Invitation to Search or Purchase? Optimal Multi-attribute Advertising

Yunfei (Jesse) Yao*

(Chinese University of Hong Kong)

August 2024

* I thank J. Miguel Villas-Boas, Yuichiro Kamada, Xinyu Cao, Ganesh Iyer, Tony Ke, Chenxi Liao, Zihao Zhou, and seminar participants at the Bay Area Marketing Symposium, the Berkeley/Columbia/Duke/MIT/Northwestern IO Theory Conference and Berkeley Econ Theory Lunch for constructive comments. All errors are my own. E-mail address: jesseyao@cuhk.edu.hk.

Abstract

When considering whether or not to buy a product, consumers often evaluate different attributes of it. This learning process requires both time and effort, which incurs costs for the consumer. To encourage consumers to either purchase the product or continue seeking information instead of abandoning their search, firms may provide information through advertising. This paper studies the firm's advertising strategy for a two-attribute product. If the firm advertises one attribute, the consumer faces a one-dimensional search problem because she is uncertain about only the other attribute. If the firm does not advertise, the consumer faces a two-dimensional search problem because she is uncertain about both attributes. In that case, it is always optimal for the consumer to search for the attribute about which the consumer has greater uncertainty. We then characterize the optimal advertising strategy. The firm does not advertise if the consumer's prior beliefs about both attributes are extreme. No advertising serves as an invitation to search when the belief is high and as an invitation to purchase when the belief is very high. Otherwise, the firm advertises the better attribute if the consumer is optimistic enough about the worse attribute, and advertises the worse attribute if the consumer is less optimistic about it. In such cases, the role of advertising is non-monotonic in the belief.

1 Introduction

When considering whether or not to buy a product, consumers often evaluate different attributes of it. For example, an incoming college student choosing a laptop can learn about the operating system, weight, exterior design, warranty, and other attributes before making a final decision. This learning process requires both time and effort, which incurs costs for the consumer. To encourage consumers to either purchase the product or continue seeking information instead of abandoning their search, firms may provide information through advertising. This leads to several managerial decisions for the firm.

Advertising can convey positive information to the consumer, thereby increasing interest in the product. For instance, a car buyer who values electronic entertainment systems might be more inclined toward Tesla if the company highlights its interior design featuring a large screen. Conversely, advertising may also create a negative impression, potentially deterring a sale. For example, a different buyer who prefers the mechanical feel of a car might lose interest in Tesla after seeing the same advertisement, due to the absence of traditional mechanical components. Therefore, the firm must first determine whether to engage in advertising at all. If the firm opts to advertise, it must carefully select the content of the advertisement. Given the limited bandwidth of advertising, the firm cannot communicate all available information to the consumer, necessitating a decision on which attributes to emphasize. Finally, advertising can serve different purposes: it might drive an immediate purchase or prompt further search. Ultimately, this paper seeks to address the following questions:

1. Whether the firm wants to advertise or not?
2. What is the optimal advertising content - which attribute should the firm advertise?
3. What is the role of advertising? Does the firm want to induce the consumer to purchase the product directly or to search for additional information after seeing the ad?

To answer the above questions, this paper considers a consumer deciding whether or not to purchase a good. The good has two attributes, each with independent values. The payoff of purchasing the good is the total value of these attributes minus the price. We focus on the horizontal match between the attributes and the consumer's tastes/needs. The value of each attribute is one if it is a good match and zero if it is a bad match. The consumer initially does not know the actual value of either attribute but holds a prior belief about the value of each. The firm can reveal the value of the attribute by informative advertising. Due to the limited bandwidth of ads, the firm can disclose the value of at most one attribute (Shapiro 2006, Bhardwaj et al. 2008, Mayzlin and Shin 2011). After seeing the ad, the consumer knows the value of the advertised attribute but still has uncertainty about the unadvertised attribute(s). She can incur a cost to search for additional information before making a decision. If the firm advertises one attribute, then the consumer may only search for information about the other attribute. If the firm chooses not to advertise, then the consumer may search for information about either attribute.

We first characterize the consumer's optimal search strategy given the firm's advertising strategy. If the firm advertises one attribute, the consumer faces a one-dimensional search problem because she is uncertain about only the other attribute. The consumer will stop searching and buy the good if she becomes sufficiently optimistic about the unadvertised attribute. Conversely, she will stop searching without purchasing if her assessment becomes too pessimistic. When her belief about the unadvertised attribute is in between, she will search for more information. If the firm does not advertise, the consumer faces a two-dimensional search problem because she is uncertain about both attributes. We characterize the search region by a set of ordinary differential equations for intermediate beliefs and by a system of equations for extreme beliefs. It is always optimal for the consumer to search for the attribute about which the consumer has greater uncertainty, due to the faster speed of learning. The consumer only searches for one attribute if she holds a strong prior belief about one of the attributes, and may search for both attributes otherwise. In the car purchasing example, a consumer might not investigate the design of a Tesla due to its well-known styling and focus instead on other attributes.[1] In contrast, a consumer considering a pre-order from Faraday Future, a new electric car manufacturer, likely faces significant uncertainty about all aspects and might investigate every attribute. We also find that it may be optimal for the consumer to revisit a previously searched attribute after searching for the other attribute.

[1] For example, the consumer may know how Tesla looks like by observing her friend's Tesla before searching for any information about it.

Given the consumer's search strategy, we then study the firm's optimal advertising strategy. The firm faces two trade-offs when choosing the advertising strategy. The first trade-off is whether or not to advertise. The firm benefits from a higher likelihood of purchase if the advertised attribute turns out to be good and moves up the consumer's overall evaluation of the product. However, the consumer will quit if the advertised attribute is bad. In such cases, the firm suffers from no chance of selling the product. So, it is not obvious whether the firm should advertise. The second trade-off is whether to advertise the better attribute (the attribute with a higher prior belief) or the worse attribute (the attribute with a lower prior belief). Because the consumer will not be interested in the product if the advertised attribute is bad, the firm has a higher chance of keeping the consumer interested by advertising the better attribute. The downside is that, conditional on the advertised attribute being good, the consumer's overall evaluation of the product is lower if the firm advertises the better attribute than if the firm advertises the worse attribute. The conditional purchasing probability is lower if the firm advertises the better attribute. It is also not obvious whether the firm should advertise the better or the worse attribute.

We find that the firm will not advertise if the consumer's prior beliefs about both attributes are extreme. If the consumer is very optimistic about both attributes, she will purchase the product for sure or with a very high likelihood. So, the firm does not have an incentive to advertise. No advertising serves as an invitation to search when the belief is high and as an invitation to purchase when the belief is very high. If the consumer is very pessimistic about both attributes, she will never purchase the product even if she knows that one attribute is good. So, the firm does not advertise either.

If the consumer's prior belief is milder, the firm benefits from advertising. It will advertise the better attribute if the consumer is optimistic enough about the worse attribute, and will advertise the worse attribute if the consumer is less optimistic about it. Suppose the consumer is more optimistic about attribute one than about attribute two. When the prior belief about attribute one is moderate, the firm advertises attribute two to invite the consumer to search for more information about attribute one. When the prior belief about attribute one is high, the role of advertising is *non-monotonic* in the prior belief about attribute two. When the prior belief is low, the firm advertises attribute two to invite the consumer to directly purchase the product. When the prior belief is higher, the firm advertises attribute one to invite the consumer to search for more information about attribute two. When the prior belief is even higher, the firm advertises attribute one to invite the consumer to directly purchase the product.

Related Literature

This paper studies a firm's information disclosure strategy. People have begun to consider the informational role of advertising since Nelson (1974). Subsequent papers study the disclosure of price (Anderson and Renault 2006) and quality (Lewis and Sappington 1994, Anderson and Renault 2009) by informative advertising. Some papers examine the strategic consideration of the firm (Liu and Dukes 2013, Dukes and Liu 2016) through product design and design, but the firm does not directly provide information to consumers before they search. Sun (2011) studies a seller's disclosure incentive for a product with multiple attributes. It shows that the unraveling result by Grossman (1981) and Milgrom (1981) will not hold if the product has a vertical attribute and a horizontal one. If the product has a high vertical quality, the seller may not disclose the product's horizontal attribute. Consumers' only source of information about the product comes from the firm in most of the existing advertising literature. In reality, consumers can search for more information after they see the ads. We take it into account by building a micro-founded consumer search model. After the firm advertises, the consumer can still search for information about any attributes. The firm anticipates it when choosing the advertising strategy.

Mayzlin and Shin (2011) are closely related to our paper. They consider a setting where the consumer can obtain an exogenously given signal by searching for information about the product quality after the firm advertises. Our paper differs from their paper in two ways. On one hand, the quality is vertical in their paper, and the advertising strategy is driven by signaling the firm's private information. We focus on horizontal quality, and the advertising strategy is driven by the difference in one-dimensional search with advertising and two-dimensional search without advertising. On the other hand, the consumer can only search once and observe an aggregate signal about the firm's quality in Mayzlin and Shin (2011). We model the search process in detail so that the consumer chooses what attribute and how long to search. This allows us to endogenize the search order and understand more about the consumer's search behavior and the firm's best response to it.

This paper is also related to the literature on how consumers with limited attention allocate their attention to different attributes or options. Existing literature mainly looks at cases where attributes or options are asymmetric (Arbatskaya 2007, Armstrong et al. 2009, Xu et al. 2010, Armstrong and Zhou 2011, Bar-Isaac et al. 2012, Bordolo et al. 2013, Kőszegi and Szeidl 2013, Branco et al. 2016, Zhu and Dukes 2017, Jeziorski and Moorthy 2018). In these papers, consumers know ex-ante that they face attributes with different prominence or importance. For example, the search order is exogenous in Arbatskaya (2007). Armstrong et al. (2009) extend Wolinsky (1986)'s symmetric search model by introducing a prominent firm that all consumers search for first. However, the reason why consumers prioritize this firm is not modeled, as its prominence is exogenously assumed. In Bordolo et al. (2013), the salient attribute of a good is the attribute furthest away from the average value of the same attribute in the choice set. In Zhu and Dukes (2017), each competing firm can promote one or both attributes of a product. Though the prominence of the product is endogenously determined by competition, it is given exogenously from the consumer's perspective. Jeziorski and Moorthy (2018) explore the role of prominence in search advertising, distinguishing between ad position prominence and advertiser prominence. They find that these are are substitutes in influencing consumer clicks. A key contribution of our paper is to endogenize the optimal attribute to search from the consumer's perspective. Instead of assuming that the consumer knows the value of each attribute or learns it instantly, as is common in this literature, the Bayesian decision-maker in our model gradually learns the value from noisy signals. So, the relative importance of the attributes may change as the consumer gathers more information. In contrast, the prominence attribute or option in the existing literature does not change over time because they impose exogenous differences on the attributes.

Lastly, this paper fits into the literature on optimal information acquisition, particularly consumer search. Following seminal papers by Stigler (1961) and Weitzman (1979), numerous papers have studied the optimal search problem under either simultaneous or sequential search (e.g., Moscarini and Smith 2001, Branco et al. 2012, Ke et al. 2016, Ke and Villas-Boas 2019, and Jerath and Ren 2023). In these studies, the relative importance of different alternatives is typically exogenous. Consumers randomly choose an attribute to search. In our model, the consumer strategically decides when to search and which attribute to focus on. Moe importantly, the information acquisition literature mainly study the decision-maker's optimal search strategy, whereas we focus on the firm's advertising strategy.

The remainder of the paper is organized as follows. Section 2 presents the main model. Section 3 solves the consumer's search strategy. Section 4 characterizes the firm's equilibrium advertising strategy. Section 5 concludes.

2 Model

There is a firm and a consumer. Both are risk-neutral. The consumer considers whether to purchase a product or not. The product has two attributes, each with independent values. The product's value to the consumer is the sum of the values of these attributes, $U = U_1 + U_2$. This paper focuses on the horizontal match between the attributes and the consumer's tastes/needs. The value of each attribute is one if it is a good match and zero if it is a bad match. Given our focus on horizontal preferences rather than vertical ones, we assume the firm does not have private information about the value of the attribute, and thereby the consumer and the firm share a common prior belief about the likelihood that attribute i is a good match, denoted by $\mu_i(0)$. The price p is given exogenously because we want to focus on the role of information in this paper. We assume that the marginal cost of producing the product is sufficiently high, and thus the price is high enough ($p \geq 3/2$), so that the consumer will decide not to purchase the product for any pair of beliefs (μ_1, μ_2) if $\mu_1 + \mu_2 \leq 1$. Therefore, we restrict our attention to the case where $\mu_1 + \mu_2 > 1$.

The firm can reveal the value of the attribute by informative advertising. Due to the limited bandwidth of ads, we assume that the firm can disclose the value of at most one attribute (Shapiro 2006, Bhardwaj et al. 2008, Mayzlin and Shin 2011). We consider the case in which the consumer knows exactly whether an attribute is a good match or not if the firm advertises that attribute. The firm can provide a noisy signal which does not fully resolve the uncertainty. However, the full disclosure setup gives us the sharpest results.

