in\mathcal{P}_T$, it also follows that $r_{0,t}$, $r_{1,t}$ are constant for all $t\geq T$. Meanwhile, $v_s^{t,x}$ is bounded inside $[\underline{V}[p_0],\bar{V}[p_0]]$. Therefore, the third and fourth terms are upper–bounded by some constant (which can be determined by the supremum of the absolute value of the integrand over the compact set $[t,T]\times[\underline{V}[p_0],\bar{V}[p_0]]$) multiple of $\mathbb{E}[\tau_\Omega^{t,x}|\mathscr{F}_t]<\infty$. Taking the limit $T'\to\infty$ using the Dominated Convergence Theorem, then set $t=0$, we obtain the expression (33) for $\mathcal{V}_1^S(0,x)$. □

Proof of Proposition 7. Equation (34) in the proposition follows from an application of Proposition 1 to the pricing strategy $p:=p_0+Kh$. It follows that as $K\to\infty$, the corresponding purchase and quitting boundaries $\bar{V}_t[p_0+Kh]$ and $\underline{V}_t[p_0+Kh]$ will monotonically decrease and increase toward p_0+Kh_t, respectively. If $x>p_0$ then only the purchasing boundary $\bar{V}_0[p_0+Kh]$ will reach x as $K\to\infty$, giving the seller the payoff p_0-g. Likewise, for $x\leq p_0$ only $\underline{V}_0[p_0+Kh]$ will reach x as $K\to\infty$ giving the seller the payoff 0.

Further, suppose that x is sufficiently high such that we can find a seller's ε–optimal pricing strategy $\tilde{p}\in\mathcal{P}_T$ satisfying $\underline{V}_t[\tilde{p}]>g$ for all $t\in[0,\infty)$. Let $\tau^*[\tilde{p}]\in\mathcal{T}$ denotes the corresponding ε–optimal buyer's stopping time to the pricing strategy $\tilde{p}$. It follows that

$$\begin{aligned}\mathcal{V}^S(x;\tau^*[\tilde{p}],\tilde{p}) &= \mathbb{E}\left[e^{-m\tau^*[\tilde{p}]}(p_{\tau^*[\tilde{p}]}-g)\cdot 1_{v_{\tau^*[\tilde{p}]}\geq\tilde{p}_{\tau^*[\tilde{p}]}}\mid v_0=x\right]\\ &\leq\mathbb{E}\left[\left(\tilde{p}_{\tau^*[\tilde{p}]}-g\right)\cdot 1_{v_{\tau^*[\tilde{p}]}\geq\tilde{p}_{\tau^*[\tilde{p}]}}\mid v_0=x\right]\leq\mathbb{E}\left[v_{\tau^*[\tilde{p}]}-g\mid v_0=x\right]=x-g.\end{aligned} \tag{59}$$

The first inequality follows from removing the discounting factor. The second inequality follows by noting that if v_t hits the purchasing boundary $\bar{V}_t[\tilde{p}]$ first we would have $v_{\tau^*[\tilde{p}]}-g\geq\tilde{p}_{\tau^*[p]}-g$, and if v_t hits the quitting boundary first we would have $v_{\tau^*[\tilde{p}]}<\tilde{p}_{\tau^*[\tilde{p}]}$, so

$v_{\tau^*[\tilde{p}]} - g = \underline{V}_{\tau^*[\tilde{p}]}[\tilde{p}] - g \geq 0 = (p_{\tau^*[\tilde{p}]} - g) \cdot 1_{v_{\tau^*[\tilde{p}]} \geq \tilde{p}_{\tau^*[\tilde{p}]}}$. The final equality followed from the Martingale stopping theorem since $\left|v_{t\wedge\tau^*[\tilde{p}]}\right|$ is bounded by $\max_{s\in[0,\infty)}\{|\underline{V}_s[\tilde{p}]|, |\bar{V}_s[\tilde{p}]|\} = \max_{s\in[0,T]}\{|\underline{V}_s[\tilde{p}]|, |\bar{V}_s[\tilde{p}]|\}$ where the latter is finite because both boundaries are continuous over $[0,T]$ and are constant over $[T,\infty)$ by the definition of $\mathcal{P}_T$. So $x - g \geq V^S(x) - \varepsilon$ for any arbitrary $\varepsilon > 0$, hence we conclude that $V^S(x) = x - g$. The claim that this supremum can be approached by the sequence $\{p_n := p_{0,n} + K_n h \in \mathcal{P}_T\}_{n\in\mathbb{Z}_{\geq 0}}$ follows from (34). $\square$

Proof of Theorem 1. We obtain the expression of $\mathcal{V}^S(x; p_0, K) = \mathcal{V}^S(x; p)$ from Proposition 6 with $h := Kt$, although strictly speaking, some justification is needed as $p \in \mathcal{P}$ instead of $p \in \mathcal{P}_T$. We will now discuss the detail. Following the proof of Proposition 6, we obtain the expression (58) for $\mathcal{V}_1^B(t,x)$ for any arbitrary large $T' \geq t$[12]. Since $m = 0$, we have from Lemma 3 that $\mathcal{V}_0^S(x; p_0) = (p_0 - g)\frac{x - \underline{V}[p_0]}{\bar{V}[p_0] - \underline{V}[p_0]}$. This means that the fourth term of (58) vanishes and (58) simplified to:

$$\mathcal{V}_1^S(t,x) = \mathbb{E}\left[\mathcal{V}_1^S(T', v_{T'}^{t,x}) \cdot 1\left\{\tau_\Omega^{t,x} \geq T'\right\}|\mathscr{F}_t\right] + \mathbb{E}\left[\tau_\Omega^{t,x} \cdot 1\left\{v_{\tau_\Omega^{t,x}}^{t,x} \geq \bar{V}[p_0], \tau_\Omega^{t,x} < T'\right\}|\mathscr{F}_t\right] - \frac{p_0 - g}{\bar{V}[p_0] - \underline{V}[p_0]}\mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}\wedge T'} (r_{1,s}' v_s^{t,x} + r_{0,s}')ds|\mathscr{F}_t\right], \quad (60)$$

$$\begin{aligned} \text{where } r_{1,t} &:= \frac{\bar{R}_t - \underline{R}_t}{\bar{V}[p_0] - \underline{V}[p_0]} = K \cdot \frac{\bar{S}_{0,0} - \underline{S}_{0,0} + (\bar{S}_{0,1} - \underline{S}_{0,1})t}{\bar{V}[p_0] - \underline{V}[p_0]} \\ r_{0,t} &:= h_t + \underline{R}_t - r_{1,t}\underline{V}[p_0] \\ &= K \cdot \left(\underline{S}_{0,0} - \frac{\bar{S}_{0,0} - \underline{S}_{0,0}}{\bar{V}[p_0] - \underline{V}[p_0]}\underline{V}[p_0]\right) + Kt \cdot \left(1 + \underline{S}_{0,1} - \frac{\bar{S}_{0,1} - \underline{S}_{0,1}}{\bar{V}[p_0] - \underline{V}[p_0]}\underline{V}[p_0]\right) \end{aligned} \quad (61)$$

are as defined in Corollary 2. We can further simplify the third term as follows:

$$\begin{aligned} &- \frac{p_0 - g}{\bar{V}[p_0] - \underline{V}[p_0]}\mathbb{E}\left[\int_t^{\tau_\Omega^x\wedge T'} (r_{1,s}' v_s^x + r_{0,s}')ds|\mathscr{F}_t\right] \\ &\quad = -\frac{p_0 - g}{\bar{V}[p_0] - \underline{V}[p_0]}\mathbb{E}\left[\int_t^{\tau_\Omega^x\wedge T'} d(r_{1,s}v_s^x) - \int_t^{\tau_\Omega^x\wedge T'} r_{1,s}dv_s^x + \int_t^{\tau_\Omega^x\wedge T'} r_{0,s}'ds|\mathscr{F}_t\right] \\ &\quad\quad = -\frac{p_0 - g}{\bar{V}[p_0] - \underline{V}[p_0]}\left(\mathbb{E}\left[r_{1,\tau_\Omega^x\wedge T'}v_{\tau_\Omega^x\wedge T'}^x + r_{0,\tau_\Omega^x\wedge T'}|\mathscr{F}_t\right] - (r_{1,t}x - r_{0,t})\right). \end{aligned}$$

We used Ito's Lemma in the first equality. For the second equality, note that $\{v_t^x\}_{t\geq 0}$ is a square–integrable martingale, hence we know that $\int_0^{\tau_\Omega^x} r_{1,s}dv_s^x$ is a continuous square–

[12] Except now $\mathcal{V}_1^S(T', x)$ is not constant in T'. Instead, since p is linear, $\mathcal{V}_1^S(T', x)$ is at most linear in T', and this will be sufficient for us to argue that the first term of (58) converges to zero as $T' \to 0$.

integrable martingale, therefore its expectation vanishes.

