Multi-attribute Search

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Abstract

When considering whether or not to buy a product, consumers often evaluate different attributes of it. Due to limited attention, they usually can search for information about only one attribute at a time. Assuming that a product has two attributes, we study the optimal search strategy of the consumer by endogenizing the optimal attribute to search for, when to keep searching for information, and when to stop and make a decision. We find that it is always optimal for the consumer to search for the attribute about which she has greater uncertainty, due to the faster speed of learning. The consumer only searches for one attribute if she holds a strong prior belief about one of the attributes, and may search for both attributes otherwise. We also characterize the marginal rate of substitution between the values of two attributes and examine how knowledge about one attribute affects the benefits of learning about the other. Lastly, we find that reducing search costs benefits the firm only if the consumer's prior belief is intermediate. In such cases, a lower search cost prevents the consumer from not searching or from prematurely ending the search after negative findings. This allows the consumer more opportunities to receive positive signals and eventually leads to a higher likelihood of purchasing the product.

1 Introduction

When considering whether or not to buy a product, consumers often evaluate different attributes of it. For example, an incoming college student choosing a laptop can learn about the operating system, weight, exterior design, warranty, and other attributes before making a final decision. Learning requires both time and effort, while consumers often have limited attention. Thus, they need to decide which attribute to focus on. Sometimes, this decision is based on exogenous reasons, such as an attribute being more prominent (Bordolo et al. 2013, Zhu and Dukes 2017) or offering a wider range of consumption utility (Kőszegi and Szeidl 2013). If one attribute is significantly more important than the other, the consumer will prioritize information about this attribute. However, when no such external differences exist among attributes, the choice of which attribute to investigate becomes more complex.

Consider a consumer deciding whether to buy a used car. She can gather information on various attributes, such as examining the car's add-on packages through review articles or purchasing a car report for accident history. Each option aids in learning more about the car and informs the decision-making process. However, it takes time and effort to search for such information. The consumer needs to decide which attribute deserves attention first.

Even after deciding which attribute to explore, the consumer won't immediately learn everything about it. Instead, she gradually gathers information about the attribute. For instance, spending half an hour researching the car's safety features might reveal that it has airbags in each seat, but this doesn't provide complete knowledge about the vehicle's safety. She could continue to investigate whether the car has an automatic braking system. Alternatively, the consumer may not want to stick to one attribute. The relative importance of attributes may shift as more information is gathered. How does the value of learning more about one attribute depends on the value of other attributes? For example, how does knowledge about a car's safety impact the benefits of learning about its design? After gathering sufficient positive information about safety, the consumer might find it a better use of her time to switch to other attributes. She may feel confident that the car is safe but remain uncertain whether she will enjoy driving in it. At some point, the consumer may switch to learning more about the car's design. When will she switch to another attribute because the relative importance of attributes evolves with the accumulation of information?

To answer the above questions, this paper considers a consumer deciding whether or not to purchase a good. The good has two attributes, each with independent values. The payoff of purchasing the good is the total value of these attributes minus the price. The consumer initially does not know the actual value of either attribute but holds a prior belief about the value of each. She can incur a cost to search for information about the attributes before making a decision. By receiving a noisy signal about an attribute from searching, she can update her belief about the value of that attribute, and consequently, about the product's overall value. By assuming that the search cost and the informativeness of the signal are the same for each attribute, we ensure that the attributes are symmetric. Therefore, the consumer's preference for searching about one attribute over the other is not influenced by exogenous reasons. The decision about which attribute to search for at any given time is endogenously determined by the expected gain from additional information regarding each attribute.

The consumer will stop searching and buy the good if she becomes sufficiently optimistic about its overall value. Conversely, she will stop searching without purchasing if her assessment becomes too pessimistic. When her belief about the product's value is in between, she will search for more information. We characterize the search region by a set of ordinary differential equations for intermediate beliefs and by a system of equations for extreme beliefs. Our findings indicate that it is always optimal for the consumer to search for the attribute about which the consumer has greater uncertainty, due to the faster speed of learning. The consumer only searches for one attribute if she holds a strong prior belief about one of the attributes, and may search for both attributes otherwise. In the car purchasing example, a consumer might not investigate the safety features of a Volvo due to its well-known safety reputation and focus instead on other attributes. In contrast, a consumer considering a pre-order from Faraday Future, a new manufacturer, likely faces significant uncertainty about all aspects and might investigate every attribute.

The gradual learning model offers valuable insights into cross-attribute dependence. In our paper, the value of learning more about one attribute is influenced by the value of the other attribute, even though the values of different attributes are independent. This cross-attribute dependence is endogenously driven by optimal learning. We characterize the marginal rate of substitution between the values of two attributes and examine how knowledge about one attribute impacts the benefits of learning about the other. To our knowledge, this learning-based endogenous cross-attribute dependenceis a novel contribution to the literature.

We study the comparative statics of the optimal search strategy. An increase in the price shifts the entire search region upwards, as the consumer needs to gain a higher value from the good to compensate for the increased price. An increase in either the search cost or the noise of the signal makes searching less attractive for the consumer, leading to a reduction in the search region.

We also investigate how the consumer's likelihood of purchasing depends on her prior belief. If the consumer is sufficiently optimistic about both attributes, she will definitely purchase the product. Conversely, if she is highly pessimistic about both attributes, she will not purchase the product. When her belief lies somewhere in between, she will purchase the product with some probability.

In practice, firms can influence the consumer's search and purchase process by modifying the search environment. They can affect the difficulty of search through website design or advertising. A consumer who is initially optimistic about the product will purchase it either for sure or with a high probability. Even if the consumer searches for information, she tend to quickly stop searching and buy the product after receiving positive signals, and is unlikely to encounter sufficient negative information to dissuade her. If a consumer is very pessimistic about the product, she is unlikely to search for or purchase it, regardless of the firm's intervention. In the above cases, the firm does not find it beneficial to alter the search environment. Reducing search costs benefits the firm if the consumer's prior belief is intermediate. In these scenarios, the consumer will either purchase the product with a low probability or not buy for sure. By reducing search costs and thus expanding the search region, the firm prevents the consumer from not searching or from prematurely ending the search after negative findings. This allows the consumer more opportunities to receive positive signals and eventually leads to a higher likelihood of purchasing the product.

This paper makes two main contributions. First, we endogenize the search order of different attributes of a product based on the consumer's optimal Bayesian learning, offering valuable insights into cross-attribute dependence in search behavior. Second, we provide practical guidance to firms on how to influence consumer search and purchasing decisions by altering the search environment.

Related Literature

This paper is related to the literature on how consumers with limited attention allocate their attention to different attributes or options. Existing literature mainly looks at cases where attributes or options are asymmetric (Arbatskaya 2007, Armstrong et al. 2009, Xu et al. 2010, Armstrong and Zhou 2011, Bar-Isaac et al. 2012, Bordolo et al. 2013, Kőszegi and Szeidl 2013, Branco et al. 2016, Zhu and Dukes 2017, Jeziorski and Moorthy 2018). In these papers, consumers know ex-ante that they face attributes with different prominence or importance. For example, the search order is exogenous in Arbatskaya (2007). Armstrong et al. (2009) extend Wolinsky (1986)'s symmetric search model by introducing a prominent firm that all consumers search for first. However, the reason why consumers prioritize this firm is not modeled, as its prominence is exogenously assumed. In Bordolo et al. (2013), the salient attribute of a good is the attribute furthest away from the average value of the same attribute in the choice set. In Zhu and Dukes (2017), each competing firm can promote one or both attributes of a product. Though the prominence of the product is endogenously determined by competition, it is given exogenously from the consumer's perspective. Jeziorski and Moorthy (2018) explore the role of prominence in search advertising, distinguishing between ad position prominence and advertiser prominence. They find that these are are substitutes in influencing consumer clicks. A key contribution of our paper is to endogenize the optimal attribute to search from the consumer's perspective. Instead of assuming that the consumer knows the value of each attribute or learns it instantly, as is common in this literature, the Bayesian decision-maker in our model gradually learns the value from noisy signals. So, the relative importance of the attributes may change as the consumer gathers more information. In contrast, the prominence attribute or option in the existing literature does not change over time because they impose exogenous differences on the attributes.

This paper also fits into the literature on optimal information acquisition, particularly consumer search. Following seminal papers by Stigler (1961) and Weitzman (1979), numerous papers have studied the optimal search problem under either simultaneous or sequential search (e.g., Moscarini and Smith 2001, Branco et al. 2012, Ke et al. 2016, and Jerath and Ren 2023). In these studies, the relative importance of different alternatives is typically exogenous. Consumers observe the distribution of the rewards before making the search decision. Similar to our work, the attributes in these papers are symmetric. However, in contrast, consumers in these models randomly choose an attribute to search. In our model, the consumer strategically decides when to search and which attribute to focus on. Ke and Villas-Boas (2019), who examine gradual information learning about multiple alternatives, are particularly relevant to our study. There are three main distinctions between their work and ours. First, in their model, the expected payoff from choosing an option depends solely on the information acquired about that option, making the search objective to differentiate between alternatives. In contrast, our paper posits that the expected payoff from adopting a product depends on information about all attributes, making the search objective to understand the overall distribution of attributes. Second, our model allows for the examination of cross-attribute dependence driven by optimal learning, where the value of learning more about one attribute is contingent on the value of the other. Third, while they focus on the decision-maker's optimal search strategy, we also study the firm's search design problem, demonstrating how a firm can influence consumer search behavior and enhance profits by changing seach costs.

Lastly, our paper relates to the literature on the design of the search environment. Various studies have examined how a firm can optimally influence consumers' search environment through information provision policies (Branco et al. 2016, Jerath and Ren 2021, Ke et al. 2023, Yao 2023, Gardete and Hunter 2024), product line design (Villas-Boas 2009, Kuksov and Vilas-Boas 2010, Guo and Zhang 2012, Liu and Dukes 2013), pricing strategies (Wee et al. 2024), and advertising (Mayzlin and Shin 2011). Our research is closely aligned with studies focusing on the design of search environments by modifying search costs. Empirical investigations in this area (Seiler 2013, Ngwe, Ferreira, and Teixeira 2019, and Ursu et al. 2020, 2023) primarily use field experiments and counterfactual analyses to study the impact of search costs on firm profits in various contexts. Dukes and Liu (2016) theoretically characterizes the intermediary's strategic choices of search costs in equilibrium. They consider simultaneous search across multiple firms, whereas our study focuses on sequential search involving multiple attributes. As our research centers on a monopoly scenario, there is no intermediary involved; instead, the seller directly designs the search environment. Bar-Isaac et al. (2010) show that a firm may prefer zero search costs if the marginal cost of production is high, infinite search costs if it is low, and may opt for intermediate search costs when facing heterogeneous consumers. Stivers and Tremblay (2005) suggest that advertising can enhance social welfare by reducing search costs, even if it results in higher prices. Both these studies do not incorporate gradual learning; instead, consumers acquire complete information in a single search instance.

The remainder of the paper is organized as follows. Section 2 presents the main model. Section 3 introduces a benchmark two-period model. Section 4 addresses the solution to the optimal search strategy in the main model. Section 5 characterizes the purchasing likelihood. Section 6 examines a firm's search design problem. Section 7 concludes.

2 Model

A consumer considers whether to purchase a product or not. The product has two attributes, each with independent values. The product's value to the consumer is the sum of the values of these attributes, $U = U_1 + U_2$. This paper focuses on the horizontal match between the attributes and the consumer's tastes/needs. The value of each attribute is one if it is a good match and zero if it is a bad match. The consumer's prior belief that attribute i is good is denoted as $\mu_i(0)$. Given our focus on horizontal preferences rather than vertical ones, we assume the firm does not have private information about the value of the attribute. The price p is given exogenously. We assume that the marginal cost of producing the product is sufficiently high, and thus the price is high enough ($p \geq 3/2$), so that the consumer will decide not to purchase the product for any pair of beliefs (μ_1, μ_2) if $\mu_1 + \mu_2 \leq 1$. Therefore, we restrict our attention to the case where $\mu_1 + \mu_2 > 1$. The consumer has the option to learn more about the attributes via costly learning before making a decision. At time t, the consumer can either make a purchasing decision or search for information. Due to limited attention, she can only search for information about one attribute at a time. Therefore, if the consumer opts to search, she also needs to decide which specific attribute to investigate. The decision-making process ends when the consumer makes a purchasing decision.

