MODULE
FORCES IN SPACE

3.1 Motivation:

- **Forces in Space:**

  Many problems in Mechanics require analysis in three dimensions, and for such problems it is often necessary to resolve a force into its three mutually perpendicular components i.e. along X, Y and Z axes in space. A force system in three dimensions is called Space Force system. A force in 3-dimensions can be expressed as vectors using X, Y and Z coordinate system. In the analysis of this force system, the vector analysis is very much involved. Thus it is required to study the representations of forces and there effects as vector quantities by using vector operations like addition, unit vectors, direction cosines, dot product, cross product etc.

- **Forces in Space:**

  Basic understanding of forces and its characteristics, types, finding resultants of various systems of forces, effects of forces like moments and couples, Equilibrium of forces arranged in different force systems are required to be known.

Objectives:

- **Forces in Space:**

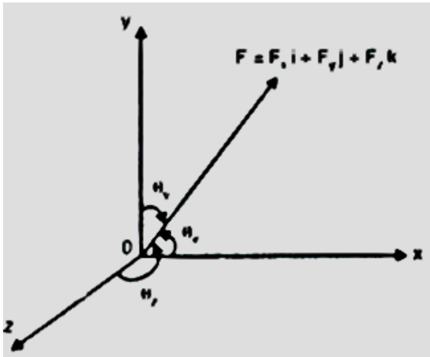
  1) Representing force in vector form, angles made by its components with rectangular axes
  2) Moment of force about a point
  3) Vector component of a force
  4) Resultant of Concurrent, parallel and general force system

Rectangular components of a force in space:

The force acting along diagonal OA can be resolved into three components.

$$\left.\begin{aligned} F_X &= F\cos\theta_X \\ F_Y &= F\cos\theta_Y \\ F_Z &= F\cos\theta_Z \end{aligned}\right\} \qquad [1]$$

Where $\theta_X, \theta_Y, \theta_Z$ are known as direction cosines of the force F.



If I, j and k are the unit vectors along x, y and z axes respectively, the force vector can be expressed as

$\bar{F} = F_X \bar{i} + F_Y \bar{j} + F_Z \bar{k}$ ----------- [2]

From equations [1], we can write direction cosines of force F as,

$$l = \cos\theta_X = \frac{F_X}{F},\ m = \cos\theta_Y = \frac{F_Y}{F} \text{ and } n = \cos\theta_Z = \frac{F_Z}{F}$$

$$\therefore l^2 + m^2 + n^2 = \cos^2\theta_X + \cos^2\theta_Y + \cos^2\theta_Z$$

$$= \left(\frac{F_X}{F}\right)^2 + \left(\frac{F_Y}{F}\right)^2 + \left(\frac{F_Z}{F}\right)^2$$

$$= \frac{F_X^2 + F_Y^2 + F_Z^2}{F^2}$$

$$\therefore l^2 + m^2 + n^2 = 1$$

[B] Unit Vector $(\bar{e})$:

A vector whose magnitude is equal to 1 and which is directed along the original force is called unit vector.

$$\bar{F} = F_X \cos\theta_X \bar{i} + F_Y \cos\theta_Y \bar{j} + F_Z \cos\theta_Z \bar{k}$$

$$= F\,(\cos\theta_X \bar{i} + \cos\theta_Y \bar{j} + \cos\theta_Z \bar{k})$$

$$\bar{F} = F\,(\bar{e})$$

Where $\bar{e}$ = Unit vector in the direction of force F

$$= (\cos\theta_X \bar{i} + \cos\theta_Y \bar{j} + \cos\theta_Z \bar{k})$$

[C] Unit Vector when force is specified by two points:

Consider a force F passing through two points A (X_1, Y_1, Z_1) and B (X_2, Y_2, Z_2) the force vector can be expressed as

$$\therefore \bar{F} = F(\bar{e})$$

$$\therefore \bar{F} = F\left[\frac{(X_2 - X_1)\bar{i} + (Y_2 - Y_1)\bar{j} + (Z_2 - Z_1)\bar{k}}{\sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2}}\right]$$

[D] Components of force when orientations of planes are given:

Consider a force F at origin O. the direction of force F with Y axis is 'θ_Y'.

$\therefore$ Y – component of force $F_Y = F\cos\theta_Y$ and component along direction 'OC' is $F_{OC} = F\sin\theta_Y$

We can resolve 'F_{OC}' along two rectangular components i.e. X and Z axes.

$F_X = F_{OC} \cos\alpha$ OR $F_X = F \sin\theta_Y \cos\alpha$

$\therefore F_Z = F_{OC} \sin\alpha$ OR $F_Z = F \sin\theta_Y \sin\alpha$

[E] Resultant of concurrent forces in space:

When a system of concurrent forces in space is given the resultant will be obtained by summing their rectangular components.

To determine resultant, resolve each force into rectangular components and express as

$$R_X \bar{i} + R_Y \bar{j} + R_X \bar{k} = \sum\left[F_X \bar{i} + F_Y \bar{j} + F_Z \bar{k}\right]$$

$$= (\textstyle\sum F_X)\bar{i} + (\sum F_Y)\bar{j} + (\sum F_Z)\bar{k}$$

$$\therefore R_X = \sum F_X \qquad R_Y = \sum F_Y \qquad R_Z = \sum F_Z$$

$$R = \sqrt{R_X^{\,2} + R_Y^{\,2} + R_Z^{\,2}}$$

$$\cos\theta_X = \frac{R_X}{R} \qquad \cos\theta_Y = \frac{R_Y}{R} \qquad \cos\theta_Z = \frac{R_Z}{R}$$

[F] Conditions of equilibrium for concurrent forces in space:

If a particle is in equilibrium the components of resultant must be equal to zero.

$$R_X = 0 \qquad R_Y = 0 \qquad R_Z = 0$$

$$\sum F_X = 0 \qquad \sum F_Y = 0 \qquad \sum F_Z = 0$$

[G] Moment of a force about a point:

Moment of a force about a point 'O' is defined as the vector product of $\bar{r}$ and $\bar{F}$.

$$\overline{M_O} = \bar{r} \times \bar{F}$$

Where $\bar{r}$ = Position vector of point of application 'O'

$\bar{F}$ = Force Vector

$$\overline{M_O} = \bar{r} \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ r_X & r_Y & r_Z \\ F_X & F_Y & F_Z \end{vmatrix}$$

[H] Moment of a force about origin:

Consider a force $\bar{F} = F_X \bar{i} + F_Y \bar{j} + F_Z \bar{k}$ passing through points P (X_1, Y_1, Z_1) and Q (X_2, Y_2, Z_2).

Moment about origin, $\overline{M_O} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ X_1 & Y_1 & Z_1 \\ F_X & F_Y & F_Z \end{vmatrix}$ OR $\overline{M_O} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ X_2 & Y_2 & Z_2 \\ F_X & F_Y & F_Z \end{vmatrix}$

[I] Moment of a force about any other point:

Consider a force $\overline{F} = F_X\,\overline{i} + F_Y\,\overline{j} + F_Z\,\overline{k}$ passing through points P (X_1, Y_1, Z_1) and Q (X_2, Y_2, Z_2).

