EXPERIMENT : No. Page No. 2 Date

1] If $u = x^3y + e^{xy^2}$, then prove that

$$u_{xy} = u_{yx}$$

Solⁿ:- Given :- $u(x, y) = x^3y + e^{xy^2}$

1° $u_x = \frac{\partial u}{\partial x}$

$\therefore u_x = \frac{\partial}{\partial x}(x^3y) + \frac{\partial}{\partial x}(e^{xy^2})$

$= 3x^2y + y^2 e^{xy^2}$

2° $u_y = \frac{\partial u}{\partial y}$

$= \frac{\partial}{\partial y}(x^3y) + \frac{\partial}{\partial y}(e^{xy^2})$

$= x^3 + 2yx e^{xy^2}$

$\because \frac{\partial(e^{xy^2})}{\partial y} = \frac{\partial e^{xy^2}}{\partial(y^2)} \cdot \frac{d(y^2)}{dy}$

3. $u_{xy} = \frac{\partial\left(\frac{\partial u}{\partial x}\right)}{\partial y}$

$= \frac{\partial}{\partial y}(3x^2y) + \frac{\partial}{\partial y}(e^{xy^2} \cdot y^2)$

$= 3x^2 + 2y \cdot e^{xy^2} + 2xy^3 e^{xy^2}$ — ①

4. $u_{yx} = \frac{\partial\left(\frac{\partial u}{\partial y}\right)}{\partial x}$

$= \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial x}(x \cdot e^{xy^2})$

$= 3x^2 + 2ye^{xy^2} + 2xy^3 e^{xy^2}$ — ②

Hence proved from ① & ② that $u_{xy} = u_{yx}$.

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Teacher's Sign. :

EXPERIMENT : No. Page No. 3 Date

2. If $u = \log r$ and $r = \sqrt{(x-a)^2 + (y-b)^2}$ where a and b are constants then prove that

$$u_{xx} = -u_{yy}$$

Soln :- $u = \log r$ $\quad r = \sqrt{(x-a)^2 + (y-b)^2}$

$$r^2 = (x-a)^2 + (y-b)^2$$

Partially differentiating w.r.t. x on both sides we get

$$\frac{\partial (r^2)}{\partial x} = \frac{\partial ((x-a)^2)}{\partial x} + \frac{\partial ((y-b)^2)}{\partial x}$$

$$2r \cdot \frac{\partial r}{\partial x} = 2(x-a)$$

$$\therefore \frac{\partial r}{\partial x} = \frac{(x-a)}{r}$$

1. $u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x}$

$$\therefore u_x = \frac{\partial (\log r)}{\partial r} \cdot \frac{(x-a)}{r}$$

$$\therefore u_x = \frac{1}{r} \cdot \frac{(x-a)}{r}$$

$$\therefore u_x = \frac{(x-a)}{(x-a)^2 + (y-b)^2}$$

2. $u_{xx} = \frac{\partial (u_x)}{\partial x} = \frac{\partial}{\partial x} \left(\frac{x-a}{(x-a)^2 + (y-b)^2} \right)$

$$\therefore u_x = \frac{(x-a)^2 + (y-b)^2 - [(x-a) 2 \cdot (x-a)]}{[(x-a)^2 + (y-b)^2]^2}$$

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Teacher's Sign. :

$$\therefore u_{xx} = \frac{-(x-a)^2 + (y-b)^2}{\left[(x-a)^2 + (y-b)^2\right]^2} \quad — (1)$$

$$r^2 = (x-a)^2 + (y-b)^2$$

$$\frac{\partial r^2}{\partial y} = \frac{\partial}{\partial y}\left((x-a)^2\right) + \frac{\partial}{\partial y}\left((y-b)^2\right)$$

$$\therefore \frac{\partial r}{\partial y} = \frac{1}{2 \cdot r} \, 2 \cdot (y-b)$$

$$\therefore \frac{\partial r}{\partial y} = \frac{y-b}{r}$$

3. $u_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y}$

$$u_y = \frac{\partial}{\partial r}(\log r) \cdot \frac{(y-b)}{r}$$

$$\therefore u_y = \frac{(y-b)}{r^2} = \frac{y-b}{(x-a)^2 + (y-b)^2}$$

4. $u_{yy} = \frac{\partial}{\partial y}(u_y)$

$$\therefore u_{yy} = \frac{\partial}{\partial y}\left[\frac{y-b}{(x-a)^2 + (y-b)^2}\right]$$

$$= \frac{(x-a)^2 + (y-b)^2 - \left[(y-b)\, 2(y-b)\right]}{\left[(x-a)^2 + (y-b)^2\right]^2}$$

$$\therefore u_{yy} = \frac{(x-a)^2 - (y-b)^2}{\left[(x-a)^2 + (y-b)^2\right]^2} \quad — (2)$$

