Lecture notes

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MATRICES

Def: Matrix

A rectangular array of mxn numbers arranged in the form $\begin{bmatrix} a_{11} & a_{12} & a_{13}... & a_{1n} \\ a_{21} & a_{22} & a_{23}... & a_{2n} \\ a_{m1} & a_{m2} & a_{m3}... & a_{mn} \end{bmatrix}$ is called an mxn matrix.

e.g. $\begin{bmatrix} 2 & 3 & 4 \\ 1 & -8 & 5 \end{bmatrix}$ is 2x3 matrix

e.g. $\begin{bmatrix} 2 \\ 7 \\ -4 \end{bmatrix}$ is 3x1 matrix

Def: Order of a Matrix

If a matrix has m rows and n columns then order of a matrix is said to be mxn

e.g. $\begin{bmatrix} 2 & 0 & 4 & 6 \\ 1 & -8 & 5 & -2 \\ 3 & 4 & 8 & 6 \end{bmatrix}$ is a matrix of order 3x4

e.g. $\begin{bmatrix} 2 & 0 & 4 \\ 1 & -8 & 5 \\ 3 & 4 & 8 \end{bmatrix}$ is a matrix of order 3

Def: Row Matrix/vector and column matrix or vector

$\begin{bmatrix} a_1 & a_2 & ... & a_n \end{bmatrix}$ is called row matrix/vector and $\begin{bmatrix} b_1 \\ b_2 \\ ... \\ b_n \end{bmatrix}$ is called column matrix/vector.

Def: Real matrix

If all the elements of given matrix are real numbers then matrix is called **Real Matrix**.

Def: Square matrix

A matrix having equal numbers of rows and columns is called a square matrix of order n

$\begin{bmatrix} a_{11} & a_{12} & a_{13}... & a_{1n} \\ a_{21} & a_{22} & a_{23}... & a_{2n} \\ a_{n1} & a_{n2} & a_{n3}... & a_{nn} \end{bmatrix}$ is called a square matrix of order n and is denoted by $(a_{ij})_{nXn}$

and $a_{11}, a_{22}, ..., a_{nn}$ is called principle diagonal.

Some Special Matrices

Def: Null Matrix

If all the elements of a given matrix is zero, the matrix is called **Zero Matrix** or **Null Matrix** and is denoted by O_{mXn}

e.g. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a zero matrix of order 2x2 and os denoted byO_{2X2}

Def: Diagonal Matrix

If all the non-diagonal elements of a given square matrix is zero, the matrix is called **Diagonal Matrix**and is denoted by D

i.e. $D = [d_{ij}] = 0$ for all $i \neq j$ and $D = [d_{ij}] \neq 0$ for all $i = j$

e.g. $\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$ is a diagonal matrix of order 2x2 and is denoted byD_{2X2}

Def: Trace of a Matrix

Sum of all the elements of a principal diagonal of the square matrix is called **Trace of a Matrix**

i.e. Trace of a Matrix A=$tra(A)$

$=\sum_{i=1}^{n} a_{ii}$

e.g. for a matrix $A=\begin{bmatrix} 2 & 3 \\ 4 & 9 \end{bmatrix}$

Trace of A=$tra(A)$=11

Def: Scalar Matrix

A diagonal matrix ,whose all the diagonal elements are equal to a nonzero scalar is called **Scalar Matrix**

i.e. For a scalar matrix $A = (a_{ij}) = 0$ for all $i \neq j$ and $A = (a_{ij}) = k$ for all $i = j$

e.g. $A=\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ is a scalar matrix of order 2

Def: Identity Matrix

A diagonal matrix ,whose all the diagonal elements are equal to 1 is called an **Identity Matrix**
and its denoted by I

i.e. For an identity matrix

$$I = (I_{ij}) = 0 forall i \neq j \tag{1}$$
$$= 1 forall i = j \tag{2}$$

Def: Triangular Matrix

A square matrix in which al the elements below or above the principal diagonal are zeros , is called **Triangular Matrix**.

There are two types of triangular Matrices

(1)Upper Triangular Matrix:

If all the elements below the leading diagonal of a square matrix is zero then it is called **Upper Triangular Matrix** and its denoted by U

i.e. For an Upper triangular matrix

$$U = (u_{ij}) = 0 foralli > j \tag{3}$$
$$\neq 0 foralli \leq j \tag{4}$$

(2)Lower Triangular Matrix:

If all the elements above the leading diagonal of a square matrix is zero then it is called **Lower Triangular Matrix** and its denoted by L

i.e. For a Lower triangular matrix

$$L = (l_{ij}) = 0 foralli < j \tag{5}$$
$$\neq 0 foralli \geq j \tag{6}$$

e.g. $\begin{bmatrix} 2 & 0 & 0 \\ 1 & -8 & 0 \\ 3 & 4 & 8 \end{bmatrix}$ is a Lower Triangular matrix of order 3

e.g. $\begin{bmatrix} 2 & 5 & 4 \\ 0 & -8 & 5 \\ 0 & 0 & 8 \end{bmatrix}$ is an Upper triangular matrix of order 3

