# Module 4:Lecture notes

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# Applications of Partial Differentiation

**Prerequisite**

- For a function of one variable, $f(x)$, we find the local maxima/minima by differentiation.
- Maxima/minima occur when $f^{'}(x) = 0$
- $x = a$ is a maximum if $f^{'}(a) = 0$ and $f^{''}(a) < 0$
- $x = a$ is a maximum if $f^{'}(a) = 0$ and $f^{''}(a) > 0$
- A point where $f^{''}(a) = 0$ and $f^{'''}(a) \neq 0$is called a point of inflection.
- Geometrically, the equation $y = f(x)$ represents a curve in the two-dimensional $(x, y)$ plane, and we call this curve the graph of the function $f(x)$.

**Maximum and Minimum of two variable Functions**

- Let $z = f(x, y)$ where $x$ and $y$ are the independent variables and $z$ is the dependent variable.
- The graph of such a function is a surface in three dimensional space.
- If $z = f(x, y)$ be some function of $x$ and $y$ then we can find **extreme values(Maxima and Minima)** of the function using partial derivatives
- **Definition**
  A function $f$ of two variables is said to have a **relative maximum (minimum)** at a point $(a, b)$ if there is a disc centred at $(a, b)$ such that $f(a, b) \geq f(x, y)(f(a, b) \leq f(x, y))$ for all points $(x, y)$ that lie inside the disc.
- **Definition**
  A function $f$ is said to have **an absolute maximum (minimum)** at $(a, b)$ if $f(a, b) \geq f(x, y)(f(a, b) \leq f(x, y))$ for all points $(x, y)$ that lie inside in the domain of $f$.

- If $f$ has a relative (absolute) maximum or minimum at $(a, b)$ then we say that $f$ has a relative (absolute) extremum at $(a, b)$

## First Partial Test

- function $f$ has a relative extremum at $(a, b)$, if the first-order derivatives of $f$ exist at this point,and

$$f_x(a,b) = 0 \ and \ f_y(a,b) = 0$$

- A point $(a, b)$ in the domain of $f(x, y)$ is called a **critical point(stationary points)** of $f$ if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if one or both partial derivatives do not exist at $(a, b)$.
- The actual value at a stationary point is called the **stationary value.**

## The second partials test

Let $f(x, y)$ have continuous second-order partial derivatives in some disc centred at a critical point $(a, b)$, and let $r = f_{xx}(a, b), t = f_{yy}(a, b) \ and \ s = f_{xy}(a, b)$ and define $D = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}(a, b)^2$

1. If $D > 0$ and $f_{xx}(a, b) > 0$, then $f$ has a **relative minimum**at$(a, b)$
2. If $D > 0$ and $f_{xx}(a, b) < 0$, then $f$ has a **relative maximum** at $(a, b)$
3. If $D < 0$, then $f$ has a saddle point at$(a, b)$.
4. If $D = 0$, then no conclusion can be drawn

# Working Rule (Type 1)

**Step (1)** For a given function $f(x, y)$, calculate

$$\frac{\partial f}{\partial x}\ ;\ \frac{\partial f}{\partial y}\ ;\ \frac{\partial^2 f}{\partial x^2}\ ;\ \frac{\partial^2 f}{\partial y^2}\ ;\ \frac{\partial^2 f}{\partial x \partial y}$$

**Step (2)** To find stationary points solve

$$\frac{\partial f}{\partial x} = 0\ ;\ \frac{\partial f}{\partial y} = 0$$

simultaneously.
This gives pair of $(x, y)$ values known as **stationary points or critical points** at which given function may take extreme values.

**Step (3)** At each stationary points find

$$r = \frac{\partial^2 f}{\partial x^2}\ ;\ t = \frac{\partial^2 f}{\partial y^2}\ ;\ s = \frac{\partial^2 f}{\partial x \partial y}$$

**Step (4)** At each stationary points,Extreme values can be decided as per following cases

**Case(i)** If $rt - s^2 > 0$ and $r < 0 (or t < 0)$ at stationary point $(a, b)$ then $f(x, y)$ is **Maximum** at $(a, b)$ and Maximum value is given by $f_{max}(a, b)$

**Case(ii)** If $rt - s^2 > 0$ and $r > 0 (or t > 0)$ at stationary point $(a, b)$ then $f(x, y)$ is **Minimum** at $(a, b)$ and Minimum value is given by $f_{min}(a, b)$