It is important to note that each attribute of the product consists of numerous sub-attributes. So, the consumer cannot learn everything about an attribute with a single evaluation. For instance, if a consumer wishes to learn about the design of a car (an attribute), she might start by looking at an image online to determine the car's exterior color (a sub-attribute). However, she will need to invest additional effort to learn about other sub-attributes, such as the wheel size or the seat material. Given the complexity of modern products, each attribute often includes so many sub-attributes that it becomes impossible for the consumer to fully learn everything. To model this gradual learning process, we assume that the consumer receives noisy signals about an attribute by incurring a flow cost of c. Let $T_i(t)$ denote the cumulative time spent searching for attribute i up to time t. We represent the signal, S_i, as a Brownian motion, where W_i are independent Wiener processes:

$$dS_i(t) = U_i dT_i(t) + \sigma dW_i(T_i(t))$$

In the above expression, the first term is driven by the true value, U_i, while the second term represents the noise. The parameter σ is a measure of the level of signal noise - a larger σ indicates a noisier signal due to the higher relative weight of the noise term. The consumer is more likely to observe a larger signal realization if the attribute is good, as the first term continuously increases over time when $U_i = 1$. Based on the received signals, the consumer continuously updates her belief about the value of each attribute according to Bayes' rule. This belief evolution can be characterized by the following ordinary differential equation (ODE):

$$d\mu_i(t) = \frac{1}{\sigma^2}\mu_i(t)[1 - \mu_i(t)]\{dS_i(t) - \mu_i dT_i(t)\}, \tag{1}$$

where μ_i is the expected value of attribute i based on the observed information up to time t. A higher σ results in slower belief updating due to noisier signals. If μ_i is closer to $1/2$, there is more uncertainty about attribute i, leading to faster updates in belief. If $dS_i(t) - \mu_i dT_i(t) > 0$, the signal's increasing speed is higher than the current belief about attribute i, and therefore the true value is more likely to be good ($U_i = 1$). So, the consumer will increase her belief about this attribute.Conversely, if the signal increases more slowly, the consumer will decrease her belief. This belief updating process also implies that the consumer's belief about an attribute remains unchanged when she searches for information about the other attribute. This continuous-time model of Bayesian learning about a binary state has been widely used to study information acquisition (Ke and Villas-Boas 2019, Morris and Strack 2019, Liao 2021), experimentation (Bolton and Harris

1999, Moscarini and Smith 2001), and decision times (Fudenberg et al. 2018). It effectively captures the gradual learning feature and offers tractable analysis.

Figure 1 illustrates a sample path of the signals and belief evolution when the first attribute is good and the second one is bad. Initially, the consumer's belief about each attribute is 1/2. She begins by searching for information about attribute 1, continuously receiving signals about it. Although these signals are predominantly positive, the consumer's belief about the first attribute gradually declines because the signal's rate of increase is slower than her belief. As attribute 2 is not initially searched, no new signals are received for it, and consequently, the consumer's belief regarding this attribute remains unchanged. After a period of time, the consumer shifts her focus to attribute 2. Her belief about this attribute first decreases and then increases, while her belief about attribute 1 stays the same. On receiving positive signals about attribute 2, she returns to investigating attribute 1. During this phase, the signal for attribute 1 increases rapidly, causing her belief to rise towards 1. Once the consumer is relatively certain that the first attribute is good, she resumes her search for information about attribute 2. She eventually stops searching and decides not to purchase the product after receiving sufficient negative signals, leading her to strongly believe that attribute 2 is bad.

At any given time, the consumer chooses among four actions: searching for attribute 1, searching for attribute 2, purchasing the product, or quitting without purchasing. The consumer's search strategy, denoted as α, maps the observed history (the signal realization) up to time t to one of these four actions, for all t. We define the stopping time τ as the first instance when the consumer makes a purchasing decision (either purchasing or quitting). The entire process ends at the stopping time. The consumer's expected payoff for a given initial belief (μ_1, μ_2) and search strategy α is:

$$J(\mu_1, \mu_2, \alpha) = \mathbb{E}\left\{\max\left[\mu_1(\tau) + \mu_2(\tau) - p, 0\right] - \tau c | (\mu_1(0), \mu_2(0)) = (\mu_1, \mu_2)\right\}$$

The value function of the consumer's problem is given by:

$$V(\mu_1, \mu_2) := \sup_{\alpha} J(\mu_1, \mu_2, \alpha)$$

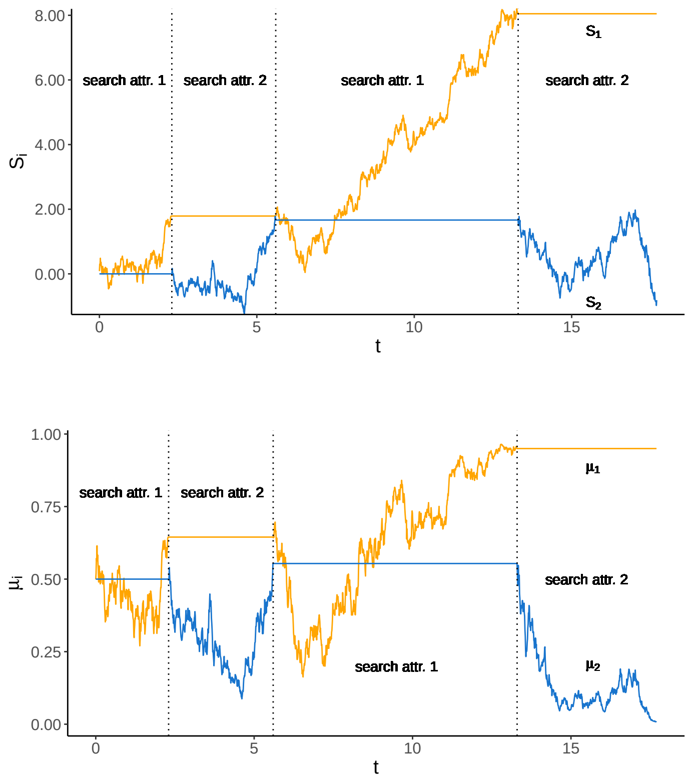

Figure 1: Sample Paths of the Signals and Beliefs for $U_1 = 1, U_2 = 0, \mu_1(0) = 0.5, \mu_2(0) = 0.5$, and $\sigma = 1$.

Since the search strategy should not depend on future information, the decision at time t must be based only on the information observed up to t. It is well established that the current belief $(\mu_1(t), \mu_2(t))$ is a sufficient statistic for the information available up to time t in this binary-valued setting. Therefore, the search strategy will depend solely on $(\mu_1(t), \mu_2(t))$. If a search strategy α^* achieves $V(\mu_1, \mu_2)$ for any given belief, it will be deemed the optimal search strategy.

$$V(\mu_1, \mu_2) = J(\mu_1, \mu_2, \alpha^*)$$

We will first present a simply two-period model to provide some intuition in the next section. Then, section 4 will characterize the consumer's value function and the optimal search strategy of the main model.

3 A Two-period One-shot Learning Model

The main model appears complex and challenging to solve. A natural question arises: Can a simpler model provide the same insights? In this section, we introduce and solve a two-period benchmark model, where there is no gradual learning. The consumer can learn everything about an attribute with a single search. We will compare the outcomes of this simpler one-shot learning model with those of the main model in the next section, and highlight the importance of explicitly modeling gradual learning in an infinite-period model.

In the two-period model, the consumer still decides whether to purchase a product. There are two periods, $t = 1, 2$. In each period, the consumer can incur a search cost c to search for information about one attribute. Upon searching attribute i, her belief about that attribute becomes 1 with probability μ_i and 0 with probability $1 - \mu_i$. The consumer can make a purchasing decision without searching, after searching once, or after searching twice. Consistent with the main model, we assume that $p \geq 3/2$, $\mu_1 + \mu_2 > 1$, and $\mu_1 \geq \mu_2$.[1] We also assume that the search cost is small, $c < (p-1)(2-p)$.[2]

Let us first consider the optimal strategy of the consumer in the second period.

Lemma 1 (Subgame)**.** *The consumer quits if the attribute she searches for in the first period is bad.*

Now suppose the consumer searches for attribute 1(2) and finds out it is good in the first period. The consumer purchases directly if $\mu_2(\mu_1) \geq 1 - c/(p-1)$, searches attribute 2(1) and buy if only if it is also good if $\mu_2(\mu_1) \in [c/(2-p), 1 - c/(p-1))$, and quits if $\mu_2(\mu_1) < c/(2-p)$.

The consumer's strategy in this subgame is intuitive. Since the consumer only receives a positive payoff if both attributes are good, she will quit if the first attribute is found to be bad. If the first attribute is good, her subsequent strategy depends on her belief about the second attribute. If she is highly optimistic about the second attribute, she can potentially earn a positive payoff with a high probability by immediately purchasing the product. This approach allows her to save the search cost while not incurring significant risk. As the search cost increases, the incentive to save

[1] The case of $\mu_1 < \mu_2$ is symmetric to the case of $\mu_1 \geq \mu_2$, so we only need to analyze one of these scenarios.
[2] The consumer will never search twice if $c \geq (p-1)(2-p)$.

on this cost becomes stronger, enlarging the belief range for this case ($\mu_2(\mu_1) \geq 1 - c/(p-1)$). If the consumer has a modest belief about the second attribute, the benefit of learning is amplified because of greater uncertainty. Consequently, she is inclined to search again and make a purchasing decision with complete information. If she is pessimistic about the second attribute, the likelihood of making a purchase after further search is too low, leading her to decide to quit. Given the consumer's search strategy in the second period, we can now proceed to characterize her strategy in the first period.

Proposition 1. *In the first period, the consumer purchases without searching if* $\begin{cases} \mu_1 + \mu_2 \geq p \\ \mu_2 \geq \max\{1 - \frac{c}{p-\mu_1}, \frac{p-\mu_1-c}{1+c-\mu_1(2-p)}\} \end{cases}$*, quits without searching if* $\begin{cases} \mu_1 + \mu_2 < p \\ \mu_1 < \min\{p - 1 + \frac{c}{\mu_2}, \frac{(1+\mu_2)c}{\mu_2(2-p)}\} \end{cases}$*, and search attribute 2 in other cases.*

A notable aspect of the optimal search strategy in this model is that the consumer always prioritizes searching for the attribute about which she has the most uncertainty, should she decide to search at all. Additionally, there is no cross-attribute dependence once the consumer conducts a single search. Since she becomes fully informed about an attribute with just one search, the only remaining option, if she chooses to search again, is to gather information about the other attribute.

4 Optimal Strategy

When the consumer searches for information about attribute one, the value function satisfies (ignoring the time index t for simplicity):

$$V(\mu_1, \mu_2) = -cdt + \mathbb{E}[V(\mu_1 + d\mu_1, \mu_2)]$$

By Taylor's expansion and Ito's lemma, we get:

$$\frac{\mu_1^2(1-\mu_1)^2}{2\sigma^2} V_{\mu_1\mu_1}(\mu_1, \mu_2) - c = 0 \tag{2}$$

Similarly, when the consumer searches for information about attribute two, we have:

$$\frac{\mu_2^2(1-\mu_2)^2}{2\sigma^2}V_{\mu_2\mu_2}(\mu_1,\mu_2)-c=0 \tag{3}$$

The HJB equation of the entire problem is:

$$\max\left\{\max_{i=1,2}\left[\frac{\mu_i^2(1-\mu_i)^2}{2\sigma^2}V_{\mu_i\mu_i}(\mu_1,\mu_2)-c\right],\max\left[\mu_1+\mu_2-p,0\right]-V(\mu_1,\mu_2)\right\}=0 \tag{$\star$}$$

A standard method for solving this kind of stochastic control problem is the"guess and verify" approach. We will show that the value function is a viscosity solution of the Hamilton-Jacobi-Bellman (HJB) equation. We will then prove that the viscosity solution is unique. Therefore, if we can find a viscosity solution, it must be the value function. To achieve this, we will construct a search strategy, which we will use to characterize the search region and the expected payoff. Finally, we will verify that the conjectured strategy generates a viscosity solution of the HJB equation, thereby implying that the conjectured search strategy is optimal. Due to symmetry, it is sufficient to consider only the case where $\mu_1 \geq \mu_2$. Analytically, we can fully characterize the optimal search strategy when the search cost is low. We do not believe that focusing on the low search cost case imposes a significant limitation, as our primary interest lies in examining the consumer's search behavior and how it can be influenced by the firm. The more compelling scenario occurs when the consumer is inclined to search more due to the low search cost. When the search cost or the price is too high, the consumer's search activity diminishes, making the problem less intriguing and relevant.

Theorem 1. *Suppose the search cost is low, $c \leq \frac{1}{2\sigma^2[\phi(1/2)-\phi(\frac{2}{3}p-\frac{1}{6})]}$. Conditional on searching, it is optimal for the consumer to search for information about attribute two (one) if $\mu_1 \geq \mu_2$ ($\mu_1 < \mu_2$).*

Figure 2 illustrates the optimal search strategy. The dashed orange line represents the quitting boundary, while the solid blue line depicts the purchasing boundary. The grey arrow indicates which attribute the consumer searches for information about, given her current belief. When the overall beliefs of the attributes are sufficiently low, the likelihood of receiving enough positive signals to warrant a purchase is too low. In this case, the consumer stops searching and quits to save on search cost. Conversely, when the overall beliefs of the attributes are high enough, purchasing the product

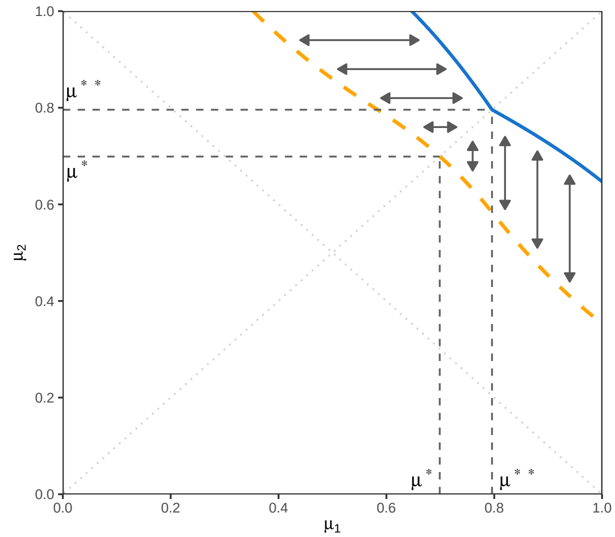

Figure 2: Optimal Search Strategy

results in a sufficiently high expected payoff, prompting the consumer to make the purchase. In other cases, the consumer searches for more information to make a better-informed decision.