Def: Transpose of a Matrix

The matrix obtained from a given matrix A by interchanging rows and columns is called **Transpose of A** and is denoted by A' or A^T

i.e. If $A = (a_{ij})$ then $A^T = (a_{ji})$

Def: Symmetric and Skew-symmetric Matrices

A square matrix $A = (a_{ij})$ is called
Symmetric if $a_{ij} = a_{ji}$ for all i, j and
Skew-symmetric if $a_{ij} = -(a_{ji})$ for all $i \neq j$ and $a_{ij} = 0$ for all $i = j$

i.e. For a Symmetric matrix $A = A^T$ and
For a Skew-symmetric Matrix $A = -A^T$

e.g. $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is Symmetric matrix of order 3

e.g. $\begin{bmatrix} 0 & h & -g \\ -h & 0 & f \\ g & -f & 0 \end{bmatrix}$ is Skew-symmetric matrix of order 3

Def:Sub Matrix

A Matrix obtained from given mXn matrix by eleminating some rows or columns or both id called **submatrix** of given matrix.

Def:Orthogonal Matrix

A square matrix A is called **Orthogonal Matrix** if $A.A^T = A^T.A = I$

(1) If A is orthogonal then $\left|A\right| = \pm 1$

(2) If A is orthogonal Matrix then A^{-1} exists and $A^T = A^{-1}$

Algebra of Matrices

Equality of Matrices

Two matrices A and B are equal if and only if they are of the same order and their corresponding elements are equal, and its denoted by $A = B$

i.e Two mXn matrices A and B are called Equal if

(a_{ij})=(b_{ij}) $\Leftrightarrow$ a_{ij}=b_{ij} for all i, j

Multiplication of a matrix by a scalar

If $k \in RorC$ is scalar and $A = (a_{ij})$ is given matrix then multiplication of A with scalar k is defined as $kA = (ka_{ij})$

Addition and subtraction of two matrices

For two mXn order matrices A and B, addition is defined as $A + B = (a_{ij} + b_{ij})$ and subtraction is defined as $A - B = (a_{ij} - b_{ij})$

Multiplication two matrices

If $A = (a_{ik})$ and $B = (b_{kj})$ are two matrices of order mXp and pXn respectively then their product matrix$C = (c_{ij})$ is defined as $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$

Properties Matrices

(1) $A + B = B + A$

(2) $A + O = A$

(3) $(A + B) + C = A + (B + C)$

(4) $A + (-A) = O$

(5) $k(A+B) = kA + kB$

(6) $(k+p)A = kA + pA$

(7) $k(pA) = (kp)A$

(8) $(A+B)^T = A^T + B^T$

(9) $(kA)^T = kA^T$

(10) $(AB)^T = B^T.A^T$

Determinant of a Matrix

Let A be a square matrix of order n. The determinant of A,denoted by $detA$ or $|A|$ is defined as follows:

(a) If $n=2$, $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

(b) If $n=3$, $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

Properties of determinant:

(1)$det(AB) = (detA)(detB)$ i.e.$|AB| = |A||B|$

(2)$|A|(|B||C|) = (|A||B|)|C|$ $(.A(BC) = (AB)C)$

(3)$|A||B| = |B||A|$ $(.AB \neq BA$ in general)

(4)$|A|(|B| + |C|) = |A||B| + |A||C|$ $(A(B+C) = AB + AC)$

Minors and Co-factors:

A minor is defined as a value computed from the determinant of a square matrix which is obtained after crossing out a row and a column corresponding to the element that is under consideration.

Let us consider that A be a square matrix.

Suppose we want to compute the value of minor for $(ij)^{th}$ element (the element in i^{th} row and j^{th} column), then we should form a determinant of a sub-matrix obtained by hiding i^{th} row and j^{th} column. This minor is known as $(ij)^{th}$ minor and is usually often denoted $M_{i,j}$

The co factor is defined as the signed minor. An $(i,j)^{th}$ co factor is computed by multiplying $(i,j)^{th}$ minor by $(-1)^{i+j}$ and is denoted by C_{ij}.

i.e. $C_{ij} = (-1)^{i+j}M_{ij}$

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then A_{ij} , the **co factor** of a_{ij} , is defined by

$A_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$; $A_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

Adjoint Matrix:

For a given square matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, if the **Co factor of matrix** A is defined as $cof(A) = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$ then **Adjoint Matrix** of A is defined as $Adj(A) = cof(A)^T = \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}$

Def: Inverse of a Matrix

(1) If a, b, c are real numbers such that$ab = c$ and b is non-zero, then $a = \frac{c}{b} = cb^{-1}$ and b^{-1} is usually called the multiplicative inverse of b.

(2) If B, C are matrices, then $\frac{C}{B}$ is undefined.

Def: Non-singular Matrix

A square matrix A of order n is said to be**non-singular or invertible** if and only if there exists a square matrix B such that $AB = BA = I$.The matrix B is called the multiplicative inverse of A, denoted by A^{-1}

i.e. $AA^{-1} = A^{-1}A = I$.

Def: Singular Matrix

If a square matrix A has an inverse, A is said to be non-singular or invertible, Otherwise, it is called **singular or non-invertible**.

i.e. $A$is non-singular if and only if A^{-1} exists.

(1)The inverse of a non-singular matrix is unique.

(2)$I^{-1} = I$, so I is always non-singular.

(3) $O.A = O \neq I$, so O is always singular.

Properties of Inverse of a Matrix

Let A, B be two non-singular matrices of the same order and k be a scalar.
(a) $(A^1)^{-1} = A$.