**Case(iii)** If $rt - s^2 < 0$ at stationary point $(a, b)$ then $f(x, y)$ has neither Maxima nor Minima. and such point $(a, b)$ is called **Saddle Point**

**Case(iv)** If $rt - s^2 = 0$ then test fails and no conclusion is drawn about maxima and minima of a function

**Examples**

**Example (Type 1)**
Find extreme values of the following function

$$x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$$

**Solution**
**(a)** Let $f(x, y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$
**Step (1)**

$$\begin{aligned} f(x, y) &= x^3 + 3xy^2 - 3x^2 - 3y^2 + 4 \\ \therefore \frac{\partial f}{\partial x} &= 3x^2 + 3y^2 - 6x \\ \frac{\partial f}{\partial y} &= 6xy - 6y \\ r = \frac{\partial^2 f}{\partial x^2} &= 6x - 6 \\ t = \frac{\partial^2 f}{\partial y^2} &= 6x - 6 \\ s = \frac{\partial^2 f}{\partial x \partial y} &= 6y \end{aligned}$$

**Step (2)** For stationary points (Critical points)

$$\frac{\partial f}{\partial x} = 0 \ and \ \frac{\partial f}{\partial y} = 0$$

$$\begin{aligned} \implies 3x^2 + 3y^2 - 6x = 0.....\mathbf{(1)} \ and \\ 6xy - 6y = 0.....\mathbf{(2)} \end{aligned}$$

By (2)

$$\begin{aligned} & 6y(x-1) = 0 \\ & \implies 6y = 0 \ or \ (x-1) = 0 \\ & \implies y = 0 \ or \ x = 1 \end{aligned}$$

**Case(i)** Using $y = 0$ in (1)

$$3x^2 + 3y^2 - 6x = 0$$
$$\implies 3x^2 - 6x = 0$$
$$\implies 3x(x-2) = 0$$
$$\implies 3x = 0 \ or \ (x-2) = 0$$
$$\implies x = 0 \ or \ x = 2$$

$\therefore$ **in this case stationary points are**

$$(0,0) \quad and \quad (2,0)$$

**Case(ii)** Using $x = 1$ in (1)

$$3x^2 + 3y^2 - 6x = 0$$
$$\implies 3 + 3y^2 - 6 = 0$$
$$\implies 3y^2 - 3 = 0$$
$$\implies y^2 - 1 = 0$$
$$\implies y = 1 \ or \ y = -1$$

$\therefore$ **in this case stationary points are**

$$(1,1) \quad and \quad (1,-1)$$

**Considering both the cases stationary points are**

$$(1,1), \quad (1,-1), \quad (0,0). \quad and \quad (2,0)$$

**Step (3)** At each stationary points (Critical points)
**(1)** At $(0,0)$

$$r = -6 < 0$$

$$s = 0$$

$$t = -6 < 0$$

$$rt - s^2 = 36 > 0$$

$\therefore$ $f(x, y)$ is **maximum** at $(0, 0)$ and Maximum value is

$$f_{max}(0, 0) = 0 + 0 - 0 - 0 + 4 = 4$$

**(2)** At $(2, 0)$

$$r = 6 > 0$$

$$s = 0$$

$$t = 6 > 0$$

$$rt - s^2 = 36 > 0$$

$\therefore$ $f(x, y)$ is **minimum** at $(2, 0)$ and Minimum value is

$$f_{min}(2, 0) = 2^3 + 0 - 3(2^2) - 0 + 4 = 0$$

**(3)** At $(1, 1)$

$$r = 0$$

$$s = 6$$

$$t = 0$$

$$rt - s^2 = -36 < 0$$

$\therefore$ $f(x, y)$ is neither maximum nor minimum at $(1, 1)$ and
$(1, 1)$ is **Saddle Point**

**(4)** At $(1, -1)$

$$r = 0$$

$$s = -6$$

$$t = 0$$

$$rt - s^2 = -36 < 0$$

$\therefore$ $f(x, y)$ is neither maximum nor minimum at $(1, -1)$ and

$(1, -1)$ is **Saddle Point**

# Working Rule (Type 2)

**Step (1)** Identify the three variable function from given data subject to the given constraint

**Step (2)** From the constraint, reduce the function into two variable function and apply maxima minima rule to find stationary points that satisfy the given constraint