The consumer has the option to learn more about the attributes via costly learning after seeing the ads. If the firm advertises one attribute, then the consumer may only search for information about the other attribute. If the firm chooses not to advertise, then the consumer may search for information about either attribute. Due to limited attention, we assume that the consumer can only search for information about one attribute at a time. If the consumer opts to search, she also needs to decide which specific attribute to investigate at a given time. The decision-making process ends when the consumer makes a purchasing decision.

Each attribute of the product consists of numerous sub-attributes. So, the consumer cannot learn everything about an attribute with a single evaluation. For instance, if a consumer wishes to learn about the design of a car (an attribute), she might start by looking at an image online to determine the car's exterior color (a sub-attribute). However, she will need to invest additional effort to learn about other sub-attributes, such as the wheel size or the seat material. Given the complexity of modern products, each attribute often includes so many sub-attributes that it becomes impossible for the consumer to fully learn everything. To model this gradual learning process, we assume that the consumer receives noisy signals about an attribute by incurring a flow cost of c. Let $T_i(t)$ denote the cumulative time spent searching for attribute i up to time t. We represent the signal, S_i, as a Brownian motion, where W_i are independent Wiener processes:

$$dS_i(t) = U_i dT_i(t) + \sigma dW_i(T_i(t))$$

In the above expression, the first term is driven by the true value, U_i, while the second term represents the noise. The parameter σ is a measure of the level of signal noise - a larger σ indicates a noisier signal due to the higher relative weight of the noise term. The consumer is more likely to observe a larger signal realization if the attribute is good, as the first term continuously increases over time when $U_i = 1$. This continuous-time model of Bayesian learning about a binary state has been widely used to study information acquisition (Ke and Villas-Boas 2019, Morris and Strack 2019, Liao 2021), experimentation (Bolton and Harris 1999, Moscarini and Smith 2001), and decision times (Fudenberg et al. 2018). It effectively captures the gradual learning feature and offers tractable analysis. The gradual learning set up in this paper is standard, except that the consumer endogenously chooses which attribute to search for at any given time rather than randomly searching for an attribute.

Based on the received signals, the consumer continuously updates her belief about the value of each attribute according to Bayes' rule. This belief evolution can be characterized by the following ordinary differential equation:

$$d\mu_i(t) = \frac{1}{\sigma^2}\mu_i(t)[1-\mu_i(t)]\{dS_i(t) - \mu_i dT_i(t)\}, \tag{1}$$

where μ_i is the expected value of attribute i based on the observed information up to time t. A higher σ results in slower belief updating due to noisier signals. If μ_i is closer to $1/2$, there is more uncertainty about attribute i, leading to faster updates in belief. If $dS_i(t) - \mu_i dT_i(t) > 0$, the signal's increasing speed is higher than the current belief about attribute i, and therefore the true value is more likely to be good ($U_i = 1$). So, the consumer will increase her belief about this attribute. Conversely, if the signal increases more slowly, the consumer will decrease her belief. This belief updating process also implies that the consumer's belief about an attribute remains unchanged when she searches for information about the other attribute.

3 Consumer's Search Strategy

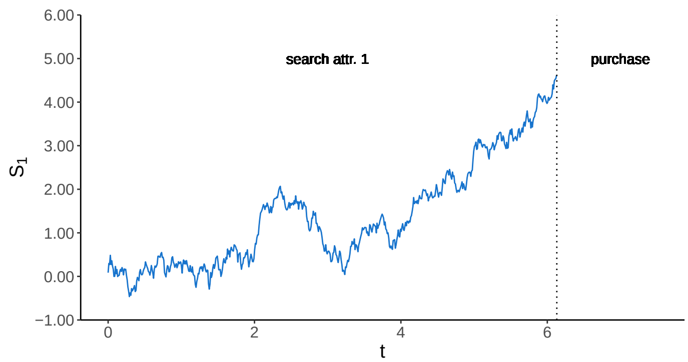
We solve the model by backward induction. In this section, we first look at the consumer search strategy for a given advertising strategy.

3.1 The Firm Advertises One Attribute

Suppose the firm advertises attribute $i \in \{1,2\}$. The value of attribute i, U_i, becomes 1 with probability μ_i and 0 with probability $1-\mu_i$.[2] The consumer may only search for information about the other attribute. She faces a one-dimensional search problem. The consumer can make a decision right away or search for information about the other attribute $j \neq i$. From the consumer's perspective, the product is equivalent to a single attribute product whose value is U_j and whose price is $p' := p - U_i$. One can see that the consumer will quit if $U_i = 0$. So, we consider the case in which $U_i = 1$ now (p' becomes $p-1$). The optimal search strategy has been shown in Ke and Villas-Boas (2019). There exists $0 < \underline{\mu}_j < \bar{\mu}_j < 1$ such that the consumer searches for more information if $\mu_j \in (\underline{\mu}_j, \bar{\mu}_j)$, purchases the product if $\mu_j \geq \bar{\mu}_j$, and quits if $\mu_j \leq \underline{\mu}_j$. Figure 1 illustrates a sample

[2] We denote $\mu_i(0)$ by μ_i to simplify the notation in this section.

path of the signals and belief evolution when both attributes are good and the firm advertises the second attribute. The consumer know from advertising that attribute two is good and has uncertainty about only attribute one. She searches for information about attribute 1, continuously receiving signals about it. Her belief goes up and down initially because she receives both positive and negative signals. After a period of time, the consumer keeps receiving positive signals about attribute 1 and she becomes confident enough that attribute one is also good. Therefore, she stops searching and purchases the product.



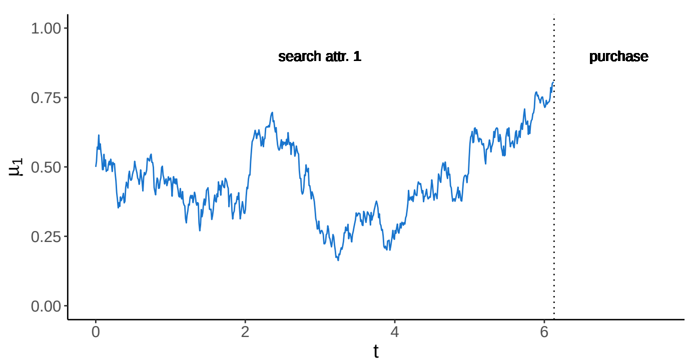


Figure 1: Sample Paths of the Signals and Beliefs When the Firm Advertises Attribute Two for $U_1 = 1, U_2 = 1, \mu_1(0) = 0.5, \mu_2(0) = 0.5$, and $\sigma = 1$.

In the search region, the value function is determined by:

$$\frac{\mu_j^2(1-\mu_j)^2}{2\sigma^2}W''(\mu_j) - c = 0$$
$$\Rightarrow \; W(\mu_j) = 2\sigma^2 c(1-2\mu_j)\ln\frac{1-\mu_j}{\mu_j} + K_1\mu_j + K_2, \;\; \mu_j \in (\underline{\mu}_j, \bar{\mu}_j),$$

where K_1 and K_2 are two unknown constants to be determined. Because $W(\underline{\mu}_j) = W'(\underline{\mu}_j) = 0$, $W(\bar{\mu}_j) = \bar{\mu}_j - p'$, and $W'(\bar{\mu}_j) = 1$, value matching and smooth pasting at $\underline{\mu}_j$ and $\bar{\mu}_j$ determine the cutoff belief:

$$\begin{cases} \phi(\underline{\mu}_j) - \phi(\bar{\mu}_j) = \frac{1}{2\sigma^2 c} \\ \psi(\underline{\mu}_j) - \psi(\bar{\mu}_j) = \frac{p-1}{2\sigma^2 c} \end{cases}, \tag{2}$$

where $\phi(x) = 2\ln\frac{1-x}{x} + \frac{1}{x} - \frac{1}{1-x}$ and $\psi(x) = \ln\frac{1-x}{x} + \frac{1-2x}{1-x}$.

3.2 The Firm Does Not Advertise

If the firm does not advertise, the consumer faces a two-dimensional search problem. At any given time, the consumer can search for information about either attribute. Figure 2 illustrates a sample path of the signals and belief evolution when the first attribute is good, the second one is bad, and the firm does not advertise. Initially, the consumer's belief about each attribute is 1/2. She begins by searching for information about attribute 1, continuously receiving signals about it. Although these signals are predominantly positive, the consumer's belief about the first attribute gradually declines because the signal's rate of increase is slower than her belief. As attribute 2 is not initially searched, no new signals are received for it, and consequently, the consumer's belief regarding this attribute remains unchanged. After a period of time, the consumer shifts her focus to attribute 2. Her belief about this attribute first decreases and then increases, while her belief about attribute 1 stays the same. On receiving positive signals about attribute 2, she returns to investigating attribute 1. During this phase, the signal for attribute 1 increases rapidly, causing her belief to rise towards 1. Once the consumer is relatively certain that the first attribute is good, she resumes her search for information about attribute 2. She eventually stops searching and decides not to purchase the product after receiving sufficient negative signals, leading her to strongly believe that attribute 2 is bad.

At any given time, the consumer chooses among four actions: searching for attribute 1, searching for attribute 2, purchasing the product, or quitting without purchasing. The consumer's search strategy, denoted as α, maps the observed history (the signal realization) up to time t to one of

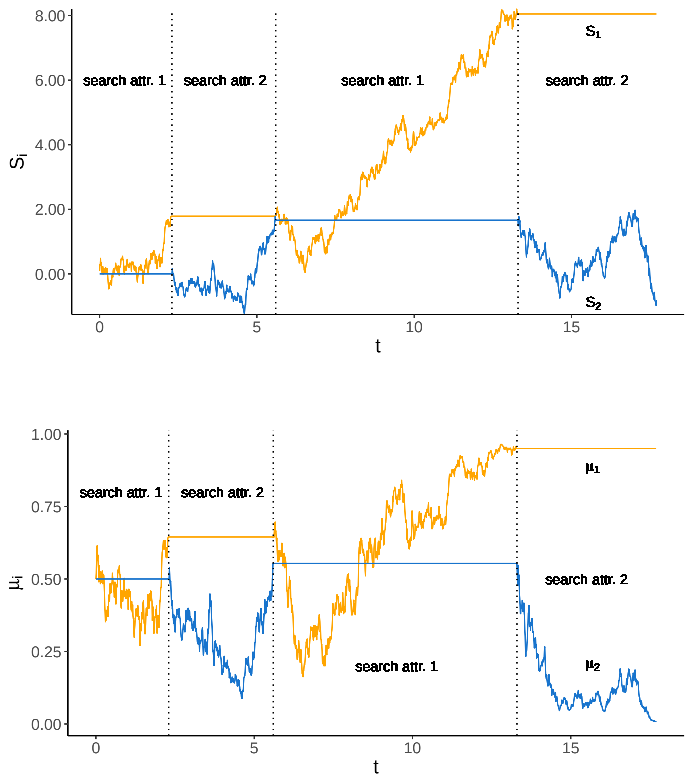


Figure 2: Sample Paths of the Signals and Beliefs When the Firm Does not Advertise for $U_1 = 1, U_2 = 0, \mu_1(0) = 0.5, \mu_2(0) = 0.5$, and $\sigma = 1$.

these four actions, for all t. We define the stopping time τ as the first instance when the consumer makes a purchasing decision (either purchasing or quitting). The entire process ends at the stopping time. The consumer's expected payoff for a given initial belief (μ_1, μ_2) and search strategy α is:

$$J(\mu_1, \mu_2, \alpha) = \mathbb{E}\left\{\max\left[\mu_1(\tau) + \mu_2(\tau) - p, 0\right] - \tau c | (\mu_1(0), \mu_2(0)) = (\mu_1, \mu_2)\right\}$$

The value function of the consumer's problem is given by:

$$V(\mu_1, \mu_2) := \sup_{\alpha} J(\mu_1, \mu_2, \alpha)$$

Since the search strategy should not depend on future information, the decision at time t must be based only on the information observed up to t. It is well established that the current belief $(\mu_1(t), \mu_2(t))$ is a sufficient statistic for the information available up to time t in this binary-valued setting. Therefore, the search strategy will depend solely on $(\mu_1(t), \mu_2(t))$. If a search strategy α^* achieves $V(\mu_1, \mu_2)$ for any given belief, it will be deemed the optimal search strategy.

$$V(\mu_1, \mu_2) = J(\mu_1, \mu_2, \alpha^*)$$

When the consumer searches for information about attribute one, the value function satisfies (ignoring the time index t for simplicity):

$$V(\mu_1, \mu_2) = -cdt + \mathbb{E}[V(\mu_1 + d\mu_1, \mu_2)]$$

By Taylor's expansion and Ito's lemma, we get:

$$\frac{\mu_1^2(1-\mu_1)^2}{2\sigma^2} V_{\mu_1\mu_1}(\mu_1, \mu_2) - c = 0 \tag{3}$$

Similarly, when the consumer searches for information about attribute two, we have:

$$\frac{\mu_2^2(1-\mu_2)^2}{2\sigma^2} V_{\mu_2\mu_2}(\mu_1, \mu_2) - c = 0 \tag{4}$$

The Hamilton-Jacobi-Bellman (HJB) equation of the entire problem is:

$$\max\left\{ \max_{i=1,2} \left[\frac{\mu_i^2(1-\mu_i)^2}{2\sigma^2} V_{\mu_i\mu_i}(\mu_1, \mu_2) - c \right], \max\left[\mu_1 + \mu_2 - p, 0\right] - V(\mu_1, \mu_2) \right\} = 0 \tag{$\star$}$$

We will show that the value function is a viscosity solution of the HJB equation. We will then prove that the viscosity solution is unique. Therefore, if we can find a viscosity solution, it must be the value function. To achieve this, we will construct a search strategy, which we will use to characterize the search region and the expected payoff. Finally, we will verify that the conjectured strategy generates a viscosity solution of the HJB equation, thereby implying that the conjectured search strategy is optimal. Due to symmetry, it is sufficient to consider only the case where $\mu_1 \geq \mu_2$. Analytically, we can fully characterize the optimal search strategy when the search cost is low. We do not believe that focusing on the low search cost case imposes a significant limitation, as our primary interest lies in examining the consumer's search behavior and how it can be influenced by the firm. The more compelling scenario occurs when the consumer is inclined to search more due to the low search cost. When the search cost or the price is too high, the consumer's search activity diminishes, making the problem less intriguing and relevant.