(b) A^T is a non-singular and$(A^{-1})^T = (A^T)^1$.

(c) A^n is a non-singular and $(A^{-1})^n = (A^n)^1$

(d) kA is a non-singular and $(kA)^{-1} = \frac{1}{k}A^{-1}$.

(e) AB is a non-singular and $(AB)^{-1} = B^{-1}A^{-1}$.

Methods to find Inverse of a Square Matrix

Inverse of a Square matrix by Determinant

Let A be a square matrix. If $detA \neq 0$, then A is non-singular and

$$A^{-1} = \frac{1}{detA}.Adj(A)$$

Inverse of a Square matrix by Row elementary transformation or Gauss Jordan Method

Complex Matrices

Def: Complex Matrix

If atleast one element of the given matrix is a complex number, then matrix is called **Complex Matrix**

e.g. $\begin{bmatrix} 2+3i & -7i \\ 5 & 1-i \end{bmatrix}$ is a complex matrix

Def: Conjugate of a complex Matrix

For a given complex matrix A, the matrix obtained by replacing all the elements,with corresponding conjugate complex numbers , is called **conjugate of complex matrix** and is denoted by $\overline{A}$

i.e. For an mXn matrix $A = [a_{ij}]$; $\overline{A} = [\overline{a_{ij}}]$

e.g. For a complex matrix $A = \begin{bmatrix} 2+3i & -7i \\ 5 & 1-i \end{bmatrix}$; $\overline{A} = \begin{bmatrix} 2-3i & 7i \\ 5 & 1+i \end{bmatrix}$

Def:Properties of Conjugate of a Matrix

If $\overline{A}$ and $\overline{B}$ are conjugate matrices of two complex matrices A and B respectively, then

(1) $\overline{(\overline{A})} = A$

(2) $\overline{(A+B)} = \overline{A} + \overline{B}$

(3) $\overline{kA} = \overline{k}.\overline{A}$; where k is any complex scalar

(4) $\overline{AB} = \overline{A}.\overline{B}$

Def:Transposed Conjugate of a Matrix

The transpose of the conjugate matrix of given complex matrix is called **Transposed conjugate or Conjugate transposed** of a matrix and is denoted by A^*,A^θ,A^H

, $\overline{A^T}$ or $(\overline{A})^T$

Note that

$\overline{(A^T)} = (\overline{A})^T = A^*$

i.e. For an mXn matrix $A = [a_{ij}]$; $A^* = [\overline{a_{ji}}]$

e.g. For a complex matrix $A = \begin{bmatrix} 2+3i & -7i \\ 5 & 1-i \end{bmatrix}$; $A^* = \begin{bmatrix} 2-3i & 5 \\ 7i & 1+i \end{bmatrix}$

Def:Properties of Transposed of Conjugate of a Matrix

If A^* and B^* are conjugate Transposed matrices of two complex matrices A and B respectively, then

(1) $(A^*)^* = A$

(2) $(A+B)^* = A^* + B^*$ where A and B are of same order

(3) $(kA)^* = \overline{k}.A^*$; where k is any complex scalar

(4) $(AB)^* = B^*A^*$

Some Complex Matrices

(1)Hermitian Matrix

A square matrix $A = [a_{ij}]$ is called **Hermitian Matrix** , if $A^* = A$

i.e. For a Hermitian Matrix $A = [a_{ij}]$; $\overline{a_{ji}} = a_{ij}$

Note: Entries on a Main diagonal of a Hermitian Matrix is real number.

e.g. $A = \begin{bmatrix} 1 & 1-i & 5 \\ 1+i & 3 & i \\ 5 & -i & 4 \end{bmatrix}$ is a Hermitian Matrix

(2)Skew-Hermitian Matrix

A square matrix $A = [a_{ij}]$ is called **Skew-Hermitian Matrix** , if $A^* = -A$

i.e. For a Hermitian Matrix $A = [a_{ij}]$; $\overline{a_{ji}} = -a_{ij}$

Note: Entries on a Main diagonal of a Skew-Hermitian Matrix is either zero or purely imaginary number.

e.g. $A = \begin{bmatrix} i & 1-i & 5 \\ -1-i & 3i & -i \\ -2 & i & 0 \end{bmatrix}$ is a Skew-Hermitian Matrix

(3)Unitary Matrix

A square matrix $A = [a_{ij}]$ is called **Unitary Matrix** , if $A^*.A = A.A^* = I$

e.g. $A = \begin{bmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ -i/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$ is a Unitary Matrix

(1) If A is Unitary Matrix then $\left|A\right|$ =1

(2) If A is Unitary Matrix then A^{-1} exists and $A^* = A^{-1}$

Important Results

Result 1
Every square matrix can be uniquely expressed as a sum of symmetric and skew-symmetric matrix.