Denote the point where the quitting boundary intersects the main diagonal as (μ^*, μ^*), and the point where the purchasing boundary intersects the main diagonal as (μ^{**}, μ^{**}). The quitting boundary for $\mu_1 \geq \mu_2$ is represented by $\underline{\mu}(\cdot)$, with a domain of $[\mu^*, 1]$ (the other half of the quitting boundary can be determined by symmetry). The purchasing boundary for $\mu_1 \geq \mu_2$ is represented by $\bar{\mu}(\cdot)$, with a domain of $[\mu^{**}, 1]$ (the other half of the purchasing boundary can also be determined by symmetry).

Intuitively, conditional on searching, the consumer prefers to search for the attribute with a higher rate of learning, since learning costs are identical. From equation (1), it is evident that the more uncertain the belief is, the faster the consumer learns about an attribute. Consequently, she always focuses on learning about the attribute with a belief closer to 1/2. The optimal search strategy implies that the consumer only searches for one attribute if she holds a strong prior belief on one of the attributes, and may search for both attributes otherwise.

The following proposition characterizes the slope of the purchasing/quitting boundary and the shape of the search region.

Proposition 2. *For $\mu \in (\mu^*, \mu^{**}]$, we have:*

$$\underline{\mu}'(\mu) = \frac{\phi(\underline{\mu}(\mu)) - \phi(\mu)}{\phi'(\underline{\mu}(\mu))[\mu - \underline{\mu}(\mu)]} \tag{D_1}$$

For $\mu \in [\mu^{**}, 1]$, we have:

$$\bar{\mu}'(\mu) = \frac{\phi(\underline{\mu}(\mu)) - \phi(\bar{\mu}(\mu))}{\phi'(\bar{\mu}(\mu))[\bar{\mu}(\mu) - \underline{\mu}(\mu)]} \tag{$\overline{D_2}$}$$

$$\underline{\mu}'(\mu) = \frac{\phi(\underline{\mu}(\mu)) - \phi(\bar{\mu}(\mu))}{\phi'(\underline{\mu}(\mu))[\bar{\mu}(\mu) - \underline{\mu}(\mu)]} \tag{$\underline{D_2}$}$$

Both $\underline{\mu}(\mu)$ and $\bar{\mu}(\mu)$ strictly decrease in μ, whereas the width of the search region, $\bar{\mu}(\mu) - \underline{\mu}(\mu)$, strictly increases in μ. In addition, if $\underline{\mu}(\mu) \geq 1/2$, then the slope of the quitting boundary is less than -1, and the slope of the purchasing boundary is greater than -1.

We find that the optimal search region exhibits a butterfly shape - the consumer searches for information in a wider region when she is more confident that the more favorable attribute is good. The underlying intuition is as follows: the product's expected value is higher when the consumer is more certain about one attribute being good, prompting her to seek information about the other attribute, even if it is associated with greater uncertainty. This is because the speed of learning is higher for a more uncertain attribute, enhancing the search's benefits while the cost of searching remains unchanged. Consequently, the consumer is motivated to engage in more extensive search activities. Furthermore, if the consumer likes an attribute more, she is more inclined to purchase the product even if she has a higher uncertainty about the other attribute. She will also be less likely to stop searching and quit. As a result, the search region shifts downwards as the belief about one attribute increases.

The slope of the search region, representing the marginal rate of substitution between the values of the first and second attributes, is also interesting. It sheds light on the learning process's cross-attribute dependence. If the slope is -1, then the two attributes are perfect substitutes. One may expect this to be the case in general because the product's value is the sum of the values of two attributes, each with independent values. However, both the slope of the quitting boundary and the slope of the purchasing boundary are not -1 in general due to the asymmetry in learning. If the quitting boundary is above 1/2, the slope of the quitting boundary is less than -1. In such cases, a one-unit increase in the belief about attribute one can compensate for more than a one-unit decrease in the belief about attribute two near the quitting boundary. The consumer will continue her search for attribute two rather than quitting, even if μ_2 declines by slightly more than one unit. This is because the consumer has more uncertainty, and hence a higher rate of learning, about attribute 2. So, the benefit of search increases while the search cost remains the same. Similarly, near the purchasing boundary, a one-unit increase in the belief about attribute one compensates for less than a one-unit decrease in the belief about attribute two. This encourages the consumer to keep searching for information about attribute two rather than making an immediate purchase, even if μ_2 decreases by slightly less than a unit.

4.1 Comparison with the Two-period Model

The benchmark model discussed in the previous section does not incorporate gradual learning. The consumer fully resolves the uncertainty about an attribute with a single search, limiting searches to a maximum of two periods. In contrast, the main model in continuous-time introduces gradual learning, where the consumer incrementally resolves uncertainty about an attribute with each search, without a predefined deadline for making a decision. The robust finding across both models is that the consumer always prioritizes searching for the attribute about which she has the most uncertainty. We now present several arguments in favor of the main model.

Firstly, the two-period model's non-stationary nature, due to the presence of a deadline, makes it harder to distinguish the main economic forces and interpret the findings. For example, it is challenging to determine if the optimal search behavior is driven by the speed of learning or the deadline effect.

Secondly, the discrete model becomes complicated quickly as the number of periods increases, with an exponentially growing number of cases to consider. This complexity poses significant challenges to extending the two-period model to an N-period framework that accommodates gradual learning.

Thirdly, partial evaluation is ubiquitous in real-world scenarios (Häubl and Trifts 2000, Hauser 2011), as evidenced by empirical studies showing that return visits account for a significant portion of all search activity (Bronnenberg et al. 2016). Such behavior is not captured in a model where complete information about an option is acquired through a single search. The gradual learning model not only accounts for return visits but also clarifies under what conditions these revisits occur, providing valuable insights for marketers on resource allocation. For instance, online car sellers might adjust their focus based on observed consumer behavior, such as a consumer's engagement with images pertaining to a car's styling, and on the prediction about whether the consumer will revisit the car design attribute.

Lastly, the gradual learning model offers valuable insights into the learning process's cross-attribute dependence, an area relatively unexplored in consumer search literature due to its inherent technical challenges. Ke and Lin (2020) allow for an exogenous correlation between attributes across different products, identifying an information complementarity effect where the lower price of one product can increase the demand for the others. In their paper, different products share some common attributes. So, the values of the product attributes are correlated. In our paper, the values of different attributes are independent, whereas the value of learning more about one attribute is influenced by the value of the other attribute. To our knowledge, this learning-based endogenous cross-attribute dependence is a novel contribution to the literature, and provides valuable managerial implications.

4.2 Comparative Statics

If the firm wants to leverage the results from our model, it needs to understand how the model primitives influence consumer search behavior. The following proposition summarizes the effects of price, search cost, and noise of the signal on the search region.

Proposition 3. *Suppose $\mu_1 \geq \mu_2$. The purchasing threshold $\bar{\mu}(\mu)$ increases in the price p, and decreases in the search cost c and the noise of the signal σ^2. The quitting threshold $\underline{\mu}(\mu)$ increases in the price p, the search cost c, and the noise of the signal σ^2.*

An increase in price results in an upward shift of the entire search region, as the consumer requires a higher value from the product to offset the increased price. For example, as Figure 3 illustrates, the consumer is willing to buy the product immediately at price $p = 1.5$ if she believes that each attribute has an 80% probability of being good, thereby achieving a positive expected payoff. However, if the price rises to 1.6, the same belief no longer justifies a purchase due to reduced expected utility. Suppose instead that the consumer believes that each attribute has a 70% probability of being good. She is willing to search for more information if the price is 1.5 despite a negative expected utility from immediate purchase, hoping to increase her valuation of the product through further information. Yet, if the price rises to 1.6, the likelihood of acquiring enough positive information to justify the higher price becomes too low, prompting her to stop searching.

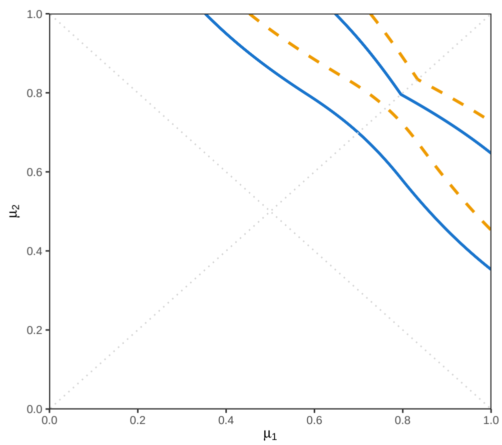

Figure 3: Optimal Search Region for $p = 1.5$ (solid blue) or 1.6 (dashed orange), $c = 0.1, \sigma^2 = 1$.

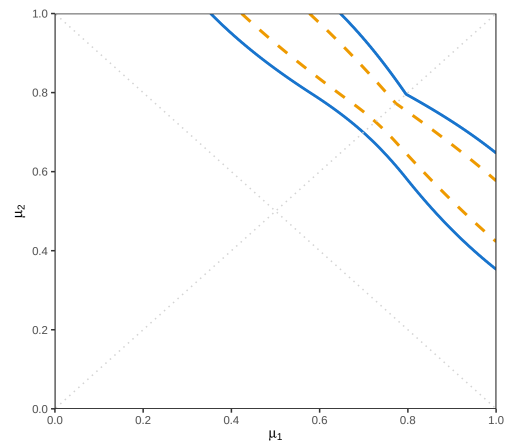

Figure 4: Optimal Search Region for $p = 1.5$, $c\sigma^2 = 0.1$ (solid blue) or 0.2 (dashed orange).

Changes in search costs and signal noise have the same effect on consumer behavior because they always appear together as a product $c\sigma^2$ in the value function. An increase in either of these factors makes searching less appealing and narrows the search region. The consumer will only search for information in a narrower range of beliefs. Figure 4 illustrates how variations in search costs and signal noise affect the search region. For example, at $p = 1.5$ and $c\sigma^2 = 0.1$, the consumer might continue searching if she believes that each attribute has a 78% chance of being good. Although she could gain a positive surplus by purchasing immediately, she might prefer to gather more information to avoid mistakenly buying a suboptimal product. However, if searching becomes more time-consuming or less accurate (e.g., $c\sigma^2 = 0.2$), the benefit of search decreases, and the consumer might opt to purchase immediately without further search.

5 Paths to Purchase and Purchasing Likelihood

We now examine the consumer's paths to purchase based on her beliefs about the product's attributes. If the consumer is highly confident about one of the attributes, she will not seek further information about it but will focus on learning about the other attribute. A decision to purchase is made once enough positive information is acquired to reach the purchasing boundary. Conversely, if sufficient negative information leads her belief to the quitting boundary, she will decide not to make a purchase. For instance, a consumer considering a Volvo may skip researching its well-regarded safety features, choosing instead to investigate other aspects of the vehicle.

In cases where the consumer has moderate beliefs about both attributes, she must gather information on both before purchasing the product.[3] Moreover, she will be equally certain about the value of each attribute upon deciding to buy. For example, a consumer considering a pre-order from Faraday Future, a new manufacturer, likely faces significant uncertainty about all aspects and might investigate every attribute. Given the consumer's optimal search strategy, we can calculate the purchasing likelihood based on the prior belief (μ_1, μ_2).

Proposition 4. *Suppose $\mu_1 \geq \mu_2$. The probability that the consumer purchases the product is:*

$$\begin{aligned} P(\mu_1,\mu_2) :=\ & \mathbb{P}[\textit{purchasing}|\textit{starting at } (\mu_1,\mu_2)] \\ =\ & \begin{cases} 1, \ \textit{if } \mu_1 \in [\mu^{**},1] \ \textit{and } \mu_2 \in [\bar{\mu}(\mu_1),\mu_1] \\ \frac{\mu_2-\underline{\mu}(\mu_1)}{\bar{\mu}(\mu_1)-\underline{\mu}(\mu_1)}, \ \textit{if } \mu_1 \in [\mu^{**},1] \ \textit{and } \mu_2 \in [\underline{\mu}(\mu_1),\bar{\mu}(\mu_1)] \\ h(\mu_1,\mu_2)\tilde{P}(\mu_1), \ \textit{if } \mu_1 \in [\mu^{*},\mu^{**}] \ \textit{and } \mu_2 \in [\underline{\mu}(\mu_1),\mu_1] \\ 0, \ \textit{if } \mu_1 \leq \mu^{*} \ \textit{or } \mu_2 \leq \bar{\mu}(\mu_1) \end{cases} \end{aligned}$$

, where $h(\mu_1,\mu_2) = \frac{\mu_2-\underline{\mu}(\mu_1)}{\mu_1-\underline{\mu}(\mu_1)}$ and $\tilde{P}(\mu) = e^{-\int_{\mu}^{\mu^{**}} \frac{2}{x-\underline{\mu}(x)}dx}$. By symmetry, $P(\mu_1,\mu_2) = P(\mu_2,\mu_1)$ if $\mu_1 < \mu_2$.