**Step (3)** At each stationary points find

$$r = \frac{\partial^2 f}{\partial x^2}\ ;\ t = \frac{\partial^2 f}{\partial y^2}\ ;\ s = \frac{\partial^2 f}{\partial x \partial y}$$

and check the signs to decide about maxima and minima

**Case(i)** If $rt - s^2 > 0$ and $r < 0 (or t < 0)$ at stationary point $(a, b)$ then $f(x, y)$ is **Maximum** at $(a, b)$ and Maximum value is given by $f_{max}(a, b)$

**Case(ii)** If $rt - s^2 > 0$ and $r > 0 (or t > 0)$ at stationary point $(a, b)$ then $f(x, y)$ is **Minimum** at $(a, b)$ and Minimum value is given by $f_{min}(a, b)$

**Step (4)** After finding require point substitute the value to find third unknown and required values

**Example (Type 2)**
A box with an open top is to have $4m^3$ capacity and be made of thin sheet metal. Calculate the dimensions of the box if it is to use the minimum possible amount of metal
**Solution**
Let $A$ be the area of the metal sheet used to make the open box
let $x$,$y$ and $z$ be the length, width and height of the box respectively.
Then

$$A = xy + 2yz + 2xz.......\mathbf{(1)}$$

Also given that capacity(Volume) of the box is

$$V = xyz = 4.......\mathbf{(2)}$$

$\implies z = \frac{4}{xy}.......\mathbf{(3)}$

Substituting (3) in (1), we have

$$A = xy + 2yz + 2xz$$

$$A = xy + 2y\left(\frac{4}{xy}\right) + 2x\left(\frac{4}{xy}\right)$$

$$A = xy + \frac{8}{x} + \frac{8}{y}.......\mathbf{(4)}$$

To find $x$,$y$ and $z$, such that $A$ defined in (4) is minimum

Now

$$\frac{\partial A}{\partial x} = \frac{-8}{x^2} + y$$

and

$$\frac{\partial A}{\partial y} = \frac{-8}{y^2} + x$$

For minimum values

$$\frac{\partial A}{\partial x} = 0$$

and

$$\frac{\partial A}{\partial y} = 0$$

$$\therefore \quad y = \frac{8}{x^2}.......\mathbf{(5)} \quad x = \frac{8}{y^2}.......\mathbf{(6)}$$

By (5) and (6), we have

$$y = \frac{y^4}{8}$$

$$y(y^3 - 8) = 0$$

$$y = 0 \ or \ y^3 = 8$$

$$y = 0 \quad y = 2 \quad ; y = pair \ of \ complex \ conjugates$$

$y = 0$ and $y = complex \ roots$ not possible as volume is given as $4m^3$

Hence $y = 2$.......**(7)**

Using (7) in (6) we have $x = \frac{8}{2^2} = \frac{8}{4} = 2$

$$x = 2.......\textbf{(8)}$$

Now for two variable function $A$ defined in(4)

$$r = \frac{\partial^2 A}{\partial x^2} = \frac{16}{x^3}$$

$$t = \frac{\partial^2 A}{\partial y^2} = \frac{16}{y^3}$$

and

$$s = \frac{\partial^2 A}{\partial x \partial y} = 1$$

$\therefore at(x, y) = (2, 2)$

$$r > 0, rt - s^2 > 0$$

Hence $A$ defined in (4) is minimum at $(2, 2)$

Hence by (3),(7) and (8), required dimensions are $(x, y, z) = (2, 2, 1)$

**Method of Lagrange Multipliers (One Constraints)(Concept)**
Consider a three variable function $u = f(x, y, z)$ whose variables are subject to a constraint $g(x, y, z) = 0$
For $u$ to have stationary points

$$\frac{\partial f}{\partial x} = 0 \ ; \ \frac{\partial f}{\partial y} = 0 \ ; \ \frac{\partial f}{\partial z} = 0$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0.........\mathbf{(1)}$$

Also differentiating $g$,we get,

$$dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy + \frac{\partial g}{\partial z}dz = 0.........\mathbf{(2)}$$

(1)+$\lambda$(2), we have

$$\left(\frac{\partial f}{\partial x} + \lambda\frac{\partial g}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial g}{\partial y}\right)dy + \left(\frac{\partial f}{\partial z} + \lambda\frac{\partial g}{\partial z}\right)dz = 0$$

This will be satisfied if

$$\left(\frac{\partial f}{\partial x} + \lambda\frac{\partial g}{\partial x}\right)dx = 0$$

$$\left(\frac{\partial f}{\partial y} + \lambda\frac{\partial g}{\partial y}\right)dy = 0$$

$$\left(\frac{\partial f}{\partial z} + \lambda\frac{\partial g}{\partial z}\right)dz = 0$$

