Precision-Recall Tradeoff in Algorithmic Targeting

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Abstract

One of the most basic uses of machine learning algorithms by firms is to enable the targeting of valuable consumers by predicting the likelihood with which they will be interested in converting to product purchase. Firms face a fundamental precision-recall trade-off in choosing their algorithmic targeting policies. They can choose to target a smaller set of consumers with a high probability of conversion (precision) but miss out on many consumers who might still be potentially interested in the product. Conversely, firms can target a larger set of consumers (recall), but this implies a greater probability that their targeting is wasted on uninterested consumers. We study this precision-recall trade-off in the context of the competition between two firms who strategically choose their algorithmic targeting policies. We show that competing firms favor a targeting policy that has higher precision but lower recall as compared to a monopoly. Under competition, firms target fewer consumers when their algorithms are more correlated. They also strategically decrease the precision of their targeting policies in order to reduce competition. If firms could endogenously choose their algorithmic correlation, then there is an equilibrium incentive to decrease the correlation between the algorithms. They may invest and earn a positive expected profit if the initial correlation is low, but may not invest and earn zero profits if the initial correlation is already high.

Keywords Precision-Recall Trade-off, Targeted Advertising, Machine Learning, Algorithms

JEL Codes D43, L13, M37

I Introduction

Targeting is fundamental to marketing strategy. One of the general challenges in targeted marketing is reducing wastage: Firms need to focus their marketing budget on consumers who are most likely to be interested in their products, which requires them to make predictions of consumer types based on observable characteristics and behavior. In the contemporary digital economy and big data environments firms have rich information on consumer characteristics, opinions, behaviors, and social interactions. Firms, as part of their data analytics strategy, can increasingly use AI and machine learning algorithms to target consumers.

At the core the prediction of consumer types through a machine learning algorithm can be interpreted as a classification problem - i.e., trying to predict whether a consumer is interested in the product and the likelihood that she will make a purchase if targeted by the firms' marketing activities. In general, any algorithm, ranging from a Logit regression to a sophisticated neural network, takes data on observed consumer characteristics and behaviors as input and produces a probability or likelihood of conversion as the output which firms can use for targeting consumers. In other words, in most contexts involving algorithmic targeting, firms don't target based on binary signals as is typically modeled in the literature. Instead, they face a *distribution of probabilities* for which they need to decide whom to target. Our framework captures this feature of the classification problem and brings it to bear on the algorithmic targeting strategies of firms.

In reality, the classification of consumers by algorithms is always imperfect because of data limitations, privacy concerns or model selection constraints. Given this, the firm faces a fundamental trade-off associated with classification problems that is relevant for all machine learning models, namely the "precision-recall" trade-off. In our context of algorithmic targeting, precision is defined as the share of interested consumers among all consumers targeted by the firm - i.e., it measures to what extent consumers targeted are indeed interested in the product. On the other hand, recall is defined as the share of interested consumers targeted out of all interested consumers.

Firms can either focus on a smaller set of consumers with high predicted probabilities of conversion (precision) so that most targeted consumers indeed turn out to be those who are interested and purchase the product, or they can target a larger set of consumers so that most of the interested consumers are targeted (recall), but not both. For those who are predicted to be moderately likely to be interested, some of them would be interested, but on whom the firm will not target and miss out if it prioritizes precision; some would be uninterested, on whom the firm will waste its marketing budget if it prioritizes recall. In practice, firms are well aware of this trade-off. For example, LinkedIn suggests advertisers to "balance precision with volume" in their targeting policies[1].

In this paper, we examine the precision-recall trade-off in competitive targeting. How should a firm choose its targeting policy, and more importantly, when firms are competing, what are the equilibrium targeting policies? What is the impact of the correlation in the firms' algorithms on their precision and recall choices?

The model considers competition between two firms for a set of consumers. Consumers have a binary type: they are either interested in the product and will make a purchase if targeted, or are not interested and will not make a purchase even if they are targeted. Firms do not know the type of consumers and they rely on

[1] https://www.linkedin.com/business/marketing/blog/content-marketing/what-is-b2b-marketing-definition-strategy-and-trends

targeting algorithms to make predictions. The algorithm takes as input the available consumer and market information and generates as output a distribution of probabilities that consumers are interested in purchasing. Firms then simultaneously decide who to target based on the predictions. Each firm makes a profit for every interested consumer that they end up targeting, and they also incur a cost for every consumer targeted. If a consumer is targeted by both firms, the profit they generate is lower than the monopoly profit when the consumer is targeted by only one firm.

For the competing firms, given possible correlation in their predictions, the equilibrium expected profit of a firm depends on the other firm's targeting policy. We first show that there exists a unique mixed-strategy symmetric equilibrium. In this equilibrium, when firms' algorithmic predictions have a high enough correlation, they will never target any consumer for sure. Consumers with higher predicted probabilities of being interested will be targeted with a higher probability. When the correlation is not that high, both firms target consumers who they predict to have a sufficiently high likelihood of being interested for sure, while mixing for consumers who have moderately high probabilities.

Competition has a nuanced effect on the precision-recall trade-off in targeting. The equilibrium targeting intensity - i.e., the overall number of consumers targeted - decreases in the correlation of firms' predictions. This is because an overlap in targeting reduces the expected profits. Thus, competition makes both firms strategically target fewer consumers as their predictions become more correlated, as this helps to soften competition.

But on the extensive margin (i.e., the lowest probability consumer being targeted), the firms become less selective as their predictions become more correlated. The intuition is as follows. Consider marginal consumers, i.e., those with a probability barely above the threshold of generating non-zero profits even if targeted by a monopoly. When the predictions are highly correlated, both firms' predictions are largely aligned on the identity of the marginal consumers. Each firm can now be more confident that those marginal consumers will be unlikely to be targeted by the other firm by mistake because the overall targeting intensity also decreases. Thus, a firm becomes more inclined to target some of those consumers. In the extreme, if both firms have exactly the same prediction, then both have incentives to target some consumers who are at the threshold probability, i.e., the expected profit from targeting equals the unit cost of targeting, as in the monopoly case. In other words, two firms with the same predictions target the same range of consumers as the monopoly.

How does competition change the precision and recall choices? We find that as firms' predictions converge, recall decreases as firms target fewer consumers. Surprisingly, precision also decreases. In other words, firms on average target consumers who are less likely to be interested. The total number of consumers being targeted generally decreases with competition even if firms' predictions are uncorrelated, implying that competition may reduce consumer welfare by creating less information value for consumers. As firms' predictions become more correlated, the information value further decreases.

We extend the model to allow for endogenous correlation of predictions. We find firms have an incentive to invest in lowering the correlation of their predictions. In another extension, we also consider asymmetric pure strategy equilibria, to show that our main result on the precision-recall trade-off under competition continues to hold.

I.A Related Research

We contribute to the targeted advertising literature (Chen et al., 2001; Chen and Iyer, 2002; Iyer et al., 2005; Bergemann and Bonatti, 2011; Johnson, 2013; Shin and Yu, 2021; Choi et al., 2023; Ning et al., 2023). Contrary to the standard assumption that firms receive a binary signal of the consumer's type, we model the micro foundation of the signal structure based on a fundamental feature of machine learning, which allows us to study the precision-recall trade-off. One way to think about our setting is that we allow the firm to optimally choose its signal structure by picking a threshold, which is not necessarily unbiased. However, our setting allows for a much richer strategy space.

To the best of our knowledge, our paper is the first to introduce the problem of precision-recall trade-offs in in choice of marketing strategy. Berman et al. (2023) uses the information design framework to study the problem of precision and recall in recommendation algorithms. However, they do not model the precision-recall trade-off in predictions and targeting which is the focus of this study.

We also contribute to the growing literature on strategic interactions of algorithms and machine learning models(Liang, 2019; Miklós-Thal and Tucker, 2019; Salant and Cherry, 2020; Calvano et al., 2020; O'Connor and Wilson, 2021; Montiel Olea et al., 2022). Closely related to our work, Iyer and Ke (2023) consider the model selection problem and the bias-variance trade-off in the context of targeting. They find competition favors bias because it can help firms to manage overlaps in targeting, thereby softening competition. We focus on a different trade-off that is also fundamental to all machine learning models. While both are fundamental trade-offs in statistical learning, the precision-recall tradeoff is not about model selection, but rather about firms' decision rule given the prediction that the model generates. In other words, Iyer and Ke (2023) studies how to pick an algorithmic model, while we focus on the micro-foundations of how to deploy algorithmic targeting - i.e., who to target given the model's predictions. This is a feature which applies in general to the deployment of any machine learning model design that the firm may end up choosing.

II Model Setup

II.A Basic Setup

There is a market with up to two firms and a unit mass of consumers. Consumers may be of two types: interested or uninterested. Interested consumers, if targeted, will make a purchase. Uninterested consumers, as well as interested consumers who are not targeted by the firm, will not make a purchase.[2] The prior probability of a consumer being interested is μ_0. Firms don't know the consumer types. Instead, they rely on predictions from a machine learning algorithm to identify interested consumers. An algorithm, e.g., Logit regression or a neural network, uses data on consumer and market characteristics, and yields a prediction for each consumer, which is the likelihood with which the consumer is interested in the product. Accordingly, we assume that the prediction for consumer i is the probability p_i of that consumer i being interested. The distribution of the predictions $f(p)$, is defined on the support $[0, 1]$. This distribution characterizes the informativeness of the algorithm, which is determined by the limits of the data and the sophistication of the algorithm. The distribution is Bayesian consistent, i.e., the expected probability of a consumer being interested is $\int pf(p)\mathrm{d}p = \mu_0$, the prior probability. At one extreme, a perfect algorithm has a bimodal distribution with two mass points at $p = 0$ and $p = 1$. Conversely, a fully uninformative algorithm has an unimodal distribution with one mass point at $p = \mu_0$.

Firms produce competing products. The only way they reach consumers is through targeted advertising. If an interested consumer is targeted by only one firm, they will generate value v for the firm. If both firms target the same consumer, they will generate an expected value $w < v$ for each firm. The unit cost of targeting is c.

The firm decides which consumers to target based on the predictions of its algorithm. Specifically, consider a targeting policy of $q(p) \in [0, 1]$, in which $q(p)$ is

[2] A standard interpretation of this assumption is advertising as information - that advertising creates awareness of the existence of the product.

the share of probability-p consumers being targeted. Therefore, we allow firms to use mixed strategies which can be interpreted in the standard manner as different advertising intensities (for example, across time or media channels).