Solution
Let A be any Square Matrix
To Prove

(1) $A = B + C$ **where** B **is symmetric and** C **is Skew Symmetric**

(2) Above expression of A **is unique**

Now , let A^T be a transpose matrix of given square matrix A, Then

$$
\begin{aligned}
A &= \frac{1}{2}\left[A + A^T + A - A^T\right] \\
A &= \frac{1}{2}\left[A + A^T\right] + \frac{1}{2}\left[A - A^T\right] \\
A &= B + C
\end{aligned}
$$

To Prove $B = \frac{1}{2}\left[A + A^T\right]$ is symmetric and $C = \frac{1}{2}\left[A - A^T\right]$ is skew- symmetric
i.e To prove $B^T = B$ and $C^T = -C$
Now

$$
\begin{aligned}
B^T &= \left[\frac{1}{2}\left(A + A^T\right)\right]^T \\
&= \frac{1}{2}\left(A + A^T\right)^T \\
&= \frac{1}{2}\left(A^T + (A^T)^T\right) \\
&= \frac{1}{2}\left(A^T + A\right) \\
&= \frac{1}{2}\left(A + A^T\right) \\
\therefore B^T &= B
\end{aligned}
$$

i.e $B = \frac{1}{2}\left(A + A^T\right)$ is symmetric Also

$$\begin{aligned}
C^T &= \left[\frac{1}{2}\left(A - A^T\right)\right]^T \\
&= \frac{1}{2}\left(A - A^T\right)^T \\
&= \frac{1}{2}\left(A^T - (A^T)^T\right) \\
&= \frac{1}{2}\left(A^T - A\right) \\
&= -\frac{1}{2}\left(A - A^T\right) \\
\therefore C^T &= -C
\end{aligned}$$

i.e $C = \frac{1}{2}\left(A - A^T\right)$ is skew-symmetric

Hence every square matrix can be expressed as sum of symmetric and skew-symmetric matrices

For Uniqueness , let there exists another expression of A such that $A = P + Q$ where P is symmetric $(P^T = P)$ and Q is skew-symmetric $(Q^T = -Q)$

To prove $P = B$ and $Q = C$

Now

$$\begin{aligned}
A = P + Q \implies A^T &= (P + Q)^T \\
&= P^T + Q^T \\
&= P + (-Q) \\
A^T &= P - Q
\end{aligned}$$

Hence

$$\begin{aligned}
A &= P + Q......(1) \\
A^T &= P - Q.........(2)
\end{aligned}$$

Adding (1) and (2)

$$\begin{aligned}
&A + A^T = 2P \\
&\implies P = \frac{1}{2}\left(A + A^T\right) \\
&\implies P = B
\end{aligned}$$

Subtracting (1) and (2)

$$A - A^T = 2Q$$
$$\implies Q = \frac{1}{2}\left(A - A^T\right)$$
$$\implies Q = C$$

Hence expression $A = B + C$,where B is symmetric and C is skew-symmetric is unique

Result 2 (H.W.)

Every square matrix can be uniquely expressed as a sum of Hermitian and Skew-Hermitian matrix.

Result 3

Show that Every square matrix can be uniquely expressed as $B + iC$ where B and C both are Hermitian Matrices.

Solution

Let A be any Square Matrix
To Prove

(1) $A = B + iC$ where B and C both are Hermitian Matrices

(2) Above Expression of A is unique

Now , let A^* be the conjugate transpose matrix of given square matrix A, Then

$$A = \frac{1}{2}\left[A + A^* + A - A^*\right]$$
$$A = \frac{1}{2}\left[A + A^*\right] + i\ \frac{1}{2i}\left[A - A^*\right]$$
$$A = B + iC$$

To Prove $B = \frac{1}{2}\left[A + A^*\right]$ and $C = \frac{1}{2i}\left[A - A^*\right]$ are Hermitian matrices
i.e To prove $B^* = B$ and $C^* = C$
Now

$$
\begin{aligned}
B^* &= \left[\frac{1}{2}(A + A^*)\right]^* \\
&= \frac{1}{2}(A + A^*)^* \\
&= \frac{1}{2}(A^* + (A^*)^*) \\
&= \frac{1}{2}(A^* + A) \\
&= \frac{1}{2}(A + A^*) \\
\therefore B^* &= B
\end{aligned}
$$

i.e $B = \frac{1}{2}(A + A^*)$ is Hermitian. Also

$$
\begin{aligned}
C^* &= \left[\frac{1}{2i}(A - A^*)\right]^* \\
&= \overline{\left(\frac{1}{2i}\right)}(A - A^*)^* \\
&= \frac{-1}{2i}(A^* - (A^*)^*) \\
&= \frac{-1}{2i}(A^* - A) \\
&= \frac{1}{2i}(A - A^*) \\
\therefore C^* &= C
\end{aligned}
$$

i.e $C = \frac{1}{2i}(A - A^*)$ is Hermitian

Hence every square matrix A can be expressed as $A = B + iC$ where B and C both are Hermitian Matrices

For Uniqueness , let there exists another expression of A such that $A = P + iQ$ where P is Hermitian $(P^* = P)$ and Q is also Hermitian $(Q^* = Q)$

To prove $P = B$ and $Q = C$

Now

$$
\begin{aligned}
A = P + iQ \implies A^* &= (P+iQ)^* \\
&= P^* + (iQ)^* \\
&= P^* + \bar{i}Q^* \\
A^* &= P - iQ
\end{aligned}
$$

Hence

$$A = P + iQ......(1)$$

$$A^* = P - iQ.........(2)$$

Adding (1) and (2)

$$
\begin{aligned}
&A + A^* = 2P \\
&\implies P = \frac{1}{2}(A + A^*) \\
&\implies P = B
\end{aligned}
$$

Subtracting (1) and (2)

$$
\begin{aligned}
&A - A^* = 2Q \\
&\implies Q = \frac{1}{2i}(A - A^*) \\
&\implies Q = C
\end{aligned}
$$

Hence expression $A = B + iC$,where B and C both are Hermitian matrices is unique

Result 4

Show that Every Hermitian matrix can be expressed as $B + iC$ where B is real symmetric and C is real skew-symmetric matrix.

