Non-stationary Pricing and Search

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Abstract

In the context of dynamic monopoly pricing with buyer learning, we study *non-stationary pricing strategies* – prices that evolve over time without being contingent on a buyer's current valuation – which endogenously induce non-stationarity in the buyer's search problem. Under zero search costs, a perfectly patient seller's optimal price is arbitrarily close to constant. In contrast, with discounting, non-stationary prices can outperform stationary ones. When search costs are positive, the optimal price is non-stationary even if the seller is perfectly patient. Prices increase over time when the information is too noisy or search costs are too high. When buyers are more incentivized to search, prices increase for buyers with high or low initial valuations and decrease for medium-valuation buyers.

Keywords: Intertemporal pricing, learning, stochastic control, non-stationary strategies

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1 Introduction

This paper studies dynamic monopoly pricing with buyer learning. Stokey (1979) originally considers the intertemporal price discrimination problem where a buyer (she) decides whether and when to buy a product from a monopolist seller (he), who chooses the pricing trajectory. Under the assumption that the buyer knows her valuation, which stays constant over time, Stokey (1979) shows that constant price is optimal for the seller. In reality, buyers often have uncertainty about a product's value, and often gather noisy information gradually to reduce such uncertainty before making a purchasing decision. A rich literature has explored a decision-maker's optimal information acquisition strategies across multiple alternatives or product attributes and their implications for sellers' pricing strategies (Weitzman, 1979; Wolinsky, 1986; Moscarini and Smith, 2001; Anderson and Renault, 2006; Armstrong et al., 2009; Villas-Boas, 2009; Bar-Isaac et al., 2010; Ke and Villas-Boas, 2019; Zhong, 2022; Callander, 2011; Fudenberg et al., 2018; Chaimanowong and Ke, 2022; Urgun and Yariv, 2025; Wong, 2025). Existing work of optimal pricing with buyer learning often assumes exogenous prices, endogenous constant prices, or endogenous prices contingent on the buyers' current valuation. However, recent privacy regulations have disrupted sellers' ability to track individual buyers in real time, making it challenging to tailor prices to evolving buyer beliefs. Even if a seller can track buyers' browsing behavior, it may be hard for the seller to know how buyers will interpret the information they see. For example, Tesla may be able to observe that a buyer clicks on an image of the interior design of the car, but may not know whether the buyer prefers the large screen on Tesla or the traditional dashboard. This calls into question whether the seller can track the evolution of the buyer's valuation when the buyer is searching for information, and raises an important question: can sellers benefit from dynamic pricing when the evolution of the buyer's valuation is unobservable?

Without the ability to track the evolution of the buyer's valuation, the only stationary pricing strategy is a constant price. This paper introduces a novel framework where sellers adopt *non-stationary pricing strategies* – prices that evolve over time without being contingent on the buyer's current valuation. Such pricing strategies endogenously induce non-stationarity in the buyer's search problem. We address two key questions: (1) Is a stationary pricing strategy always optimal for a seller that cannot observe the evolution of a buyer's valuation? (2) If not, what are the characteristics of the optimal non-stationary pricing strategy?

We use perturbation theory to tackle the challenge of characterizing the buyer's search strategy given the seller's non-stationary pricing strategies. Our findings challenge the conventional reliance on stationary pricing by showing that non-stationary pricing strategies can outperform stationary ones. We prove that a buyer can do almost as well by approximating any sufficiently slow-moving price with a linear price if she is sufficiently myopic. Given this result, we show that, under zero search costs, a perfectly patient seller's optimal price is arbitrarily close to constant. By contrast, with discounting, the seller may benefit from charging non-stationary prices. When search costs are positive, the optimal price is non-stationary even if the seller is perfectly patient. The price increases over time if the information is too noisy or the search cost is too high. The direction of price trajectories is more nuanced in other cases where buyers have a stronger incentive to search, with increasing prices for buyers with high or low initial valuation and decreasing prices for buyers with a medium level of initial valuation.

By incorporating non-stationary strategies into a search framework, we provide a theoretical advance in optimal control. Unlike most economic models, which impose stationarity for tractability, our results highlight that such restrictions may lead to sub-optimal outcomes. While a few earlier papers have explored non-stationarity in search problems driven by exogenous environments, such as the finite horizon and the evolving distribution of rewards (Gilbert and Mosteller, 1966; Sakaguchi, 1978; Van den Berg, 1990; Smith, 1999; Kamada and Muto, 2015), we model endogenous non-stationarity arising from sellers' strategic pricing in response to buyer search, providing a foundational step toward understanding sellers' non-stationary interventions in this context. A closely related paper, Libgober and Mu (2021), also considers dynamic monopoly pricing with buyer learning. By focusing on maximizing profits against the worst-case information arrival process when the buyer's true valuation is fixed over time, it shows that a constant price leads to robustly optimal profit. In contrast, we consider scenarios where the seller knows the information arrival process but does not know the realization of the signal, and maximizes the expected profit. We show that a constant price may no longer be optimal in such cases.

We also offer practical insights into how sellers can adapt to privacy regulations. Non-stationary pricing leverages time – a freely available and regulation-resistant resource – as an information source for pricing decisions, reducing reliance on costly tracking technologies. Our work provides guidance on how sellers can proactively adjust their pricing strategies to thrive in a privacy-conscious environment.

2 The Model

A seller offers a product with marginal cost g and sets its price. A buyer then decides whether to purchase the product. The buyer's initial valuation is v_0, which is common knowledge. Before making a purchase decision, the buyer can gradually learn about various product attributes and update her belief about the product's value. The buyer's discount rate is r and the seller's discount rate is m. We focus on the learning processes that arise within the general non-linear optimal filtering framework (Liptser and Shiryaev, 2013, Chapter 8).

Assume that the buyer's total utility from consuming the product is given by an unobservable process $\{\pi_t\}_{t\geq 0}$. To learn about π_t, the buyer pays a flow search cost of c per unit of time and observes a process $\{S_t\}_{t\geq 0}$, which generates a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ such that $v_t := \mathbb{E}[\pi_t|\mathscr{F}_t]$ is a continuous martingale.[1] We denote the valuation process as $\{v_s^{t,x}\}_{s\geq t}$ when emphasizing the initial condition $v_t^{t,x} = x$, or as $\{v_t^x\}_{t\geq 0}$ when the initial condition $v_0^x = x$ is specified at $t = 0$. When the initial value is not of central importance, we simply write v_t. We assume that $\{v_s^{t,x}\}_{s\geq t}$ is the unique strong solution to:

$$dv_s^{t,x} = \mu(v_s^{t,x}, \pi_s)ds + \sigma(v_s^{t,x})dW_s, \qquad v_t^{t,x} = x, \tag{1}$$

where $\{W_t\}_{t\in\mathbb{R}_{\geq 0}}$ is the standard Brownian motion adapted to the filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in\mathbb{R}_{\geq 0}}, \mathbb{P})$. In particular, we have $\mathbb{E}[dv_t|\mathscr{F}_t] = 0$ and $\mathbb{E}[(dv_t)^2|\mathscr{F}_t] = \sigma(v_t)^2 dt$. We impose the following assumptions on $\mu(.)$ and $\sigma(.)$:

Assumption 1. *Let $\underline{\pi}, \bar{\pi} \in \mathbb{R} \cup \{\pm\infty\}$ be such that $\underline{\pi} \leq v_t \leq \bar{\pi}$ a.e., for all $t \in \mathbb{R}$. We assume that:*

- *$\mu(., \pi), \sigma(.) \in C^\infty(\underline{\pi}, \bar{\pi})$, and $\sigma(x) > 0$ for all $x \in (\underline{\pi}, \bar{\pi})$.*
- *The global Lipschitz condition holds for some constant $L \geq 0$, for all $t \in \mathbb{R}$ and $x, y \in [\underline{\pi}, \bar{\pi}]$:*
$$|\mu(x, \pi_t) - \mu(y, \pi_t)| + |\sigma(x) - \sigma(y)| \leq L|x - y|. \tag{2}$$

The constants $\underline{\pi}$ and $\bar{\pi}$ represent the highest and lowest possible values of the product. We allow for the possibility that $\bar{\pi} = +\infty$, $\underline{\pi} = -\infty$. Given $\mathbf{x} = (t, x) \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}]$, we will sometimes write $\mu(\mathbf{x})$ and $\sigma(\mathbf{x})$ instead of $\mu(x)$ and $\sigma(x)$ when it is more convenient to use vector notation, even though $\mu(.)$ and $\sigma(.)$ do not explicitly depend on t. The assumption that $\{v_t\}_{t\geq 0}$ is a strong solution to the SDE (1) implies that $\{v_t\}_{t\geq 0}$ is a square-integrable martingale, i.e., $\mathbb{E}[v_t^2] < \infty$ for all $t \geq 0$. The global Lipschitz condition plays a role of controlling the growth rate of the square integral ($\mathbb{E}[v_t^2] = O(e^{L^2 t})$). Assumption 1 ensures that the buyer's expected payoff is well-defined.

Previous work studying the seller's endogenous pricing strategy in the presence of buyer gradual learning assumes either that the seller perfectly observes the evolution of the buyer's

[1] A sufficient condition for $\{v_t\}_{t\geq 0}$ to be martingale with respect to $\{\mathscr{F}_t\}_{t\geq 0}$ is that $\{\pi_t\}_{t\geq 0}$ is martingale with respect to the filtration generated by $\{(S_t, \pi_t)\}_{t\geq 0}$.

valuation (v_t is common knowledge) and can condition the price on the buyer's current valuation, or that the seller charges a constant price over time. Suppose we define the state variable by the buyer's current valuation, as is standard in the literature. The seller's problem in the first scenario is to choose the optimal *stationary strategy* because the strategy does not explicitly depend on the time, but only on the current state. This setup does not always fit real-world examples. Recent privacy regulations have disrupted sellers' ability to track individual buyers in real time. Even if a seller can track buyers' browsing behavior, it may be hard for the seller to know how buyers will interpret the information they see. Moreover, in many offline settings, individual-level tracking is not feasible.

Without the ability to observe the evolution of the buyer's valuation and thereby to tailor prices based on v_t, the only stationary pricing strategy is a constant price. Is a stationary pricing strategy always optimal for a seller in such cases? The major innovation of this paper is to consider *non-stationary pricing strategies* – prices that evolve over time without being contingent on a buyer's current valuation. Such pricing strategies endogenously induce non-stationarity in the buyer's search problem. Formally, the seller can commit to a pricing scheme $p := \{p_t\}_{t\geq 0} \in \mathcal{P}$, where $\mathcal{P}$ is the set of *admissible* pricing strategies, which is a subset of smooth functions on $[0,\infty)$, $\mathcal{P} \subset C^\infty[0,\infty)$.[2] This pricing strategy is a *non-stationary strategy* because p_t depends explicitly on time. It is widely known in optimal control that it is much harder to characterize *non-stationary strategies* than *stationary strategies*.

The buyer's search strategy consists of choosing an appropriate stopping time. Denote by $\mathcal{T}$ the set of all stopping times adapted to $\{\mathscr{F}_t\}_{t\in\mathbb{R}_{\geq 0}}$. The timing of the game is as follows.

1. At $t = 0$, the seller commits to a pricing strategy $p \in \mathcal{P} \subset C^\infty[0,\infty)$.
2. At any $t > 0$, the buyer decides whether to purchase the product, quit, or search.
3. The game ends when the buyer makes a purchase or quits.

The only knowledge the seller has about the buyer is their initial valuation, v_0. Importantly, when the buyer decides whether to purchase the product, quit, or keep searching at any given time, she takes into account both the current price and the future price trajectory. For any $p \in \mathcal{P}$ and $\tau \in \mathcal{T}$, we define the buyer's and seller's expected payoffs as:

$$\mathcal{V}^B(t,x;\tau,p) := \mathbb{E}\left[e^{-r(\tau-t)}\max\{v_\tau^{t,x} - p_\tau, 0\} - \int_t^\tau ce^{-r(s-t)}ds \mid \mathscr{F}_t\right], \tag{3}$$

$$\mathcal{V}^S(x;\tau,p) := \mathbb{E}\left[e^{-m\tau}(p_\tau - g)\cdot 1_{v_\tau^x \geq p_\tau}\right]. \tag{4}$$

A sufficient condition for the buyer's expected payoff (3) to be well-defined is

[2] For simplicity, we use p to denote $\{p_t\}_{t\geq 0}$ whenever this does not cause confusion.

$\sup_{\tau\in\mathcal{T}} \mathbb{E}\left[e^{-2r\tau}v_\tau^2\right] < \infty$. This condition holds as long as the global Lipschitz constant L in (2) is less than $\sqrt{r}$. For the remainder of this work, we will assume $L < \sqrt{r}$, which is satisfied in many applications such as when v_t is the standard Brownian motion, when v_t is bounded, or when the buyer has a high discount rate.

Solution Concept

Definition 1. *A subgame perfect ε-equilibrium (ε-SPE) consists of $(\{\tau^*[p] \in \mathcal{T}\}_{p\in\mathcal{P}}, p^* \in \mathcal{P})$ such that, for all $p \in \mathcal{P}$, $\mathcal{V}^B(t,x;\tau^*[p],p) \geq \mathcal{V}^B(t,x;\tau,p) - \varepsilon, \quad \forall\tau \in \mathcal{T}$, and $\mathcal{V}^S(x;\tau^*[p^*],p^*) \geq \mathcal{V}^S(x;\tau^*[p],p) - \varepsilon, \quad \forall p \in \mathcal{P}$.*

The buyer's value function given the seller's pricing strategy p is:

$$V^B(t,x;p) := \sup_{\tau\in\mathcal{T}} \mathcal{V}^B(t,x;\tau,p). \tag{5}$$

When there is no ambiguity, we will compactly write $V^B(t,x) = V^B(t,x;p)$. Analogously, we define the seller's value function as:

$$V^S(x) := \sup_{p\in\mathcal{P}} \mathcal{V}^S(x;\tau^*[p],p) \tag{6}$$

We work with subgame perfect ε-equilibrium rather than the usual subgame perfect equilibrium for tractability reasons. Specifically, it is challenging to characterize the buyer's search strategy given any non-stationary pricing strategies by the seller. To deal with this technical difficulty, a key idea of this paper is that, if a pricing strategy p is a small perturbation from a pricing strategy with a known solution, then the solution to p could also be a small perturbation from the known solution. The use of sub-game perfect ε-equilibrium allows us to formalize this idea using perturbation theory to the order of ε. The choice of ε can be very close to zero and is not the economic force behind the results.

3 Buyer's Strategy

The buyer faces an optimal stopping problem. She needs to determine the purchasing and quitting boundaries at any time. When the price is non-stationary, her purchasing and quitting boundaries are also time-contingent. This time-varying property makes her optimal stopping problem challenging, even if we fix a pricing scheme. To illustrate the impact of non-stationary pricing on the buyer's problem, we first review the constant-price benchmark.

3.1 Benchmark: Stationary Pricing

When the price is constant, $p_t = p_0 \in \mathbb{R}$, the buyer's search strategy does not depend on time. In particular, we have a time-independent value function $V^B(t, x; p_0) = V_0^B(x; p_0)$, purchasing threshold $\bar{V}_t = \bar{V}[p_0]$, and quitting threshold $\underline{V}_t = \underline{V}[p_0]$. The value function of the buyer satisfies the Hamilton-Jacobi-Bellman (HJB) equation:

$$\frac{\sigma(x)^2}{2}\partial_x^2 V_0^B - rV_0^B - c = 0, \tag{7}$$

subject to the value-matching condition and smooth pasting conditions:

$$\begin{aligned} V_0^B(\bar{V}[p_0]; p_0) &= \bar{V}[p_0] - p_0, \quad \partial_x V_0^B(\bar{V}[p_0]; p_0) = 1, \\ V_0^B(\underline{V}[p_0]; p_0) &= 0, \quad \partial_x V_0^B(\underline{V}[p_0]; p_0) = 0. \end{aligned}$$

We refer to Strulovici and Szydlowski (2015) for the derivation of the free-boundary ODE problem from the optimal stopping problem along with the results, which guarantees the existence and uniqueness of the solution in our setting.[3] In particular, V_0^B is continuously differentiable for all $x \in \mathbb{R}$ and twice continuously differentiable for all $x \in \mathbb{R} \setminus \{\underline{V}[p_0], \bar{V}[p_0]\}$.

We now consider two learning structures commonly used in the literature.

3.1.1 Product attributes learning

We first consider the learning process studied in Ke et al. (2022), where a buyer gradually learns about the ground-truth value of a product, π_t, which evolves over time according to $d\pi_t = \sigma dW_t^\pi$. The buyer learns about $\{\pi_t\}_{t\geq 0}$ by observing the signal $\{S_t\}_{t\geq 0}$, where $dS_t := \pi_t dt + \sigma_S dW_t$. The constant σ_S represents the information quality of the signal or the amount of attention the buyer pays. In this case, the ground-truth volatility σ and the observation noise σ_S together give the variance an asymptote of $\sigma\sigma_S > 0$. Assuming a normal prior belief $\pi_0 \sim \mathcal{N}(v_0, \sigma\sigma_S)$, the variance is constant over time and the valuation process has a constant volatility $dv_t = \frac{\sigma}{\sigma_S}(\pi_t - v_t)dt + \sigma dW_t$, or simply:[4]

[3] Unlike in Strulovici and Szydlowski (2015) our $\mu(.)$ also depends on another unobservable process $\{\pi_t\}_{t\geq 0}$. One can consider the process $X_t := (v_t, \pi_t)$ that generates the full information filtration $\{\mathscr{F}_t^X\}_{t\geq 0}$, following Strulovici and Szydlowski (2015), then take another expectation condition on $\{\mathscr{F}_t\}_{t\geq 0}$ at the end. Later, we will study the general version of this problem rigorously via the viscosity solutions framework.

[4] An alternative interpretation that gives equivalent technical results (see Branco et al. (2012))is that a buyer gradually learns about various product attributes to update her belief about the product's value before making a purchase decision. Each attribute i has a ground-truth utility of x_i. The product's total expected utility relative to the outside option (which is normalized to zero), given t searched attributes, is $v_t := \sum_{i=0}^t x_i$. When there are an infinite number of attributes, each with a very small weight in the valuation, v_t becomes a Brownian motion: $dv_t = \sigma dW_t^\pi$, for some constant σ.

$$dv_t = \sigma dW_t^v \tag{8}$$

by Lévy characterization, where $\{W_t^v\}_{t\geq 0}$ is a standard Brownian motion adapted to $\{\mathscr{F}_t\}_{t\geq 0}$. We write W_t^v as W_t hereinafter for convenience. In this example, $\underline{\pi} = -\infty$ and $\bar{\pi} = +\infty$. Branco et al. (2012) have characterized closed-form expressions for the value function:

$$V_0^B(x; p_0) = \frac{c}{r}\left[\cosh \frac{\sqrt{2r}}{\sigma}(x - \underline{V} - p_0) - 1\right], \tag{9}$$

and for the purchasing and quitting boundaries $\bar{V}[p_0] := p_0 + \bar{V}$, $\underline{V}[p_0] := p_0 + \underline{V}$, where:

$$\bar{V} := \sqrt{\frac{c^2}{r^2} + \frac{\sigma^2}{2r}} - \frac{c}{r},\ \underline{V} := \left(\sqrt{\frac{c^2}{r^2} + \frac{\sigma^2}{2r}} - \frac{c}{r}\right) - \frac{\sigma}{\sqrt{2r}} \log\left(\sqrt{\frac{r\sigma^2}{2c^2}} + \sqrt{1 + \frac{r\sigma^2}{2c^2}}\right). \tag{10}$$

3.1.2 Binary classification

We also consider the learning process of classifying the product's ground-truth value, which is a time-independent binary random variable $\pi_t = \pi \in \{0, 1\}$. A Bayesian decision-maker makes a purchase decision by learning whether the product has a *high* value ($\pi = \bar{\pi} = 1$) or a *low* value ($\pi = \underline{\pi} = 0$). Given the initial expectation $v_0 = \mathbb{E}[\pi|\mathscr{F}_0] \in [0, 1]$, the buyer can further learn the value of π by observing the signal $\{S_t\}_{t\geq 0}$, where $dS_t := \pi dt + \sigma_S dW_t$. Then, the valuation $v_t = \mathbb{E}[\pi|\mathscr{F}_t]$ is updated according to:

$$dv_t = \frac{v_t(1 - v_t)}{\sigma_S^2}\left[(\pi - v_t)dt + \sigma_S dW_t\right]. \tag{11}$$

The resulting free-boundary ODE problem has been considered in Ke and Villas-Boas (2019) in the non-discounting case: $r = 0$. In the context of our work, the solution with positive discounting $r > 0$ is more relevant, and is given as follows:

$$V_0^B(x; p_0) = A_+ x^{m_+}(1 - x)^{m_-} + A_- x^{m_-}(1 - x)^{m_+} - \frac{c}{r}, \tag{12}$$

where $m_\pm := \frac{1 \pm \sqrt{1 + 8r\sigma_S^2}}{2}$, $A_\pm := \frac{\bar{V}[p_0](1 - \bar{V}[p_0]) + (\bar{V}[p_0] - p_0 + c/r)(\bar{V}[p_0] - m_\mp)}{(m_\pm - m_\mp)\bar{V}[p_0]^{m_\pm}(1 - \bar{V}[p_0])^{m_\mp}}$, and $\bar{V}[p_0]$, $\underline{V}[p_0]$ are determined by:

$$\frac{\bar{V}[p_0](1 - \bar{V}[p_0]) + (\bar{V}[p_0] - p_0 + c/r)(\bar{V}[p_0] - m_\mp)}{\bar{V}[p_0]^{m_\pm}(1 - \bar{V}[p_0])^{m_\mp}} = \frac{(c/r)(\underline{V}[p_0] - m_\mp)}{\underline{V}[p_0]^{m_\pm}(1 - \underline{V}[p_0])^{m_\mp}} \tag{13}$$

3.1.3 Comparison Between Our Problem and the Benchmarks

Comparing the benchmarks with our problem, we can see that stationarity simplifies the problem significantly. In the benchmark model, the buyer's entire optimal stopping strategy can be summarized by **two unknowns**: $\bar{V}[p_0]$ and $\underline{V}[p_0]$. The buyer will purchase the product at any time during the search if her valuation reaches the purchasing threshold and will quit searching at any time if her valuation reaches the quitting threshold. In contrast, the buyer's entire optimal stopping strategy consists of **an infinite number of unknowns**. The buyer's decision at any time depends on the current price and the future trajectory of the prices. Knowing that prices change over time, the buyer's purchasing and quitting thresholds also evolve. These time-dependent thresholds significantly complicate our problem.

3.2 Buyer's Strategy under Non-Stationary Pricing

The set of admissible pricing strategies that we consider is:

$$\mathcal{P} := \{p \in C^\infty[0,\infty) \mid p_t' + r(\bar{\pi} - p_t) + c > 0, p_t > \underline{\pi}, \text{ for all } t \in [0,\infty)\}.$$

The conditions on p_t and p_t' ensure that it is optimal for a buyer with $v_t = \bar{\pi}$ or $v_t = \underline{\pi}$ either to immediately make a purchase or to quit. Note that any constant pricing policy $p_0 \in [\underline{\pi}, \bar{\pi}]$ is contained in $\mathcal{P}$, and that the condition on p_t' also controls the price growth rate, i.e., $\lim_{t\to\infty} e^{-rt} p_t = 0$. For a learning process such that $\bar{\pi} = +\infty$, $\underline{\pi} = -\infty$, all the conditions are trivial; therefore, we can take $\mathcal{P} := C^\infty[0,\infty)$. We will also work with the subset $\mathcal{P}_T \subset \mathcal{P}$ of strategies that are constant after some amount of time $T > 0$:

$$\mathcal{P}_T := \{p \in \mathcal{P} \mid p_t = p_T, \forall t \geq T\},$$

which is helpful in some existence and uniqueness arguments. The idea is to first establish technical results for a finite $T > 0$, and then to take the limit $T \to \infty$ to establish the result for strategies in $\mathcal{P}$. We start with the following intuitive characterization of $V^B(t, x; p)$:

Lemma 1. *Let $p \in \mathcal{P}_T$ be a pricing strategy.*

1. *$V^B(t, x; p)$ is monotonically increasing in x for any fixed t. Moreover, if $V^B(t, x; p) > 0$, then we have a strict inequality: $V^B(t, x'; p) > V^B(t, x; p)$ for any $x' > x$.*

2. *Let $q \in \mathcal{P}_T$ be another pricing strategy such that $q_t \leq p_t$ for all $t \in \mathbb{R}$; then, $V^B(t, x; q) \leq V^B(t, x; p)$ for all fixed $(t, x) \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}]$. Moreover, if $q_t < p_t$ for all $t > 0$, and $V^B(t, x; q) > 0$ for any fixed $(t, x) \in \mathbb{R}_{\geq 0} \times [\underline{\pi}, \bar{\pi}]$, then we have the strict inequality: $V^B(t, x; q) < V^B(t, x; p)$.*

Instead of directly finding the optimal $\tau^*[p] \in \mathcal{T}$ to the optimization problem (5), it is often more convenient to characterize the learning strategy in terms of the *moving* purchasing and quitting thresholds given by a pair of continuously differentiable functions $\bar{V}[p], \underline{V}[p] : \mathbb{R} \to [\underline{\pi}, \bar{\pi}]$ satisfying $\bar{V}_t[p] \geq \underline{V}_t[p]$. The following results characterize some properties of the purchasing and quitting boundaries.