II.B Predictions of Competing Firms

When both firms are predicting consumer types, their predictions may be correlated. In extreme cases, the predictions from two perfect (or uninformative) algorithms can be perfectly correlated. Our analysis focuses on the case where the two algorithms have identically distributed predictions but may be positively correlated. This allows us to derive general results on how correlations in predictions affect targeting policies for all algorithms.

Specifically, we assume that for any given consumer, if the prediction from firm 1 is p, then with probability ρ, the prediction from firm 2 is also p, and with probability $1-\rho$, the prediction is uniformly drawn from $f(p)$ In other words, for any given consumer, the prediction of firm 2 agrees with firm 1 with probability ρ. Otherwise, firm 2 will assign this consumer a probability according to the overall distribution of its prediction. We will first consider the case of exogenous ρ in the main analysis and then subsequently endogenize ρ in an extension.

III Analysis

III.A Monopoly Benchmark

The monopoly firm's targeting strategy will be to choose a threshold policy $\underline{p}_m \in (0,1)$, such that it targets all consumers with $p \geq \underline{p}_m$: $q(p) = 1_{p \geq \underline{p}_m}$. At the threshold $\underline{p}_m$, the marginal revenue of targeting $\underline{p}_m v$ equals the targeting cost c, thus $\underline{p}_m = c/v$.

In monopoly, the precision of the optimal targeting policy is:

$$\frac{\int_{\underline{p}_m}^1 pf(p)dp}{\int_{\underline{p}_m}^1 f(p)dp} = \frac{\int_{\underline{p}_m}^1 pf(p)dp}{a_m} \tag{1}$$

The recall of the optimal targeting policy is:

$$\frac{\int_{\underline{p}_m}^1 pf(p)dp}{\int_0^1 pf(p)dp} = \frac{\int_{\underline{p}_m}^1 pf(p)dp}{\mu_0} \tag{2}$$

Both the precision and recall depend on the firm's targeting policy $\underline{p}_m$ and the informativeness of the algorithm, as characterized by the distribution of the predictions $f(\cdot)$. Notice that higher precision implies that the algorithm targets more interested consumers rather than uninterested ones, while higher recall implies that the algorithm targets more of the interested consumers. Given the algorithm, the precision increases while the recall decreases in the targeting threshold $\underline{p}_m$. By targeting only a small subset of consumers who are highly likely to be interested, the firm generates a low false negative rate and achieves a high precision. However, the firm sacrifices recall because it does not target many consumers who are moderately likely to be interested. So, the firm generates a high false negative rate and ends up with a low recall. In one extreme, as $\underline{p}_m \to 1$, Precision $\to 1$, and Recall $\to 0$. The opposite applies when the targeting threshold $\underline{p}_m$ is low. In the other extreme, as $\underline{p}_m \to 0$, Precision $\to \mu_0$, and Recall $\to 1$.

To what extent the firm's optimal targeting policy prioritizes precision vs. recall depends on the profits and cost of targeting. If the targeting cost is high relative to its benefit (high c/v), the monopoly only wants to target high-probability consumers to save targeting costs (precision over recall). In contrast, if the targeting cost is low relative to its benefit (low c/v), the monopoly does not want to miss out on consumers who may be moderately likely to be interested (recall over precision).

III.B Main Analysis: Competitive Algorithmic Targeting

In this section, we consider the case of the competing firms choosing their targeting policies $q_i(p)$ simultaneously. In the main analysis, we focus on the symmetric equilibrium while leaving the asymmetric equilibria case to the extension.

We first note that there does not exist a symmetric pure strategy equilibrium for any positive ρ.[3] We thus consider the *symmetric mixed-strategy equilibrium*. Suppose each firm targets probability p consumers with probability $q(p)$. Let $\underline{p} = \inf\{p : q(p) > 0\}$, i.e., the lowest probability consumer that firms target for strictly positive probability.

The expected payoff of a firm from targeting a probability p consumer is:

$$[\rho q(p) + (1-\rho)a]pw + [1 - \rho q(p) - (1-\rho)a]pv - c$$

where $a = \int_{\underline{p}}^{1} f(p)q(p)dp$, the aggregate targeting intensity given $q(p)$. $\rho q(p) + (1-\rho)a$ is the probability that a probability p-consumer is targeted by both firms, in which case the focal firm gets a payoff of pw. There are two such cases: i) when the predictions coincide and the other firm also targets this consumer, which happens with probability $\rho q(p)$, or ii) when the two predictions differ but the algorithm of the other firm assigns a sufficiently high probability (i.e., above the threshold $\underline{\mathrm{p}}$) and targets this consumer regardless, which happens with probability $(1-\rho)a$. With the complementary probability, the consumer is targeted by only one firm, which will result in a payoff of pv.

Assumption 1 $0 \le w < c < v$. *A firm prefers to target an interested consumer by*

[3] Suppose otherwise there is such equilibrium. Consider $\underline{p}$ as the lowest probability for which firms target. Clearly, $\underline{p} \ge \underline{p}_m$, the monopoly targeting threshold. We now have $\underline{p}\{[\rho + (1-\rho)a]w + [1-\rho-(1-\rho)a]v\} - c \ge 0$. By targeting probability $p - \epsilon$ consumer, the firm obtains an expected payoff of $(\underline{p}-\epsilon)\{[(1-\rho)a]w + [1-(1-\rho)a]v\} - c = \underline{p}\{[\rho + (1-\rho)a]w + [1-\rho-(1-\rho)a]v\} - c + \rho(v-w)\underline{p} - \epsilon\{[(1-\rho)a]w + [1-(1-\rho)a]v\} \ge 0 + \rho(v-w)\underline{p}_m - \epsilon > 0$ for ϵ small enough. Therefore, firms have an incentive to deviate.

itself, but does not prefer to target an interested consumer if it is together with the competitor.

In equilibrium, there are two types of targeting strategies. Either the firm never targets any consumer for sure, or it targets a set of the highest probability consumers for sure, and the lower probability consumers partially.

Lemma 1 *In any symmetric equilibrium, either $q(p) \in (0,1), \forall p \in (\underline{p}, 1]$ or there exists $\overline{p} \in (\underline{p}, 1)$ such that $q(p) = 1, \forall p > \overline{p}$, and $q(p) \in (0,1), \forall p \in (\underline{p}, \overline{p})$.*

For notational ease, we let $\overline{p} = 1$ if the firm never targets any consumer for sure. Under competition, the precision of the optimal targeting policy is:

$$\frac{\int_{\underline{p}}^{1} pq(p)f(p)dp}{\int_{\underline{p}}^{1} q(p)f(p)dp} = \frac{\int_{\underline{p}}^{1} pq(p)f(p)dp}{a} \tag{3}$$

The recall of the optimal targeting policy is:

$$\frac{\int_{\underline{p}}^{1} pq(p)f(p)dp}{\int_{0}^{1} pf(p)dp} = \frac{\int_{\underline{p}}^{1} pq(p)f(p)dp}{\mu_0} \tag{4}$$

The firm is indifferent between targeting or not for $p \in (\underline{p}, \overline{p})$. The firm's expected profit for those consumers is 0, its payoff from not targeting.

$$[\rho q(p) + (1-\rho)a]pw + [1 - \rho q(p) - (1-\rho)a]pv - c = 0$$

$$\Rightarrow \quad q(p) = \frac{v - (1-\rho)(v-w)a - c/p}{\rho(v-w)}, \ \forall p \in (\underline{p}, \overline{p}) \tag{5}$$

$$q(\underline{p}) = 0$$

$$\Rightarrow \quad \underline{p} = \frac{c}{v - (1-\rho)(v-w)a} \tag{6}$$

The firm strictly prefers targeting to not targeting for $p > \overline{p}$. So, it targets those high-probability consumers for sure and obtains a positive profit.[4] To emphasize that $\underline{p}$ and a depend on ρ, we use the notation $\underline{p}(\rho)$ and $a(\rho)$ when necessary to avoid confusion. Assumption 1 implies that $\underline{p}(1) = c/v \in (0,1)$. Lemma 2 establishes that such an equilibrium exists.

[4]This set is empty if $\overline{p} = 1$.

Lemma 2 *There exists a unique mixed-strategy symmetric equilibrium. The firm targets some consumers with a positive probability in equilibrium. Furthermore, there exists* $\widehat{\rho} \in [0, \frac{v-c}{v-w}]$ *such that firms never target any consumer for sure if and only if* $\rho \geq \widehat{\rho}$. *Otherwise, the firm targets consumers with probabilities* $p \geq \overline{p}$ *for sure and mix for consumers with probabilities between* $\underline{p}$ *and* $\overline{p}$.

Figure 1 illustrates the targeting probability of the optimal targeting policy for different correlations ρ. When the correlation is high, a firm never targets any consumer for sure because the rival firm has a high likelihood of having the same prediction for the consumer. In contrast, when the correlation is low, a firm targets high-probability consumers for sure because the other firm has a high chance of having a different prediction and not targeting that consumer. However, for lower-probability consumers, the potential gain for a firm from targeting decreases. At the same time, the probability of overlapping in targeting stays the same if firms were to target that consumer for sure. Thus, when the predicted probability for a consumer is below a threshold, the firms in equilibrium mix between targeting and not targeting that consumer in order to soften the competition.

We now proceed to establish one of the main results of this paper by comparing the equilibrium targeting policies of the monopoly benchmark with the duopoly case in the following proposition. It shows that compared with monopoly, firms lean towards precision under competition.

Proposition 1 *The duopoly's optimal targeting policy has a higher precision and a lower recall than the monopoly's.*

This proposition connects the precision-recall trade-off that is fundamental to machine-learning algorithms to the equilibrium incentives of competitive firms. Relative to a monopoly, competition favors precision over recall in algorithmic targeting. The intuition is that competition reduces the expected profit from targeting due to overlaps in targeted consumers between the rival firms. Firms therefore strategically

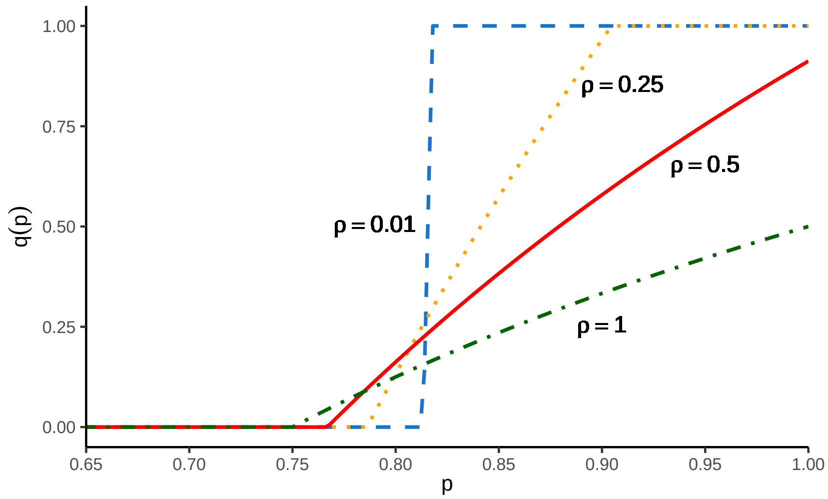


Figure 1: Targeting probability of the Optimal Targeting Policy for $c = .75, v = 1, w = .5, f \sim U[0,1]$.

respond in equilibrium by concentrating their targeting efforts on consumers who are predicted by their algorithms to have high likelihood of conversion, thereby leading to an increase in the precision of their targeting.