Since h is linear in t, we find that $\mathcal{V}_1^S(T',x)$ is at most linear in T', and we have already seen from (61) that $r_{1,t}$, $r_{0,t}$ are linear in t. Meanwhile, $v_s^{t,x}$ is bounded inside $[\underline{V}[p_0],\bar{V}[p_0]]$. Therefore, the first term is upper–bounded by $\sup_{x\in[\underline{V}[p_0],\bar{V}[p_0]]}\mathcal{V}_1^S(T',x)\mathbb{P}\left[\tau_\Omega^{t,x}\geq T'|\mathscr{F}_t\right]\to 0$ as $T'\to\infty$ since $\mathbb{P}\left[\tau_\Omega^{t,x}\geq T'|\mathscr{F}_t\right]\to 0$ exponentially according to footnote 6. Both the second and third terms are upper–bounded by some constant multiple of $\mathbb{E}[\tau_\Omega^{t,x}|\mathscr{F}_t]<\infty$. Taking the limit $T'\to\infty$ using the Dominated Convergence Thoerem, then set $t=0$, we obtain the expression for $\mathcal{V}_1^S(0,x)$. Substituting the expression for $\mathcal{V}_1^S(0,x)$ into the perturbative expansion (32), we obtain $\mathcal{V}^S(x;p_0,K)=(p_0-g)\frac{x-\underline{V}[p_0]}{\bar{V}[p_0]-\underline{V}[p_0]}+\sqrt{\varepsilon}K\mathbb{E}\left[\tau_\Omega^x\cdot 1\left\{v_{\tau_\Omega^x}^x\geq\bar{V}[p_0]\right\}\right]-\frac{\sqrt{\varepsilon}(p_0-g)}{\bar{V}[p_0]-\underline{V}[p_0]}\mathbb{E}\left[r_{1,\tau_\Omega^x}v_{\tau_\Omega^x}^x+r_{0,\tau_\Omega^x}\right]+O(\varepsilon)$, which leads to (35) after some simplifications and substitution of (61). □

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Online Appendix for Gradual Learning and Non-stationary Pricing

Viscosity Solution and Perturbation Theory

As a standard practice in optimal stopping theory, rather than directly finding the optimal $\tau^*[p] \in \mathcal{T}$ to the optimization problem (5), it is often more analytically tractable to consider the corresponding Hamilton–Jacobi–Bellman (HJB) equation:

$$H(t, x, V, \nabla V, \Delta V) = 0, \tag{62}$$

where $H : (\mathbb{R} \times [\underline{\pi}, \bar{\pi}]) \times \mathbb{R} \times \mathbb{R}^2 \times \mathcal{S}_2(\mathbb{R}) \to \mathbb{R}$ is given by

$$H(t, x, V, \nabla V, \Delta V) := \min \left\{ c + rV - \partial_t V - \frac{\sigma(x)^2}{2} \partial_x^2 V, V - x + p_t, V \right\},$$

with $\mathcal{S}_2(\mathbb{R})$ denoted the space of 2×2 symmetric matrices. Since $p \in \mathcal{P}_T$ is only defined for $t \geq 0$, to discuss the solution on $\mathbb{R} \times [\underline{\pi}, \bar{\pi}]$, we extend it by defining $p_t = p_0$ for all $t < 0$. We consider the solution $V : \mathbb{R} \times [\underline{\pi}, \bar{\pi}] \to \mathbb{R}$ subject to the following asymptotic boundary conditions:

$$\begin{aligned} V(t, x) &= V_0^B(x; p_T),\ \forall t \geq T, \qquad && \lim_{t \to -\infty} V(t, x) = V_0^B(x; p_0) \\ V(t, x) &= x - p_t,\ \forall x \geq \bar{V}_t[p], \qquad && V(t, x) = 0,\ \forall x \leq \underline{V}_t[p] \end{aligned} \tag{63}$$

for some functions $\bar{V}[p], \underline{V}[p] : \mathbb{R} \to [\underline{\pi}, \bar{\pi}]$, depending on $p \in \mathcal{P}_T$ and $\bar{V}_t[p] \geq \underline{V}_t[p], \forall t \in \mathbb{R}$. The purchase and quitting boundaries, $\bar{V}[p]$, and $\underline{V}[p]$, provide a simple characterization of the learning strategy. Note that by definition of $\mathcal{P}$, the range of $\bar{V}[p]$ and $\underline{V}[p]$ are contained in $[\underline{\pi}, \bar{\pi}]$.

We need to establish the existence and uniqueness of the solution to (62) subjects to the boundary condition (63). Since the classical solution does not always exist, we will work with a relaxed notion of a *viscosity* solution:[1]

Definition 2. *Let $D \subset \mathbb{R}^n$ and $H : D \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}_n(\mathbb{R}) \to \mathbb{R}$ be a continuous function satisfying the* properness *condition: $H(\mathbf{x}, v, p, X) \geq H(\mathbf{x}, u, p, X)$ if $v \geq u$, and the* degenerate ellipticity *condition: $H(\mathbf{x}, v, p, X) \geq H(\mathbf{x}, v, p, Y)$ if $Y \geq X$.*

A continuous function $v : D \to \mathbb{R}$ is a viscosity subsolution *if for any $\mathbf{x}_0 \in D$ and any twice continuously differentiable function ϕ such that $\mathbf{x}_0$ is a local maximum of $v - \phi$ we have $H(\mathbf{x}_0, v(\mathbf{x}_0), \nabla \phi(\mathbf{x}_0), \Delta \phi(\mathbf{x}_0)) \leq 0$.*

[1] Crandall et al. (1992) provides a detailed description of the viscosity solution.

A continuous function $v : D \to \mathbb{R}$ is a viscosity supersolution *if for any $\mathbf{x}_0 \in D$ and any twice continuously differentiable function ϕ such that $\mathbf{x}_0$ is a local minimum of $v - \phi$ we have $H(\mathbf{x}_0, v(\mathbf{x}_0), \nabla\phi(\mathbf{x}_0), \Delta\phi(\mathbf{x}_0)) \geq 0$.*

A continuous function $v : D \to \mathbb{R}$ is a viscosity *solution if it is both a viscosity subsolution and supersolution.*

The following existence and uniqueness result relates the buyer's value function V^B to the viscosity solution over the domain $D := \mathbb{R} \times [\underline{\pi}, \bar{\pi}]$.

Lemma 5. *For a given $p \in \mathcal{P}_T$, the buyer's value function V^B is the unique viscosity solution to (62) subject to the asymptotic boundary conditions (63).*

Working directly with the viscosity solution via Definition 2 can still be challenging, thus we alternatively consider the following free–boundary backward parabolic PDE initial–value problem: Find $V : \Omega \to \mathbb{R}$, and continuously differentiable functions $\bar{V}[p], \underline{V}[p] : \mathbb{R} \to [\underline{\pi}, \bar{\pi}]$ satisfying $\bar{V}_t[p] \geq \underline{V}_t[p]$, such that

$$\begin{cases} \dfrac{\sigma(x)^2}{2}\partial_x^2 V(t,x) + \partial_t V(t,x) - rV(t,x) - c = 0, & (t,x) \in \Omega \\ V(t, \bar{V}_t[p]) = \bar{V}_t[p] - p_t, & V(t, \underline{V}_t[p]) = 0, \\ \partial_x V(t, \bar{V}_t[p]) = 1, & \partial_x V(t, \underline{V}_t[p]) = 0, \\ V(T,x) = V_0^B(x; p_T), & \end{cases} \tag{64}$$

where

$$\Omega := \{(t,x) \in (-\infty, T] \times [\underline{\pi}, \bar{\pi}] \mid \underline{V}_t[p] < x < \bar{V}_t[p]\}.$$

The upshot is that, any *exact* solution to the problem (64), subject to some additional mild conditions, will also be the viscosity solution of the HJB (62) and thus the consumer's value function by Lemma 5. Meanwhile, (64) enables us to naturally use a perturbation technique to obtain the consumer value function up to any $\varepsilon^{(k+1)/2}$–order by *solving (64) up to the $\varepsilon^{(k+1)/2}$–order*, since the solution up to the $\varepsilon^{(k+1)/2}$–order of (64), subject to some additional mild conditions, also agree with the viscosity solution to (62) up to the $\varepsilon^{(k+1)/2}$–order. We discuss the details in the following.

Given a solution V to (64) on Ω with the specified boundary conditions, we can extend it to $\widetilde{V}$, a function continuously differentiable on $\mathbb{R} \times [\underline{\pi}, \bar{\pi}]$, and twice continuously differentiable in x on $\mathbb{R} \times [\underline{\pi}, \bar{\pi}] \setminus \partial\Omega$, by defining $\widetilde{V}(t,x) = \max\{x - p_t, 0\}$ if $t \leq T$ and $x \notin (\underline{V}_t[p], \bar{V}_t[p])$, and $\widetilde{V}(t,x) = V_0^B(x; p_T)$ if $t > T$. This extension is rather natural, therefore, we will abuse the notation and simply refer to $\widetilde{V}$ as V. We will state formally in Lemma 6 that the solution V will coincide with the buyer's value function V^B. This justifies that the constant price benchmark solutions in §3.1 are the viscosity solutions, and therefore the value functions of their respective buyer's problems.