Proposition 1. *Let $p \in \mathcal{P}_T$ be a pricing strategy and let $h : \mathbb{R} \to \mathbb{R}_{\geq 0} \in \mathcal{P}_T$ be strictly increasing over $[0, T)$.*

1. *Suppose that $h_0 = 0$; then, at $t = 0$, the purchasing and quitting boundaries under the pricing strategy $\tilde{p} := p + h$ satisfy $\bar{V}_0[\tilde{p}] < \bar{V}_0[p]$, and $\underline{V}_0[\tilde{p}] \geq \underline{V}_0[p]$.*

2. *Let $K \in \mathbb{R}$ be a constant; then, under the pricing strategy $\tilde{p} := p + Kh$, for any given $t \in [0, T)$, $\bar{V}_t[\tilde{p}] \searrow \max\{\tilde{p}_t, \underline{\pi}\}$, $\underline{V}_t[\tilde{p}] \nearrow \min\{\tilde{p}_t, \bar{\pi}\}$, as $K \to +\infty$ and $\bar{V}_t[\tilde{p}] \nearrow \bar{\pi}$, $\underline{V}_t[\tilde{p}] \searrow \underline{\pi}$, as $K \to -\infty$.*

The first part of Proposition 1 implies that, if $\tilde{p} := p+h$ and $h : \mathbb{R} \to \mathbb{R}_{\leq 0} \in \mathcal{P}_T$ is strictly monotonically decreasing, then $\bar{V}_0[\tilde{p}] \geq \bar{V}_0[p]$, and $\underline{V}_0[\tilde{p}] < \underline{V}_0[p]$. Note that the inequality for the quitting boundary may not be strict; for example, when the search cost is zero, we can have $\underline{V}_t[p] = \underline{\pi}$ for all $p \in \mathcal{P}_T$.

For a given $p \in \mathcal{P}_T$, the thresholds $\bar{V}[p], \underline{V}[p] : \mathbb{R} \to [\underline{\pi}, \bar{\pi}]$, along with the value function function $V^B(.,.;p)$, can be determined by solving the corresponding free-boundary backward parabolic PDE initial-value problem: Find $V : \Omega \to \mathbb{R}$, and continuously differentiable functions $\bar{V}[p], \underline{V}[p] : \mathbb{R} \to [\underline{\pi}, \bar{\pi}]$ satisfying $\bar{V}_t[p] \geq \underline{V}_t[p]$, such that

$$\begin{cases} \dfrac{\sigma(x)^2}{2}\partial_x^2 V(t,x) + \partial_t V(t,x) - rV(t,x) - c = 0, & (t,x) \in \Omega \\ V(t, \bar{V}_t[p]) = \bar{V}_t[p] - p_t, & V(t, \underline{V}_t[p]) = 0, \\ \partial_x V(t, \bar{V}_t[p]) = 1, & \partial_x V(t, \underline{V}_t[p]) = 0, \\ V(T,x) = V_0^B(x; p_T), \end{cases} \tag{14}$$

where

$$\Omega := \{(t,x) \in [0,T] \times [\underline{\pi}, \bar{\pi}] \mid \underline{V}_t[p] < x < \bar{V}_t[p]\}.$$

This PDE connects us back to the constant price benchmark in §3.1, except that now we have the *moving* purchasing and quitting boundaries $\bar{V}[p]$ and $\underline{V}[p]$ instead of the fixed counterparts that we saw in §3.1. The second and third lines of (14) amount to the value-matching and the smooth-pasting conditions at the purchasing and quitting boundaries, respectively. It can be shown that, if V satisfies the free-boundary backward parabolic PDE initial-value problem (14) with the pricing policy $p \in \mathcal{P}_T$, such that $V(t,x) \geq \max\{x - p_t, 0\}$, and $p'_t + r(\bar{V}_t[p] - p_t) + c \geq 0$ for all $(t,x) \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}]$, then V coincides with the buyer's value function: $V^B = V$ (see Lemma 6 in the Appendix).

Solving (14) in full generality is beyond the scope of this research. For an arbitrary given pricing policy $p \in \mathcal{P}_T$, there may not exist an analytical closed-form solution. However, if p is a small perturbation from a policy with a known solution, then we expect the solution corresponding to p to be a small perturbation from the known solution. The PDE formulation of the problem enables us to employ the perturbation theory.

Suppose we know that the value function $V^B(.,.;p)$ for a given $p \in \mathcal{P}_T$ is a solution to (14), and we would like to compute $V^B(.,.;p+\sqrt{\varepsilon}h)$ for some $h \in \mathcal{P}_T$ and a small $\varepsilon > 0$. The idea of perturbation theory is to proceed by writing $V^B(.,.;p+\sqrt{\varepsilon}h) = V_0(.,.) + V_1(.,.)\sqrt{\varepsilon} + V_2(.,.)\varepsilon + \cdots$, where $V_0(.,.) := V^B(.,.;p)$, and $\bar{V}_t[p+\sqrt{\varepsilon}h] = \bar{V}_{0,t} + \bar{V}_{1,t}\sqrt{\varepsilon} + \bar{V}_{2,t}\varepsilon + \cdots$, $\underline{V}_t[p+\sqrt{\varepsilon}h] = \underline{V}_{0,t} + \underline{V}_{1,t}\sqrt{\varepsilon} + \underline{V}_{2,t}\varepsilon + \cdots$, where $\bar{V}_{0,t} := \bar{V}_t[p]$, $\underline{V}_{0,t} := \underline{V}_t[p]$. By substituting these expansions into (14) and comparing the $\varepsilon^{k/2}$ terms for $k = 1, 2, \cdots$, we can solve for $V_k, \bar{V}_k, \underline{V}_k$ because we know the value of $V_{k'}, \bar{V}_{k'}, \underline{V}_{k'}$ for $k' = 0, \cdots, k-1$. In the Appendix (Lemma 6), we argue that such a technique is valid for all sufficiently small $\varepsilon > 0$. We shall assume this to be the case for the remainder of the work.

The Appendix provides a more thorough technical discussion of the connection between the problems (5) and (14), as well as further discussion of the validity of the perturbation technique. We will apply the perturbation technique to solve (14) only up to the ε-order, to be consistent with the ε-equilibrium concept. In other words, we can stop at $k = 1$ in the process described, and we will have $V^B(t,x;p+\sqrt{\varepsilon}h) = V^B(t,x;p) + V_1(t,x)\sqrt{\varepsilon} + O(\varepsilon)$. Then, we can take the buyer's learning policy to be given by the corresponding boundaries $\bar{V}^{\varepsilon}_{\leq 1}[p+\sqrt{\varepsilon}h] = \bar{V}[p] + \bar{V}_1\sqrt{\varepsilon}$, and $\underline{V}^{\varepsilon}_{\leq 1}[p+\sqrt{\varepsilon}h] = \underline{V}[p] + \underline{V}_1\sqrt{\varepsilon}$. The following proposition provides a characterization of the $\sqrt{\varepsilon}$-order perturbed boundaries in terms of the zero-th order solution.

Proposition 2. *Let $p \in \mathcal{P}_T$ be a given pricing strategy such that the buyer's value function $V^B(.,.;p)$ is a solution to the PDE (14), which is C^∞-smooth on $\Omega := \{(t,x) \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}] \mid \underline{V}_t[p] < x < \bar{V}_t[p]\}$, with C^∞-smooth corresponding purchasing and quitting boundaries $\bar{V}[p], \underline{V}[p] : \mathbb{R} \to (\underline{\pi}, \bar{\pi})$.*[5] *Let $h \in \mathcal{P}_T$ be arbitrary; then, under the pricing strategy $\tilde{p} := p + \sqrt{\varepsilon}h$, we can find an ε-optimal value function taking the form:*

$$V^B(t,x;\tilde{p}) = V^B(t, x - \sqrt{\varepsilon}h_t; p) + \sqrt{\varepsilon}V_1^B(t,x) + O(\varepsilon), \tag{15}$$

[5] We impose the C^∞-smoothness assumptions for simplicity, which are not necessary conditions. The results hold as long as the classical solution to the $\sqrt{\varepsilon}$-order PDE boundary-value problem $V_1(t,x) \in C^{1,2}(\Omega)$ exists.

where $V_1^B(.,.) : \Omega \to \mathbb{R}$ is given by:

$$V_1^B(t,x) = -\mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}} h'_s e^{-r(s-t)} \partial_x V^B(s, v_s^{t,x}; p) ds | \mathscr{F}_t\right] \\ + \mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}} h_s e^{-r(s-t)} \sigma(v_s^{t,x}) \sigma'(v_s^{t,x}) \partial_x^2 V^B(s, v_s^{t,x}; p) ds | \mathscr{F}_t\right], \quad (16)$$

where $\tau_\Omega^{t,x} := \inf\{t' \geq t \mid (t', v_{t'}^{t,x}) \notin \Omega\}$ is the exit time. The ε-optimal purchasing and quitting boundaries taking the form:

$$\begin{aligned} \bar{V}[\tilde{p}] &= (\bar{V}[p] + \sqrt{\varepsilon} h) + \sqrt{\varepsilon} \bar{R} + O(\varepsilon) \\ \underline{V}[\tilde{p}] &= (\underline{V}[p] + \sqrt{\varepsilon} h) + \sqrt{\varepsilon} \underline{R} + O(\varepsilon) \end{aligned} \quad (17)$$

for functions $\bar{R} : \mathbb{R} \to \mathbb{R}$, and $\underline{R} : \mathbb{R} \to \mathbb{R}$ given by:

$$\bar{R}_t = -\frac{\partial_x V_1^B(t, \bar{V}_t[p])}{\partial_x^2 V^B(t, \bar{V}_t[p]; p)}, \qquad \underline{R}_t = -\frac{\partial_x V_1^B(t, \underline{V}_t[p])}{\partial_x^2 V^B(t, \underline{V}_t[p]; p)} \quad (18)$$

Corollary 1. *Under the setting of Proposition 2, if $\sigma'(.) = O(\varepsilon)$ (stable volatility), and if $h := K\tilde{h}$ for some monotonically increasing $\tilde{h} \in \mathcal{P}_T$ and a constant $K \in \mathbb{R} \setminus \{0\}$, then $\bar{S}_t := \bar{R}_t / K \leq 0$ and $\underline{S}_t := \underline{R}_t / K \geq 0$ for all $t \in \mathbb{R}$.*

Lemma 6 in the Appendix shows that an $\sqrt{\varepsilon}$-order change in p will result in an $\sqrt{\varepsilon}$-order change in the value of V, and in the boundaries $\bar{V}[p], \underline{V}[p]$. The following result gives a more concrete upper bound:

Lemma 2. *Let $p, q \in \mathcal{P}$; then, $|V^B(t,x;p) - V^B(t,x;q)| \leq \max_{s \geq t} e^{-r(s-t)} |p_s - q_s|$ for all $(t,x) \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}]$.*

Lemma 2 shows that, for buyers who are not perfectly patient, any changes in price in the far future do not have much effect in the present. This enables us to extend our buyer response results for $p \in \mathcal{P}_T$ to an arbitrary $p \in \mathcal{P}$. Note that, by definition, any $p \in \mathcal{P}$ satisfies the asymptotic condition, $\lim_{t \to \infty} e^{-rt} p_t = 0$. Let p^T be given by p over $[0, T - \varepsilon]$, constant for all $t \geq T$, and some in-between smooth transition for $t \in (T - \varepsilon, T)$. We find the solution $V(.,.;p^T)$ of the free-boundary PDE initial value problem (14) corresponding to $p^T \in \mathcal{P}_T$, which coincides with the value function $V^B(.,.;p^T)$ according to Lemma 6. Then, for all sufficiently large T, we have

$$\left|V\left(t,x;p^T\right) - V^B\left(t,x;p\right)\right| = \left|V^B\left(t,x;p^T\right) - V^B\left(t,x;p\right)\right| < \varepsilon \quad (19)$$

for an arbitrarily given $\varepsilon > 0$. This proves that the sequence of the solutions $\left\{V(.,.;p^T)\right\}_{T\geq 0}$ uniformly converges to the value function $V^B(.,.;p)$ of an infinite horizon pricing strategy p on any compact subset of $\mathbb{R} \times [\underline{\pi}, \bar{\pi}]$.

In this paper, we pay special attention to pricing policies that are linear in time. Such linear pricing p does not belong to $\mathcal{P}_T$ for any $T > 0$. However, this is not a problem according to Lemma 2. By choosing a sufficiently large T, an ε-optimal buyer will not differentiate between p and $p^T \in \mathcal{P}_T$. This enables us to utilize the theory we have developed so far for $\mathcal{P}_T$ on linear pricing.

As it turns out, when p is linear in t, the buyer's value function admits an analytic closed-form under some simple learning settings, such as when $\{v_t\}_{t\geq 0}$ is a vanilla Brownian motion. It is also simpler to analyze the seller's strategies when they are restricted to the space of linear pricing. The fact that the linear pricing space is much smaller than the general pricing space also simplifies the problem, especially when searching for the seller's optimal pricing strategy, which we will do in §4.

Consideration of linear pricing may seem restrictive. However, the following proposition, which is an application of Lemma 2, shows that, for myopic enough ε-optimal buyers, any pricing strategy that is sufficiently slow-moving can be approximated by linear pricing. Intuitively, unless the price changes very drastically in the far future, such as growing super-exponentially, myopic buyers do not look too far into the future, and any differentiable functions *look like* a linear function over any sufficiently short time interval.

Proposition 3. *(Near-optimality of linear price approximation) Let $p \in \mathcal{P}$ be an admissible pricing policy with $\sup_{t\in\mathbb{R}} |p''_t| \leq M$. At any $\mathbf{x} = (t, x) \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}]$, we consider the linear approximation pricing policy $l_{\mathbf{x}} \in \mathcal{P} : s \mapsto l_{\mathbf{x},s} := p_t + p'_t \cdot (s - t)$.*[6] *Let the buyer's optimal learning strategy given the linear pricing $l_{\mathbf{x}}$ be $\tau^*[l_{\mathbf{x}}] \in \mathcal{T}$. If the buyer is sufficiently myopic: $r > e^{-1}\sqrt{2M/\varepsilon}$, then $\tau^*[l_{\mathbf{x}}]$ is also the buyer's ε-optimal stopping time under the p pricing strategy:*

$$\mathcal{V}^B(t, x; \tau^*[l_{\mathbf{x}}]; p) \geq V^B(t, x; p) - \varepsilon.$$

The following corollary shows that linear perturbation is particularly simple.

Corollary 2. *Consider a linear pricing strategy $p : t \mapsto p_0 + \sqrt{\varepsilon}Kt \in \mathcal{P}$, where p_0 is a constant. Suppose that the constant price buyer's value function $V_0^B(.;p_0)$ is a solution to the PDE (14), which is C^∞-smooth on $\Omega := \{(t, x) \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}] \mid \underline{V}[p_0] < x < \bar{V}[p_0]\}$, where*

[6] It is also possible to apply Lemma 2 to the constant price approximation, i.e., we assume $\sup_{t\in R} |p'_t| \leq M$ and consider $p_0 \in \mathcal{P} : t \mapsto p_0$ for all $t \in \mathbb{R}$. We have $\mathcal{V}^B(t, x; \tau^*[p_0]; p) \geq V^B(t, x; p) - \varepsilon$ if $r > e^{-1}M/\varepsilon$. In other words, if the buyer is very myopic, $r = O(1/\varepsilon)$, then every pricing $p \in \mathcal{P}$ can be treated as constant, which is a trivial result.

$\bar{V}[p_0], \underline{V}[p_0] \in (\underline{\pi}, \bar{\pi})$ *are the corresponding constant purchasing and quitting boundaries. Then, we can find an* ε*-optimal value function given by:*

$$V^B(t,x;p) = V_0^B(x - \sqrt{\varepsilon}Kt; p_0) + \sqrt{\varepsilon}V_1^B(t,x) + O(\varepsilon)$$

where $V_1^B(t,x) = V_{1,0}^B(x) + tV_{1,1}^B(x)$ *is linear in* t*, with* $V_{1,1}^B$ *being the unique solution to the ODE boundary-value problem:*

$$\frac{\sigma(x)^2}{2}\partial_x^2 V_{1,1}^B(x) - rV_{1,1}^B(x) + K\sigma(x)\sigma'(x)\partial_x^2 V_0^B(x;p_0) = 0, \quad V_{1,1}^B(\bar{V}[p_0]) = V_{1,1}^B(\underline{V}[p_0]) = 0, \tag{20}$$

and $V_{1,0}^B$ *being the unique solution to the ODE boundary-value problem:*

$$\frac{\sigma(x)^2}{2}\partial_x^2 V_{1,0}^B(x) - rV_{1,0}^B(x) + V_{1,1}^B(x) - K\partial_x V_0^B(x;p_0) = 0, \quad V_{1,0}^B(\bar{V}[p_0]) = V_{1,0}^B(\underline{V}[p_0]) = 0. \tag{21}$$

The ε*-optimal purchasing and quitting boundaries:* $\bar{V}_t[p] = \bar{V}[p_0] + \sqrt{\varepsilon}Kt + \sqrt{\varepsilon}\bar{R}_t + O(\varepsilon)$*, and* $\underline{V}_t[p] = \underline{V}[p_0] + \sqrt{\varepsilon}Kt + \sqrt{\varepsilon}\underline{R}_t + O(\varepsilon)$*, where*

$$\bar{R}_t = -\frac{\partial_x V_1^B(t, \bar{V}[p_0])}{\partial_x^2 V_0^B(\bar{V}[p_0]; p_0)} =: K\bar{S}_{0,0} + K\bar{S}_{0,1}t, \quad \underline{R}_t = -\frac{\partial_x V_1^B(t, \underline{V}[p_0])}{\partial_x^2 V_0^B(\underline{V}[p_0]; p_0)} =: K\underline{S}_{0,0} + K\underline{S}_{0,1}t$$

are linear in t*, for some constants* $\bar{S}_{0,0} \leq 0, \underline{S}_{0,0} \geq 0$*,* $\bar{S}_{0,1}, \underline{S}_{0,1}$*.*

We now revisit the two buyer learning processes considered in §3.1.1 and §3.1.2, obtaining the closed-form expression for the value function given the first learning process and the perturbative solution up to the ε-order given the second learning process.

Solution: Product attributes learning

The learning process in §3.1.1 is a rare example where the free-boundary PDE (14) can be solved exactly. The main reason is that $\sigma(.)$ is a constant in this case; thus, the probability measure of $\{v_s^{t,x}\}_{s\geq t}$ is x-translation-invariant. Therefore, we can transform the original problem to a simpler problem where the price is fixed at p_0, while the buyer valuation process is a drifted Brownian motion $v_t = -\sqrt{\varepsilon}Kt + \sigma W_t$. The transformed problem is stationary in time, with the corresponding HJB: $\frac{\sigma^2}{2}\partial_x^2 V(x) - \sqrt{\varepsilon}K\partial_x V(x) - rV(x) - c = 0$. We first solve this HJB equation, and then make an inverse transformation back to the original problem.

Proposition 4. *Consider the buyer's learning process as in §3.1.1. Under a linear pricing* *strategy* $p : t \mapsto p_0 + \sqrt{\varepsilon}Kt \in \mathcal{P}$*, the buyer's value function is given by*

$$V^B(t,x) = A_1 e^{\frac{\sqrt{\varepsilon}K - \sqrt{\varepsilon K^2 + 2r\sigma^2}}{\sigma^2}(x - p_0 - \sqrt{\varepsilon}Kt)} + A_2 e^{\frac{\sqrt{\varepsilon}K + \sqrt{\varepsilon K^2 + 2r\sigma^2}}{\sigma^2}(x - p_0 - \sqrt{\varepsilon}Kt)} - \frac{c}{r} \tag{22}$$

with purchasing and quitting boundaries given by

$$\bar{V}_t = p_0 + \bar{V}[\sqrt{\varepsilon}K] + \sqrt{\varepsilon}Kt, \qquad \underline{V}_t = p_0 + \underline{V}[\sqrt{\varepsilon}K] + \sqrt{\varepsilon}Kt \tag{23}$$

where the constants $\bar{V}[\sqrt{\varepsilon}K]$*,* $\underline{V}[\sqrt{\varepsilon}K], A_1$*, and* A_2 *are determined by boundary conditions (see the corresponding proof in the Appendix). To the* $\sqrt{\varepsilon}$*-order,* $\bar{V}[\sqrt{\varepsilon}K]$ *and* $\underline{V}[\sqrt{\varepsilon}K]$ *take the following analytical form,*

$$\bar{V}[\sqrt{\varepsilon}K] = \bar{V} + \sqrt{\varepsilon}\bar{R} + O(\varepsilon), \qquad \underline{V}[\sqrt{\varepsilon}K] = \underline{V} + \sqrt{\varepsilon}\underline{R} + O(\varepsilon), \tag{24}$$

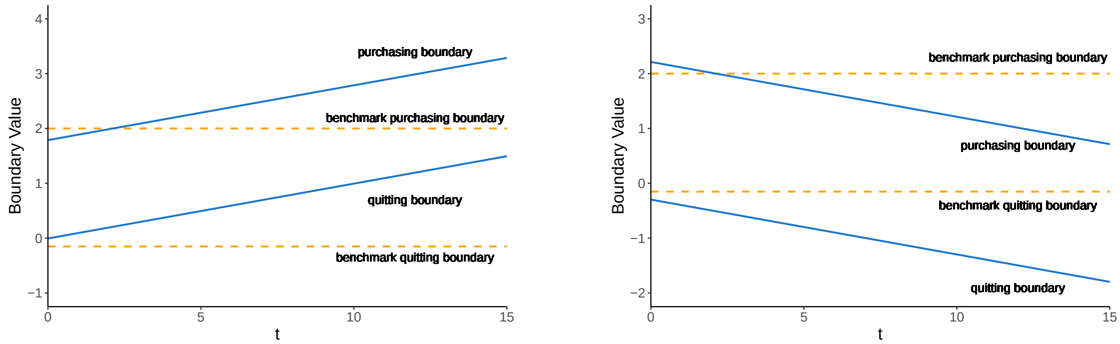
where $\bar{V}, \underline{V}$ *are given by* (10)*, and*

$$\underline{S} := \frac{\underline{R}}{K} = \left(\frac{\bar{V} - \underline{V}}{\sigma^2}\right)\left(\bar{V} + \frac{c}{r}\right) - \frac{1}{2r} = \frac{\sqrt{\frac{c^2}{r^2} + \frac{\sigma^2}{2r}}}{\sigma\sqrt{2r}} \log\left(\sqrt{\frac{r\sigma^2}{2c^2}} + \sqrt{1 + \frac{r\sigma^2}{2c^2}}\right) - \frac{1}{2r} > 0,$$

$$\bar{S} := \frac{\bar{R}}{K} = \underline{S} - \frac{1}{2r} \cdot \frac{\bar{V} - \underline{V}}{\bar{V} + c/r} = \frac{1/(\sigma\sqrt{2r})}{\sqrt{\frac{c^2}{r^2} + \frac{\sigma^2}{2r}}} \cdot \frac{c^2}{r^2} \log\left(\sqrt{\frac{r\sigma^2}{2c^2}} + \sqrt{1 + \frac{r\sigma^2}{2c^2}}\right) - \frac{1}{2r} < 0.$$

Compared to the result of Proposition 2, $\bar{R}$ and $\underline{R}$ are constants in this case. Compared to the constant price benchmark, an increasing pricing scheme ($K > 0$) with the same initial price has two impacts on the purchasing threshold. On the one hand, the benefit of learning becomes lower because the buyer will have to pay more in the future if she receives positive information that causes her to like the product more. Rationally anticipating this, the buyer has a lower incentive to search and is more inclined to purchase now, which reduces the purchasing threshold (captured by the negative $\sqrt{\varepsilon}K\bar{S}$ term). On the other hand, a higher price makes the buyer less willing to purchase, which raises the purchasing threshold (captured by the positive $\sqrt{\varepsilon}Kt$ term). Because the first effect remains stable while the second effect increases over time, the purchasing threshold is lower than the benchmark threshold at the beginning but eventually exceeds the benchmark threshold as the price keeps increasing.