[3] The consumer may have searched for only one attribute if she decide not to purchase the product after receiving enough negative information about that attribute.

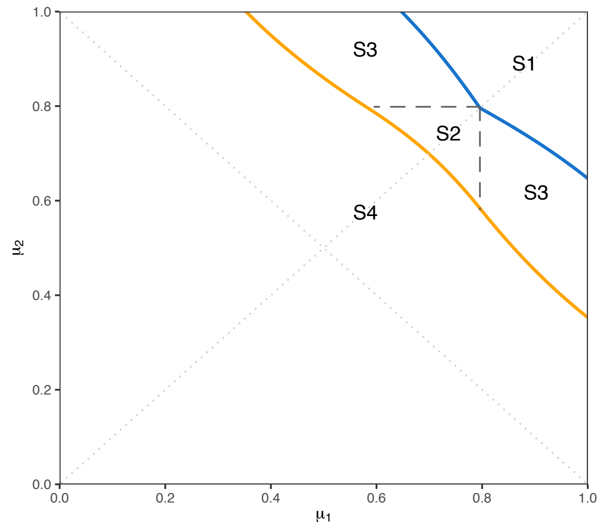

Figure 5: Four Regions for Purchase

Figure 5 delineates four regions that outline the consumer's purchasing strategy based on her initial beliefs. In region $S1$, the consumer purchases immediately, while in region$S4$, she quits without searching. Beliefs that fall within the intermediate regions, S_2 and S_3, indicate a high value of information, prompting further search before a decision is made.

Specifically, in region $S3$, the consumer strongly believes that the first attribute is good and focuses her search on the second, more uncertain attribute. Suffcient positive information about the second attributeleads to a purchase, while enough negative information results in quitting. Due to the low information value about the first attribute, the consumer never switches her focus back to it regardless of the outcomes of her search regarding the second attribute. In this scenario, the second attribute is pivotal in determining the consumer's purchasing decision.

In region $S2$, the consumer is quite uncertain about the values of both attributes. She will initially focus her search on attribute two, given her greater uncertainty about it compared to attribute one. However, she is not confident about the value of attribute one either. Consequently, if she receives sufficient positive signals about attribute two, she will then shift her focus to searching for information about attribute one. If subsequent positive signals about attribute one are received, she may toggle her search focus back to attribute two, and this pattern of switching may continue until she gains enough confidence in both attributes to make a purchasing decision. As shown in

Figure 5, the belief must reach (μ^{**}, μ^{**}) for the consumer to make a purchasing decision. Conversely, if she encounters sufficient negative information about either attribute, she will discontinue her search and opt not to purchase the product.

6 Search Design

Having established the probability of purchasing based on the prior belief, we now turn our attention to the consumer's search environment, which is typically considered as given. However, in practice, firms have significant control over how difficult the search process is, through mechanisms like website design or advertising. For example, many websites allow consumers to filter search results, but the degree of precision in these filters can vary widely. Some companies offer extensive control, allowing consumers to tailor the information presented to them closely, while others provide only basic filtering options. These variations in design can effectively alter the search costs or the noise associated with the search process. Additionally, firms can facilitate easier access to pertinent information for consumers through targeted advertising, which has been empirically shown to reduce search costs (De Corniere, 2016).

Since only the product of the search cost and signal noise, $c\sigma^2$, are identifiable in our model, we consider search design as a firm's decision to adjust the search cost without loss of generality.

6.1 Symmetric Case

The firm is endowed with a default search cost c, and decides whether to reduce it to $\tilde{c} < c$.[4] Denote the purchasing and quitting boundaries under the reduced search cost by $\tilde{\bar{\mu}}$ and $\underline{\tilde{\mu}}$, respectively. Proposition 3 and Figure 4 have shown that a lower search cost broadens the range within which the consumer will search. The critical question is whether this adjustment benefits the firm. The next proposition asserts that reducing the search cost is beneficial only if the consumer's prior belief is intermediate.

Proposition 5. *Suppose $\mu_1 \geq \mu_2$. There exists $\tilde{\mu}(\mu_1)$ such that $\underline{\mu}(\mu_1) < \tilde{\mu}(\mu_1) < \bar{\mu}(\mu_1)$ if $\mu_1 \geq \mu^{**}$ and $\underline{\mu}(\mu_1) < \tilde{\mu}(\mu_1) \leq \mu_1$ if $\mu_1 < \mu^{**}$. The firm reduces the search cost if and only if $\mu_2 \in$*

[4] The results of increasing the search cost are symmetric to the ones of decreasing the search cost. So, we only discuss the case of reduction. Additionally, we need to consider the firm's strategy only for $\mu_1 \geq \mu_2$ due to symmetry.

$(\underline{\tilde{\mu}}(\mu_1), \tilde{\mu}(\mu_1))$.

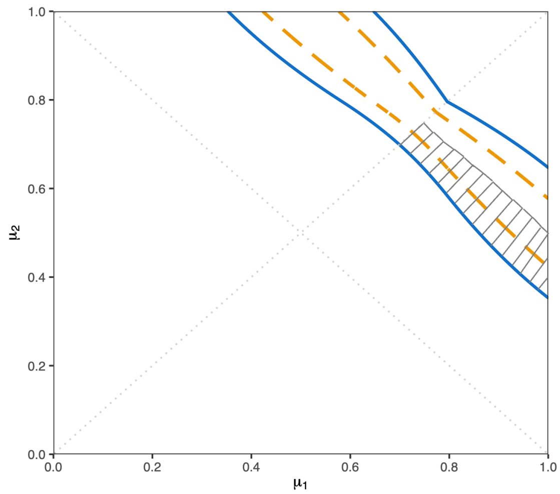

Figure 6: Illustration of Search Design for the Symmetric Case. Dashed line: stopping boundary under the default search cost. Solid line: stopping boundary under the reduced search cost. Striped shaded region: the firm prefers reducing search costs.

A consumer with a sufficiently high prior belief will purchase the product immediately, eliminating incentives for the firm to lower search costs. Even if the consumer does not make a purchase directly, the firm does not want to reduce the search cost. Maintaining a tighter search region ensures that the consumer will quickly stop searching and buy the product after receiving positive signals, minimizing the chance of learning enough negative information and quit.

A consumer will neither search nor purchase the product if she is very pessimistic about the product, regardless of firm intervention.

Reducing search costs benefits the firm only if the consumer's prior belief is intermediate. The firm has an incentive to reduce search costs if the consumer never buys under the default search cost but may buy under the reduced search cost. Even if the consumer buys with a low probability without firm intervention, by reducing search costs, the firm can enlarge the search region and prevent the consumer from prematurely ending the search after negative findings. This allows the consumer more opportunities to receive positive signals and eventually leads to a higher likelihood of purchasing the product. So, the firm is better off reducing search costs.

Figure 6 illustrates the firm's strategy. The dashed line represents the stopping boundary under the default search cost. The firm can expand the search region to the area within the solid lines by reducing search costs. It does so when the belief falls within the striped shaded region.

6.2 Asymmetric Case

It is also interesting to consider the case where the firm can reduce the search cost from c to $\tilde{c}$ for one attribute. A complete characterization of the optimal strategy is beyond the scope of this paper. Instead, we discuss below some insights from the symmetric case that will extend to the asymmetric case.

Suppose the firm reduces the search cost for attribute 2, the attractiveness of searching for information about this attribute increases for the consumer, as the relative benefit of searching either attribute remains unchanged. Consequently, the consumer may prioritize searching for attribute 2 over attribute 1 in some situations where $\mu_2 > \mu_1$. In this case, the consumer exclusively searches for attribute 2 if $\mu_1 \geq \underline{\mu^{**}}(< \mu^{**})$, and for attribute 1 if $\mu_2 \geq \overline{\mu^{**}}(> \mu^{**})$. Suppose instead the firm reduces the search cost for attribute 1. By symmetry, one can see that the consumer exclusively searches for attribute 1 if $\mu_2 \geq \underline{\mu^{**}}(< \mu^{**})$, and for attribute 2 if $\mu_1 \geq \overline{\mu^{**}}(> \mu^{**})$.

Consider the case where $\mu_1 \geq \overline{\mu^{**}}$. The consumer always searches for information about attribute 2 regardless of firm intervention. The firm, therefore, has no incentive to lower the cost for attribute 1. Therefore, it either reduces the search cost for attribute 2 or does not adjust the search environment. Since the consumer only searches for attribute 2, the effect of reducing its search cost mirrors that of reducing costs for both attributes. The result in Proposition 5 for this case stays the same: the firm reduces the search cost for attribute 2 if and only if $\mu_2 \in (\underline{\tilde{\mu}}(\mu_1), \tilde{\mu}(\mu_1))$ for the same $\underline{\tilde{\mu}}$ and $\tilde{\mu}$ as in the symmetric case. When $\mu_2 \geq \overline{\mu^{**}}$, by symmetry, the firm reduces the search cost for attribute 1 if and only if $\mu_1 \in (\underline{\tilde{\mu}}(\mu_2), \tilde{\mu}(\mu_2))$.

Now consider the case where $\mu_2 \leq \mu_1 < \overline{\mu^{**}}$. Similar arguments as the proof of Proposition 5 imply that the firm has an incentive to reduce the search cost for attribute 2 for intermediate μ_2 such that the consumer purchases the product with zero or a low probability under the default search cost, whereas buy with a positive probability under the reduced search cost. This is also consistent with insights from the symmetric case.[5]

[5] The determination of whether reducing the search cost for attribute 1 is beneficial in this case remains an open question, meriting further investigation in future research.

7 Conclusion

Understanding how consumers decide which attribute to pay more attention to has important managerial implications. It helps the firm decide how to design the product and allocate marketing resources. In this paper, we study the optimal search strategy of a Bayesian decision-maker by endogenizing the optimal attribute to search for, when to keep searching, and when to stop and make a decision. We characterize the search region by a set of ordinary differential equations for moderate beliefs and by a system of equations for extreme beliefs. We find that it is always optimal for the consumer to search for the attribute about which she has greater uncertainty, due to the faster speed of learning. The consumer only searches for one attribute if she holds a strong prior belief about one of the attributes, and may search for both attributes otherwise. We also characterize the marginal rate of substitution between the values of two attributes and examine how knowledge about one attribute affects the benefits of learning about the other. Finally, we study the firm's search design problem. In practice, it can affect the difficulty of search by website design or advertising. We find that reducing search costs benefits the firm only if the consumer's prior belief is intermediate. In such cases, a lower search cost prevents the consumer from not searching or from prematurely ending the search after negative findings. This allows the consumer more opportunities to receive positive signals and eventually leads to a higher likelihood of purchasing the product.

There are some limitations to this paper. The consumer only considers one product in our model. If there are multiple products, the consumer needs to make two decisions - which product to search for and which attribute of the product to search for. Studying this richer problem can lead to interesting findings. It will also be interesting to extend the number of attributes beyond two and see whether the consumer still searches for the attribute with the highest uncertainty due to the fastest learning speed. Lastly, we consider an exogenous price throughout the paper to focus on the role of information. Future research can study the optimal pricing of the product given the consumer's optimal search strategy.

Appendix

Proof of Proposition 2. We have derived (D_1) in the main text. It implies immediately that $\underline{\mu}'(\mu) < 0$ for $\mu \in (\mu^*, \mu^{**}]$. For $\mu \in [\mu^{**}, 1]$, by the implicit function theorem, we have:

$$\begin{bmatrix} \bar{\mu}'(\mu) \\ \underline{\mu}'(\mu) \end{bmatrix} = - \begin{bmatrix} -\phi'(\bar{\mu}(\mu)) & \phi'(\underline{\mu}(\mu)) \\ -\psi'(\bar{\mu}(\mu)) & \psi'(\underline{\mu}(\mu)) \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 \\ \frac{1}{2\sigma^2 c} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{2\sigma^2 c} \frac{1}{\phi'(\bar{\mu}(\mu))[\bar{\mu}(\mu) - \underline{\mu}(\mu)]} \\ \frac{1}{2\sigma^2 c} \frac{1}{\phi'(\underline{\mu}(\mu))[\bar{\mu}(\mu) - \underline{\mu}(\mu)]} \end{bmatrix} = \begin{bmatrix} \frac{\phi(\underline{\mu}(\mu)) - \phi(\bar{\mu}(\mu))}{\phi'(\bar{\mu}(\mu))[\bar{\mu}(\mu) - \underline{\mu}(\mu)]} & < 0 \\ \frac{\phi(\underline{\mu}(\mu)) - \phi(\bar{\mu}(\mu))}{\phi'(\underline{\mu}(\mu))[\bar{\mu}(\mu) - \underline{\mu}(\mu)]} & < 0 \end{bmatrix}$$

This gives us the expression for $(\overline{D_2})$ and $(\underline{D_2})$. One can see from the negative sign of the derivative that both $\underline{\mu}(\mu)$ and $\bar{\mu}(\mu)$ strictly decrease in μ.

