Module 3:Lecture notes and Practice examples 1

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Partial Differentiation

Concept

Partial Derivatives First order $\frac{\partial z}{\partial x}$ Let $z = f(x, y)$ be a bivariate function continuous in its domain.

Then the rate of change of z with respect to x keeping y constant, is called **Partial derivative of z with respect to** $x$and is denoted by any of the following symbols:

$$\frac{\partial z}{\partial x}, \frac{\partial f}{\partial x}, f_x(x, y), f_1(x, y)$$

Here

$$\frac{\partial z}{\partial x} = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

Partial Derivatives First order $\frac{\partial z}{\partial y}$ Similarly,

the rate of change of z with respect to y keeping x constant, is called Partial derivative of z with respect to y and is denoted by any of the following symbols:

$$\frac{\partial z}{\partial y}, \frac{\partial f}{\partial y}, f_y(x, y), f_2(x, y)$$

Here

$$\frac{\partial z}{\partial y} = \lim_{k \to 0} \frac{f(x, y + k) - f(x, y)}{k}$$

$\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are also called First order partial derivatives of z.

Rules of Partial Differentiation

Derivative of Sum/difference

$$\frac{\partial}{\partial x}(u \pm v) = \frac{\partial u}{\partial x} \pm \frac{\partial u}{\partial x}$$

$$\frac{\partial}{\partial y}(u \pm v) = \frac{\partial u}{\partial y} \pm \frac{\partial u}{\partial y}$$

Derivative of Product

$$\frac{\partial}{\partial x}(uv) = u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}$$

$$\frac{\partial}{\partial y}(uv) = u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}$$

Derivative of Quotient

$$\frac{\partial}{\partial x}\left(\frac{u}{v}\right) = \frac{v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}}{v^2}$$

$$\frac{\partial}{\partial y}\left(\frac{u}{v}\right) = \frac{v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}}{v^2}$$

Derivative of a constant

$$\frac{\partial}{\partial x}(k) = 0$$
$$\frac{\partial}{\partial y}(k) = 0$$

- If k is constant, then

$$\frac{\partial}{\partial x}(k.u) = k.\frac{\partial u}{\partial x}$$
$$\frac{\partial}{\partial y}(k.u) = k.\frac{\partial u}{\partial y}$$

- $$\frac{\partial}{\partial x}[f(x,y,z)]^n = n.[f(x,y,z)]^{n-1}.\frac{\partial f}{\partial x}$$

- $$\frac{\partial}{\partial y}[f(x,y,z)]^n = n.[f(x,y,z)]^{n-1}.\frac{\partial f}{\partial y}$$

- $$\frac{\partial}{\partial z}[f(x,y,z)]^n = n.[f(x,y,z)]^{n-1}.\frac{\partial f}{\partial z}$$

Partial Derivatives Higher order

Let $z = f(x, y)$ be a bi variate function continuous in its domain then $\frac{\partial z}{\partial x} = f_x$ and $\frac{\partial z}{\partial y} = f_y$ are called **First order partial derivatives of** z
where $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are themselves functions of x and y.

Hence **Second order Partial derivatives of** z are given by,

- $f_{xx} = \dfrac{\partial}{\partial x}(\dfrac{\partial z}{\partial x}) = \dfrac{\partial^2 z}{\partial x^2}$ or $\dfrac{\partial^2 f}{\partial x^2}$ or z_{xx}
- $f_{xy} = \dfrac{\partial}{\partial y}(\dfrac{\partial z}{\partial x}) = \dfrac{\partial^2 z}{\partial y \partial x}$ or $\dfrac{\partial^2 f}{\partial y \partial x}$ or z_{xy}
- $f_{yx} = \dfrac{\partial}{\partial x}(\dfrac{\partial z}{\partial y}) = \dfrac{\partial^2 z}{\partial x \partial y}$ or $\dfrac{\partial^2 f}{\partial x \partial y}$ or z_{yx}
- $f_{yy} = \dfrac{\partial}{\partial y}(\dfrac{\partial z}{\partial y}) = \dfrac{\partial^2 z}{\partial y^2}$ or $\dfrac{\partial^2 f}{\partial y^2}$ or z_{yy}

z_{xy} and z_{yx} are called **mixed partial derivatives** and they are not always equal

Examples (Type 1)

(A) Find First and Second Order Partial Derivatives
$f(x, y, z) = \dfrac{y}{x + y + z}$
Solution

Given $f(x, y, z) = \dfrac{y}{x + y + z}$
(1)

$$
\begin{aligned}
f_x &= \frac{\partial f}{\partial x} \\
&= \frac{\partial}{\partial x}\left(\frac{y}{x+y+z}\right) \\
&= y\ \frac{\partial}{\partial x}\left(\frac{1}{x+y+z}\right) \\
&= y\ \left(\frac{-1}{(x+y+z)^2}\right) \\
f_x &= \ \left(\frac{-y}{(x+y+z)^2}\right)
\end{aligned}
$$