We now discuss the details of equilibrium and the implications of the important comparative static predictions. To begin with, consider the effect of the correlation in the firms' predictions on the equilibrium profits: When the correlation is sufficiently high ($\rho > \hat{\rho}$) then as shown in 2 the equilibrium targeting policy involves firms not targeting any consumers for sure. This implies that the equilibrium profits are zero. Conversely, when the correlation is lower a set of high probability consumers will be targeted for sure and this implies positive equilibrium profits. Further, in this case lower values of ρ are associated with increasing profits. This suggests that in terms of the profit impact greater correlation in the firms algorithmic predictions is analogous to that of more intense competition between the firms. Thus there is an implication that firms will choose uncorrelated algorithms ($\rho = 0$) if it is costless. In section

IV.A, we will consider an extension analysis that endogenizes ρ.

Next, as illustrated in Figure 2, as ρ increases and the predictions of the firm converge, firms end up targeting fewer consumers and the recall decreases. As the predictions converge, the targeting of the firms is more likely to coincide, reducing the returns from targeting. Interestingly, and somewhat counter to expectation, we find that as long as the level of correlation is high enough ($\rho > \hat{\rho}$), the precision also decreases as ρ increases. As the correlation increases, the high-probability consumers become less attractive because both firms increasingly identify these consumers as the more valuable high-probability consumers and compete by setting greater targeted share. Conversely, the more moderate-probability consumers become relatively more attractive to firms because they are now less likely to be targeted. With higher correlation, when a firm predicts a moderate probability consumer, it can expect that the consumer is less likely to be predicted as a high probability consumer by the rival, implying that the rival will have the incentive to target moderate consumers less aggressively. Overall, this leads to $q(p)$ becoming flatter, decreasing the precision.

And the targeting threshold $\underline{p}$ also decreases in ρ - on the extensive margin, firms become less selective as their predictions converge. In the extreme, when $\rho = 1$, both predictions will always coincide. If, on the margin, one firm targets $\underline{p}$ consumers with infinitesimal probability, the other firm will be a de facto monopoly. Indeed, when $\rho = 1$, we have $\underline{p} = c/v$, which is the same as in the monopoly case.

The following proposition summarizes the comparative statics results.

Proposition 2 *The following comparative statics apply*

1. *As firms' predictions become more correlated, recall, the overall targeting probability, a, and the lowest type firms target, $\underline{p}$, decrease.*

2. *The precision decreases in ρ for $\rho > \widehat{\rho}$. For $\rho < \widehat{\rho}$, the lowest type firms target for sure, $\overline{p}$, increases in ρ.*

3. *Firms' profits are zero when the correlation is high such that firms don't target*

any consumer for sure. Otherwise, the profits are positive and decreasing in ρ.

4. *The overall targeting probability a decreases in the unit cost of targeting c.*

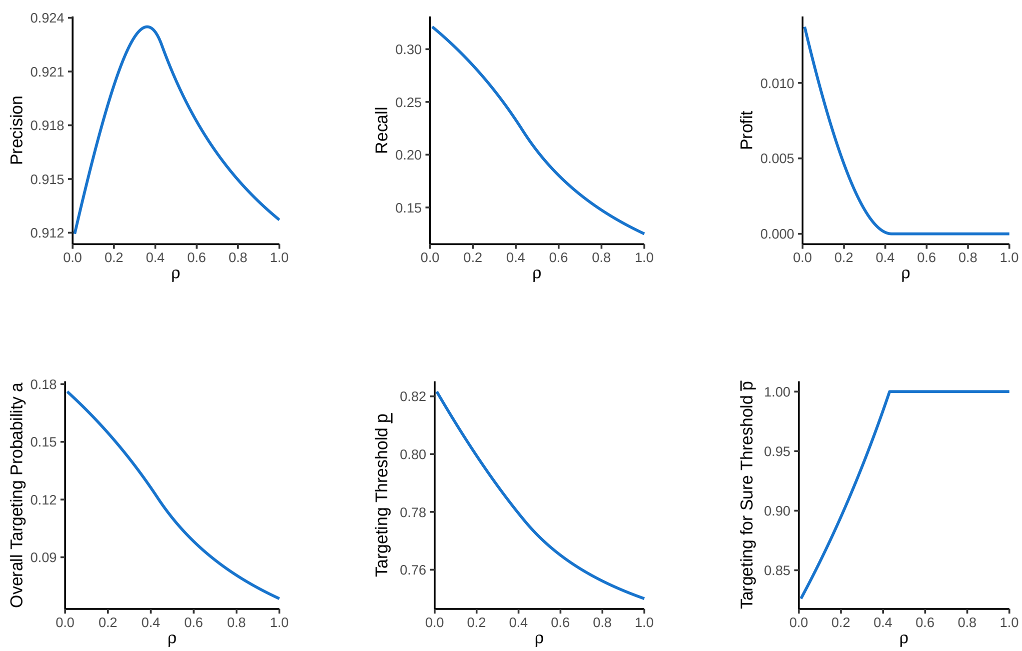


Figure 2: Comparative Statics with regard to ρ for $c = .75, v = 1, w = .5, f \sim U[0,1]$.

III.B.1 Consumer Surplus

Targeting has two effects on consumer surplus. On one hand, it provides information value to consumers if it targets interested consumers. Targeted consumers who find the product fits their needs will buy and obtain a positive surplus. The total information value of a targeting policy is proportional to $\int_{\underline{p}}^{1} pq(p)f(p)dp$. On the other hand, mis-targeting is costly to consumers by wasting their time. Consumers will obtain a negative surplus if they see information about some uninterested products. The total mis-targeting cost of a targeting policy is proportional to $\int_{\underline{p}}^{1}(1-p)q(p)f(p)dp$. We separately discuss the welfare implications of these two components because it is not obvious how to aggregate them.

We find that a higher correlation decreases the information value because fewer interested consumers end up being targeted. Since $\int_{\underline{p}}^{1} pq(p)f(p)dp$ is the recall multiplied by the prior probability, μ_0, and the prior probability does not depend on the targeting policy, the information value of a targeting policy is proportional to its recall. The comparative statics of the information value of the optimal targeting policy is the same as the comparative statics of the recall, which decreases in ρ.

Note that $\int_{\underline{p}}^{1}(1-p)q(p)f(p)dp = a - \int_{\underline{p}}^{1} pq(p)f(p)dp = a \cdot (1 - \text{precision})$. When the precision increases in ρ (for small ρ), one can see that the total mis-targeting cost of the optimal targeting policy decreases in ρ because the overall targeting probability, a, always decreases in ρ. When the precision decreases in ρ (for large ρ), $1 - \text{precision}$ increases in ρ while a decreases in ρ. It is ambiguous whether the total mis-targeting cost of the optimal targeting policy increases or decreases in ρ.

IV Extensions

IV.A Endogenous Prediction Correlations

We now examine the implications when firms are able to endogenously choose the correlations in their predictions. When firms rely on public data or common data analytics providers or tools, their targeting predictions are likely to be more correlated. Conversely, a firm may also invest in internal analytics organization and private consumer and market data, which may lead to a decrease in the prediction correlations between the firms. For example, for a fixed size of the training sample and thus the probability distribution f, a firm may use more private data as opposed to publicly available data in order to differentiate their prediction models.

Consider the following model that endogenizes the prediction correlation ρ:

In the first period, both firms simultaneously make costly investments in lowering ρ. This can be seen as each firm's decision to invest in private data and analytics within the organization. Nevertheless, such investments will be a public good since any firm's investment will help determine the prediction correlation between the firms. Specifically, the correlation ρ is decreasing in the investments of both firms: $\rho = \rho(I_1, I_2)$, in which I_1, I_2 are the first-period investment costs of firms 1 and 2. Denote the initial correlation as $\rho_0 := \rho(0, 0)$. In the second period, both firms simultaneously choose the targeting policy exactly as in the main model.

We make the following assumptions on the investment technology.

Assumption 2 *$\rho(I_1, I_2)$ is smooth and strictly decreasing in $I_j, j = 1, 2$. $\rho(I_1, I_2) > 0,\ \forall I_1, I_2 < v$. Define $\rho_1(I_1) = \rho_0 - \rho(I_1, 0)$, $\rho_2(I_2) = \rho_0 - \rho(0, I_2)$, and $K_j(\Delta\rho) = \rho_j^{-1}(\Delta\rho)$. $K_j'(0) = 0, j = 1, 2$.*

$\rho_j(I_j)$ measures how much correlation a firm can unilaterally reduce by investing I_j when the other firm incurs zero costs. $K_j(\Delta\rho)$ is its inverse and measures how much costs a firm needs to incur to reduce the correlation by $\Delta\rho$ when the other firm does not invest and incurs zero costs. The assumptions $K_j'(0) = 0$ and $\rho(I_1, I_2) > 0,\ \forall\ I_1, I_2 < v$ mean that it is very easy to reduce the correlation by a little bit, but very costly to reduce the correlation all the way to zero. In reality, firms can easily reduce ρ a bit by using a little more private data. However, it is almost impossible for firms to achieve zero correlation in their predictions in a competitive market, regardless of how they try to differentiate.

Proposition 3 *Firms invest and earn a positive expected profit if $\rho_0 < \widehat{\rho}$. They will not invest and earn zero profits if $\rho_0 > \widehat{\rho}$.*

One may think that firms have a stronger incentive to invest in reducing the correlation if the initial correlation is high because otherwise they will get zero profits without the investment. It turns out to be the opposite. Firms will invest if the initial correlation is already low enough, such that they will earn a positive profit without investment. The low-cost assumption for small investment does not directly imply that firms will invest because the benefit of a small investment may also be very low. We prove that the benefit of investing a little bit is roughly linear in the investment cost, which outweighs the investment cost. In contrast, as Figure 2 shows, the profit is zero for an interval of $\rho \in [\widehat{\rho}, 1]$. So, a lower ρ will increase the profit (not taking into account the investment costs) only if firms invest a lot such that $\rho < \widehat{\rho}$. In that case, the investment cost may be so high that firms prefer not to invest.