Proposition 1. *Suppose the firm does not advertise and the search cost is low, $c \leq \frac{1}{2\sigma^2[\phi(1/2)-\phi(\frac{2}{3}p-\frac{1}{6})]}$. Conditional on searching, it is optimal for the consumer to search for information about attribute two (one) if $\mu_1 \geq \mu_2$ ($\mu_1 < \mu_2$).*

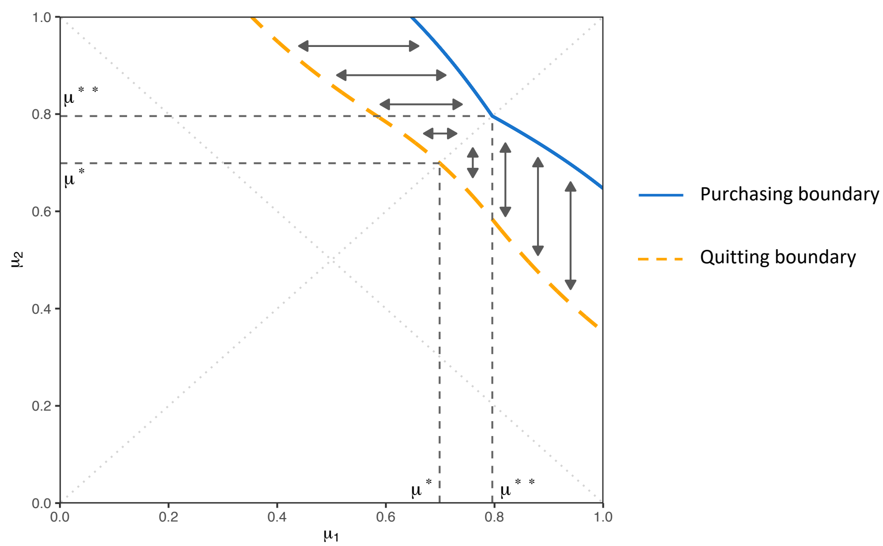


Figure 3: Optimal Search Strategy

Figure 3 illustrates the optimal search strategy if the firm does not advertise. The dashed orange line represents the quitting boundary, while the solid blue line depicts the purchasing boundary. The grey arrow indicates which attribute the consumer searches for information about, given her current belief. When the overall beliefs of the attributes are sufficiently low, the likelihood of receiving enough positive signals to warrant a purchase is too low. In this case, the consumer stops searching and quits to save on search cost. Conversely, when the overall beliefs of the attributes are high enough, purchasing the product results in a sufficiently high expected payoff, prompting the consumer to make the purchase. In other cases, the consumer searches for more information to make a better-informed decision.

Denote the point where the quitting boundary intersects the main diagonal as (μ^*, μ^*), and the point where the purchasing boundary intersects the main diagonal as (μ^{**}, μ^{**}). The quitting boundary for $\mu_1 \geq \mu_2$ is represented by $\underline{\mu}(\cdot)$, with a domain of $[\mu^*, 1]$ (the other half of the quitting boundary can be determined by symmetry). The purchasing boundary for $\mu_1 \geq \mu_2$ is represented by $\bar{\mu}(\cdot)$, with a domain of $[\mu^{**}, 1]$ (the other half of the purchasing boundary can also be determined by symmetry).

Intuitively, conditional on searching, the consumer prefers to search for the attribute with a higher rate of learning, since learning costs are identical. From equation (1), it is evident that the more uncertain the belief is, the faster the consumer learns about an attribute. Consequently, she always focuses on learning about the attribute with a belief closer to 1/2. The optimal search strategy implies that the consumer only searches for one attribute if she holds a strong prior belief on one of the attributes, and may search for both attributes otherwise.

The following proposition characterizes the slope of the purchasing/quitting boundary and the shape of the search region.

Proposition 2. *For $\mu \in (\mu^*, \mu^{**}]$, we have:*

$$\underline{\mu}'(\mu) = \frac{\phi(\underline{\mu}(\mu)) - \phi(\mu)}{\phi'(\underline{\mu}(\mu))[\mu - \underline{\mu}(\mu)]} \tag{D_1}$$

For $\mu \in [\mu^{**}, 1]$, we have:

$$\bar{\mu}'(\mu) = \frac{\phi(\underline{\mu}(\mu)) - \phi(\bar{\mu}(\mu))}{\phi'(\bar{\mu}(\mu))[\bar{\mu}(\mu) - \underline{\mu}(\mu)]} \tag{$\overline{D_2}$}$$

$$\underline{\mu}'(\mu) = \frac{\phi(\underline{\mu}(\mu)) - \phi(\bar{\mu}(\mu))}{\phi'(\underline{\mu}(\mu))[\bar{\mu}(\mu) - \underline{\mu}(\mu)]} \tag{$\underline{D_2}$}$$

Both $\underline{\mu}(\mu)$ and $\bar{\mu}(\mu)$ strictly decrease in μ, whereas the width of the search region, $\bar{\mu}(\mu) - \underline{\mu}(\mu)$, strictly increases in μ. In addition, if $\underline{\mu}(\mu) \geq 1/2$, then the slope of the quitting boundary is less than -1, and the slope of the purchasing boundary is greater than -1.

We find that the optimal search region exhibits a butterfly shape - the consumer searches for information in a wider region when she is more confident that the more favorable attribute is good. The underlying intuition is as follows: the product's expected value is higher when the consumer is more certain about one attribute being good, prompting her to seek information about the other attribute, even if it is associated with greater uncertainty. This is because the speed of learning is higher for a more uncertain attribute, enhancing the search's benefits while the cost of searching remains unchanged. Consequently, the consumer is motivated to engage in more extensive search activities. Furthermore, if the consumer likes an attribute more, she is more inclined to purchase the product even if she has a higher uncertainty about the other attribute. She will also be less likely to stop searching and quit. As a result, the search region shifts downwards as the belief about one attribute increases.

The slope of the search region, representing the marginal rate of substitution between the values of the first and second attributes, is also interesting. It sheds light on the learning process's cross-attribute dependence. If the slope is -1, then the two attributes are perfect substitutes. One may expect this to be the case in general because the product's value is the sum of the values of two attributes, each with independent values. However, both the slope of the quitting boundary and the slope of the purchasing boundary are not -1 in general due to the asymmetry in learning. If the quitting boundary is above $1/2$, the slope of the quitting boundary is less than -1. In such cases, a one-unit increase in the belief about attribute one can compensate for more than a one-unit decrease in the belief about attribute two near the quitting boundary. The consumer will continue her search for attribute two rather than quitting, even if μ_2 declines by slightly more than one unit. This is because the consumer has more uncertainty, and hence a higher rate of learning, about attribute 2. So, the benefit of search increases while the search cost remains the same. Similarly, near the purchasing boundary, a one-unit increase in the belief about attribute one compensates for less than a one-unit decrease in the belief about attribute two. This encourages the consumer to keep searching for information about attribute two rather than making an immediate purchase, even if μ_2 decreases by slightly less than a unit.

4 Firm's Advertising Strategy

After characterizing the consumer search strategy for a given advertising strategy, we now study the firm's optimal advertising strategy.

4.1 Advertising One Attribute

We first consider the purchasing probability if the firm advertises one attribute. We know from the last section that the consumer will quit if she knows that one attribute is bad. Therefore, the consumer's purchase probability is the multiplication of two terms: the likelihood that the advertised attribute is good and the conditional purchasing probability given that the advertised attribute is good.

We first look at the case where the firm advertises attribute one. In this case, the consumer will know that the first attribute is bad with probability $1 - \mu_1$. She will quit after receiving such information. With the complementary probability μ_1, the consumer will learn that attribute one is good and may purchase the product directly, searching for more information about the second attribute, or quit, depending on her belief about attribute two. The following proposition characterizes the overall purchasing probability if the firm advertises attribute one.

Proposition 3. *Suppose the firm advertises attribute one. The probability that the consumer purchases the product is:*

$$\begin{aligned} P_1(\mu_1, \mu_2) &:= \mathbb{P}[\textit{purchasing}|\textit{firm advertises attribute 1, prior belief } (\mu_1, \mu_2)] \\ &= \begin{cases} \mu_1, & \textit{if } \mu_2 \geq \bar{\mu}(1) \\ \mu_1 \cdot \frac{\mu_2 - \underline{\mu}(1)}{\bar{\mu}(1) - \underline{\mu}(1)}, & \textit{if } \mu_2 \in [\underline{\mu}(1), \bar{\mu}(1)] \\ 0, & \textit{if } \mu_2 \leq \underline{\mu}(1) \end{cases} \end{aligned}$$

The case of advertising attribute two is symmetric to the case of advertising attribute one. One can obtain a similar result about the consumer's purchasing likelihood given the advertising strategy.

Corollary 1. *Suppose the firm advertises attribute two. The probability that the consumer pur-*

chases the product is:

$$
\begin{aligned}
P_2(\mu_1,\mu_2) := \ & \mathbb{P}[\textit{purchasing}|\textit{firm advertises attribute 2, prior belief } (\mu_1,\mu_2)] \\
= \ & \begin{cases} \mu_2, & \textit{if } \mu_1 \geq \bar{\mu}(1) \\ \mu_2 \cdot \frac{\mu_1 - \underline{\mu}(1)}{\bar{\mu}(1)-\underline{\mu}(1)}, & \textit{if } \mu_1 \in [\underline{\mu}(1), \bar{\mu}(1)] \\ 0, & \textit{if } \mu_1 \leq \underline{\mu}(1) \end{cases}
\end{aligned}
$$

4.2 Not Advertising

The consumer may search for information about either attribute if the firm does not advertise. If the consumer is highly confident about one of the attributes, she will not seek further information about it but will focus on learning about the other attribute. A decision to purchase is made once enough positive information is acquired to reach the purchasing boundary. Conversely, if sufficient negative information leads her belief to the quitting boundary, she will decide not to make a purchase. For instance, a consumer considering a Tesla might skip researching its design because she has seen the body styling of her friend's Tesla, choosing instead to investigate other aspects of the vehicle.

In cases where the consumer has moderate beliefs about both attributes, she must gather information on both before purchasing the product.[3] Moreover, she will be equally certain about the value of each attribute upon deciding to buy. For example, a consumer considering a pre-order from Faraday Future, a new manufacturer, likely faces significant uncertainty about all aspects and might investigate every attribute. Given the consumer's optimal search strategy, we can calculate the purchasing likelihood if the firm advertises neither attribute.

Proposition 4. *Suppose the firm does not advertise. If $\mu_1 \geq \mu_2$, the probability that the consumer*

[3] The consumer may have searched for only one attribute if she decide not to purchase the product after receiving enough negative information about that attribute.

purchases the product is:

$$P(\mu_1,\mu_2) := \mathbb{P}[\textit{purchasing}|\textit{firm does not advertise, prior belief } (\mu_1,\mu_2)]$$
$$= \begin{cases} 1, & \textit{if } \mu_1 \in [\mu^{**},1] \textit{ and } \mu_2 \in [\bar{\mu}(\mu_1),\mu_1] \\ \frac{\mu_2-\underline{\mu}(\mu_1)}{\bar{\mu}(\mu_1)-\underline{\mu}(\mu_1)}, & \textit{if } \mu_1 \in [\mu^{**},1] \textit{ and } \mu_2 \in [\underline{\mu}(\mu_1),\bar{\mu}(\mu_1)] \\ h(\mu_1,\mu_2)\tilde{P}(\mu_1), & \textit{if } \mu_1 \in [\mu^{*},\mu^{**}] \textit{ and } \mu_2 \in [\underline{\mu}(\mu_1),\mu_1] \\ 0, & \textit{if } \mu_1 \le \mu^{*} \textit{ or } \mu_2 \le \bar{\mu}(\mu_1) \end{cases}$$

, *where* $h(\mu_1,\mu_2) = \frac{\mu_2-\underline{\mu}(\mu_1)}{\mu_1-\underline{\mu}(\mu_1)}$ *and* $\tilde{P}(\mu) = e^{-\int_{\mu}^{\mu^{**}} \frac{2}{x-\underline{\mu}(x)}dx}$. *By symmetry,* $P(\mu_1,\mu_2) = P(\mu_2,\mu_1)$ *if* $\mu_1 < \mu_2$.