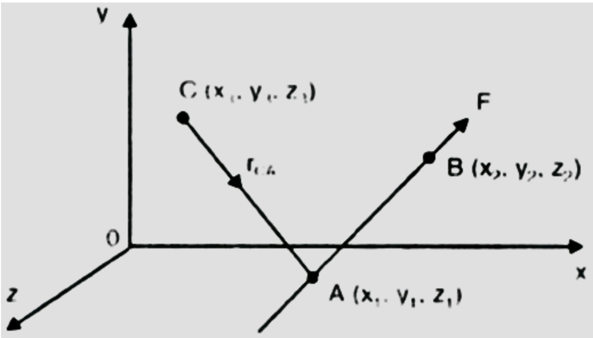
The moment of this force about a point C (X_3, Y_3, Z_3)

$$\overline{M_C} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ (X_1 - X_3) & (Y_1 - Y_3) & (Z_1 - Z_3) \\ F_X & F_Y & F_Z \end{vmatrix} \quad \text{OR} \quad \overline{M_C} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ (X_2 - X_3) & (Y_2 - Y_3) & (Z_2 - Z_3) \\ F_X & F_Y & F_Z \end{vmatrix}$$

Steps to find moment of a force about a point:

Let force 'F' is passing through points A(X_1, Y_1, Z_1) and B (X_2, Y_2, Z_2) on its line of action.

Let C(X_3, Y_3, Z_3) be the moment centre.



Step 1: Put force in vector form i.e.

$$\overline{F} = F_X\,\overline{i} + F_Y\,\overline{j} + F_Z\,\overline{k}$$

$$= F\,.\,\overline{e}$$

Step 2: Find position vector extending from moment centre to any other point on the force. i.e.

$$\overline{r} = r_X\,\overline{i} + r_Y\,\overline{j} + r_Z\,\overline{k}$$

Step 3: Perform cross product of the position vector and the force vector to get moment vector.

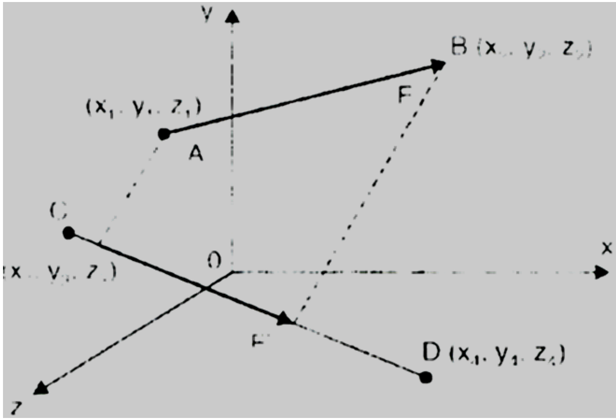
$$\overline{M_{Point}} = \overline{r} \times \overline{F}$$

$$= \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ r_X & r_Y & r_Z \\ F_X & F_Y & F_Z \end{vmatrix}$$

[M] Steps to find Vector component of force:

Let **F** be the given force and **F'** is said to be its vector component.

Let force **F** passes through points A and B.



Step 1: Put the force in vector form i.e. $\overline{F}$

Step 2: Find the unit vector of the line along which the vector component is required i.e. $\overline{e_{Line}}$

Step 3: Perform the dot product of the force and the unit vector of the line to get the magnitude of the vector component i.e. **F '** = $\overline{e_{Line}} \cdot \overline{F}$

Step 4: Multiply magnitude of the force component with unit vector of the line to get vector form of this component i.e. $\overline{F'} = \overline{e_{Line}}\ F'$

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problems:

Forces in Space:

1) A 150 kN force acts at P(8, 12, 0) and passes through Q (2, 0, 4). Put the force in vector form.

Solution: We know that $\overline{F} = F\,(\overline{e})$ where F = 150 kN and $\overline{e}$ = Unit vector in the direction of force F.

Let $(X_1, Y_1, Z_1) \equiv (8, 12, 0)$ and $(X_2, Y_2, Z_2) \equiv (2, 0, 4)$

$$\therefore \overline{F} = F\left[\frac{(X_2 - X_1)\,\overline{i} + (Y_2 - Y_1)\,\overline{j} + (Z_2 - Z_1)\,\overline{k}}{\sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2}}\right]$$

$$\therefore \overline{F} = 150\left[\frac{(2-8)\,\overline{i} + (0-12)\,\overline{j} + (4-0)\,\overline{k}}{\sqrt{(2-8)^2 + (0-12)^2 + (4-0)^2}}\right]$$

$$\therefore \overline{F} = (-64.2857\ i\ -\ 128.5714\ j\ +\ 42.85714\ k)\ \ kN$$

2) If $\overline{F}$ = - 238 I + 157 j +312 k kN, determine the magnitude and directions of the force.

Solution: Magnitude of the force,

$$F = \sqrt{F_X^{\,2} + F_Y^{\,2} + F_Z^{\,2}}$$

$$\therefore\ F = \sqrt{(-238)^2 + (157)^2 + (312)^2}$$

$$\therefore\ F = 422.6547\ \text{kN}$$

Direction of the force,

$F_X = F\ \cos\theta_X$

$\therefore\ -238 = 422.6547\ \cos\ \theta_X$

$\therefore\ \theta_X = 124.27^0$

$F_Y = F\ \cos\theta_Y$

$\therefore\ 157 = 422.6547\ \cos\ \theta_Y$

$\therefore\ \theta_Y = 68.19^0$

$F_Z = F\ \cos\theta_Z$

$\therefore\ 312 = 422.6547\ \cos\ \theta_Z$

$\therefore\ \theta_Z = 42.42^0$

3) A force $\overline{F}$ = (3 i – 4 j + 12 k) N acts at a point P (1, -2, 3) m. Find a) moment of the force about origin b) moment of the force about point Q (2, 1, 2) m and c) **Solution:**

a) Moment of the force about origin 'O':

$$\overline{M_O} = \overline{r_{PO}}\ \text{x}\ \overline{F} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ r_X & r_Y & r_Z \\ F_X & F_Y & F_Z \end{vmatrix} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ 1 & -2 & 3 \\ 3 & -4 & 12 \end{vmatrix}$$

$$\therefore\ \overline{M_O} = [(12 \text{ x } -2) - (3 \text{ x } -4)]\ i - [(1 \text{ x } 12) - (3 \text{ x } 3)]\ j + [(1 \text{ x } -4) - (3 \text{ x } -2)]\ k$$

$$\therefore\ \overline{M_O} = -12\ i - 3\ j + 2\ k\ \ \text{Nm}$$

b) Moment of the force about point Q (2, 1, 2) m:

$$\overline{M_Q} = \overline{r_{PQ}}\ \text{x}\ \overline{F} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ r_X & r_Y & r_Z \\ F_X & F_Y & F_Z \end{vmatrix} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ (1-2) & (-2-1) & (3-2) \\ 3 & -4 & 12 \end{vmatrix} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ -1 & -3 & 1 \\ 3 & -4 & 12 \end{vmatrix}$$

$$\therefore\ \overline{M_Q} = [(12 \text{ x } -3) - (1 \text{ x } -4)]\ i - [(-1 \text{ x } 12) - (3 \text{ x } 1)]\ j + [(-1 \text{ x } -4) - (3 \text{ x } -3)]\ k$$

$$\therefore\ \overline{M_Q} = -32\ i + 15\ j + 13\ k\ \ \text{Nm}$$

$$\therefore\ \overline{F'} = -6.3315\ \ (0.3015\ i + 0.9045\ j - 0.3015\ k)$$

$$\therefore\ \overline{F'} = -1.9089\ i - 5.7268\ j + 1.9089\ k\ \ \text{N}$$

4) A force acts at the origin in a direction defined by the angles $\theta_X = 56^0$ and $\theta_Y = 35^0$. Knowing that the Z component of the force is 345 N. Determine a) the other components b) the magnitude of the force c) the value of θ_Z.