Solution

Let A be any Hermitian Matrix
$\therefore A^* = \overline{(A^T)} = (\overline{A})^T = A$
To Prove $A = B + iC$ where B is real symmetric and C is real skew-symmetric matrix
Now , let $\overline{A}$ be the conjugate matrix of given Hermitian matrix A, Then

$$
\begin{aligned}
A &= \frac{1}{2}\left[A+\overline{A}+A-\overline{A}\right] \\
A &= \frac{1}{2}\left[A+\overline{A}\right]+i\ \frac{1}{2i}\left[A-\overline{A}\right] \\
A &= B+iC
\end{aligned}
$$

To Prove $B=\frac{1}{2}\left[A+\overline{A}\right]$ is real symmetric and $C=\frac{1}{2i}\left[A-\overline{A}\right]$is real skew symmetric matrix

Now from complex numbers we know that if $z=x+iy$ ans $\overline{z}=x-iy$ are complex conjugates of each other then

$$\frac{1}{2}(z+\overline{z})=x$$

is real and

$$\frac{1}{2i}(z-\overline{z})=y$$

is also real

Hence

$$B=\frac{1}{2}\left[A+\overline{A}\right]$$

and

$$C=\frac{1}{2i}\left[A-\overline{A}\right]$$

are real matrices

Now

$$
\begin{aligned}
B^T &= \left[\frac{1}{2}\left(A+\overline{A}\right)\right]^T \\
&= \frac{1}{2}\left(A^T+\left(\overline{A}\right)^T\right) \\
&= \frac{1}{2}\left[\left((\overline{A})^T\right)^T+\left(\overline{A}\right)^T\right] \\
&= \frac{1}{2}\left(\overline{A}+A\right) \\
\therefore B^T &= B
\end{aligned}
$$

i.e $B = \frac{1}{2}\left(A + \overline{A}\right)$ is symmetric Also

$$\begin{aligned}
C^T &= \left[\frac{1}{2i}\left(A - \overline{A}\right)\right]^T \\
&= \frac{1}{2i}\left(A^T - \left(\overline{A}\right)^T\right) \\
&= \frac{1}{2i}\left[\left((\overline{A})^T\right)^T - \left(\overline{A}\right)^T\right] \\
&= \frac{-1}{2i}\left(A - \overline{A}\right) \\
\therefore C^T &= -C
\end{aligned}$$

i.e $C = \frac{1}{2i}\left(A - \overline{A}\right)$ is skew-symmetric

Hence every Hermitian matrix A can be expressed as $A = B + iC$ where B is real symmetric and C is real skew-symmetric matrix

Result 5 (H.W.)

Show that every skew-hermitian matrix can be expressed as $B + iC$ where B is real skew-symmetric and C is real symmetric matrix.

Elementary transformation of a Matrix

The following three operations with rows or columns are known as **elementary transformations**:

(1) The interchange of any two rows(or columns). The interchange of i^{th} and j^{th} row is denoted by $R_i \leftrightarrow R_j$

(2) The multiplication of any row(or column)by a nonzero number. The multiplication of i^{th} row by a non-zero number k is denoted by kR_i

(3) The addition of a constant multiple of the elements of any row (or column)to the corresponding element of any other row(or column). The addition of k-times the elements of j^{th} row to the corresponding elements of i^{th} row is denoted by $R_i + kR_j$

Def: Elementary Matrix

A Matrix obtained from a Unit(Identity) matrix of same order by performing a single elementary row(or column)transformation is called an **Elementary Matrix**

Def: Equivalent Matrices If a matrix B is obtained from matrix A by performing one or more elementary row(or column)transformation, then B is said to be equivalent to A and is denoted by $A \sim B$

Rank of a Matrix

The matrix A_{mXn} is said to be of rank r if,

(i) At least one minor of order r is non vanishing

(ii) Every minors of order $(r+1)$ vanishes.

i.e. The rank of a matrix A is the highest order of its non vanishing minor and it is denoted by $r(A)$ or $\rho(A)$

NOTE:

- If the order of matrix A is mXn , then $r(A) \leq min(m,n)$
- If a matrix A has nonzero minor of order r then $r(A) \geq r$
- If a matrix A has all the minors of order $(r+1)$ are zero then $r(A) \geq r$
- Elementary transformation does not alter rank of a matrix
  hence if A and B are equivalent matrices then $r(A) = r(B)$
- Rank of a non-singular matrix of order r is r
- Rank of a null matrix is 0
- Rank of an identity matrix is equal to the order of a matrix
- $r(A) = r(A^T)$
- $r(AB) \leq r(A)$ and $r(AB) \leq r(B)$
- the rank of matrix A and A^{-1} (if exists) is same.

Echelon form of a Matrix:

The Echelon form of a matrix A is an equivalent matrix C of rank r with the following properties:

(1) At least one element in each of the first r rows is non-zero and the Elements in the remaining rows are zero

(2) In the first r rows the first element in each row is nonzero(known as Pivot element) and it appears in the column right to the first nonzero element of the preceding row.