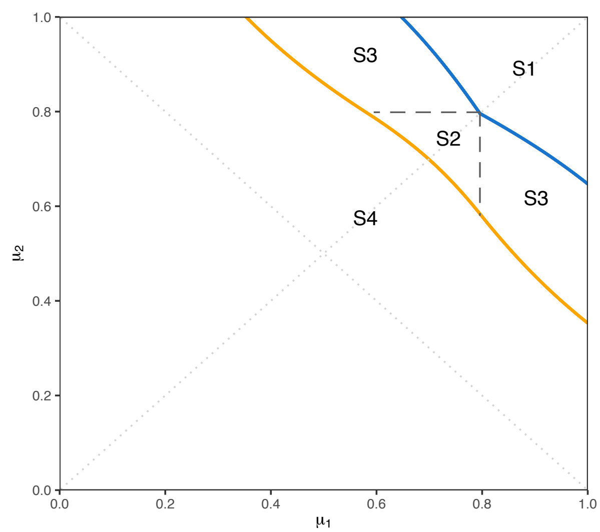


Figure 4: Four Regions for Purchase

Figure 4 delineates four regions that outline the consumer's purchasing strategy based on her initial beliefs. In region $S1$, the consumer purchases immediately, while in region$S4$, she quits without searching. Beliefs that fall within the intermediate regions, S_2 and S_3, indicate a high value of information, prompting further search before a decision is made.

Specifically, in region $S3$, the consumer strongly believes that the first attribute is good and focuses her search on the second, more uncertain attribute. Suffcient positive information about the second attributeleads to a purchase, while enough negative information results in quitting. Due to the low information value about the first attribute, the consumer never switches her focus back to it regardless of the outcomes of her search regarding the second attribute. In this scenario, the second attribute is pivotal in determining the consumer's purchasing decision.

In region $S2$, the consumer is quite uncertain about the values of both attributes. She will initially focus her search on attribute two, given her greater uncertainty about it compared to attribute one. However, she is not confident about the value of attribute one either. Consequently, if she receives sufficient positive signals about attribute two, she will then shift her focus to searching for information about attribute one. If subsequent positive signals about attribute one are received, she may toggle her search focus back to attribute two, and this pattern of switching may continue until she gains enough confidence in both attributes to make a purchasing decision. As shown in Figure 4, the belief must reach (μ^{**}, μ^{**}) for the consumer to make a purchasing decision. Conversely, if she encounters sufficient negative information about either attribute, she will discontinue her search and opt not to purchase the product.

4.3 Optimal Advertising Strategy

When choosing the advertising strategy, the firm faces tough trade-offs. The first trade-off is whether or not to advertise. The firm benefits from a higher likelihood of purchase if the advertised attribute turns out to be good and moves up the consumer's overall evaluation of the product. However, the consumer will quit if the advertised attribute is bad. In such cases, the firm suffers from no chance of selling the product. So, it is not obvious whether the firm should advertise. The second trade-off is whether to advertise the better attribute (the attribute with a higher prior belief) or the worse attribute (the attribute with a lower prior belief). Because the consumer will not be interested in the product if the advertised attribute is bad, the firm has a higher chance of keeping the consumer interested by advertising the better attribute. The downside is that, conditional on the advertised attribute being good, the consumer's overall evaluation of the product is lower if the firm advertises the better attribute than if the firm advertises the worse attribute. The conditional purchasing probability is lower if the firm advertises the better attribute. It is also not obvious whether the firm should advertise the better or the worse attribute.

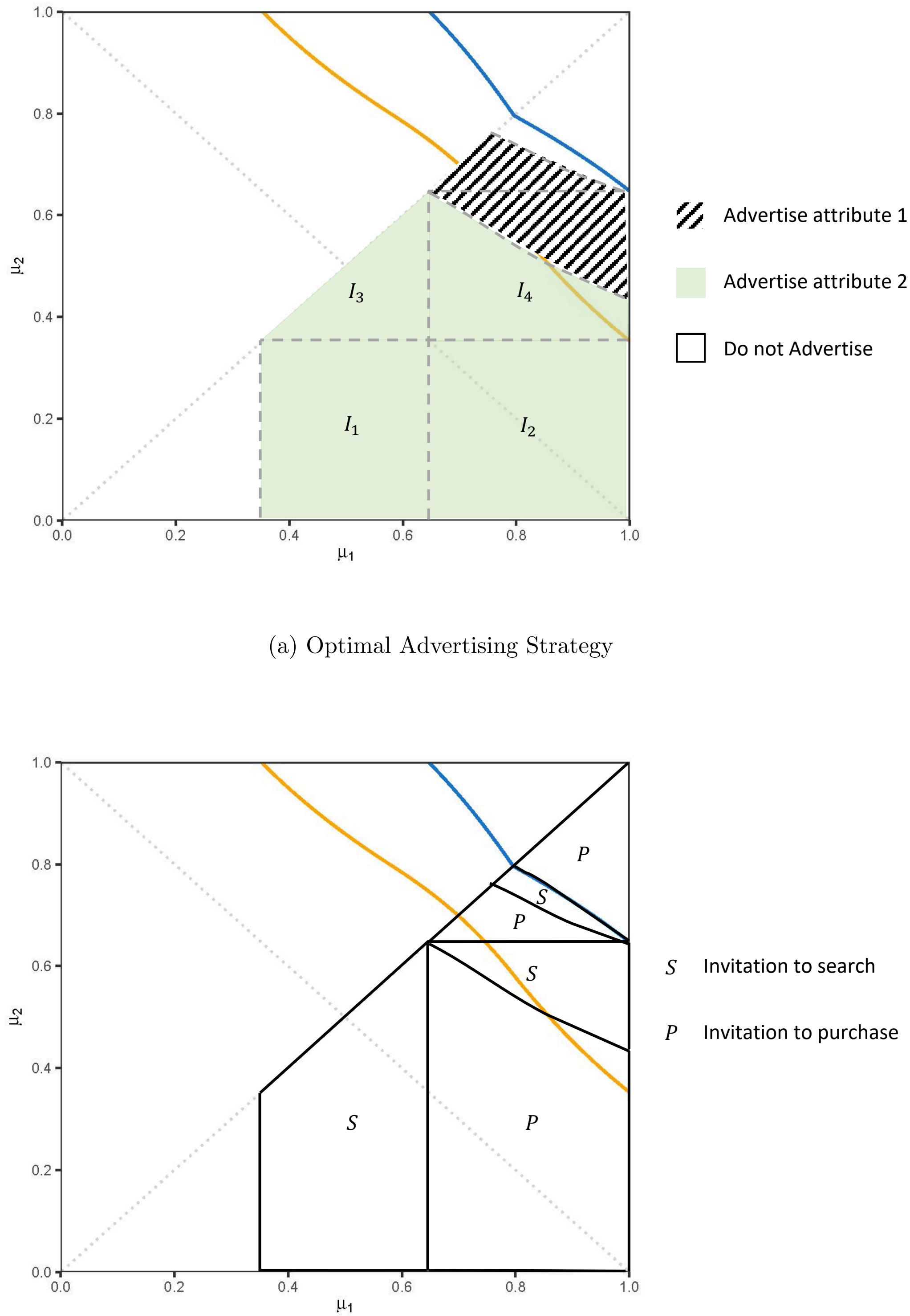
It turns out that any of the options, advertising the better attribute, advertising the worse attribute, and not advertising, may be optimal. By symmetry, we only need to consider the firm's advertising strategy when $\mu_1 \geq \mu_2$. The following proposition summarizes the main result of this paper.

Proposition 5. *Suppose $\mu_1 \geq \mu_2$. There exists $\tilde{\mu}(\mu_1)$ and $\widehat{\mu}(\mu_1)$ such that $\underline{\mu}(1) < \tilde{\mu}(\mu_1) \leq \widehat{\mu}(\mu_1) < \bar{\mu}(\mu_1)$ and $\tilde{\mu}(\mu_1)$ decreases in μ_1. The firm does not advertise if $\mu_1 \leq \underline{\mu}(1)$ or $\mu_2 \geq \widehat{\mu}(\mu_1)$, advertises attribute two if $\mu_1 \in (\underline{\mu}(1), \bar{\mu}(1)]$, or $\mu_1 > \bar{\mu}(1)$ and $\mu_2 \leq \tilde{\mu}(\mu_1)$, advertises attribute one if $\mu_1 > \bar{\mu}(1)$ and $\mu_2 \in (\tilde{\mu}(\mu_1), \widehat{\mu}(\mu_1))$.*

Advertising serves as an invitation to search if $\mu_1 \in (\underline{\mu}(1), \bar{\mu}(1)]$ or if $\mu_1 > \bar{\mu}(1)$ and $\mu_2 \in (\tilde{\mu}(\mu_1), \bar{\mu}(\mu_1))$, and serves as an invitation to purchase if $\mu_1 > \bar{\mu}(1)$ and $\mu_2 < \tilde{\mu}(\mu_1)$ or if $\mu_1 > \bar{\mu}(1)$ and $\mu_2 \in (\bar{\mu}(1), \widehat{\mu}(\mu_1))$. No advertising serves as an invitation to search if $\mu_1 > \bar{\mu}(1)$ and $\mu_2 \in (\widehat{\mu}(\mu_1), \bar{\mu}(\mu_1))$, and serves as an invitation to purchase if $\mu_1 > \bar{\mu}(1)$ and $\mu_2 > \bar{\mu}(\mu_1)$.

Figure 5a illustrates the optimal advertising strategy. The firm advertises attribute one in the diagonal striped black region, attribute two in the solid green region, and does not advertise in the blank region. Figure 5b presents the implication of the optimal advertising strategy. When the prior belief about attribute one is moderate, the firm advertises the other attribute to invite the consumer to search for more information about attribute one. When the prior belief about attribute one is high, the role of advertising is *non-monotonic* in the prior belief about attribute two. When μ_2 is low, the firm advertises attribute two to invite the consumer to directly purchase the product. When μ_2 is higher, the firm advertises attribute one to invite the consumer to search for more information about attribute two. When μ_2 is even higher, the firm advertises attribute one to invite the consumer to directly purchase the product. The firm stops advertising as μ_2 further increases. No advertising serves as an invitation to search when μ_2 is high and as an invitation to purchase when μ_2 is very high.

We will now discuss the mechanism and intuition of the optimal advertising strategy. If the consumer's prior beliefs about both attributes are too low, the product will not be attractive to the consumer even if she knows that one attribute is good. The consumer will neither search for information nor purchase the product even if the firm advertises. So, the firm does not advertise. If the consumer has high enough prior beliefs about both attributes, she will purchase the product



(a) Optimal Advertising Strategy

(b) Consumer Behavior Induced by the Advertising Strategy

Figure 5: Optimal Advertising Strategy and the Induced Consumer Behavior

without searching. The firm also has no incentive to advertise. Even if the consumer's belief is within the search region, she will purchase the product after receiving a little positive information as long as her belief is close to the purchasing boundary. The purchasing probability is close to 1. In contrast, if the firm advertises, the consumer will quit for sure if she finds out that one attribute is bad. So, the purchasing probability is lower. The firm is better off by not advertising. The intuition is the following. If the consumer finds out that one attribute is good from advertising, her belief about the product value will be higher than what is needed for her to purchase the product immediately. Such excessive belief is wasteful from the firm's standpoint. If the firm does not advertise, the consumer will be just indifferent between searching for more information and purchasing the product after receiving a little positive information. The firm does not waste any belief. Therefore, the consumer will be more likely to purchase the product without advertising. Therefore, the firm does not advertise in the white region.

Now let's consider the solid green region and the diagonal striped black region. We divide the solid green region into four sub-regions. If the belief lies in the region I_1 or I_2, the consumer is very pessimistic about the second attribute. Even if she knows for sure that the first attribute is good, she needs to receive a lot of positive signals about attribute two to purchase the product. The search cost outweighs the benefit of the search. So, she will not search for information. The only way of inducing the consumer to search is to advertise attribute two. With a high probability, the consumer will find out that attribute two is bad and quit. However, if the consumer finds out that attribute two is good, she needs fewer positive signals to purchase the product by searching for attribute one. The benefit of search outweighs the search cost. So, the consumer will search for information about attribute one and purchase the product with a positive probability. Therefore, the firm advertises attribute two.

If the belief lies in the region I_3, the consumer will never purchase the product without advertising but may purchase the product if the firm advertises either attribute. So, the firm advertises. On one hand, the consumer is more optimistic about attribute one and will be more likely to search for information if the firm advertises attribute one rather than two. On the other hand, she needs more positive signals to purchase the product if the firm advertises attribute one. So, the conversion rate conditional on searching is lower. Because it is costless for the consumer to receive ads but costly for her to search for information, the firm can extract more surplus from the consumer by reducing the expected search time. Therefore, the firm prefers to have a lower search probability but a higher purchasing probability conditional on searching and advertises attribute two.