Suppose we know the value function $V^B(.,.;p)$ for a given $p \in \mathcal{P}_T$ is a solution to (64), and we would like to compute $V^B(.,.;p+\sqrt{\varepsilon}h)$ for some $h \in \mathcal{P}_T$ and a small $\varepsilon > 0$ up to the $\varepsilon^{(k+1)/2}$–order. By Lemma 6, we aim to solve for the corresponding solution $V(.,.;p+\sqrt{\varepsilon}h)$ to (14) up to the $\varepsilon^{(k+1)/2}$–order. The idea of perturbation theory is to consider the solution ansatz

$$\begin{aligned} V^\varepsilon_{\le k,t}(.,.) = V^\varepsilon_{\le k}(.,.;p+\sqrt{\epsilon}h) &:= V_0(.,.) + V_1(.,.)\sqrt{\varepsilon} + \cdots + V_k(.,.)\varepsilon^{k/2} \\ \bar{V}^\varepsilon_{\le k,t} = \bar{V}^\varepsilon_{\le k,t}[p+\sqrt{\varepsilon}h] &:= \bar{V}_{0,t} + \bar{V}_{1,t}\sqrt{\varepsilon} + \cdots + \bar{V}_{k,t}\varepsilon^{k/2} \\ \underline{V}^\varepsilon_{\le k,t} = \underline{V}^\varepsilon_{\le k,t}[p+\sqrt{\varepsilon}h] &:= \underline{V}_{0,t} + \underline{V}_{1,t}\sqrt{\varepsilon} + \cdots + \underline{V}_{k,t}\varepsilon^{k/2}. \end{aligned}$$

where $V_0(.,.) := V_0^B(.,.;p)$, $\bar{V}_{0,t} := \bar{V}_t[p]$, and $\underline{V}_{0,t} := \underline{V}_t[p]$, satisfying (64) over $\Omega^\varepsilon_{\le k} := \{(t,x) \in (-\infty, T] \times [\underline{\pi}, \bar{\pi}] | \underline{V}^\varepsilon_{\le k,t} < x < \bar{V}^\varepsilon_{\le k,t}\}$ up to the $\varepsilon^{(k+1)/2}$–order, i.e.

$$\begin{cases} \dfrac{\sigma(x)^2}{2}\partial_x^2 V^\varepsilon_{\le k,t}(t,x) + \partial_t V^\varepsilon_{\le k,t}(t,x) - rV^\varepsilon_{\le k,t}(t,x) - c = O(\varepsilon^{(k+1)/2}), \quad (t,x) \in \Omega^\varepsilon_{\le k} \\ V^\varepsilon_{\le k,t}(t, \bar{V}^\varepsilon_{\le k,t}) = \bar{V}^\varepsilon_{\le k,t} - (p_t + \sqrt{\varepsilon}h_t) + O(\varepsilon^{(k+1)/2}), \\ V^\varepsilon_{\le k,t}(t, \underline{V}^\varepsilon_{\le k,t}) = O(\varepsilon^{(k+1)/2}), \\ \partial_x V^\varepsilon_{\le k,t}(t, \bar{V}^\varepsilon_{\le k,t}) = 1 + O(\varepsilon^{(k+1)/2}), \\ \partial_x V^\varepsilon_{\le k,t}(t, \underline{V}^\varepsilon_{\le k,t}) = O(\varepsilon^{(k+1)/2}), \\ V^\varepsilon_{\le k,t}(T,x) = V_0^B(x;p_T) + O(\varepsilon^{(k+1)/2}) \end{cases}.$$

By substituting the expression of $V^\varepsilon_{\le k,t}(.,.)$, $\bar{V}^\varepsilon_{\le k,t}$, and $\underline{V}^\varepsilon_{\le k,t}$ into (64) and comparing the $\varepsilon^{k'/2}$ terms for $k' = 1, 2, \cdots, k$, we can solve for $V_{k'}, \bar{V}_{k'}, \underline{V}_{k'}$ using the knowledge of $V_{k''}, \bar{V}_{k''}, \underline{V}_{k''}$ for $k'' = 0, \cdots, k'-1$. Note that although both the value–matching and smooth–pasting conditions are only satisfied up to the $\varepsilon^{(k+1)/2}$–order, it is possible to find a twice continuously differentiable function $\chi : \mathbb{R} \times [\underline{\pi}, \bar{\pi}] \setminus \Omega^\varepsilon_{\le k} \to \mathbb{R}$ which continuously differentiably transitions from $V^\varepsilon_{\le k}$ at $\partial\Omega^\varepsilon_{\le k}$ to $\max\{x - p_t, 0\}$ for all (t,x) some distance $R > 0$ away from $\Omega^\varepsilon_{\le k}$, e.g. a smooth 'bump' function. In particular, we have $\chi = V^\varepsilon_{\le k}$ and $\nabla\chi = \nabla V^\varepsilon_{\le k}$ on $\partial\Omega^\varepsilon_{\le k}$. We also require that $|\partial_t \chi(t,x) - p'_t - \sqrt{\varepsilon}h'_t| = O(\varepsilon^{(k+1)/2})$, $|\partial_x^2 \chi| = O(\varepsilon^{(k+1)/2})$, and that the asymptotic boundary conditions (63) are met. We extend $V^\varepsilon_{\le k}$ to $\widetilde{V}^\varepsilon_{\le k}$, a function continuously differentiable on $\mathbb{R} \times [\underline{\pi}, \bar{\pi}]$, and twice continuously differentiable in x on $\mathbb{R} \times [\underline{\pi}, \bar{\pi}] \setminus \partial\Omega^\varepsilon_{\le k}$, by defining $\widetilde{V}^\varepsilon_{\le k}(t,x) = \chi(t,x)$ if $t \le T$ and $x \notin (\underline{V}^\varepsilon_{\le k,t}, \bar{V}^\varepsilon_{\le k,t})$, $\widetilde{V}^\varepsilon_{\le k}(t,x) = V_0^B(x; p_T + \sqrt{\varepsilon}h_T)$ if $t > T$, and $\widetilde{V}^\varepsilon_{\le k}(t,x) = V^\varepsilon_{\le k}(t,x)$ otherwise. We will abuse the notation and refer to $\widetilde{V}^\varepsilon_{\le k}$ as $V^\varepsilon_{\le k}$.

Lemma 6. *Consider pricing strategies $p, h \in \mathcal{P}_T$ and a given $\varepsilon > 0$.*

1. *If V satisfies the free–boundary backward parabolic PDE initial–value problem (64) with the pricing policy $p \in \mathcal{P}_T$, such that $V(t,x) \geq \max\{x - p_t, 0\}$, and $p'_t + r(\bar{V}_t[p] - p_t) + c \geq 0$ for all $(t,x) \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}]$, then V is a viscosity solution to (62). In particular, the buyer's value function is given by $V^B = V$.*

2. *If $V^\varepsilon_{\leq k}$ satisfies the free–boundary backward parabolic PDE initial–value problem (64) up to the $\varepsilon^{(k+1)/2}$–order with the pricing policy $p + \sqrt{\varepsilon}h \in \mathcal{P}_T$, such that $V^\varepsilon_{\leq k}(t,x) \geq \max\{x - p_t, 0\} + O(\varepsilon^{(k+1)/2})$, and $p'_t + \sqrt{\varepsilon}h'_t + r(\bar{V}^\varepsilon_{\leq k,t}[p + \sqrt{\varepsilon}h] - p_t - \sqrt{\varepsilon}h_t) + c \geq O(\varepsilon^{(k+1)/2})$ for all $(t,x) \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}]$, then $V^B = V^\varepsilon_{\leq k} + O(\varepsilon^{(k+1)/2})$.*

The conditions on p' and h' in Lemma 6 should not be particularly restrictive. For example, if we can check that $\lim_{x \nearrow \bar{V}_t[p]} \partial_x^2 V(t,x) \geq 0$, or $\lim_{x \nearrow \bar{V}^\varepsilon_{\leq k,t}[p+\sqrt{\varepsilon}h]} \partial_x^2 V^\varepsilon_{\leq k,t}(t,x) \geq O(\varepsilon^{(k+1)/2})$ then the conditions on p' and h' are automatically satisfied. In the context of our work, these conditions are easily satisfied when we focus on small perturbations from the known constant price solution and investigate the direction of buyers' reactions. Typically, if our zero–th order perturbation for $p + \sqrt{\varepsilon}h$ is given by the buyer's value function: $V_0(.,.) = V^B(.,.;p)$, and the boundaries $\bar{V}_{0,t} = \bar{V}_t[p]$, $\underline{V}_{0,t} = \underline{V}_t[p]$, corresponding to p, then $p'_t + r(\bar{V}_t[p] - p_t) + c \geq 0$ by Lemma 5. Then it follows that $p'_t + \sqrt{\varepsilon}h'_t + r(V^\varepsilon_{\leq k,t}[p + \sqrt{\varepsilon}h] - p_t - \sqrt{\varepsilon}h_t) + c \geq 0$ is satisfied for all sufficiently small $\varepsilon > 0$. To avoid unnecessary technical complications, for the remainder we shall assume that all the conditions in Lemma 6 are satisfied whenever it is used.