The proposition provides a structure to consider the industry scenarios which are likely to encourage investments by firms in private data analytics. For example, when there is already a well-entrenched public data system which is used by the firms, the incentive to invest within the organization's data analytics is attenuated. Indeed, given that profits with a high correlation are zero, this would precisely be the situation in which such investments by a firm would have been beneficial. In a sense, there is a suggestion in the model that there can be under-investment in private data because the investment by a firm to soften competition through reduced correlation is a public good.

IV.B Robustness: Pure Strategy Asymmetric Equilibria

In our main analysis, we focus on the symmetric mixed-strategy equilibrium - this is because both firms are symmetric, and thus it has a natural interpretation. We also note that there are also a multiplicity of asymmetric pure strategy equilibria. Our objective in this section is to show the robustness and the generality of our main result on the precision-recall under competition to the asymmetric equilibria. While this makes it difficult to analyze them and study the comparative statics, we nonetheless characterize some common features of those equilibria.

Define $\underline{p} := \inf\{p \in [0,1] : q_1(p) > 0 \textit{ or } q_2(p) > 0\}$, where $q_i(p)$ is firm i's targeting probability of a type p consumer. Then, one can see that at least one firm targets at any $p > \underline{p}$. In addition, pure strategy symmetric equilibrium does not exist. If both firms target consumers whose probability is larger than or equal to $\underline{p}$, then each firm wants to deviate by targeting $\underline{p} - \epsilon$. Lastly, $\underline{p} \geq \underline{p}_m$. This is because a monopoly does not target below $\underline{p}_m$, and the presence of competition makes targeting at any probability less attractive.

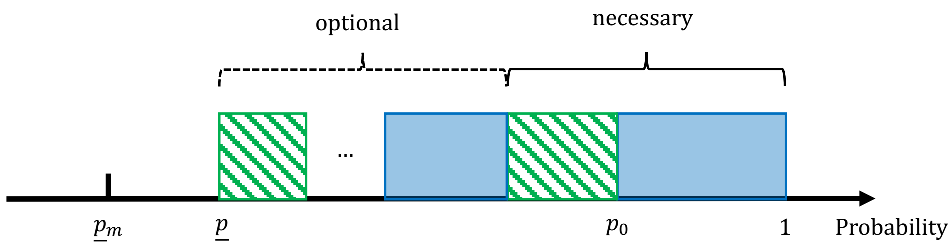
Proposition 4 *There exists an equilibrium in which one firm acts as a monopoly if* $\rho \geq \frac{a_m w + (1-a_m)v - c}{(1-a_m)(v-w)}$. *In any other pure strategy asymmetric equilibria, there exists* $p_0 \in (\underline{p}, 1)$ *such that exactly one firm targets consumers whose* $p \in (\underline{p}, p_0)$. *Either one or both firms target every consumer whose* $p > p_0$. *The firm who targets probability* $\underline{p}$ *consumer has a higher overall targeting probability.*

The recall of either firm is lower than the monopoly case. The precision of at least one firm is higher than the monopoly case.

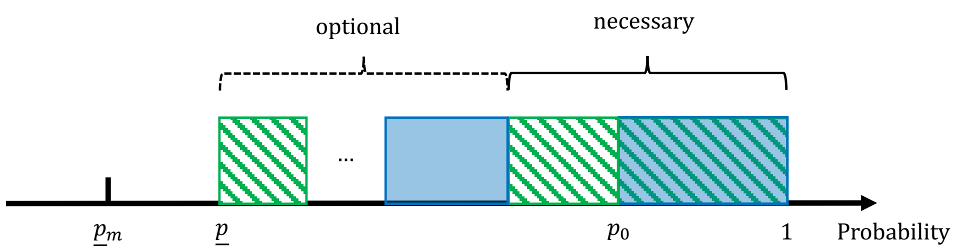
Figure 3 illustrates the equilibrium strategy. When the prediction correlation is high (Figure 3a), it is possible that a firm acts as a monopoly and drives the opponent out of the market. Even if the opponent predicts a consumer to have probability one, it knows that the monopoly is highly likely to target that consumer as well. The expected payoff of targeting is negative, and thus, it does not enter the market. Other than this special case, both firms target some consumers. For consumers moderately interested, firms partition their targeting region to soften competition. For really valuable consumers (high p), either one (Figure 3b) or both firms (Figure 3c) target them. Whether one or both firms target those consumers depends on the correlation of their prediction and the overall targeting probability of each firm. There is at least one partitioned region where only one firm targets consumers, and there can be more than one partitioned region.

The main message that firms prioritize precision over recall under competition still holds in all asymmetric equilibria. Except for the really valuable consumers, firms do not simultaneously target consumers predicted to have the same probability, which

(a) Monopoly



(b) No overlapping (partition)



(c) Overlapping at high-probability consumers

Figure 3: Pure Strategy Asymmetric Equilibria

avoids overlaps in targeting the same consumer due to having the same predictions.[5] Thus, firms reduce their recall and focus on precision instead.

V Concluding Remarks

It is hard to overstate the importance of targeting in marketing. While the advances in data analytics allow firms to have unprecedented and at-scale targeting abilities, algorithmic targeting predictions may nevertheless not be perfect. Thus, firms invariably have to choose between precision - targeting a few consumers with high probabilities of conversion, or recall - targeting a large set of consumers, many of whom may not convert. This trade-off is a fundamental feature of all machine learning algorithms.

This paper introduces a micro-founded model to study competitive targeting policies in the presence of the precision-recall trade-off. We find that compared to a monopoly, the competition between firms lowers recall but increases precision. However, as competing firms have more correlated predictions, they tend to not only target fewer consumers as the return to targeting decreases (lower recall) but to also target less probable consumers (lower precision) in order to avoid head-to-head competition. This may also decrease consumer welfare through fewer interested consumers ending up being targeted, as well as through more uninterested consumers receiving ads. Firms have an incentive to invest to decrease the correlations of their predictions, but such investments, being a public good, also suffer from the free-rider problem.

The results have some important managerial and policy implications. For firms, the results prescribe optimal targeting policies based on profit maximization instead of ad hoc rules. Our main result on the precision-recall trade-off provides prescriptions on how firms should deploy their data analytics operations under competition. Furthermore, the equilibrium targeting policies under competition imply

[5] Notice that there are still overlaps in targeting even for moderately interested consumers because firms may have different predictions for the same consumer.

that firms will need to account for how much the predictions differ across competing firms in designing their targeting policies. Policy-wise, the results suggest that privacy regulations may have an unintended side effect of preventing competing firms from using private data to differentiate their predictions. This could have negative consequences both for the industry and consumers because firms may strategically change their targeting policies using less precision and recall.

While we focus on targeted advertising, the framework can be potentially applied to other marketing decisions such as product design, promotion, and pricing. For example, it would be interesting to study whether competition as measured by firms' predictions favors more distinct product designs. Another potential generalization is to extend the model beyond the binary type assumption. Finally, our precision-recall targeting framework can be used to study algorithmic discrimination in targeting in markets where the protected characteristics (e.g., race or gender) of consumers are salient and can be discriminated against.

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Appendix

Proof of Lemma 1. The lemma is equivalent to the following two statements:

Claim 1: If firms target a type p_1 consumer with a positive probability ($q(p_1) > 0$), then they target any higher-type consumer with a positive probability ($q(p) > 0,\ \forall p > p_1$).

Claim 2: If firms target a type p_2 consumer for sure ($q(p_2) = 1$), then they target any higher-type consumer for sure ($q(p) = 1,\ \forall p > p_2$).

Proof of Claim 1: If firms target a type p_1 consumer with a positive probability, then the expected payoff from targeting such a consumer is no less than the expected payoff from not targeting: $[\rho q(p_1) + (1-\rho)a]p_1 w + [1 - \rho q(p_1) - (1-\rho)a]p_1 v - c \geq 0$. Suppose they do not target a type $p > p_1$ consumer at all, $q(p) = 0$. By deviating and targeting a type p consumer for sure, the deviating firm can obtain an expected payoff of $[(1-\rho)a]pw + [1-(1-\rho)a]pv - c > [\rho q(p_1) + (1-\rho)a]p_1 w + [1 - \rho q(p_1) - (1-\rho)a]p_1 v - c \geq 0$. So, firms will deviate. A contradiction.

Proof of Claim 2: If firms target a type p_2 consumer for sure, then the expected payoff from targeting such a consumer is no less than the expected payoff from not targeting: $[\rho q(p_2) + (1-\rho)a]p_2 w + [1 - \rho q(p_2) - (1-\rho)a]p_2 v - c \geq 0$. Suppose they do not target a type $p > p_2$ consumer for sure, $q(p) < 1$. By deviating and targeting a type p consumer for sure, the deviating firm can obtain an expected payoff of $[(1-\rho)a]pw + [1-(1-\rho)a]pv - c > [\rho q(p_2) + (1-\rho)a]p_2 w + [1 - \rho q(p_2) - (1-\rho)a]p_2 v - c \geq 0$. So, firms will deviate. A contradiction. ■

Proof of Lemma 2.

Existence:

The theorem in Becker and Damianov (2006) states that any symmetric game with a finite number of players, whose strategy space is compact and Hausdorff and payoff function is continuous, has a symmetric Nash equilibrium in mixed strategies. It is natural to define the (pure) strategy space of the firm as the standard topology on

[0,1], which is compact and Hausdorff. Also, the firm's payoff function is continuous. Hence, there exists a symmetric Nash equilibrium in mixed strategies. One can see that no firm targeting cannot be a Nash Equilibrium, and the equilibrium strategy must be a cutoff strategy (target probability p consumers if and only if p is higher than a cutoff value). So, there exists a solution $\underline{p} < 1$ and $a^* > 0$.