Lastly, we consider the case where the belief lies in the region I_4 or the diagonal striped black region. If the consumer has a sufficiently high belief about attribute two, she will purchase the product immediately if she knows for sure that either attribute is good. One can see that the firm always prefers to advertise attribute one to attribute two because of the higher purchasing probability. If the consumer's belief about attribute two is lower, the decision between advertising attribute one or two becomes more complex. If the firm advertises attribute one and the consumer finds out it is good, the consumer will always search for information about attribute two before making a decision. In contrast, the consumer will be very positive about the product value if the firm advertises attribute two and the consumer knows that attribute two is good. In that case, she will purchase the product immediately. So, some beliefs are "wasted" - the consumer will purchase the product immediately even if her belief is lower. The more optimistic she is about the first attribute, the more beliefs are wasted. So, the firm will be more likely to advertise attribute one.

In sum, the firm will not advertise if the consumer's prior beliefs about both attributes are extreme and will advertise if the consumer's prior belief is milder. In that case, the firm will advertise the better attribute if the consumer is optimistic enough about the worse attribute, and will advertise the worse attribute if the consumer is less optimistic about it.

4.4 Advertising Costs

In the previous discussion, we did not consider the advertising costs. In reality, the firm needs to incur a cost to advertise. Our framework can incorporate this cost, but the analysis will be more tedious. So, we abstract away the advertising costs in the previous analysis. We briefly discuss what happens if we take into account the advertising costs. Suppose the firm needs to incur a cost c_A to advertise attribute i. The comparison between advertising attribute one and two will not change because both require an extra cost, c_A. However, whether the firm prefers to advertise or not may change. If the prior belief of the consumer without advertising is close to the purchasing boundary, then the firm will not advertise. Even without advertising, the consumer will purchase the product with a high probability. By not advertising, the firm saves advertising costs. The firm will also not advertise if the belief about one of the attributes is too low. Even if the firm can raise the purchasing probability above zero by advertising, the purchasing likelihood is very low. The profit will be negative because of the advertising costs. So, the firm will not advertise, and the consumer will neither search nor purchase. For all other beliefs, the firm's advertising strategy is the same as the case without advertising costs.

5 Conclusion

Choosing the optimal advertising content is an important managerial decision faced by all firms. This paper studies the firm's advertising strategy for a two-attribute product. If the firm advertises one attribute, the consumer faces a one-dimensional search problem because she is uncertain about only the other attribute. If the firm does not advertise, the consumer faces a two-dimensional search problem because she is uncertain about both attributes. In that case, it is always optimal for the consumer to search for the attribute about which the consumer has greater uncertainty. The consumer only searches for one attribute if she holds a strong prior belief about one of the attributes, and may search for both attributes otherwise.

We then characterize the optimal advertising strategy. The firm does not advertise if the consumer's prior beliefs about both attributes are extreme. No advertising serves as an invitation to search when the belief is high and as an invitation to purchase when the belief is very high. Otherwise, the firm advertises the better attribute if the consumer is optimistic enough about the worse attribute, and advertises the worse attribute if the consumer is less optimistic about it. When the prior belief about the better attribute is moderate, the firm advertises the other attribute to invite the consumer to search for more information. When the prior belief about the better attribute is high, the role of advertising is *non-monotonic* in the prior belief about attribute two. When the belief about the worse attribute is low, the firm advertises that attribute to invite the consumer to directly purchase the product. When the belief about the worse attribute is higher, the firm advertises the other attribute to invite the consumer to search for more information. When the belief about the worse attribute is even higher, the firm advertises the other attribute to invite the consumer to directly purchase the product.

There are some limitations to this paper. We consider a monopoly in this paper. Introducing competition can lead to interesting findings. It will also be interesting to extend the number of attributes beyond two and see whether the consumer still searches for the attribute with the highest uncertainty due to the fastest learning speed. Lastly, we consider an exogenous price throughout the paper to focus on the role of information. Future research can study the optimal pricing of the product given the consumer's optimal search strategy.

Appendix

Proof of Proposition 1. By symmetry, we only need to prove the case of $\mu_1 \geq \mu_2$. We first show that the viscosity solution of the HJB equation ($\star$) exists and is unique. Since the value function is a viscosity solution of ($\star$), the viscosity solution of ($\star$) must be the value function by uniqueness. We then conjecture an optimal search strategy and characterize its properties. Lastly, we verify that the conjectured strategy indeed generates a viscosity solution to ($\star$). So, the conjectured strategy is optimal.

Lemma 1. *The viscosity solution of the HJB equation* ($\star$) *exists and is unique.*

Proof. Since the consumer can guarantee a payoff of zero by quitting immediately and cannot achieve a payoff higher than $\sup\{\mu_1 + \mu_2 - p\} = 1 + 1 - p \leq 2$, the value function is bounded and thus exists. This implies the existence of the viscosity solution because the value function is a viscosity solution to ($\star$).

The proof of the uniqueness uses a modification of a comparison principle in Crandall et al. (1992). Given that it very much resembles the proof of Lemma 1 in Ke and Villas-Boas (2019), we refer the reader to their proof. □

Conjecture:

Conditional on searching, it is optimal for the consumer to search for information about attribute two (one) if $\mu_1 \geq \mu_2$ ($\mu_1 < \mu_2$).

Given this conjecture, we now characterize the search region (illustrated in Figure 3).

The PDE when the consumer searches attribute two, equation (4), has the following general solution (B_1 and B_2 are unknown functions to be determined):

$$V(\mu_1, \mu_2) = 2\sigma^2 c(1 - 2\mu_2) \ln \frac{1 - \mu_2}{\mu_2} + B_1(\mu_1)\mu_2 + B_2(\mu_1), \mu_1 \in [\mu^*, 1]$$

We also have $V(\mu_1, \mu_2) = 0$ at the quitting boundary $\mu_2 = \underline{\mu}(\mu_1)$. For the value function in the search region, value matching and smooth pasting (wrt μ_2) at the quitting boundary $(\mu_1, \underline{\mu}(\mu_1))$

imply:[4]

$$\frac{V(\mu_1,\mu_2)}{2\sigma^2 c} = (1-2\mu_2)\ln\frac{1-\mu_2}{\mu_2} + \phi(\underline{\mu}(\mu_1))\mu_2 - \psi(\underline{\mu}(\mu_1)) \tag{5}$$

, where $\phi(x) = 2\ln\frac{1-x}{x} + \frac{1}{x} - \frac{1}{1-x}$ and $\psi(x) = \ln\frac{1-x}{x} + \frac{1-2x}{1-x}$.

By symmetry, for $\mu_1 < \mu_2$, the value function in the search region satisfies:

$$\frac{V(\mu_1,\mu_2)}{2\sigma^2 c} = (1-2\mu_1)\ln\frac{1-\mu_1}{\mu_1} + \phi(\underline{\mu}(\mu_2))\mu_1 - \psi(\underline{\mu}(\mu_2)) \tag{6}$$

Equation (5) characterizes the value function for beliefs $\mu_1 \geq \mu_2$. Equation (6) characterizes the value function for beliefs $\mu_1 < \mu_2$. The two regions are separated by the main diagonal $\{(\mu_1,\mu_2) : \mu_1 = \mu_2\}$. Continuity of $V_{\mu_1}(\mu_1,\mu_2)$ at this boundary implies that:

$$\underline{\mu}'(\mu) = \frac{\phi(\underline{\mu}(\mu)) - \phi(\mu)}{\phi'(\underline{\mu}(\mu))[\mu - \underline{\mu}(\mu)]}, \text{ for } \mu \in (\mu^*, \mu^{**}] \tag{D_1}$$

For $\mu_1 \in [\mu^{**}, 1]$, $V(\mu_1,\mu_2) = \mu_1 + \mu_2 - p$ at the purchasing boundary $\mu_2 = \bar{\mu}(\mu_1)$. Value matching and smooth pasting (w.r.t. μ_2) at the purchasing boundary $(\mu_1, \underline{\mu}(\mu_1))$ imply (in the search region):

$$\frac{V(\mu_1,\mu_2)}{2\sigma^2 c} = (1-2\mu_2)\ln\frac{1-\mu_2}{\mu_2} + \phi(\bar{\mu}(\mu_1))\mu_2 - \psi(\bar{\mu}(\mu_1)) + \frac{\mu_1 - \mu_2 - p}{2\sigma^2 c} \tag{7}$$

Equation (5) and (7) use the quitting boundary and the purchasing boundary to pin down the value function, respectively. The resulting expression should be equivalent in the common domain $\mu_1 \in [\mu^{**}, 1]$. By equalizing V and V_{μ_2} of equation (5) and (7), we obtain the following system of equations:

$$\begin{cases} \phi(\underline{\mu}(\mu)) - \phi(\bar{\mu}(\mu)) = \frac{1}{2\sigma^2 c} \\ \psi(\underline{\mu}(\mu)) - \psi(\bar{\mu}(\mu)) = \frac{p-\mu}{2\sigma^2 c} \end{cases}, \text{ for } \mu \in [\mu^{**}, 1] \tag{8}$$

[4] For technical details, please refer to Dixit (1993).

For each belief, μ, the system of equations above consists of two unknowns ($\bar{\mu}(\mu)$ and $\underline{\mu}(\mu)$) and two equations. They uniquely determine the function for the purchasing boundary $\bar{\mu}(\mu)$ and the function for the quitting boundary $\underline{\mu}(\mu)$, for $\mu \in [\mu^{**}, 1]$, given a cutoff belief μ^{**}.

Instead of determining $\bar{\mu}(\mu)$ and $\underline{\mu}(\mu)$ by a system of equations (8), we can also implicitly determine $\bar{\mu}(\mu)$ and $\underline{\mu}(\mu)$ in two separate equations. Representing $\bar{\mu}(\mu)$ by $\underline{\mu}(\mu)$ from the first equation of (8), we have:

$$\bar{\mu}(\mu) = \phi^{-1}\left[\phi(\underline{\mu}(\mu)) - \frac{1}{2\sigma^2 c}\right]$$

Plugging it into the second equation of (8), we have:

$$\underline{\mu}(\mu) = \psi^{-1}\left\{\psi\left(\phi^{-1}\left[\phi(\underline{\mu}(\mu)) - \frac{1}{2\sigma^2 c}\right]\right) + \frac{p-\mu}{2\sigma^2 c}\right\}$$

The equation above implicitly determines $\underline{\mu}(\mu)$, for $\mu \in [\mu^{**}, 1]$. Similarly, we can implicitly determine $\bar{\mu}(\mu)$ by the following equation:

$$\bar{\mu}(\mu) = \psi^{-1}\left\{\psi\left(\phi^{-1}\left[\phi(\bar{\mu}(\mu)) + \frac{1}{2\sigma^2 c}\right]\right) - \frac{p-\mu}{2\sigma^2 c}\right\}$$

We now solve for the cutoff belief at the intersection of the purchasing boundary and the main diagonal, μ^{**}. Since (μ^{**}, μ^{**}) is on the purchasing boundary, we have $\mu^{**} = \bar{\mu}(\mu^{**})$, μ^{**} is determined by:

$$\begin{cases} \phi(\underline{\mu}(\mu^{**})) - \phi(\mu^{**}) = \frac{1}{2\sigma^2 c} \\ \psi(\underline{\mu}(\mu^{**})) - \psi(\mu^{**}) = \frac{p-\mu^{**}}{2\sigma^2 c} \end{cases} \tag{9}$$

The system of equations above consists of two unknowns (μ^{**} and $\underline{\mu}(\mu^{**})$ and two equations. They uniquely determine the cutoff belief μ^{**} via the following equations:

$$\phi^{-1}\left[\phi(\mu^{**}) + \frac{1}{2\sigma^2 c}\right] = \psi^{-1}\left[\psi(\mu^{**}) + \frac{p-\mu^{**}}{2\sigma^2 c}\right] \tag{I^{**}}$$

We have pinned down the cutoff belief μ^{**}. Given this cutoff beliefs, we have determined the purchasing boundary $(\mu, \bar{\mu}(\mu))$ and the quitting boundary $(\mu, \underline{\mu}(\mu))$, for $\mu \in [\mu^{**}, 1]$.

Equation (D_1) and the initial condition (I^{**}) implicitly determine the function for the quitting boundary $\underline{\mu}(\mu)$, for $\mu \in (\mu^*, \mu^{**}]$, given a cutoff belief μ^*.

We now solve for the cutoff belief at the intersection of the quitting boundary and the main diagonal, μ^*. Since (μ^*, μ^*) is on the quitting boundary, we have $\mu^* = \underline{\mu}(\mu^*)$. This initial condition determines μ^*.

In sum, we have pinned down the cutoff belief μ^* and the quitting boundary $(\mu, \underline{\mu}(\mu))$, for $\mu \in [\mu^*, \mu^{**}]$.