(2)

$$
\begin{aligned}
f_y &= \frac{\partial f}{\partial y} \\
&= \frac{\partial}{\partial y}\left(\frac{y}{x+y+z}\right) \\
&= \frac{(x+y+z)(1) - y(1)}{(x+y+z)^2} \\
f_y &= \frac{x+z}{(x+y+z)^2}
\end{aligned}
$$

(3)

$$
\begin{aligned}
f_z &= \frac{\partial f}{\partial z} \\
&= \frac{\partial}{\partial z}\left(\frac{y}{x+y+z}\right) \\
&= y\ \frac{\partial}{\partial z}\left(\frac{1}{x+y+z}\right) \\
&= y\ \left(\frac{-1}{(x+y+z)^2}\right) \\
f_z &= \ \left(\frac{-y}{(x+y+z)^2}\right)
\end{aligned}
$$

(4)

$$
\begin{aligned}
f_{xz} &= \frac{\partial f_x}{\partial z} \\
&= \frac{\partial}{\partial z}\left(\frac{-y}{(x+y+z)^2}\right) \\
&= -y\ \frac{\partial}{\partial z}\left(\frac{1}{x+y+z}\right) \\
&= -y\ \left(\frac{-2}{(x+y+z)^3}\right) \\
f_{xz} &= \ \left(\frac{2y}{(x+y+z)^3}\right)
\end{aligned}
$$

In the same manner we can find, $f_{xy}, f_{xx}, f_{yz}, f_{yx}, f_{yy}, f_{zx}, f_{zy}, f_{zz}$

Examples(Type 2)

If $z = x^2\ tan^{-1}\left(\frac{y}{x}\right) - y^2\ tan^{-1}\left(\frac{x}{y}\right)$, prove that

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = \frac{x^2 - y^2}{x^2 + y^2}$$

Solution

Given

$$z = x^2\ tan^{-1}(\frac{y}{x}) - y^2\ tan^{-1}(\frac{x}{y})............\mathbf{(1)}$$

Differentiating (1) w.r.t x

$$\begin{aligned}
\frac{\partial z}{\partial x} &= 2x\ tan^{-1}\left(\frac{y}{x}\right) + x^2\ \left(\frac{1}{1+\frac{y^2}{x^2}}\right)\ \frac{\partial}{\partial x}\left(\frac{y}{x}\right) - y^2\ \left(\frac{1}{1+\frac{x^2}{y^2}}\right)\frac{\partial}{\partial x}\left(\frac{x}{y}\right) \\
&= 2x\ tan^{-1}\left(\frac{y}{x}\right) + x^2\ \left(\frac{1}{1+\frac{y^2}{x^2}}\right)\ \left(\frac{-y}{x^2}\right) - y^2\ \left(\frac{1}{1+\frac{x^2}{y^2}}\right)\ \left(\frac{1}{y}\right) \\
&= 2x\ tan^{-1}\left(\frac{y}{x}\right) + \left(\frac{x^4}{x^2+y^2}\right)\ \left(\frac{-y}{x^2}\right) - \left(\frac{y^4}{x^2+y^2}\right)\ \left(\frac{1}{y}\right) \\
&= 2x\ tan^{-1}\left(\frac{y}{x}\right) - \frac{x^2y}{x^2+y^2} - \frac{y^3}{x^2+y^2} \\
\frac{\partial z}{\partial x} &= 2x\ tan^{-1}\left(\frac{y}{x}\right) - y \text{..........}\mathbf{(2)}
\end{aligned}$$

Differentiating (2) w.r.t y

$$\begin{aligned}
\frac{\partial^2 z}{\partial y \partial x} &= 2x\ \frac{\partial}{\partial y}\left(tan^{-1}\left(\frac{y}{x}\right)\right) - 1 \\
&= 2x\ \left(\frac{1}{1+\frac{y^2}{x^2}}\right)\ \left(\frac{1}{x}\right) - 1 \\
&= 2x\left(\frac{x^2}{x^2+y^2}\right)\ \left(\frac{1}{x}\right) - 1 \\
&= \frac{2x^2}{x^2+y^2} - 1 \\
&= \frac{2x^2 - x^2 - y^2}{x^2+y^2} \\
\frac{\partial^2 z}{\partial y \partial x} &= \frac{x^2-y^2}{x^2+y^2}\text{.........}\mathbf{(3)}
\end{aligned}$$