Solution: We know that $\cos^2\theta_X + \cos^2\theta_Y + \cos^2\theta_Z = 1$

$$\cos^2 56 + \cos^2 35 + \cos^2 \theta_z = 1$$

$$\therefore \cos^2 \theta_z = 0.01629$$

$$\therefore \cos \theta_z = \pm 0.1276$$

$$\therefore \theta_z = 82.66^0 \text{ or } \theta_z = 97.33^0$$

Since $F_Z = 345 \text{ N}$, force component is directed towards Positive Z-direction.

So $\theta_Z = 82.66^0$

Using $F_Z = F \cos \theta_Z$

$\therefore$ 345 = F cos 82.66

$\therefore$ F = 2700.43 N

Using $F_Y = F \cos \theta_Y$

$\therefore F_Y = 2700.43 \cos 35$

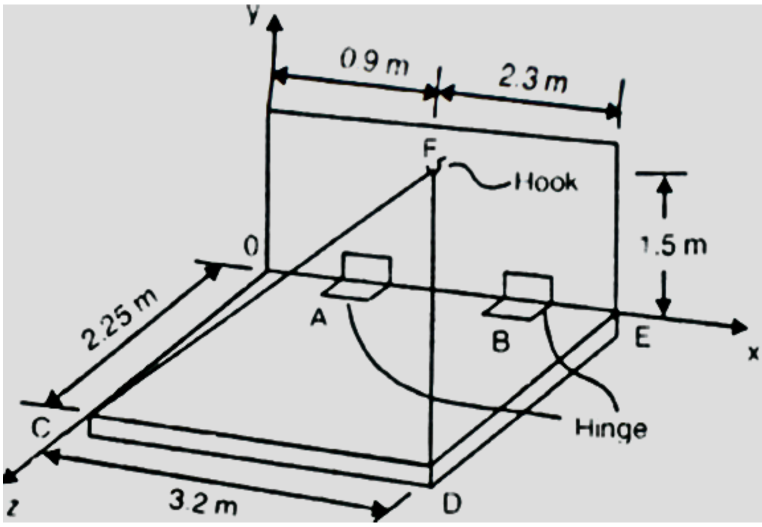
$\therefore F_Y = 2212.06$ N

Using $F_X = F \cos \theta_X$

$\therefore F_X = 2700.43 \cos 56$

$\therefore F_Y = 1510.06$ N

5) The rectangular platform OCDE is hinged to a vertical wall at A and B and supported by a cable which passes over a smooth hook at F. If the tension in the cable is 280 N, find the moment about each of the coordinate axes of the force exerted by the cable at D.



Solution: Referring the diagram, D ≡ (X_1, Y_1, Z_1) ≡ (3.2, 0, 2.25) , F ≡ (X_2, Y_2, Z_2) ≡ (0.9, 1.5, 0)

Tension in the cable will be from D to F = 280 N

Tension (Force) Vector,

$$\overline{T} = T \; \overline{e_{DF}} \text{ .... (Multiplication)}$$

$$= 280 \left[\frac{(0.9-3.2)\,\overline{i} + (1.5-0)\,\overline{j} + (0-2.25)\,\overline{k}}{\sqrt{(0.9-3.2)^2 + (1.5-0)^2 + (0-2.25)^2}} \right] = 280 \left[\frac{-2.3\,\overline{i} + 1.5\,\overline{j} - 2.25\,\overline{k}}{3.55} \right]$$

$$= 280 \; (-0.6478 \text{ i} + 0.4225 \text{ j} - 0.6338 \text{ k})$$

$$= -181.384 \text{ i} + 118.3 \text{ j} - 177.464 \text{ k N}$$

Moment of force $\overline{T}$ about origin 'O'

$$\overline{M_O} = \overline{r_{DO}} \times \overline{T} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ r_X & r_Y & r_Z \\ T_X & T_Y & T_Z \end{vmatrix} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 3.2 & 0 & 2.25 \\ -181.384 & 118.3 & -177.464 \end{vmatrix}$$

$$\therefore \overline{M_O} = [(-177.464 \times 0) - (118.3 \times 2.25)]\, i - [(-177.464 \times 3.2) - (-181.384 \times 2.25)]\, j + [(118.3 \times 3.2) - (-181.384 \times 0)]\, k$$

$$\therefore \overline{M_O} = -266.175\ i + 159.7708\ j + 378.56\ k \ \text{Nm}$$

$$\therefore \overline{M_{O_X}} = -266.175\ i \ \text{Nm} \qquad \overline{M_{O_Y}} = 159.7708\ j \ \text{Nm} \qquad \text{and} \qquad \overline{M_{O_Z}} = 378.56\ k \ \text{Nm}$$

6) Determine moment due to 100 N force 'F_{AB}' about a line in X-Y plane, passing from origin and acting away. It is making an angle of 30^0 w.r.t. positive X axis. Position vectors for A and B are $2\,i - 3\,j + 4\,k$ and $-3\,i + k$ respectively.

Solution:

Step 1: Force 100 N passes through line AB

So finding force in vector form

$$\overline{F} = F\ \overline{e_{AB}} \ \text{.............. (Multiplication)}$$

$$= 100 \left[\frac{(-3-2)\,\bar{i} + [0-(-3)]\,\bar{j} + (1-4)\,\bar{k}}{\sqrt{(-3-2)^2 + [0-(-3)]^2 + (1-4)^2}} \right] = 100 \left[\frac{-5\,\bar{i} + 3\,\bar{j} - 3\,\bar{k}}{6.5574} \right]$$

$$= 100\ (-0.7624\ i + 0.4574\ j - 0.4574\ k)$$

$$= -76.24\ i + 45.74\ j - 45.74\ k \ \ \text{N}$$

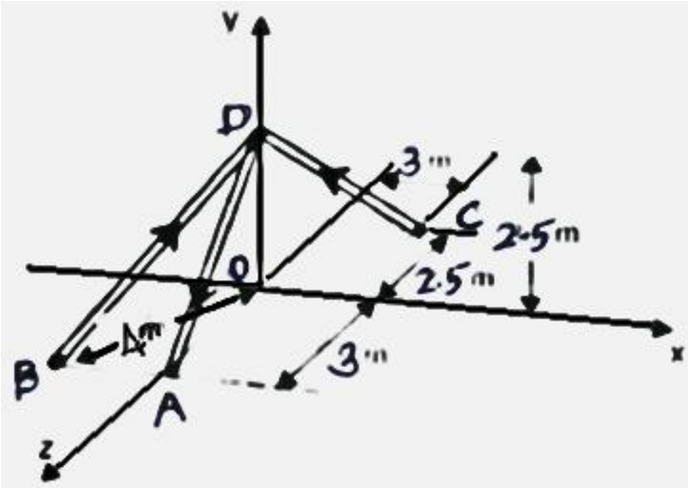
Step 2: Moment of the force about any point on line AB say point B (-3, 0, 1) i.e. $\overline{M_B}$

$$\overline{M_B} = \overline{r_{AB}} \times \overline{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ r_X & r_Y & r_Z \\ F_X & F_Y & F_Z \end{vmatrix} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -3 & 0 & 1 \\ -76.254 & 45.752 & -45.752 \end{vmatrix}$$

$$\therefore \overline{M_B} = [(0 \times -45.752) - (1 \times 45.752)]\ i - [(-3 \times -45.752) - (1 \times -76.254)]\ j + [(-3 \times 45.752) - (0 \times -76.254)]\ k$$

$$\therefore \overline{M_B} = -45.752\ i - 213.51\ j - 137.256\ k \ \ \text{Nm}$$

7) Three forces are concurrent at a point D. If the force in member AD is 1.29 kN, determine the force in members BD and CD if the resultant of the three forces is vertical. Also find the resultant force. Points A, B and C lie in XZ plane.