These equations together with $g = 0$ determine the values of $x, y, z$ *and* $\lambda$

**Method of Lagrange Multipliers (One Constraints)(Working Rule)**

- Let $u = f(x, y, z)$ subject to a constraint $g(x, y, z) = 0$

- Define Lagrange function $L(x, y, z, \lambda) = f(x, y, z) + \lambda g(x, y, z)$

- Equate
$$\frac{\partial L}{\partial x} = 0; \frac{\partial L}{\partial y} = 0; \frac{\partial L}{\partial z} = 0$$

- Solve above equation subject to the constraint $g(x, y, z) = 0$

- values $x, y, z$ obtained are the stationary values of $u = f(x, y, z)$

**Examples**
**Example (Type 3)**
Find greatest and the smallest values that function $f(x, y) = xy$ takes on the ellipse $\frac{x^2}{8} + \frac{y^2}{2} = 1$ **Solution**
To find extreme values of $f(x, y) = xy$ subject to the constraint $g(x, y) = \dfrac{x^2}{8} + \dfrac{y^2}{2} - 1 = 0$
Define Lagrange function

$$L(x, y, \lambda) = f(x, y) + \lambda g(x, y)$$

$$L = xy + \lambda \left( \frac{x^2}{8} + \frac{y^2}{2} - 1 \right)$$

For stationary Points
$$\frac{\partial L}{\partial x} = 0 \ ; \frac{\partial L}{\partial y} = 0$$

$$\implies y + \frac{x\lambda}{4} = 0.....\mathbf{(1)}$$

$$x + \lambda y = 0.....\mathbf{(2)}$$

Using (2) in (1)

$$y + \frac{(-\lambda y)\lambda}{4} = 0$$
$$\implies y - \frac{\lambda^2 y}{4} = 0$$
$$\implies y\left(1 - \frac{\lambda^2}{4}\right) = 0$$
$$\implies y = 0 \; or \; \frac{\lambda^2}{4} = 1$$
$$\implies y = 0 \; or \; \lambda = \pm 2$$

**Case(1)**$y = 0$
In this case from eq (2), $x = 0$ which gives stationary point $(0, 0)$ which does not lie on an ellipse
So this case is not possible
**Case(2)**$y \neq 0$ and $\lambda = \pm 2$

substituting in (2) we have $x = \pm 2y$
Substituting this values in constraint $g(x, y) = 0$ we have

$$\frac{(\pm 2y)^2}{8} + \frac{y^2}{2} = 1$$
$$\implies 4y^2 + 4y^2 = 8$$
$$\implies y^2 = 1$$
$$\implies y = \pm 1 and\; x \qquad = \pm 2$$

Hence function $f(x, y) = xy$ takes on its extreme values on ellipse at four Points $(2, 1), (-2, 1), (2, -1), (-2, -1)$ and extreme values are $f_{max} = 2$ and $f_{min} = -2$

**Example 2 (Type 3)**
Find the point on the surface $z^2 = xy + 1$ at a least distance from the origin
**Solution**
Let $(x, y, z)$ be any point on the given surface $z^2 = xy + 1$
To find $(x, y, z)$ such that their distance $d = \sqrt{x^2 + y^2 + z^2}$ from the origin is minimum subject to the constraint $g(x, y, z) = z^2 - xy - 1 = 0$
We minimize $f(x, y, z) = d^2 = x^2 + y^2 + z^2$......**(1)** subject to a constraint $g(x, y, z) =$

$z^2 - xy - 1 = 0$ using Lagrange Multiplier Method

Define Lagrange Function

$$L(x, y, z, \lambda) = f(x, y, z) + \lambda\ g(x, y, z)$$
$$L = x^2 + y^2 + z^2 + \lambda(z^2 - xy - 1)$$