If the non-zero element (pivot element) is 1, then echelon form is known as Row echelon form.

e.g. $A = \begin{pmatrix} 2 & 5 & 7 & 9 & 2 \\ 0 & 0 & 2 & -2 & 5 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ is in Echelon form

e.g. $B = \begin{pmatrix} 1 & 5 & 7 & 9 & 2 \\ 0 & 0 & 1 & -2 & 5 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ is in Row Echelon form

Methods to find Rank of a Matrix

(1) Reducing into Echelon form

If a matrix A_{mxn} is reduced into an echelon (or Row echelon Form)R then rank of a matrix is equal to total number of nonzero rows in R
i.e. rank of A= Total number of rows in A- Number of zero rows in Echelon form of A

Working Rule

Step (1) Find first **Pivot Element** of a given matrix (Preferably a_{11} and check whether it is zero or nonzero)

Step (2)
Case(i) If Pivot element is nonzero, then using the pivot Row R_1 make other than pivot entries of the pivot column C_1 zero, by performing Row elementary transformations only

Case(ii) If Pivot element is zero then check whether atleast one entry below that element in that column is zero or nonzero

If nonzero , replace the R_1 with the row containing the nonzero entry.

If all below entries zero, move to next column in the same row for pivot entry

Step (3) Once you finish step 1 and step 2, repeat the same process for next pivot entry (Preferably a_{22})

Step (4) Repeat the process for each pivot position till you get the required row echelon form of a matrix

Step (5) The rank of a matrix is equal to total number of nonzero rows in echelon form of a matrix

NOTE
If you are reducing a matrix in **Row echelon Form** make sure pivot entry is 1

EXAMPLE

Find the rank of a matrix by reducing into an Echelon form

(1)

$$A = \begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix}$$

Solution
Method 1
Given

$$A = \begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix}$$

$R_2 - \frac{2}{3}\ R_1$

$$\sim \begin{pmatrix} 3 & -3 & 4 \\ 0 & -1 & \frac{4}{3} \\ 0 & -1 & 1 \end{pmatrix}$$

$R_3 - R_2$

$$\sim \begin{pmatrix} 3 & -3 & 4 \\ 0 & -1 & \frac{4}{3} \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}$$

This is called **echelon form** of a matrix A
Since there are three non zero rows in the reduced matrix
Rank of a matrix $r(A) = 3$

Method 2
Given

$$A = \begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix}$$

$R_1(\frac{1}{3})$

$$\sim \begin{pmatrix} 1 & -1 & \frac{4}{3} \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix}$$

$R_2 - 2R_2$

$$\sim \begin{pmatrix} 1 & -1 & \frac{4}{3} \\ 0 & -1 & \frac{4}{3} \\ 0 & -1 & 1 \end{pmatrix}$$

$R_3(-1)$

$$\sim \begin{pmatrix} 1 & -1 & \frac{4}{3} \\ 0 & 1 & -\frac{4}{3} \\ 0 & -1 & 1 \end{pmatrix}$$

$R_3 + R_2$

$$\sim \begin{pmatrix} 1 & -1 & \frac{4}{3} \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}$$

$R_3(-3)$

$$\sim \begin{pmatrix} 1 & -1 & \frac{4}{3} \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & 1 \end{pmatrix}$$

This is called **Row echelon form** of a matrix A
Since there are three non zero rows in the reduced matrix
Rank of a matrix $r(A) = 3$

(2) Reducing into Normal form

A matrix A_{mxn} of rank r can be reduced by performing row and column elementary transformations in one of the following form $\left[I_r\right]$; $\left[I_r \quad 0\right]$; $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$; $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ known as Normal form of matrix A, where I_r is identity matrix of order r and 0 is null matrix of suitable order.

Working Rule

Step (1) Find first **Pivot Element** of a given matrix (Preferably a_{11} and make that element 1)

Step (2) Using the pivot Row R_1 make other than pivot entries of the pivot column C_1 zero, by performing Row elementary transformations only

Step (3) Using the pivot column C_1 make other than pivot entries of the pivot row R_1 zero, by performing Column elementary transformations only

Step (4) Once you finish step 1 ,2 and 3, repeat the process for next pivot entry (Preferably a_{22})

Step (5) Repeat the process for each pivct entry of given matrix and get the matrix in one of the following **Normal Form**

$\left[I_r\right]$; $\left[I_r \quad 0\right]$; $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$; $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

Step (6) The rank of a matrix in normal form is $r(A) = O(I_r) = r$ or number of non zero rows in the reduced matrix

EXAMPLE

Find the rank of a matrix by reducing into an Normal form

(1)

$$A = \begin{pmatrix} 2 & -1 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 3 & 3 & 3 & 1 \\ 1 & 4 & 2 & 0 \end{pmatrix}$$

Solution

Given

$$A = \begin{pmatrix} 2 & -1 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 3 & 3 & 3 & 1 \\ 1 & 4 & 2 & 0 \end{pmatrix}$$

$R_2 \leftrightarrow R_1$

$$\sim \begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & -1 & 1 & 1 \\ 3 & 3 & 3 & 1 \\ 1 & 4 & 2 & 0 \end{pmatrix}$$

$R_2 - 2R_1 \quad ; R_3 - 3R_1 \quad ; R_4 - R_1$

$$\sim \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 3 & 0 & -5 \\ 0 & 4 & 1 & -2 \end{pmatrix}$$