We have fully characterized the purchasing boundary $(\mu, \bar{\mu}(\mu))$ and the quitting boundary $(\mu, \underline{\mu}(\mu))$, for $\mu_1 \geq \mu_2$. The other case in which $\mu_1 < \mu_2$ is readily determined by symmetry.

Verification:

To verify that the conjectured strategy indeed generates a viscosity solution to the HJB equation $(\star)$:

$$\max\left\{\max_{i=1,2}\left[\frac{\mu_i^2(1-\mu_i)^2}{2\sigma^2}V_{\mu_i\mu_i}(\mu_1,\mu_2)-c\right], \max\left[\mu_1+\mu_2-p, 0\right]-V(\mu_1,\mu_2)\right\}=0$$

We just need to show that (everything else holds by our construction):

$$\begin{aligned} &\frac{\mu_1^2(1-\mu_1)^2}{2\sigma^2}V_{\mu_1\mu_1}(\mu_1,\mu_2)-c\leq 0 \\ \Leftrightarrow\ &\mu_1^2(1-\mu_1)^2V_{\mu_1\mu_1}(\mu_1,\mu_2)/2\sigma^2c\leq 1 \end{aligned} \tag{10}$$

if $\mu_1+\mu_2>1$, $\mu_1\geq\mu_2$, and $\underline{\mu}(\mu_1)<\mu_2<\bar{\mu}(\mu_1)$.

For $\mu_1 \in (\mu^*, \mu^{**}]$, we have

$$\begin{aligned} &V_{\mu_1}(\mu_1,\mu_2)/2\sigma^2c \\ =&\phi'(\underline{\mu}(\mu_1))\underline{\mu}'(\mu_1)[\mu_2-\underline{\mu}(\mu_1)] \\ \overset{(D_1)}{=}&\frac{\phi(\underline{\mu}(\mu_1))-\phi(\mu_1)}{\mu_1-\underline{\mu}(\mu_1)}[\mu_2-\underline{\mu}(\mu_1)] \\ \Rightarrow\ &V_{\mu_1\mu_1}(\mu_1,\mu_2)/2\sigma^2c \\ =&\phi'(\underline{\mu}(\mu_1))\underline{\mu}'(\mu_1)[\mu_2-\underline{\mu}(\mu_1)] \end{aligned}$$

$$
\begin{aligned}
&\overset{(D_1)}{=}\frac{\phi'(\underline{\mu}(\mu_1))\underline{\mu}'(\mu_1)-\phi'(\mu_1)}{\mu_1-\underline{\mu}(\mu_1)}[\mu_2-\underline{\mu}(\mu_1)]+[\phi(\underline{\mu}(\mu_1))-\phi(\mu_1)]\frac{(\mu_2-\mu_1)\underline{\mu}'(\mu_1)+\underline{\mu}(\mu_1)-\mu_2}{[\mu_1-\underline{\mu}(\mu_1)]^2}\\
&=-\frac{\phi'(\mu_1)[\mu_2-\underline{\mu}(\mu_1)]}{\mu_1-\underline{\mu}(\mu_1)}+(\mu_2-\mu_1)\frac{[\phi(\underline{\mu}(\mu_1))-\phi(\mu_1)]^2}{\phi'(\underline{\mu}(\mu_1))[\mu_1-\underline{\mu}(\mu_1)]^3}\\
&\Rightarrow\ \mu_1^2(1-\mu_1)^2V_{\mu_1\mu_1}(\mu_1,\mu_2)/2\sigma^2c\\
&=\frac{\mu_2-\underline{\mu}(\mu_1)}{\mu_1-\underline{\mu}(\mu_1)}+(\mu_1-\mu_2)\mu_1^2(1-\mu_1)^2\frac{\underline{\mu}(\mu_1)^2[1-\underline{\mu}(\mu_1)]^2}{[\mu_1-\underline{\mu}(\mu_1)]^3}[\phi(\underline{\mu}(\mu_1))-\phi(\mu_1)]^2
\end{aligned}
$$

So,

$$
\begin{aligned}
&\mu_1^2(1-\mu_1)^2V_{\mu_1\mu_1}(\mu_1,\mu_2)/2\sigma^2c\le 1\\
\Leftrightarrow&\mu_1^2(1-\mu_1)^2\frac{\underline{\mu}(\mu_1)^2[1-\underline{\mu}(\mu_1)]^2}{[\mu_1-\underline{\mu}(\mu_1)]^2}[\phi(\underline{\mu}(\mu_1))-\phi(\mu_1)]^2\le 1\\
\Leftrightarrow&\mu_1(1-\mu_1)\frac{\underline{\mu}(\mu_1)^2[1-\underline{\mu}(\mu_1)]}{[\mu_1-\underline{\mu}(\mu_1)]}[\phi(\underline{\mu}(\mu_1))-\phi(\mu_1)]\le 1\\
\Leftrightarrow&H(\mu_1):=\mu_1(1-\mu_1)[\phi(\underline{\mu}(\mu_1))-\phi(\mu_1)]-\frac{\mu_1-\underline{\mu}(\mu_1)}{\underline{\mu}(\mu_1)[1-\underline{\mu}(\mu_1)]}\le 0
\end{aligned}
\tag{11}
$$

Observe that $H(\mu^*)=0$. Ignoring the subscript 1 for notational ease, we have:

$$
\begin{aligned}
H'(\mu)=&(1-2\mu)[\phi(\underline{\mu}(\mu))-\phi(\mu)]+\frac{\mu(1-\mu)}{\mu-\underline{\mu}(\mu)}[\phi(\underline{\mu}(\mu))-\phi(\mu)]+\frac{1}{\mu(1-\mu)}\\
&-\frac{1}{\underline{\mu}(\mu)(1-\underline{\mu}(\mu))}+\frac{\phi(\underline{\mu}(\mu))-\phi(\mu)}{\mu-\underline{\mu}(\mu)}[-\mu+2\mu\underline{\mu}(\mu)-\underline{\mu}(\mu)^2]\\
=&[1-3\mu+\underline{\mu}(\mu)][\phi(\underline{\mu}(\mu))-\phi(\mu)]+\frac{1}{\mu(1-\mu)}-\frac{1}{\underline{\mu}(\mu)(1-\underline{\mu}(\mu))}
\end{aligned}
$$

Suppose (11) does not hold. There would exist $\widehat{\mu}$ such that $H(\widehat{\mu})=0$ and $H'(\widehat{\mu})>0$.

$$
(11)\Rightarrow\phi(\underline{\mu}(\widehat{\mu}))-\phi(\widehat{\mu})=\frac{\widehat{\mu}-\underline{\mu}(\widehat{\mu})}{\widehat{\mu}(1-\widehat{\mu})\underline{\mu}(\widehat{\mu})[1-\underline{\mu}(\widehat{\mu})]}
$$

Hence, we get an expression for $\frac{1}{\widehat{\mu}(1-\widehat{\mu})}$ and $\frac{1}{\underline{\mu}(\mu)[1-\underline{\mu}(\mu)]}$. Plugging these expressions into the previous expression for $H'(\mu)$, we have:

$$
H'(\widehat{\mu})=-2[\phi(\underline{\mu}(\mu))-\phi(\mu)][\mu-\underline{\mu}(\mu)]\le 0
$$

A contradiction! So, (11) and thus (10) hold, $\forall \mu_1 \in [\mu^*, \mu^{**}]$.

For $\mu_1 \in [\mu^{**}, 1]$, we have

$$\begin{aligned}
V_{\mu_1}(\mu_1, \mu_2)/2\sigma^2 c =& \phi'(\underline{\mu}(\mu_1))\underline{\mu}'(\mu_1)[\mu_2 - \underline{\mu}(\mu_1)] \\
\overset{(D_2)}{=}& \frac{\mu_2 - \underline{\mu}(\mu_1)}{\bar{\mu}(\mu_1) - \underline{\mu}(\mu_1)} \\
V_{\mu_1\mu_1}(\mu_1, \mu_2)/2\sigma^2 c =& \frac{-\underline{\mu}'(\mu_1)[\bar{\mu}(\mu_1) - \underline{\mu}(\mu_1)] - [\bar{\mu}'(\mu_1) - \underline{\mu}'(\mu_1)][\mu_2 - \underline{\mu}(\mu_1)]}{[\bar{\mu}(\mu_1) - \underline{\mu}(\mu_1)]^2} \\
=& \frac{1}{2\sigma^2 c} \frac{1}{[\bar{\mu}(\mu_1) - \underline{\mu}(\mu_1)]^3} [\frac{\mu_2 - \bar{\mu}(\mu_1)}{\phi'(\underline{\mu}(\mu_1))} - \frac{\mu_2 - \underline{\mu}(\mu_1)}{\phi'(\bar{\mu}(\mu_1))}] \\
\Rightarrow V_{\mu_1\mu_1}(\mu_1, \mu_2) =& \frac{1}{[\bar{\mu}(\mu_1) - \underline{\mu}(\mu_1)]^3} [\frac{\mu_2 - \bar{\mu}(\mu_1)}{\phi'(\underline{\mu}(\mu_1))} - \frac{\mu_2 - \underline{\mu}(\mu_1)}{\phi'(\bar{\mu}(\mu_1))}]
\end{aligned}$$

Since $\frac{\partial V_{\mu_1\mu_1}(\mu_1,\mu_2)}{\partial \mu_2} < 0$, we only need to show that (10) holds for $\mu_2 = \underline{\mu}(\mu_1)$:

$$\begin{aligned}
& \mu_1^2(1-\mu_1)^2 V_{\mu_1\mu_1}(\mu_1, \underline{\mu}(\mu_1))/2\sigma^2 c \le 1 \\
\Leftrightarrow & \frac{\mu_1^2(1-\mu_1)^2}{[\bar{\mu}(\mu_1) - \underline{\mu}(\mu_1)]^2} \frac{-1}{\phi'(\underline{\mu}(\mu_1))} \le 1
\end{aligned} \tag{12}$$

Let's first show that $\underline{\mu}(\mu^{**}) \le 1/2$ by contradiction. Suppose instead $\underline{\mu}(\mu^{**}) > 1/2$.

$$\begin{aligned}
& p - \mu^{**} = \frac{\bar{\mu}(\mu^{**}) + \underline{\mu}(\mu^{**})}{2} \\
\Leftrightarrow \ & p - \mu^{**} = \frac{\mu^{**} + \underline{\mu}(\mu^{**})}{2} \\
\Leftrightarrow \ & \underline{\mu}(\mu^{**}) = 2p - 3\mu^{**}
\end{aligned}$$

Hence, $2p - 3\mu^{**} > 1/2 \Rightarrow \mu^{**} < \frac{2}{3}p - \frac{1}{6}$. Since $\phi(x)$ is strictly decreasing in x, the first equation of (9) implies

$$\begin{aligned}
\frac{1}{2\sigma^2 c} =& \phi(\underline{\mu}(\mu^{**})) - \phi(\mu^{**}) \\
<& \phi(1/2) - \phi(\frac{2}{3}p - \frac{1}{6}) \\
\Leftrightarrow c >& \frac{1}{2\sigma^2[\phi(1/2) - \phi(\frac{2}{3}p - \frac{1}{6})]}
\end{aligned}$$

A contradiction! Therefore, $\underline{\mu}(\mu^{**}) \leq 1/2$. Since $\underline{\mu}(\mu_1)$ is decreasing in μ_1, we have $\underline{\mu}(\mu_1) \leq 1/2$, $\forall \mu \in [\mu^{**}, 1]$. One can see that the LHS of (12), $\frac{\mu_1^2(1-\mu_1)^2}{[\bar{\mu}(\mu_1)-\underline{\mu}(\mu_1)]^2} \frac{-1}{\phi'(\underline{\mu}(\mu_1))}$, decreases in $\mu_1 \in [\mu^{**}, 1]$. And we know that (12) holds for $\mu_1 = \mu^{**}$ (we have shown that (10) and thus (12) hold for $\forall \mu_1 \in [\mu^*, \mu^{**}]$). Therefore, (12) and thus (10) hold for $\forall \mu_1 \in [\mu^{**}, 1]$.

□

Proof of Proposition 2. We have derived (D_1) in the proof of Proposition 1. It implies immediately that $\underline{\mu}'(\mu) < 0$ for $\mu \in (\mu^*, \mu^{**}]$. For $\mu \in [\mu^{**}, 1]$, by the implicit function theorem, we have:

$$\begin{bmatrix} \bar{\mu}'(\mu) \\ \underline{\mu}'(\mu) \end{bmatrix} = - \begin{bmatrix} -\phi'(\bar{\mu}(\mu)) & \phi'(\underline{\mu}(\mu)) \\ -\psi'(\bar{\mu}(\mu)) & \psi'(\underline{\mu}(\mu)) \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 \\ \frac{1}{2\sigma^2 c} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{2\sigma^2 c} \frac{1}{\phi'(\bar{\mu}(\mu))[\bar{\mu}(\mu)-\underline{\mu}(\mu)]} \\ \frac{1}{2\sigma^2 c} \frac{1}{\phi'(\underline{\mu}(\mu))[\bar{\mu}(\mu)-\underline{\mu}(\mu)]} \end{bmatrix} = \begin{bmatrix} \frac{\phi(\underline{\mu}(\mu))-\phi(\bar{\mu}(\mu))}{\phi'(\bar{\mu}(\mu))[\bar{\mu}(\mu)-\underline{\mu}(\mu)]} & < 0 \\ \frac{\phi(\underline{\mu}(\mu))-\phi(\bar{\mu}(\mu))}{\phi'(\underline{\mu}(\mu))[\bar{\mu}(\mu)-\underline{\mu}(\mu)]} & < 0 \end{bmatrix}$$

This gives us the expression for $(\overline{D_2})$ and $(\underline{D_2})$. One can see from the negative sign of the derivative that both $\underline{\mu}(\mu)$ and $\bar{\mu}(\mu)$ strictly decrease in μ.