We now look at the width of the search region.

$$\begin{aligned} & [\bar{\mu}(\mu) - \underline{\mu}(\mu)]' \\ = & \frac{\phi(\underline{\mu}(\mu)) - \phi(\bar{\mu}(\mu))}{\phi'(\bar{\mu}(\mu))[\bar{\mu}(\mu) - \underline{\mu}(\mu)]} - \frac{\phi(\underline{\mu}(\mu)) - \phi(\bar{\mu}(\mu))}{\phi'(\underline{\mu}(\mu))[\bar{\mu}(\mu) - \underline{\mu}(\mu)]} \\ = & \frac{\phi(\underline{\mu}(\mu)) - \phi(\bar{\mu}(\mu))}{\bar{\mu}(\mu) - \underline{\mu}(\mu)} \left[1/\phi'(\bar{\mu}(\mu)) - 1/\phi'(\underline{\mu}(\mu)) \right] \\ = & \frac{\phi(\underline{\mu}(\mu)) - \phi(\bar{\mu}(\mu))}{\bar{\mu}(\mu) - \underline{\mu}(\mu)} \left[\underline{\mu}(\mu)^2 (1 - \underline{\mu}(\mu))^2 - \bar{\mu}(\mu)^2 (1 - \bar{\mu}(\mu))^2 \right] \end{aligned}$$

One can see that $\frac{\phi(\underline{\mu}(\mu)) - \phi(\bar{\mu}(\mu))}{\bar{\mu}(\mu) - \underline{\mu}(\mu)} > 0$. So, $[\bar{\mu}(\mu) - \underline{\mu}(\mu)]' > 0 \Leftrightarrow \underline{\mu}(\mu)^2(1 - \underline{\mu}(\mu))^2 > \bar{\mu}(\mu)^2(1 - \bar{\mu}(\mu))^2 \Leftrightarrow \underline{\mu}(\mu)(1 - \underline{\mu}(\mu)) > \bar{\mu}(\mu)(1 - \bar{\mu}(\mu)) \Leftrightarrow |\underline{\mu}(\mu) - 1/2| < |\bar{\mu}(\mu) - 1/2|$. Thus, the width of the search region, $\bar{\mu}(\mu) - \underline{\mu}(\mu)$, increases in the belief, μ, if and only if the quitting boundary is closer to 1/2 than the purchasing boundary. We know that $\forall \mu \geq \mu^{**},\ p = \mu + \frac{\bar{\mu}(\mu) + \underline{\mu}(\mu)}{2}$ due to the symmetry of the one-dimensional learning problem.[6] Therefore,

[6] More specifically, the sum of the purchasing and quitting thresholds is zero when the price is zero in the one-dimensional optimal search strategy, as shown by Branco et al. (2012). It implies that the price equals to the average of the two boundaries. In our two-dimensional problem, the consumer only searches the more uncertain attribute when $\mu \geq \mu^{**}$. So, it can be translated to a one-dimensional search problem with the price p normalized to $p - \mu$.

$$\begin{aligned}
&\frac{\bar{\mu}(\mu)+\underline{\mu}(\mu)}{2} = p-\mu \geq 3/2-1 = 1/2 \\
\Rightarrow\ &\frac{\bar{\mu}(\mu)+\underline{\mu}(\mu)}{2} \geq 1/2 \\
\Leftrightarrow\ &\bar{\mu}(\mu)+\underline{\mu}(\mu) > 1 \\
\Leftrightarrow\ &|\underline{\mu}(\mu)-1/2| < |\bar{\mu}(\mu)-1/2|,\ \forall \mu \geq \mu^{**}
\end{aligned}$$

Thus, the width of search region, $\bar{\mu}(\mu)-\underline{\mu}(\mu)$, always increases in the belief μ.

Now suppose that $\underline{\mu}(\mu) \geq 1/2$, then $\forall \mu \in (\mu^*, \mu^{**}]$, we have

$$\begin{aligned}
\underline{\mu}'(\mu) &\overset{(D_1)}{=} \frac{\phi(\underline{\mu}(\mu))-\phi(\mu)}{\phi'(\underline{\mu}(\mu))[\mu-\underline{\mu}(\mu)]} \\
&= \frac{-\phi'(\xi_1(\mu))[\mu-\underline{\mu}(\mu)]}{\phi'(\underline{\mu}(\mu))[\mu-\underline{\mu}(\mu)]}\ (\xi_1(\mu) \in (\underline{\mu}(\mu), \mu)) \\
&= -\frac{\phi'(\xi_1(\mu)}{\phi'(\underline{\mu}(\mu))} \\
&< -1
\end{aligned}$$

, where the last inequality comes from the fact that the absolute value of $\phi'(x) = -\frac{1}{x^2(1-x)^2}$ is strictly increasing in x for $x \geq 1/2$. Similarly, $\forall \mu \in [\mu^{**}, 1]$, we have

$$\begin{aligned}
\underline{\mu}'(\mu) &\overset{(\underline{D_2})}{=} \frac{\phi(\underline{\mu}(\mu))-\phi(\bar{\mu}(\mu))}{\phi'(\underline{\mu}(\mu))[\bar{\mu}(\mu)-\underline{\mu}(\mu)]} \\
&= \frac{-\phi'(\xi_2(\mu))[\bar{\mu}(\mu)-\underline{\mu}(\mu)]}{\phi'(\underline{\mu}(\mu))[\bar{\mu}(\mu)-\underline{\mu}(\mu)]}\ (\xi_2(\mu) \in (\underline{\mu}(\mu), \bar{\mu}(\mu))) \\
&= -\frac{\phi'(\xi_2(\mu)}{\phi'(\underline{\mu}(\mu))} < -1 \\
\bar{\mu}'(\mu) &\overset{(\overline{D_2})}{=} \frac{\phi(\underline{\mu}(\mu))-\phi(\bar{\mu}(\mu))}{\phi'(\bar{\mu}(\mu))[\bar{\mu}(\mu)-\underline{\mu}(\mu)]} \\
&= \frac{-\phi'(\xi_3(\mu))[\bar{\mu}(\mu)-\underline{\mu}(\mu)]}{\phi'(\bar{\mu}(\mu))[\bar{\mu}(\mu)-\underline{\mu}(\mu)]}\ (\xi_3(\mu) \in (\underline{\mu}(\mu), \bar{\mu}(\mu))) \\
&= -\frac{\phi'(\xi_3(\mu))}{\phi'(\bar{\mu}(\mu))} > -1
\end{aligned}$$

□

Proof of Theorem 1. By symmetry, we only need to prove the case of $\mu_1 \geq \mu_2$. We first show that the viscosity solution of the HJB equation ($\star$) exists and is unique. Since the value function is a viscosity solution of ($\star$), the viscosity solution of ($\star$) must be the value function by uniqueness. We then conjecture an optimal search strategy and characterize its properties. Lastly, we verify that the conjectured strategy indeed generates a viscosity solution to ($\star$). So, the conjectured strategy is optimal.

Lemma 2. *The viscosity solution of the HJB equation* ($\star$) *exists and is unique.*

Proof. Since the consumer can guarantee a payoff of zero by quitting immediately and cannot achieve a payoff higher than $\sup\{\mu_1 + \mu_2 - p\} = 1 + 1 - p \leq 2$, the value function is bounded and thus exists. This implies the existence of the viscosity solution because the value function is a viscosity solution to ($\star$).

The proof of the uniqueness uses a modification of a comparison principle in Crandall et al. (1992). Given that it very much resembles the proof of Lemma 1 in Ke and Villas-Boas (2019), we refer the reader to their proof. □

Conjecture:

Conditional on searching, it is optimal for the consumer to search for information about attribute two (one) if $\mu_1 \geq \mu_2$ ($\mu_1 < \mu_2$).

Given this conjecture, we now characterize the search region (illustrated in Figure 2).

The PDE when the consumer searches attribute two, equation (3), has the following general solution:

$$V(\mu_1, \mu_2) = 2\sigma^2 c(1 - 2\mu_2) \ln \frac{1 - \mu_2}{\mu_2} + B_1(\mu_1)\mu_2 + B_2(\mu_1), \mu_1 \in [\mu^*, 1]$$

We also have $V(\mu_1, \mu_2) = 0$ at the quitting boundary $\mu_2 = \underline{\mu}(\mu_1)$. For the value function in the search region, value matching and smooth pasting (wrt μ_2) at the quitting boundary $(\mu_1, \underline{\mu}(\mu_1))$ imply:[7]

$$\frac{V(\mu_1, \mu_2)}{2\sigma^2 c} = (1 - 2\mu_2) \ln \frac{1 - \mu_2}{\mu_2} + \phi(\underline{\mu}(\mu_1))\mu_2 - \psi(\underline{\mu}(\mu_1)) \tag{4}$$

[7] For technical details, please refer to Dixit (1993).

, where $\phi(x) = 2\ln\frac{1-x}{x} + \frac{1}{x} - \frac{1}{1-x}$ and $\psi(x) = \ln\frac{1-x}{x} + \frac{1-2x}{1-x}$.

By symmetry, for $\mu_1 < \mu_2$, the value function in the search region satisfies:

$$\frac{V(\mu_1,\mu_2)}{2\sigma^2 c} = (1-2\mu_1)\ln\frac{1-\mu_1}{\mu_1} + \phi(\underline{\mu}(\mu_2))\mu_1 - \psi(\underline{\mu}(\mu_2)) \tag{5}$$

Equation (4) characterizes the value function for beliefs $\mu_1 \geq \mu_2$. Equation (5) characterizes the value function for beliefs $\mu_1 < \mu_2$. The two regions are separated by the main diagonal $\{(\mu_1,\mu_2) : \mu_1 = \mu_2\}$. Continuity of $V_{\mu_1}(\mu_1,\mu_2)$ at this boundary implies that:

$$\underline{\mu}'(\mu) = \frac{\phi(\underline{\mu}(\mu)) - \phi(\mu)}{\phi'(\underline{\mu}(\mu))[\mu - \underline{\mu}(\mu)]}, \text{ for } \mu \in (\mu^*, \mu^{**}] \tag{D_1}$$

For $\mu_1 \in [\mu^{**}, 1]$, $V(\mu_1,\mu_2) = \mu_1 + \mu_2 - p$ at the purchasing boundary $\mu_2 = \bar{\mu}(\mu_1)$. Value matching and smooth pasting (w.r.t. μ_2) at the purchasing boundary $(\mu_1, \underline{\mu}(\mu_1))$ imply (in the search region):

$$\frac{V(\mu_1,\mu_2)}{2\sigma^2 c} = (1-2\mu_2)\ln\frac{1-\mu_2}{\mu_2} + \phi(\bar{\mu}(\mu_1))\mu_2 - \psi(\bar{\mu}(\mu_1)) + \frac{\mu_1 - \mu_2 - p}{2\sigma^2 c} \tag{6}$$

Equation (4) and (6) use the quitting boundary and the purchasing boundary to pin down the value function, respectively. The resulting expression should be equivalent in the common domain $\mu_1 \in [\mu^{**}, 1]$. By equalizing V and V_{μ_2} of equation (4) and (6), we obtain the following system of equations:

$$\begin{cases} \phi(\underline{\mu}(\mu)) - \phi(\bar{\mu}(\mu)) = \frac{1}{2\sigma^2 c} \\ \psi(\underline{\mu}(\mu)) - \psi(\bar{\mu}(\mu)) = \frac{p-\mu}{2\sigma^2 c} \end{cases}, \text{ for } \mu \in [\mu^{**}, 1] \tag{7}$$

For each belief, μ, the system of equations above consists of two unknowns ($\bar{\mu}(\mu)$ and $\underline{\mu}(\mu)$) and two equations. They uniquely determine the function for the purchasing boundary $\bar{\mu}(\mu)$ and the function for the quitting boundary $\underline{\mu}(\mu)$, for $\mu \in [\mu^{**}, 1]$, given a cutoff belief μ^{**}.