Uniqueness:

1. Firms never target any consumer for sure ($\overline{p} = 1$)

$$
\begin{aligned}
a &= \int_{\underline{p}}^{1} f(p)q(p)dp \\
&= \int_{\frac{c}{v-(1-\rho)(v-w)a}}^{1} f(p)\frac{-c/p + (1-\rho)aw + [1-(1-\rho)a]v}{\rho(v-w)}dp \\
&= \int_{\frac{c}{v-(1-\rho)(v-w)a}}^{1} \frac{f(p)}{\rho(v-w)}[-c/p + v - (1-\rho)(v-w)a]dp
\end{aligned}
$$

For any fixed ρ, define:

$$G(a) := \int_{\frac{c}{v-(1-\rho)(v-w)a}}^{1} \frac{f(p)}{\rho(v-w)}[-c/p + v - (1-\rho)(v-w)a]dp - a$$

Then, $G(0) = \int_{\frac{c}{v}}^{1} \frac{f(p)}{\rho(v-w)}[v - c/p]dp - a > 0$

$$
\begin{aligned}
G'(a) &= \int_{\frac{c}{v-(1-\rho)(v-w)a}}^{1} -(1-\rho)(v-w)\frac{f(p)}{\rho(v-w)}dp - \\
&\quad c\left(-\frac{-(1-\rho)(v-w)}{[v-(1-\rho)(v-w)a]^2}\right)\left[v-(1-\rho)(v-w)a - \frac{c}{\frac{c}{v-(1-\rho)(v-w)a}}\right]\frac{f\left(\frac{c}{v-(1-\rho)(v-w)a}\right)}{\rho(v-w)} \\
&= -\frac{1-\rho}{\rho}\left[1 - F\left(\frac{c}{v-(1-\rho)(v-w)a}\right)\right] - 1 < 0
\end{aligned}
$$

Uniqueness then follows.

2. Firms target high-probability consumers for sure ($\overline{p} < 1$)

$$
\begin{aligned}
&[\rho + (1-\rho)a(\rho)]\overline{p}w + [1-\rho-(1-\rho)a(\rho)]\overline{p}v - c = 0 \\
\Rightarrow \ &\overline{p} = \frac{c}{[\rho + (1-\rho)a(\rho)]w + [1-\rho-(1-\rho)a(\rho)]v}
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
a &= \int_{\overline{p}}^{1} f(p)dp + \int_{\underline{p}}^{\overline{p}} f(p)q(p)dp \\
&= 1 - F(\overline{p}) + \int_{\underline{p}}^{\overline{p}} f(p)q(p)dp
\end{aligned}
$$

For any fixing ρ, define:

$$
H(a) := 1 - F(\overline{p}) + \int_{\underline{p}}^{\overline{p}} f(p)q(p)dp - a
$$

Then, $H(0) = 1 - F(\overline{p}) + \int_{c/v}^{\overline{p}} f(p)q(p)dp > 0$

$$
\begin{aligned}
H'(a) =& c\frac{-(1-\rho)(v-w)}{[v-(1-\rho)(v-w)a]^2} q\left(\frac{c}{v-(1-\rho)(v-w)a}\right) f\left(\frac{c}{v-(1-\rho)(v-w)a}\right) + \\
& \int_{\frac{c}{v-(1-\rho)(v-w)a}}^{\overline{p}} -(1-\rho)(v-w)\frac{f(p)}{\rho(v-w)}dp - 1 \\
=& -(1-\rho)(v-w)\left[\frac{cq(\frac{c}{v-(1-\rho)(v-w)a})f(\frac{c}{v-(1-\rho)(v-w)a})}{[v-(1-\rho)(v-w)a]^2} + \frac{F(\overline{p}) - F(\frac{c}{v-(1-\rho)(v-w)a})}{\rho(v-w)}\right] - 1 \\
& < 0
\end{aligned}
$$

Uniqueness then follows.

To show that there exists $\widehat{\rho} \in [0, \frac{v-c}{v-w}]$ such that firms never target any consumer for sure if and only if $\rho \geq \widehat{\rho}$, we just need to show the following claim:

If firms never target any consumer for sure for $\rho = \rho_s$, then they also never target any consumer for sure for any $\rho_l > \rho_s$.

Suppose not. When $\rho = \rho_l$, Lemma 1 implies that there exists $\overline{p}_l$ and $\underline{p}_l$ such that firms target consumers with probabilities $p \geq \overline{p}_l$ for sure and mix for consumers with probabilities between $\underline{p}_l$ and $\overline{p}_l$. Thus, $q_l(p) = 1 > q_s(p),\ \forall p \geq \overline{p}_l$. The proof of Proposition 2 will show that $q_l'(p) < q_s'(p),\ \forall p \in (\underline{p}_s, \overline{p}_l)$. So, $q_l(p) > q_s(p),\ \forall p \in [\underline{p}_s, 1] \Rightarrow a_l > a_s$. A contradiction to the comparative statics in Proposition 2. ■

Proof of Proposition 1. The result of the recall is implied by the fact that each duopolistic firm only targets a subset of the consumers that the monopoly targets. We now show that each duopolistic firm has a higher targeting precision than the monopoly.

The precision of the monopoly's optimal targeting policy is:

$$\frac{\int_{\underline{p}_m}^1 pf(p)dp}{\int_{\underline{p}_m}^1 f(p)dp} = \int_{\underline{p}_m}^1 p\frac{f(p)}{\int_{\underline{p}_m}^1 f(p)dp}dp$$

Observe that $\int_{\underline{p}_m}^1 \frac{f(p)}{\int_{\underline{p}_m}^1 f(p)dp}dp = 1$. Hence, $f_m(p) := \frac{q_s(p)f(p)}{\int_{\underline{p}_s}^1 q_s(p)f(p)dp}$ is a p.d.f. of a random variable $X_m \in [\underline{p}_m, 1]$. The monopoly's precision is the expectation of X_m.

The precision of the duopoly's optimal targeting policy is:

$$\frac{\int_{\underline{p}_d}^1 pq(p)f(p)dp}{\int_{\underline{p}_d}^1 q(p)f(p)dp} = \frac{\int_{\underline{p}_m}^1 pq(p)f(p)dp}{\int_{\underline{p}_m}^1 q(p)f(p)dp} = \int_{\underline{p}_m}^1 p\frac{q(p)f(p)}{\int_{\underline{p}_m}^1 q(p)f(p)dp}dp$$

, where the first equality comes from the fact that $\underline{p}_d \geq \underline{p}_m$ and $q(p) = 0,\ \forall p \in [\underline{p}_m, \underline{p}_d)$.

Observe that $\int_{\underline{p}_m}^1 \frac{q(p)f(p)}{\int_{\underline{p}_m}^1 q(p)f(p)dp}dp = 1$. Hence, $f_d(p) := \frac{q(p)f(p)}{\int_{\underline{p}_m}^1 q(p)f(p)dp}$ is a p.d.f. of a random variable $X_d \in [\underline{p}_m, 1]$. The duopoly's precision is the expectation of X_d.

We proceed by showing that X_d first-order stochastic dominates X_m. As a result, the expectation of X_d is larger than the expectation of X_m, which concludes the proof.

Lemma 3 *Given two continuously distributed random variables Z_1 and $Z_2 \in [\underline{z}, \overline{z}] \subset \mathbb{R}$, whose p.d.f. (c.d.f.) are f_1 and f_2 (F_1 and F_2), respectively. Z_1 first-order stochastic dominates Z_2 if $h(z) := f_1(z)/f_2(z)$ increases in z. The stochastic dominance is strict if $h(z)$ is not a constant.*

Proof of Lemma 3. We have $1 = \int_{\underline{z}}^{\overline{z}} f_1(z)dz = \int_{\underline{z}}^{\overline{z}} h(z)f_2(z)dz$.

If $h(z)$ is a constant, $h(z) = h_0$, then $1 = \int_{\underline{z}}^{\overline{z}} h(z)f_2(z)dz = h_0 \int_{\underline{z}}^{\overline{z}} f_2(z)dz = h_0 \Rightarrow f_1(z) = f_2(z),\ \forall z$. So, Z_1 first-order stochastic dominates Z_2.

If $h(z)$ is not a constant, then $h(\underline{z}) < h(\overline{z})$. Since $h(z)$ increases in z, we have $h(\underline{z}) = h(\underline{z}) \int_{\underline{z}}^{\overline{z}} f_2(z)dz < \int_{\underline{z}}^{\overline{z}} h(z)f_2(z)dz = 1 < h(\overline{z}) \int_{\underline{z}}^{\overline{z}} f_2(z)dz = h(\overline{z})$. Therefore, there exists $\hat{z}_1 \leq \hat{z}_2 \in (\underline{z}, \overline{z})$ such that $h(z) < 1$ if $z < \hat{z}_1$ and $h(z) > 1$ if $z > \hat{z}_2$.

For any $z \in (\underline{z}, \overline{z})$, we want to show that $F_1(z) \leq F_2(z)$.

Define $H(z) := F_2(z) - F_1(z) = \int_{\underline{z}}^{z} f_2(z)dz - \int_{\underline{z}}^{z} f_1(z)dz = \int_{\underline{z}}^{z} f_2(z)[1-h(z)]dz$.

One can see that $H(\underline{z}) = H(\overline{z}) = 0$. $H'(z) = f_2(z)[1-h(z)] \begin{cases} > 0, & if\ z \in (\underline{z}, \hat{z}_1) \\ = 0, & if\ z \in (\hat{z}_1, \hat{z}_2) \\ < 0, & if\ z \in (\hat{z}_2, \overline{z}) \end{cases}$.

So, $H(z) = F_2(z) - F_1(z) > 0, \forall z \in (\underline{z}, \overline{z})$. For any $z \in [\underline{z}, \overline{z}]$, one can see that $F_2(z) \geq F_1(z)$, with the inequality strict when $z \notin \{\underline{z}, \overline{z}\} \Rightarrow Z_1$ strictly first-order stochastic dominates Z_2. ■

Note that $q(p)$ weakly increases in p according to our characterization of the optimal targeting policy. Therefore, $\frac{q(p)f(p)}{\int_{\underline{p}}^{1} q(p)f(p)dp} / \frac{f(p)}{\int_{\underline{p}}^{1} f(p)dp} = q(p)\frac{\int_{\underline{p}}^{1} f(p)dp}{\int_{\underline{p}}^{1} q(p)f(p)dp}$ weakly increases in p. Lemma 3 then implies that X_d first-order stochastic dominates X_m. As a result, the expectation of X_d is larger than the expectation of X_m, which concludes the proof. ■

Proof of Proposition 2.