We now look at the width of the search region.

$$\begin{aligned} & [\bar{\mu}(\mu) - \underline{\mu}(\mu)]' \\ =& \frac{\phi(\underline{\mu}(\mu)) - \phi(\bar{\mu}(\mu))}{\phi'(\bar{\mu}(\mu))[\bar{\mu}(\mu) - \underline{\mu}(\mu)]} - \frac{\phi(\underline{\mu}(\mu)) - \phi(\bar{\mu}(\mu))}{\phi'(\underline{\mu}(\mu))[\bar{\mu}(\mu) - \underline{\mu}(\mu)]} \\ =& \frac{\phi(\underline{\mu}(\mu)) - \phi(\bar{\mu}(\mu))}{\bar{\mu}(\mu) - \underline{\mu}(\mu)} \left[1/\phi'(\bar{\mu}(\mu)) - 1/\phi'(\underline{\mu}(\mu))\right] \\ =& \frac{\phi(\underline{\mu}(\mu)) - \phi(\bar{\mu}(\mu))}{\bar{\mu}(\mu) - \underline{\mu}(\mu)} \left[\underline{\mu}(\mu)^2(1 - \underline{\mu}(\mu))^2 - \bar{\mu}(\mu)^2(1 - \bar{\mu}(\mu))^2\right] \end{aligned}$$

One can see that $\frac{\phi(\underline{\mu}(\mu))-\phi(\bar{\mu}(\mu))}{\bar{\mu}(\mu)-\underline{\mu}(\mu)} > 0$. So, $[\bar{\mu}(\mu) - \underline{\mu}(\mu)]' > 0 \Leftrightarrow \underline{\mu}(\mu)^2(1 - \underline{\mu}(\mu))^2 > \bar{\mu}(\mu)^2(1 - \bar{\mu}(\mu))^2 \Leftrightarrow \underline{\mu}(\mu)(1 - \underline{\mu}(\mu)) > \bar{\mu}(\mu)(1 - \bar{\mu}(\mu)) \Leftrightarrow |\underline{\mu}(\mu) - 1/2| < |\bar{\mu}(\mu) - 1/2|$. Thus, the width of the search region, $\bar{\mu}(\mu) - \underline{\mu}(\mu)$, increases in the belief, μ, if and only if the quitting boundary is closer to 1/2 than the purchasing boundary. We know that $\forall \mu \geq \mu^{**}$, $p = \mu + \frac{\bar{\mu}(\mu)+\underline{\mu}(\mu)}{2}$ due to the symmetry of the one-dimensional learning problem.[5] Therefore,

[5] More specifically, the sum of the purchasing and quitting thresholds is zero when the price is zero in the one-

$$
\begin{aligned}
&\frac{\bar{\mu}(\mu)+\underline{\mu}(\mu)}{2}=p-\mu\geq 3/2-1=1/2\\
\Rightarrow\ &\frac{\bar{\mu}(\mu)+\underline{\mu}(\mu)}{2}\geq 1/2\\
\Leftrightarrow\ &\bar{\mu}(\mu)+\underline{\mu}(\mu)>1\\
\Leftrightarrow\ &|\underline{\mu}(\mu)-1/2|<|\bar{\mu}(\mu)-1/2|,\ \forall\mu\geq\mu^{**}
\end{aligned}
$$

Thus, the width of the search region, $\bar{\mu}(\mu)-\underline{\mu}(\mu)$, always increases in the belief μ.

Now suppose that $\underline{\mu}(\mu)\geq 1/2$, then $\forall\mu\in(\mu^*,\mu^{**}]$, we have

$$
\begin{aligned}
\underline{\mu}'(\mu)\overset{(D_1)}{=}&\frac{\phi(\underline{\mu}(\mu))-\phi(\mu)}{\phi'(\underline{\mu}(\mu))[\mu-\underline{\mu}(\mu)]}\\
=&\frac{-\phi'(\xi_1(\mu))[\mu-\underline{\mu}(\mu)]}{\phi'(\underline{\mu}(\mu))[\mu-\underline{\mu}(\mu)]}\ (\xi_1(\mu)\in(\underline{\mu}(\mu),\mu))\\
=&-\frac{\phi'(\xi_1(\mu)}{\phi'(\underline{\mu}(\mu))}\\
<&-1,
\end{aligned}
$$

where the last inequality comes from the fact that the absolute value of $\phi'(x)=-\frac{1}{x^2(1-x)^2}$ is strictly increasing in x for $x\geq 1/2$. Similarly, $\forall\mu\in[\mu^{**},1]$, we have

$$
\begin{aligned}
\underline{\mu}'(\mu)\overset{(\underline{D_2})}{=}&\frac{\phi(\underline{\mu}(\mu))-\phi(\bar{\mu}(\mu))}{\phi'(\underline{\mu}(\mu))[\bar{\mu}(\mu)-\underline{\mu}(\mu)]}\\
=&\frac{-\phi'(\xi_2(\mu))[\bar{\mu}(\mu)-\underline{\mu}(\mu)]}{\phi'(\underline{\mu}(\mu))[\bar{\mu}(\mu)-\underline{\mu}(\mu)]}\ (\xi_2(\mu)\in(\underline{\mu}(\mu),\bar{\mu}(\mu)))\\
=&-\frac{\phi'(\xi_2(\mu)}{\phi'(\underline{\mu}(\mu))}<-1\\
\bar{\mu}'(\mu)\overset{(\overline{D_2})}{=}&\frac{\phi(\underline{\mu}(\mu))-\phi(\bar{\mu}(\mu))}{\phi'(\bar{\mu}(\mu))[\bar{\mu}(\mu)-\underline{\mu}(\mu)]}\\
=&\frac{-\phi'(\xi_3(\mu))[\bar{\mu}(\mu)-\underline{\mu}(\mu)]}{\phi'(\bar{\mu}(\mu))[\bar{\mu}(\mu)-\underline{\mu}(\mu)]}\ (\xi_3(\mu)\in(\underline{\mu}(\mu),\bar{\mu}(\mu)))\\
=&-\frac{\phi'(\xi_3(\mu))}{\phi'(\bar{\mu}(\mu))}>-1
\end{aligned}
$$

dimensional optimal search strategy, as shown by Branco et al. (2012). It implies that the price equals to the average of the two boundaries. In our two-dimensional problem, the consumer only searches the more uncertain attribute when $\mu\geq\mu^{**}$. So, it can be translated to a one-dimensional search problem with the price p normalized to $p-\mu$.

$\square$

Proof of Proposition 3. We have shown that the consumer will quit if the firm advertises attribute one and the attribute is bad. It happens with probability $1-\mu_1$. With the complementary probability, the consumer purchases the product if $\mu_2 \geq \bar{\mu}_2$, searches for more information about attribute 2 if $\underline{\mu}_2 < \mu_2 < \bar{\mu}_2$, and quit if $\mu_2 \leq \underline{\mu}_2$, according to section 3.1. When the consumer searches for more information about attribute 2, by Dynkin's lemma, one can see that the conditional purchasing probability is $\frac{\mu_2-\underline{\mu}_2}{\bar{\mu}_2-\underline{\mu}_2}$. So, the probability that the consumer purchases the product is:

$$P_1(\mu_1,\mu_2) = \begin{cases} \mu_1, & \text{if } \mu_2 \geq \bar{\mu}_2 \\ \mu_1 \cdot \frac{\mu_2-\underline{\mu}_2}{\bar{\mu}_2-\underline{\mu}_2}, & \text{if } \mu_2 \in [\underline{\mu}_2, \bar{\mu}_2] \\ 0, & \text{if } \mu_2 \leq \underline{\mu}_2 \end{cases}$$

Comparing equation (2) with equation (8), one can see that $\underline{\mu}_2 = \underline{\mu}(1)$ and that $\bar{\mu}_2 = \bar{\mu}(1)$. $\square$

Proof of Proposition 4. We first consider $\mu_1 \in [\mu^{**}, 1]$ and $\mu_1 \geq \mu_2$. Under this circumstance, the consumer only learns about attribute two until μ_2 hits either the purchasing boundary or the quitting boundary. As μ_2 is a martingale, by Dynkin's formula, we get:

$$P(\mu_1,\mu_2) := \mathbb{P}[\text{purchasing}|\text{starting at } (\mu_1,\mu_2)] = \frac{\mu_2 - \underline{\mu}(\mu_1)}{\bar{\mu}(\mu_1) - \underline{\mu}(\mu_1)}$$

Now we consider $\mu_1 \in [\mu^*, \mu^{**}]$ and $\mu_1 \geq \mu_2$. The belief either hits (μ^{**}, μ^{**}) and the consumer purchases the good or the belief hits $\{(x, \underline{\mu}(x)) : x \in [\mu_1, \mu^{**})\} \cup \{(\underline{\mu}(x), x) : x \in [\mu_1, \mu^{**})\}$ and the consumer quits. To calculate the purchasing likelihood, let's first calculate the likelihood of the belief hitting $(\mu_1, \underline{\mu}(\mu_1))$ before hitting the main diagonal (μ_1, μ_1), $q(\mu_1, \mu_2)$.

$$q(\mu_1,\mu_2) = \frac{\mu_1 - \mu_2}{\mu_1 - \underline{\mu}(\mu_1)}$$

Now we calculate the probability of purchasing given belief (μ, μ), $\tilde{P}(\mu)$ by considering the infinitesimal learning on attribute two. Noticing that $q(\mu,\mu) = 0$, $\frac{\partial q}{\partial \mu_1}|_{\mu_1=\mu_2=\mu} = \frac{1}{\mu-\underline{\mu}(\mu)}$, $\frac{\partial q}{\partial \mu_2}|_{\mu_1=\mu_2=\mu} = -\frac{1}{\mu-\underline{\mu}(\mu)}$, we have:

$$
\begin{aligned}
\tilde{P}(\mu) &= \frac{1}{2}\mathbb{P}[\text{purchasing}|(\mu,\mu), d\mu \geq 0] + \frac{1}{2}\mathbb{P}[\text{purchasing}|(\mu,\mu), d\mu < 0] \\
&= \frac{1}{2}[1 - q(\mu + |d\mu|, \mu)]\tilde{P}(\mu + |d\mu|) + \frac{1}{2}[1 - q(\mu - |d\mu|, \mu)]\tilde{P}(\mu) \\
&= \tilde{P}(\mu) + \frac{|d\mu|}{2}\tilde{P}'(\mu) + |d\mu|\frac{\tilde{P}(\mu)}{\underline{\mu}(\mu) - \mu} + o(d\mu) \\
\Rightarrow 0 &= \frac{|d\mu|}{2}\left[\tilde{P}'(\mu) + 2\frac{\tilde{P}(\mu)}{\underline{\mu}(\mu) - \mu}\right] + o(d\mu) \\
\Rightarrow \frac{\tilde{P}'(\mu)}{\tilde{P}(\mu)} &= -\frac{2}{\underline{\mu}(\mu) - \mu}, \ \forall \mu \in (\mu^*, \mu^{**})
\end{aligned}
$$

, where the last equality comes from dividing the previous equation by $|d\mu|$ and take the limit of $d\mu$ to 0. Together with the initial condition $\tilde{P}(\mu^{**}) = 1$, we obtain:

$$\tilde{P}(\mu) = e^{-\int_{\mu}^{\mu^{**}} \frac{2}{x - \underline{\mu}(x)} dx}$$

In sum, the purchasing likelihood when $\mu_1 \geq \mu_2$ and $\mu_1 \in (\mu^*, \mu^{**})$ is:

$$P(\mu_1, \mu_2) = \mathbb{P}[\text{purchasing}|\text{starting at } (\mu_1, \mu_2)] = [1 - q(\mu_1, \mu_2)]\tilde{P}(\mu_1) = h(\mu_1, \mu_2)\tilde{P}(\mu_1)$$

, where $h(\mu_1, \mu_2) = \frac{\mu_2 - \underline{\mu}(\mu_1)}{\mu_1 - \underline{\mu}(\mu_1)}$.