For Minimum values

$$\frac{\partial L}{\partial x} = 0 = \frac{\partial}{\partial x}\left[x^2 + y^2 + z^2 + \lambda(z^2 - xy - 1)\right]$$
$$\frac{\partial L}{\partial y} = 0 = \frac{\partial}{\partial y}\left[x^2 + y^2 + z^2 + \lambda(z^2 - xy - 1)\right]$$
$$\frac{\partial L}{\partial z} = 0 = \frac{\partial}{\partial z}\left[x^2 + y^2 + z^2 + \lambda(z^2 - xy - 1)\right]$$

subject to constraint

$$z^2 - xy - 1 = 0$$

$\therefore$ we have

$$2x + \lambda(-y) = 0......\mathbf{(2)}$$
$$2y + \lambda(-x) = 0......\mathbf{(3)}$$
$$2z + \lambda(2z) = 0......\mathbf{(4)}$$

subject to constraint

$$z^2 - xy - 1 = 0......\mathbf{(5)}$$

By (4)

$$2z(1 + \lambda) = 0$$
$$\implies z = 0 \ or \ \lambda = -1$$

**Case (1)** $z = 0$

Substituting in (5) we have

$$xy = -1 \implies x = -\frac{1}{y} \text{......}\mathbf{(6)}$$

Using (6) in (2) and (3)

$$\begin{aligned} 2x + \lambda(-y) &= 0 \\ \implies 2(-\frac{1}{y}) + \lambda(-y) &= 0 \\ \implies \frac{-2}{y} - \lambda y &= 0 \\ \implies \lambda = -\frac{2}{y^2} \end{aligned}$$

Also

$$\begin{aligned} 2y + \lambda(-x) &= 0 \\ \implies 2y + \lambda(-(-\frac{1}{y})) &= 0 \\ \implies 2y^2 + \lambda &= 0 \\ \implies \lambda &= -2y^2 \end{aligned}$$

Hence

$$\begin{aligned} -2y^2 &= \frac{2}{y^2} \\ y^4 &= 1 \\ (y^2 - 1)(y^2 + 1) &= 0 \end{aligned}$$

which gives $y = \pm 1$ and $x = \mp 1$

So this case gives stationary point $(1, -1, 0) and (-1, 1, 0)$

**Case (2)** $\lambda = -1$

Substituting in (2) and (3) we have

$$\begin{aligned} 2x + (-1)(-y) &= 0 \\ \implies 2x + y &= 0 \end{aligned}$$

Also

$$\begin{aligned} 2y + (-1)(-x) &= 0 \\ \implies 2y + x &= 0 \end{aligned}$$

Solving these two equations we have

$$x = \pm y......\mathbf{(7)}$$

Using(7) in (5), we have
$z^2 = \pm 1$ which gives $x = y = 0$
Hence required points which are at least distance from the origin on the surface $z^2 = xy + 1$ are $(0, 0, 1)$ and $(0, 0, -1)$

**Examples**

**Example (Type 1)**

Find extreme values of the following functions

(a) $f(x,y) = xy(3-x-y)$

(b) $f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

(c) $f(x,y) = x^3y^2(1-x-y)$

(d) $f(x,y) = x^4 + y^4 - x^2 - y^2 + 1$

(e) $f(x,y) = x^4 + y^4 + 4xy$

(f) $f(x,y) = 5xy - 7x^2 + 3x - 6y + 5 + 2$

(g) $f(x,y) = xy + a^3\left(\frac{1}{x} + \frac{1}{y}\right) \quad a > 0$

(h) $f(x,y) = x^3 + y^3 - 63(x+y) + 12xy$

(i) $f(x,y) = sin\ x\ sin\ y\ sin(xy)$

**Example (Type 2) and (Type 3)**

(1) Divide 24 into three parts such that continued product of the first, square of second and cube of third is maximum

(2) Find three positive numbers the sum of which is 27, such that the sum of their squares is as small as possible

(3) Find the point $P(x, y, z)$ closest to the origin on the plane $2x + y - z - 5 = 0$

(4) Divide 120 into 3 parts such that the sum of their product taken two at a time is maximum

(5) Find the shortest and longest distance from a point $(1, 2, -1)$ to the sphere $x^2 + y^2 + z^2 = 24$

(6) Find the maximum and minimum value of the function $f(x, y) = 3x + 4y$ on the circle $x^2 + y^2 = 1$

(7) Find the point $P(x, y, z)$ on the plane $x + 2y + 3z - 13 = 0$ closest to the point $(1, 1, 1)$

(8) Find the rectangle of largest are with sides parallel to coordinate axes that can be inscribed in an ellipse $x^2 + 2y^2 = 1$

(9) Find the maximum and minimum of $x^2 - 10x - y^2$ on an ellipse $x^2 + 4y^2 = 16$

(10) Find the Points on the sphere $x^2 + y^2 + z^2 = 1$ closest to and farthest from $(1, 2, 2)$