Instead of determining $\bar{\mu}(\mu)$ and $\underline{\mu}(\mu)$ by a system of equations (7), we can also implicitly determine $\bar{\mu}(\mu)$ and $\underline{\mu}(\mu)$ in two separate equations. Representing $\bar{\mu}(\mu)$ by $\underline{\mu}(\mu)$ from the first equation of (7), we have:

$$\bar{\mu}(\mu) = \phi^{-1}\left[\phi(\underline{\mu}(\mu)) - \frac{1}{2\sigma^2 c}\right]$$

Plugging it into the second equation of (7), we have:

$$\underline{\mu}(\mu) = \psi^{-1}\left\{\psi\left(\phi^{-1}\left[\phi(\underline{\mu}(\mu)) - \frac{1}{2\sigma^2 c}\right]\right) + \frac{p-\mu}{2\sigma^2 c}\right\}$$

The equation above implicitly determines $\underline{\mu}(\mu)$, for $\mu \in [\mu^{**}, 1]$. Similarly, we can implicitly determine $\bar{\mu}(\mu)$ by the following equation:

$$\bar{\mu}(\mu) = \psi^{-1}\left\{\psi\left(\phi^{-1}\left[\phi(\bar{\mu}(\mu)) + \frac{1}{2\sigma^2 c}\right]\right) - \frac{p-\mu}{2\sigma^2 c}\right\}$$

We now solve for the cutoff belief at the intersection of the purchasing boundary and the main diagonal, μ^{**}. Since (μ^{**}, μ^{**}) is on the purchasing boundary, we have $\mu^{**} = \bar{\mu}(\mu^{**})$, μ^{**} is determined by:

$$\begin{cases} \phi(\underline{\mu}(\mu^{**})) - \phi(\mu^{**}) = \frac{1}{2\sigma^2 c} \\ \psi(\underline{\mu}(\mu^{**})) - \psi(\mu^{**}) = \frac{p-\mu^{**}}{2\sigma^2 c} \end{cases} \tag{8}$$

The system of equations above consists of two unknowns (μ^{**} and $\underline{\mu}(\mu^{**})$ and two equations. They uniquely determine the cutoff belief μ^{**} via the following equations:

$$\phi^{-1}\left[\phi(\mu^{**}) + \frac{1}{2\sigma^2 c}\right] = \psi^{-1}\left[\psi(\mu^{**}) + \frac{p-\mu^{**}}{2\sigma^2 c}\right] \tag{I^{**}}$$

We have pinned down the cutoff belief μ^{**}. Given this cutoff beliefs, we have determined the purchasing boundary $(\mu, \bar{\mu}(\mu))$ and the quitting boundary $(\mu, \underline{\mu}(\mu))$, for $\mu \in [\mu^{**}, 1]$.

The ODE (D_1) and the initial condition (I^{**}) implicitly determine the function for the quitting boundary $\underline{\mu}(\mu)$, for $\mu \in (\mu^*, \mu^{**}]$, given a cutoff belief μ^*.

We now solve for the cutoff belief at the intersection of the quitting boundary and the main diagonal, μ^*. Since (μ^*, μ^*) is on the quitting boundary, we have $\mu^* = \underline{\mu}(\mu^*)$. This initial condition determines μ^*.

In sum, we have pinned down the cutoff belief μ^* and the quitting boundary $(\mu, \underline{\mu}(\mu))$, for $\mu \in [\mu^*, \mu^{**}]$.

We have fully characterized the purchasing boundary $(\mu, \bar{\mu}(\mu))$ and the quitting boundary $(\mu, \underline{\mu}(\mu))$, for $\mu_1 \geq \mu_2$. The other case in which $\mu_1 < \mu_2$ is readily determined by symmetry.

Verification:

To verify that the conjectured strategy indeed generates a viscosity solution to the HJB equation $(\star)$:

$$\max\left\{\max_{i=1,2}\left[\frac{\mu_i^2(1-\mu_i)^2}{2\sigma^2}V_{\mu_i\mu_i}(\mu_1,\mu_2)-c\right], \max\left[\mu_1+\mu_2-p, 0\right]-V(\mu_1,\mu_2)\right\}=0$$

We just need to show that (everything else holds by our construction):

$$\begin{aligned} &\frac{\mu_1^2(1-\mu_1)^2}{2\sigma^2}V_{\mu_1\mu_1}(\mu_1,\mu_2)-c \leq 0 \\ \Leftrightarrow\ &\mu_1^2(1-\mu_1)^2 V_{\mu_1\mu_1}(\mu_1,\mu_2)/2\sigma^2 c \leq 1 \end{aligned} \tag{9}$$

if $\mu_1 + \mu_2 > 1$, $\mu_1 \geq \mu_2$, and $\underline{\mu}(\mu_1) < \mu_2 < \bar{\mu}(\mu_1)$.

For $\mu_1 \in (\mu^*, \mu^{**}]$, we have

$$\begin{aligned} &V_{\mu_1}(\mu_1,\mu_2)/2\sigma^2 c \\ =&\phi'(\underline{\mu}(\mu_1))\underline{\mu}'(\mu_1)[\mu_2-\underline{\mu}(\mu_1)] \\ \overset{(D_1)}{=}&\frac{\phi(\underline{\mu}(\mu_1))-\phi(\mu_1)}{\mu_1-\underline{\mu}(\mu_1)}[\mu_2-\underline{\mu}(\mu_1)] \\ \Rightarrow\ &V_{\mu_1\mu_1}(\mu_1,\mu_2)/2\sigma^2 c \\ =&\phi'(\underline{\mu}(\mu_1))\underline{\mu}'(\mu_1)[\mu_2-\underline{\mu}(\mu_1)] \\ \overset{(D_1)}{=}&\frac{\phi'(\underline{\mu}(\mu_1))\underline{\mu}'(\mu_1)-\phi'(\mu_1)}{\mu_1-\underline{\mu}(\mu_1)}[\mu_2-\underline{\mu}(\mu_1)]+[\phi(\underline{\mu}(\mu_1))-\phi(\mu_1)]\frac{(\mu_2-\mu_1)\underline{\mu}'(\mu_1)+\underline{\mu}(\mu_1)-\mu_2}{[\mu_1-\underline{\mu}(\mu_1)]^2} \\ =&-\frac{\phi'(\mu_1)[\mu_2-\underline{\mu}(\mu_1)]}{\mu_1-\underline{\mu}(\mu_1)}+(\mu_2-\mu_1)\frac{[\phi(\underline{\mu}(\mu_1))-\phi(\mu_1)]^2}{\phi'(\underline{\mu}(\mu_1))[\mu_1-\underline{\mu}(\mu_1)]^3} \\ \Rightarrow\ &\mu_1^2(1-\mu_1)^2 V_{\mu_1\mu_1}(\mu_1,\mu_2)/2\sigma^2 c \end{aligned}$$

$$=\frac{\mu_2 - \underline{\mu}(\mu_1)}{\mu_1 - \underline{\mu}(\mu_1)} + (\mu_1 - \mu_2)\mu_1^2(1-\mu_1)^2\frac{\underline{\mu}(\mu_1)^2[1-\underline{\mu}(\mu_1)]^2}{[\mu_1 - \underline{\mu}(\mu_1)]^3}[\phi(\underline{\mu}(\mu_1)) - \phi(\mu_1)]^2$$

So,

$$\begin{aligned}
&\mu_1^2(1-\mu_1)^2 V_{\mu_1\mu_1}(\mu_1,\mu_2)/2\sigma^2 c \leq 1 \\
\Leftrightarrow &\mu_1^2(1-\mu_1)^2\frac{\underline{\mu}(\mu_1)^2[1-\underline{\mu}(\mu_1)]^2}{[\mu_1 - \underline{\mu}(\mu_1)]^2}[\phi(\underline{\mu}(\mu_1)) - \phi(\mu_1)]^2 \leq 1 \\
\Leftrightarrow &\mu_1(1-\mu_1)\frac{\underline{\mu}(\mu_1)^2[1-\underline{\mu}(\mu_1)]}{[\mu_1 - \underline{\mu}(\mu_1)]}[\phi(\underline{\mu}(\mu_1)) - \phi(\mu_1)] \leq 1 \\
\Leftrightarrow &H(\mu_1) := \mu_1(1-\mu_1)[\phi(\underline{\mu}(\mu_1)) - \phi(\mu_1)] - \frac{\mu_1 - \underline{\mu}(\mu_1)}{\underline{\mu}(\mu_1)[1-\underline{\mu}(\mu_1)]} \leq 0 \qquad (10)
\end{aligned}$$

Observe that $H(\mu^*) = 0$. Ignoring the subscript 1 for notational ease, we have:

$$\begin{aligned}
H'(\mu) =&(1-2\mu)[\phi(\underline{\mu}(\mu)) - \phi(\mu)] + \frac{\mu(1-\mu)}{\mu - \underline{\mu}(\mu)}[\phi(\underline{\mu}(\mu)) - \phi(\mu)] + \frac{1}{\mu(1-\mu)} \\
&- \frac{1}{\underline{\mu}(\mu)(1-\underline{\mu}(\mu))} + \frac{\phi(\underline{\mu}(\mu)) - \phi(\mu)}{\mu - \underline{\mu}(\mu)}[-\mu + 2\mu\underline{\mu}(\mu) - \underline{\mu}(\mu)^2] \\
=&[1 - 3\mu + \underline{\mu}(\mu)][\phi(\underline{\mu}(\mu)) - \phi(\mu)] + \frac{1}{\mu(1-\mu)} - \frac{1}{\underline{\mu}(\mu)(1-\underline{\mu}(\mu))}
\end{aligned}$$

Suppose (10) does not hold. There would exist $\widehat{\mu}$ such that $H(\widehat{\mu}) = 0$ and $H'(\widehat{\mu}) > 0$.

$$(10) \Rightarrow \phi(\underline{\mu}(\widehat{\mu})) - \phi(\widehat{\mu}) = \frac{\widehat{\mu} - \underline{\mu}(\widehat{\mu})}{\widehat{\mu}(1-\widehat{\mu})\underline{\mu}(\widehat{\mu})[1-\underline{\mu}(\widehat{\mu})]}$$

Hence, we get an expression for $\frac{1}{\widehat{\mu}(1-\widehat{\mu})}$ and $\frac{1}{\underline{\mu}(\mu)[1-\underline{\mu}(\mu)]}$. Plugging these expressions into the previous expression for $H'(\mu)$, we have:

$$H'(\widehat{\mu}) = -2[\phi(\underline{\mu}(\mu)) - \phi(\mu)][\mu - \underline{\mu}(\mu)] \leq 0$$

A contradiction! So, (10) and thus (9) hold, $\forall \mu_1 \in [\mu^*, \mu^{**}]$.

For $\mu_1 \in [\mu^{**}, 1]$, we have

$$
\begin{aligned}
V_{\mu_1}(\mu_1,\mu_2)/2\sigma^2 c =& \phi'(\underline{\mu}(\mu_1))\underline{\mu}'(\mu_1)[\mu_2 - \underline{\mu}(\mu_1)] \\
\overset{(D_2)}{=}& \frac{\mu_2 - \underline{\mu}(\mu_1)}{\bar{\mu}(\mu_1) - \underline{\mu}(\mu_1)} \\
V_{\mu_1\mu_1}(\mu_1,\mu_2)/2\sigma^2 c =& \frac{-\underline{\mu}'(\mu_1)[\bar{\mu}(\mu_1) - \underline{\mu}(\mu_1)] - [\bar{\mu}'(\mu_1) - \underline{\mu}'(\mu_1)][\mu_2 - \underline{\mu}(\mu_1)]}{[\bar{\mu}(\mu_1) - \underline{\mu}(\mu_1)]^2} \\
=& \frac{1}{2\sigma^2 c}\frac{1}{[\bar{\mu}(\mu_1) - \underline{\mu}(\mu_1)]^3}[\frac{\mu_2 - \bar{\mu}(\mu_1)}{\phi'(\underline{\mu}(\mu_1))} - \frac{\mu_2 - \underline{\mu}(\mu_1)}{\phi'(\bar{\mu}(\mu_1))}] \\
\Rightarrow V_{\mu_1\mu_1}(\mu_1,\mu_2) =& \frac{1}{[\bar{\mu}(\mu_1) - \underline{\mu}(\mu_1)]^3}[\frac{\mu_2 - \bar{\mu}(\mu_1)}{\phi'(\underline{\mu}(\mu_1))} - \frac{\mu_2 - \underline{\mu}(\mu_1)}{\phi'(\bar{\mu}(\mu_1))}]
\end{aligned}
$$

Since $\frac{\partial V_{\mu_1\mu_1}(\mu_1,\mu_2)}{\partial \mu_2} < 0$, we only need to show that (9) holds for $\mu_2 = \underline{\mu}(\mu_1)$:

$$
\begin{aligned}
& \mu_1^2(1-\mu_1)^2 V_{\mu_1\mu_1}(\mu_1, \underline{\mu}(\mu_1))/2\sigma^2 c \le 1 \\
\Leftrightarrow & \frac{\mu_1^2(1-\mu_1)^2}{[\bar{\mu}(\mu_1) - \underline{\mu}(\mu_1)]^2}\frac{-1}{\phi'(\underline{\mu}(\mu_1))} \le 1
\end{aligned} \tag{11}
$$

Let's first show that $\underline{\mu}(\mu^{**}) \le 1/2$ by contradiction. Suppose instead $\underline{\mu}(\mu^{**}) > 1/2$.

$$
\begin{aligned}
& p - \mu^{**} = \frac{\bar{\mu}(\mu^{**}) + \underline{\mu}(\mu^{**})}{2} \\
\Leftrightarrow\ & p - \mu^{**} = \frac{\mu^{**} + \underline{\mu}(\mu^{**})}{2} \\
\Leftrightarrow\ & \underline{\mu}(\mu^{**}) = 2p - 3\mu^{**}
\end{aligned}
$$

Hence, $2p - 3\mu^{**} > 1/2 \Rightarrow \mu^{**} < \frac{2}{3}p - \frac{1}{6}$. Since $\phi(x)$ is strictly decreasing in x, the first equation of (8) implies

$$
\begin{aligned}
\frac{1}{2\sigma^2 c} =& \phi(\underline{\mu}(\mu^{**})) - \phi(\mu^{**}) \\
<& \phi(1/2) - \phi(\frac{2}{3}p - \frac{1}{6}) \\
\Leftrightarrow c >& \frac{1}{2\sigma^2[\phi(1/2) - \phi(\frac{2}{3}p - \frac{1}{6})]}
\end{aligned}
$$

A contradiction! Therefore, $\underline{\mu}(\mu^{**}) \le 1/2$. Since $\underline{\mu}(\mu_1)$ is decreasing in μ_1, we have $\underline{\mu}(\mu_1) \le$

$1/2$, $\forall\mu \in [\mu^{**}, 1]$. One can see that the LHS of (11), $\frac{\mu_1^2(1-\mu_1)^2}{[\bar{\mu}(\mu_1)-\underline{\mu}(\mu_1)]^2}\frac{-1}{\phi'(\underline{\mu}(\mu_1))}$, decreases in $\mu_1 \in [\mu^{**}, 1]$. And we know that (11) holds for $\mu_1 = \mu^{**}$ (we have shown that (9) and thus (11) hold for $\forall\mu_1 \in [\mu^*, \mu^{**}]$). Therefore, (11) and thus (9) hold for $\forall\mu_1 \in [\mu^{**}, 1]$.