Again Differentiating (1) w.r.t y

$$
\begin{aligned}
\frac{\partial z}{\partial y} &= x^2\left(\frac{1}{1+\frac{y^2}{x^2}}\right)\frac{\partial}{\partial y}\left(\frac{y}{x}\right) - y^2\left(\frac{1}{1+\frac{x^2}{y^2}}\right)\frac{\partial}{\partial y}\left(\frac{x}{y}\right) - 2y\ tan^{-1}\left(\frac{x}{y}\right)\\
&= x^2\left(\frac{1}{1+\frac{y^2}{x^2}}\right)\left(\frac{1}{x}\right) - y^2\left(\frac{1}{1+\frac{x^2}{y^2}}\right)\ \left(\frac{-x}{y^2}\right) - 2y\ tan^{-1}\left(\frac{x}{y}\right)\\
&= \left(\frac{x^4}{x^2+y^2}\right)\ \left(\frac{1}{x}\right) - \left(\frac{y^4}{x^2+y^2}\right)\ \left(\frac{-x}{y^2}\right) - 2y\ tan^{-1}\left(\frac{x}{y}\right)\\
&= \left(\frac{x^3}{x^2+y^2}\right) + \left(\frac{xy^2}{x^2+y^2}\right) - 2y\ tan^{-1}\left(\frac{x}{y}\right)\\
&= x - 2y\ tan^{-1}\left(\frac{x}{y}\right) \text{...........}\mathbf{(4)}
\end{aligned}
$$

Differentiating (4) w.r.t x

$$
\begin{aligned}
\frac{\partial^2 z}{\partial x \partial y} &= 1 - 2y\ \frac{\partial}{\partial x}\left(tan^{-1}\left(\frac{y}{x}\right)\right)\\
&= 1 - 2y\ \left(\frac{1}{1+\frac{y^2}{x^2}}\right)\left(\frac{1}{y}\right)\\
&= 1 - 2y\ \left(\frac{y^2}{x^2+y^2}\right)\left(\frac{1}{y}\right)\\
&= \frac{x^2+y^2-2y^2}{x^2+y^2}\\
\frac{\partial^2 z}{\partial x \partial y} &= \frac{x^2-y^2}{x^2+y^2}\text{.........}\mathbf{(5)}
\end{aligned}
$$

$$
\therefore \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y} = \frac{x^2-y^2}{x^2+y^2}\text{.......(}\mathbf{By\ (3)\ and\ (5)}\text{)}
$$

Partial Differentiation of composite functions

Case(1)
If $z = f(r)$ where $r = g(x, y)$ then z becomes composite function of x and y.
Thus,

$$\frac{\partial z}{\partial x} = \frac{dz}{dr}.\frac{\partial r}{\partial x} \ or \ f'(r).\frac{\partial r}{\partial x}$$

and

$$\frac{\partial z}{\partial y} = \frac{dz}{dr}.\frac{\partial r}{\partial y} \ or \ f'(r).\frac{\partial r}{\partial y}$$

This can also be expressed using **Dependency chart**(Tree diagram)

Case(2)
$r = h(x, y, z)$ then u becomes composite function of x, y, z
Then in this case ,

$$\frac{\partial u}{\partial x} = \frac{du}{dr}.\frac{\partial r}{\partial x} \ ;$$

$$\frac{\partial u}{\partial y} = \frac{du}{dr}.\frac{\partial r}{\partial y}$$

$$\frac{\partial u}{\partial z} = \frac{du}{dr}.\frac{\partial r}{\partial z}$$

This can also be expressed using **Dependency chart**(Tree diagram)

Case(3)
If $p = f(x, y)$ where $x = g(t)$; $y = h(t)$ then p becomes composite function of t
Then in this case chain rule of partial derivative is ,

$$\frac{dp}{dt} = \frac{\partial p}{\partial x}.\frac{dx}{dt} + \frac{\partial p}{\partial y}.\frac{dy}{dt}$$

This can also be expressed using **Dependency chart**(Tree diagram)

Case(4)
If $z = f(x, y)$ where $x = g(u, v)$; $y = h(u, v)$ then z becomes composite function of

u, v
Then in this case chain rule of partial derivative is ,

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial u} \ ;$$

and

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial v}$$

This can also be expressed using **Dependency chart**(Tree diagram)

Case(5)
If $p = f(x, y)$ where $x = g(u, v, w)$; $y = h(u, v, w)$ then p becomes composite function of u, v, w
Then in this case chain rule of partial derivative is ,

$$\frac{\partial p}{\partial u} = \frac{\partial p}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial p}{\partial y}.\frac{\partial y}{\partial u};$$

and

$$\frac{\partial p}{\partial v} = \frac{\partial p}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial p}{\partial y}.\frac{\partial y}{\partial v} \ ;$$

and

$$\frac{\partial p}{\partial w} = \frac{\partial p}{\partial x}.\frac{\partial x}{\partial w} + \frac{\partial p}{\partial y}.\frac{\partial y}{\partial w}$$