An increasing pricing scheme also has two impacts on the quitting threshold. Both a lower benefit of searching and a higher price make it more likely for the buyer to quit. So, the quitting threshold is always higher than the benchmark threshold. We also find that the



(a) Increasing price $K = 1$

(b) Decreasing price $K = -1$

Figure 1: Purchasing and quitting boundaries when $c = .2, p = 1, r = .1, \sigma = 1$, and $\epsilon = 0.01$

buyer searches in a narrower region (smaller $\bar{V}_t - \underline{V}_t$) if the price increases rather than staying constant because of the lower benefit of searching. Figure 1a illustrates the purchasing and quitting boundaries in this case, under both non-stationary pricing and constant price.

A decreasing pricing scheme ($K < 0$) has the opposite impact on the purchasing and quitting thresholds. The purchasing threshold is higher than the benchmark threshold at the beginning because the buyer has a stronger incentive to search and is less inclined to purchase immediately. It eventually falls below the benchmark threshold as the price keeps decreasing. The quitting threshold is always lower than the benchmark threshold because the benefit of both searching and purchasing is higher. Also, the buyer searches in a broader region. Figure 1b illustrates the purchasing and quitting boundaries in this case, under both non-stationary pricing and constant price.

Solution: Binary classification

We can obtain the buyer's value function, as well as the purchasing and quitting boundaries, in terms of the constant price parameters $\bar{V}[p_0]$, $\underline{V}[p_0]$, up to the ε-order, by evaluating the elementary integrals and solving the system of linear equations associated with the boundary conditions.

Proposition 5. *Consider the buyer's binary classification process, as in §3.1.2, under a linear pricing strategy $p : t \mapsto p_0 + \sqrt{\varepsilon}Kt \in \mathcal{P}$. Let $\bar{V}[p_0], \underline{V}[p_0] \in (0,1)$ be the constant price p_0 purchasing and quitting boundaries as specified by the solution to (13). For convenience, let us define: $u_\pm(x) := x^{m_\pm}(1-x)^{m_\mp}$. According to Corollary 2 there is an ε-optimal buyer learning strategy with the value function, purchase, and quitting boundaries taking the form:*

$$V^B(t,x) = V_0^B(x - \sqrt{\varepsilon}Kt; p_0) + \sqrt{\varepsilon}V_1^B(t,x) + O(\varepsilon),$$

$\bar{V}_t[p] = \bar{V}[p_0] + \sqrt{\varepsilon}Kt + \sqrt{\varepsilon}\bar{R}_t + O(\varepsilon)$, and $\underline{V}_t[p] = \underline{V}[p_0] + \sqrt{\varepsilon}Kt + \sqrt{\varepsilon}\underline{R}_t + O(\varepsilon)$, respectively, where $V_0^B(x;p_0) = A_+u_+(x) + A_-u_-(x) - \frac{c}{r}$ is given by (12); $V_1^B(t,x) := V_{1,0}^B(x) + tV_{1,1}^B(x)$ where

$$V_{1,0}^B(x) := \left(B_+ + \frac{2\sigma_S^2}{\sqrt{1+8r\sigma_S^2}} \int \frac{K\partial_x V_0^B(x;p_0) - V_{1,1}^B(x)}{x^{2-m_-}(1-x)^{2-m_+}} dx\right) u_+(x) + \left(B_- - \frac{2\sigma_S^2}{\sqrt{1+8r\sigma_S^2}} \int \frac{K\partial_x V_0^B(x;p_0) - V_{1,1}^B(x)}{x^{2-m_+}(1-x)^{2-m_-}} dx\right) u_-(x), \quad (25)$$

$$V_{1,1}^B(x) := \left(C_+ - \frac{4r\sigma_S^2 K}{\sqrt{1+8r\sigma_S^2}} \int \frac{1-2x}{x^{3-m_-}(1-x)^{3-m_+}} \left(V_0(x) + \frac{c}{r}\right) dx\right) u_+(x) + \left(C_- + \frac{4r\sigma_S^2 K}{\sqrt{1+8r\sigma_S^2}} \int \frac{1-2x}{x^{3-m_+}(1-x)^{3-m_-}} \left(V_0(x) + \frac{c}{r}\right) dx\right) u_-(x), \quad (26)$$

and $\bar{R}_t = K\bar{S}_{0,0} + K\bar{S}_{0,1}t$, $\underline{R}_t = K\underline{S}_{0,0} + K\underline{S}_{0,1}t$ are given in terms of $V_0^B(.,.;p_0)$, $V_{1,0}^B(.)$, and $V_{1,1}^B(.)$, as in Corollary 2. The constants $B_\pm$ and $C_\pm$ are determined by the boundary conditions: $V_{1,0}^B(\bar{V}[p_0]) = V_{1,0}^B(\underline{V}[p_0]) = 0$ and $V_{1,1}^B(\bar{V}[p_0]) = V_{1,1}^B(\underline{V}[p_0]) = 0$, respectively.

3.3 Generalizability of the Results for the Buyer's Problem

The previous section has derived many analytical results under a slow-moving linear pricing strategy. Proposition 3 indicates that ε-optimal buyers will respond to broader classes of non-linear pricing p as if they were linear under the following assumptions.

Assumption 2. *For a given $\varepsilon > 0$, we assume that the buyer is ε-optimal and sufficiently myopic (sufficiently large r), and that the seller adjusts the price slowly over time: $|p'_t| = O(\sqrt{\varepsilon})$, such that the conditions for Proposition 3 are satisfied.*

Assumption 2 clarifies when we can apply our results under linear pricing to more general non-linear pricing strategies. For a given $p \in \mathcal{P}_T$ that satisfies the assumption, the buyer will derive the learning strategy from the linear pricing approximation:

$$t \mapsto p_0 + \sqrt{\varepsilon}Kt, \qquad \sqrt{\varepsilon}K := p'_0 = O(\sqrt{\varepsilon}). \quad (27)$$

4 Seller's Strategy

4.1 Seller's Expected Payoff

The expected payoff for a seller implementing the pricing strategy $p \in \mathcal{P}$ is given by $\mathcal{V}^S(x; \tau^*[p], p)$, where $\tau^*[p] \in \mathcal{T}$ denotes the buyer's ε-optimal response to p. We will denote $\mathcal{V}^S(x; \tau^*[p], p)$ by $\mathcal{V}^S(x; p)$ hereinafter for simplicity. To characterize the optimal pricing, we need to compute $\mathcal{V}^S(x; p)$ for a given pricing strategy p.

4.1.1 Constant Price

In the simplest cases of a constant price, the seller's expected payoff can be derived from the properties of martingales.

Lemma 3. *Consider a constant pricing $p = p_0 \in \mathbb{R}$. Suppose that the constant purchasing and quitting boundaries $\bar{V}[p_0], \underline{V}[p_0] \in (\underline{\pi}, \bar{\pi})$ are finite. For any given $x \in [\underline{V}[p_0], \bar{V}[p_0]]$:*

1. *If $m = 0$, then $\mathcal{V}^S(x; p_0) = (p_0 - g)\frac{x - \underline{V}[p_0]}{\bar{V}[p_0] - \underline{V}[p_0]}$.*

2. *If the volatility is constant: $\sigma(x)^2 = \sigma^2$, then $\mathcal{V}^S(x; p_0) = (p_0 - g)\frac{\sinh \frac{\sqrt{2m}}{\sigma}(x - \underline{V}[p_0])}{\sinh \frac{\sqrt{2m}}{\sigma}(\bar{V}[p_0] - \underline{V}[p_0])}$.*

4.1.2 General Price

For a buyer with initial valuation x, let $\bar{V}[p], \underline{V}[p] : [0, \infty) \to \mathbb{R}$ denote the buyer's stopping boundaries corresponding to the $\tau^*[p]$ learning strategy and let $\Omega := \{(t, v) \in [0, \infty) \times \mathbb{R} \mid \underline{V}_t[p] < v < \bar{V}_t[p]\}$. We consider $U(s, v; t, x)$, the transition probability density of a particle starting from x at time t to some point v at a later time $s \geq t$, as described by the process $\{v_s^{t,x}\}_{s \geq t}$, without leaving the domain Ω. For any fixed $(s, v) \in \Omega$, $U(s, v; t, x)$ satisfies the Kolmogorov backward equation with absorbing boundary condition:

$$
\begin{cases}
\dfrac{\sigma(x)^2}{2}\partial_x^2 U(s, v; t, x) + \partial_t U(s, v; t, x) = 0, & (t, x) \in \Omega \\
U(s, v; t, \bar{V}_t[p]) = 0, & U(s, v : t, \underline{V}_t[p]) = 0 \, , \\
U(s, v; t = s, x) = \delta(v - x)
\end{cases}
\tag{28}
$$

where $\delta(v - x)$ denotes the Dirac-Delta distribution concentrated at v. When it is clear from the context, we denote $U(t, v; t_0 = 0, x)$ simply as $U(t, v)$.

The existence and properties of the solution $U(s, v; t, x)$ depend on the smoothness conditions of $\bar{V}[p]$, $\underline{V}[p]$ (Friedman, 2008, Chapter 3). We assume all necessary conditions are satisfied so that the solution $U(s,v;t,x) \in C^{1,2}(\Omega)$ exists. The probability flux of the buyer hitting the moving purchasing boundary, and thus getting absorbed, at time s is:

$$-\frac{1}{2}\partial_v\left[\sigma(v)^2 U(s,v;t,x)\right]|_{v=\bar{V}_s[p]} - \bar{V}_s'[p]\cdot U(s,\bar{V}_s;t,x) = -\frac{1}{2}\partial_v\left[\sigma(v)^2 U(s,v;t,x)\right]|_{v=\bar{V}_s[p]}.$$

In the above equation, the term $\bar{V}_s'[p]$ is needed to take into account the boundary movement, which nevertheless vanishes because of the boundary condition: $U(s,\bar{V}_s;t,x)=0$. Hence, for a pricing policy $p \in \mathcal{P}$, the seller's expected payoff from a buyer starting at time t with valuation x is:

$$\mathcal{V}^S(t,x;p) = -\frac{1}{2}\int_t^\infty e^{-m(s-t)}(p_s - g)\partial_v\left[\sigma(v)^2 U(s,v;t,x)\right]|_{v=\bar{V}_s[p]}ds, \text{ if } x \in (\underline{V}_t[p], \bar{V}_t[p]), \tag{29}$$

and $\mathcal{V}^S(t,x;p) = (p_t - g)1_{x\geq \bar{V}_t[p]}$ otherwise. An important case is $t=0$, in which we write $\mathcal{V}^S(x;p) = \mathcal{V}^S(t=0,x;p)$. If the seller's expected payoff is known to be given by some $\mathcal{V}_T^S(.) : [\underline{V}_T[p], \bar{V}_T[p]] \to \mathbb{R}$ then (29) and (28) imply that $\mathcal{V}^S$ satisfies the following backward parabolic PDE initial boundary value problem:

$$\begin{cases} \dfrac{\sigma(x)^2}{2}\partial_x^2\mathcal{V}^S(t,x;p) + \partial_t\mathcal{V}^S(t,x;p) - m\mathcal{V}^S(t,x;p) = 0, & (t,x)\in\Omega \\ \mathcal{V}^S(t,\bar{V}_t[p];p) = p_t - g, & \mathcal{V}_0^S(t,\underline{V}_t[p];p) = 0\ , \\ \mathcal{V}^S(T,x;p) = \mathcal{V}_T^S(x). \end{cases} \tag{30}$$

For example, if $p \in \mathcal{P}_T$, then we have $\mathcal{V}_T^S(.) = \mathcal{V}_0^S(.;p_T)$, the seller's payoff under the constant price policy: $p_t = p_T$ for all $t \geq T$ which is time-independent and can be determined by the ODE: $\frac{\sigma^2(x)}{2}\partial_x^2\mathcal{V}_0^S - m\mathcal{V}_0^S = 0$.

4.2 Seller's Linear Perturbative Pricing Strategy

Motivated by §3.3, we focus on the implementation of slow-moving linear pricing by the seller. Specifically, we consider the following set of admissible pricing:

$$\mathcal{P}_{lin}^\varepsilon := \left\{t \mapsto p_0 + \sqrt{\varepsilon}Kt \mid p_0 \in \mathbb{R}, K \in [-1,+1]\right\} \subset C^\infty[0,\infty).$$

Within $\mathcal{P}_{lin}^\varepsilon$, we denote the expected payoff by $\mathcal{V}^S(x;p_0,K)$. The seller only needs to determine the optimal $(p_0,K) = (p_0^*,K^*)$. Considering linear pricing from the seller's perspective is not without loss of generality, but it is sufficient to answer the economically relevant questions of whether a constant price is always optimal when the seller cannot track the evolution of the buyer's valuation, and what would be the profitable direction of a slow-moving price otherwise. In particular, denote by $\hat{p}_0 := \hat{p}_0(x)$ the optimal constant price given the buyer's initial valuation x. By computing $\frac{\partial \mathcal{V}^S}{\partial K}(x; p_0 = \hat{p}_0, K = 0)$, we can determine whether $K^* > 0$, $K^* < 0$, or $K^* = 0$, which characterizes (p_0^*, K^*), the optimal policy in some vicinity of $K = 0$. The above analysis provides normative guidance to a seller initially using optimal constant pricing $\hat{p}_0$ on how it can improve its profit with non-stationary pricing.

The discussion of linear pricing also serves as a template for understanding pricing strategies in more general settings, where $\mathcal{P}$ could include non-linear pricing strategies as long as Assumption 2 holds. With these assumptions, the buyer's ε-optimal learning decision in response to any $p \in \mathcal{P}$ is entirely determined by the value of p_t and its slope p_t' at any given time t according to Proposition 3. Because $|p_t'| = O(\sqrt{\varepsilon})$, we are able to utilize our linear perturbation framework. In practice, such assumptions hold if the seller's buyers are impatient and impulsive in their purchasing decisions, and if, due to regulation, the seller is restricted on how quickly he can change the price over time. We focus on the case where the seller is perfectly patient because, as the next section suggests, the seller is more inclined to charge non-stationary prices if it discounts the future, and we will show in this section that even a perfectly patient seller will charge non-stationary prices. We start with the following result on linear perturbation from an arbitrary constant price p_0.

Theorem 1. *Consider a linear pricing strategy $p : t \mapsto p_0 + \sqrt{\varepsilon}Kt \in \mathcal{P}$ of a perfectly patient seller ($m = 0$) and the buyer ε–optimal response according to Corollary 2. Let $\bar{V}[p_0], \underline{V}[p_0] \in (\underline{\pi}, \bar{\pi})$ be the purchasing and quitting boundaries corresponding to the constant price p_0 strategy. For all sufficiently small $\varepsilon > 0$, the seller's $\sqrt{\varepsilon}$–order expected payoff from the buyer with initial valuation $x \in (\underline{V}[p_0], \bar{V}[p_0])$ under the pricing strategy p is given by:*

$$
\begin{aligned}
\mathcal{V}^S(x; p_0, K) = &\frac{(p_0 - g)(x - \underline{V}[p_0])}{\bar{V}[p_0] - \underline{V}[p_0]} + \sqrt{\varepsilon}K\mathbb{E}\left[\tau_\Omega^x \cdot 1\left\{v_{\tau_\Omega^x}^x \geq \bar{V}[p_0]\right\}\right] \\
&- \sqrt{\varepsilon}K \cdot \frac{p_0 - g}{\bar{V}[p_0] - \underline{V}[p_0]}\left((1 + \underline{S}_{0,1})\mathbb{E}\left[\tau_\Omega^x\right] + (\bar{S}_{0,1} - \underline{S}_{0,1})\mathbb{E}\left[\tau_\Omega^x \cdot 1\left\{v_{\tau_\Omega^x}^x \geq \bar{V}[p_0]\right\}\right]\right) \\
&- \sqrt{\varepsilon}K \cdot \frac{p_0 - g}{\bar{V}[p_0] - \underline{V}[p_0]}\left(\underline{S}_{0,0} + (\bar{S}_{0,0} - \underline{S}_{0,0})\mathbb{P}\left[v_{\tau_\Omega^x}^x \geq \bar{V}[p_0]\right]\right) + O(\varepsilon) \qquad (31)
\end{aligned}
$$

where $\tau_\Omega^x := \inf\{t \geq 0 \mid (t, v_t^x) \notin \Omega\}$ is the stopping time, $\Omega := \{(t, x) \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}] \mid \underline{V}[p_0] < x < \bar{V}[p_0]\}$, and the constants $\bar{S}_{0,0}, \bar{S}_{0,1}, \underline{S}_{0,0}, \underline{S}_{0,1}$ determine the $\sqrt{\varepsilon}$-order buyer's response to p as given in Corollary 2.

The first term of the expression in (31) represents the seller's payoff from a constant price policy p_0. Below, we consider each of the following terms.

Change to the profit per purchase: $+\sqrt{\varepsilon}K\mathbb{E}\left[\tau_\Omega^x \cdot 1\left\{v_{\tau_\Omega^x}^x \geq \bar{V}[p_0]\right\}\right]$

This term is related to the expected change in price at the time of purchase. If the price is increasing over time ($K > 0$), the buyer will pay a price higher than the initial price p_0 if she ends up buying the product. If the price is decreasing over time ($K < 0$), the seller can only extract a lower profit if the buyer buys after searching. Note that $\mathbb{E}\left[\tau_\Omega^x \cdot 1\left\{v_{\tau_\Omega^x}^x \geq \bar{V}[p_0]\right\}\right]$ satisfies the ODE $\frac{1}{2}\sigma(x)^2 w''(x) = -\frac{x-\underline{V}[p_0]}{\bar{V}[p_0]-\underline{V}[p_0]}$ with $w(\bar{V}[p_0]) = w(\underline{V}[p_0]) = 0$. Solving the ODE with the boundary condition gives us: $\mathbb{E}\left[\tau_\Omega^x \cdot 1\left\{v_{\tau_\Omega^x}^x \geq \bar{V}[p_0]\right\}\right] = \frac{2(x-\underline{V}[p_0])}{(\bar{V}[p_0]-\underline{V}[p_0])^2}\int_{\underline{V}[p_0]}^{\bar{V}[p_0]} \frac{(\bar{V}[p_0]-z)(z-\underline{V}[p_0])}{\sigma(z)^2}dz - \frac{2}{\bar{V}[p_0]-\underline{V}[p_0]}\int_{\underline{V}[p_0]}^{x} \frac{(x-z)(z-\underline{V}[p_0])}{\sigma(z)^2}dz.$

Change to purchase probability due to rescaling of the search interval:

$$-\sqrt{\varepsilon}K \cdot \frac{p_0 - g}{\bar{V}[p_0] - \underline{V}[p_0]}\left(\underline{S}_{0,0} + (\bar{S}_{0,0} - \underline{S}_{0,0})\mathbb{P}\left[v_{\tau_\Omega^x}^x \geq \bar{V}[p_0]\right]\right)$$

Expecting the price to change over time rather than stay constant, the buyer will adjust the search region. An increasing price trajectory shrinks the search region, whereas a decreasing price trajectory enlarges the search region. This economic force affects the search interval even at time 0 when the initial prices are identical. Consider an increasing price. Observing that $\mathbb{P}\left[v_{\tau_\Omega^x}^x \geq \bar{V}[p_0]\right]$ increases as the buyer's initial valuation becomes higher, and that $\bar{S}_{0,0} - \underline{S}_{0,0} \leq 0$, one can see that an increasing price increases the probability of purchase if the buyer has a high initial valuation, and reduces the probability of purchase if she has a low initial valuation. We can evaluate this term using the fact that $\mathbb{P}\left[v_{\tau_\Omega^x}^x \geq \bar{V}[p_0]\right] = \frac{x-\underline{V}[p_0]}{\bar{V}[p_0]-\underline{V}[p_0]}$, as shown in the proof of Lemma 3.

Change to purchase probability due to moving boundaries and price:

$$\begin{aligned}
&-\sqrt{\varepsilon}K \cdot \frac{p_0 - g}{\bar{V}[p_0] - \underline{V}[p_0]}\left((1+\underline{S}_{0,1})\mathbb{E}\left[\tau_\Omega^x\right] + (\bar{S}_{0,1} - \underline{S}_{0,1})\mathbb{E}\left[\tau_\Omega^x \cdot 1\left\{v_{\tau_\Omega^x}^x \geq \bar{V}[p_0]\right\}\right]\right) \\
=&-\sqrt{\varepsilon}K \cdot \frac{p_0 - g}{\bar{V}[p_0] - \underline{V}[p_0]}\left(\mathbb{E}\left[\tau_\Omega^x\right] + \bar{S}_{0,1}\mathbb{E}\left[\tau_\Omega^x \cdot 1\left\{v_{\tau_\Omega^x}^x \geq \bar{V}[p_0]\right\}\right] + \underline{S}_{0,1}\mathbb{E}\left[\tau_\Omega^x \cdot 1\left\{v_{\tau_\Omega^x}^x \leq \underline{V}[p_0]\right\}\right]\right).
\end{aligned}$$

The term $\mathbb{E}[\tau_\Omega^x]$ reflects that the probability of purchase is affected by the expected amount of price change over the entire search process. The buyer is less likely to make a purchase if the price increases over time, and is more likely to make a purchase if the price decreases over time. The term $\bar{S}_{0,1}\mathbb{E}\left[\tau_\Omega^x \cdot 1\left\{v_{\tau_\Omega^x}^x \geq \bar{V}[p_0]\right\}\right]$ accounts for the fact that the purchasing boundary would have moved a certain distance by the time a buyer reaches the original purchasing boundary. Analogously, the term $\underline{S}_{0,1}\mathbb{E}\left[\tau_\Omega^x \cdot 1\left\{v_{\tau_\Omega^x}^x \leq \underline{V}[p_0]\right\}\right]$ captures that the quitting boundary has moved a certain distance by the time a buyer reaches the original quitting boundary. To evaluate the above formula, it remains to compute $\mathbb{E}[\tau_\Omega^x]$, which satisfies the ODE $\frac{1}{2}\sigma(x)^2 w''(x) = -1$ with $w(\bar{V}[p_0]) = w(\underline{V}[p_0]) = 0$. Solving the ODE with the boundary condition gives us: $\mathbb{E}[\tau_\Omega^x] = \frac{2(x-\underline{V}[p_0])}{\bar{V}[p_0]-\underline{V}[p_0]} \int_{\underline{V}[p_0]}^{\bar{V}[p_0]} \frac{\bar{V}[p_0]-z}{\sigma(z)^2} dz - 2\int_{\underline{V}[p_0]}^{x} \frac{x-z}{\sigma(z)^2} dz$.

The derivative $\frac{\partial \mathcal{V}^S}{\partial K}(x; p_0, K=0)$ **:**

Theorem 1 allows us to compute the exact value of $\frac{\partial \mathcal{V}^S}{\partial K}(x; p_0, K=0)$ at any arbitrary $x \in [\underline{V}[p_0], \bar{V}[p_0]]$ and p_0, in terms of the buyer's $\sqrt{\varepsilon}$-order response characterized by the parameters: $\underline{V}[p_0], \bar{V}[p_0], \underline{S}_{0,0}, \bar{S}_{0,0}, \underline{S}_{0,1}, \bar{S}_{0,1}$.

For convenience, we denote by $q := \frac{x-\underline{V}[\hat{p}_0]}{\bar{V}[\hat{p}_0]-\underline{V}[\hat{p}_0]}$ the buyer's initial valuation relative to the purchasing and exiting boundaries under the optimal static price $\hat{p}_0$. We can equivalently consider $\frac{\partial \mathcal{V}^S}{\partial K}(x; p_0, K=0)$ or $\frac{\partial \mathcal{V}^S}{\partial K}(q; p_0, K=0)$.

Importantly, if $\frac{\partial \mathcal{V}^S}{\partial K}(q; p_0 = \hat{p}_0, K=0)$ is non-zero for a given $q \in [0,1]$, then the optimal K^* will be bounded away from 0. The seller can improve its expected profit by setting $K \gtrsim 0$ if $\frac{\partial \mathcal{V}^S}{\partial K}(q; p_0 = \hat{p}, K=0) > 0$, and by setting $K \lesssim 0$ if $\frac{\partial \mathcal{V}^S}{\partial K}(q; p_0 = \hat{p}, K=0) < 0$.

For the product attributes learning process in §3.1.1, the optimal static price is:

$$\hat{p}_0 = \hat{p}_0(x) = \begin{cases} \frac{x+g-\underline{V}}{2}, & \underline{V} + g < x < 2\bar{V} - \underline{V} + g \\ x - \bar{V}, & x \geq 2\bar{V} - \underline{V} + g \end{cases},$$

where $\bar{V}, \underline{V}$ are given by (10). In this case, $q = \frac{x-\underline{V}-g}{2(\bar{V}-\underline{V})}$. Substituting the buyer's linear perturbation solution from Proposition 4 into Theorem 1 leads to:

$$\frac{1}{\sqrt{\varepsilon}} \frac{\partial \mathcal{V}^S}{\partial K}(q; p_0 = \hat{p}_0, K=0) = \frac{(\bar{V}-\underline{V})^2}{3\sigma^2}(1-2q)q(1-q) - (\bar{S}q + \underline{S}(1-q))q. \tag{32}$$

Because the sign of the above expression depends only on q, c/r, and σ^2/r, in Figure 2, we illustrate the direction of a slow-moving price that improves the seller's expected profit over the optimal constant price strategy as a function of q, c/r, and σ^2/r.