EXPERIMENT : No.

from ① and ②

$$U_{xx} = -U_{yy}$$

Hence proved

3] If $u = f(r, s)$ where $r = x^2 + y^2$ and $s = x^2 - y^2$ then show that

a] $y \cdot \frac{\partial u}{\partial x} + x \frac{\partial u}{\partial y} = 4xy \cdot \frac{\partial u}{\partial r}$

b] $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = 4r\left[\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial s}\right)^2\right] + 8s\left[\frac{\partial u}{\partial r} \cdot \frac{\partial u}{\partial s}\right]$

Solⁿ:- $u = f(r, s)$ $\quad r = x^2 + y^2$ $\quad s = x^2 - y^2$

a] 1. $\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial x}$

$= \frac{\partial u}{\partial r} \cdot \frac{\partial (x^2+y^2)}{\partial x} + \frac{\partial u}{\partial s} \cdot \frac{\partial (x^2-y^2)}{\partial x}$

$\therefore \frac{\partial u}{\partial x} = 2x \cdot \frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} \cdot (2x)$

$\therefore \frac{\partial u}{\partial x} = 2x\left[\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s}\right]$ — (1)

$y \cdot \frac{\partial u}{\partial x} = 2xy\left[\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s}\right]$ — (2)

2. $\therefore \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial y}$

$= \frac{\partial u}{\partial r} \cdot \frac{\partial (x^2+y^2)}{\partial y} + \frac{\partial u}{\partial s} \cdot \frac{\partial (x^2-y^2)}{\partial y}$

$= 2y \cdot \frac{\partial u}{\partial r} - 2y \cdot \frac{\partial u}{\partial s}$

$\therefore \frac{\partial u}{\partial y} = 2y\left[\frac{\partial u}{\partial r} - \frac{\partial u}{\partial s}\right]$ — (3)

$x \frac{\partial u}{\partial y} = 2xy\left[\frac{\partial u}{\partial r} - \frac{\partial u}{\partial s}\right]$ — (4)

Adding ② and ④, we get

$$y\cdot\frac{\partial u}{\partial x} + x\cdot\frac{\partial u}{\partial y} = 2xy\left[\frac{\partial u}{\partial r}+\frac{\partial u}{\partial s}\right] + 2xy\left[\frac{\partial u}{\partial r}-\frac{\partial u}{\partial s}\right]$$

$$\therefore y\frac{\partial u}{\partial x} + x\cdot\frac{\partial u}{\partial y} = 4xy\cdot\frac{\partial u}{\partial r}$$

Hence proved

b] 1. Squaring equation ①, we get

$$\left(\frac{\partial u}{\partial x}\right)^2 = \left[\left[\frac{\partial u}{\partial r}+\frac{\partial u}{\partial s}\right]2x\right]^2$$

$$\therefore \left(\frac{\partial u}{\partial x}\right)^2 = 4x^2\left(\frac{\partial u}{\partial r}\right)^2 + 4x^2\cdot\left(\frac{\partial u}{\partial s}\right)^2 + 8x^2\cdot\frac{\partial u}{\partial r}\cdot\frac{\partial u}{\partial s} \quad -⑤$$

2. Squaring equation ③, we get

$$\left(\frac{\partial u}{\partial y}\right)^2 = \left(2y\left[\frac{\partial u}{\partial r}-\frac{\partial u}{\partial s}\right]\right)^2$$

$$\therefore \left(\frac{\partial u}{\partial y}\right)^2 = 4y^2\left(\frac{\partial u}{\partial r}\right)^2 + 4y^2\left(\frac{\partial u}{\partial s}\right)^2 - 8y^2\cdot\frac{\partial u}{\partial r}\cdot\frac{\partial u}{\partial s} \quad -⑥$$

Adding ⑤ and ⑥, we get

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = 4x^2\left(\frac{\partial u}{\partial r}\right)^2 + 8x^2\frac{\partial u}{\partial r}\cdot\frac{\partial u}{\partial s} + 4x^2\cdot\left(\frac{\partial u}{\partial s}\right)^2$$
$$+ 4y^2\left(\frac{\partial u}{\partial r}\right)^2 - 8y^2\frac{\partial u}{\partial r}\cdot\frac{\partial u}{\partial s} + 4y^2\left(\frac{\partial u}{\partial s}\right)^2$$

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = 4\left(\frac{\partial u}{\partial r}\right)^2 (x^2+y^2) + 4\left(\frac{\partial u}{\partial s}\right)^2 (x^2+y^2)$$

$$+ 8 \cdot \frac{\partial u}{\partial r} \cdot \frac{\partial u}{\partial s} (x^2 - y^2)$$