For compatibility with the ε–equilibrium concept, we will only use Lemma 6 with $k = 1$ in all of our applications. However, it is important to remark that Lemma 6 does not directly claim that: $\bar{V}[p + \sqrt{\varepsilon}h] = \bar{V}^\varepsilon_{\leq 1}[p + \sqrt{\varepsilon}h] + O(\varepsilon)$ or $\underline{V}[p + \sqrt{\varepsilon}h] = \underline{V}^\varepsilon_{\leq 1}[p + \sqrt{\varepsilon}h] + O(\varepsilon)$, instead, some extra cares is needed which we outline as follows. The solution $V^\varepsilon_{\leq 1}$ can be interpreted (up to $O(\varepsilon)$), via the probabilistic Feynman–Kac expression, as the expected discounted value of purchasing when the valuation process $v_s^{t,x}$ reaches $\bar{V}^\varepsilon_{\leq 1,s}$ and exiting when $v_s^{t,x}$ reaches $\underline{V}^\varepsilon_{\leq 1,s}$, under the flow cost c. Since $V^B = V^\varepsilon_{\leq 1} + O(\varepsilon)$ according to Lemma 6, the learning strategy characterized by $\bar{V}^\varepsilon_{\leq 1}[p + \sqrt{\varepsilon}h]$ and $\underline{V}^\varepsilon_{\leq 1}[p + \sqrt{\varepsilon}h]$ are considered ε–optimal. Given this understanding, we shall slightly abuse our notation for convenience by writing $\bar{V}[p + \sqrt{\varepsilon}h] = \bar{V}^\varepsilon_{\leq 1}[p + \sqrt{\varepsilon}h] + O(\varepsilon)$ and $\underline{V}[p + \sqrt{\varepsilon}h] = \underline{V}^\varepsilon_{\leq 1}[p + \sqrt{\varepsilon}h] + O(\varepsilon)$.

Omitted Proofs

Proof of Lemma 5. For convenience, in the following we will use $A_1(\mathbf{x}, V, \nabla V, \Delta V) := c + rV - \partial_t V - \frac{\sigma(\mathbf{x})^2}{2}\partial_x^2 V$, $A_2(\mathbf{x}, V, \nabla V, \Delta V) := V - x + p_t$, $A_3(\mathbf{x}, V, \nabla V, \Delta V) := V$, so that

$H := \min_{i=1,2,3} A_i$. We divide the proof in to two parts, first we show that the viscosity solution to (62) subject to the specified boundary conditions is unique, then we show that the value function is a viscosity solution.

Part 1 (viscosity solution is unique): We show that the viscosity solution to (62) subjects to the specified condition is unique. Although, this is mostly an application of the comparison principle (Crandall et al., 1992, Theorem 3.3), in our context the domain is unbounded, hence we layout the detail for completeness. Let $u : \mathbb{R} \times [\underline{\pi}, \bar{\pi}] \to \mathbb{R}$ and $v : \mathbb{R} \times [\underline{\pi}, \bar{\pi}] \to \mathbb{R}$ be viscosity subsolution and supersolution to (62), respectively, and suppose that $\lim_{t \to \pm\infty}(u - v) \leq 0$, $\lim_{x \to \underline{\pi}}(u - v) \leq 0$, and $\lim_{x \to \bar{\pi}}(u - v) \leq 0$. We claim that $u \leq v$ everywhere on $\mathbb{R} \times [\underline{\pi}, \bar{\pi}]$. To show this let us assume the contrary that there exists $\hat{\mathbf{x}} \in \mathbb{R} \times (\underline{\pi}, \bar{\pi})$ such that $u(\hat{\mathbf{x}}) - v(\hat{\mathbf{x}}) = \max_{\mathbf{x} \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}]} (u(\mathbf{x}) - v(\mathbf{x})) > 0$. Consider the function: $w_\alpha(\mathbf{x}, \mathbf{y}) := u(\mathbf{x}) - v(\mathbf{y}) - (\alpha/2)\|\mathbf{x} - \mathbf{y}\|_2^2$ for some constant $\alpha \geq 0$. The assumption on the boundary conditions of u and v implies that for any $\alpha \geq 0$ there exists a local maximum $(\mathbf{x}_\alpha, \mathbf{y}_\alpha) \in (\mathbb{R} \times [\underline{\pi}, \bar{\pi}])^2$ of w_α, and by (Crandall et al., 1992, Lemma 3.1):

$$\lim_{\alpha \to \infty} \alpha \|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2^2 = 0, \quad \lim_{\alpha \to \infty} \left(u(\mathbf{x}_\alpha) - v(\mathbf{x}_\alpha) - \frac{\alpha}{2} \|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2^2 \right) = u(\hat{\mathbf{x}}) - v(\hat{\mathbf{x}}).$$

By our assumption, we can find $\delta > 0$ such that $u(\mathbf{x}_\alpha) - v(\mathbf{y}_\alpha) \geq \delta$ for all $\alpha \geq 0$. We can apply (Crandall et al., 1992, Theorem 3.2) since $\mathbb{R} \times [\underline{\pi}, \bar{\pi}]$ is locally compact, and we find $X, Y \in \mathcal{S}_2(\mathbb{R})$ such that

$$-3\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \tag{65}$$

with $\mathbf{x}_\alpha$ a local maximum of $u(\mathbf{x}) - \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha)^\intercal (\mathbf{x} - \mathbf{x}_\alpha) - \frac{1}{2}(\mathbf{x} - \mathbf{x}_\alpha)^\intercal X (\mathbf{x} - \mathbf{x}_\alpha)$ and $\mathbf{y}_\alpha$ a local minimum of $v(\mathbf{y}) - \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha)^\intercal (\mathbf{y} - \mathbf{y}_\alpha) - \frac{1}{2}(\mathbf{y} - \mathbf{y}_\alpha)^\intercal Y (\mathbf{y} - \mathbf{y}_\alpha)$. Since u and v are subsolution and supersolution, respectively, we have:

$$H(\mathbf{x}_\alpha, u(\mathbf{x}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X) \leq 0 \leq H(\mathbf{y}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), Y). \tag{66}$$

From (65) we have:

$$\begin{aligned}
&A_1(\mathbf{y}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), Y) - A_1(\mathbf{x}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X) \\
&\qquad = \frac{\sigma(\mathbf{x}_\alpha)^2}{2} X_{xx} - \frac{\sigma(\mathbf{y}_\alpha)^2}{2} Y_{xx} = \begin{pmatrix} \sigma(\mathbf{x}_\alpha) & \sigma(\mathbf{y}_\alpha) \end{pmatrix} \begin{pmatrix} X_{xx} & 0 \\ 0 & -Y_{xx} \end{pmatrix} \begin{pmatrix} \sigma(\mathbf{x}_\alpha) \\ \sigma(\mathbf{y}_\alpha) \end{pmatrix} \\
&\quad \leq 3\alpha \begin{pmatrix} \sigma(\mathbf{x}_\alpha) & \sigma(\mathbf{y}_\alpha) \end{pmatrix} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \begin{pmatrix} \sigma(\mathbf{x}_\alpha) \\ \sigma(\mathbf{y}_\alpha) \end{pmatrix} = 3\alpha(\sigma(\mathbf{x}_\alpha) - \sigma(\mathbf{y}_\alpha))^2 \leq 3\alpha L^2 \|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2^2
\end{aligned}$$

where we used the condition (2) for σ in the last inequality. Similarly, we can check that

$$
\begin{aligned}
A_2(\mathbf{y}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), Y) - A_2(\mathbf{x}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X) &\leq \|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2 + |p_{t_x} - p_{t_y}| \\
&\leq \left(1 + \max_{t\in[0,T]} |p'_t|\right) \|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2,
\end{aligned}
$$

and $A_3(\mathbf{y}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), Y) - A_3(\mathbf{x}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X) = 0$. Let us define $\omega(r) := \max\left\{3L^2, 1 + \max_{t\in[0,T]} |p'_t|\right\} \cdot r$ and $i^* := \operatorname{argmin}_{i=1,2,3} A_i(\mathbf{x}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X)$, then