For the learning process of binary classification in §3.1.2, we can only evaluate $\hat{p}_0$ and $\bar{V}[\hat{p}_0]$, $\underline{V}[\hat{p}_0]$ numerically because $\bar{V}[p_0]$ and $\underline{V}[p_0]$ are only implicitly specified through a system of non-linear algebraic equations. Figure 3 presents the direction of a slow-moving price that improves the seller's expected profit over the optimal constant price strategy.

Figures 2 and 3 are qualitatively similar. We classify each plot into four regions.

I (Low incentive to search) When the search cost c is too high, the buyer has a low incentive to search for information. The seller needs to give the buyer a high surplus to encourage her to search, which hurts the profit. So, it becomes more attractive for the seller to convince the buyer to purchase the product at the beginning, based on the initial valuation and the price trajectory. For any given initial price, by charging an increasing

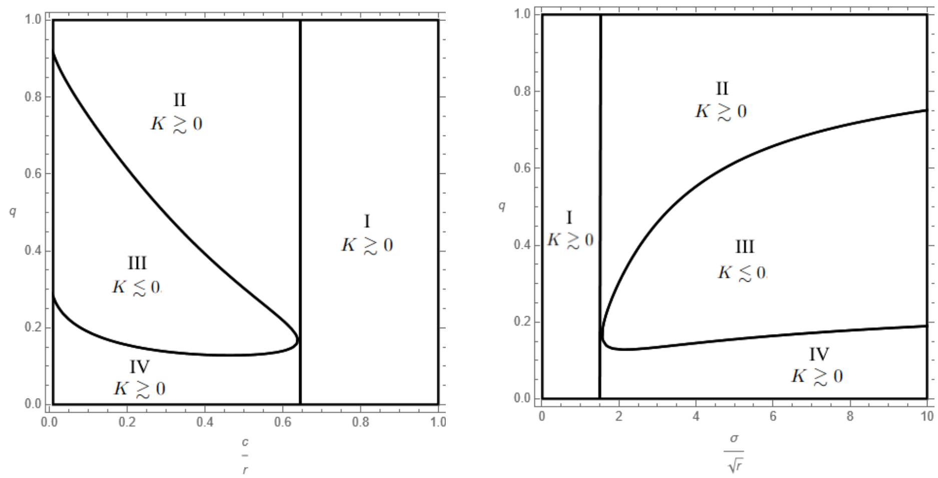


Figure 2: Direction of the price for a perfectly patient seller where the buyer follows the product attributes learning process, $\sigma^2/r = 1$ in the left plot, and $c/r = 1$ in the right plot.

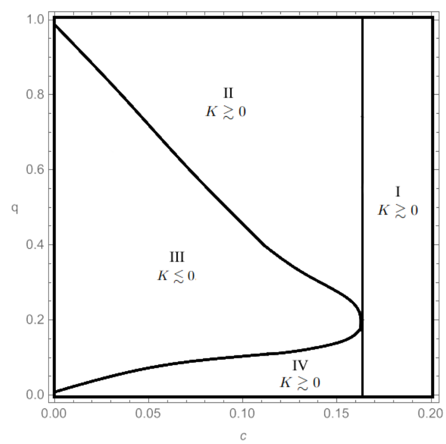


Figure 3: Direction of the price for a perfectly patient seller where the buyer follows the binary classification process, $g = 0.3$, $r = 1$, and $\sigma_S = 1$.

price over time, the seller lowers the purchasing threshold at the beginning by making it more desirable for the buyer to make an immediate decision. Compared with the stationary pricing strategy of charging a lower constant price, this non-stationary pricing strategy moves the purchasing threshold in the same direction (downward) without sacrificing the profit conditional on purchase. In other words, it increases the probability of purchase without reducing the profit per purchase.

II (High-valuation buyer) When the buyer has a high initial valuation, she is too valuable to lose from the seller's perspective. Therefore, the seller wants to increase the purchasing probability in this case. Moreover, a high-valuation buyer can earn a positive payoff from purchasing immediately, which decreases over time because of discounting. Thus, the seller also wants the buyer to make a quick purchase. An increasing pricing strategy reduces the benefits of searching and encourages the buyer to buy quickly and with a higher likelihood.

III (Medium-valuation buyer) When the buyer has a moderate interest in the product, an increase in price does not suffice to convince her to purchase quickly without learning much additional information. Instead, it reduces the benefit of searching because the buyer knows she will have to pay a higher price if she learns positive things. Therefore, an increasing price will lead to a quick exit rather than a quick purchase.

In this case, the seller can benefit from a decreasing price, which helps keep the buyer engaged in the search process, even if she receives some negative information early on. As a result, it increases the purchasing likelihood. Because of the moderate initial valuation, the seller can still obtain a decent profit at a lower price. This pricing strategy protects the seller from missing potentially valuable buyers.

IV (Low-valuation buyer) It is not worthwhile for the seller to reduce the price over time for two reasons. First, the profit from an immediate purchase is already low when the buyer has a low initial valuation. The seller will obtain an even lower profit from an eventual purchase if the buyer searches for a while and eventually buys from the seller at a lower price. Second, due to the low initial valuation, the buyer must accumulate a lot of positive information before purchasing. The purchasing probability will still be low even if the price is slightly reduced over time, and cannot offset the cost of a lower profit per purchase.

In this case, the seller quickly filters out many buyers by implementing an increasing pricing strategy. On the one hand, the loss from not making a deal with these people is limited due to the low profit per purchase and the low purchasing probability. On the other hand, the benefits of charging a higher price to the remaining buyers are high. Any buyers who do not quit despite the increasing price must have learned positive information and therefore are more valuable to the seller.

4.3 Constant Volatility

In this section, we consider the learning process in §3.1.1 with constant volatility $\sigma(x)^2 = \sigma^2$. In this setting, the exact seller's payoff $\mathcal{V}^S(x; p_0, K)$ under any linear pricing strategy $p : t \mapsto p_0 + \sqrt{\varepsilon}Kt \in \mathcal{P}^{\varepsilon}_{lin}$, for any $p_0, K \in \mathbb{R}$ and $\varepsilon > 0$ can be derived analytically.[7] The

[7] In this section, we allow for an arbitrary pricing slope $\sqrt{\varepsilon}K \in \mathbb{R}$. Hence, $\sqrt{\varepsilon}$ can be omitted throughout the section. We keep the term for consistency.

key reason is that, in this case, Proposition 4 has shown that the resulting purchasing and quitting boundaries are linear in t. The series solution to the heat equation $U(s, v; t, x)$ with two absorbing fixed boundaries has been well-studied, and we can use the Girsanov Theorem to transform such a solution to the solution to the equation with two linearly moving absorbing boundaries. By substituting this solution into (29), we can obtain a closed form expression for $\mathcal{V}^S(x; p_0, K)$. We first consider the special case with zero search costs. Then, we study the more interesting case with positive search costs.

4.3.1 Zero Search Costs

When the buyer has zero search costs, the continuation value of searching is positive, whereas the payoff from quitting is zero. In this case, she would never quit searching without purchasing the product. Therefore, her optimal search strategy is characterized by a single boundary, the purchasing boundary $\bar{V}[p]$.

If the seller is perfectly patient, a purchase at any time yields the same payoff for him. Hence, he does not have an incentive to increase the price over time to push the buyer to make an early decision. In addition, the buyer's quitting threshold is $-\infty$. So, the seller has no incentive to reduce the price over time to prevent the buyer from quitting. In sum, the seller has little incentive to charge non-stationary prices. The following result shows that the optimal price is arbitrarily close to constant when the seller is perfectly patient. In contrast, the seller may benefit from charging non-stationary prices if he discounts the future.

Proposition 6. *Suppose the search cost is zero, $c = 0$.*

1. *When the seller is perfectly patient ($m = 0$), for any fixed initial price p_0, the seller can approach the profit supremum by choosing $K > 0$ as close to zero as possible, $V^S(x) = \sup_{K \searrow 0} \mathcal{V}^S(x; p_0, K) = 2p_0 + \frac{\sigma}{\sqrt{2r}} - g - x$.*
2. *When the seller discounts the future ($m > 0$) – in particular, when $m >> 0$ or $m \sim 0$ – then the slope K of the optimal linear pricing is bounded away from zero.*

4.3.2 Positive Search Costs

When the buyer has a positive search cost, the continuation value of searching may be negative. Hence, both the purchasing and quitting boundaries are finite. Similar to the reason in the previous section, we focus on the case of a perfectly patient seller.

Proposition 7. *Suppose the search cost is positive $c > 0$, and the seller is perfectly patient ($m = 0$). The seller's expected profit from a buyer whose initial valuation is x is:*

$$\mathcal{V}^S(x;p_0,K) = \frac{p_0 - g + (\bar{V}_0 + x - 2\underline{V}_0)}{1 - \exp\left(+\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0 - \underline{V}_0)\right)} - \frac{2(\bar{V}_0 - \underline{V}_0)}{\left(1 - \exp\left(+\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0 - \underline{V}_0)\right)\right)^2}$$
$$- \frac{(p_0 - g + (\bar{V}_0 - x))\exp\left(+\frac{2\sqrt{\varepsilon}K}{\sigma^2}(x - \underline{V}_0)\right)}{1 - \exp\left(+\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0 - \underline{V}_0)\right)} + \frac{2(\bar{V}_0 - \underline{V}_0)\exp\left(+\frac{2\sqrt{\varepsilon}K}{\sigma^2}(x - \underline{V}_0)\right)}{\left(1 - \exp\left(+\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0 - \underline{V}_0)\right)\right)^2} \quad (33)$$

if $x \in (\underline{V}_0, \bar{V}_0)$ *and* $K \neq 0$, *and* $\mathcal{V}^S(x;p_0,K) = (p_0 - g)\left(\frac{x - \underline{V}_0}{\bar{V}_0 - \underline{V}_0}\right)$ *if* $x \in (\underline{V}_0, \bar{V}_0)$ *and* $K = 0$. $\mathcal{V}^S(x;p_0,K) = 0$ *if* $x \leq \underline{V}_0$. $\mathcal{V}^S(x;p_0,K) = p_0 - g$ *if* $x \geq \bar{V}_0$.

By taking the derivative of (33) at $K = 0$, one obtains equation (32), derived in the previous perturbative analysis. Unlike the no-search-cost case, here the slope K^* of the optimal pricing strategy can be bounded away from zero even if $m = 0$. From (33) we can find the optimal initial price $p_0^* = p_0^*(x, K)$ that maximizes $\mathcal{V}^S(x;.,K)$ for any $K \neq 0$ by solving $\frac{\partial \mathcal{V}^S}{\partial p_0}(x;p_0^*,K) = 0$:

$$p_0^*(x,K) := \frac{x+g}{2} + \frac{\sigma^2}{2\sqrt{\varepsilon}K} - \frac{\underline{V}[\sqrt{\varepsilon}K]}{2}\left(1 - \coth\frac{\sqrt{\varepsilon}K}{\sigma^2}\left(\bar{V}[\sqrt{\varepsilon}K] - \underline{V}[\sqrt{\varepsilon}K]\right)\right)$$
$$- \frac{\bar{V}[\sqrt{\varepsilon}K]}{2}\coth\frac{\sqrt{\varepsilon}K}{\sigma^2}\left(\bar{V}[\sqrt{\varepsilon}K] - \underline{V}[\sqrt{\varepsilon}K]\right), \quad (34)$$

for $x \in (\underline{V}_0, \bar{V}_0)$. One can verify that $\lim_{K\to 0} p_0^*(x,K) = \hat{p}_0(x) = \frac{x+g-V}{2}$, as expected.

Since (33) is valid for all $\sqrt{\varepsilon}K \in \mathbb{R}$, we can find the globally optimal linear pricing strategy $t \mapsto p_0 + Kt$ for each given x by maximizing $\mathcal{V}^S(x;p_0,K)$ over all $(p_0, K) \in \mathbb{R}^2$. This gives us a unified picture that within the linear pricing class under constant volatility σ, there are two types of strategies: the perturbative strategy as studied in §4.2, and the fast–rising price (take it or leave it) strategy where the seller forces an immediate decision from the buyer by threatening a rapid price increase. We show results for some representative choices of x and other parameters in Figure 4. In can be shown, via the Martingale Stopping Theorem, that the seller can achieve the payoff supremum $x - g$ over any arbitrary pricing strategies (linear or not) for any sufficiently high value, or impatient buyers (sufficiently high x or c). On the other hand, when x is not too high, we find from the Figure 4 that the global maximum lies within the perturbative regime.

Using (34), we can further characterize (p_0^*, K_0^*) when it lies in the perturbative regime:

Lemma 4. *Consider a perfectly patient seller ($m = 0$). Suppose that $r, \sigma, c > 0$; then, there exists $\varepsilon > 0$ sufficiently small such that the seller's profit maximizing linear pricing strategy (p_0^*, K^*) satisfies either $p_0^* < \hat{p}_0, K^* \gtrsim 0$ or $p_0^* > \hat{p}_0, K^* \lesssim 0$.*

This is consistent with Figure 4, where the optimal strategy with increasing price is

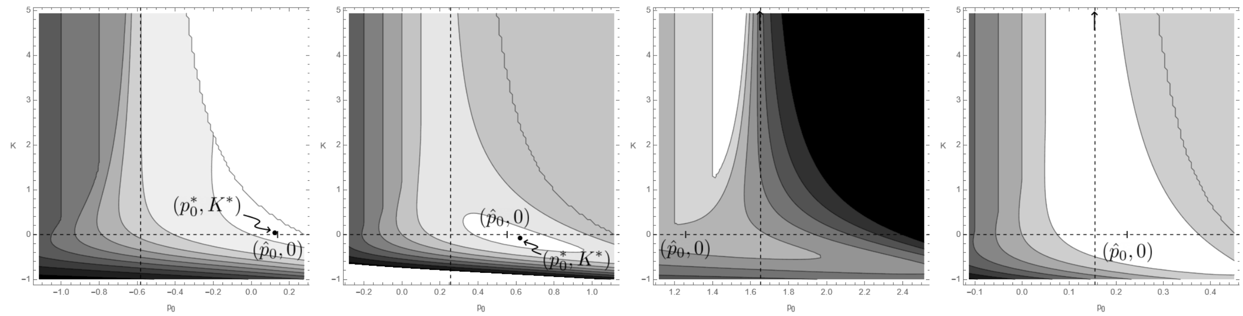


Figure 4: Contour plots of the seller's expected profit from the linear pricing strategy $t \mapsto p_0 + Kt$ when $r = \sigma = \varepsilon = 1$. The dashed vertical line illustrates the initial valuation x. $\hat{p}_0$: the optimal constant pricing. First figure: $q = 0.1$, $c = 0.2$, with the global maximum (p_0^*, K^*) in the perturbative regime where $p_0^* > \hat{p}_0$, $K^* \lesssim 0$. Second figure: $q = 0.4$, $c = 0.2$, with the global maximum (p_0^*, K^*) in the perturbative regime where $p_0^* < \hat{p}_0$, $K^* \gtrsim 0$. Third figure: $q = 0.9$, $c = 0.2$. Fourth figure: $q = 0.4$, $c = 0.8$.

always coupled with a lower initial price, and the optimal strategy with decreasing price is always coupled with a higher initial price. Going from the first to the second figure in Figure 4 corresponds to crossing the boundary from region IV to region III, as q increases, in Figure 2.

4.4 More General Pricing Strategy

Consider a more general set of admissible pricing strategies:

$$\mathcal{P}^{\varepsilon,M} := \left\{ p \in C^{\infty}[0,\infty) \mid \sup_{t \in \mathbb{R}_{>0}} |p'_t| \leq \sqrt{\varepsilon}, \sup_{t \in \mathbb{R}_{>0}} |p''_t| \leq M \right\} \text{ for some } \varepsilon > 0 \text{ and } M > 0.$$

It includes non-linear pricing strategies that are not *too* far away from linear strategies. In practice, sellers may be restricted by regulations or reputational concerns in how quickly they can change the price over time, indicating that ε is not too large. According to Proposition 3, a myopic buyer with $r > e^{-1}\sqrt{2M/\varepsilon}$ can make an ε-optimal learning decision at any time t by approximating the pricing strategy $p \in \mathcal{P}^{\varepsilon,M}$ with the linear pricing $l_{\mathbf{x}} : s \mapsto p_t + p'_t \cdot (s - t)$, i.e., the learning decision at any time t is entirely determined by p_t and p'_t.

From the seller's perspective, suppose that the search process is very informative, $\sigma(x)^2 \geq \underline{\sigma}^2$, for all x over the relevant learning region and for some constant $\underline{\sigma}^2 >> 0$. Suppose also that the search cost is sufficiently expensive $c >> 0$ to keep the purchasing and quitting boundaries bounded. In such cases, the buyer updates her valuation and reaches a purchasing decision quickly. We can argue that it is also ε-optimal for the seller to approximate any $p \in \mathcal{P}^{\varepsilon,M}$ with linear pricing. Let $\tau^*[l] \in \mathcal{T}$ denote the ε-optimal buyer's stopping time, corresponding to $p \in \mathcal{P}^{\varepsilon,M}$, by the linear approximation $l : t \mapsto p_0 + p_0' t$. Then, for any $\delta > 0$, we have:

$$V^S(x) = \sup_{p \in \mathcal{P}^{\varepsilon,M}} \left[\mathbb{E}\left[e^{-m\tau^*[l]} (p_{\tau^*[l]} - g) \cdot 1\left\{ v^x_{\tau^*[l]} \geq p_{\tau^*[l]}, \tau^*[l] < \delta \right\} \right] + e^{-m\delta} \mathcal{V}^S\left(U(\delta, .; 0, x); p_{.+\delta} \right) \right], \quad (35)$$

where $p_{.+\delta} : t \mapsto p_{t+\delta}$ is the corresponding time-shifted pricing strategy, and $U(\delta, .; 0, x)$ is the transition probability density (satisfying equation (28)). Given a finite $M > 0$, we can choose $\delta = O(\sqrt{\varepsilon})$, so that the first term of (35) can be approximated with a linear pricing l up to an order of some $\varepsilon > 0$, i.e., $\mathbb{E}\left[e^{-m\tau^*[l]} (p_{\tau^*[l]} - l_{\tau^*[l]}) \cdot 1_{\{v^x_{\tau^*[l]} \geq p_{\tau^*[l]}, \tau^*[l] < \delta\}} \right] < \frac{1}{2} M \delta^2 = O(\varepsilon)$.

The survival probability $\mathbb{P}[\tau^*[l] > \delta]$ can be upper-bounded by the survival probability $O\left(\frac{1}{\underline{\sigma}\sqrt{\delta}}\right)$ of the Brownian motion with volatility $\underline{\sigma}$ and a single absorbing boundary. Given that $\underline{\sigma}^2 >> 0$, we may argue that the second term of (35) is $O(\varepsilon)$, hence it is ε-optimal for the seller to plan only for the $\delta = O(\sqrt{\varepsilon})$ time ahead using the linear perturbation theory for $\mathcal{P}^{\varepsilon}_{lin}$, which we studied to specify $l : t \mapsto p_0 + Kt$, i.e., specifying (p_0^*, K^*). After time δ, the seller can repeat the process to further improve profit by maximizing $\mathcal{V}^S(\phi^1; p_{.+\delta})$, where $\phi^1 := U(\delta, .; 0, x)$, over the next δ-period using a linear approximation. More generally, the seller can specify the ε-optimal pricing strategy over the interval $t \in [k\delta, (k+1)\delta)$ for any $k \geq 1$ using the linear approximation $l^{k+1} : t \mapsto l^k_{k\delta} + K^{k+1} \cdot (t - k\delta) \in \mathcal{P}^{\varepsilon}_{lin}$, which can be specified from Theorem 1 integrated against ϕ^k. By patching together the linear segments, we obtain the piecewise linear pricing strategy, which should approximate the optimal $p \in \mathcal{P}^{\varepsilon,M}$[8].

5 Conclusion

This paper introduces a novel framework where sellers adopt non-stationary pricing strategies. We use perturbation theory to tackle the challenge of characterizing the buyer's search strategy given the seller's non-stationary pricing strategies. Our findings challenge the conventional reliance on stationary pricing by showing that non-stationary pricing strategies can outperform stationary ones. We provide a theoretical advance in optimal control by incorporating non-stationary strategies into a buyer search framework. Unlike previous work, the non-stationarity in the buyer's search problem arises endogenously from sellers' strategic pricing in response to buyers' gradual learning.

[8] Although the assumption of Proposition 3 is not entirely satisfied for piecewise linear pricing strategies, due to the lack of second derivatives, this would not be an issue as long as we impose the constraint: $(K^{k+1} - K^k)/\delta \leq M$ for all $k \geq 1$. Intuitively, linear approximation is a better approximation to a piecewise linear function than to a non-linear function.

Appendix

Proof of Lemma 1. Part 1: Consider any $x, x' \in \mathbb{R}$, and suppose that $x' > x$. Let $\{v_s^{t,x}\}_{s\geq t}$ and $\{v_s^{t,x'}\}_{s\geq t}$ be the two strong solutions of the SDE (1), and we have $v_s^{t,x'} > v_s^{t,x}$ a.e., for all $s \geq t$. This can be seen by using the Lipschitz condition (2) to analyze the difference process $d_s := v_s^{t,x'} - v_s^{t,x}$. It follows that $\mathcal{V}^B(t, x'; \tau, p) \geq \mathcal{V}^B(t, x; \tau, p)$ for all $\tau \in \mathcal{T}$. For any $\varepsilon > 0$, we can find $\tau_{t,x} \in \mathcal{T}$ such that $\mathcal{V}^B(t, x; \tau_{t,x}, p) \geq V(t, x; p) - \varepsilon$. Then, $V^B(t, x'; p) \geq \mathcal{V}^B(t, x'; \tau_{t,x}, p) \geq \mathcal{V}^B(t, x; \tau_{t,x}, p) \geq V(t, x; p) - \varepsilon$. Since $\varepsilon > 0$ is arbitrarily small, hence $V^B(t, x'; p) \geq V^B(t, x; p)$ as claimed. Further, if $V^B(t, x; p) > 0$, then either we can find $\tau_{t,x} \in \mathcal{T}$ such that $\mathbb{P}[\tau_{t,x} > t] > 0$ for any given $\varepsilon > 0$, or $V^B(t, x; p) = x - p_t$. In both cases, we have $\mathcal{V}^B(t, x'; \tau_{t,x}, p) > \mathcal{V}^B(t, x; \tau_{t,x}, p)$, which implies the strict inequality: $V^B(t, x'; p) > V^B(t, x; p)$ for any $x' > x$.

Part 2: It is clear from (3) that $\mathcal{V}^B(t, x; \tau, q) \leq \mathcal{V}^B(t, x; \tau, p)$ for all $\tau \in \mathcal{T}$, hence following the similar argument as in the previous part, we get $V^B(t, x; q) \leq V^B(t, x; p)$. Further, if $V^B(t, x; q) > 0$ then we can find $\tau_{t,x} \in \mathcal{T}$ such that $V^B(t, x; q) - \varepsilon \leq \mathcal{V}^B(t, x; \tau_{t,x}, q)$ for any given $\varepsilon > 0$, and either $\mathbb{P}[\tau_{t,x} > t] > 0$ or $V^B(t, x; q) = x - q_t$. In both cases, we have $\mathcal{V}^B(t, x; \tau_{t,x}, q) < \mathcal{V}^B(t, x; \tau_{t,x}, p)$, proving the strict inequality $V^B(t, x; q) < V^B(t, x; p)$. □

Proof of Proposition 1. Part 1: From Lemma 1, we already know that $V^B(0, x; \tilde{p}) \leq V^B(0, x; p)$. In fact, we have $V^B(0, x; \tilde{p}) < V^B(0, x; p)$ at any x such that $V^B(0, x; \tilde{p}) > 0$. Meanwhile, we have $\max\{x - \tilde{p}_0, 0\} = \max\{x - p_0, 0\}$ from the assumption $h_0 = 0$. It follows that $\bar{V}_0[\tilde{p}] = \sup\{x \in \mathbb{R} \mid V^B(0, x; \tilde{p}) > \max\{x - p_0, 0\}\} < \sup\{x \in \mathbb{R} \mid V^B(0, x; p) > \max\{x - p_0, 0\}\} = \bar{V}_0[p]$, and similarly $\underline{V}_0[\tilde{p}] = \inf\{x \in \mathbb{R} \mid V^B(0, x; \tilde{p}) > \max\{x - p_0, 0\}\} \geq \inf\{x \in \mathbb{R} \mid V^B(0, x; p) > \max\{x - p_0, 0\}\} = \underline{V}_0[p]$, which proves the claim.