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = 4r\left[\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial s}\right)^2\right] + 8s\left(\frac{\partial u}{\partial r} \cdot \frac{\partial u}{\partial s}\right)$$

Hence proved

4] If $u = \frac{1}{r} \cdot f(\theta)$; $x = r\cos\theta$; $y = r\sin\theta$

then find the value of

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y}$$

Soln:- $x = r\cos\theta$ $\quad$ $y = r\sin\theta$

$$x^2 + y^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2$$

$$\therefore r^2 = x^2 + y^2$$

$$\tan\theta = \frac{y}{x}$$

$$\therefore \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$u = \frac{1}{r} f(\theta)$ is given

$$\therefore u = f\left(\frac{\tan^{-1}(y/x)}{\sqrt{x^2+y^2}}\right)$$

$$u(x, y) = f\left(\frac{\tan^{-1}(y/x)}{\sqrt{x^2+y^2}}\right)$$

$$u(\lambda x, \lambda y) = f\left(\frac{\tan^{-1}(\lambda y/\lambda x)}{\sqrt{(\lambda x)^2+(\lambda y)^2}}\right)$$

$$\therefore u(\lambda x, \lambda y) = \cancel{f}\left(\frac{\tan^{-1}(y/x)}{\lambda\sqrt{x^2+y^2}}\right)$$

$$\therefore u(\lambda x, \lambda y) = \lambda^{-1} \cdot u(x, y)$$

Hence u is a homogeneous function of degree $n = -1$

By Euler's theorem

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = nu$$

$$\therefore \; x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -u$$

EXPERIMENT : No.

5] If $u = \sin\left[\frac{xy}{x^2+y^2}\right] + \sqrt{x^2+y^2} + \frac{x^2y}{x+y}$

then find the value of

$$x\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y}$$

Soln:- $u = \sin\left[\frac{xy}{x^2+y^2}\right] + \sqrt{x^2+y^2} + \frac{x^2y}{x+y}$

Let $u = p + q + r$

where

$p = \sin\left[\frac{xy}{x^2+y^2}\right]$; $q = \sqrt{x^2+y^2}$; ~~$r = \sqrt{}$~~

$r = \frac{x^2y}{x+y}$

1. $p = \sin\left[\frac{xy}{x^2+y^2}\right]$

$\therefore \sin^{-1}p = \frac{xy}{x^2+y^2}$

Let $a = \sin^{-1}p$ $\therefore a = f(p)$

$p(x,y) = \frac{xy}{x^2+y^2}$

$p(\lambda x, \lambda y) = \frac{(\lambda x)(\lambda y)}{(\lambda x)^2+(\lambda y)^2}$

$p(\lambda x, \lambda y) = \frac{xy}{x^2+y^2}$

$\therefore p(\lambda x, \lambda y) = p(x,y)$

Hence, p is a homogeneous function of degree $n = 0$

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By Euler's theorem,

$$x \cdot \frac{\partial p}{\partial x} + y \cdot \frac{\partial p}{\partial y} = np$$

$$\therefore x \cdot \frac{\partial p}{\partial x} + y \cdot \frac{\partial p}{\partial y} = 0 \quad - (1)$$

2. $q = \sqrt{x^2 + y^2}$

$$q(x, y) = \sqrt{x^2 + y^2}$$

$$\therefore q(\lambda x, \lambda y) = \sqrt{(\lambda x)^2 + (\lambda y)^2}$$

$$\therefore q(\lambda x, \lambda y) = \lambda \sqrt{x^2 + y^2}$$

$$\therefore q(\lambda x, \lambda y) = \lambda \cdot q(x, y)$$

Hence, q is a homogeneous function of degree $n = 1$

By Euler's theorem,

$$x \cdot \frac{\partial q}{\partial x} + y \cdot \frac{\partial q}{\partial y} = nq$$

$$\therefore x \cdot \frac{\partial q}{\partial x} + y \cdot \frac{\partial q}{\partial y} = q \quad - (3)$$

3. $u = p + q + r$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial p}{\partial x} + \frac{\partial q}{\partial x} + \frac{\partial r}{\partial x} \qquad \therefore \frac{\partial u}{\partial y} = \frac{\partial p}{\partial y} + \frac{\partial q}{\partial y} + \frac{\partial r}{\partial y}$$