$$
\begin{aligned}
&H(\mathbf{y}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), Y) - H(\mathbf{x}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X) \\
&\qquad \leq A_{i^*}(\mathbf{y}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), Y) - A_{i^*}(\mathbf{x}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X) \\
&\qquad\qquad \leq \omega\left(\alpha\|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2^2 + \|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
0 < \min\{1, r\}\delta &\leq \min\{1, r\}(u(\mathbf{x}_\alpha) - v(\mathbf{y}_\alpha)) \\
&\leq H(\mathbf{x}_\alpha, u(\mathbf{x}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X) - H(\mathbf{x}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X) \\
&= H(\mathbf{x}_\alpha, u(\mathbf{x}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X) - H(\mathbf{y}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), Y) \\
&+ H(\mathbf{y}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), Y) - H(\mathbf{x}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X) \\
&\leq \omega\left(\alpha\|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2^2 + \|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2\right) \quad (67)
\end{aligned}
$$

for all $\alpha \geq 0$, where we used (66) to replace the first two terms with zero in the last inequality. By taking the $\alpha \to \infty$ limit, $\omega\left(\alpha\|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2^2 + \|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2\right) \to 0$, while the inequality above specifies that it is bounded away from zero by $\min\{1, r\}\delta$, which is a contradiction. In other words, we have $u \leq v$ over the entire $\mathbb{R} \times [\underline{\pi}, \bar{\pi}]$. Therefore, if $u : \mathbb{R} \times [\underline{\pi}, \bar{\pi}] \to \mathbb{R}$ and $v : \mathbb{R} \times [\underline{\pi}, \bar{\pi}] \to \mathbb{R}$ are both viscosity solution to (62) with the specified boundary conditions: $\lim_{t\to\pm\infty}(u - v) = 0$, $\lim_{x\to\underline{\pi}}(u - v) = 0$, and $\lim_{x\to\bar{\pi}}(u - v) = 0$, then $u = v$ over the entire $\mathbb{R} \times [\underline{\pi}, \bar{\pi}]$.

Part 2 (the value function is a viscosity solution): First, we show that V^B is continuous. Suppose that $\mathbf{x}_0 = (t, x_0), \mathbf{x}_1 = (t, x_1) \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}]$ are given. For any $\varepsilon > 0$, we can find $\tau_a \in \mathcal{T}$ for $a = 0, 1$ such that: $V^B(\mathbf{x}_a) \leq \mathcal{V}^B(\mathbf{x}_a; \tau_a, p) + \varepsilon$, while $V^B(\mathbf{x}_a) \geq \mathcal{V}^B(\mathbf{x}_a; \tau_{1-a}, p)$ by definition. It follows that:

$$
\left|V^B(\mathbf{x}_0) - V^B(\mathbf{x}_1)\right| \leq \max_{a=0,1} \mathbb{E}\left[e^{-r(\tau_a - t)}\left(\max\{v_{\tau_a}^{t,x_a} - p_{\tau_a}, 0\} - \max\{v_{\tau_a}^{t,x_{1-a}} - p_{\tau_a}, 0\}\right) |\mathscr{F}_t\right] + \varepsilon.
$$

Using the Lipschitz condition (2) with Gronwall's inequality to upper bounds the growth of the SDE solutions $\{v_s^{t,x_a}\}_{s\geq t}$, it is possible to show that the RHS approaches zero as $x_1 \to x_0$. Suppose that $\mathbf{x}_0 = (t_0, x), \mathbf{x}_1 = (t_1, x) \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}]$ are given, for some $t_1 > t_0$. Then by the optimality principle, we can find $\tau \in \mathcal{T}$ such that

$$
\begin{aligned}
&\mathbb{E}\left[e^{-r(t_1\wedge\tau-t)}V^B(t_1\wedge\tau, v_{t_1\wedge\tau}^{t_0,x}) - V^B(\mathbf{x}_1) - \int_{t_0}^{t_1\wedge\tau} ce^{-r(s-t_0)}ds|\mathscr{F}_{t_0}\right] + \varepsilon(t_1 - t_0) \\
&\qquad \geq V^B(\mathbf{x}_0) - V^B(\mathbf{x}_1) \geq \mathbb{E}\left[e^{-r(t_1-t)}V^B(t_1, v_{t_1}^{t_0,x}) - V^B(\mathbf{x}_1) - \int_{t_0}^{t_1} ce^{-r(s-t_0)}ds|\mathscr{F}_{t_0}\right].
\end{aligned}
$$

Using the continuity of V^B in x we have previously proven, we have a pointwise convergence to zero for both integrands as $t_1 \searrow t_0$. We can conclude using the Dominated Convergence Theorem that both expected values converge to zero as $t_1 \searrow t_0$. The convergence to zero as $t_1 \nearrow t_0$ can be obtained similarly. Putting both results together, we find that V^B is continuous in both t and x. Next, we will show that V^B is both a viscosity subsolution and supersolution using the standard argument (e.g. see Yong and Zhou (2012)).

Given $\mathbf{x} = (t, x) \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}]$ and a twice continuously differentiable function ϕ such that $\mathbf{x}$ is a local maximum of $V^B - \phi$. Let us assume that $V^B(t, x) > \max\{x - p_t, 0\}$, i.e. $\mathbf{x}$ is in the learning region, otherwise we have $\min_{i=2,3} A_i(t, x, V^B(t, x), \nabla\phi, \Delta\phi) \leq 0$ then it is trivial that $H(t, x, V^B(t, x), \nabla\phi, \Delta\phi) \leq 0$. Let $t' > t \geq 0$, we can find $\tau \in \mathcal{T}$ such that $V^B(t, x) \leq \mathbb{E}\left[e^{-r(t'\wedge\tau-t)}V^B(t'\wedge\tau, v_{t'\wedge\tau}^{t,x}) - \int_t^{t'\wedge\tau} ce^{-r(s-t)}ds|\mathscr{F}_t\right] + \varepsilon(t' - t)$. Then for all t' sufficiently close to t, we have

$$
\begin{aligned}
&0 \leq \mathbb{E}\left[V^B(t, x) - \phi(t, x) - V^B(t'\wedge\tau, v_{t'\wedge\tau}^{t,x}) + \phi(t'\wedge\tau, v_{t'\wedge\tau}^{t,x}) \mid \mathscr{F}_t\right] \\
&\leq \mathbb{E}\left[\phi(t'\wedge\tau, v_{t'\wedge\tau}^{t,x}) - \phi(t, x) - \int_t^{t'\wedge\tau} ce^{-r(s-t)}ds - \left(1 - e^{-r(t'\wedge\tau-t)}\right)V^B(t'\wedge\tau, v_{t'\wedge\tau}^{t,x}) \mid \mathscr{F}_t\right] \\
&\quad + \varepsilon(t' - t) = \mathbb{E}\left[\int_t^{t'\wedge\tau}\left(\partial_t\phi(s, v_s^{t,x}) + \frac{\sigma(v_s)^2}{2}\partial_x^2\phi(s, v_s^{t,x})\right)ds + \int_t^{t'\wedge\tau}\partial_x\phi(s, v_s^{t,x})dv_s^{t,x}\right. \\
&\qquad\qquad \left.- \int_t^{t'\wedge\tau} ce^{-r(s-t)}ds - \left(1 - e^{-r(t'\wedge\tau-t)}\right)V^B(t'\wedge\tau, v_{t'\wedge\tau}^{t,x}) \mid \mathscr{F}_t\right] + \varepsilon(t' - t).
\end{aligned}
$$

Since $\{v_t^{t,x}\}_{t\geq 0}$ is a square–integrable martingale, we know that $\int_t^{t'\wedge\tau}\partial_x\phi(s, v_s^{t,x})dv_s^{t,x}$ is also a continuous square–integrable martingale (see Karatzas and Shreve (2012)[2]), hence the second term vanishes by the Martingale Stopping Theorem. Dividing both–sides by $t' - t$, taking the limit $t' \to t$ and apply the Dominated Convergence Theorem,

[2] We also need a square–integrability condition: $\mathbb{E}\left[\int_t^{t'\wedge\tau}\left(\sigma(v_s^{t,x})\partial_x\phi(s, v_s^{t,x})\right)^2 ds\right] < \infty$, but we can always ensure this by choosing a more appropriate ϕ with the same $\nabla\phi$ and $\Delta\phi$ at (t, x).

we get: $A_1(t,x,V^B(t,x),\nabla\phi,\Delta\phi) \le \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we conclude that $A_1(t,x,V^B(t,x),\nabla\phi,\Delta\phi) \le 0$. Thus, we have $H(t,x,V^B(t,x),\nabla\phi,\Delta\phi) \le 0$, so V^B is a viscosity subsolution of (62).