Part 2: Without the loss of generality, let's only consider $t = 0$ and h such that $h_0 = 0$, we can always redefine t and shift the x–axis by a constant, otherwise. Let us suppose for a contradiction that there exists $x \in [\underline{\pi}, \bar{\pi}]$ where it is optimal to continue learning for any $K \geq 0$, i.e. $V^B(0, x; \tilde{p}) > 0$ is bounded away from zero for all $K \geq 0$. In other words, for any $\varepsilon, \varepsilon' > 0$, we can choose $\{\tau[K]\}_{K\geq 0} \subset \mathcal{T}$ and $\delta > 0$ such that $\sup_{K\geq 0} \mathbb{E}[1_{\tau[K]<\delta}] < \varepsilon'^2$ and:

$$V^B(0, x; \tilde{p}) \leq \mathbb{E}\left[e^{-r\delta\wedge\tau[K]} V^B(\delta \wedge \tau[K], v^x_{\delta\wedge\tau[K]}; \tilde{p}) - \int_0^{\delta\wedge\tau[K]} ce^{-rs} ds\right] + \varepsilon\delta, \quad \forall K \geq 0.$$

We can separate the expression above further into two terms corresponding to the events $\tau[K] \geq \delta$ and $\tau[K] < \delta$. Then, using Lemma 1: $V^B(t, x; \tilde{p}) \leq V^B(t, x; p)$ for all $(t, x) \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}]$, and $V^B(\delta, x; \tilde{p}) \leq V^B(\delta, x; p_\delta + Kh_\delta)$, where $p_\delta + Kh_\delta$ denotes a constant pricing policy (i.e. the buyer is better–off if $\tilde{p}_t$ stopped increasing after $t = \delta$), we obtain:

$$\begin{aligned} V^B(0,x;\tilde{p}) &\leq \mathbb{E}\left[e^{-r\tau[K]}V^B(\tau[K],v^x_{\tau[K]};\tilde{p})\cdot 1_{\tau[K]<\delta}\right] + \mathbb{E}\left[V^B(\delta,v^x_\delta;\tilde{p})\cdot 1_{\tau[K]\geq\delta}\right] + \varepsilon\delta \\ &\leq \mathbb{E}\left[e^{-r\tau[K]}V^B(\tau[K],v^x_{\tau[K]};p)\cdot 1_{\tau[K]<\delta}\right] + \mathbb{E}\left[V^B(\delta,v^x_\delta;p_\delta+Kh_\delta)\cdot 1_{\tau[K]\geq\delta}\right] + \varepsilon\delta. \end{aligned}$$

We bound the first term using the restriction on the growth–rate of the square–integral of the process $v^{t,x}_s$ implied by Assumption 1, and (37) the asymptotically linear condition in x for $V^B(t,x;p)$, we have:

$$\mathbb{E}\left[e^{-r\tau[K]}V^B(\tau[K],v^x_{\tau[K]};p)\cdot 1_{\tau[K]<\delta}\right] \leq \sup_{\tau\in\mathcal{T}}\mathbb{E}\left[e^{-2r\tau}V^B(\tau,v^x_\tau;p)^2\right]^{1/2}\cdot\sup_{K\geq 0}\mathbb{E}\left[1_{\tau[K]<\delta}\right]^{1/2},$$

where the first factor is finite, and the second factor is $< \varepsilon'$ by our choice of δ. For the second term, we know from the result on constant pricing policy value function that $V^B(\delta,v^x_\delta;p_\delta + Kh_\delta) = 0$ for all sufficiently large $K > 0$, giving a pointwise convergence of $V^B(\delta,v^x_\delta;p_\delta + Kh_\delta)\cdot 1_{\tau[K]\geq\delta}$ in the probability space. Thus, given any $\varepsilon'' > 0$, we can find a sufficiently large $K > 0$ such that $\mathbb{E}\left[V^B(\delta,v^x_\delta;p_\delta+Kh_\delta)\cdot 1_{\tau[K]\geq\delta}\right] < \varepsilon''$ by the Dominated Convergence Theorem. Overall, we have

$$V^B(0,x;\tilde{p}) \leq \varepsilon'\cdot\sup_{\tau\in\mathcal{T}}\mathbb{E}\left[e^{-2r\tau}V^B(\tau,v^x_\tau;p)^2\right]^{1/2} + \varepsilon'' + \varepsilon\delta.$$

Since $\varepsilon,\varepsilon',\varepsilon'' > 0$ are arbitrarily small, we conclude that $V^B(0,x;\tilde{p}) \leq 0$, a contradiction. Therefore, for any $x \in [\underline{\pi},\bar{\pi}]$, for all sufficiently large $K \geq 0$, either it is optimal to purchase immediately ($x > \tilde{p}_0$), or exit immediately ($x < \tilde{p}_0$). It must be the case that: $\bar{V}_0[\tilde{p}] \searrow \max\{\tilde{p}_0,\underline{\pi}\}$, $\underline{V}_0[\tilde{p}] \nearrow \min\{\tilde{p}_0,\bar{\pi}\}$ as $K \to +\infty$.

Suppose that $K < 0$, consider any $x \in [\underline{\pi},\bar{\pi}]$, we note that

$$V^B(0,x;\tilde{p}) \geq \mathcal{V}^B(0,x;\delta,\tilde{p}) \geq e^{-r\delta}\mathbb{E}\left[\max\left\{v^x_\delta - p_\delta - Kh_\delta, 0\right\}\right] - c\delta \geq -e^{-r\delta}(Kh_\delta + p_\delta) - c\delta$$

where δ denotes the simple policy of stopping exactly at some time $\delta > 0$ regardless of the valuation, and the first inequality followed from the sub–optimality of δ. Therefore, for all sufficiently negative $K << 0$, we have $V^B(0,x;\tilde{p}) > 0$, thus it is optimal to continue searching: i.e. $\underline{V}_0[\tilde{p}] < x < \bar{V}_0[\tilde{p}]$ for any $x \in [\underline{\pi},\bar{\pi}]$, proving $\bar{V}_0[\tilde{p}] \nearrow \bar{\pi}$ and $\underline{V}_0[\tilde{p}] \searrow \underline{\pi}$ as $K \to -\infty$. $\square$

Viscosity Solution and Perturbation Theory

As a standard practice in optimal stopping theory, rather than directly finding the optimal $\tau^*[p] \in \mathcal{T}$ to the optimization problem (5), it is often more analytically tractable to consider the corresponding Hamilton–Jacobi–Bellman (HJB) equation:

$$H(t, x, V, \nabla V, \Delta V) = 0, \tag{36}$$

where $H : (\mathbb{R} \times [\underline{\pi}, \bar{\pi}]) \times \mathbb{R} \times \mathbb{R}^2 \times \mathcal{S}_2(\mathbb{R}) \to \mathbb{R}$ is given by $H(t, x, V, \nabla V, \Delta V) := \min\left\{c + rV - \partial_t V - \frac{\sigma(x)^2}{2}\partial_x^2 V, V - x + p_t, V\right\}$, with $\mathcal{S}_2(\mathbb{R})$ denoted the space of 2×2 symmetric matrices. Since $p \in \mathcal{P}_T$ is only defined for $t \geq 0$, to discuss the solution on $\mathbb{R} \times [\underline{\pi}, \bar{\pi}]$, we extend it by defining $p_t = p_0$ for all $t < 0$. We consider the solution $V : \mathbb{R} \times [\underline{\pi}, \bar{\pi}] \to \mathbb{R}$ subject to the following asymptotic boundary conditions:

$$\begin{aligned} V(t,x) &= V_0^B(x; p_T),\ \forall t \geq T, &\qquad \lim_{t \to -\infty} V(t,x) &= V_0^B(x; p_0) \\ V(t,x) &= x - p_t,\ \forall x \geq \bar{V}_t[p], &\qquad V(t,x) &= 0,\ \forall x \leq \underline{V}_t[p] \end{aligned} \tag{37}$$

for some functions $\bar{V}[p], \underline{V}[p] : \mathbb{R} \to [\underline{\pi}, \bar{\pi}]$, depending on $p \in \mathcal{P}_T$ and $\bar{V}_t[p] \geq \underline{V}_t[p], \forall t \in \mathbb{R}$. The purchase and quitting boundaries, $\bar{V}[p]$, and $\underline{V}[p]$, provide a simple characterization of the learning strategy. By definition of $\mathcal{P}$, the range of $\bar{V}[p]$ and $\underline{V}[p]$ are contained in $[\underline{\pi}, \bar{\pi}]$.

We need to establish the existence and uniqueness of the solution to (36) subjects to the boundary condition (37). Since the classical solution does not always exist, we will work with a relaxed notion of a *viscosity* solution:[9]

Definition 2. *Let $D \subset \mathbb{R}^n$ and $H : D \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}_n(\mathbb{R}) \to \mathbb{R}$ be a continuous function satisfying the* properness *condition: $H(\mathbf{x}, v, p, X) \geq H(\mathbf{x}, u, p, X)$ if $v \geq u$, and the* degenerate ellipticity *condition: $H(\mathbf{x}, v, p, X) \geq H(\mathbf{x}, v, p, Y)$ if $Y \geq X$.*

A continuous function $v : D \to \mathbb{R}$ is a viscosity subsolution *if for any $\mathbf{x}_0 \in D$ and any twice continuously differentiable function ϕ such that $\mathbf{x}_0$ is a local maximum of $v - \phi$ we have $H(\mathbf{x}_0, v(\mathbf{x}_0), \nabla\phi(\mathbf{x}_0), \Delta\phi(\mathbf{x}_0)) \leq 0$.*

A continuous function $v : D \to \mathbb{R}$ is a viscosity supersolution *if for any $\mathbf{x}_0 \in D$ and any twice continuously differentiable function ϕ such that $\mathbf{x}_0$ is a local minimum of $v - \phi$ we have $H(\mathbf{x}_0, v(\mathbf{x}_0), \nabla\phi(\mathbf{x}_0), \Delta\phi(\mathbf{x}_0)) \geq 0$.*

A continuous function $v : D \to \mathbb{R}$ is a viscosity *solution if it is both a viscosity subsolution and supersolution.*

The following existence and uniqueness result relates the buyer's value function V^B to

[9] Crandall et al. (1992) provides a detailed description of the viscosity solution.

the viscosity solution over the domain $D := \mathbb{R} \times [\underline{\pi}, \bar{\pi}]$.

Lemma 5. *For a given $p \in \mathcal{P}_T$, the buyer's value function V^B is the unique viscosity solution to (36) subject to the asymptotic boundary conditions (37).*

Proof of Lemma 5. The proof that the value function V^B is a viscosity solution to (36) is standard (e.g. see Yong and Zhou (2012)) and we shall omit the details. The fact that V^B satisfies the boundary condition (37) is enforced by the definition of $\mathcal{P}_T$. It remains for us to check the uniqueness of the viscosity solution to (36) with boundary condition (37). This is mostly an application of the comparison principle (Crandall et al., 1992, Theorem 3.3). However, unlike in the standard setup, our domain is unbounded, therefore we provide the details for completeness. For convenience, in the following we will use $A_1(\mathbf{x}, V, \nabla V, \Delta V) := c + rV - \partial_t V - \frac{\sigma(\mathbf{x})^2}{2}\partial_x^2 V$, $A_2(\mathbf{x}, V, \nabla V, \Delta V) := V - x + p_t$, $A_3(\mathbf{x}, V, \nabla V, \Delta V) := V$, so that $H := \min_{i=1,2,3} A_i$.

Let $u : \mathbb{R} \times [\underline{\pi}, \bar{\pi}] \to \mathbb{R}$ and $v : \mathbb{R} \times [\underline{\pi}, \bar{\pi}] \to \mathbb{R}$ be viscosity subsolution and supersolution to (36), respectively, and suppose that $\lim_{t\to\pm\infty}(u-v) \le 0$, $\lim_{x\to\underline{\pi}}(u-v) \le 0$, and $\lim_{x\to\bar{\pi}}(u-v) \le 0$. We claim that $u \le v$ everywhere on $\mathbb{R}\times[\underline{\pi}, \bar{\pi}]$. To show this let us assume the contrary that there exists $\hat{\mathbf{x}} \in \mathbb{R} \times (\underline{\pi}, \bar{\pi})$ such that $u(\hat{\mathbf{x}}) - v(\hat{\mathbf{x}}) = \max_{\mathbf{x}\in\mathbb{R}\times[\underline{\pi},\bar{\pi}]} (u(\mathbf{x}) - v(\mathbf{x})) > 0$. Consider the function: $w_\alpha(\mathbf{x}, \mathbf{y}) := u(\mathbf{x}) - v(\mathbf{y}) - (\alpha/2)\|\mathbf{x} - \mathbf{y}\|_2^2$ for some constant $\alpha \ge 0$. The assumption on the boundary conditions of u and v implies that for any $\alpha \ge 0$, there exists a local maximum $(\mathbf{x}_\alpha, \mathbf{y}_\alpha) \in (\mathbb{R} \times [\underline{\pi}, \bar{\pi}])^2$ of w_α, and by (Crandall et al., 1992, Lemma 3.1):

$$\lim_{\alpha\to\infty} \alpha\|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2^2 = 0, \quad \lim_{\alpha\to\infty} \left(u(\mathbf{x}_\alpha) - v(\mathbf{x}_\alpha) - \frac{\alpha}{2}\|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2^2 \right) = u(\hat{\mathbf{x}}) - v(\hat{\mathbf{x}}).$$

By our assumption, we can find $\delta > 0$ such that $u(\mathbf{x}_\alpha) - v(\mathbf{y}_\alpha) \ge \delta$ for all $\alpha \ge 0$. We can apply (Crandall et al., 1992, Theorem 3.2) since $\mathbb{R} \times [\underline{\pi}, \bar{\pi}]$ is locally compact, and we find $X, Y \in \mathcal{S}_2(\mathbb{R})$ such that

$$-3\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \tag{38}$$

with $\mathbf{x}_\alpha$ a local maximum of $u(\mathbf{x}) - \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha)^\intercal(\mathbf{x} - \mathbf{x}_\alpha) - \frac{1}{2}(\mathbf{x} - \mathbf{x}_\alpha)^\intercal X(\mathbf{x} - \mathbf{x}_\alpha)$ and $\mathbf{y}_\alpha$ a local minimum of $v(\mathbf{y}) - \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha)^\intercal(\mathbf{y} - \mathbf{y}_\alpha) - \frac{1}{2}(\mathbf{y} - \mathbf{y}_\alpha)^\intercal Y(\mathbf{y} - \mathbf{y}_\alpha)$. Since u and v are subsolution and supersolution, respectively, we have:

$$H(\mathbf{x}_\alpha, u(\mathbf{x}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X) \le 0 \le H(\mathbf{y}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), Y). \tag{39}$$

From (38) we have:

$$
\begin{aligned}
&A_1(\mathbf{y}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), Y) - A_1(\mathbf{x}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X) \\
&\qquad = \frac{\sigma(\mathbf{x}_\alpha)^2}{2} X_{xx} - \frac{\sigma(\mathbf{y}_\alpha)^2}{2} Y_{xx} = \begin{pmatrix} \sigma(\mathbf{x}_\alpha) & \sigma(\mathbf{y}_\alpha) \end{pmatrix} \begin{pmatrix} X_{xx} & 0 \\ 0 & -Y_{xx} \end{pmatrix} \begin{pmatrix} \sigma(\mathbf{x}_\alpha) \\ \sigma(\mathbf{y}_\alpha) \end{pmatrix} \\
&\quad \leq 3\alpha \begin{pmatrix} \sigma(\mathbf{x}_\alpha) & \sigma(\mathbf{y}_\alpha) \end{pmatrix} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \begin{pmatrix} \sigma(\mathbf{x}_\alpha) \\ \sigma(\mathbf{y}_\alpha) \end{pmatrix} = 3\alpha(\sigma(\mathbf{x}_\alpha) - \sigma(\mathbf{y}_\alpha))^2 \leq 3\alpha L^2 \|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2^2
\end{aligned}
$$

where we used the condition (2) for σ in the last inequality. Similarly, we can check that

$$
\begin{aligned}
A_2(\mathbf{y}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), Y) - A_2(\mathbf{x}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X) &\leq \|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2 + |p_{t_x} - p_{t_y}| \\
&\leq \left(1 + \max_{t \in [0,T]} |p_t'|\right) \|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2,
\end{aligned}
$$

and $A_3(\mathbf{y}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), Y) - A_3(\mathbf{x}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X) = 0$. Let us define $\omega(r) := \max\left\{3L^2, 1 + \max_{t \in [0,T]} |p_t'|\right\} \cdot r$ and $i^* := \operatorname{argmin}_{i=1,2,3} A_i(\mathbf{x}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X)$, then

$$
\begin{aligned}
&H(\mathbf{y}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), Y) - H(\mathbf{x}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X) \\
&\qquad \leq A_{i^*}(\mathbf{y}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), Y) - A_{i^*}(\mathbf{x}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X) \\
&\qquad \leq \omega\left(\alpha \|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2^2 + \|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2\right).
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow 0 < \min\{1, r\}\delta &\leq \min\{1, r\}(u(\mathbf{x}_\alpha) - v(\mathbf{y}_\alpha)) \\
&\leq H(\mathbf{x}_\alpha, u(\mathbf{x}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X) - H(\mathbf{x}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X) \\
&= H(\mathbf{x}_\alpha, u(\mathbf{x}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X) - H(\mathbf{y}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), Y) \\
&+ H(\mathbf{y}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), Y) - H(\mathbf{x}_\alpha, v(\mathbf{y}_\alpha), \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X) \\
&\leq \omega\left(\alpha \|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2^2 + \|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2\right) \qquad (40)
\end{aligned}
$$

for all $\alpha \geq 0$, where we used (39) to replace the first two terms with zero in the last inequality. By taking the $\alpha \to \infty$ limit, $\omega\left(\alpha \|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2^2 + \|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2\right) \to 0$, while the inequality above specifies that it is bounded away from zero by $\min\{1, r\}\delta$, which is a contradiction. In other words, we have $u \leq v$ over the entire $\mathbb{R} \times [\underline{\pi}, \bar{\pi}]$. Therefore, if $u : \mathbb{R} \times [\underline{\pi}, \bar{\pi}] \to \mathbb{R}$ and $v : \mathbb{R} \times [\underline{\pi}, \bar{\pi}] \to \mathbb{R}$ are both viscosity solution to (36) with the specified boundary conditions: $\lim_{t \to \pm\infty}(u - v) = 0$, $\lim_{x \to \underline{\pi}}(u - v) = 0$, and $\lim_{x \to \bar{\pi}}(u - v) = 0$, then $u = v$ over the entire $\mathbb{R} \times [\underline{\pi}, \bar{\pi}]$. □

Working directly with the viscosity solution via Definition 2 can still be challenging, thus we alternatively consider the following free–boundary backward parabolic PDE initial–value problem: Find $V : \Omega \to \mathbb{R}$, and continuously differentiable functions $\bar{V}[p], \underline{V}[p] : \mathbb{R} \to [\underline{\pi}, \bar{\pi}]$ satisfying $\bar{V}_t[p] \geq \underline{V}_t[p]$, such that

$$\begin{cases} \dfrac{\sigma(x)^2}{2}\partial_x^2 V(t,x) + \partial_t V(t,x) - rV(t,x) - c = 0, & (t,x) \in \Omega \\ V(t, \bar{V}_t[p]) = \bar{V}_t[p] - p_t, & V(t, \underline{V}_t[p]) = 0, \\ \partial_x V(t, \bar{V}_t[p]) = 1, & \partial_x V(t, \underline{V}_t[p]) = 0, \\ V(T,x) = V_0^B(x; p_T), \end{cases} \tag{41}$$

where

$$\Omega := \{(t,x) \in (-\infty, T] \times [\underline{\pi}, \bar{\pi}] \mid \underline{V}_t[p] < x < \bar{V}_t[p]\}.$$

Any *exact* solution to the problem (41), subject to some additional mild conditions, will also be the viscosity solution of the HJB (36) and thus the consumer's value function by Lemma 5. Meanwhile, (41) enables us to use a perturbation technique to obtain the consumer value function up to any $\varepsilon^{(k+1)/2}$–order by *solving (41) up to the* $\varepsilon^{(k+1)/2}$*–order*, since the solution up to the $\varepsilon^{(k+1)/2}$–order of (41), subject to some additional mild conditions, also agree with the viscosity solution to (36) up to the $\varepsilon^{(k+1)/2}$–order.

Given a solution V to (41) on Ω with the specified boundary conditions, we can extend it to $\widetilde{V}$, a function continuously differentiable on $\mathbb{R} \times [\underline{\pi}, \bar{\pi}]$, and twice continuously differentiable in x on $\mathbb{R} \times [\underline{\pi}, \bar{\pi}] \setminus \partial\Omega$, by defining $\widetilde{V}(t,x) = \max\{x - p_t, 0\}$ if $t \leq T$ and $x \notin (\underline{V}_t[p], \bar{V}_t[p])$, and $\widetilde{V}(t,x) = V_0^B(x; p_T)$ if $t > T$. This extension is rather natural, therefore, we will abuse the notation and simply refer to $\widetilde{V}$ as V. We will state formally in Lemma 6 that the solution V will coincide with the buyer's value function V^B. This justifies that the constant price benchmark solutions in §3.1 are the viscosity solutions, and therefore the value functions of their respective buyer's problems.

Suppose we know that $V^B(.,.;p)$ for a given $p \in \mathcal{P}_T$ is a solution to (41), and we would like to compute $V^B(.,.;p + \sqrt{\varepsilon}h)$ for some $h \in \mathcal{P}_T$ and a small $\varepsilon > 0$ up to the $\varepsilon^{(k+1)/2}$–order. By Lemma 6, we aim to solve for the corresponding solution $V(.,.;p + \sqrt{\varepsilon}h)$ to (14) up to the $\varepsilon^{(k+1)/2}$–order. The idea of perturbation theory is to consider the solution ansatz

$$\begin{aligned} V^{\varepsilon}_{\leq k,t}(.,.) = V^{\varepsilon}_{\leq k}(.,.;p + \sqrt{\epsilon}h) &:= V_0(.,.) + V_1(.,.)\sqrt{\varepsilon} + \cdots + V_k(.,.)\varepsilon^{k/2} \\ \bar{V}^{\varepsilon}_{\leq k,t} = \bar{V}^{\varepsilon}_{\leq k,t}[p + \sqrt{\varepsilon}h] &:= \bar{V}_{0,t} + \bar{V}_{1,t}\sqrt{\varepsilon} + \cdots + \bar{V}_{k,t}\varepsilon^{k/2} \\ \underline{V}^{\varepsilon}_{\leq k,t} = \underline{V}^{\varepsilon}_{\leq k,t}[p + \sqrt{\varepsilon}h] &:= \underline{V}_{0,t} + \underline{V}_{1,t}\sqrt{\varepsilon} + \cdots + \underline{V}_{k,t}\varepsilon^{k/2}, \end{aligned}$$

where $V_0(.,.) := V_0^B(.,.;p)$, $\bar{V}_{0,t} := \bar{V}_t[p]$, and $\underline{V}_{0,t} := \underline{V}_t[p]$, satisfying (41) over $\Omega^{\varepsilon}_{\leq k} :=$

$\{(t,x)\in(-\infty,T]\times[\underline{\pi},\bar{\pi}]|\underline{V}^{\varepsilon}_{\leq k,t}<x<\bar{V}^{\varepsilon}_{\leq k,t}\}$ up to the $\varepsilon^{(k+1)/2}$–order, i.e.

$$\begin{cases}\dfrac{\sigma(x)^2}{2}\partial_x^2 V^{\varepsilon}_{\leq k,t}(t,x)+\partial_t V^{\varepsilon}_{\leq k,t}(t,x)-rV^{\varepsilon}_{\leq k,t}(t,x)-c=O(\varepsilon^{(k+1)/2}), & (t,x)\in\Omega^{\varepsilon}_{\leq k}\\ V^{\varepsilon}_{\leq k,t}(t,\bar{V}^{\varepsilon}_{\leq k,t})=\bar{V}^{\varepsilon}_{\leq k,t}-(p_t+\sqrt{\varepsilon}h_t)+O(\varepsilon^{(k+1)/2}), & \\ V^{\varepsilon}_{\leq k,t}(t,\underline{V}^{\varepsilon}_{\leq k,t})=O(\varepsilon^{(k+1)/2}), \partial_x V^{\varepsilon}_{\leq k,t}(t,\bar{V}^{\varepsilon}_{\leq k,t})=1+O(\varepsilon^{(k+1)/2}), & \\ \partial_x V^{\varepsilon}_{\leq k,t}(t,\underline{V}^{\varepsilon}_{\leq k,t})=O(\varepsilon^{(k+1)/2}), V^{\varepsilon}_{\leq k,t}(T,x)=V_0^B(x;p_T)+O(\varepsilon^{(k+1)/2}) & \end{cases}.$$

By substituting the expression of $V^{\varepsilon}_{\leq k,t}(.,.)$, $\bar{V}^{\varepsilon}_{\leq k,t}$, and $\underline{V}^{\varepsilon}_{\leq k,t}$ into (41) and comparing the $\varepsilon^{k'/2}$ terms for $k'=1,2,\cdots,k$, we can solve for $V_{k'},\bar{V}_{k'},\underline{V}_{k'}$ using the knowledge of $V_{k''},\bar{V}_{k''},\underline{V}_{k''}$ for $k''=0,\cdots,k'-1$. Note that although both the value–matching and smooth–pasting conditions are only satisfied up to the $\varepsilon^{(k+1)/2}$–order, it is possible to find a twice continuously differentiable function $\chi:\mathbb{R}\times[\underline{\pi},\bar{\pi}]\setminus\Omega^{\varepsilon}_{\leq k}\to\mathbb{R}$ which continuously differentiably transitions from $V^{\varepsilon}_{\leq k}$ at $\partial\Omega^{\varepsilon}_{\leq k}$ to $\max\{x-p_t,0\}$ for all (t,x) some distance $R>0$ away from $\Omega^{\varepsilon}_{\leq k}$, e.g. a smooth 'bump' function. In particular, we have $\chi=V^{\varepsilon}_{\leq k}$ and $\nabla\chi=\nabla V^{\varepsilon}_{\leq k}$ on $\partial\Omega^{\varepsilon}_{\leq k}$. We also require that $|\partial_t\chi(t,x)-p'_t-\sqrt{\varepsilon}h'_t|=O(\varepsilon^{(k+1)/2})$, $|\partial_x^2\chi|=O(\varepsilon^{(k+1)/2})$, and that the asymptotic boundary conditions (37) are met. We extend $V^{\varepsilon}_{\leq k}$ to $\widetilde{V}^{\varepsilon}_{\leq k}$, a function continuously differentiable on $\mathbb{R}\times[\underline{\pi},\bar{\pi}]$, and twice continuously differentiable in x on $\mathbb{R}\times[\underline{\pi},\bar{\pi}]\setminus\partial\Omega^{\varepsilon}_{\leq k}$, by defining $\widetilde{V}^{\varepsilon}_{\leq k}(t,x)=\chi(t,x)$ if $t\leq T$ and $x\notin(\underline{V}^{\varepsilon}_{\leq k,t},\bar{V}^{\varepsilon}_{\leq k,t})$, $\widetilde{V}^{\varepsilon}_{\leq k}(t,x)=V_0^B(x;p_T+\sqrt{\varepsilon}h_T)$ if $t>T$, and $\widetilde{V}^{\varepsilon}_{\leq k}(t,x)=V^{\varepsilon}_{\leq k}(t,x)$ otherwise. We abuse the notation and refer to $\widetilde{V}^{\varepsilon}_{\leq k}$ as $V^{\varepsilon}_{\leq k}$.