□

Proof of Proposition 4. We first consider $\mu_1 \in [\mu^{**}, 1]$ and $\mu_1 \geq \mu_2$. Under this circumstance, the consumer only learns about attribute two until μ_2 hits either the purchasing boundary or the quitting boundary. As μ_2 is a martingale, by Dynkin's formula, we get:

$$P(\mu_1, \mu_2) := \mathbb{P}[\text{purchasing}|\text{starting at } (\mu_1, \mu_2)] = \frac{\mu_2 - \underline{\mu}(\mu_1)}{\bar{\mu}(\mu_1) - \underline{\mu}(\mu_1)}$$

Now we consider $\mu_1 \in [\mu^*, \mu^{**}]$ and $\mu_1 \geq \mu_2$. The belief either hits (μ^{**}, μ^{**}) and the consumer purchases the good or the belief hits $\{(x, \underline{\mu}(x)) : x \in [\mu_1, \mu^{**})\} \cup \{(\underline{\mu}(x), x) : x \in [\mu_1, \mu^{**})\}$ and the consumer quits. To calculate the purchasing likelihood, let's first calculate the likelihood of the belief hitting $(\mu_1, \underline{\mu}(\mu_1))$ before hitting the main diagonal (μ_1, μ_1), $q(\mu_1, \mu_2)$.

$$q(\mu_1, \mu_2) = \frac{\mu_1 - \mu_2}{\mu_1 - \underline{\mu}(\mu_1)}$$

Now we calculate the probability of purchasing given belief (μ, μ), $\tilde{P}(\mu)$ by consider the infinitesimal learning on attribute two. Noticing that $q(\mu, \mu) = 0$, $\frac{\partial q}{\partial \mu_1}|_{\mu_1=\mu_2=\mu} = \frac{1}{\mu-\underline{\mu}(\mu)}$, $\frac{\partial q}{\partial \mu_2}|_{\mu_1=\mu_2=\mu} = -\frac{1}{\mu-\underline{\mu}(\mu)}$, we have:

$$\begin{aligned}
\tilde{P}(\mu) &= \frac{1}{2}\mathbb{P}[\text{purchasing}|(\mu, \mu), d\mu \geq 0] + \frac{1}{2}\mathbb{P}[\text{purchasing}|(\mu, \mu), d\mu < 0] \\
&= \frac{1}{2}[1 - q(\mu + |d\mu|, \mu)]\tilde{P}(\mu + |d\mu|) + \frac{1}{2}[1 - q(\mu - |d\mu|, \mu)]\tilde{P}(\mu) \\
&= \tilde{P}(\mu) + \frac{|d\mu|}{2}\tilde{P}'(\mu) + |d\mu|\frac{\tilde{P}(\mu)}{\underline{\mu}(\mu) - \mu} + o(d\mu) \\
\Rightarrow 0 &= \frac{|d\mu|}{2}\left[\tilde{P}'(\mu) + 2\frac{\tilde{P}(\mu)}{\underline{\mu}(\mu) - \mu}\right] + o(d\mu) \\
\Rightarrow \frac{\tilde{P}'(\mu)}{\tilde{P}(\mu)} &= -\frac{2}{\underline{\mu}(\mu) - \mu}, \ \forall\mu \in (\mu^*, \mu^{**})
\end{aligned}$$

, where the last equality comes from dividing the previous equation by $|d\mu|$ and take the limit of

$d\mu$ to 0. Together with the initial condition $\tilde{P}(\mu^{**}) = 1$, we obtain:

$$\tilde{P}(\mu) = e^{-\int_{\mu}^{\mu^{**}} \frac{2}{x-\underline{\mu}(x)} dx}$$

In sum, the purchasing likelihood when $\mu_1 \geq \mu_2$ and $\mu_1 \in (\mu^*, \mu^{**})$ is:

$$P(\mu_1, \mu_2) = \mathbb{P}[\text{purchasing}|\text{starting at } (\mu_1, \mu_2)] = [1 - q(\mu_1, \mu_2)]\tilde{P}(\mu_1) = h(\mu_1, \mu_2)\tilde{P}(\mu_1)$$

, where $h(\mu_1, \mu_2) = \frac{\mu_2 - \underline{\mu}(\mu_1)}{\mu_1 - \underline{\mu}(\mu_1)}$.

By symmetry, the purchasing likelihood when $\mu_1 < \mu_2$ and $\mu_2 \in (\mu^*, \mu^{**})$ is:

$$P(\mu_1, \mu_2) = P(\mu_2, \mu_1) = [1 - q(\mu_2, \mu_1)]\tilde{P}(\mu_2) = h(\mu_2, \mu_1)\tilde{P}(\mu_2)$$

□

Proof of Proposition 5. We first define $\tilde{\mu}^*$ by $\underline{\tilde{\mu}}(\tilde{\mu}^*) = \tilde{\mu}^*$ and $\tilde{\mu}^{**}$ by $\tilde{\bar{\mu}}(\tilde{\mu}^{**}) = \tilde{\mu}^{**}$.

If $\mu_2 \leq \underline{\tilde{\mu}}(\mu_1)$ or $\mu_2 \geq \tilde{\bar{\mu}}(\mu_1)$, the purchasing probability is 0 regardless the search costs. Also, the purchasing probability is positive under the reduced search cost $\tilde{c}$ and 0 under the default search cost c if $\underline{\tilde{\mu}}(\mu_1) < \mu_2 \leq \underline{\mu}(\mu_1)$. So, it is always better for the firm to reduce search costs in that region. The purchasing probability is less than 1 under the reduced search cost $\tilde{c}$ and 1 under the default search cost c if $\bar{\mu}(\mu_1) \leq \mu_2 < \tilde{\bar{\mu}}(\mu_1)$. So, it is always worse for the firm to reduce search costs in that region. We restrict our attention to the remaining case where $\underline{\mu}(\mu_1) < \mu_2 < \bar{\mu}(\mu_1)$ in the subsequent analyses. Consider three cases.

1. $\mu_1 \geq \tilde{\mu}^{**}$

   The purchasing probability under the default search cost c is:

$$P(\mu_1, \mu_2) = \frac{\mu_2 - \underline{\mu}(\mu_1)}{\bar{\mu}(\mu_1) - \underline{\mu}(\mu_1)}$$

The purchasing probability under the reduced search cost $\tilde{c}$ is:

$$\tilde{P}(\mu_1, \mu_2) = \frac{\mu_2 - \tilde{\underline{\mu}}(\mu_1)}{\tilde{\bar{\mu}}(\mu_1) - \tilde{\underline{\mu}}(\mu_1)}$$

We also know that $\mu_1 + \frac{\bar{\mu}(\mu_1)+\underline{\mu}(\mu_1)}{2} = \mu_1 + \frac{\tilde{\bar{\mu}}(\mu_1)+\tilde{\underline{\mu}}(\mu_1)}{2} = p \Rightarrow \bar{\mu}(\mu_1)+\underline{\mu}(\mu_1) = \tilde{\bar{\mu}}(\mu_1)+\tilde{\underline{\mu}}(\mu_1) = 2(p-\mu_1)$. Since both $P(\mu_1, \mu_2)$ and $\tilde{P}(\mu_1, \mu_2)$ are linear in μ_2, $P(\mu_1, p-\mu_1) = \tilde{P}(\mu_1, p-\mu_1) = 1/2$, and $P(\mu_1, \mu_2)$ has a lower slope than $\tilde{P}(\mu_1, \mu_2)$ as a linear function of μ_2, one can see that $P(\mu_1, \mu_2) < \tilde{P}(\mu_1, \mu_2)$ if and only if $\mu_2 < p - \mu_1$. Therefore, $\tilde{\mu}(\mu_1) = p - \mu_1 \in (\underline{\mu}(\mu_1), \bar{\mu}(\mu_1))$ in this case.

2. $\mu^{**} < \mu_1 < \tilde{\mu}^{**}$

   The purchasing probability under the default search cost c is still:

$$P(\mu_1, \mu_2) = \frac{\mu_2 - \underline{\mu}(\mu_1)}{\bar{\mu}(\mu_1) - \underline{\mu}(\mu_1)}$$

   The purchasing probability under the reduced search cost $\tilde{c}$ is now:

$$\tilde{P}(\mu_1, \mu_2) = \frac{\mu_2 - \tilde{\underline{\mu}}(\mu_1)}{\mu_1 - \tilde{\underline{\mu}}(\mu_1)} e^{-\int_{\mu_1}^{\tilde{\mu}^{**}} \frac{2}{x - \tilde{\underline{\mu}}(x)} dx}$$

   Both $P(\mu_1, \mu_2)$ and $\tilde{P}(\mu_1, \mu_2)$ are linear in μ_2. Observe that $P(\mu_1, \underline{\mu}(\mu_1)) = 0 < \tilde{P}(\mu_1, \underline{\mu}(\mu_1))$ and $P(\mu_1, \bar{\mu}(\mu_1)) = 1 > \tilde{P}(\mu_1, \bar{\mu}(\mu_1))$, one can see that there exists an unique $\tilde{\mu}(\mu_1) \in (\underline{\mu}(\mu_1), \bar{\mu}(\mu_1))$ such that $P(\mu_1, \mu_2) < \tilde{P}(\mu_1, \mu_2)$ if and only if $\mu_2 < \tilde{\mu}(\mu_1)$.

3. $\mu^{*} < \mu_1 < \mu^{**}$

   The purchasing probability under the default search cost c is:

$$P(\mu_1, \mu_2) = \frac{\mu_2 - \underline{\mu}(\mu_1)}{\mu_1 - \underline{\mu}(\mu_1)} e^{-\int_{\mu_1}^{\mu^{**}} \frac{2}{x - \underline{\mu}(x)} dx}$$

   The purchasing probability under the reduced search cost $\tilde{c}$ is:

$$\tilde{P}(\mu_1, \mu_2) = \frac{\mu_2 - \tilde{\underline{\mu}}(\mu_1)}{\mu_1 - \tilde{\underline{\mu}}(\mu_1)} e^{-\int_{\mu_1}^{\tilde{\mu}^{**}} \frac{2}{x - \tilde{\underline{\mu}}(x)} dx}$$

Hence,

$$
\begin{aligned}
& P(\mu_1,\mu_2) < \tilde{P}(\mu_1,\mu_2) \\
\Leftrightarrow\ & \frac{\mu_2 - \underline{\mu}(\mu_1)}{\mu_2 - \underline{\tilde{\mu}}(\mu_1)} < \frac{\mu_1 - \underline{\mu}(\mu_1)}{\mu_1 - \underline{\tilde{\mu}}(\mu_1)} e^{\int_{\mu_1}^{\mu^{**}} \frac{2}{x-\underline{\mu}(x)}dx - \int_{\mu_1}^{\tilde{\mu}^{**}} \frac{2}{x-\underline{\tilde{\mu}}(x)}dx}
\end{aligned}
$$

The left-hand side strictly increases in μ_2 whereas the right-hand side does not depend on μ_2. Furthermore, observe that the left-hand side $\to 0$ as $\mu_2 \to \underline{\mu}(\mu_1)$ while the right-hand side is positive, one can see that there exists an unique $\tilde{\mu}(\mu_1) \in (\underline{\mu}(\mu_1), \mu_1] \subset (\underline{\mu}(\mu_1), \mu^{**})$ such that $P(\mu_1,\mu_2) < \tilde{P}(\mu_1,\mu_2)$ if and only if $\mu_2 < \tilde{\mu}(\mu_1)$.[8]

In sum, there exists $\tilde{\mu}(\mu_1)$ such that $\underline{\mu}(\mu_1) < \tilde{\mu}(\mu_1) < \bar{\mu}(\mu_1)$ if $\mu_1 \geq \mu^{**}$ and $\underline{\mu}(\mu_1) < \tilde{\mu}(\mu_1) \leq \mu_1$ if $\mu_1 < \mu^{**}$. The firm reduces the search cost if and only if $\mu_2 \in (\underline{\tilde{\mu}}(\mu_1), \tilde{\mu}(\mu_1))$. □

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[8] For μ_1 close to μ^*, $\tilde{\mu}(\mu_1)$ may equal μ_1.