This can also be expressed using **Dependency chart**(Tree diagram)

Case(6)
If $p = f(x, y, z)$ where $x = g_1(t)$; $y = g_2(t)$ and $z = g_3(t)$ then p becomes composite function of t
Then in this case chain rule of partial derivative is ,

$$\frac{dp}{dt} = \frac{\partial p}{\partial x}.\frac{dx}{dt} + \frac{\partial p}{\partial y}.\frac{dy}{dt} + \frac{\partial p}{\partial z}.\frac{dz}{dt}$$

This can also be expressed using **Dependency chart**(Tree diagram)

Case(7)
If $p = f(x, y, z)$ where $x = g_1(u, v)$; $y = g_2(u, v)$ and $z = g_3(u, v)$ then p becomes composite function of u, v
Then in this case chain rule of partial derivative is ,

$$\frac{\partial p}{\partial u} = \frac{\partial p}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial p}{\partial y}.\frac{\partial y}{\partial u} + \frac{\partial p}{\partial z}.\frac{\partial z}{\partial u}$$

and

$$\frac{\partial p}{\partial v} = \frac{\partial p}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial p}{\partial y}.\frac{\partial y}{\partial v} + \frac{\partial p}{\partial z}.\frac{\partial z}{\partial u}$$

This can also be expressed using **Dependency chart**(Tree diagram)

Case(8)
If $p = f(x, y, z)$ where $x = g_1(u, v, w)$; $y = g_2(u, v, w)$ and $z = g_3(u, v, w)$ then p becomes composite function of u, v, w
Then in this case chain rule of partial derivative is ,

$$\frac{\partial p}{\partial u} = \frac{\partial p}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial p}{\partial y}.\frac{\partial y}{\partial u} + \frac{\partial p}{\partial z}.\frac{\partial z}{\partial u} \ ;$$

and

$$\frac{\partial p}{\partial v} = \frac{\partial p}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial p}{\partial y}.\frac{\partial y}{\partial v} + \frac{\partial p}{\partial z}.\frac{\partial z}{\partial v} \ ;$$

and

$$\frac{\partial p}{\partial w} = \frac{\partial p}{\partial x}.\frac{\partial x}{\partial w} + \frac{\partial p}{\partial y}.\frac{\partial y}{\partial w} + \frac{\partial p}{\partial z}.\frac{\partial z}{\partial w}$$

This can also be expressed using **Dependency chart**(Tree diagram)

Case(9)
If $p = f(x, y)$ where $y = g(x)$ then p becomes composite function of x
Then in this case chain rule of partial derivative is ,

$$\frac{dp}{dx} = \frac{\partial p}{\partial x}.1 + \frac{\partial p}{\partial y}.\frac{dy}{dx}$$

This can also be expressed using **Dependency chart**(Tree diagram)

Examples (Type 3)(Using Chain Rules)

Example 1

Find $\frac{du}{dt}$ if $u = tan^{-1}\left(\frac{y}{x}\right)$ and $x = e^t - e^{-t}$ and $y = e^t + e^{-t}$

Solution

Given

$$u = tan^{-1}\left(\frac{y}{x}\right)$$
$$x = e^t - e^{-t}$$
$$y = e^t + e^{-t}$$

$\therefore$ by chain rule

$$\begin{aligned}
\frac{du}{dt} &= \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} \\
\frac{du}{dt} &= \frac{\partial}{\partial x}\left[tan^{-1}\left(\frac{y}{x}\right)\right]\frac{d}{dt}\left(e^t - e^{-t}\right) + \frac{\partial}{\partial y}\left[tan^{-1}\left(\frac{y}{x}\right)\right]\frac{d}{dt}\left(e^t + e^{-t}\right) \\
&= \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{-y}{x^2}\right)\left(e^t + e^{-t}\right) + \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right)\left(e^t - e^{-t}\right) \\
&= \frac{-y}{x^2+y^2}(e^t + e^{-t}) + \frac{x}{x^2+y^2}(e^t - e^{-t}) \\
&= \frac{-y}{x^2+y^2}(y) + \frac{x}{x^2+y^2}(x) \\
&= \frac{x^2-y^2}{x^2+y^2} \\
&= \frac{(e^t - e^{-t})^2 - (e^t + e^{-t})^2}{(e^t - e^{-t})^2 + (e^t + e^{-t})^2} = \frac{-4}{2e^{2t} + 2e^{-2t}} \\
\frac{du}{dt} &= \frac{-2}{e^{2t} + e^{-2t}}
\end{aligned}$$