$C_3 - C_1 \quad ; C_4 - 2C_1$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -3 \\ 0 & 3 & 0 & -5 \\ 0 & 4 & 1 & -2 \end{pmatrix}$$

$R_2(-1)$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 3 & 0 & -5 \\ 0 & 4 & 1 & -2 \end{pmatrix}$$

$R_3 - 3R_2 \;\; ; R_4 - 4R_2 \;\; ;$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -3 & -14 \\ 0 & 0 & -3 & -14 \end{pmatrix}$$

$C_3 - C_2 \;\; ; C_4 - 3C_2 \;\; ;$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & -14 \\ 0 & 0 & -3 & -14 \end{pmatrix}$$

$C_3(-\frac{1}{3}) \;\; ; C_4(-\frac{1}{14})$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$R_4 - R_3$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$C_4 - C_3$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} I_3 & \theta \\ \theta & \theta \end{pmatrix}$$

This is called **Normal form** of a matrix A
Rank of a matrix $r(A) = O(I_3) = 3$

(3) Reducing into PAQ form

If A_{mXn} is a matrix of rank r,there exist two non singular matrices P and Q such that PAQ is in normal form, For that

Rewriting A as $A = I_m A I_n$
Reducing A on L.H.S. in normal form , while performing row operation on I_m and performing column operations on I_n only

Note:
If A is non-singuar matrix, then $A^{-1} = QP$
For any matrix A , in reduced normal form PAQ, P and Q are not unique.

EXAMPLE

Find the rank of a matrix by reducing into an PAQ form

(A) Non- Homogeneous system of equations

$$A = \begin{pmatrix} 2 & 1 & 4 \\ 3 & 2 & 2 \\ 7 & 4 & 10 \\ 8 & 5 & 8 \end{pmatrix}$$

Solution

Given

$$A = \begin{pmatrix} 2 & 1 & 4 \\ 3 & 2 & 2 \\ 7 & 4 & 10 \\ 8 & 5 & 8 \end{pmatrix}$$

$\therefore$ Rewritting A as $A = I_4 A I_3$

$$\therefore \begin{pmatrix} 2 & 1 & 4 \\ 3 & 2 & 2 \\ 7 & 4 & 10 \\ 8 & 5 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ A \ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$C_2 \leftrightarrow C_1$

$$\begin{pmatrix}1 & 2 & 4\\ 2 & 3 & 2\\ 4 & 7 & 10\\ 5 & 8 & 8\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \ A \ \begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$R_2 - 2R_1 \ \ ; R_3 - 4R_1 \ \ ; R_4 - 5R_1$

$$\begin{pmatrix}1 & 2 & 4\\ 0 & -1 & -6\\ 0 & -1 & -6\\ 0 & -2 & -12\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\ -2 & 1 & 0 & 0\\ -4 & 0 & 1 & 0\\ -5 & 0 & 0 & 1\end{pmatrix} \ A \ \begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$C_2 - 2C_1 \ \ ; C_3 - 4C_1$

$$\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & -6\\ 0 & -1 & -6\\ 0 & -2 & -12\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\ -2 & 1 & 0 & 0\\ -4 & 0 & 1 & 0\\ -5 & 0 & 0 & 1\end{pmatrix} \ A \ \begin{pmatrix}0 & 1 & 0\\ 1 & -2 & -4\\ 0 & 0 & 1\end{pmatrix}$$

$R_2(-1)$

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 6\\ 0 & -1 & -6\\ 0 & -2 & -12\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\ 2 & -1 & 0 & 0\\ -4 & 0 & 1 & 0\\ -5 & 0 & 0 & 1\end{pmatrix} \ A \ \begin{pmatrix}0 & 1 & 0\\ 1 & -2 & -4\\ 0 & 0 & 1\end{pmatrix}$$

$R_3 + R_2 \ \ R_4 + 2R_2$

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 6\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\ 2 & -1 & 0 & 0\\ -2 & -1 & 1 & 0\\ -1 & -2 & 0 & 1\end{pmatrix} \ A \ \begin{pmatrix}0 & 1 & 0\\ 1 & -2 & -4\\ 0 & 0 & 1\end{pmatrix}$$

$C_3 - 6C_2$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ -2 & -1 & 1 & 0 \\ -1 & -2 & 0 & 1 \end{pmatrix} A \begin{pmatrix} 0 & 1 & -6 \\ 1 & -2 & 8 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence Matrix A on L.H.S. is in Normal form $\sim \begin{pmatrix} I_2 & \theta \\ \theta & \theta \end{pmatrix}$, therefore $r(A) = 2$ and required Matrices P and Q are $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ -2 & -1 & 1 & 0 \\ -1 & -2 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 & -6 \\ 1 & -2 & 8 \\ 0 & 0 & 1 \end{pmatrix}$ respectively.