1. Firms never target any consumer for sure ($\overline{p} = 1$)

   Comparative statics of a w.r.t. ρ:

   In equilibrium, $G(a) = 0$. By the implicit function theorem, $\frac{\partial a}{\partial \rho} = -\frac{\frac{\partial G}{\partial \rho}}{\frac{\partial G}{\partial a}}$. We have shown that $\frac{\partial G}{\partial a}$ is negative.

$$
\begin{aligned}
\frac{\partial G}{\partial \rho} =& \int_{\frac{c}{v-(1-\rho)(v-w)a}}^{1} \frac{f(p)}{v-w}[-c/p + v - (1-\rho)(v-w)a](-1/\rho^2) + \frac{f(p)a}{\rho} dp \\
=& \int_{\frac{c}{v-(1-\rho)(v-w)a}}^{1} \frac{f(p)}{\rho^2(v-w)}[c/p - v + (1-\rho)(v-w)a + \rho a(v-w)]dp \\
=& \int_{\frac{c}{v-(1-\rho)(v-w)a}}^{1} \frac{f(p)}{\rho^2(v-w)}[c/p - v - (v-w)a]dp \\
<& 0
\end{aligned}
$$

, where the last inequality holds because $c/p \leq c/[\frac{c}{v-(1-\rho)(v-w)a}] = v - (1-\rho)(v-w)a < v$. Therefore, $\frac{\partial a}{\partial \rho} < 0$.

Comparative statics of $\underline{p}$ w.r.t. ρ:

We have shown that the overall targeting probability a decreases when ρ increases. Therefore, $v - (1-\rho)(v-w)a$ increases. Equation (6) then implies that $\underline{p}$ decreases in ρ.

Comparative statics of the precision wrt ρ :

Suppose $\rho_l > \rho_s > \frac{v-c}{v-w}$. The precision of the optimal targeting policy when $\rho = \rho_s$ is:

$$
\frac{\int_{\underline{p}_s}^{1} p q_s(p) f(p) dp}{\int_{\underline{p}_s}^{1} q_s(p) f(p) dp} = \int_{\underline{p}_s}^{1} p \frac{q_s(p) f(p)}{\int_{\underline{p}_s}^{1} q_s(p) f(p) dp} dp
$$

Observe that $\int_{\underline{p}_s}^{1} \frac{q_s(p)f(p)}{\int_{\underline{p}_s}^{1} q_s(p)f(p)dp} dp = 1$. Hence, $f_s(p) := \frac{q_s(p)f(p)}{\int_{\underline{p}_s}^{1} q_s(p)f(p)dp}$ is a p.d.f. of a random variable $X_s \in [\underline{p}_s, 1]$. The precision is the expectation of X_s.

The precision of the optimal targeting policy when $\rho = \rho_l$ is:

$$\begin{aligned}
\frac{\int_{\underline{p}_l}^1 pq_l(p)f(p)dp}{\int_{\underline{p}_l}^1 q_l(p)f(p)dp} &= \frac{\int_{\underline{p}_l}^{\underline{p}_s} pq_l(p)f(p)dp + \int_{\underline{p}_s}^1 pq_l(p)f(p)dp}{\int_{\underline{p}_l}^{\underline{p}_s} q_l(p)f(p)dp + \int_{\underline{p}_s}^1 q_l(p)f(p)dp} \\
&< \frac{\int_{\underline{p}_s}^1 pq_l(p)f(p)dp}{\int_{\underline{p}_s}^1 q_l(p)f(p)dp} \\
&= \int_{\underline{p}_s}^1 p\frac{q_l(p)f(p)}{\int_{\underline{p}_s}^1 q_l(p)f(p)dp}dp
\end{aligned}$$

, where the inequality comes from the fact that $\frac{\int_{\underline{p}_l}^{\underline{p}_s} pq_l(p)f(p)dp}{\int_{\underline{p}_l}^{\underline{p}_s} q_l(p)f(p)dp} < \underline{p}_s < \frac{\int_{\underline{p}_s}^1 pq_l(p)f(p)dp}{\int_{\underline{p}_s}^1 q_l(p)f(p)dp}$. Observe that $\int_{\underline{p}_s}^1 \frac{q_l(p)f(p)}{\int_{\underline{p}_s}^1 q_l(p)f(p)dp}dp = 1$. Hence, $f_l(p) := \frac{q_l(p)f(p)}{\int_{\underline{p}_s}^1 q_l(p)f(p)dp}$ is a p.d.f. of a random variable $X_l \in [\underline{p}_s, 1]$. The upper bound of the precision is the expectation of X_l.

We proceed by showing that X_s first-order stochastic dominates X_l. As a result, the expectation of X_s is larger than the expectation of X_l, which concludes the proof.

In order to apply lemma 3 to obtain the final result, we need to show that $f_s(p)/f_l(p) = \frac{\frac{q_s(p)f(p)}{\int_{\underline{p}_s}^1 q_s(p)f(p)dp}}{\frac{q_l(p)f(p)}{\int_{\underline{p}_s}^1 q_l(p)f(p)dp}} = \frac{q_s(p)}{q_l(p)}\frac{\int_{\underline{p}_s}^1 q_l(p)f(p)dp}{\int_{\underline{p}_s}^1 q_s(p)f(p)dp}$ increases in p. Since $\frac{\int_{\underline{p}_s}^1 q_l(p)f(p)dp}{\int_{\underline{p}_s}^1 q_s(p)f(p)dp}$ is a constant, we only need to show that $\frac{q_s(p)}{q_l(p)}$ strictly increases in p.

According to (5), $q_s'(p) = \frac{c}{p^2\rho_s(v-w)} > \frac{c}{p^2\rho_l(v-w)} = q_l'(p), \forall p \in (\underline{p}_s, 1)$. One can see that $\int_{\underline{p}_s}^1 q_s(p)f(p)dp = a_s > a_l = \int_{\underline{p}_l}^1 q_l(p)f(p)dp > \int_{\underline{p}_s}^1 q_l(p)f(p)dp$. Since $q_l(\underline{p}_s) > q_l(\underline{p}_l) = 0 = q_s(\underline{p}_s)$, there must exist $\tilde{p} \in (\underline{p}_s, 1)$ such that

$$\begin{cases} q_s(p) > q_l(p), \ if \ p \in (\tilde{p}, 1] \\ \tilde{q} := q_s(\tilde{p}) = q_l(\tilde{p}) \\ q_s(p) < q_l(p), \ if \ p \in [\underline{p}_s, \tilde{p}) \end{cases}.$$

$$
\begin{aligned}
\Xi(p) :=& \frac{q_s(p)}{q_l(p)} \overset{(5)}{=} \frac{\tilde{q} + \frac{c}{\rho_s(v-w)}(\frac{1}{\tilde{p}} - \frac{1}{p})}{\tilde{q} + \frac{c}{\rho_l(v-w)}(\frac{1}{\tilde{p}} - \frac{1}{p})} \\
\Rightarrow Sgn(\Xi'(p)) =& Sgn(\frac{1}{\rho_s} - \frac{1}{\rho_l}) > 0
\end{aligned}
$$

Therefore, $\frac{q_s(p)}{q_l(p)}$ strictly increases in p. Lemma 3 then implies that X_s first-order stochastic dominates X_l. As a result, the expectation of X_s is larger than the expectation of X_l, which concludes the proof.

Proof of the Comparative statics of the recall wrt ρ :

Suppose $\rho > \frac{v-c}{v-w}$. We have shown that $\frac{\int_{\underline{p}_s}^1 pq_s(p)f(p)dp}{a_s} > \frac{\int_{\underline{p}_l}^1 pq_l(p)f(p)dp}{a_l}$. Since $a_s > a_l$, we have $\frac{\int_{\underline{p}_l}^1 pq_l(p)f(p)dp}{a_l} = \frac{\int_{\underline{p}_l}^1 pq_l(p)f(p)dp}{a_s}\frac{a_s}{a_l} > \frac{\int_{\underline{p}_l}^1 pq_l(p)f(p)dp}{a_s} \Rightarrow \frac{\int_{\underline{p}_s}^1 pq_s(p)f(p)dp}{a_s} > \frac{\int_{\underline{p}_l}^1 pq_l(p)f(p)dp}{a_s} \Rightarrow \frac{\int_{\underline{p}_s}^1 pq_s(p)f(p)dp}{\mu_0} > \frac{\int_{\underline{p}_l}^1 pq_l(p)f(p)dp}{\mu_0}$.

2. Firms target high-probability consumers for sure ($\overline{p} < 1$)

   In this case, the precision of the optimal targeting policy is:

$$
\frac{\int_{\underline{p}}^{\overline{p}} pq(p)f(p)dp + \int_{\overline{p}}^1 pf(p)dp}{\int_{\underline{p}}^{\overline{p}} q(p)f(p)dp + \int_{\overline{p}}^1 f(p)dp} = \frac{\int_{\underline{p}}^{\overline{p}} pq(p)f(p)dp + \int_{\overline{p}}^1 pf(p)dp}{a}
$$

   The recall of the optimal targeting policy is:

$$
\frac{\int_{\underline{p}}^{\overline{p}} pq(p)f(p)dp + \int_{\overline{p}}^1 pf(p)dp}{\int_0^1 pf(p)dp} = \frac{\int_{\underline{p}}^{\overline{p}} pq(p)f(p)dp + \int_{\overline{p}}^1 pf(p)dp}{\mu_0}
$$

   Comparative statics of a w.r.t. ρ:

   In equilibrium, $H(a) = 0$. By the implicit function theorem, $\frac{\partial a}{\partial \rho} = -\frac{\frac{\partial H}{\partial \rho}}{\frac{\partial H}{\partial a}}$. We have shown that $\frac{\partial H}{\partial a}$ is negative.

$$
\begin{aligned}
\frac{\partial H}{\partial \rho} =& - f(\overline{p})\frac{\partial \overline{p}}{\partial \rho} + \frac{\partial \overline{p}}{\partial \rho} q(\overline{p})f(\overline{p}) - \\
& \frac{c}{[v-(1-\rho)(v-w)a]^2}(v-w)aq(\frac{c}{v-(1-\rho)(v-w)a})f(\frac{c}{v-(1-\rho)(v-w)a}) \\
=& - \frac{c}{[v-(1-\rho)(v-w)a]^2}(v-w)aq(\frac{c}{v-(1-\rho)(v-w)a})f(\frac{c}{v-(1-\rho)(v-w)a}) \\
<& 0
\end{aligned}
$$

Therefore, $\frac{\partial a}{\partial \rho} < 0$.

Comparative statics of $\underline{p}$ w.r.t. ρ:

We have shown that the overall targeting probability a decreases when ρ increases. Therefore, $v - (1-\rho)(v-w)a$ increases. Equation (6) then implies that $\underline{p}$ decreases in ρ.