Example 2

If $z = f(x, y)$ and $x = e^u + e^{-v}$ and $y = e^{-u} - e^v$ then show that

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

Solution
Given

$$z = f(x, y)$$
$$x = e^u + e^{-v}$$
$$y = e^{-u} - e^v$$

$\therefore$ by chain rule

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$
$$= \frac{\partial z}{\partial x}\frac{\partial}{\partial u}\left(e^u + e^{-v}\right) + \frac{\partial z}{\partial y}\frac{\partial}{\partial u}\left(e^{-u} + e^v\right)$$
$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}(e^u) + \frac{\partial z}{\partial y}(-e^{-u}) \text{....................}\mathbf{(1)}$$
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$
$$= \frac{\partial z}{\partial x}\frac{\partial}{\partial v}\left(e^u + e^{-v}\right) + \frac{\partial z}{\partial y}\frac{\partial}{\partial v}\left(e^{-u} + e^v\right)$$
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(-e^{-v}) + \frac{\partial z}{\partial y}(-e^v) \text{....................}\mathbf{(2)}$$

From (1) and (2)

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(e^u) + \frac{\partial z}{\partial y}(-e^{-u}) - \frac{\partial z}{\partial x}(-e^{-v}) + \frac{\partial z}{\partial y}(-e^v)$$

$$= (e^u + e^{-v})\frac{\partial z}{\partial x} - (e^{-u} + e^v)\frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

Example 3
If $u = f(2x - 3y, 3y - 4z, 4z - 2x)$ then show that

$$\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = 0$$

Solution
Given

$$u = f(2x - 3y, 3y - 4z, 4z - 2x)$$

Let

$$p = 2x - 3y \; ; q = 3y - 4z \; ; r = 4z - 2x$$

Then

$$u = f(p, q, r)$$

By chain rule

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} \\ &= \frac{\partial u}{\partial p}(2) + \frac{\partial u}{\partial q}(0) + \frac{\partial u}{\partial r}(-2) \\ &= 2\ \frac{\partial u}{\partial p} - 2\ \frac{\partial u}{\partial r}.........\mathbf{(1)}\end{aligned}$$

Also By chain rule

$$\begin{aligned}\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial y} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} \\ &= \frac{\partial u}{\partial p}(-3) + \frac{\partial u}{\partial q}(3) + \frac{\partial u}{\partial r}(0) \\ &= 3\ \frac{\partial u}{\partial q} - 3\ \frac{\partial u}{\partial p}.........\mathbf{(2)}\end{aligned}$$

Also By chain rule

$$\begin{aligned}
\frac{\partial u}{\partial z} &= \frac{\partial u}{\partial p}\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} \\
&= \frac{\partial u}{\partial p}(0) + \frac{\partial u}{\partial q}(-4) + \frac{\partial u}{\partial r}(4) \\
&= 4\ \frac{\partial u}{\partial r} - 4\ \frac{\partial u}{\partial q} ....\mathbf{(3)}
\end{aligned}$$

By (1),(2) and (3)

$$\begin{aligned}
\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} &= \frac{1}{2}\left[2\ \frac{\partial u}{\partial p} - 2\ \frac{\partial u}{\partial r}\right] + \frac{1}{3}\left[3\ \frac{\partial u}{\partial r} - 3\ \frac{\partial u}{\partial p}\right] \\
&+\ \frac{1}{4}\left[4\ \frac{\partial u}{\partial r} - 4\ \frac{\partial u}{\partial q}\right] \\
&= \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r} + \frac{\partial u}{\partial q} - \frac{\partial u}{\partial p} + \frac{\partial u}{\partial r} - \frac{\partial u}{\partial q} \\
\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} &= 0
\end{aligned}$$

Hence Proved.

Examples(Type 4)

Example 1

If $z = xlog(x+r) - r$ where $r^2 = x^2 + y^2$ then show that

(a)

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{x+r}$$

(b)

$$\frac{\partial^3 z}{\partial x^3} = -\frac{x}{r^3}$$

Solution

Given

$$r^2 = x^2 + y^2$$
$$\therefore 2r\frac{\partial r}{\partial x} = 2x$$
$$\therefore \frac{\partial r}{\partial x} = \frac{x}{r}.....\mathbf{(1)}$$
$$and$$
$$r^2 = x^2 + y^2$$
$$\therefore 2r\frac{\partial r}{\partial y} = 2y$$
$$\therefore \frac{\partial r}{\partial y} = \frac{y}{r}.....\mathbf{(2)}$$