Lemma 6. *Consider pricing strategies $p,h\in\mathcal{P}_T$ and a given $\varepsilon>0$.*

1. *If V satisfies the free–boundary backward parabolic PDE initial–value problem (41) with the pricing policy $p\in\mathcal{P}_T$, such that $V(t,x)\geq\max\{x-p_t,0\}$, and $p'_t+r(\bar{V}_t[p]-p_t)+c\geq 0$ for all $(t,x)\in\mathbb{R}\times[\underline{\pi},\bar{\pi}]$, then V is a viscosity solution to (36). In particular, the buyer's value function is given by $V^B=V$.*

2. *If $V^{\varepsilon}_{\leq k}$ satisfies the free–boundary backward parabolic PDE initial–value problem (41) up to the $\varepsilon^{(k+1)/2}$–order with the pricing policy $p+\sqrt{\varepsilon}h\in\mathcal{P}_T$, such that $V^{\varepsilon}_{\leq k}(t,x)\geq\max\{x-p_t,0\}+O(\varepsilon^{(k+1)/2})$, and $p'_t+\sqrt{\varepsilon}h'_t+r(\bar{V}^{\varepsilon}_{\leq k,t}[p+\sqrt{\varepsilon}h]-p_t-\sqrt{\varepsilon}h_t)+c\geq O(\varepsilon^{(k+1)/2})$ for all $(t,x)\in\mathbb{R}\times[\underline{\pi},\bar{\pi}]$, then $V^B=V^{\varepsilon}_{\leq k}+O(\varepsilon^{(k+1)/2})$.*

The conditions on p' and h' in Lemma 6 should not be particularly restrictive. For example, if we can check that $\lim_{x\nearrow\bar{V}_t[p]}\partial_x^2 V(t,x)\geq 0$, or $\lim_{x\nearrow\bar{V}^{\varepsilon}_{\leq k,t}[p+\sqrt{\varepsilon}h]}\partial_x^2 V^{\varepsilon}_{\leq k,t}(t,x)\geq O(\varepsilon^{(k+1)/2})$ then the conditions on p' and h' are automatically satisfied. In the context of our work, these conditions are easily satisfied when we focus on small perturbations from the known constant price solution and investigate the direction of buyers' reactions.

Typically, if our zero–th order perturbation for $p+\sqrt{\varepsilon}h$ is given by the buyer's value function: $V_0(.,.) = V^B(.,.;p)$, and the boundaries $\bar{V}_{0,t} = \bar{V}_t[p]$, $\underline{V}_{0,t} = \underline{V}_t[p]$, corresponding to p, then $p'_t + r(\bar{V}_t[p] - p_t) + c \geq 0$ by Lemma 5. Then it follows that $p'_t + \sqrt{\varepsilon}h'_t + r(V^{\varepsilon}_{\leq k,t}[p+\sqrt{\varepsilon}h] - p_t - \sqrt{\varepsilon}h_t) + c \geq 0$ is satisfied for all sufficiently small $\varepsilon > 0$. To avoid unnecessary technical complications, for the remainder we shall assume that all the conditions in Lemma 6 are satisfied whenever it is used.

For compatibility with the ε–equilibrium concept, we will only use Lemma 6 with $k = 1$ in all of our applications. However, it is important to remark that Lemma 6 does not directly claim that: $\bar{V}[p+\sqrt{\varepsilon}h] = \bar{V}^{\varepsilon}_{\leq 1}[p+\sqrt{\varepsilon}h] + O(\varepsilon)$ or $\underline{V}[p+\sqrt{\varepsilon}h] = \underline{V}^{\varepsilon}_{\leq 1}[p+\sqrt{\varepsilon}h] + O(\varepsilon)$, instead, some extra cares is needed which we outline as follows. The solution $V^{\varepsilon}_{\leq 1}$ can be interpreted (up to $O(\varepsilon)$), via the probabilistic Feynman–Kac expression, as the expected discounted value of purchasing when the valuation process $v^{t,x}_s$ reaches $\bar{V}^{\varepsilon}_{\leq 1,s}$ and exiting when $v^{t,x}_s$ reaches $\underline{V}^{\varepsilon}_{\leq 1,s}$, under the flow cost c. Since $V^B = V^{\varepsilon}_{\leq 1} + O(\varepsilon)$ according to Lemma 6, the learning strategy characterized by $\bar{V}^{\varepsilon}_{\leq 1}[p+\sqrt{\varepsilon}h]$ and $\underline{V}^{\varepsilon}_{\leq 1}[p+\sqrt{\varepsilon}h]$ are considered ε–optimal. Given this understanding, we shall slightly abuse our notation for convenience by writing $\bar{V}[p+\sqrt{\varepsilon}h] = \bar{V}^{\varepsilon}_{\leq 1}[p+\sqrt{\varepsilon}h] + O(\varepsilon)$ and $\underline{V}[p+\sqrt{\varepsilon}h] = \underline{V}^{\varepsilon}_{\leq 1}[p+\sqrt{\varepsilon}h] + O(\varepsilon)$.

Proof of Lemma 6. Part 1: Let such a solution V to (14) be given. Since we have assumed $V(t,x) \geq \max\{x - p_t, 0\}$ and $p'_t + r(\bar{V}_t[p] - p_t) + c \geq 0$, then $V - \max\{x - p_t, 0\} \geq 0$ for all $\mathbf{x} \in \Omega$, and $c + rV - \partial_t V - \frac{\sigma(\mathbf{x})^2}{2}\partial^2_x V \geq 0$ for all $\mathbf{x} \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}] \setminus \Omega$. By the value–matching, the smooth pasting conditions, and the assumption that $p \in \mathcal{P}_T$ is smooth, we have that V is continuously differentiable[10]. Moreover, V is twice continuously differentiable in x on $\mathbb{R} \times [\underline{\pi}, \bar{\pi}] \setminus \partial\Omega$, as it is a (classical) solution to the PDE on Ω, and $\max\{x - p_t, 0\}$ is twice continuously differentiable in x on $\mathbb{R} \times [\underline{\pi}, \bar{\pi}] \setminus \Omega$. Therefore, we have $H(\mathbf{x}, V, \nabla V, \Delta V) = 0$ classically on $\mathbb{R} \times [\underline{\pi}, \bar{\pi}] \setminus \partial\Omega$. Thus, for any twice continuously differentiable ϕ and any $\mathbf{x}_0 \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}]$, we have $\nabla\phi(\mathbf{x}_0) = \nabla V(\mathbf{x}_0)$, and we can find $\{\mathbf{x}_i\}_{i=0}^{\infty} \subset \mathbb{R} \times [\underline{\pi}, \bar{\pi}] \setminus \partial\Omega$ converging to $\mathbf{x}_0$. If $\mathbf{x}_0$ is a local maximum of $V - \phi$ then $\partial^2_x\phi(\mathbf{x}_0) \geq \lim_{i\to\infty} \partial^2_x V(\mathbf{x}_i)$ which implies $H(\mathbf{x}_0, V(\mathbf{x}_0), \nabla\phi(\mathbf{x}_0), \Delta\phi(\mathbf{x}_0)) \leq \lim_{i\to\infty} H(\mathbf{x}_i, V(\mathbf{x}_i), \nabla V(\mathbf{x}_i), \Delta V(\mathbf{x}_i)) = 0$. Similarly, if $\mathbf{x}_0$ is a local minimum of $V - \phi$ then $\partial^2_x\phi(\mathbf{x}_0) \leq \lim_{i\to\infty} \partial^2_x V(\mathbf{x}_i)$ which implies $H(\mathbf{x}_0, V(\mathbf{x}_0), \nabla\phi(\mathbf{x}_0), \Delta\phi(\mathbf{x}_0)) \geq \lim_{i\to\infty} H(\mathbf{x}_i, V(\mathbf{x}_i), \nabla V(\mathbf{x}_i), \Delta V(\mathbf{x}_i)) = 0$.

Part 2: Repeat the argument from the previous part with the perturbed pricing policy $p+\sqrt{\varepsilon}h \in \mathcal{P}_T$, we have that $H(\mathbf{x}, V^{\varepsilon}_{\leq k}, \nabla V^{\varepsilon}_{\leq k}, \Delta V^{\varepsilon}_{\leq k}) = O(\varepsilon^{(k+1)/2})$ classically on $\mathbb{R} \times [\underline{\pi}, \bar{\pi}] \setminus \partial\Omega^{\varepsilon}_{\leq k}$. Moreover, for any twice continuously differentiable ϕ and any $\mathbf{x}_0 \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}]$, if $\mathbf{x}_0$ is a local maximum of $V^{\varepsilon}_{\leq k} - \phi$ then $H(\mathbf{x}_0, V^{\varepsilon}_{\leq k}, \nabla\phi, \Delta\phi) \leq O(\varepsilon^{(k+1)/2})$, and if $\mathbf{x}_0$ is a local

[10] To get the continuity of t derivative across the boundary, consider the defining equation: $V^B(t, \bar{V}_t[p]; p) = \bar{V}_t[p] - p_t$. Differentiating with respect to t gives: $\bar{V}'_t[p] \cdot \partial_x V^B(t, \bar{V}_t[p]; p) + \partial_t V^B(t, \bar{V}_t[p]; p) = \bar{V}'_t[p] - p'_t$, or $\partial_t V^B(t, \bar{V}_t[p]; p) = -p'_t$. Similarly, we have $\partial_t V^B(t, \underline{V}_t[p]; p) = 0$

minimum of $V^\varepsilon_{\le k}-\phi$ then $H(\mathbf{x}_0, V^\varepsilon_{\le k}, \nabla\phi, \Delta\phi) \ge O(\varepsilon^{(k+1)/2})$. Since $V^\varepsilon_{\le k}$ satisfies the same asymptotic boundary conditions as the value function V^B, we can repeat the comparison principle argument in the proof of Lemma 5. In particular, setting $u := V^\varepsilon_{\le k}, v := V^B$ we have (39) becomes $H(\mathbf{x}_\alpha, V^\varepsilon_{\le k}, \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X) + O(\varepsilon^{(k+1)/2}) \le 0 \le H(\mathbf{y}_\alpha, V^B, \alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), Y)$, which means (40) becomes $\min\{1, r\}(V^\varepsilon_{\le k}(\mathbf{x}_\alpha) - V^B(\mathbf{y}_\alpha)) \le \omega(\alpha\|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2^2 + \|\mathbf{x}_\alpha - \mathbf{y}_\alpha\|_2) + O(\varepsilon^{(k+1)/2})$. Taking the limit $\alpha\to\infty$, we find that $\sup_{\mathbf{x}\in\mathbb{R}\times[\underline{\pi},\bar{\pi}]}\big(V^\varepsilon_{\le k}(\mathbf{x}) - V^B(\mathbf{x})\big) \le O(\varepsilon^{(k+1)/2})$, in other words: $V^\varepsilon_{\le k}(\mathbf{x}) \le V^B(\mathbf{x}) + O(\varepsilon^{(k+1)/2})$. On the other hand, setting $u := V^B, v := V^\varepsilon_{\le k}$ yields $V^B(\mathbf{x}) \le V^\varepsilon_{\le k}(\mathbf{x}) + O(\varepsilon^{(k+1)/2})$, thus we have $V^B(\mathbf{x}) = V^\varepsilon_{\le k}(\mathbf{x}) + O(\varepsilon^{(k+1)/2})$ as claimed. □

Proof of Proposition 2. We note that $V^B(t, x-\sqrt{\varepsilon}h_t; p)$ is simply the solution $V^B(t,x;p)$ shifted according to $\sqrt{\varepsilon}Kh$ which satisfies the value–matching and smooth–pasting conditions at $\bar{V}[p]+\sqrt{\varepsilon}h$ and $\underline{V}[p]+\sqrt{\varepsilon}h$, but does not satisfies the PDE, hence the $\sqrt{\varepsilon}V_1^B$ correction is needed. By adding $\sqrt{\varepsilon}V_1^B$ correction, we further need a $\sqrt{\varepsilon}$–order correction to the purchase and quitting boundaries $\bar{V}[p]+\sqrt{\varepsilon}h$ and $\underline{V}[p]+\sqrt{\varepsilon}h$ which take the form (17). We find the equation for V_1^B by substituting the ansatz (15) into the PDE for $V^B(.,.;\tilde{p})$ and collecting the $\sqrt{\varepsilon}$–order terms:

$$\frac{\sigma(x)^2}{2}\partial_x^2 V_1^B(t,x) + \partial_t V_1^B(t,x) - rV_1^B(t,x) - h_t'\partial_x V^B(t,x;p) + h_t\sigma(x)\sigma'(x)\partial_x^2 V^B(t,x;p) = 0. \quad (42)$$

To study $\bar{R}$ and $\underline{R}$ we analyze the boundary conditions of $V^B(.,.;\tilde{p})$ to the first–order in $\sqrt{\varepsilon}$. Note that $V^B(t, x-\sqrt{\varepsilon}h_t; p)$ automatically satisfies the value–matching conditions at $\bar{V}[\tilde{p}]$ and $\underline{V}[\tilde{p}]$, as we will conseller below, because $\partial_x V^B(t, \bar{V}_t[p]; p) = 1$ and $\partial_x V^B(t, \underline{V}_t[p]; p) = 0$. We have by substituting the ansatz (15) and (17) into the boundary conditions and comparing the $\sqrt{\varepsilon}$–order terms:

$$\begin{aligned}
&V^B(t, \bar{V}_t[\tilde{p}]; \tilde{p}) = \bar{V}_t[\tilde{p}] - \tilde{p}_t \\
&\qquad\qquad \Longrightarrow\ V^B(t, \bar{V}_t[p]+\sqrt{\varepsilon}\bar{R}_t; p) + \sqrt{\varepsilon}V_1^B(t, \bar{V}_t[p]) = \bar{V}_t[p] - p_t + \sqrt{\varepsilon}\bar{R}_t \\
&\qquad\qquad\qquad V_1^B(t, \bar{V}_t[p]) = -\bar{R}_t\partial_x V^B(t, \bar{V}_t[p]; p) + \bar{R}_t \ \Longrightarrow\ V_1^B(t, \bar{V}_t[p]) = 0. \quad (43)
\end{aligned}$$

$$\begin{aligned}
&\partial_x V^B(t, \bar{V}_t[\tilde{p}]; \tilde{p}) = 1 \ \Longrightarrow\ \partial_x V^B(t, \bar{V}_t[p]+\sqrt{\varepsilon}\bar{R}_t; p) + \sqrt{\varepsilon}\partial_x V_1^B(t, \bar{V}_t[p]) = 1 \\
&\qquad\qquad \partial_x V_1^B(t, \bar{V}_t[p]) = -\bar{R}_t\partial_x^2 V^B(t, \bar{V}_t[p]; p) \ \Longrightarrow\ \bar{R}_t = -\frac{\partial_x V_1^B(t, \bar{V}_t[p])}{\partial_x^2 V^B(t, \bar{V}_t[p]; p)}. \quad (44)
\end{aligned}$$

$$\begin{aligned}
&V^B(t, \underline{V}_t[\tilde{p}]; \tilde{p}) = 0 \ \Longrightarrow\ V^B(t, \underline{V}_t[p]+\sqrt{\varepsilon}\underline{R}_t; p) + \sqrt{\varepsilon}V_1^B(t, \underline{V}_t[p]) = 0 \\
&\qquad\qquad\qquad\qquad \Longrightarrow\ V_1^B(t, \underline{V}_t[p]) = 0. \quad (45)
\end{aligned}$$

$$\partial_x V^B(t, \underline{V}_t[\tilde{p}]; \tilde{p}) = 0 \implies \partial_x V^B(t, \underline{V}_t[p] + \sqrt{\varepsilon}\underline{R}_t; p) + \sqrt{\varepsilon}\partial_x V_1^B(t, \underline{V}_t[p]) = 0$$

$$\partial_x V_1^B(t, \underline{V}_t[p]) = -\underline{R}_t \partial_x^2 V^B(t, \underline{V}_t[p]; p) \implies \underline{R}_t = -\frac{\partial_x V_1^B(t, \underline{V}_t[p])}{\partial_x^2 V^B(t, \underline{V}_t[p]; p)}. \quad (46)$$

Since $p, h \in \mathcal{P}_T$, they are constant for all $t \geq T$, therefore we have the terminal condition at any $T' \geq T$: $V^B(T', x; \tilde{p}) = V_0^B(x; \tilde{p}_T)$ and $V^B(T', x; p) = V_0^B(x; p_T)$, giving the terminal condition for V_1^B:

$$V_1^B(T', x) = V_1^B(T, x) = \frac{1}{\sqrt{\varepsilon}} \left(V_0^B(x; p_T + \sqrt{\varepsilon}h_T) - V_0^B(x - \sqrt{\varepsilon}h_T; p_T) \right) + O(\sqrt{\varepsilon}). \quad (47)$$

We recognize the PDE (42) with (43). (45), and (47) as a backward parabolic (fixed) boundary–value problem. We may transform the problem into the more standard parabolic form for: $\widetilde{V}_1^B(t', x') := V_1^B\left(T - t', \underline{V}_{T-t'}[p] + (\bar{V}_{T-t'}[p] - \underline{V}_{T-t'}[p])x'\right)$ on $\widetilde{\Omega} := [0, \infty) \times [0, 1]$ with smooth coefficients $(a_{ij}(.), b_i(.), c(.))$, according to our smoothness assumptions on $\bar{V}[p]$, $\underline{V}[p]$, and $\sigma(.)$. Since $-(h)'_t \partial_x V^B(.,.;p) + h_t \sigma(.)\sigma'(.)\partial_x^2 V^B(.,.;p)$ is assumed smooth on $\widetilde{\Omega}$, and $V_1^B(T,.)$ is smooth on $\{0\} \times [0, 1]$, we can apply (Evans, 2022, Chapter 7.1, Theorem 7) or (Friedman, 2008, Chapter 3, §5, Corollary 2) to conclude the existence of the smooth solution $\widetilde{V}_1^B$ to the parabolic initial boundary–value problem. Transforming back to the original problem, we get the smooth solution $V_1^B(.,.)$. The solution is unique, and admits a probabilistic expression via the semi–elliptic version of Feynman–Kac formula (Øksendal, 2003, Theorem 9.1.1):

$$V_1^B(t, x) = \mathbb{E}\left[e^{-r(T'-t)} V_1^B(T', v_{T'}^{t,x}) \cdot 1\left\{\tau_\Omega^{t,x} \geq T'\right\} | \mathscr{F}_t\right] - \mathbb{E}\left[\int_t^{\tau_\Omega^{t,x} \wedge T'} h'_s e^{-r(s-t)} \partial_x V^B(s, v_s^{t,x}; p) ds | \mathscr{F}_t\right]$$
$$+ \mathbb{E}\left[\int_t^{\tau_\Omega^{t,x} \wedge T'} h_s e^{-r(s-t)} \sigma(v_s^{t,x})\sigma'(v_s^{t,x}) \partial_x^2 V^B(s, v_s^{t,x}; p) ds | \mathscr{F}_t\right], \quad (48)$$

The first term is upper–bounded by $\sup_{x \in [\underline{V}_T[p], \bar{V}_T[p]]} V_1^B(T, x) e^{-r(T'-t)} \to 0$ as $T' \to \infty$. Since $p, h \in \mathcal{P}_T$, we have that h_t, $V^B(t, x; p)$, and the boundaries $\bar{V}_t[p], \underline{V}_t[p]$ are constant in t for $t \geq T$. Meanwhile, $v_s^{t,x}$ is bounded inside $[\inf_{s \in [t,T]} \underline{V}_s[p], \sup_{s \in [t,T]} \bar{V}_s[p]]$. Therefore, the third and fourth terms are upper–bounded by some constant (which can be determined by the supremum of the absolute value of the integrand over the compact set $[t, T] \times [\inf_{s \in [t,T]} \underline{V}_s[p], \sup_{s \in [t,T]} \bar{V}_s[p]])$ multiple of $\int_t^\infty e^{-r(s-t)} ds = 1/r < \infty$. Taking the limit $T' \to \infty$ of (48) using the Dominated Convergence Theorem for the right–hand–side while noting that the left–hand–side is independent of T', we obtain expression (16). □

Proof of Corollary 1. Suppose that $h := K\tilde{h}$, where $\tilde{h} \in \mathcal{P}_T$ is monotonically increasing in t, and that $\sigma'(.) = O(\varepsilon)$. We define $\bar{S} := \bar{R}/K : \mathbb{R} \to \mathbb{R}$, and $\underline{S} := \underline{R}/K : \mathbb{R} \to \mathbb{R}$. It remains to show that $\bar{S}_t \le 0$ and $\underline{S}_t \ge 0$. From our assumption that $\sigma'(.) = O(\varepsilon)$, we may ignore the third term in the $\sqrt{\varepsilon}$–order equation (16). Moreover, $\sigma'(.) = O(\varepsilon)$ implies $V_0^B(x; p_T + \varepsilon h_T) = V_0^B(x - \varepsilon h_T; p_T) + O(\varepsilon)$, hence we can also ignore the first term in (16). Since $V^B(t, .; p)$ is monotonically increasing in x from Lemma 1, it follows from the second term of (16) that $V_1^B(t, x)/K \le 0$ for any $(t, x) \in \Omega$. In particular, $\partial_x V_1^B(t, \bar{V}_t[p])/K \ge 0$ and $\partial_x V_1^B(t, \underline{V}_t[p])/K \le 0$.

Now, let us show that $\partial_x^2 V^B(t, \bar{V}_t[p]; p) \ge 0$. Let $\mathbf{x}_0 = (t, \bar{V}_t[p]) \in \partial\Omega$ be a point on the purchasing boundary, then we can find sequences $\{\mathbf{x}_i^+ = (t_i, x_i^+)\}_{i=0}^\infty$ and $\{\mathbf{x}_i^- = (t_i, x_i^-)\}_{i=0}^\infty \subset \Omega$ converging to $\mathbf{x}_0$ such that $x_i^- \le \bar{V}_{t_i}[p] \le x_i^+$ for all $i \ge 0$. Since $V^B(., .; p)$ is the viscosity solution, we have $c + rV^B(\mathbf{x}_i^+; p) - \partial_t V^B(\mathbf{x}_i^+; p) - \frac{\sigma(\mathbf{x}_i^+)^2}{2}\partial_x^2 V^B(\mathbf{x}_i^+; p) \ge 0$, while $c + rV^B(\mathbf{x}_i^-; p) - \partial_t V^B(\mathbf{x}_i^-; p) - \frac{\sigma(\mathbf{x}_i^-)^2}{2}\partial_x^2 V^B(\mathbf{x}_i^-; p) = 0$ for all $i \ge 0$. But $V^B(\mathbf{x}_i^+; p) = x_i^+ - p_{t_i}$, so $\frac{\sigma(\mathbf{x}_i^+)^2}{2}\partial_x^2 V^B(\mathbf{x}_i^+; p) = 0$, hence it follows from the continuous differentiability of $V^B(., .; p)$ across the boundary $\partial\Omega$ that $\partial_x^2 V^B(t, \bar{V}_t[p]; p) \ge 0$. Similarly, we have that $\partial_x^2 V^B(t, \underline{V}_t[p]; p) \ge 0$.