Solution:

This is a concurrent space force system of three forces. To put the forces in vector form, we need the coordinates of the points through which the forces pass. From the figure the coordinates are: A (0, 0, 3) m, B (- 2.645, 0, 3) m, C (3, 0, - 2.5) m, D (0, 2.5, 0) m.

Putting the forces in vector form

$$\overline{F_1} = F_1 \; \overline{e_{DA}} \text{ .... (Multiplication)}$$

$$= 1.29 \left[\frac{(0-0)\,\bar{i} + (0-2.5)\,\bar{j} + (3-0)\,\bar{k}}{\sqrt{(0-0)^2 + (0-2.5)^2 + (3-0)^2}} \right] = 1.29 \left[\frac{-2.5\,\bar{j} + 3\,\bar{k}}{3.905} \right]$$

$$= 1.29 \;(-0.6402\, j + 0.7682\, k)$$

$$= -0.8258\, j + 0.9910\, k \text{ kN}$$

$$\overline{F_2} = F_2 \; \overline{e_{BD}} \text{ .... (Multiplication)}$$

$$= F_2 \left[\frac{[0-(-2.645)]\,\bar{i} + (2.5-0)\,\bar{j} + (0-3)\,\bar{k}}{\sqrt{[0-(-2.645)]^2 + (2.5-0)^2 + (0-3)^2}} \right] = F_2 \left[\frac{2.645\,\bar{i} + 2.5\,\bar{j} - 3\,\bar{k}}{4.7165} \right]$$

$$= F_2 \;(0.5607\, i + 0.5300\, j - 0.6360\, k) \text{ kN}$$

$$\overline{F_3} = F_3 \; \overline{e_{CD}} \text{ .... (Multiplication)}$$

$$= F_3 \left[\frac{(0-3)\,\bar{i} + (2.5-0)\,\bar{j} + [0-(-2.5)]\,\bar{k}}{\sqrt{(0-3)^2 + (2.5-0)^2 + [0-(-2.5)]^2}} \right] = F_3 \left[\frac{-3\,\bar{i} + 2.5\,\bar{j} + 2.5\,\bar{k}}{4.6368} \right]$$

$$= F_3 \;(-0.6469\, i + 0.5391\, j + 0.5391\, k) \text{ kN}$$

The resultant force $\overline{R} = \overline{F_1} + \overline{F_2} + \overline{F_3}$

$$\overline{R} = (-0.8258\, j + 0.9910\, k) + [\, F_2 \;(0.5607\, i + 0.5300\, j - 0.6360\, k)\,] + [\, F_3 \;(-0.6469\, i + 0.5391\, j + 0.5391\, k)\,]$$

$$= (0.5607\, F_2 - 0.6469\, F_3)\, i + (-0.8258 + 0.5300\, F_2 + 0.5391\, F_3)\, j + (0.9910 - 0.6360\, F_2 + 0.5391\, F_3)\, k$$

As resultant is vertical, $R_X = 0$, $R_Y = R$ and $R_Z = 0$

Also $\theta_X = 90^0$, $\theta_Y = 0^0$, $\theta_Z = 90^0$

$R = R_X\ i + R_Y\ j + R_Z\ k$

$\therefore\ R_X = 0.5607\ F_2 - 0.6469\ F_3 = 0$ ------------------------------ (1)

$R_Y = -\ 0.8258 + 0.5300\ F_2 + 0.5391\ F_3 = R$ --------------- (2)

$R_Z = 0.9910 - 0.6360\ F_2 + 0.5391\ F_3 = 0$ ------------------- (3)

solving equations (1) and (3), we get

$F_2 = 5.873$ kN and $F_3 = 5.09$ kN

substituting above values of F_2 and F_3 in equation (2)

we get R = 5.03 j kN

8) A force P_1 = 10 N in magnitude acts along direction AB whose coordinates of points A and B are (3, 2, -1) m and (8, 5, 3) m respectively. Another force P_2 = 5 N in magnitude acts along BC where C has coordinates (-2, 11, -5) m. Determine **a)** the resultant of P_1 and P_2 in its vector form **b)** the moment of the resultant about a point D (1, 1, 1) m c) the magnitude of the component of the resultant along the line BK where the coordinates of the point K are (5, 8, 3) m.

Solution:

$P_1 = 10$ N, $P_2 = 5$ N

Coordinates, A $\equiv$ (3, 2, -1), B $\equiv$ (8, 5, 3), C $\equiv$ (-2, 11, -5), D $\equiv$ (1, 1, 1) and K $\equiv$ (5, 8, 3)

a) Putting forces in vector form

$\overline{P_1} = P_1\ \overline{e_{AB}}$ .... (Multiplication)

$$= 10\left[\frac{(8-3)\bar{i} + (5-2)\bar{j} + [3-(-1)]\bar{k}}{\sqrt{(8-3)^2 + (5-2)^2 + [3-(-1)]^2}}\right] = 10\left[\frac{5\,\bar{i} + 3\,\bar{j} + 4\,\bar{k}}{7.071}\right]$$

$= 10\ (0.7071\ i + 0.4242\ j + 0.5656\ k)$

$= 7.071\ i + 4.242\ j + 5.656\ k$ N

$\overline{P_2} = P_2\ \overline{e_{BC}}$ .... (Multiplication)

$$= 5\left[\frac{(-2-8)\bar{i} + (11-5)\bar{j} + (-5-3)\bar{k}}{\sqrt{(-2-8)^2 + (11-5)^2 + (-5-3)^2}}\right] = 5\left[\frac{-10\,\bar{i} + 6\,\bar{j} - 8\,\bar{k}}{14.142}\right]$$

$= 5\ (-\ 0.7071\ i + 0.4242\ j - 0.5656\ k)$

$= -\ 3.5355\ i + 2.121\ j - 2.828\ k$ N

$\therefore$ Resultant Force, $\overline{R} = \overline{P_1} + \overline{P_2}$

$= (7.071\ i + 4.242\ j + 5.656\ k) + (-\ 3.5355\ i + 2.121\ j - 2.828\ k)$

$\therefore\ \overline{R} = 3.5355\ i + 6.363\ j + 2.828\ k$ N

b) Forces are concurrent at point B, so resultant also passes through this point.

Moment of the resultant about point D (1, 1, 1) = $\overline{M_D} = \overline{r_{DB}} \times \overline{R}$ ....(Cross Product)

$\overline{r_{DB}}$ = Position Vector extending from point D to point B = (8 - 1) i + (5 - 1) j + (3 - 1) k

$\therefore\ \overline{r_{DB}} = 7\,i + 4\,j + 2\,k$ m

$$\text{So, } \overline{M_D} = \begin{vmatrix} i & j & k \\ 7 & 4 & 2 \\ 3.5355 & 6.363 & 2.828 \end{vmatrix}$$

$\therefore\ \overline{M_D} = -1.414\,i - 12.725\,j + 30.399\,k$ Nm

c) Component of resultant along line BK

$\overline{R} = 3.5355\,i + 6.363\,j + 2.828\,k$ N

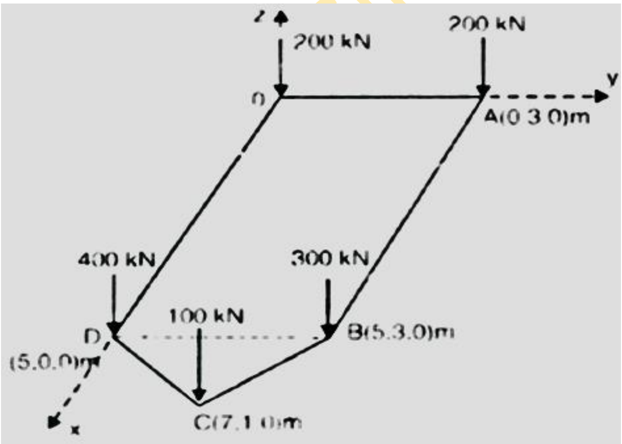
Unit vector of line BK, $\overline{e_{BK}} = \dfrac{(5-8)\,\bar{i} + (8-5)\,\bar{j} + (3-3)\,\bar{k}}{\sqrt{(5-8)^2 + (8-5)^2 + (3-3)^2}} = \dfrac{-3\,\bar{i} + 3\,\bar{j}}{4.2426} = -0.7071\,i + 0.7071\,j$

The scalar component of resultant, $R' = \overline{R} \cdot \overline{e_{BK}}$ .... (Dot Product)

$\therefore\ R' = (3.5355\,i + 6.363\,j + 2.828\,k) \cdot (-0.7071\,i + 0.7071\,j)$

$\therefore\ R' = 1.9993$ N ≈ 2 N

9) A plate foundation is subjected to five vertical forces as shown. Replace these five forces by means of a single vertical force and find the point of replacement.