Comparative statics of $\overline{p}$ w.r.t. ρ:

Consider any given $\rho_l > \rho_s$ such that the corresponding $\overline{p}_l$ and $\overline{p}_s$ are lower than 1. We prove by contradiction. suppose $\overline{p}_l \leq \overline{p}_s$. The overall targeting probability corresponding to ρ_l is $a_l = \int_{\overline{p}_l}^{1} f(p)dp + \int_{\underline{p}_l}^{\overline{p}_l} q_l(p)f(p)dp$. The overall targeting probability corresponding to ρ_s is $a_s = \int_{\overline{p}_s}^{1} f(p)dp + \int_{\underline{p}_s}^{\overline{p}_s} q_s(p)f(p)dp$. We first show that $q_l(p) > q_s(p), \forall p \in (\underline{p}_s, \overline{p}_l)$.

Observe that $q_l(\overline{p}_l) = 1 > q_s(\overline{p}_l)$. For any $p \in (\underline{p}_s, \overline{p}_l)$, we have

$$
\begin{aligned}
q_l(p) =& q_l(\overline{p}_l) + \frac{c}{\rho_l(v-w)}(1/\overline{p}_l - 1/p) \\
>& q_s(\overline{p}_l) + \frac{c}{\rho_s(v-w)}(1/\overline{p}_l - 1/p) \\
=& q_s(\overline{p}_l)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
a_l =& \int_{\overline{p}_s}^{1} f(p)dp + \int_{\overline{p}_l}^{\overline{p}_s} f(p)dp + \int_{\underline{p}_l}^{\overline{p}_l} q_l(p)f(p)dp \\
>& \int_{\overline{p}_s}^{1} f(p)dp + \int_{\overline{p}_l}^{\overline{p}_s} q_s(p)f(p)dp + \int_{\underline{p}_s}^{\overline{p}_l} q_l(p)f(p)dp \\
>& \int_{\overline{p}_s}^{1} f(p)dp + \int_{\overline{p}_l}^{\overline{p}_s} q_s(p)f(p)dp + \int_{\underline{p}_s}^{\overline{p}_l} q_s(p)f(p)dp \\
=& a_s
\end{aligned}
$$

But we have shown that a decreases in ρ. A contradiction. Therefore, $\overline{p}_l > \overline{p}_s$.

Comparative statics of the profit w.r.t. ρ:

Denote the firm's total profit by Π and the per-unit profit for probability p consumer by $\pi(p)$. Then, $\Pi = \int_{\overline{p}}^{1} \pi(p) f(p) dp$, where $\pi(p) = p\{[\rho + (1-\rho)a]w + [1-\rho-(1-\rho)a]v\} - c$. One can see that $\pi(p)$ is strictly increasing and linear in p for $p \in [\overline{p}, 1]$. Consider any given $\rho_l > \rho_s$ such that the corresponding $\overline{p}_l$ and $\overline{p}_s$ are lower than 1. We have shown that $\overline{p}_s < \overline{p}_l$. Therefore, $\pi_s(\overline{p}_l) > \pi_s(\overline{p}_s) = 0 = \pi_l(\overline{p}_l) \Rightarrow [\rho_s + (1-\rho_s)a_s]w + [1-\rho_s-(1-\rho_s)a_s]v > [\rho_l + (1-\rho_l)a_l]w + [1-\rho_l-(1-\rho_l)a_l]v \Rightarrow \pi_s(p) > \pi_l(p),\ \forall p \in [\overline{p}_l, 1] \Rightarrow \Pi_s > \Pi_l$.[6]

■

Proof of Proposition 3. We first consider the case in which $\rho_0 > \widehat{\rho}$. We have shown that the profit is zero for any $\rho \geq \widehat{\rho}$. Therefore, if firms invest in equilibrium, it must be that $\rho(I_1, I_2) < \widehat{\rho}$. The investment cost may be higher than the benefit. For example, suppose $\rho(v, v) \geq \widehat{\rho}$. Then no firm will invest.

We then show that no firm investing is not an equilibrium if $\rho_0 < \widehat{\rho}$. Suppose no firm investing is an equilibrium. We want to show that either firm has an incentive to deviate. Consider without loss of generality firm 1.

We first compute how much firm 1's payoff in the second period increases as ρ decreases.

Equation (7) implies that:

$$Q(\rho, \overline{p}) := \overline{p}\{[\rho + (1-\rho)a(\rho)]w + [1-\rho-(1-\rho)a(\rho)]v\} - c = 0$$

By implicit function theorem,

$$\begin{aligned}\frac{\partial \overline{p}}{\partial \rho} &= -\frac{\partial Q/\partial \rho}{\partial Q/\partial \overline{p}} \\ &= \frac{\overline{p}[1 - a + (1-\rho)\frac{\partial a}{\partial \rho}](v-w)}{[\rho + (1-\rho)a]w + [1-\rho-(1-\rho)a]v}\end{aligned}$$

[6]From the expression of $\pi(p)$ and $\pi(\overline{p}) = 0$, one can derive that $\pi(p) = \frac{p-\overline{p}}{\overline{p}}c,\ \forall p \geq \overline{p}$.

We have shown in Proposition 2 that $\overline{p}$ strictly increases in ρ for $\rho < \widehat{\rho}$. Therefore, $\overline{p}[1 - a + (1-\rho)\frac{\partial a}{\partial \rho}](v - w) > 0$. Denote $\frac{\partial \overline{p}}{\partial \rho}|_{\rho=\rho_0}$ by D_0 ($D_0 > 0$). Consider an investment by firm 1 such that the correlation decreases from ρ_0 to $\rho_1 := \rho_0 - \epsilon$. Denote the $\overline{p}$ corresponding to ρ_0 by $\overline{p}_0$ and the $\overline{p}$ corresponding to ρ_1 by $\overline{p}_1$. We use similar notations for π and Π.[7]

By Taylor expansion, $\overline{p}_1 = \overline{p}_0 - D_0\epsilon + o(\epsilon)$. Firm 1's increase in profit (not taking into account the investment cost) is:

$$
\begin{aligned}
\Pi_1 - \Pi_0 &= \int_{\overline{p}_1}^{1} \pi_1(p)f(p)dp - \int_{\overline{p}_0}^{1} \pi_0(p)f(p)dp \\
&> \int_{\overline{p}_0}^{1} \pi_1(p)f(p)dp - \int_{\overline{p}_0}^{1} \pi_0(p)f(p)dp \\
&= \int_{\overline{p}_0}^{1} [\pi_1(p) - \pi_0(p)]f(p)dp \\
&= \int_{\overline{p}_0}^{1} [\frac{p - \overline{p}_1}{\overline{p}_1}c - \frac{p - \overline{p}_0}{\overline{p}_0}c]f(p)dp \\
&= \int_{\overline{p}_0}^{1} p\frac{\overline{p}_0 - \overline{p}_1}{\overline{p}_0\overline{p}_1}cf(p)dp \\
&> \int_{\overline{p}_0}^{1} \overline{p}_0\frac{D_0\epsilon + o(\epsilon)}{\overline{p}_0\overline{p}_1}cf(p)dp \\
&= \frac{D_0\epsilon + o(\epsilon)}{\overline{p}_1}c\int_{\overline{p}_0}^{1} f(p)dp \\
&= \frac{D_0\epsilon + o(\epsilon)}{\overline{p}_1}c[1 - F(\overline{p}_0)] \\
&= \frac{D_0c[1 - F(\overline{p}_0)]}{\overline{p}_1}\epsilon + o(\epsilon)
\end{aligned}
$$

Thus, the profit increase is linear in the decrease of ρ, ϵ, if we ignore the higher-order term $o(\epsilon)$.

We then compute firm 1's investment cost in the first period. One can see that

[7] Both have been defined in the proof of Proposition 2.

$K_j(0) = 0$.

$$\begin{aligned} K_1(\epsilon) =& K_1(0) + K_1'(0)\epsilon + o(\epsilon) \\ =& 0 + 0 \cdot \epsilon + o(\epsilon) \\ =& o(\epsilon) \end{aligned}$$

Therefore, $\Pi_1 - \Pi_0 > K_1(\epsilon)$ for ϵ small enough. It shows that firm 1 will deviate if no firm invests. ■

Proof of Proposition 4. In order for a firm acting as a monopoly to be an equilibrium, the other firm must not want to target probability 1 consumer, which is equivalent to:

$$\begin{aligned} & \rho w + (1-\rho)[a_m w + (1-a_m)v] - c \leq 0 \\ \Leftrightarrow \;\; & \rho \geq \frac{a_m w + (1-a_m)v - c}{(1-a_m)(v-w)} \end{aligned}$$

The statement that in any other pure strategy asymmetric equilibria, there exists $p_0 \in (\underline{p}, 1)$ such that exactly one firm targets consumers whose $p \in (\underline{p}, p_0)$, and either one or both firms target every consumer whose $p > p_0$ is equivalent to the following two claims.

Claim 1: At least one firm targets consumers whose $p \in (\underline{p}, 1]$.

Claim 2: If both firms target a probability p' consumer, then they also target any consumer whose $p > p'$.

Proof of Claim 1: For any $p \in (\underline{p}, 1]$, there exists $\tilde{p} \in [\underline{p}, p)$ such that $q_1(\tilde{p}) > 0$ or $q_2(\tilde{p}) > 0$. Assume without loss of generality that $q_1(\tilde{p}) > 0$. Firm 1's expected payoff from targeting such a consumer is no less than the expected payoff from not targeting: $[\rho + (1-\rho)a_2]\tilde{p}w + [1 - \rho - (1-\rho)a_2]\tilde{p}v - c \geq 0$. Suppose neither firm targets a type $p > \tilde{p}$ consumer. By deviating and targeting that consumer, the deviating firm can obtain an expected payoff of $[(1-\rho)a_2]pw + [1-(1-\rho)a_2]pv - c > [\rho + (1-\rho)a_2]\tilde{p}w + [1-\rho-(1-\rho)a_2]\tilde{p}v - c \geq 0$. So, firms will deviate. A contradiction.

Proof of Claim 2: If both firms target a probability p' consumer, then firm i's expected payoff from targeting such a consumer is no less than the expected payoff from not targeting: $[\rho + (1-\rho)a_j]p'w + [1-\rho-(1-\rho)a_j]p'v - c \geq 0$, where $j \neq i$. Claim 1 says that at least one firm targets any consumer whose $p > p'$. Suppose one of the firms does not target probability p consumer. Assume without loss of generality that firm 1 does not target that consumer. By deviating and targeting that consumer, it can obtain an expected payoff of $[(1-\rho)a_2]pw + [1-(1-\rho)a_2]pv - c > [\rho+(1-\rho)a_2]p'w + [1-\rho-(1-\rho)a_2]p'v - c \geq 0$. So, firm 1 will deviate. A contradiction.