System of Linear Algebraic Equations

Consider a system of m linear equations with n unknowns $x_1, x_2, ..., x_n$ as

$$a_{11}x_1 + a_{12}x_2 + ...a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + ...a_{2n}x_n = b_2$$

$$a_{m1}x_1 + a_{m2}x_2 + ...a_{mn}x_n = b_m$$

Writing in matrix notation
$AX = B$

$$\begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ \\ a_{m1} & a_{m2} & ... & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \\ b_m \end{pmatrix}$$

where
A is coefficient matrix,
X is Column matrix of unknowns
B is Column matrix of constants

Non Homogeneous System of Equation

If atleast one $b_i \neq 0$, in B,system is called **Non Homogeneous system of equations**

Working Rule
1)Write the given system in Matrix Notation $AX = B$

2)Reduced Augmented matrix $[A|B]$ into Echelon form
3)**A Non-Homogeneous system** is either **consistent or inconsistent** and has three types of solution

- If $r(A) = r[A|B]$=number of unknowns , then system has **unique solution** and system is called **consistent**

- If $r(A) = r[A|B] <$ number of unknowns,then system has **infinitely many solution** and system is called **consistent**

- If $r(A) \neq r[A|B]$ then system has **no solution** and system is called **inconsistent**

EXAMPLES

Solve the following system of equations by Matrix Method.

(1)

$$\begin{aligned} x + y + z &= 6 \\ x - y + 2z &= 5 \\ 3x + y + z &= 8 \\ 2x - 2y + 3z &= 7 \end{aligned}$$

Solution
Rewriting a system in a matrix Form

$$AX = B$$
$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 1 \\ 2 & -2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ 8 \\ 7 \end{pmatrix}$$

$\therefore$ Augmented Matrix

$$[A:B] = \begin{pmatrix} 1 & 1 & 1 & | & 6 \\ 1 & -1 & 2 & | & 5 \\ 3 & 1 & 1 & | & 8 \\ 2 & -2 & 3 & | & 7 \end{pmatrix}$$

$$R_2 - R_1 \; ; R_3 - 3R_1 \; ; R_4 - 2R_1$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & | & 6 \\ 0 & -2 & 1 & | & -1 \\ 0 & -2 & -2 & | & -10 \\ 0 & -4 & 1 & | & -5 \end{pmatrix}$$

$$R_3 - R_2 \; ; R_4 - 2R_2$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & | & 6 \\ 0 & -2 & 1 & | & -1 \\ 0 & 0 & -3 & | & -9 \\ 0 & 0 & -1 & | & -3 \end{pmatrix}$$

$$R_4 - (\frac{1}{3})R_3$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & | & 6 \\ 0 & -2 & 1 & | & -1 \\ 0 & 0 & -3 & | & -9 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

Here $r(A) = r(A|B) = 3$
Total number of unknowns=3
i.e $r(A) = r(A|B) = 3 =$ no of unknowns
$\therefore$ Given system is consistent and has a unique solution
Using back substitution solution is obtained as

$$\begin{aligned} x + y + z &= 6 \\ -2y + z &= -1 \\ -3z &= -9 \end{aligned}$$

$$
\begin{aligned}
\therefore z &= 3\\
\implies y &= \frac{1}{-2}(-1-z)\\
&- \frac{1}{-2}(-1-3)\\
y &= 2\\
\implies x &= 6-y-z\\
&= 6-2-3\\
x &= 1
\end{aligned}
$$

Hence Unique solution is $(x, y, z) = (1, 2, 3)$

Homogeneous System of Equation

If all $b_i = 0$,in B, system is called **Homogeneous system of Equations** and
Working Rule

1)Write the given system in Matrix Notation $AX = \theta$

2)Reduced Augmented matrix $[A|\theta]$ into Echelon form
3)**A Homogeneous system** is always consistent and has two type of solution

- If $r(A) = r[A|\theta]$=number of unknowns , then system has **trivial or zero solution**

- If $r(A) = r[A|\theta] <$ number of unknowns,then system has **non-trivial or infinite solution**

(B) Homogeneous system of equations

$$
\begin{aligned}
x + y + 2z &= 0\\
x + 2y + 3z &= 0\\
x + 3y + 4z &= 0\\
3x + 4y + 7z &= 0
\end{aligned}
$$

Solution
Rewriting a system in a matrix Form

$$AX = \theta$$
$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \\ 3 & 4 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\therefore$ Augmented Matrix

$$[A : \theta] = \begin{pmatrix} 1 & 1 & 2 & | & 0 \\ 1 & 2 & 3 & | & 0 \\ 1 & 3 & 4 & | & 0 \\ 3 & 4 & 7 & | & 0 \end{pmatrix}$$
$$R_2 - R_1 \; ; R_3 - R_1 \; ; R_4 - 3R_1$$
$$\sim \begin{pmatrix} 1 & 1 & 2 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 2 & 2 & | & 0 \\ 0 & 1 & 1 & | & 0 \end{pmatrix}$$
$$R_3 - 2R_2 \; ; R_4 - R_2$$
$$\sim \begin{pmatrix} 1 & 1 & 2 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

Here $r(A) = r(A|B) = 2$
Total number of unknowns=3
i.e $r(A) = r(A|B) = 3 <$ no of unknowns
$\therefore$ Given homogeneous system is consistent and has a non-trivial solution
Using back substitution solution is obtained as

$$x + y + 2z = 0$$
$$y + z = 0$$

Let $z = t; t \in R$ then solution is obtained as

$$\begin{aligned}
\therefore z &= t \\
\implies y &= -t \\
\implies x &= -y - 2z \\
&= t - 2t \\
x &= -t
\end{aligned}$$

Hence Non trivial solution is $(x, y, z) = (-t, -t, t); t \in R$