Solution:

The given system is a parallel force system of five forces. The coordinates through which the forces act are,

A $\equiv$ (0, 3, 0), B $\equiv$ (5, 3, 0), C $\equiv$ (7, 1, 0), D $\equiv$ (5, 0, 0), D $\equiv$ (0, 0, 0)

Putting forces in vector form

Let F_1 = 200 N $\quad F_2$ = 300 N $\quad F_3$ = 100 N $\quad F_4$ = 400 N

$\overline{F_1} = -200\,k$ kN $\quad \overline{F_2} = -300\,k$ kN $\quad \overline{F_3} = -100\,k$ kN $\quad \overline{F_4} = -400\,k$ kN

F_5 = 200 N

$\overline{F_5} = -200\,k$ kN

$\therefore$ Resultant Force, $\overline{R} = \overline{F_1} + \overline{F_2} + \overline{F_3} + \overline{F_4} + \overline{F_5}$

$= [(-200\,k) + (-300\,k) + (-100\,k) + (-400\,k) + (-200\,k)]$

$\therefore\ \overline{R} = -1200\,k$ kN

Point of application of Resultant:

Let the resultant act at a point P(x, y, 0) m in the plane of the plate.

To use Varignon's theorem, we need to find moments of all the forces and also of the resultant about point 'O'.

$\overline{M}_O^{F_1} = \overline{r_{OA}} \times \overline{F_1}$ ....where $\overline{r_{OA}} = 3\ \overline{j}$ m

$= 3\ \overline{j} \times (-200)\ \overline{k}$

$= -\ 600\ i$ kN m

$\overline{M}_O^{F_2} = \overline{r_{OB}} \times \overline{F_2}$ ....where $\overline{r_{OB}} = 5\ \overline{i} + 3\ \overline{j}$ m

$= (5\ \overline{i} + 3\ \overline{j}) \times (-300)\ \overline{k}$

$= (-900\ i + 1500\ j)$ kN m

$\overline{M}_O^{F_3} = \overline{r_{OC}} \times \overline{F_3}$ ....where $\overline{r_{OC}} = 7\ \overline{i} + \overline{j}$ m

$= (7\ \overline{i} + \overline{j}) \times (-100)\ \overline{k}$

$= (-100\ i + 700\ j)$ kN m

$\overline{M}_O^{F_4} = \overline{r_{OD}} \times \overline{F_4}$ ....where $\overline{r_{OD}} = 5\ \overline{i}$ m

$= (5\ \overline{i}) \times (-400)\ \overline{k}$

$= 2000\ j$ kN m

$\overline{M}_O^{R} = \overline{r_{OP}} \times \overline{R}$ ....where $\overline{r_{OP}} = x\ \overline{i} + y\ \overline{j}$ m

$= (x\ \overline{i} + y\ \overline{j}) \times (-1200)\ \overline{k}$

$= (-1200\ y)\ i + (1200\ x)\ j$ kN m

Using Varignaon's Theorem, $\sum \overline{M}_O^{F} = \sum \overline{M}_O^{R}$

$\therefore\ \overline{M}_O^{F_1} + \overline{M}_O^{F_2} + \overline{M}_O^{F_3} + \overline{M}_O^{F_4} + \overline{M}_O^{F_5} = \overline{M}_O^{R}$

$\therefore\ (-\ 600\ i) + (-900\ i + 1500\ j) + (-100\ i + 700\ j) + (2000\ j) + 0 = (-1200\ y)\ i + (1200\ x)\ j$

$\therefore\ -\ 1600\ i + 4200\ j = (-1200\ y)\ i + (1200\ x)\ j$

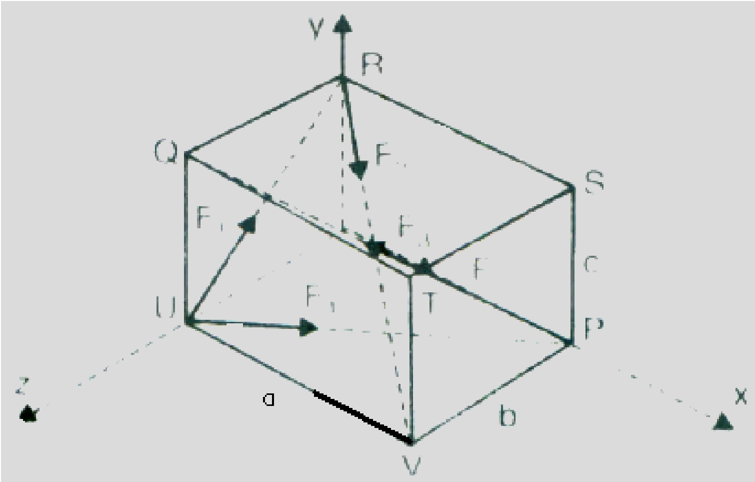
equating the coefficient

$-1600 = -1200\ y$ $\qquad$ $4200 = 1200\ x$

$\therefore\ y = 1.33$ m $\qquad$ $\therefore\ x = 3.5$ m

So resultant $\overline{R} = -1200\ k$ acts at point P $\equiv$ (3.5, 1.33, 0) m.

10) Forces act along the lines joining the parallelepiped. Find the resultant force and the moment the resultant coupled at the origin. $F_1 = 1$ N, $F_2 = 3$ N, $F_3 = 2$ N, $F_4 = 5$ N, $F_5 = 2$ N, a = 2.5 m, b = 2 m and c = 1.5 m. Note that F_3 is directed from P towards Q and F_5 directed from S towards T.