Now we prove that the firm that targets probability $\underline{p}$ consumer has a higher overall targeting probability. Assume without loss of generality that firm 1 targets probability $\underline{p}$ consumer. Its payoff from targeting that consumer is $[(1-\rho)a_2]\underline{p}w + [1-(1-\rho)a_2]\underline{p}v - c \geq 0$. Suppose $a_1 < a_2$. By deviating and targeting probability $\underline{p} - \epsilon$ consumer, firm 2 obtains an expected payoff of $[(1-\rho)a_1](\underline{p}-\epsilon)w + [1-(1-\rho)a_1](\underline{p}-\epsilon)v - c = [(1-\rho)a_1]\underline{p}w + [1-(1-\rho)a_1]\underline{p}v - c - \epsilon\{[(1-\rho)a_1]w + [1-(1-\rho)a_1]v\} = [(1-\rho)a_2]\underline{p}w + [1-(1-\rho)a_2]\underline{p}v - c + (a_2 - a_1)(1-\rho)(v-w)\underline{p} - \epsilon\{[(1-\rho)a_1]w + [1-(1-\rho)a_1]v\} \geq 0 + (a_2 - a_1)(1-\rho)(v-w)\underline{p} - \epsilon\{[(1-\rho)a_1]w + [1-(1-\rho)a_1]v\} > 0$ for ϵ small enough. So, firm 2 will deviate. A contradiction. Therefore, $a_1 \geq a_2$.

Lastly, we prove that the recall of either firm is lower than the monopoly case, and the precision of at least one firm is higher than the monopoly case.

The monopoly equilibrium is straightforward. Now let's look at other equilibria where both firms target some consumers (the case of Figure 3b and 3c). $\underline{p} \geq \underline{p}_m$ implies that the recall of either firm is lower than the monopoly case.

The precision of firm i's targeting policy is:

$$\frac{\int_{\underline{p}}^{1} p q_i(p) f(p) dp}{\int_{\underline{p}}^{1} q_i(p) f(p) dp} = \frac{\int_{\underline{p}}^{1} p q_i(p) f(p) dp}{a_i}$$

We first consider the case where there is no overlap in firms' targeting regions (the case of Figure 3b). In that case, $q_1(p) + q_2(p) = 1, \forall p \in (\underline{p}, 1]$. Since both firms target some consumers, one can see that $\underline{p} > \underline{p}_m$. We have:

$$
\begin{aligned}
&\frac{\int_{\underline{p}}^{1} pq_1(p)f(p)dp + \int_{\underline{p}}^{1} pq_2(p)f(p)dp}{\int_{\underline{p}}^{1} q_1(p)f(p)dp + \int_{\underline{p}}^{1} q_2(p)f(p)dp} \\
=&\frac{\int_{\underline{p}}^{1} pf(p)dp}{\int_{\underline{p}}^{1} f(p)dp} \\
=&\int_{\underline{p}}^{1} p\frac{f(p)}{\int_{\underline{p}}^{1} f(p)dp}dp \\
=&\int_{\underline{p}}^{1} p\frac{f(p)}{\int_{\underline{p}}^{1} f(p)dp}dp + \int_{\underline{p}_m}^{\underline{p}} p \cdot 0 \ dp
\end{aligned}
$$

The above formula is the expectation of a random variable $X \in [\underline{p}_m, 1]$ with a p.d.f. of 0 for $p \in [\underline{p}_m, \underline{p}]$ and a p.d.f. of $\frac{f(p)}{\int_{\underline{p}}^{1} f(p)dp}$ for $p \in (\underline{p}, 1]$.

The precision of the monopoly's targeting policy is:

$$
\frac{\int_{\underline{p}_m}^{1} pf(p)dp}{\int_{\underline{p}_m}^{1} f(p)dp} = \int_{\underline{p}_m}^{1} p\frac{f(p)}{\int_{\underline{p}_m}^{1} f(p)dp}dp
$$

The above formula is the expectation of a random variable $Y \in [\underline{p}_m, 1]$ with a p.d.f. of $\frac{f(p)}{\int_{\underline{p}_m}^{1} f(p)dp}$ for $p \in [\underline{p}_m, 1]$. Lemma 3 implies that X strictly first-order stochastic dominates Y. Therefore, $\frac{\int_{\underline{p}}^{1} pq_1(p)f(p)dp+\int_{\underline{p}}^{1} pq_2(p)f(p)dp}{\int_{\underline{p}}^{1} q_1(p)f(p)dp+\int_{\underline{p}}^{1} q_2(p)f(p)dp} > \frac{\int_{\underline{p}_m}^{1} pf(p)dp}{\int_{\underline{p}_m}^{1} f(p)dp}$.

Now suppose $\frac{\int_{\underline{p}}^{1} pq_1(p)f(p)dp}{\int_{\underline{p}}^{1} q_1(p)f(p)dp} \leq \frac{\int_{\underline{p}_m}^{1} pf(p)dp}{\int_{\underline{p}_m}^{1} f(p)dp}$ and $\frac{\int_{\underline{p}}^{1} pq_2(p)f(p)dp}{\int_{\underline{p}}^{1} q_2(p)f(p)dp} \leq \frac{\int_{\underline{p}_m}^{1} pf(p)dp}{\int_{\underline{p}_m}^{1} f(p)dp}$. We then have:

$$
\begin{aligned}
&\frac{\int_{\underline{p}}^{1} pq_1(p)f(p)dp + \int_{\underline{p}}^{1} pq_2(p)f(p)dp}{\int_{\underline{p}}^{1} q_1(p)f(p)dp + \int_{\underline{p}}^{1} q_2(p)f(p)dp} \\
=&\frac{\int_{\underline{p}}^{1} pq_1(p)f(p)dp}{\int_{\underline{p}}^{1} q_1(p)f(p)dp + \int_{\underline{p}}^{1} q_2(p)f(p)dp} + \frac{\int_{\underline{p}}^{1} pq_2(p)f(p)dp}{\int_{\underline{p}}^{1} q_1(p)f(p)dp + \int_{\underline{p}}^{1} q_2(p)f(p)dp} \\
=&\frac{\int_{\underline{p}}^{1} q_1(p)f(p)dp}{\int_{\underline{p}}^{1} q_1(p)f(p)dp + \int_{\underline{p}}^{1} q_2(p)f(p)dp}\frac{\int_{\underline{p}}^{1} pq_1(p)f(p)dp}{\int_{\underline{p}}^{1} q_1(p)f(p)dp} + \\
&\frac{\int_{\underline{p}}^{1} q_2(p)f(p)dp}{\int_{\underline{p}}^{1} q_1(p)f(p)dp + \int_{\underline{p}}^{1} q_2(p)f(p)dp}\frac{\int_{\underline{p}}^{1} pq_2(p)f(p)dp}{\int_{\underline{p}}^{1} q_2(p)f(p)dp} \\
\leq&\frac{\int_{\underline{p}}^{1} q_1(p)f(p)dp}{\int_{\underline{p}}^{1} q_1(p)f(p)dp + \int_{\underline{p}}^{1} q_2(p)f(p)dp}\frac{\int_{\underline{p}_m}^{1} pf(p)dp}{\int_{\underline{p}_m}^{1} f(p)dp} + \\
&\frac{\int_{\underline{p}}^{1} q_2(p)f(p)dp}{\int_{\underline{p}}^{1} q_1(p)f(p)dp + \int_{\underline{p}}^{1} q_2(p)f(p)dp}\frac{\int_{\underline{p}_m}^{1} pf(p)dp}{\int_{\underline{p}_m}^{1} f(p)dp} \\
=&\frac{\int_{\underline{p}_m}^{1} pf(p)dp}{\int_{\underline{p}_m}^{1} f(p)dp}
\end{aligned}
$$

A contradiction. So, the precision of at least one firm is higher than the monopoly case.

We then consider the case where there is overlap in firms' targeting regions (the case of Figure 3c). In that case, $q_1(p)+q_2(p) = 1, \forall p \in (\underline{p}, p_0)$ and $q_1(p)+q_2(p) = 2, \forall p > p_0$.

The following lemma relates the overlapping case with the non-overlapping case so that the previous argument applies.

Lemma 4 *Suppose $a, b, A, B > 0$ and $A/a < B/b$, then $\frac{A+2B}{a+2b} > \frac{A+B}{a+b}$.*

Proof of Lemma 4.

$$\begin{aligned}\frac{A+2B}{a+2b} =& \frac{A+B}{a+2b} + \frac{B}{a+2b} \\ =& \frac{a+b}{a+2b}\frac{A+B}{a+b} + \frac{b}{a+2b}\frac{B}{b} \\ <& \frac{a+b}{a+2b}\frac{B}{b} + \frac{b}{a+2b}\frac{B}{b} \\ =& \frac{B}{b}\end{aligned}$$

, where the last inequality holds because $\frac{B}{b} = \frac{a\frac{B}{b}+B}{a+b} > \frac{a\frac{A}{a}+B}{a+b} = \frac{A+B}{a+b}$. ■

Let $A = \int_{\underline{p}}^{p_0} pf(p)dp, B = \int_{p_0}^{1} pf(p)dp, a = \int_{\underline{p}}^{p_0} f(p)dp, b = \int_{p_0}^{1} f(p)dp$. Notice that $A/a < p_0 < B/b$. So, Lemma 4 implies that

$$\frac{A+2B}{a+2b} > \frac{A+B}{a+b}$$

Notice that $\frac{\int_{\underline{p}}^{1} pq_1(p)f(p)dp+\int_{\underline{p}}^{1} pq_2(p)f(p)dp}{\int_{\underline{p}}^{1} q_1(p)f(p)dp+\int_{\underline{p}}^{1} q_2(p)f(p)dp} = \frac{A+2B}{a+2b}$ in the overlapping case. We have shown in the non-overlapping case that the RHS is larger than the precision of the monopoly. So, in the overlapping case, $\frac{\int_{\underline{p}}^{1} pq_1(p)f(p)dp+\int_{\underline{p}}^{1} pq_2(p)f(p)dp}{\int_{\underline{p}}^{1} q_1(p)f(p)dp+\int_{\underline{p}}^{1} q_2(p)f(p)dp}$ is also larger than the precision of the monopoly. By the same proof-by-contradiction argument as before, one can see that the precision of at least one firm is higher than the monopoly case. ■