It follows from (44) and (46) that the sign of $\bar{S}_t$ and $\underline{S}_t$ are opposite to the sign of $\partial_x V_1^B(t, \bar{V}_t[p])/K$ and $\partial_x V_1^B(t, \underline{V}_t[p])/K$, respectively. So, $\bar{S}_t \le 0$ and $\underline{S}_t \ge 0$ for $t \in \mathbb{R}$. $\square$

Proof of Lemma 2. Consider a fixed $(t, x) \in \mathbb{R} \times [\underline{\pi}, \bar{\pi}]$, and suppose that $V^B(t, x; q) \le V^B(t, x; p)$. For an arbitrary $\varepsilon > 0$, let $\tau_{t,x,\varepsilon}[p] \in \mathcal{T}$ be such that $\mathcal{V}^B(t, x; \tau_{t,x,\varepsilon}[p], p) \ge V^B(t, x; p) - \varepsilon$, then $V^B(t, x; q) \ge \mathcal{V}^B(t, x; \tau_{t,x,\varepsilon}[p], q) > \mathcal{V}^B(t, x; \tau_{t,x,\varepsilon}[p]; p) - \max_{s \ge t} e^{-r(s-t)}|p_s - q_s| \ge V^B(t, x; p) - \max_{s \ge t} e^{-r(s-t)}|p_s - q_s| - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, it must be the case that $V^B(t, x; p) \ge V^B(t, x; q) \ge V^B(t, x; p) - \max_{s \ge t} e^{-r(s-t)}|p_s - q_s|$.

If $V^B(t, x; q) \ge V^B(t, x; p)$, then we simply switch the role of p, q and follow through with the above argument, hence we get that $|V^B(t, x; p) - V^B(t, x; q)| \le \max_{s \ge t} e^{-r(s-t)}|p_s - q_s|$, which proves the result. $\square$

Proof of Proposition 3. Let $p^T, l_{\mathbf{x}}^T \in \mathcal{P}_T$ be given by some pricing strategies which coincide with $p, l_{\mathbf{x}}$ over $[0, T - \varepsilon]$ and constant for all $t \ge T$. By Lemma 2, we have $|V^B(t, x; p^T) - V^B(t, x; l_{\mathbf{x}}^T)| \le \max_{s \in [t,T]} e^{-r(s-t)}|p_s - l_{\mathbf{x},s}| \le \max_{s \ge t} e^{-r(s-t)}|p_s - l_{\mathbf{x},s}|$. Since this inequality holds for all T, we conclude that $|V^B(t, x; p) - V^B(t, x; l_{\mathbf{x}})| \le \max_{s \ge t} e^{-r(s-t)}|p_s - l_{\mathbf{x},s}|$. But from Taylor's Theorem, we have $|p_s - l_{\mathbf{x},s}| \le \frac{M}{2}(s - t)^2$ for all $s \ge t$. It follows that $\max_{s \ge t} e^{-r(s-t)}|p_s - l_{\mathbf{x},s}| \le \frac{M}{2} \max_{s \ge t}(s - t)^2 e^{-r(s-t)} = \frac{2M}{r^2}e^{-2}$. Therefore, $|V^B(t, x; p) - V^B(t, x; l_{\mathbf{x}})| < \varepsilon$ if $r > e^{-1}\sqrt{2M/\varepsilon}$. $\square$

Proof of Corollary 2. The existence and uniqueness of the ODE boundary value problems (20) and (21) follows from the standard theory (Agarwal et al., 2008, Lecture 40). In order to make use of Proposition 2, let us first fix a large $T \ge 0$ and consider $p^T = p_0 + \sqrt{\varepsilon}h^T \in \mathcal{P}_T$

where $h^T \in \mathcal{P}_T$ is given by $h^T_t = Kt$ for $t \in [0, T-\varepsilon]$, constant $h^T_t = KT$ for $t \geq T$, and some in–between smooth transition for $t \in (T-\varepsilon, T)$. We shall assume that $|(h^T)'_t| \leq 1$ for $t \in (T-\varepsilon, T)$. From Proposition 2, we have the following probabilistic expression:

$$\begin{aligned} V_1^B(t,x;p^T) = &-\mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}} (h^T)'_s e^{-r(s-t)} \partial_x V_0^B(v_s^{t,x};p_0) ds | \mathscr{F}_t\right] \\ &+ \mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}} h^T_s e^{-r(s-t)} \sigma(v_s^{t,x})\sigma'(v_s^{t,x}) \partial_x^2 V_0^B(v_s^{t,x};p_0) ds | \mathscr{F}_t\right], \end{aligned} \tag{49}$$

where $\tau_\Omega^{t,x} := \inf\{t' \geq t \mid (t', v_{t'}^{t,x}) \notin \Omega\}$ is the stopping time. We note that $v_s^{t,x}$ is bounded inside $[\underline{V}[p_0], \bar{V}[p_0]]$, while $\int_0^\infty |h^T_s e^{-r(s-t)}| ds \leq Ke^{rt}/r^2$, and $\int_0^\infty |(h^T)'_s e^{-r(s-t)}| ds \leq Ke^{rt}/r$ by construction for all $T \geq 0$. Therefore, by taking the limit $T \to \infty$ of (49), we have by Lemma 2 and inquality (19) that $V_1^B(t,x;p^T) \to V_1^B(t,x)$, and by applying the Dominated Convergence Theorem to the right–hand–side with $h^T_s \to Ks, (h^T)'_s \to K$, we obtain:

$$\begin{aligned} V_1^B(t,x) = &-K \cdot \mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}} e^{-r(s-t)} \partial_x V_0^B(v_s^{t,x};p_0) ds | \mathscr{F}_t\right] \\ &+ K \cdot \mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}} s e^{-r(s-t)} \sigma(v_s^{t,x})\sigma'(v_s^{t,x}) \partial_x^2 V_0^B(v_s^{t,x};p_0) ds | \mathscr{F}_t\right] \end{aligned} \tag{50}$$

We rewrite this further as follows:

$$\begin{aligned} V_1^B(t,x) = &-K \cdot \mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}} e^{-r(s-t)} \partial_x V_0^B(v_s^{t,x};p_0) ds | \mathscr{F}_t\right] \\ &+ K \cdot \mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}} (s-t) e^{-r(s-t)} \sigma(v_s^{t,x})\sigma'(v_s^{t,x}) \partial_x^2 V_0^B(v_s^{t,x};p_0) ds | \mathscr{F}_t\right] \\ &+ Kt \cdot \mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}} e^{-r(s-t)} \sigma(v_s^{t,x})\sigma'(v_s^{t,x}) \partial_x^2 V_0^B(v_s^{t,x};p_0) ds | \mathscr{F}_t\right] \\ = &\, K \cdot \mathbb{E}\left[\int_0^{\tau_\Omega^{x}} s e^{-rs} \sigma(v_s^{x})\sigma'(v_s^{x}) \partial_x^2 V_0^B(v_s^{x};p_0) ds | \mathscr{F}_0\right] - K \cdot \mathbb{E}\left[\int_0^{\tau_\Omega^{x}} e^{-rs} \partial_x V_0^B(v_s^{x};p_0) ds | \mathscr{F}_0\right] \\ &+ Kt \cdot \mathbb{E}\left[\int_0^{\tau_\Omega^{x}} e^{-rs} \sigma(v_s^{x})\sigma'(v_s^{x}) \partial_x^2 V_0^B(v_s^{x};p_0) ds | \mathscr{F}_0\right] =: V_1^B(0,x) + t\widetilde{V}_{1,1}^B(x). \end{aligned} \tag{51}$$

Note that the above expression is linear in t, in particular, the first two terms $V_1^B(0,x)$ and the factor $\widetilde{V}_{1,1}^B(x)$ of t in the last term are functions of x only. The boundary conditions $V_1^B(0, \bar{V}[p_0]) = V_1^B(0, \underline{V}[p_0]) = 0$ and $\widetilde{V}_{1,1}^B(\bar{V}[p_0]) = \widetilde{V}_{1,1}^B(\underline{V}[p_0]) = 0$ are evident from their probabilistic expression definitions. We recognize the probabilistic expression for $\widetilde{V}^B_{1,1}$ as that of the solution to the boundary–value problem (20), thus, we have $\widetilde{V}^B_{1,1} = V^B_{1,1}$. Let us define:

$$V^B_{1,1}(x;\beta) := K \cdot \mathbb{E}\left[\int_0^{\tau^x_\Omega} e^{-\beta s}\sigma(v^x_s)\sigma'(v^x_s)\partial^2_x V^B_0(v^x_s;p_0)ds\right],$$

which satisfies the ODE:

$$\frac{\sigma(x)^2}{2}\partial^2_x V^B_{1,1}(x;\beta) - \beta V^B_{1,1}(x;\beta) + K\sigma(x)\sigma'(x)\partial^2_x V^B_0(x;p_0) = 0. \tag{52}$$

We can see that $V^B_{1,1}(x) := V^B_{1,1}(x;\beta = r)$, and that the first term of (51) is given by $-\partial_\beta V^B_{1,1}(x;\beta)|_{\beta=r}$. Therefore, $V^B_1(0,x)+\partial_\beta V^B_{1,1}(x;\beta)|_{\beta=r} = -K\cdot\mathbb{E}\left[\int_0^{\tau^x_\Omega} e^{-rs}\partial_x V^B_0(v^x_s;p_0)ds\right]$. Substituting this into the corresponding ODE of the probability expression on the RHS of the above, we get:

$$\frac{\sigma(x)^2}{2}\partial^2_x\left(V^B_1(0,x)+\partial_\beta V^B_{1,1}(x;\beta)|_{\beta=r}\right) - r\left(V^B_1(0,x)+\partial_\beta V^B_{1,1}(x;\beta)|_{\beta=r}\right) - K\partial_x V^B_0(x;p_0) = 0$$
$$\implies \frac{\sigma(x)^2}{2}\partial^2_x V^B_1(0,x) - rV^B_1(0,x) + V^B_{1,1}(x) - K\partial_x V^B_0(x;p_0) = 0,$$

where we used $V^B_{1,1}(x) = \frac{\sigma(x)^2}{2}\partial^2_x\partial_\beta V^B_{1,1}(x;\beta)|_{\beta=r} - \beta\partial_\beta V^B_{1,1}(x;\beta)|_{\beta=r}$, which is obtained by differentiating (52) at $\beta = r$. Note that although $V^B_0(x;p_0)$ depends on r, it does not depends on β, hence its β derivative vanishes. Therefore, $V^B_1(0,.)$ satisfies the ODE (21) with the specified boundary conditions, therefore it must coincide with $V_{1,0}$. $\square$

Proof of Proposition 4. In the special case of linear pricing $t \mapsto p_t := p_0 + \sqrt{\varepsilon}Kt$ the value function takes the form (22) over Ω as we can directly check that it satisfies the PDE of (14). Let's define $K_\pm := \frac{\sqrt{\varepsilon}K\pm\sqrt{\varepsilon K^2+2r\sigma^2}}{\sigma^2}$ for convenience. The purchase and quitting boundaries ansatz take the form (23). We determine the unknown $A_1, A_2, \bar{V}[\sqrt{\varepsilon}K]$, and $\underline{V}[\sqrt{\varepsilon}K]$ from the boundary conditions

$$V^B(t,\bar{V}_t) = \bar{V}_t - p_t \implies A_1 e^{K_-\bar{V}[\sqrt{\varepsilon}K]} + A_2 e^{K_+\bar{V}[\sqrt{\varepsilon}K]} - \frac{c}{r} = \bar{V}[\sqrt{\varepsilon}K] \tag{53}$$

$$\partial_x V^B(t,\bar{V}_t) = 1 \implies A_1K_- e^{K_-\bar{V}[\sqrt{\varepsilon}K]} + A_2K_+ e^{K_+\bar{V}[\sqrt{\varepsilon}K]} = 1 \tag{54}$$

$$V^B(t,\underline{V}_t) = 0 \implies A_1 e^{K_-\underline{V}[\sqrt{\varepsilon}K]} + A_2 e^{K_+\underline{V}[\sqrt{\varepsilon}K]} - \frac{c}{r} = 0 \tag{55}$$

$$\partial_x V^B(t,\bar{V}_t) = 0 \implies A_1K_- e^{K_-\underline{V}[\sqrt{\varepsilon}K]} + A_2K_+ e^{K_+\underline{V}[\sqrt{\varepsilon}K]} = 0 \tag{56}$$

From (55) and (56) we find that

$$A_1 = \frac{c}{r}\left(\frac{K_+}{K_+ - K_-}\right)e^{-K_-\underline{V}[\sqrt{\varepsilon}K]}, \quad A_2 = \frac{c}{r}\left(\frac{K_-}{K_- - K_+}\right)e^{-K_+\underline{V}[\sqrt{\varepsilon}K]}. \tag{57}$$

Substituting (57) back into (54), we obtain the equation to be solved for $(\bar{V}[\sqrt{\varepsilon}K]-\underline{V}[\sqrt{\varepsilon}K])$:

$$e^{K_+(\bar{V}[\sqrt{\varepsilon}K]-\underline{V}[\sqrt{\varepsilon}K])} - e^{K_-(\bar{V}[\sqrt{\varepsilon}K]-\underline{V}[\sqrt{\varepsilon}K])} = \frac{r}{c}\cdot\frac{K_- - K_+}{K_-K_+}, \tag{58}$$

we note that the LHS is an increasing function, hence the solution always exists. Finally, we find $\bar{V}[\sqrt{\varepsilon}K]$ by substituting (57) back into (53) and simplify:

$$\bar{V}[\sqrt{\varepsilon}K] = \frac{1}{K_-} + \frac{c}{r}\left(e^{K_+(\bar{V}[\sqrt{\varepsilon}K]-\underline{V}[\sqrt{\varepsilon}K])} - 1\right) \tag{59}$$

from this it is simple to find $\underline{V}[\sqrt{\varepsilon}K]$. Equation (58) and (59) is equivalent to the following non–linear system of equations:

$$\begin{cases} e^{\frac{\sqrt{\varepsilon}K+\sqrt{\varepsilon K^2+2r\sigma^2}}{\sigma^2}(\bar{V}[\sqrt{\varepsilon}K]-\underline{V}[\sqrt{\varepsilon}K])} - e^{\frac{\sqrt{\varepsilon}K-\sqrt{\varepsilon K^2+2r\sigma^2}}{\sigma^2}(\bar{V}[\sqrt{\varepsilon}K]-\underline{V}[\sqrt{\varepsilon}K])} & = \dfrac{\sqrt{\varepsilon K^2+2r\sigma^2}}{c} \\ \dfrac{c}{r}\left(e^{\frac{\sqrt{\varepsilon}K+\sqrt{\varepsilon K^2+2r\sigma^2}}{\sigma^2}(\bar{V}[\sqrt{\varepsilon}K]-\underline{V}[\sqrt{\varepsilon}K])} - 1\right) - \bar{V}[\sqrt{\varepsilon}K] & = \dfrac{\sqrt{\varepsilon}K+\sqrt{\varepsilon K^2+2r\sigma^2}}{2r} \end{cases} \tag{60}$$

When $\sqrt{\varepsilon}K \sim 0$, we may obtain a simple expression for $\bar{V}[\sqrt{\varepsilon}K]$ and $\underline{V}[\sqrt{\varepsilon}K]$ to the ε–order. We substituting the ansatz (24) into (58), (59), and comparing the zeroth–order and $\sqrt{\varepsilon}$–order terms we get the claimed expression for $\bar{S} := \bar{R}/K, \underline{S} := \underline{R}/K$. The signs of $\bar{S}$ and $\underline{S}$ followed from the Proposition 2, but one can also verify explicitly. □

Proof of Proposition 5. The solution (25) and (26) to (21) and (20) can be obtained using standard ODE solving techniques such as the "variation of parameters". The rest of the results are taken care of by Corollary 2. □

Proof of Lemma 3. Part 1: Assumption 1 implies that $\sigma(x) \geq \underline{\sigma}, \forall x \in [\underline{V}[p_0], \bar{V}[p_0]]$, for some constant $\underline{\sigma} > 0$. Using the Dubins–Schwarz theorem, by computing the survival probability of the standard Brownian motion from the Heat equation series solution, we have

$$\mathbb{P}\left[\tau^*[p_0] > T\right] \leq C\cdot\exp\left(-\frac{\pi^2\underline{\sigma}^2}{2(\bar{V}[p_0]-\underline{V}[p_0])^2}\cdot T\right), \text{ for some constant } C,\ \forall T > 0. \tag{61}$$

This implies that $\mathbb{P}[\tau^*[p_0] < \infty] = 1$ and $\mathbb{E}[\tau^*[p_0]] < \infty$. Therefore, $\mathbb{P}[v^x_{\tau^*[p_0]} \geq \bar{V}[p_0]] + \mathbb{P}[v^x_{\tau^*[p_0]} \leq \bar{V}[p_0]] = 1$. Since $\{v^x_{t\wedge\tau^*[p_0]}\}_{t\geq 0}$ is a uniformly integrable martingale, by the

Martingale Stopping Theorem, $x = v_0^x = \mathbb{E}[v^x_{\tau^*[p_0]}] = \bar{V}[p_0]\mathbb{P}[v^x_{\tau^*[p_0]} \geq \bar{V}[p_0]] + \underline{V}[p_0]\mathbb{P}[v^x_{\tau^*[p_0]} \leq \bar{V}[p_0]]$. If $m = 0$ then $\mathcal{V}^S(x; p_0) = (p_0 - g)\mathbb{P}[v^x_{\tau^*[p_0]} \geq \bar{V}[p_0]]$. One can prove the first part by solving the system of linear equations for $\mathbb{P}[v^x_{\tau^*[p_0]} \geq \bar{V}[p_0]]$.

Part 2: We consider $v_t = \sigma W_t$, where $\{W_t\}_{t\geq 0}$ is a standard Brownian motion. Then, $\mathcal{V}^S(x; p_0) = (p_0 - g)\mathbb{E}[e^{-m\tau^*[p_0]} \cdot 1\{v^x_{\tau^*[p_0]} = \bar{V}[p_0]\}]$, which can be evaluated using the standard technique involving Martingale Stopping Theorem (see Karatzas and Shreve (2012)). □

Proof of Theorem 1. Let us first fix a large $T \geq 0$, and let us assume that $\mathcal{V}^S_T(.) := \mathcal{V}^S(T, .; p) : [\underline{V}_T[p], \bar{V}_T[p]] \to \mathbb{R}$ is known and can be used as the terminal condition. We would like to solve the PDE initial boundary value problem (30) up to the ε–order. The idea is similar to the proof of Proposition 2 but simpler since the boundaries $\bar{V}_t[p] = \bar{V}[p_0] + \sqrt{\varepsilon}Kt + \sqrt{\varepsilon}\bar{R}_t$, and $\underline{V}_t[p] = \underline{V}[p_0] + \sqrt{\varepsilon}Kt + \sqrt{\varepsilon}\underline{R}_t$ are already determined for us by Corollary 2. The claim is that if $\mathcal{V}^S(., .; p)$ solves (30) exactly for the given $\bar{V}[p], \underline{V}[p]$, and if $\mathcal{V}^\varepsilon_{\leq k}(., .; p)$ solves (30) up to the $\varepsilon^{(k+1)/2}$–order with the same given $\bar{V}[p], \underline{V}[p]$, then by comparing their corresponding Feynman–Kac expressions, we have $\mathcal{V}^S = \mathcal{V}^\varepsilon_{\leq k} + O(\varepsilon^{(k+1)/2})$. We omit further details, and proceed with $k = 1$ to obtain the seller's expected payoff up to the ε–order.

Consider $p := p_0 + \sqrt{\varepsilon}h$, where $h_t = Kt$. We propose the perturbation ansatz:

$$
\begin{aligned}
\mathcal{V}^S(t, x; p) &= \mathcal{V}^S_0\left(\frac{\bar{V}[p_0] - \underline{V}[p_0]}{\bar{V}_t[p] - \underline{V}_t[p]}(x - \underline{V}_t[p]) + \underline{V}[p_0]; p_0\right) + \sqrt{\varepsilon}\mathcal{V}^S_1(t, x) + O(\varepsilon) \\
&= \mathcal{V}^S_0\left((1 - \sqrt{\varepsilon}r_{1,t})x - \sqrt{\varepsilon}r_{0,t}; p_0\right) + \sqrt{\varepsilon}\mathcal{V}^S_1(t, x) + O(\varepsilon),
\end{aligned}
\tag{62}
$$

where, in the second equality, we expanded the argument of $\mathcal{V}^S_0(.; p_0)$ to the first order in $\sqrt{\varepsilon}$ and we define

$$
\begin{aligned}
r_{1,t} &:= \frac{\bar{R}_t - \underline{R}_t}{\bar{V}[p_0] - \underline{V}[p_0]} = K \cdot \frac{\bar{S}_{0,0} - \underline{S}_{0,0} + (\bar{S}_{0,1} - \underline{S}_{0,1})t}{\bar{V}[p_0] - \underline{V}[p_0]} \\
r_{0,t} &:= h_t + \underline{R}_t - r_{1,t}\underline{V}[p_0] \\
&= K \cdot \left(\underline{S}_{0,0} - \frac{\bar{S}_{0,0} - \underline{S}_{0,0}}{\bar{V}[p_0] - \underline{V}[p_0]}\underline{V}[p_0]\right) + Kt \cdot \left(1 + \underline{S}_{0,1} - \frac{\bar{S}_{0,1} - \underline{S}_{0,1}}{\bar{V}[p_0] - \underline{V}[p_0]}\underline{V}[p_0]\right)
\end{aligned}
\tag{63}
$$

where $\bar{S}_{0,0}, \bar{S}_{0,1}, \underline{S}_{0,0}, \underline{S}_{0,1}$ are as defined in Corollary 2. The first term represents a naive rescaling of the constant price solution according to the buyer's response moving boundaries. Substituting the ansatz into the PDE (30) and collect the $\sqrt{\varepsilon}$–terms, we obtain the PDE for $\mathcal{V}^S_1$:

$$\frac{\sigma(x)^2}{2}\partial_x^2\mathcal{V}_1^S(t,x) + \partial_t\mathcal{V}_1^S(t,x) - m\mathcal{V}_1^S(t,x) + \left(\sigma(x)\sigma'(x)(r_{1,t}x + r_{0,t}) - \sigma(x)^2 r_{1,t}\right)\partial_x^2\mathcal{V}_0^S(x;p_0) - (r'_{1,t}x + r'_{0,t})\partial_x\mathcal{V}_0^S(x;p_0) = 0,$$

along with the boundary conditions up to the ε–order:

$$\mathcal{V}^S(t,\bar{V}_t[p];p) = p_t - g \implies \mathcal{V}_0^S(\bar{V}[p_0];p_0) + \sqrt{\varepsilon}\mathcal{V}_1^S(t,\bar{V}[p_0]) + O(\varepsilon) = p_0 + \sqrt{\varepsilon}h_t - g \implies \mathcal{V}_1^S(t,\bar{V}[p_0]) = h_t$$

$$\mathcal{V}^S(t,\underline{V}_t[p];p) = 0 \implies \mathcal{V}_0^S(\underline{V}[p_0];p_0) + \sqrt{\varepsilon}\mathcal{V}_1^S(t,\underline{V}[p_0]) + O(\varepsilon) = 0 \implies \mathcal{V}_1^S(t,\underline{V}[p_0]) = 0,$$

and finally the terminal condition at T: $\mathcal{V}^S(T,x;p) = \mathcal{V}_T^S(x)$, gives

$$\mathcal{V}_1^S(T,x) = \frac{1}{\sqrt{\varepsilon}}\left(\mathcal{V}_T^S(x) - \mathcal{V}^S((1-\sqrt{\varepsilon}r_{1,T})x - \sqrt{\varepsilon}r_{0,T};p_0)\right) + O(\sqrt{\varepsilon}).$$