Solution: The given force system is a general force system of five forces. The coordinates of the various points through which the forces pass are as follows:

$P \equiv (2.5, 0, 0)$, $Q \equiv (0, 1.5, 2)$, $R \equiv (0, 1.5, 0)$, $S \equiv (2.5, 1.5, 0)$,
$T \equiv (2.5, 1.5, 2)$, $U \equiv (0, 0, 2)$, $V \equiv (2.5, 0, 2)$,

Putting forces in vector form,

1) $F_1 = F_{UR} = 1$ kN

$\overline{F_1} = F_1 \ \overline{e_{UR}}$ .... (Multiplication)

$$= 1\left[\frac{(0-0)\bar{i}+(1.5-0)\bar{j}+(0-2)\bar{k}}{\sqrt{(0-0)^2+(1.5-0)^2+(0-2)^2}}\right] = 1\left[\frac{1.5\,\bar{j}-2\,\bar{k}}{2.5}\right]$$

$= (0.6\,j - 0.8\,k)$ kN

2) $F_2 = F_{RV} = 3$ kN

$\overline{F_2} = F_2 \ \overline{e_{RV}}$ .... (Multiplication)

$$= 3\left[\frac{(2.5-0)\bar{i}+(0-1.5)\bar{j}+(2-0)\bar{k}}{\sqrt{(2.5-0)^2+(0-1.5)^2+(2-0)^2}}\right] = 3\left[\frac{2.5\,\bar{i}-1.5\,\bar{j}+2\,\bar{k}}{3.5355}\right]$$

$= 3\,(0.7071\,i - 0.4242\,j + 0.5656\,k)$

$= (2.1213\,i - 1.2726\,j + 1.6968\,k)$ kN

3) $F_3 = F_{PQ} = 2$ kN

$\overline{F_3} = F_3 \ \overline{e_{PQ}}$ .... (Multiplication)

$$= 2\left[\frac{(0-2.5)\bar{i}+(1.5-0)\bar{j}+(2-0)\bar{k}}{\sqrt{(0-2.5)^2+(1.5-0)^2+(2-0)^2}}\right] = 2\left[\frac{-2.5\,\bar{i}+1.5\,\bar{j}+2\,\bar{k}}{3.5355}\right]$$

$= 2\,(-0.7071\,i + 0.4242\,j + 0.5656\,k)$

$= (-1.4142\,i + 0.8484\,j + 1.1312\,k)$ kN

4) $F_4 = F_{UP} = 5$ kN

$\overline{F_4} = F_4 \ \overline{e_{UP}}$ .... (Multiplication)

$$= 5\left[\frac{(2.5-0)\,\overline{i} + (0-0)\,\overline{j} + (0-2)\,\overline{k}}{\sqrt{(2.5-0)^2 + (0-0)^2 + (0-2)^2}}\right] = 5\left[\frac{2.5\,\overline{i} - 2\,\overline{k}}{3.2015}\right]$$

$= 5\,(0.7808\, i - 0.6247\, k)$

$= (3.904\, i - 3.1235\, k)$ kN

5) $F_5 = F_{ST} = 2$ kN

$\overline{F_5} = F_5 \ \overline{e_{ST}}$ .... (Multiplication)

$$= 2\left[\frac{(2.5-2.5)\,\overline{i} + (1.5-1.5)\,\overline{j} + (2-0)\,\overline{k}}{\sqrt{(2.5-2.5)^2 + (1.5-1.5)^2 + (2-0)^2}}\right] = 2\left[\frac{2\,\overline{k}}{2}\right]$$

$= (2\, k)$ kN

The resultant force,

$$\overline{R} = \overline{F_1} + \overline{F_2} + \overline{F_3} + \overline{F_4} + \overline{F_5}$$

$= (0.6\, j - 0.8\, k) + (2.1213\, i - 1.2726\, j + 1.6968\, k) +$

$(-1.4142\, i + 0.8484\, j + 1.1312\, k) + (3.904\, i - 3.1235\, k) + 2\, k$

$\therefore \ \overline{R} = 4.6111\, i + 0.1758\, j + 0.9045\, k$ kN

Taking moments of all forces about origin,

$\overline{M_O^{F_1}} = \overline{r_{OU}} \times \overline{F_1}$ ....where $\overline{r_{OU}} = 2\,\overline{k}$ m

$= (2\, k) \times (0.6\, j - 0.8\, k)$

$= (-1.2\, i)$ kN m

$\overline{M_O^{F_2}} = \overline{r_{OR}} \times \overline{F_2}$ ....where $\overline{r_{OR}} = 1.5\,\overline{j}$ m

$= (1.5\, j) \times (2.1213\, i - 1.2726\, j + 1.6968\, k)$

$= (2.5452\, i - 3.1819\, k)$ kN m

$\overline{M_O^{F_3}} = \overline{r_{OP}} \times \overline{F_3}$ ....where $\overline{r_{OP}} = 2.5\,\overline{i}$ m

$= (2.5\, i) \times (-1.4142\, i + 0.8484\, j + 1.1312\, k)$

$= (-2.828\, j + 2.121\, k)$ kN m

$\overline{M_O^{F_4}} = \overline{r_{OP}} \times \overline{F_4}$ ....where $\overline{r_{OP}} = 2.5\,\overline{i}$ m

$= (2.5\,\overline{i}) \times (3.904\, i - 3.1235\, k)$

$= (7.8087\, j)$ kN m

$\overline{M_O^{F_5}} = \overline{r_{OS}} \times \overline{F_5}$ ....where $\overline{r_{OS}} = 2.5\,\overline{i} + 1.5\,\overline{j}$ m

$= (2.5\,\overline{i} + 1.5\,\overline{j}) \times (2\, k)$

$= (3\, i - 5\, j)$ kN m

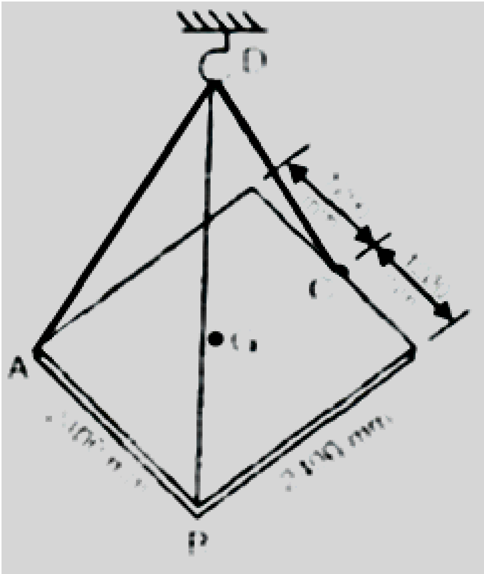
The resultant moment about the origin,

$$\overline{M_O} = \overline{M_O^{F_1}} + \overline{M_O^{F_2}} + \overline{M_O^{F_3}} + \overline{M_O^{F_4}} + \overline{M_O^{F_5}}$$

$$= (-1.2\,i) + (2.5452\,i - 3.1819\,k) + (-2.828\,j + 2.121\,k) + (7.8087\,j) + (3\,i - 5\,j)$$

$$\therefore\ \overline{M_O} = (4.3452\,i - 0.0193\,j - 1.0609\,k)\ \text{kN}$$

11) A square steel plate 2400 mm X 2400 mm has a mass of 1800 kg with mass centre at G. Calculate the tension in each of the three cables with which the plate is lifted while remaining horizontal. Length DG = 2400 mm.



Solution:

At support D, tensions in the cables and reaction R_D form a concurrent system in equilibrium. Drawing FBD of the joint D and finding the coordinates of the points through which these forces pass.

A ≡ (-1200, 0, 1200), B ≡ (1200, 0, 1200), C ≡ (0, 0, -1200), D ≡ (0, 2400, 0)

Putting forces in vector form,

1) $\overline{F_{DA}} = F_{DA}\ \overline{e_{DA}}$ .... (Multiplication)

$$= F_{DA}\left[\frac{(-1200-0)\bar{i} + (0-2400)\bar{j} + (1200-0)\bar{k}}{\sqrt{(-1200-0)^2 + (0-2400)^2 + (1200-0)^2}}\right] = F_{DA}\left[\frac{-1200\ \bar{i}\ - 2400\ \bar{j}\ + 1200\ \bar{k}}{2939.387}\right]$$

$$= F_{DA}\ (-0.4082\,i - 0.8164\,j + 0.4082\,k)\ \text{kN}$$

2) $\overline{F_{DB}} = F_{DB}\ \overline{e_{DB}}$ .... (Multiplication)

$$= F_{DB}\left[\frac{(1200-0)\bar{i} + (0-2400)\bar{j} + (1200-0)\bar{k}}{\sqrt{(1200-0)^2 + (0-2400)^2 + (1200-0)^2}}\right] = F_{DB}\left[\frac{1200\ \bar{i} - 2400\ \bar{j}\ + 1200\ \bar{k}}{6.7082}\right]$$

$$= F_{DB}\ (0.4082\,i - 0.8164\,j + 0.4082\,k)\ \text{kN}$$

3) $\overline{F_{DC}} = F_{DC}\ \overline{e_{DC}}$ .... (Multiplication)

$$= F_{DC}\left[\frac{(0-0)\bar{i} + (0-2400)\bar{j} + (-1200-0)\bar{k}}{\sqrt{(0-0)^2 + (0-2400)^2 + (-1200-0)^2}}\right] = F_{DC}\left[\frac{-2400\ \bar{j}\ - 1200\ \bar{k}}{2683.281}\right]$$

$$= F_{DC}\ (-0.8944\,j - 0.4472\,k)\ \text{kN}$$

$$4)\ \overline{R_D} = (1800 \times 9.81)\ N$$
$$\therefore\ \overline{R_D} = 17658\ j\ \ N$$

Since reaction RD would be equal to the weight of the steel plate in magnitude and direction and opposite in sense.