4 $r = \frac{x^2 y}{x+y}$

$$r(x,y) = \frac{x^2 y}{x+y}$$

$$r(\lambda x, \lambda y) = \frac{(\lambda x)^2 (\lambda y)}{(\lambda x) + (\lambda y)}$$

$$r(\lambda x, \lambda y) = \lambda^2 \left(\frac{x^2 y}{x+y} \right)$$

$$\therefore r(\lambda x, \lambda y) = \lambda^2 r$$

Hence, r is a homogeneous function of degree n=2

By Euler's theorem,

$$x \cdot \frac{\partial r}{\partial x} + y \cdot \frac{\partial r}{\partial y} = n\cancel{r}$$

$$\therefore x \frac{\partial r}{\partial x} + y \cdot \frac{\partial r}{\partial y} = 2r \quad -(4)$$

Adding (1), (2) and (4), we get

$$x \frac{\partial p}{\partial x} + y \cdot \frac{\partial p}{\partial y} + x \frac{\partial q}{\partial x} + y \frac{\partial q}{\partial y} + x \cdot \frac{\partial r}{\partial x} + y \cdot \frac{\partial r}{\partial y} = q + 2\cancel{r}$$

$$x \left[\frac{\partial p}{\partial x} + \frac{\partial q}{\partial x} + \frac{\partial r}{\partial x} \right] + y \left[\frac{\partial p}{\partial y} + \frac{\partial q}{\partial y} + \frac{\partial r}{\partial y} \right] = q + 2\cancel{r}$$

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = \sqrt{x^2+y^2} + \frac{2x^2 y}{x+y}$$

6] If $u = \frac{1}{x^2} + \frac{1}{y^2} + \left(\frac{\log x - \log y}{x^2+y^2}\right)$ then prove that

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = -2u$$

Soln:-

$$u = \frac{1}{x^2} + \frac{1}{y^2} + \frac{\log x - \log y}{x^2+y^2}$$

$$\therefore u(x,y) = \frac{1}{x^2} + \frac{1}{y^2} + \frac{\log x - \log y}{x^2+y^2}$$

$$\therefore u(x,y) = \frac{1}{x^2} + \frac{1}{y^2} + \frac{\log(x/y)}{x^2+y^2}$$

$$u(\lambda x, \lambda y) = \frac{1}{(\lambda x)^2} + \frac{1}{(\lambda y)^2} + \frac{\log(\lambda x/\lambda y)}{(\lambda x)^2 + (\lambda y)^2}$$

$$u(\lambda x, \lambda y) = \frac{1}{\lambda^2}\left[\frac{1}{x^2} + \frac{1}{y^2} + \frac{\log(x/y)}{x^2+y^2}\right]$$

$$\therefore u(\lambda x, \lambda y) = \lambda^{-2} u(x,y)$$

Hence, u is a homogeneous function of degree $n = -2$

By Euler's theorem,

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = nu$$

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = -2u$$

Hence proved.

7. If $u = \operatorname{cosec}^{-1}\sqrt{\dfrac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}$ then show that

$$x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = \frac{1}{144}\tan u\left[13 + \tan^2 u\right]$$

Soln:- $u = \operatorname{cosec}^{-1}\sqrt{\dfrac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}$

$$\therefore \operatorname{cosec} u = \sqrt{\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}$$

Let $p = \operatorname{cosec} u$

$$p(x,y) = \sqrt{\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}$$

$$p(\lambda x, \lambda y) = \sqrt{\frac{(\lambda x)^{1/2}+(\lambda y)^{1/2}}{(\lambda x)^{1/3}+(\lambda y)^{1/3}}}$$

$$p(\lambda x, \lambda y) = \sqrt{\lambda^{\frac{1}{2}-\frac{1}{3}}\frac{(x^{1/2}+y^{1/2})}{(x^{1/3}+y^{1/3})}}$$

$$\therefore p(\lambda x, \lambda y) = \lambda^{1/12}\sqrt{\frac{(x^{1/2}+y^{1/2})}{(x^{1/3}+y^{1/3})}}$$

$$\therefore p(\lambda x, \lambda y) = \lambda^{1/12}\, p(x,y)$$

Hence p is a homogeneous function with degree $n = \frac{1}{12}$

1. By deduction to Euler's theorem

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} = g(u)$$

$$= n \frac{\operatorname{cosec} u}{-\cot u \operatorname{cosec} u}$$

$$= -n \tan u$$

$$\therefore g(u) = \frac{-1 \tan u}{12}$$

2. By deduction to Euler's theorem

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \cdot \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = g(u)\left(g(u) - 1\right)$$

$$= -\frac{\tan u}{12}\left[\frac{-\sec^2 u}{12} - 1\right]$$

$$\therefore x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = \frac{\tan u}{12}\left(\frac{12 + \sec^2 u}{12}\right)$$

$$= \frac{\tan u}{144}\left(12 + 1 + \tan^2 u\right)$$

$$\therefore x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = \frac{\tan u}{144}\left(13 + \tan^2 u\right)$$

Hence proved