(a) Given that

$$z = xlog(x+r) - r..............\mathbf{(3)}$$

Partially differentiating (3) w.r.t. x

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{\partial}{\partial x}\left[xlog(x+r)-r\right] \\
&= x\frac{\partial}{\partial x}\left[log(x+r)\right]+log(x+r)-\frac{\partial r}{\partial x} \\
&= x\frac{1}{x+r}\frac{\partial}{\partial x}(x+r)+log(x+r)-\frac{\partial r}{\partial x} \\
&= \frac{x}{x+r}\left(1+\frac{\partial r}{\partial x}\right)+log(x+r)-\frac{\partial r}{\partial x} \\
&= \frac{x}{x+r}\left(1+\frac{x}{r}\right)+log(x+r)-\frac{\partial r}{\partial x} \\
\frac{\partial z}{\partial x} &= log(x+r)
\end{aligned}$$

Differentiating above result w.r.t. x

$$\begin{aligned}
\frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x}\log(x+r) \\
&= \frac{1}{x+r}\frac{\partial}{\partial x}(x+r) \\
&= \frac{1}{x+r}\left(1+\frac{\partial r}{\partial x}\right) \\
&= \frac{1}{x+r}\left(1+\frac{x}{r}\right) \\
\frac{\partial^2 z}{\partial x^2} &= \frac{1}{r}\text{...........}\mathbf{(4)}
\end{aligned}$$

Again partially differentiating (3) w.r.t. y

$$\begin{aligned}
\frac{\partial z}{\partial y} &= \frac{\partial}{\partial y}\left[xlog(x+r)-r\right] \\
&= x\frac{\partial}{\partial y}\left[log(x+r)\right]-\frac{\partial r}{\partial y} \\
&= \frac{x}{x+r}\frac{\partial}{\partial y}(x+r)-\frac{\partial r}{\partial y} \\
&= \frac{x}{x+r}\left(0+\frac{\partial r}{\partial y}\right)-\frac{\partial r}{\partial y} \\
&= \frac{x}{x+r}\ \frac{y}{r}-\frac{y}{r} \\
&= \frac{y}{r}\left[\frac{x}{x+r}-1\right] \\
\frac{\partial z}{\partial y} &= \frac{-y}{x+r}\text{...........}\mathbf{(5)}
\end{aligned}$$

Differentiating above result w.r.t. y

$$\begin{aligned}
\frac{\partial^2 z}{\partial y^2} &= \frac{\partial}{\partial y}\left[\frac{-y}{x+r}\right] \\
&= \frac{(x+r)(-1)+y\left(\frac{y}{r}\right)}{(x+r)^2} \\
&= \frac{-r(x+r)+y^2}{r(x+r)^2} \\
&= \frac{-rx-r^2+y^2}{r(x+r)^2} \\
&= \frac{-rx-x^2}{r(x+r)^2} \\
&= \frac{-x(x+r)}{r(x+r)^2} \\
\frac{\partial^2 z}{\partial y^2} &= \frac{-x}{r(x+r)}\text{...........}\mathbf{(6)}
\end{aligned}$$

Adding (4) and (6)

$$
\begin{aligned}
\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} &= \frac{1}{r} - \frac{-x}{r(x+r)} \\
&= \frac{x + r - x}{r(x+r)} \\
&= \frac{r}{r(x+r)} \\
\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} &= \frac{1}{x+r}
\end{aligned}
$$

Hence Proved

(b) By (4) we have

$$
\begin{aligned}
\frac{\partial^2 z}{\partial x^2} &= \frac{1}{r} \\
\therefore \frac{\partial^3 z}{\partial x^3} &= \frac{\partial}{\partial x}\left(\frac{1}{r}\right) \\
&= \frac{-1}{r^2}\frac{\partial r}{\partial x} \\
&= \frac{-1}{r^2}\frac{x}{r} \\
\frac{\partial^3 z}{\partial x^3} &= -\frac{x}{r^3}
\end{aligned}
$$

Hence Proved.

Example 2

Find the value of n so that $u = r^n(3\ cos^2\ \theta - 1)$satisfies the differential equation

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{sin\ \theta}\ \frac{\partial}{\partial \theta}\left(sin\ \theta\frac{\partial u}{\partial \theta}\right) = 0$$

Solution
Given

$$u = r^n(3\ cos^2\ \theta - 1)..........\mathbf{(1)}$$

Partially differentiating (1) w.r.t r we have

$$\frac{\partial u}{\partial r} = nr^{n-1}(3\ cos^2\ \theta - 1)$$
$$\therefore r^2\frac{\partial u}{\partial r} = nr^{n+1}(3\ cos^2\ \theta - 1)$$
$$\therefore \frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = n(n+1)r^n(3\ cos^2\ \theta - 1)....\mathbf{(2)}$$