Applying Conditions of equilibrium

$$\sum F_X = 0$$
$$\therefore\ -0.4082\ F_{DA} + 0.4082\ F_{DB} = 0$$

$$\sum F_Y = 0$$
$$\therefore\ -0.8164\ F_{DA} - 0.8164\ F_{DB} - 0.8944\ F_{DC} + 17658 = 0$$

$$\sum F_Z = 0$$
$$\therefore\ 0.4082\ F_{DA} + 0.4082\ F_{DB} - 0.4472\ F_{DC} = 0$$

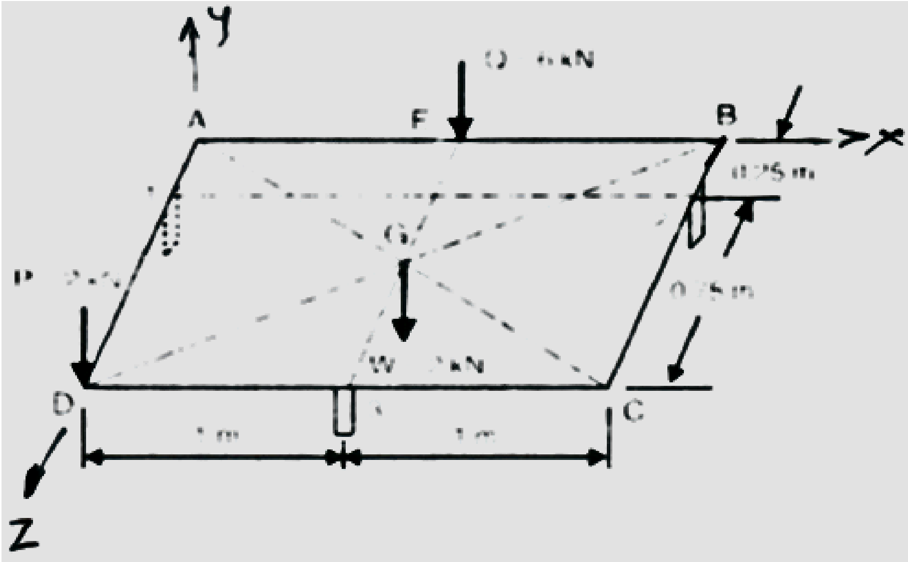
solving above equations, we get

$$F_{DA} = -5407.27\ N$$
$$\therefore\ F_{AB} = 5407.27\ N\ \ (\text{Compressive})$$

$$F_{DB} = -5407.27\ N$$
$$\therefore\ F_{DB} = 5407.27\ N\ \ (\text{Compressive})$$

$$F_{DC} = -9871.422\ N$$
$$\therefore\ F_{DC} = 9871.422\ N\ \ (\text{Compressive})$$

12) A rectangular table 1 m X 2 m is mounted on three equal supports to 1, 2 and 3. The table weighs 2kN which acts at the CG of the table. If two vertical loads 2 kN and 6 kN are applied on the surface of the table at D and E as shown, find the reactions at the supports.



Solution:

The given system is a parallel force system of five forces. Let R_1, R_2 and R_3 be the reactions at supports 1, 2 and 3 respectively.FBD of the plate is as shown in the figure.

$1 \equiv (0, 0, 0.25)$, $2 \equiv (2, 0, 0.25)$, $3 \equiv (1, 0, 1)$,
$G \equiv (1, 0, 0.5)$, $D \equiv (0, 0, 1)$ and $E \equiv (1, 0, 0)$, $B \equiv (2, 0, 0)$

Putting forces in vector form,

1) $\overline{F_Q} = -6\,j$ kN 4) $\overline{R_1} = R_1\,j$ kN

2) $\overline{F_P} = -2\,j$ kN 5) $\overline{R_2} = R_2\,j$ kN

3) $\overline{W} = -2\,j$ kN 6) $\overline{R_3} = R_3\,j$ kN

Taking moments of all forces about any point say 'E',

$$\overline{M_B^{F_Q}} = \overline{r_{BE}} \times \overline{F_Q} \text{ ....where } \overline{r_{BE}} = -\,\overline{i} \text{ m}$$
$$= (-\,i) \times (-\,6\,j)$$
$$= (6\,k) \text{ N m}$$

$$\overline{M_B^{F_P}} = \overline{r_{BD}} \times \overline{F_P} \text{ ....where } \overline{r_{BD}} = -\,2\,\overline{i} + \overline{k} \text{ m}$$
$$= (-\,2\,\overline{i} + \overline{k}) \times (-\,2\,j)$$
$$= (2\,i + 4\,k) \text{ N m}$$

$$\overline{M_B^{W}} = \overline{r_{BG}} \times \overline{W} \text{ ....where } \overline{r_{BG}} = -\,\overline{i} + 0.5\,\overline{k} \text{ m}$$
$$= (-\,\overline{i} + 0.5\,\overline{k}) \times (-\,2\,j)$$
$$= (i + 2\,k) \text{ N m}$$

$$\overline{M_B^{R_1}} = \overline{r_{B1}} \times \overline{R_1} \text{ ....where } \overline{r_{B1}} = -\,2\,\overline{i} + 0.25\,\overline{k} \text{ m}$$
$$= (-\,2\,i + 0.25\,k) \times (R_1\,j)$$
$$= (-\,0.25\,R_1\,i - 2\,R_1\,k) \text{ N m}$$

$$\overline{M_B^{R_2}} = \overline{r_{B2}} \times \overline{R_2} \text{ ....where } \overline{r_{B2}} = 0.25\,\overline{k} \text{ m}$$
$$= (0.25\,k) \times (R_2\,j)$$
$$= (-\,0.25\,R_2\,i) \text{ N m}$$

$$\overline{M_B^{R_3}} = \overline{r_{B3}} \times \overline{R_3} \text{ ....where } \overline{r_{B3}} = -\,\overline{i} + \overline{k} \text{ m}$$
$$= (-\,i + k) \times (R_3\,j)$$
$$= (-\,R_3\,i - R_3\,k) \text{ N m}$$