By symmetry, the purchasing likelihood when $\mu_1 < \mu_2$ and $\mu_2 \in (\mu^*, \mu^{**})$ is:

$$P(\mu_1, \mu_2) = P(\mu_2, \mu_1) = [1 - q(\mu_2, \mu_1)]\tilde{P}(\mu_2) = h(\mu_2, \mu_1)\tilde{P}(\mu_2)$$

□

Proof of Proposition 5. One can see that the consumer will not purchase the product if $\mu_1 \leq \underline{\mu}(1)$, even if the firm advertises one attribute which turns out to be good. So, the firm does not advertise if $\mu_1 \leq \underline{\mu}(1)$. Also, the consumer will purchase the product for sure if $\mu_2 \geq \bar{\mu}(\mu_1)$ without advertising. So, the firm does not advertise if $\mu_2 \geq \bar{\mu}(\mu_1)$. We now look at other cases.

(1) $\mu_1 > \underline{\mu}(1)$ and $\mu_2 \leq \underline{\mu}(1)$ (Region I_1 and I_2)

The consumer will never purchase the product if the firm advertises attribute one or does not advertise. In contrast, the consumer may purchase theproduct if the firm advertises on attribute two. The consumer will not purchase if attribute two is bad. However, if attribute two is good, the consumer will purchase the product immediately in the region I_2, and will search for information about attribute one in the region I_1. In the region I_1, the consumer will purchase the product after receiving enough positive information. So, the purchasing likelihood is strictly positive. Hence, the firm advertises attribute two.

(2) $\mu_1 \in (\underline{\mu}(1), \bar{\mu}(1)]$ and $\mu_2 > \underline{\mu}(1)$ (Region I_3)

The purchasing probability is zero if the firm does not advertise, and is positive if the firm advertises either attriutes. Thus, we need to compare the purchasing likelihoods between advertising attribute one and attribute two.

$$\begin{aligned} P_1(\mu_1, \mu_2) &= \mu_1 \cdot \frac{\mu_2 - \underline{\mu}(1)}{\bar{\mu}(1) - \underline{\mu}(1)} \\ P_2(\mu_1, \mu_2) &= \mu_2 \cdot \frac{\mu_1 - \underline{\mu}(1)}{\bar{\mu}(1) - \underline{\mu}(1)} \\ &\overset{\mu_1 \geq \mu_2}{\geq} P_1(\mu_1, \mu_2), \end{aligned}$$

where the inequality is strict if $\mu_1 > \mu_2$. So, the firm advertises attribute two.

(3) $\mu_1 > \bar{\mu}(1)$ and $\mu_2 \in (\underline{\mu}(1), \bar{\mu}(\mu))$ (Region I_4, the diagonal striped black region, and the white search region)

To characterize the advertising strategy, we need to determine two things. First, whether the firm wants to advertise. Second, whether the firm prefers advertising attribute one or two, conditional on advertising.

We first compare advertising attribute one and two.

$$\begin{aligned} P_1(\mu_1, \mu_2) &= \mu_1 \cdot \frac{\mu_2 - \underline{\mu}(1)}{\bar{\mu}(1) - \underline{\mu}(1)} \\ P_2(\mu_1, \mu_2) &= \mu_2 \end{aligned}$$

$$P_1(\mu_1, \mu_2) > P_1(\mu_1, \mu_2) \Leftrightarrow \frac{\mu_2 - \underline{\mu}(1)}{\bar{\mu}(1) - \underline{\mu}(1)} > \frac{\mu_2}{\mu_1}$$
$$\Leftrightarrow \mu_2 > \frac{\underline{\mu}(1)\mu_1}{\mu_1 - \bar{\mu}(1) + \underline{\mu}(1)} := \tilde{\mu}(\mu_1)$$

So, the firm prefers advertising attribute one to advertising attribute two if and only if $\mu_2 > \tilde{\mu}(\mu_1)$. One can see that $\tilde{\mu}(\mu_1)$ decreases in μ_1.

We then determine whether the firm wants to advertise or not. If the belief is below the purchasing boundary, the firm always prefers advertising because the consumer will never purchase without advertising. Now suppose the belief is in the search region, $\mu_1 \in [\mu^{**}, 1]$ and $\mu_2 \in [\bar{\mu}(\mu_1), \mu_1]$. According to Proposition 4, the purchasing likelihood without advertising is:

$$P(\mu_1, \mu_2) = \frac{\mu_2 - \underline{\mu}(\mu_1)}{\bar{\mu}(\mu_1) - \underline{\mu}(\mu_1)}$$

If the firm advertises attribute one, the purchasing likelihood is:

$$P_1(\mu_1, \mu_2) = \begin{cases} \mu_1 \cdot \frac{\mu_2 - \underline{\mu}(1)}{\bar{\mu}(1) - \underline{\mu}(1)}, \text{ if } \mu_2 < \bar{\mu}(1) \\ \mu_1, \text{ if } \mu_2 \geq \bar{\mu}(1) \end{cases}$$

If the firm advertises attribute two, the purchasing likelihood is:

$$P_2(\mu_1, \mu_2) = \mu_2$$

Observe that $P(\mu_1, \underline{\mu}(\mu)) = 0$, $P(\mu_1, \underline{\mu}(\mu)) = 1$, $P_1(\mu_1, \underline{\mu}(1)) = 0$, $P_1(\mu_1, \bar{\mu}(\mu)) = \mu_1$, and $\underline{\mu}(1) \leq \underline{\mu}(\mu)$. By (quasi-) linearity of the purchasing likelihood, one can see that $P(\mu_1, \mu_2)$ crosses $\max\{P_1(\mu_1, \mu_2), P_2(\mu_1, \mu_2)\}$ once as μ_2 increases, fixing a μ_1. Hence, there exists $\widehat{\mu}(\mu_1) \in [\tilde{\mu}(\mu_1), \bar{\mu}(\mu_1))$ such that the firm does not advertise if and only if $\mu_2 \geq \widehat{\mu}(\mu_1)$.

□

References

Anderson, S. P., Renault, R. (2006). Advertising content. American Economic Review, 96(1), 93-113.

Anderson, S. P., Renault, R. (2009). Comparative advertising: disclosing horizontal match information. The RAND Journal of Economics, 40(3), 558-581.

Arbatskaya, M. (2007). Ordered search. The RAND Journal of Economics, 38(1), 119-126.

Armstrong, M., Zhou, J. (2011). Paying for prominence. The Economic Journal, 121(556), F368-F395.

Armstrong, M., Vickers, J., Zhou, J. (2009). Prominence and consumer search. The RAND Journal of Economics, 40(2), 209-233.

Bar-Isaac, H., Caruana, G., & Cuñat, V. (2010). Information gathering and marketing. Journal of Economics & Management Strategy, 19(2), 375-401.

Bar-Isaac, H., Caruana, G., & Cuñat, V. (2012). Information gathering externalities for a multi-attribute good. The Journal of Industrial Economics, 60(1), 162-185.

Bhardwaj, P., Chen, Y., & Godes, D. (2008). Buyer-initiated vs. seller-initiated information revelation. Management Science, 54(6), 1104-1114.

Bolton, P., & Harris, C. (1999). Strategic experimentation. Econometrica, 67(2), 349-374.

Bordalo, P., Gennaioli, N., Shleifer, A. (2013). Salience and consumer choice. Journal of Political Economy, 121(5), 803-843.

Branco, F., Sun, M., Villas-Boas, J. M. (2012). Optimal search for product information. Management Science, 58(11), 2037-2056.

Branco, F., Sun, M., Villas-Boas, J. M. (2016). Too much information? Information provision and search costs. Marketing Science, 35(4), 605-618.

Bronnenberg, B. J., Kim, J. B., & Mela, C. F. (2016). Zooming in on choice: How do consumers search for cameras online?. Marketing science, 35(5), 693-712.

Chaimanowong, W., Chen Q., & Yao, Y. (2024). Consumer Gradual Learning and Firm Non-stationary Pricing. Working Paper

Crandall, M. G., Ishii, H., Lions, P. L. (1992). User's guide to viscosity solutions of second order partial differential equations. Bulletin of the American mathematical society, 27(1), 1-67.

De Corniere, A. (2016). Search advertising. American Economic Journal: Microeconomics, 8(3), 156-188.

Dixit, A. (1993). The art of smooth pasting. Harwood Academic Publishers: Chur, Switzerland.

Dukes, A., & Liu, L. (2016). Online shopping intermediaries: The strategic design of search environments. Management Science, 62(4), 1064-1077.

Fudenberg, D., Strack, P., & Strzalecki, T. (2018). Speed, accuracy, and the optimal timing of choices. American Economic Review, 108(12), 3651-3684.

Gardete, P., & Hunter, M., (2024). Multiattribute Search: Empirical Evidence and Information Design. Available at SSRN: http://dx.doi.org/10.2139/ssrn.3180811

Grossman, S. J. (1981). The informational role of warranties and private disclosure about product quality. The Journal of Law and Economics, 24(3), 461-483.

Guo, L., & Zhang, J. (2012). Consumer deliberation and product line design. Marketing Science, 31(6), 995-1007.

Häubl, G., & Trifts, V. (2000). Consumer decision making in online shopping environments: The effects of interactive decision aids. Marketing science, 19(1), 4-21.

Hauser, J. (2011). A marketing science perspective on recognition-based heuristics (and the fast-and-frugal paradigm). Judgment and Decision Making, 6(5), 396-408.

Jerath, K., & Ren, Q. (2021). Consumer rational (in) attention to favorable and unfavorable product information, and firm information design. Journal of Marketing Research, 58(2), 343-362.

Jerath, K., & Ren, Q. (2023). Consumer search and product returns. Working paper.

Jeziorski, P., Moorthy, S. (2018). Advertiser prominence effects in search advertising. Management science, 64(3), 1365-1383.

Ke, T. T., & Lin, S. (2020). Informational complementarity. Management Science, 66(8), 3699-3716.

Ke, T. T., Lin, S., Lu, M. Y. (2023). Information design of online platforms. Working Paper.

Ke, T. T., Shen, Z. J. M., Villas-Boas, J. M. (2016). Search for information on multiple products. Management Science, 62(12), 3576-3603.

Ke, T. T., Villas-Boas, J. M. (2019). Optimal learning before choice. Journal of Economic

Theory, 180, 383-437.

Kőszegi, B., Szeidl, A. (2013). A model of focusing in economic choice. The Quarterly journal of economics, 128(1), 53-104.

Kuksov, D., Villas-Boas, J. M. (2010). When more alternatives lead to less choice. Marketing Science, 29(3), 507-524.

Lewis, T. R., Sappington, D. E. (1994). Supplying information to facilitate price discrimination. International Economic Review, 309-327.

Liao, X. (2021). Bayesian persuasion with optimal learning. Journal of Mathematical Economics, 97, 102534.

Liu, L., & Dukes, A. (2013). Consideration set formation with multiproduct firms: The case of within-firm and across-firm evaluation costs. Management Science, 59(8), 1871-1886.

Mayzlin, D., Shin, J. (2011). Uninformative advertising as an invitation to search. Marketing science, 30(4), 666-685.

Milgrom, P. R. (1981). Good news and bad news: Representation theorems and applications. The Bell Journal of Economics, 380-391.

Morris, S., & Strack, P. (2019). The wald problem and the relation of sequential sampling and ex-ante information costs. Available at SSRN 2991567.

Moscarini, G., Smith, L. (2001). The optimal level of experimentation. Econometrica, 69(6), 1629-1644.

Nelson, P. (1974). Advertising as information. Journal of political economy, 82(4), 729-754.

Ngwe, D., Ferreira, K. J., & Teixeira, T. (2019). The impact of increasing search frictions on online shopping behavior: Evidence from a field experiment. Journal of Marketing Research, 56(6), 944-959.

Seiler, S. (2013). The impact of search costs on consumer behavior: A dynamic approach. Quantitative Marketing and Economics, 11, 155-203.

Shapiro, J. M. (2006). A 'memory-jamming' theory of advertising. Available at SSRN 903474.

Stigler, G. J. (1961). The economics of information. Journal of political economy, 69(3), 213-225.

Stivers, A., & Tremblay, V. J. (2005). Advertising, search costs, and social welfare. Information

Economics and Policy, 17(3), 317-333.

Sun, M. (2011). Disclosing multiple product attributes. Journal of Economics & Management Strategy, 20(1), 195-224.

Ursu, R. M., Wang, Q., Chintagunta, P. K. (2020). Search duration. Marketing Science, 39(5), 849-871.

Ursu, R. M., Zhang, Q., & Honka, E. (2023). Search gaps and consumer fatigue. Marketing Science, 42(1), 110-136.

Villas-Boas, J. M. (2009). Product variety and endogenous pricing with evaluation costs. Management Science, 55(8), 1338-1346.

Weitzman, M. L. (1979). Optimal search for the best alternative. Econometrica: Journal of the Econometric Society, 641-654.

Wolinsky, A. (1986). True monopolistic competition as a result of imperfect information. The Quarterly Journal of Economics, 101(3), 493-511.

Xu, L., Chen, J., Whinston, A. (2010). Oligopolistic pricing with online search. Journal of Management information systems, 27(3), 111-142.

Yao, Y. (2023). Dynamic persuasion and strategic search. Management Science.

Zhu, Y., Dukes, A. (2017). Prominent attributes under limited attention. Marketing Science, 36(5), 683-698.