Again partially differentiating (1) w.r.t. θ

$$\frac{\partial u}{\partial \theta} = \frac{\partial}{\partial \theta}r^n\left[(3\ cos^2\ \theta - 1)\right]$$
$$= r^n(-6\ cos\ \theta\ sin\ \theta)$$
$$\therefore sin\ \theta\ \frac{\partial u}{\partial \theta} = r^n(-6\ cos\ \theta\ sin^2\ \theta)$$
$$\therefore \frac{\partial}{\partial \theta}\left(sin\ \theta\ \frac{\partial u}{\partial \theta}\right) = r^n(6\ sin^3\ \theta\ \ - 12cos^2\ \theta\ sin\ \theta)$$
$$\therefore \frac{1}{sin\ \theta}\ \frac{\partial}{\partial \theta}\left(sin\ \theta\ \frac{\partial u}{\partial \theta}\right) = r^n(6\ sin^2\ \theta\ \ - 12cos^2\ \theta)$$
$$\therefore \frac{1}{sin\ \theta}\ \frac{\partial}{\partial \theta}\left(sin\ \theta\ \frac{\partial u}{\partial \theta}\right) = r^n(6 - 18\ cos^2\ \theta)$$
$$\therefore \frac{1}{sin\ \theta}\ \frac{\partial}{\partial \theta}\left(sin\ \theta\ \frac{\partial u}{\partial \theta}\right) = -6\ r^n(3cos^2\ \theta - 1)..........\mathbf{(3)}$$

Given that u satisfies the equation

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{sin\ \theta}\ \frac{\partial}{\partial \theta}\left(sin\ \theta\frac{\partial u}{\partial \theta}\right) = 0$$

by (2) and (3)

$$\begin{aligned}(n+1)r^n(3\ cos^2\ \theta-1)+-6\ r^n(3cos^2\ \theta-1)&=0\\(n^2+n-6)r^n(3\ cos^2\ \theta-1)&=0\\(n+3)(n-2)r^n(3\ cos^2\ \theta-1)&=0\\n&=2,-3\end{aligned}$$

Hence Proved.

Example 3
If $u=f(r)$ and $r=\sqrt{x^2+y^2+z^2}$ then ,P.T.

$$u_{xx}+u_{yy}+u_{zz}=f^{''}(r)+\frac{2}{r}f^{'}(r)$$

Solution

Given

$$\begin{aligned}r&=\sqrt{x^2+y^2+z^2}\\\therefore r^2&=x^2+y^2+z^2\\\therefore 2r\frac{\partial r}{\partial x}&=2x\\\therefore \frac{\partial r}{\partial x}&=\frac{x}{r}.....\mathbf{(1)}\\and&\\r^2&=x^2+y^2+z^2\\\therefore 2r\frac{\partial r}{\partial y}&=2y\\\therefore \frac{\partial r}{\partial y}&=\frac{y}{r}.....\mathbf{(2)}\\and&\\r^2&=x^2+y^2+z^2\\\therefore 2r\frac{\partial r}{\partial z}&=2z\\\therefore \frac{\partial r}{\partial z}&=\frac{z}{r}.....\mathbf{(3)}\end{aligned}$$

Given that

$$u = f(r)\text{.............}\mathbf{(4)}$$

Partially differentiating (3) w.r.t. x

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial x}\\ &= f^{'}(r)\left(\frac{x}{r}\right)\end{aligned}$$

Again differentiating w.r.t x

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left[f^{'}(r)\left(\frac{x}{r}\right)\right]\\ &= f^{'}(r)\left[\frac{r(1) - x\left(\frac{x}{r}\right)}{r^2}\right] + \left(\frac{x}{r}\right)f^{''}(r)\left(\frac{x}{r}\right)\\ \therefore \frac{\partial^2 u}{\partial x^2} &= f^{'}(r)\left[\frac{r^2 - x^2}{r^3}\right] + f^{''}(r)\left(\frac{x^2}{r^2}\right)\text{......}\mathbf{(4)}\end{aligned}$$

Again Partially differentiating (3) w.r.t. y

$$\begin{aligned}\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial y}\\ &= f^{'}(r)\left(\frac{y}{r}\right)\end{aligned}$$

Again differentiating w.r.t y

$$\begin{aligned}\frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y}\left[f^{'}(r)\left(\frac{y}{r}\right)\right]\\ &= f^{'}(r)\left[\frac{r(1) - y\left(\frac{y}{r}\right)}{r^2}\right] + \left(\frac{y}{r}\right)f^{''}(r)\left(\frac{y}{r}\right)\\ \therefore \frac{\partial^2 u}{\partial y^2} &= f^{'}(r)\left[\frac{r^2 - y^2}{r^3}\right] + f^{''}(r)\left(\frac{y^2}{r^2}\right)\text{......}\mathbf{(5)}\end{aligned}$$

Partially differentiating (3) w.r.t. z

$$\begin{aligned}\frac{\partial u}{\partial z} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} \\ &= f^{'}(r)\left(\frac{z}{r}\right)\end{aligned}$$