Applying Conditions of equilibrium

$$\sum M_X = 0$$
$$2 + 1 - 0.25\,R_1 - 0.25\,R_2 - R_3 = 0$$

$$\sum M_Z = 0$$
$$6 + 4 + 2 - 2\,R_1 - R_3 = 0$$

$$\sum F_Y = 0$$
$$\therefore\ -6 - 2 - 2 + R_1 + R_2 + R_3 = 0$$

Solving above equations, we get

$R_1 = 5.67$ N

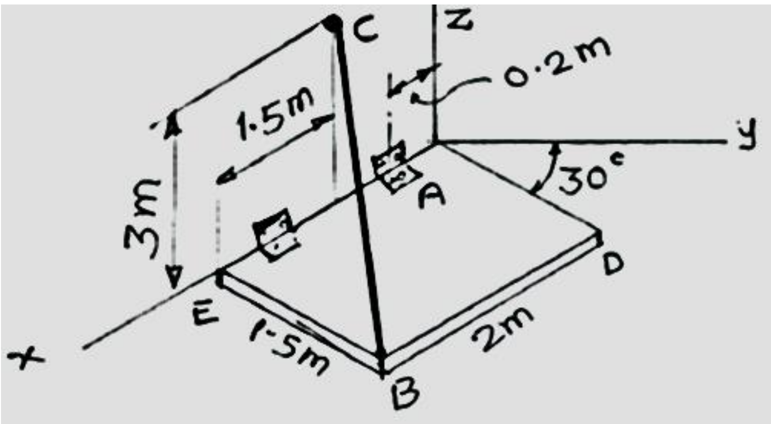
$R_2 = 3.67$ N

$R_3 = 0.67$ N

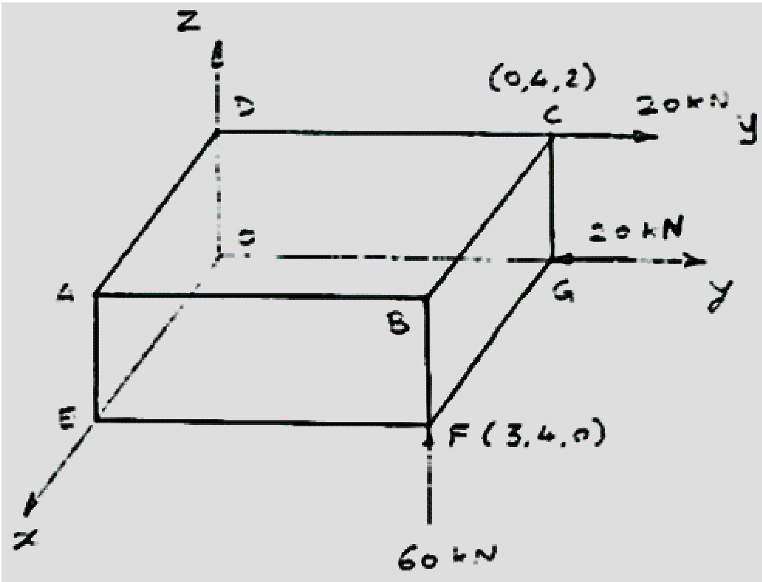
University Problems:

Forces in Space:

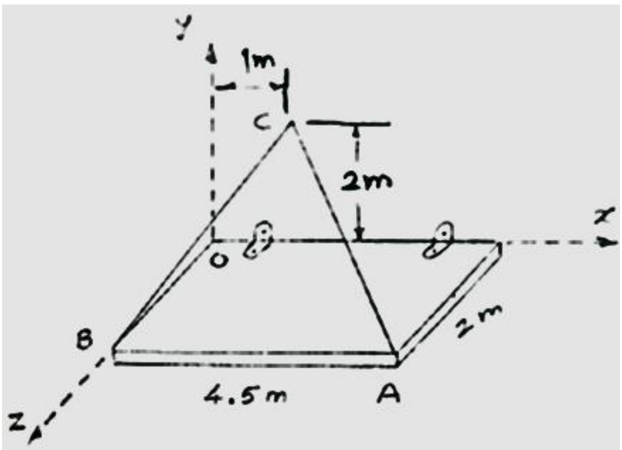
1) Determine the magnitude and the direction of the force F = (34 i + 10 j – 90 k) N.
2) A force of 1000 N forms angle forces of 60^0, 45^0 and 120^0 with X, Y and Z axes respectively. Write the equation of the force in the vector form. Find the equation of the unit vector, λ in the direction of the force.
3) Force vector F = 20 i – 30 j + 60 k N. Find θ_X, θ_Y and θ_Z.
4) The coordinates initial and terminal points of a force vector F = 100 N are (2, 4, 3) and (1, -6, 2) respectively. Determine the components of the force and its angles with the axes. Specify the force vector.
5) A force F =80 I + 50 j – 60 k passes through a point B (6, 3, 7), compute its moment about a point M (5, 8, 4).
6) A force 35 kN acts through a point C (3, -4, -6) in the direction of vector 6 i + 5 j – 3 k. find the moment of the force about the point (4, -7, 2).
7) A force acts at the origin in a direction defined by the angles $\theta_Y = 65^0$ and $\theta_Z = 40^0$. Knowing that the X component of the force is -750 N, determine:-
   (i) The other components
   (ii) Magnitude of the force and
   (iii) The value of θ_X.
8) The rope BC exerts a force of 350 N on the lid at B determine the moment of this force about the hinge at A.



9) A weightless box of size 3 x 4 x 2 m is loaded by three forces as shown in figure. Find the moment of the each about the solid diagonal OB due to the three forces.

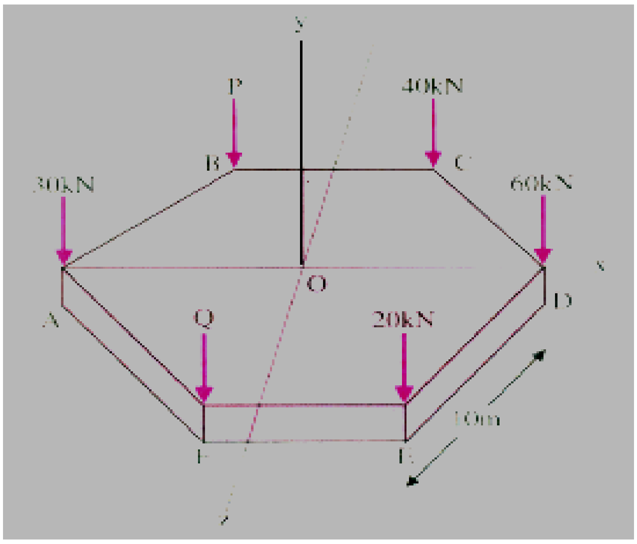


10) Given that the tension in cable AC is 900 N, determine the components of TAC exerted on point C.

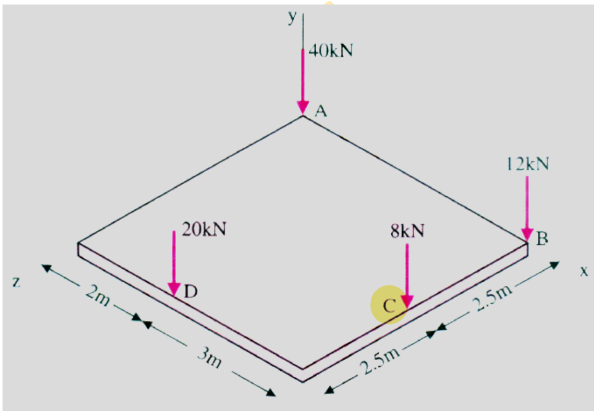


- **Resultant of Parallel force system:**

11) A concrete mat in the shape of regular hexagon of side 10 m, supports four loads as shown in figure. Determine loads P and Q if resultant of six loads is to pass through centre of the mat.



12) A square foundation supports four loads as shown in Figure. Determine magnitude, direction and point of application of resultant of four forces.



13) The forces of 20 N, 10 N and 30 N are as shown in figure. Forces are acting in the x-z plane at coordinates (x, z) are (2, 3), (4, 2) and (7, 4) respectively. Determine the resultant and locate it.