Given $\mathbf{x} = (t,x) \in \mathbb{R} \times [\underline{\pi},\bar{\pi}]$ and a twice continuously differentiable function ϕ such that $\mathbf{x}$ is a local minimum of $V^B - \phi$. Then for all $t' > t \ge 0$ sufficiently close, we have

$$\begin{aligned}
0 &\ge \frac{1}{t'-t}\mathbb{E}\left[V^B(t,x) - \phi(t,x) - V^B(t', v_{t'}^{t,x}) + \phi(t', v_{t'}^{t,x}) \mid \mathscr{F}_t\right] \\
&\ge \frac{1}{t'-t}\mathbb{E}\left[\phi(t', v_{t'}^{t,x}) - \phi(t,x) - \int_t^{t'} ce^{-r(s-t)}ds - \left(1 - e^{-r(t'-t)}\right) V^B(t', v_{t'}^{t,x}) \mid \mathscr{F}_t\right] \\
&= \frac{1}{t'-t}\mathbb{E}\Bigg[\int_t^{t'} \left(\partial_t\phi(s, v_s^{t,x}) + \frac{\sigma(v_s^{t,x})^2}{2}\partial_x^2\phi(s, v_s^{t,x})\right) ds + \int_t^{t'} \partial_x\phi(s, v_s^{t,x})dv_s^{t,x} \\
&\qquad - \int_t^{t'} ce^{-r(s-t)}ds - \left(1 - e^{-r(t'-t)}\right) V^B(t', v_{t'}^{t,x}) \mid \mathscr{F}_t\Bigg].
\end{aligned}$$

The second inequality followed from the optimality principle: $V^B(t,x) \ge \mathbb{E}\left[e^{-r(t'-t)}V^B(t', v_{t'}^{t,x}) - \int_t^{t'} ce^{-r(s-t)}ds \mid \mathscr{F}_t\right]$. The second term is a continuous square–integrable martingale, hence vanishes as explained. Taking limit $t' \to t$ and apply the Dominated Convergence Theorem, we get: $A_1(t,x,V^B(t,x),\nabla\phi,\Delta\phi) \ge 0$. The conditions $A_{i=2,3}(t,x,V^B(t,x),\nabla\phi,\Delta\phi) \ge 0$ only depends on $V^B(t,x) \ge \max\{x - p_t, 0\}$, hence they are trivial. Thus, we have shown that $H(t,x,V^B(t,x),\nabla\phi,\Delta\phi) \ge 0$, so V^B is a viscosity supersolution of (62). □

Proof of Lemma 6. Part 1: Let such a solution V to (14) be given. Since we have assumed $V(t,x) \ge \max\{x - p_t, 0\}$ and $p'_t + r(\bar{V}_t[p] - p_t) + c \ge 0$, then $V - \max\{x - p_t, 0\} \ge 0$ for all $\mathbf{x} \in \Omega$, and $c + rV - \partial_t V - \frac{\sigma(\mathbf{x})^2}{2}\partial_x^2 V \ge 0$ for all $\mathbf{x} \in \mathbb{R} \times [\underline{\pi},\bar{\pi}] \setminus \Omega$. By the value–matching, the smooth pasting conditions, and the assumption that $p \in \mathcal{P}_T$ is smooth, we have that V is continuously differentiable[3]. Moreover, V is twice continuously differentiable in x on $\mathbb{R} \times [\underline{\pi},\bar{\pi}] \setminus \partial\Omega$, as it is a (classical) solution to the PDE on Ω, and $\max\{x - p_t, 0\}$ is twice continuously differentiable in x on $\mathbb{R} \times [\underline{\pi},\bar{\pi}] \setminus \Omega$. Therefore, we have $H(\mathbf{x}, V, \nabla V, \Delta V) = 0$ classically on $\mathbb{R} \times [\underline{\pi},\bar{\pi}] \setminus \partial\Omega$. Thus, for any twice continuously differentiable ϕ and any $\mathbf{x}_0 \in \mathbb{R} \times [\underline{\pi},\bar{\pi}]$, we have $\nabla\phi(\mathbf{x}_0) = \nabla V(\mathbf{x}_0)$, and we can find $\{\mathbf{x}_i\}_{i=0}^\infty \subset \mathbb{R} \times [\underline{\pi},\bar{\pi}] \setminus \partial\Omega$ converging to $\mathbf{x}_0$. If $\mathbf{x}_0$ is a local maximum of $V - \phi$ then $\partial_x^2\phi(\mathbf{x}_0) \ge \lim_{i\to\infty}\partial_x^2 V(\mathbf{x}_i)$ which implies $H(\mathbf{x}_0, V(\mathbf{x}_0), \nabla\phi(\mathbf{x}_0), \Delta\phi(\mathbf{x}_0)) \le \lim_{i\to\infty} H(\mathbf{x}_i, V(\mathbf{x}_i), \nabla V(\mathbf{x}_i), \Delta V(\mathbf{x}_i)) = 0$.

[3] To get the continuity of t derivative across the boundary, consider the defining equation: $V^B(t, \bar{V}_t[p]; p) = \bar{V}_t[p] - p_t$. Differentiating with respect to t gives: $\bar{V}'_t[p] \cdot \partial_x V^B(t, \bar{V}_t[p]; p) + \partial_t V^B(t, \bar{V}_t[p]; p) = \bar{V}'_t[p] - p'_t$, or $\partial_t V^B(t, \bar{V}_t[p]; p) = -p'_t$. Similarly, we have $\partial_t V^B(t, \underline{V}_t[p]; p) = 0$

Similarly, if $\mathbf{x}_0$ is a local minimum of $V - \phi$ then $\partial_x^2\phi(\mathbf{x}_0) \le \lim_{i\to\infty} \partial_x^2 V(\mathbf{x}_i)$ which implies $H(\mathbf{x}_0, V(\mathbf{x}_0), \nabla\phi(\mathbf{x}_0), \Delta\phi(\mathbf{x}_0)) \ge \lim_{i\to\infty} H(\mathbf{x}_i, V(\mathbf{x}_i), \nabla V(\mathbf{x}_i), \Delta V(\mathbf{x}_i)) = 0$.

Part 2: Repeat the argument from the previous part with the perturbed pricing policy $p + \sqrt{\varepsilon}h \in \mathcal{P}_T$, we have that $H(\mathbf{x}, V^\varepsilon_{\le k}, \nabla V^\varepsilon_{\le k}, \Delta V^\varepsilon_{\le k}) = O(\varepsilon^{(k+1)/2})$ classically on $\mathbb{R} \times [\underline{\pi}, \bar{\pi}] \setminus \partial\Omega^\varepsilon_{\le k}$. Moreover, for any twice continuously differentiable ϕ and any $\mathbf{x}_0 \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}]$, if $\mathbf{x}_0$ is a local maximum of $V^\varepsilon_{\le k} - \phi$ then $H(\mathbf{x}_0, V^\varepsilon_{\le k}, \nabla\phi, \Delta\phi) \le O(\varepsilon^{(k+1)/2})$, and if $\mathbf{x}_0$ is a local minimum of $V^\varepsilon_{\le k} - \phi$ then $H(\mathbf{x}_0, V^\varepsilon_{\le k}, \nabla\phi, \Delta\phi) \ge O(\varepsilon^{(k+1)/2})$. Since $V^\varepsilon_{\le k}$ satisfies the same asymptotic boundary conditions as the value function V^B, we can repeat the comparison principle argument in the proof of Lemma 5. In particular, setting $u := V^\varepsilon_{\le k}, v := V^B$ we have (66) becomes $H(\mathbf{x}_\alpha, V^\varepsilon_{\le k}, \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X) + O(\varepsilon^{(k+1)/2}) \le 0 \le H(\mathbf{y}_\alpha, V^B, \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), Y)$, which means (67) becomes $\min\{1, r\}(V^\varepsilon_{\le k}(\mathbf{x}_\alpha) - V^B(\mathbf{y}_\alpha)) \le \omega(\alpha\|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2^2 + \|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2) + O(\varepsilon^{(k+1)/2})$. Taking the limit $\alpha \to \infty$, we find that $\sup_{\mathbf{x}\in\mathbb{R}\times[\underline{\pi},\bar{\pi}]} \left(V^\varepsilon_{\le k}(\mathbf{x}) - V^B(\mathbf{x})\right) \le O(\varepsilon^{(k+1)/2})$, in other words: $V^\varepsilon_{\le k}(\mathbf{x}) \le V^B(\mathbf{x}) + O(\varepsilon^{(k+1)/2})$. On the other hand, setting $u := V^B, v := V^\varepsilon_{\le k}$ yields $V^B(\mathbf{x}) \le V^\varepsilon_{\le k}(\mathbf{x}) + O(\varepsilon^{(k+1)/2})$, thus we have $V^B(\mathbf{x}) = V^\varepsilon_{\le k}(\mathbf{x}) + O(\varepsilon^{(k+1)/2})$ as claimed. □

Proof of Proposition 8. Part 1:

From the first equation of (57), when $c \searrow 0$, the RHS becomes large which means $\bar{V}[\sqrt{\varepsilon}K] - \underline{V}[\sqrt{\varepsilon}K]$ becomes large, and the LHS is $\sim e^{\frac{\sqrt{\varepsilon}K + \sqrt{\varepsilon K^2 + 2r\sigma^2}}{\sigma^2}(\bar{V}[\sqrt{\varepsilon}K] - \underline{V}[\sqrt{\varepsilon}K])}$. Therefore, the second equation of (57) together with (23) gives:

$$\bar{V}_t = p_0 + \sqrt{\varepsilon}Kt + \frac{\sqrt{\varepsilon K^2 + 2r\sigma^2} - \sqrt{\varepsilon}K}{2r}.$$

and $\underline{V}_t = -\infty$. Therefore, we only have one linearly moving boundary $\bar{V}_t$. Let's assume throughout also that $p_0 \ge g$. The solution $U(t, v)$ to the heat equation with the single linearly moving absorbing boundary with initial condition $U(t = 0, v) = \delta(v - x), x \le \bar{V}_0$, is well–known:

$$U(t, v) = \frac{\exp\left(-\frac{\sqrt{\varepsilon}K}{\sigma^2}(v - x - \sqrt{\varepsilon}Kt) - \frac{\varepsilon K^2}{2\sigma^2}t\right)}{\sigma\sqrt{2\pi t}} \left(e^{-\frac{(v - \sqrt{\varepsilon}Kt - x)^2}{2t\sigma^2}} - e^{-\frac{(v - \sqrt{\varepsilon}Kt + x - 2\bar{V}_0)^2}{2t\sigma^2}}\right).$$

Therefore, the purchase probability flux is:

$$-\frac{\sigma^2}{2}\partial_v U(t, \bar{V}_t) = \frac{\bar{V}_0 - x}{\sigma\sqrt{2\pi t^3}} \exp\left(-\frac{\left(\bar{V}_t - x\right)^2}{2t\sigma^2}\right).$$

It is now straightforward to compute the expected seller's payoff at $t = 0$:

$$\begin{aligned}
\mathcal{V}^S(x;p_0,K) &:= -\frac{\sigma^2}{2}\int_0^\infty e^{-ms}(p_s-g)\partial_v U(s,\bar{V}_s)ds \\
&= \left(p_0 - g + \frac{\sqrt{\varepsilon}K}{\sqrt{2m\sigma^2+\varepsilon K^2}}\left(p_0 - x + \frac{\sqrt{\varepsilon K^2+2r\sigma^2}-\sqrt{\varepsilon}K}{2r}\right)\right) \\
&\quad \times \exp\left(-\left(\frac{\sqrt{\varepsilon}K+\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}\right)\left(p_0 - x + \frac{\sqrt{\varepsilon K^2+2r\sigma^2}-\sqrt{\varepsilon}K}{2r}\right)\right), \quad (68)
\end{aligned}$$

for $x \le \bar{V}_0$, otherwise if $x > \bar{V}_0$ then we have $\mathcal{V}^S(x;p_0,K) = p_0 - g$. In the special case where $m = 0$, we have

$$\mathcal{V}^S(x;p_0,K) = \begin{cases} \left(2p_0 - g - x + \frac{\sqrt{\varepsilon K^2+2r\sigma^2}-\sqrt{\varepsilon}K}{2r}\right) & \\ \quad \times \exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^2}\left(p_0 - x + \frac{\sqrt{\varepsilon K^2+2r\sigma^2}-\sqrt{\varepsilon}K}{2r}\right)\right), & \sqrt{\varepsilon}K > 0 \\ p_0 - g, & \sqrt{\varepsilon}K = 0 \\ x - g - \frac{\sqrt{\varepsilon K^2+2r\sigma^2}-\sqrt{\varepsilon}K}{2r}, & \sqrt{\varepsilon}K < 0 \end{cases}.$$

For any fixed p_0, we can approach the supremum $2p_0 + \frac{\sigma}{\sqrt{2r}} - g - x \ge p_0 - g$ of $\mathcal{V}^S$ by choosing $\sqrt{\varepsilon}K \gtrsim 0$ as close to 0 as possible, and earning an extra of $\left(2p_0 + \frac{\sigma}{\sqrt{2r}} - g - x\right) - (p_0 - g) = p_0 - x + \frac{\sigma}{\sqrt{2r}}$. If we can also vary the initial price p_0, then it is optimal to set p_0 as large as possible, i.e. the optimal price is unbounded.

Part 2:

For $m > 0$, the optimal K is now bounded from 0. This can also be seen for a general p_0 by computing:

$$\begin{aligned}
\frac{\partial\mathcal{V}^S}{\partial K}(x;p_0,K=0) &= e^{-\frac{\sqrt{2m}}{\sigma}\left(p_0 - x + \frac{\sigma}{\sqrt{2r}}\right)} \\
&\quad \times \left(\frac{p_0 - x + \sigma/\sqrt{2r}}{\sigma\sqrt{2m}} - (p_0 - g)\left(\frac{p_0 - x + \sigma/\sqrt{2r} - (\sigma/r)\sqrt{m/2}}{\sigma^2}\right)\right),
\end{aligned}$$

we can see that this is always > 0 for sufficiently small and sufficiently large $m > 0$. $\square$

Proof of Proposition 9. The standard solution U_0 to the heat equation (29) with 2 absorbing non–moving boundaries at $\bar{V}_0 := p_0 + \bar{V}[\sqrt{\varepsilon}K]$, $\underline{V}_0 := p_0 + \underline{V}[\sqrt{\varepsilon}K]$, and the initial condition

$U_0(0,v) = \delta(v-x)$ is given by Karatzas and Shreve (2012):

$$U_0(t,v) = \frac{1}{\sigma\sqrt{2\pi t}} \sum_{k=-\infty}^{+\infty} \left[e^{-\frac{(v-x+2k(\bar{V}_0-\underline{V}_0))^2}{2t\sigma^2}} - e^{-\frac{(v+x-2\underline{V}_0+2k(\bar{V}_0-\underline{V}_0))^2}{2t\sigma^2}} \right]. \tag{69}$$

Equivalently:

$$U_0(t,v)dv = \mathbb{P}\left[x+\sigma W_t \in dv, \underline{V}_0 < x+\sigma W_s < \bar{V}_0, s \in [0,t]\right].$$

Instead of moving the boundary according to $\sqrt{\varepsilon}Kt$ we may consider the buyer valuation process to be the Brownian process with drift starting at x: $\tilde{v}_t = x - \sqrt{\varepsilon}Kt + \sigma W_t$ with fixed absorbing boundaries at $\bar{V}_0, \underline{V}_0$. By Girsanov Theorem, if $\{W_t\}$ is the standard Brownian process on $(\Omega, \mathscr{F}, \Sigma, \mathbb{P})$ then $\{x+\sigma W_t\}$ is the Brownian process with drift starting at x, i.e. $\{\tilde{v}_t\}$ on $(\Omega, \mathscr{F}, \Sigma, \mathbb{Q})$ where

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathscr{F}_t} = \exp\left(-\frac{\sqrt{\varepsilon}K}{\sigma}W_t - \frac{\varepsilon K^2}{2\sigma^2}t\right).$$

Consequently, we have that the solution U to the heat equation (29) with moving boundaries $\bar{V}_t, \underline{V}_t$ is given by

$$\begin{aligned} U(t,v)dv &= \mathbb{P}\left[\tilde{v}_t \in v - \sqrt{\varepsilon}Kt + dv, \underline{V}_0 < \tilde{v}_s < \bar{V}_0, s \in [0,t]\right] \\ &= \mathbb{Q}\left[x+\sigma W_t \in v - \sqrt{\varepsilon}Kt + dv, \underline{V}_0 < x+\sigma W_s < \bar{V}_0, s \in [0,t]\right] \\ &= \exp\left(-\frac{\sqrt{\varepsilon}K}{\sigma^2}(v-x-\sqrt{\varepsilon}Kt) - \frac{\varepsilon K^2}{2\sigma^2}t\right) U_0(t, v-\sqrt{\varepsilon}Kt)dv \end{aligned}$$

Therefore, the purchase probability flux is:

$$-\frac{\sigma^2}{2}\partial_v U(t,\bar{V}_t) = \sum_{k=-\infty}^{+\infty} \frac{(2k+1)(\bar{V}_0-\underline{V}_0)-(x-\underline{V}_0)}{\sigma\sqrt{2\pi t^3}} e^{\frac{2\sqrt{\varepsilon}Kk}{\sigma^2}(\bar{V}_0-\underline{V}_0)} e^{-\frac{((2k+1)(\bar{V}_0-\underline{V}_0)-(x-\underline{V}_0)+\sqrt{\varepsilon}Kt)^2}{2t\sigma^2}}. \tag{70}$$