We can reverse the time–axis, then apply (Evans, 2022, Chapter 7, Theorem 7) or (Friedman, 2008, Chapter 3, §5, Corollary 2) to conclude the existence of the smooth solution $\mathcal{V}_1^S$ to the parabolic initial boundary–value problem with fixed boundaries $\bar{V}[p_0], \underline{V}[p_0]$. The solution is unique and admits the following probabilistic expression via the semi–elliptic version of Feynman–Kac Formula (Øksendal, 2003, Theorem 9.1.1):

$$\begin{aligned}\mathcal{V}_1^S(t,x) = \mathbb{E}&\left[e^{-m(T-t)}\mathcal{V}_1^S(T,v_T^{t,x})\cdot 1\left\{\tau_\Omega^{t,x} \geq T\right\}|\mathscr{F}_t\right] \\ &+ \mathbb{E}\left[h_{\tau_\Omega^{t,x}}e^{-m(\tau_\Omega^{t,x}-t)}\cdot 1\left\{v^{t,x}_{\tau_\Omega^{t,x}} \geq \bar{V}[p_0], \tau_\Omega^{t,x} < T\right\}|\mathscr{F}_t\right] \\ &- \mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}\wedge T} e^{-m(s-t)}(r'_{1,s}v_s^{t,x} + r'_{0,s})\partial_x\mathcal{V}_0^S(v_s^{t,x};p_0)ds|\mathscr{F}_t\right] \\ + \mathbb{E}&\left[\int_t^{\tau_\Omega^{t,x}\wedge T} e^{-m(s-t)}\left(\sigma(v_s^{t,x})\sigma'(v_s^{t,x})(r_{1,s}v_s^{t,x} + r_{0,s}) - \sigma(v_s^{t,x})^2 r_{1,s}^T\right)\partial_x^2\mathcal{V}_0^S(v_s^{t,x};p_0)ds|\mathscr{F}_t\right]. \end{aligned} \quad (64)$$

Under the assumption that the seller is perfectly patient: $m = 0$, we have from Lemma 3 that $\mathcal{V}_0^S(x;p_0) = (p_0 - g)\frac{x - \underline{V}[p_0]}{\bar{V}[p_0] - \underline{V}[p_0]}$. This means that the fourth term of (64) vanishes and (64) simplified to:

$$\begin{aligned}\mathcal{V}_1^S(t,x) = \mathbb{E}\left[\mathcal{V}_1^S(T,v_T^{t,x})\cdot 1\left\{\tau_\Omega^{t,x} \geq T\right\}|\mathscr{F}_t\right] + \mathbb{E}\left[h_{\tau_\Omega^{t,x}}\cdot 1\left\{v^{t,x}_{\tau_\Omega^{t,x}} \geq \bar{V}[p_0], \tau_\Omega^{t,x} < T\right\}|\mathscr{F}_t\right] \\ - \frac{p_0 - g}{\bar{V}[p_0] - \underline{V}[p_0]}\mathbb{E}\left[\int_t^{\tau_\Omega^{t,x}\wedge T'}(r'_{1,s}v_s^{t,x} + r'_{0,s})ds|\mathscr{F}_t\right], \end{aligned} \quad (65)$$

We can further simplify the third term as follows:

$$
\begin{aligned}
&-\frac{p_0-g}{\bar{V}[p_0]-\underline{V}[p_0]}\mathbb{E}\left[\int_t^{\tau_\Omega^x\wedge T}(r'_{1,s}v_s^x+r'_{0,s})ds|\mathscr{F}_t\right]\\
&\qquad=-\frac{p_0-g}{\bar{V}[p_0]-\underline{V}[p_0]}\mathbb{E}\left[\int_t^{\tau_\Omega^x\wedge T}d(r_{1,s}v_s^x)-\int_t^{\tau_\Omega^x\wedge T}r_{1,s}dv_s^x+\int_t^{\tau_\Omega^x\wedge T}r'_{0,s}ds|\mathscr{F}_t\right]\\
&\qquad\qquad=-\frac{p_0-g}{\bar{V}[p_0]-\underline{V}[p_0]}\left(\mathbb{E}\left[r_{1,\tau_\Omega^x\wedge T}v^x_{\tau_\Omega^x\wedge T}+r_{0,\tau_\Omega^x\wedge T}|\mathscr{F}_t\right]-(r_{1,t}x-r_{0,t})\right).
\end{aligned}
$$

We used Ito's Lemma in the first equality. For the second equality, note that $\{v_t^x\}_{t\geq 0}$ is a square–integrable martingale, hence we know that $\int_0^{\tau_\Omega^x} r_{1,s}dv_s^x$ is a continuous square–integrable martingale, therefore its expectation vanishes.

Since h is linear in t, we find that $\mathcal{V}_1^S(T,x)$ is at most linear in T, and we have already seen from (63) that $r_{1,t}$, $r_{0,t}$ are linear in t. Meanwhile, $v_s^{t,x}$ is bounded inside $[\underline{V}[p_0],\bar{V}[p_0]]$. Therefore, the first term is upper–bounded by $\sup_{x\in[\underline{V}[p_0],\bar{V}[p_0]]}\mathcal{V}_1^S(T,x)\mathbb{P}\left[\tau_\Omega^{t,x}\geq T|\mathscr{F}_t\right]\to 0$ as $T\to\infty$ since $\mathbb{P}\left[\tau_\Omega^{t,x}\geq T|\mathscr{F}_t\right]\to 0$ exponentially according to equation (61). Both the second and third terms are upper–bounded by some constant multiple of $\mathbb{E}[\tau_\Omega^{t,x}|\mathscr{F}_t]<\infty$. Taking the limit $T\to\infty$ using the Dominated Convergence Thoerem, then set $t=0$, we obtain the expression for $\mathcal{V}_1^S(0,x)$. Substituting the expression for $\mathcal{V}_1^S(0,x)$ into the perturbative expansion (62) at $t=0$, we obtain $\mathcal{V}^S(x;p_0,K)=(p_0-g)\frac{x-\underline{V}[p_0]}{\bar{V}[p_0]-\underline{V}[p_0]}+\sqrt{\varepsilon}K\mathbb{E}\left[\tau_\Omega^x\cdot 1\left\{v^x_{\tau_\Omega^x}\geq\bar{V}[p_0]\right\}\right]-\frac{\sqrt{\varepsilon}(p_0-g)}{\bar{V}[p_0]-\underline{V}[p_0]}\mathbb{E}\left[r_{1,\tau_\Omega^x}v^x_{\tau_\Omega^x}+r_{0,\tau_\Omega^x}\right]+O(\varepsilon)$, which leads to (31) after some simplifications and substitution of (63). □

Proof of Proposition 6. Part 1:

From the first equation of (60), when $c\searrow 0$, the RHS becomes large which means $\bar{V}[\sqrt{\varepsilon}K]-\underline{V}[\sqrt{\varepsilon}K]$ becomes large, and the LHS is $\sim e^{\frac{\sqrt{\varepsilon}K+\sqrt{\varepsilon K^2+2r\sigma^2}}{\sigma^2}(\bar{V}[\sqrt{\varepsilon}K]-\underline{V}[\sqrt{\varepsilon}K])}$. Therefore, the second equation of (60) together with (23) gives: $\bar{V}_t=p_0+\sqrt{\varepsilon}Kt+\frac{\sqrt{\varepsilon K^2+2r\sigma^2}-\sqrt{\varepsilon}K}{2r}$ and $\underline{V}_t=-\infty$. Therefore, we only have one linearly moving boundary $\bar{V}_t$. Let's assume throughout also that $p_0\geq g$. The solution $U(t,v)$ to the heat equation with the single linearly moving absorbing boundary with initial condition $U(t=0,v)=\delta(v-x), x\leq\bar{V}_0$, is well–known: $U(t,v)=\frac{\exp\left(-\frac{\sqrt{\varepsilon}K}{\sigma^2}(v-x-\sqrt{\varepsilon}Kt)-\frac{\varepsilon K^2}{2\sigma^2}t\right)}{\sigma\sqrt{2\pi t}}\left(e^{-\frac{(v-\sqrt{\varepsilon}Kt-x)^2}{2t\sigma^2}}-e^{-\frac{(v-\sqrt{\varepsilon}Kt+x-2\bar{V}_0)^2}{2t\sigma^2}}\right)$. Therefore, the purchase probability flux is $-\frac{\sigma^2}{2}\partial_vU(t,\bar{V}_t)=\frac{\bar{V}_0-x}{\sigma\sqrt{2\pi t^3}}\exp[-\frac{\left(\bar{V}_t-x\right)^2}{2t\sigma^2}]$.

It is now straightforward to compute the expected seller's payoff at $t=0$:

$$
\mathcal{V}^S(x;p_0,K):=-\frac{\sigma^2}{2}\int_0^\infty e^{-ms}(p_s-g)\partial_vU(s,\bar{V}_s)ds
$$

$$= \left(p_0 - g + \frac{\sqrt{\varepsilon}K}{\sqrt{2m\sigma^2 + \varepsilon K^2}} \left(p_0 - x + \frac{\sqrt{\varepsilon K^2 + 2r\sigma^2} - \sqrt{\varepsilon}K}{2r}\right)\right)$$
$$\times \exp\left(-\left(\frac{\sqrt{\varepsilon}K + \sqrt{2m\sigma^2 + \varepsilon K^2}}{\sigma^2}\right)\left(p_0 - x + \frac{\sqrt{\varepsilon K^2 + 2r\sigma^2} - \sqrt{\varepsilon}K}{2r}\right)\right), \quad (66)$$

for $x \leq \bar{V}_0$, and $\mathcal{V}^S(x; p_0, K) = p_0 - g$ if $x > \bar{V}_0$.

In the special case where $m = 0$, we have $\mathcal{V}^S(x; p_0, K) = (2p_0 - g - x + \frac{\sqrt{\varepsilon K^2+2r\sigma^2}-\sqrt{\varepsilon}K}{2r}) \exp[-\frac{2\sqrt{\varepsilon}K}{\sigma^2}(p_0 - x + \frac{\sqrt{\varepsilon K^2+2r\sigma^2}-\sqrt{\varepsilon}K}{2r})]$, if $\sqrt{\varepsilon}K > 0$, $\mathcal{V}^S(x; p_0, K) = p_0 - g$ if $\sqrt{\varepsilon}K = 0$, and $\mathcal{V}^S(x; p_0, K) = x - g - \frac{\sqrt{\varepsilon K^2+2r\sigma^2}-\sqrt{\varepsilon}K}{2r}$ if $\sqrt{\varepsilon}K < 0$. For any fixed p_0, we can approach the supremum $2p_0 + \frac{\sigma}{\sqrt{2r}} - g - x \geq p_0 - g$ of $\mathcal{V}^S$ by choosing $\sqrt{\varepsilon}K \gtrsim 0$ as close to 0 as possible, and earning an extra of $\left(2p_0 + \frac{\sigma}{\sqrt{2r}} - g - x\right) - (p_0 - g) = p_0 - x + \frac{\sigma}{\sqrt{2r}}$.

Part 2:

For $m > 0$, the optimal K is now bounded from 0. This can be seen by computing:

$$\frac{\partial \mathcal{V}^S}{\partial K}(x; p_0, K = 0) = e^{-\frac{\sqrt{2m}}{\sigma}\left(p_0 - x + \frac{\sigma}{\sqrt{2r}}\right)}$$
$$\times \left(\frac{p_0 - x + \sigma/\sqrt{2r}}{\sigma\sqrt{2m}} - (p_0 - g)\left(\frac{p_0 - x + \sigma/\sqrt{2r} - (\sigma/r)\sqrt{m/2}}{\sigma^2}\right)\right),$$

we can see that this is always > 0 for sufficiently small and sufficiently large $m > 0$. □

Proof of Proposition 7. It is known that, in addition to the Kolmogorov backward equation (28), the transition probability $U(t, v) = U(t, v; t = 0, x)$ satisfies the backward equation: $\frac{1}{2}\partial_v^2(\sigma(v)^2 U(t, v)) - \partial_t U(t, v) = 0$ with 2 absorbing boundaries $\bar{V}_t = \bar{V}_0 + \sqrt{\varepsilon}Kt, \underline{V}_t = \underline{V}_0 + \sqrt{\varepsilon}Kt$. In the context of this section, we simply have a constant $\sigma(v)^2 = \sigma^2$, hence such an equation is simply the heat equation. The standard solution U_0 to the heat equation with 2 absorbing non–moving boundaries at $\bar{V}_0 := p_0 + \bar{V}[\sqrt{\varepsilon}K]$, $\underline{V}_0 := p_0 + \underline{V}[\sqrt{\varepsilon}K]$, and the initial condition $U_0(0, v) = \delta(v - x)$ is given by Karatzas and Shreve (2012):

$$U_0(t, v) = \frac{1}{\sigma\sqrt{2\pi t}} \sum_{k=-\infty}^{+\infty} \left[e^{-\frac{(v-x+2k(\bar{V}_0-\underline{V}_0))^2}{2t\sigma^2}} - e^{-\frac{(v+x-2\underline{V}_0+2k(\bar{V}_0-\underline{V}_0))^2}{2t\sigma^2}}\right]. \quad (67)$$

By the standard application of Girsanov Theorem, if $\{W_t\}_{t=0}^{\infty}$ is the standard Brownian process on $(\Omega, \mathscr{F}, \Sigma, \mathbb{P})$ then $\{x + \sigma W_t\}_{t=0}^{\infty}$ is the Brownian process with drift starting at x on $(\Omega, \mathscr{F}, \Sigma, \mathbb{Q})$ where $\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathscr{F}_t} = \exp\left(-\frac{\sqrt{\varepsilon}K}{\sigma}W_t - \frac{\varepsilon K^2}{2\sigma^2}t\right)$. Consequently, we have that the solution U to the heat equation with moving boundaries $\bar{V}_t, \underline{V}_t$ are given by: $U(t, v)dv =$

$\exp[-\frac{\sqrt{\varepsilon}K}{\sigma^2}(v-x-\sqrt{\varepsilon}Kt)-\frac{\varepsilon K^2}{2\sigma^2}t]U_0(t,v-\sqrt{\varepsilon}Kt)dv$. So, the purchase probability flux is:

$$-\frac{\sigma^2}{2}\partial_v U(t,\bar{V}_t) = \sum_{k=-\infty}^{+\infty} \frac{(2k+1)(\bar{V}_0-\underline{V}_0)-(x-\underline{V}_0)}{\sigma\sqrt{2\pi t^3}} e^{\frac{2\sqrt{\varepsilon}Kk}{\sigma^2}(\bar{V}_0-\underline{V}_0)} e^{-\frac{((2k+1)(\bar{V}_0-\underline{V}_0)-(x-\underline{V}_0)+\sqrt{\varepsilon}Kt)^2}{2t\sigma^2}}. \tag{68}$$

The term–by–term differentiation is justified at $v=\bar{V}_t$ for any fixed $x\in(\underline{V}_0,\bar{V}_0)$ because $0<|\bar{V}_0-x|<|\bar{V}_0-\underline{V}_0|$, hence the series representation of $U_0(t,v-\sqrt{\varepsilon}Kt)$, and the derivative series both converge absolutely and uniformly for all v in some neighborhoods of $\bar{V}_t$ and $t\in[0,\infty)$. We now compute the seller's expected profit:

Claim 1. *The seller's expected profit from the buyer initially at $x\in(\underline{V}_0,\bar{V}_0)$ is:*

$$\begin{aligned}
\mathcal{V}^S(x;p_0,K) = {} & \frac{\left(p_0-g-\frac{\sqrt{\varepsilon}K}{\sqrt{2m\sigma^2+\varepsilon K^2}}(\bar{V}_0+x-2\underline{V}_0)\right)e^{-\frac{\sqrt{\varepsilon}K+\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(\bar{V}_0-x)}}{1-e^{-\frac{2\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(\bar{V}_0-\underline{V}_0)}} \\
& + \frac{\frac{2\sqrt{\varepsilon}K}{\sqrt{2m\sigma^2+\varepsilon K^2}}(\bar{V}_0-\underline{V}_0)e^{-\frac{\sqrt{\varepsilon}K+\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(\bar{V}_0-x)}}{\left(1-e^{-\frac{2\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(\bar{V}_0-\underline{V}_0)}\right)^2} \\
& - \frac{\left(p_0-g-\frac{\sqrt{\varepsilon}K}{\sqrt{2m\sigma^2+\varepsilon K^2}}(\bar{V}_0-x)\right)e^{-\frac{\sqrt{\varepsilon}K+\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(\bar{V}_0-\underline{V}_0)}e^{+\frac{\sqrt{\varepsilon}K-\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(x-\underline{V}_0)}}{1-e^{-\frac{2\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(\bar{V}_0-\underline{V}_0)}} \\
& - \frac{\frac{2\sqrt{\varepsilon}K}{\sqrt{2m\sigma^2+\varepsilon K^2}}(\bar{V}_0-\underline{V}_0)e^{-\frac{\sqrt{\varepsilon}K+\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(\bar{V}_0-\underline{V}_0)}e^{+\frac{\sqrt{\varepsilon}K-\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(x-\underline{V}_0)}}{\left(1-e^{-\frac{2\sqrt{2m\sigma^2+\varepsilon K^2}}{\sigma^2}(\bar{V}_0-\underline{V}_0)}\right)^2},
\end{aligned} \tag{69}$$

if $m>0$ or $K\neq 0$, and $\mathcal{V}^S(x;p_0,K)=(p_0-g)\left(\frac{x-\underline{V}_0}{\bar{V}_0-\underline{V}_0}\right)$ if $m=0,K=0$. On the other hand, if $x\leq\underline{V}_0$ then $\mathcal{V}^S(x;p_0,K)=0$, and if $x\geq\bar{V}_0$ then $\mathcal{V}^s(x;p_0,K)=p_0-g$.

Proof. Let us only consider the non–trivial case when $x\in(\underline{V}_0,\bar{V}_0)$. The result when $m=0,K=0$ has been covered by Lemma 3. For the cases where $m>0$ or $K\neq 0$, we compute $\mathcal{V}^S(x;p_0,K)$ by substituting (68) into (29), after switching the order of summation and integration, which can be justified by Fubini's theorem when $m>0$ or $K\neq 0$, the resulting infinite series is a standard geometric series which can easily be evaluated to gives (69). □

In the limit $\underline{V}_0\to-\infty$, (69) reduces to (66) we previously studied. Unlike in the single boundary case, in the presence of the quitting boundary, the expected seller's profit is not only continuous at $K=0$, but also differentiable, even when $m=0$, as we will show below.

Consider the case of $m = 0$. According to (69), $\mathcal{V}^S(x; p_0, K < 0)|_{m=0}$ is given by (33), and:

$$\begin{aligned}
&\mathcal{V}^S(x; p_0, K > 0)|_{m=0} = \\
&\quad \frac{(p_0 - g - (\bar{V}_0 + x - 2\underline{V}_0)) \exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0 - x)\right)}{1 - \exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0 - \underline{V}_0)\right)} + \frac{2(\bar{V}_0 - \underline{V}_0) \exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0 - x)\right)}{\left(1 - \exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0 - \underline{V}_0)\right)\right)^2} \\
&- \frac{(p_0 - g - (\bar{V}_0 - x)) \exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0 - \underline{V}_0)\right)}{1 - \exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0 - \underline{V}_0)\right)} - \frac{2(\bar{V}_0 - \underline{V}_0) \exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0 - \underline{V}_0)\right)}{\left(1 - \exp\left(-\frac{2\sqrt{\varepsilon}K}{\sigma^2}(\bar{V}_0 - \underline{V}_0)\right)\right)^2}. \quad (70)
\end{aligned}$$

Both (33) and (70) are valid expressions for all $K \neq 0$, and with some works, we can show them to be equal for all $K \neq 0$. This proves $\mathcal{V}^S(x; p_0, K)$ is given by (33) for all $K \neq 0$. □

Proof of Lemma 4. One can see that $\frac{\partial p_0^*}{\partial K}(x, K = 0) = \frac{1}{12r\sigma}[3\sigma - 3\sqrt{\frac{2c^2}{r} + \sigma^2} \sinh^{-1}\sqrt{\frac{r\sigma^2}{2c^2}} - \sigma(\sinh^{-1}\sqrt{\frac{r\sigma^2}{2c^2}})^2] \leq 0$, where the inequality is strict everywhere except when $r\sigma^2/c^2 = 0$. Given that $r\sigma^2/c^2 > 0$, we can find a sufficiently small $\varepsilon > 0$ such that $p_0^*(x, .)$ is a decreasing function for $K \in [-1, +1]$. Any local maximum point of $\mathcal{V}^S(x; ., .)$ would take the form $(p_0^*(x; K^*), K^*)$ where $K^* := \arg\max_K \mathcal{V}^S(x; p_0^*(x, K), K)$. Hence, for all sufficiently small $\varepsilon > 0$, (p_0^*, K^*) in $\mathcal{P}_{lin}^{\varepsilon}$ either satisfies $p_0^* < \hat{p}_0, K^* \gtrsim 0$, or $p_0^* > \hat{p}_0, K^* \lesssim 0$. □

References

Agarwal, R. P., O'Regan, D., et al. (2008). *An introduction to ordinary differential equations.* Springer.

Anderson, S. P. and Renault, R. (2006). Advertising content. *American Economic Review*, 96(1):93–113.

Armstrong, M., Vickers, J., and Zhou, J. (2009). Prominence and consumer search. *The RAND Journal of Economics*, 40(2):209–233.

Bar-Isaac, H., Caruana, G., and Cuñat, V. (2010). Information gathering and marketing. *Journal of Economics & Management Strategy*, 19(2):375–401.

Branco, F., Sun, M., and Villas-Boas, J. M. (2012). Optimal search for product information. *Management Science*, 58(11):2037–2056.

Callander, S. (2011). Searching and learning by trial and error. *American Economic Review*, 101(6):2277–2308.

Chaimanowong, W. and Ke, T. T. (2022). A simple micro-founded model of repeat buying based on continuous information tracking. *Available at SSRN 3954939.*

Crandall, M. G., Ishii, H., and Lions, P.-L. (1992). User's guide to viscosity solutions of second order partial differential equations. *Bulletin of the American mathematical society*, 27(1):1–67.

Evans, L. C. (2022). *Partial differential equations*, volume 19. American Mathematical Society.

Friedman, A. (2008). *Partial differential equations of parabolic type*. Courier Dover Publications.

Fudenberg, D., Strack, P., and Strzalecki, T. (2018). Speed, accuracy, and the optimal timing of choices. *American Economic Review*, 108(12):3651–3684.

Gilbert, J. P. and Mosteller, F. (1966). Recognizing the maximum of a sequence. *Journal of the American Statistical Association*, 61(313):35–73.

Kamada, Y. and Muto, N. (2015). Multi-agent search with deadline. *working paper*.

Karatzas, I. and Shreve, S. (2012). *Brownian motion and stochastic calculus*, volume 113. Springer Science & Business Media.

Ke, T. T., Tang, W., Villas-Boas, J. M., and Zhang, Y. P. (2022). Parallel search for information in continuous time—optimal stopping and geometry of the pde. *Applied Mathematics & Optimization*, 85(2):3.

Ke, T. T. and Villas-Boas, J. M. (2019). Optimal learning before choice. *Journal of Economic Theory*, 180:383–437.

Libgober, J. and Mu, X. (2021). Informational robustness in intertemporal pricing. *The Review of Economic Studies*, 88(3):1224–1252.

Liptser, R. S. and Shiryaev, A. N. (2013). *Statistics of random processes: I. General theory*, volume 5. Springer Science & Business Media.

Moscarini, G. and Smith, L. (2001). The optimal level of experimentation. *Econometrica*, 69(6):1629–1644.

Øksendal, B. (2003). *Stochastic differential equations*. Springer.

Sakaguchi, M. (1978). When to stop: randomly appearing bivariate target values. *Journal of the Operations Research Society of Japan*, 21(1):45–58.

Smith, L. (1999). Optimal job search in a changing world. *Mathematical Social Sciences*, 38(1):1–9.

Stokey, N. L. (1979). Intertemporal price discrimination. *The Quarterly Journal of Economics*, 93(3):355–371.

Strulovici, B. and Szydlowski, M. (2015). On the smoothness of value functions and the existence of optimal strategies in diffusion models. *Journal of Economic Theory*, 159:1016–1055.

Urgun, C. and Yariv, L. (2025). Contiguous search: Exploration and ambition on uncharted terrain. *Journal of Political Economy*, 133(2):522–567.

Van den Berg, G. J. (1990). Nonstationarity in job search theory. *The review of economic studies*, 57(2):255–277.

Villas-Boas, J. M. (2009). Product variety and endogenous pricing with evaluation costs. *Management Science*, 55(8):1338–1346.

Weitzman, M. L. (1979). Optimal search for the best alternative. *Econometrica: Journal of the Econometric Society*, pages 641–654.

Wolinsky, A. (1986). True monopolistic competition as a result of imperfect information. *The Quarterly Journal of Economics*, 101(3):493–511.

Wong, Y. F. (2025). Forward-looking experimentation of correlated alternatives. *Theoretical Economics*, 20(3):883–909.

Yong, J. and Zhou, X. Y. (2012). *Stochastic controls: Hamiltonian systems and HJB equations*, volume 43. Springer Science & Business Media.

Zhong, W. (2022). Optimal dynamic information acquisition. *Econometrica*, 90(4):1537–1582.