Again differentiating w.r.t z

$$\begin{aligned}\frac{\partial^2 u}{\partial z^2} &= \frac{\partial}{\partial z}\left[f^{'}(r)\left(\frac{z}{r}\right)\right] \\ &= f^{'}(r)\left[\frac{r(1) - z\left(\frac{z}{r}\right)}{r^2}\right] + \left(\frac{z}{r}\right)f^{''}(r)\left(\frac{z}{r}\right) \\ \therefore \frac{\partial^2 u}{\partial z^2} &= f^{'}(r)\left[\frac{r^2 - z^2}{r^3}\right] + f^{''}(r)\left(\frac{z^2}{r^2}\right) \text{......}\mathbf{(6)}\end{aligned}$$

Adding (4),(5) and (6), we have

$$\begin{aligned}u_{xx} + u_{yy} + u_{zz} &= f^{'}(r)\left[\frac{r^2 - x^2}{r^3}\right] + f^{''}(r)\left(\frac{x^2}{r^2}\right) + f^{'}(r)\left[\frac{r^2 - y^2}{r^3}\right] + f^{''}(r)\left(\frac{y^2}{r^2}\right) + \\ &\quad f^{'}(r)\left[\frac{r^2 - z^2}{r^3}\right] + f^{''}(r)\left(\frac{z^2}{r^2}\right) \\ &= f^{'}(r)\left[\frac{r^2 - x^2 + r^2 - y^2 + r^2 - z^2}{r^3}\right] + f^{''}(r)\left(\frac{x^2 + y^2 + z^2}{r^2}\right) \\ &= f^{'}(r)\left[\frac{3r^2 - r^2}{r^3}\right] + f^{''}(r)\left(\frac{r^2}{r^2}\right) \\ u_{xx} + u_{yy} + u_{zz} &= f^{''}(r) + \left(\frac{2}{r}\right)f^{'}(r)\end{aligned}$$

Hence Proved

EXAMPLES

Type 1

Find all possible Partial Derivatives

(1) $f(x,y) = (1-2xy+y^2)^{-1}$

(2) $f(x,y,z) = (x^3+y^3+z^3)^3$

(3) $f(x,y,z) = \dfrac{xyz}{x+y+z}$

Type 2

(1)If $u = tan^{-1}\left(\dfrac{xy}{\sqrt{1+x+y}}\right)$, show that

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{(1+x+y)^{\frac{3}{2}}}$$

(2) If $u = e^{xyz}$, P.T.

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = (1+3xyz+x^2y^2z^2)e^{xyz}$$

(3) If $u = z.tan^{-1}(\dfrac{y}{x})$; Find the value of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

(4) If $u = log(x^3 + y^3 - x^2y - xy^2)$; Prove that

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = -\frac{4}{(x+y)^2}$$

(5) If $z(x + y) = (x^2 + y^2)$; Prove that

$$\left[\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right]^2 = 4\left[1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right]$$

(6) If $z = x^y + y^x$, verify that

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

(7) If $u = log(x^3 + y^3 + z^3 - 3xyz)$; Prove that

$$[\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}]^2 u = \frac{-9}{(x+y+z)^2}$$

(8) If $u = \dfrac{e^{x+y+z}}{e^x + e^y + e^z}$; Prove that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} u = 2u$$

(9) If $u = e^x(xcosy - ysiny)$, verify that

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

(10) If $u = r^m$ and $r = \sqrt{x^2 + y^2 + z^2}$ then ,P.T.

$$u_{xx} + u_{yy} + u_{zz} = m(m+1)r^{m-2}$$

(11) If $u = f(\frac{x}{y}, \frac{y}{z}, \frac{z}{x})$ then ,P.T.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$

(12) If $z = f(x, y)$; $x = e^u + e^{-v}$; $y = e^{-u} - e^v$ then ,P.T.

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

(13) If $u = tan^{-1}(\frac{y}{x})$; $x = e^t + e^{-t}$; $y = e^t + e^{-t}$ then ,Find $\frac{du}{dt}$

(14) If $z = f(u, v)$ where $u = x^2 + y^2; v = 2xy$; then ,show that

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 2(\sqrt{u^2 - v^2})\frac{\partial z}{\partial u}$$

(15) If $u = log\ r$ where $r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$; then ,show that

$$u_{xx} + u_{yy} + u_{zz} = \frac{1}{r^2}$$

(16) If $z = f(u, v)$ where $u = x^2 - y^2; v = 2xy$; then ,show that

$$\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = 4(u^2 + v^2)^{\frac{1}{2}} \left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$$

(17) If $u = (x^2 - y^2)f(r)$ where $r = xy$; then ,show that

$$u_{xy} = (x^2 - y^2)\left[3f^{'}(r) + rf^{''}(r)\right]$$