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Online Appendix for Multi-attribute Search

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1 Proofs of the Benchmark Model

Proof of Lemma 1. If one attribute is bad, the consumer's payoff from purchasing is negative regardless of the other attribute's value. So, the consumer will quit directly.

Now suppose the consumer searches attribute 1 and finds out it is good in the first period. Conditional on the first-period action and outcome, the consumer's second-period utility is $1+\mu_2-p$ if she purchases without searching again, $-c+\mu_2(2-p)$ if she searches attribute 2 and buy if only if it is also good, and 0 if she quits.

The consumer prefers searching again to quitting if and only if $-c+\mu_2(2-p) \geq 0 \Leftrightarrow \mu_2 > c/(2-p)$. She prefers purchasing directly to searching again if and only if $1+\mu_2-p \geq -c+\mu_2(2-p) \Leftrightarrow \mu_2 \geq 1-c/(p-1)$. The assumption $c < (p-1)[1-\sqrt{(3-p)(p-1)}]/(2-p)$ implies that $c/(2-p) < 1-c/(p-1)$. Hence, the consumer purchases directly if $\mu_2 \geq 1-c/(p-1)$, searches attribute 2 and buy if only if it is also good if $\mu_2 \in [c/(2-p), 1-c/(p-1))$, and quits if $\mu_2 < c/(2-p)$.

The case of searching attribute 2 in the first period is symmetric to that of searching attribute 1 in the first period. □

Proof of Proposition 1. The consumer's expected utility is $\mu_1+\mu_2-p$ if she buys without searching, $-c+\mu_1(1+\mu_2-p)$ if she searches attribute 1 and makes a decision, $-c+\mu_2(1+\mu_1-p)$ if she searches attribute 2 and makes a decision, $-(1+\mu_1)c+\mu_1\mu_2(2-p)$ if she searches attribute 1 and then attribute 2 when attribute 1 is good, and $-(1+\mu_2)c+\mu_2\mu_1(2-p)$ if she searches attribute 2 and then attribute 1 when attribute 2 is good.

The consumer will buy without searching if
$$\begin{cases} \mu_1 + \mu_2 - p \geq 0 \\ \mu_1 + \mu_2 - p \geq -c + \mu_1(1 + \mu_2 - p) \\ \mu_1 + \mu_2 - p \geq -c + \mu_2(1 + \mu_1 - p) \\ \mu_1 + \mu_2 - p \geq -(1 + \mu_1)c + \mu_1\mu_2(2 - p) \\ \mu_1 + \mu_2 - p \geq -(1 + \mu_2)c + \mu_2\mu_1(2 - p) \end{cases}$$

One can see that the third inequality implies the second inequality, and that the fifth inequality implies the fourth inequality. Therefore, the above conditions are equivalent to

$$\begin{cases} \mu_1 + \mu_2 - p \geq 0 \\ \mu_1 + \mu_2 - p \geq -c + \mu_2(1 + \mu_1 - p) \\ \mu_1 + \mu_2 - p \geq -(1 + \mu_2)c + \mu_2\mu_1(2 - p) \end{cases} \Leftrightarrow \begin{cases} \mu_1 + \mu_2 \geq p \\ \mu_2 \geq \max\{1 - \frac{c}{p-\mu_1}, \frac{p-\mu_1-c}{1+c-\mu_1(2-p)}\} \end{cases}$$

Similarly, The consumer will buy without searching if
$$\begin{cases} \mu_1 + \mu_2 - p < 0 \\ -c + \mu_1(1 + \mu_2 - p) < 0 \\ -c + \mu_2(1 + \mu_1 - p) < 0 \\ -(1 + \mu_1)c + \mu_1\mu_2(2 - p) < 0 \\ -(1 + \mu_2)c + \mu_2\mu_1(2 - p) < 0 \end{cases}$$

One can see that the third inequality implies the second inequality, and that the fifth inequality implies the fourth inequality. Therefore, the above conditions are equivalent to

$$\begin{cases} \mu_1 + \mu_2 - p < 0 \\ -c + \mu_2(1 + \mu_1 - p) < 0 \\ -(1 + \mu_2)c + \mu_2\mu_1(2 - p) < 0 \end{cases} \Leftrightarrow \begin{cases} \mu_1 + \mu_2 < p \\ \mu_1 < \min\{p - 1 + \frac{c}{\mu_2}, \frac{(1+\mu_2)c}{\mu_2(2-p)}\} \end{cases}$$

Now consider the case in which the consumer searches first. There are four possibilities before pinning down which attribute the consumer will search first.

1. The consumer will make the purchasing decision without searching again if she searches either attribute first.

2. The consumer will search again if she searches either attribute first and it turns out to be good.

3. The consumer will make the purchasing decision without searching again if she searches attribute 1 first, but will search again if she searches attribute 2 first and it turns out to be good.

4. The consumer will make the purchasing decision without searching again if she searches attribute 2 first, but will search again if she searches attribute 1 first and it turns out to be good.

By comparing the expected utility from searching one of the attributes first, one can see that the consumer will search attribute 2 first in the first two cases. By lemma 1, one can see that the third case is impossible, because it requires that $\mu_2 > 1 - c/(p-1) > \mu_1$, condtradicted with the assumption $\mu_1 \geq \mu_2$. We now show that the consumer searches attribute 2 first in the last case. Therefore, the consumer never searches attribute 1 first. The necessary and sufficient conditions for the last case to hold and for the consumer to prefer searching attribute 1 first are:

$$\begin{cases} \mu_1 > 1 - \frac{c}{p-1} \ (lemma1) \\ \mu_2 \in (\frac{c}{2-p}, 1 - \frac{c}{p-1}) \ (lemma1) \\ -(1+\mu_1)c + \mu_1\mu_2(2-p) > \mu_1 + \mu_2 - p \text{ (searching attribute 1 first is better than buying directly)} \\ -(1+\mu_1)c + \mu_1\mu_2(2-p) > 0 \text{ (searching attribute 1 first is better than quitting directly)} \\ -(1+\mu_1)c + \mu_1\mu_2(2-p) > -c + \mu_2(1+\mu_1-p) \\ \text{(searching attribute 1 first is better than searching attribute 2 first)} \end{cases}$$

However, we have:

$$\begin{aligned} -(1+\mu_1)c + \mu_1\mu_2(2-p) &\overset{\mu_1 \geq \mu_2}{\leq} -(1+\mu_2)c + \mu_1\mu_2(2-p) \\ &\overset{\mu_1 > 1 - \frac{c}{p-1}}{<} -c + \mu_2(1+\mu_1-p) \end{aligned}$$

, which contradicts with the last condition. $\square$

2 Proof of the Comparative Statics

Proof of Proposition 3.

(1) Comparative statics w.r.t. p

We first consider $\bar{\mu}(\mu)$. Fixing an arbitrary $\mu \in (\mu^{**}, 1]$, recall the system of equations (7):

$$\begin{aligned} \phi(\underline{\mu}(\mu)) - \phi(\bar{\mu}(\mu)) &= \frac{1}{2\sigma^2 c} \\ \psi(\underline{\mu}(\mu)) - \psi(\bar{\mu}(\mu)) &= \frac{p-\mu}{2\sigma^2 c} \end{aligned}$$

By the implicit function theorem, we obtain:

$$\begin{bmatrix} \frac{\partial \bar{\mu}(\mu)}{\partial p} \\ \frac{\partial \underline{\mu}(\mu)}{\partial p} \end{bmatrix} = - \begin{bmatrix} -\phi'(\bar{\mu}(\mu)) & \phi'(\underline{\mu}(\mu)) \\ -\psi'(\bar{\mu}(\mu)) & \psi'(\underline{\mu}(\mu)) \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 \\ -\frac{1}{2\sigma^2 c} \end{bmatrix} = \begin{bmatrix} -\frac{\phi(\bar{\mu}(\mu)) - \phi(\bar{\mu}(\mu))}{\phi'(\underline{\mu}(\mu))[\bar{\mu}(\mu) - \underline{\mu}(\mu)]} & > 0 \\ -\frac{\phi(\underline{\mu}(\mu)) - \phi(\bar{\mu}(\mu))}{\phi'(\underline{\mu}(\mu))[\bar{\mu}(\mu) - \underline{\mu}(\mu)]} & > 0 \end{bmatrix}$$

We now consider $\underline{\mu}(\mu)$. Suppose there exists $p_1 > p_2$ with the corresponding quitting boundaries $(\mu_1, \underline{\mu}_{p_1}(\mu_1))$ and $(\mu_1, \underline{\mu}_{p_2}(\mu_1))$, respectively. Denote the cutoff beliefs by $(\mu^*_{p_1}, \mu^{**}_{p_1})$ for price p_1 and by $(\mu^*_{p_2}, \mu^{**}_{p_2})$ for price p_2. Fixing an arbitrary $\mu_1 \in (\mu^*_{p_1}, 1]$, we know that the consumer is indifferent between quitting and searching for information when her belief is $(\mu_1, \underline{\mu}_{p_1}(\mu_1))$ and the price is p_1. Since $p_2 < p_1$, one can see that the value of searching for information when her belief is $(\mu_1, \underline{\mu}_{p_1}(\mu_1))$ and the price is p_2 is strictly higher than zero. So, the consumer will keep searching for information. Thus, $\underline{\mu}_{p_2}(\mu_1) < \underline{\mu}_{p_1}(\mu_1)$.

Therefore, the entire search region shifts upwards as the price increases.

(2) Comparative statics w.r.t. c

We first consider $\bar{\mu}(\mu)$. Fixing an arbitrary $\mu \in (\mu^{**}, 1]$, recall the system of equations (7):

$$\begin{aligned} \phi(\underline{\mu}(\mu)) - \phi(\bar{\mu}(\mu)) &= \frac{1}{2\sigma^2 c} \\ \psi(\underline{\mu}(\mu)) - \psi(\bar{\mu}(\mu)) &= \frac{p-\mu}{2\sigma^2 c} \end{aligned}$$

By the implicit function theorem, we obtain:

$$\begin{bmatrix} \frac{\partial \bar{\mu}(\mu)}{\partial c} \\ \frac{\partial \underline{\mu}(\mu)}{\partial c} \end{bmatrix} = - \begin{bmatrix} -\phi'(\bar{\mu}(\mu)) & \phi'(\underline{\mu}(\mu)) \\ -\psi'(\bar{\mu}(\mu)) & \psi'(\underline{\mu}(\mu)) \end{bmatrix}^{-1} \cdot \begin{bmatrix} \frac{1}{2\sigma^2 c^2} \\ \frac{p-\mu}{2\sigma^2 c^2} \end{bmatrix}$$

$$= \frac{1}{2\sigma^2 c^2 \phi'(\bar{\mu}(\mu))\phi'(\underline{\mu}(\mu))[\bar{\mu}(\mu) - \underline{\mu}(\mu)]} \cdot \begin{bmatrix} \phi'(\underline{\mu}(\mu))(p - \mu - \underline{\mu}(\mu)) \\ \phi'(\bar{\mu}(\mu))(p - \mu - \bar{\mu}(\mu)) \end{bmatrix}$$

The consumer purchases the product when the belief is $(\mu, \bar{\mu}(\mu))$. So, $\mu + \bar{\mu}(\mu) - p > 0$. The consumer stops searching and does not purchase the product when the belief is $(\mu, \underline{\mu}(\mu))$. So, $\mu + \underline{\mu}(\mu) - p < 0$. We also have $\phi'(x) = -\frac{1}{x^2(1-x)^2} \Rightarrow \phi'(x) < 0, \forall x$. Thus, we obtain:

$$\begin{bmatrix} \frac{\partial \bar{\mu}(\mu)}{\partial c} < 0 \\ \frac{\partial \underline{\mu}(\mu)}{\partial c} > 0 \end{bmatrix}$$

We now consider $\underline{\mu}(\mu)$. Suppose there exists $c_1 > c_2$ with the corresponding quitting boundaries $(\mu_1, \underline{\mu}_{c_1}(\mu_1))$ and $(\mu_1, \underline{\mu}_{c_2}(\mu_1))$, respectively. Denote the cutoff beliefs by $(\mu^*_{c_1}, \mu^{**}_{c_1})$ for price c_1 and by $(\mu^*_{c_2}, \mu^{**}_{c_2})$ for price c_2. Fixing an arbitrary $\mu_1 \in (\mu^*_{c_1}, 1]$, we know that the consumer is indifferent between quitting and searching for information when her belief is $(\mu_1, \underline{\mu}_{c_1}(\mu_1))$ and the price is c_1. Since $c_2 < c_1$, one can see that the value of searching for information when her belief is $(\mu_1, \underline{\mu}_{c_1}(\mu_1))$ and the price is c_2 is strictly higher than zero. So, the consumer will keep searching for information. Thus, $\underline{\mu}_{c_2}(\mu_1) < \underline{\mu}_{c_1}(\mu_1)$.

(3) Comparative statics w.r.t. σ^2

c and σ^2 always appear together as $2\sigma^2 c$ in the equations. So, the qualitative result of the comparative statics w.r.t. σ^2 is the same as the comparative statics w.r.t. c.

□