The term–by–term differentiation is justified at $v = \bar{V}_t$ for any fixed $x \in (\underline{V}_0, \bar{V}_0)$ because $0 < |\bar{V}_0 - x| < |\bar{V}_0 - \underline{V}_0|$, hence the series representation of $U_0(t, v-\sqrt{\varepsilon}Kt)$, and the derivative series both converge absolutely and uniformly for all v in some neighborhoods of $\bar{V}_t$ and $t \in [0,\infty)$. We now compute the seller's expected profit:

Claim 1. *The seller's expected profit from the buyer initially at $x \in (\underline{V}_0, \bar{V}_0)$ is:*

$$\mathcal{V}^S(x;p_0,K) = \frac{\left(p_0 - g - \frac{\sqrt{\varepsilon}K}{\sqrt{2m\sigma^2+\varepsilon K^2}}(\bar{V}_0 + x - 2\underline{V}_0)\right) e^{-\frac{\sqrt{\varepsilon}K+\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(\bar{V}_0 - x)}}{1 - e^{-\frac{2\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(\bar{V}_0-\underline{V}_0)}}$$

$$+ \frac{\frac{2\sqrt{\varepsilon}K}{\sqrt{2m\sigma^2+\varepsilon K^2}}(\bar{V}_0 - \underline{V}_0) e^{-\frac{\sqrt{\varepsilon}K+\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(\bar{V}_0 - x)}}{\left(1 - e^{-\frac{2\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(\bar{V}_0-\underline{V}_0)}\right)^2}$$

$$- \frac{\left(p_0 - g - \frac{\sqrt{\varepsilon}K}{\sqrt{2m\sigma^2+\varepsilon K^2}}(\bar{V}_0 - x)\right) e^{-\frac{\sqrt{\varepsilon}K+\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(\bar{V}_0-\underline{V}_0)} e^{+\frac{\sqrt{\varepsilon}K-\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(x-\underline{V}_0)}}{1 - e^{-\frac{2\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(\bar{V}_0-\underline{V}_0)}}$$

$$- \frac{\frac{2\sqrt{\varepsilon}K}{\sqrt{2m\sigma^2+\varepsilon K^2}}(\bar{V}_0 - \underline{V}_0) e^{-\frac{\sqrt{\varepsilon}K+\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(\bar{V}_0-\underline{V}_0)} e^{+\frac{\sqrt{\varepsilon}K-\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(x-\underline{V}_0)}}{\left(1 - e^{-\frac{2\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(\bar{V}_0-\underline{V}_0)}\right)^2}, \quad (71)$$

if $m > 0$ or $K \neq 0$, and $\mathcal{V}^S(x;p_0,K) = (p_0 - g)\left(\frac{x-\underline{V}_0}{\bar{V}_0-\underline{V}_0}\right)$ if $m = 0, K = 0$. On the other hand, if $x \leq \underline{V}_0$ then $\mathcal{V}^S(x;p_0,K) = 0$, and if $x \geq \bar{V}_0$ then $\mathcal{V}^s(x;p_0,K) = p_0 - g$.

Proof. We shall only cover the non–trivial case where $x \in (\underline{V}_0, \bar{V}_0)$. First, let's assume that either $m > 0$ or $K \neq 0$. We compute $\mathcal{V}^S(x;p_0,K)$ by substituting (70) into (30):

$$\begin{aligned}
\mathcal{V}^S(x;p_0,K) &= -\frac{\sigma^2}{2}\int_0^\infty e^{-ms}(p_s - g)\partial_v U(s,\bar{V}_s)ds \\
&= \sum_{k=-\infty}^{+\infty} \big((2k+1)(\bar{V}_0 - \underline{V}_0) - (x - \underline{V}_0)\big)\, e^{\frac{2\sqrt{\varepsilon}Kk}{\sigma^2}(\bar{V}_0-\underline{V}_0)} \\
&\quad \times \int_0^{+\infty} \frac{(p_0 + \sqrt{\varepsilon}Ks - g)}{\sigma\sqrt{2\pi s^3}} e^{-ms-\frac{\left((2k+1)(\bar{V}_0-\underline{V}_0)-(x-\underline{V}_0)+\sqrt{\varepsilon}Ks\right)^2}{2s\sigma^2}}ds \\
&= \sum_{k=0}^{+\infty}\left(p_0 - g + \frac{\sqrt{\varepsilon}K}{\sqrt{2m\sigma^2+\varepsilon K^2}}\big((2k+1)(\bar{V}_0 - \underline{V}_0) - (x - \underline{V}_0)\big)\right) \\
&\times \exp\left(+\frac{\sqrt{\varepsilon}K}{\sigma^2}\cdot 2k(\bar{V}_0 - \underline{V}_0) - \frac{\sqrt{\varepsilon}K + \sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}\big((2k+1)(\bar{V}_0 - \underline{V}_0) - (x - \underline{V}_0)\big)\right) \\
&\quad - \sum_{k=1}^{+\infty}\left(p_0 - g + \frac{\sqrt{\varepsilon}K}{\sqrt{2m\sigma^2+\varepsilon K^2}}\big((2k-1)(\bar{V}_0 - \underline{V}_0) + (x - \underline{V}_0)\big)\right) \\
&\times \exp\left(-\frac{\sqrt{\varepsilon}K}{\sigma^2}\cdot 2k(\bar{V}_0 - \underline{V}_0) + \frac{\sqrt{\varepsilon}K - \sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}\big((2k-1)(\bar{V}_0 - \underline{V}_0) + (x - \underline{V}_0)\big)\right)
\end{aligned}$$

In the second equality, we switched the order of summation and integration, which can be justified by Fubini's theorem for $m > 0$ or $K \neq 0$. The resulting infinite series can be evaluated using standard geometric series results to yield (71). If $m = 0$ and $K = 0$, then it is known (see Branco et al. (2012)) that the seller's expected profit is $(p_0 - g)\left(\frac{x - \underline{V}_0}{\bar{V}_0 - \underline{V}_0}\right)$. $\square$

In the limit $\underline{V}_0 \to -\infty$ (i.e. the limit $c \to 0$) (71) reduces to (68) we previously studied. Unlike in the single boundary case, in the presence of the quitting boundary, the expected seller's profit is not only continuous at $K = 0$, but also differentiable, even when $m = 0$, as we will show below. We now focus on the $m = 0$ case.

From (71) we have that $\mathcal{V}^S(x; p_0, K < 0)|_{m=0}$ is given by (37), and that:

$$\mathcal{V}^S(x; p_0, K > 0)|_{m=0} =$$
$$\frac{(p_0 - g - (\bar{V}_0 + x - 2\underline{V}_0)) \exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0 - x)\right)}{1 - \exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0 - \underline{V}_0)\right)} + \frac{2(\bar{V}_0 - \underline{V}_0) \exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0 - x)\right)}{\left(1 - \exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0 - \underline{V}_0)\right)\right)^2}$$
$$- \frac{(p_0 - g - (\bar{V}_0 - x)) \exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0 - \underline{V}_0)\right)}{1 - \exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0 - \underline{V}_0)\right)} - \frac{2(\bar{V}_0 - \underline{V}_0) \exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0 - \underline{V}_0)\right)}{\left(1 - \exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0 - \underline{V}_0)\right)\right)^2}. \quad (72)$$

Both (37) and (72) are valid expressions for all $K \neq 0$, and with some works, we can show them to be equal for all $K \neq 0$. This proves $\mathcal{V}^S(x; p_0, K)$ is given by (37) for all $K \neq 0$. $\square$

Proof of Lemma 4. We can compute that

$$\frac{\partial p_0^*}{\partial K}(x, K = 0) = \frac{1}{12r\sigma}\left(3\sigma - 3\sqrt{\frac{2c^2}{r} + \sigma^2} \sinh^{-1}\sqrt{\frac{r\sigma^2}{2c^2}} - \sigma\left(\sinh^{-1}\sqrt{\frac{r\sigma^2}{2c^2}}\right)^2\right) \leq 0$$

where the inequality is strict everywhere except when $r\sigma^2/c^2 = 0$. Given that $r\sigma^2/c^2 > 0$, we can find a sufficiently small $\varepsilon > 0$ such that $p_0^*(x, .)$ is a decreasing function for $K \in [-1, +1]$. Any local maximum point of $\mathcal{V}^S(x; ., .)$ would take the form $(p_0^*(x; K^*), K^*)$ where $K^* := \arg\max_K \mathcal{V}^S(x; p_0^*(x, K), K)$. Hence, for all sufficiently small $\varepsilon > 0$, the profit maximizer (p_0^*, K^*) in $\mathcal{P}^\varepsilon_{lin}$ either satisfies $p_0^* < \hat{p}_0, K^* \gtrsim 0$, or $p_0^* > \hat{p}_0, K^* \lesssim 0$. $\square$