JESSY

MODULE 3
ELECTRODYNAMICS

Syllabus:

Scalar and vector field, Physical significance of gradient, curl and divergence in cartesian co-ordinate system, Gauss's law for electrostatics, Gauss's law for magnetoslatics, Faraday's law and Ampere's circuital law. Maxwell's equations (Free space & time varying fields)

Scalar and Vector Fields

Field: A field is a function that describes the behaviour of a physical quantity at all points in a given region of space. There are two types of fields namely scalar field and vector field based on the nature of physical quantity

Scalar Field: — A scalar field is specified by a scalar value to everypoint in space. It has only one specification, i.e, magnitude

Examples: Temperature, pressure, Electric potential, energy etc

Vector Field: — A vector field is specified by assigning a vector value to every point in space. It has two specifications, i.e magnitude and direction

Examples: Velocity, acceleration, force, electric field etc.

② ## The Del operator $\vec{\nabla}$

The del operator is a vector differential operator, which is defined as

$$\boxed{\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}}$$

The del is not a vector quantity, it is only a vector operator.

When $\vec{\nabla}$ operates on a

(i) scalar function $\longrightarrow$ Gradient
(ii) vector function via dot product $\longrightarrow$ Divergence
(iii) vector function via cross product $\longrightarrow$ Curl.

1 ## Gradient

Gradient is a mathematical operation performed by $\vec{\nabla}$ on a scalarfield which result in to a vectorfield.

Let $\phi = \phi(x, y, z)$ be a scalar function, then

$$\text{grad } \phi = \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

Physical significance

Gradient is a directional derivative.
The gradient of a scalar function is a vector, whose magnitude is equal to maximum rate of change of that function and whose direction is along that maximum rate of change.

e.g: Temperature gradient in a metal bar is the rate of change of temperature along bar.

2 DIVERGENCE

When Del operates on a vector field via dot product, the resultant is called divergence and is a scalar.

Let $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$, Then

$$\vec{\nabla}\cdot\vec{A} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(A_x\hat{i} + A_y\hat{j} + A_z\hat{k}\right)$$

$$= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Physical Significance

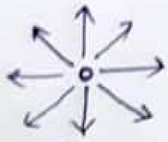
Divergence can be considered as quantitative measure of how much a vector field diverges (spreads out) at any given point.

It is the outward flow per unit volume over a closed surface.

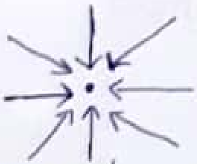
e.g: A fluid with vector field represents the velocity of fluid at each point. Divergence of fluid velocity measures the quantity of fluid flowing out of a given point

(i) If the field is spreaded outward ⇒ Positive divergence called source

(ii) If the field is converging inward ⇒ Negative divergence called sink



+ve divergence (source)



−ve divergence (sink)

zero divergence

If the divergence of a vector function is zero ($\nabla\cdot A = 0$), then it is called Solenoidal vector.

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3. CURL

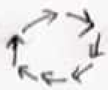
When del operator acts on a vector field via cross product, the resultant is called curl & is a vector.

The curl of a vector field $\vec{A}$ is

$$\text{curl } \vec{A} = \vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \hat{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

Physical significance



- The curl of a vector field at any point signifies how much the vector quantity curls or twist around that points.
- ∴ Curl of a vector can be interpreted as rotation or circulation.
- The magnitude of this rotational vector will be the maximum circulation and its direction is perpendicular to the plane of area.
- If curl of a vector field is zero, then it is called irrotational
- The conservative vector fields have zero curl. eg: Electrostatic field.

Combination of grad, div & Curl

curl grad $\phi = \nabla \times \nabla \phi = 0$

div curl A $= \nabla \cdot (\nabla \times \vec{A}) = 0$

Divergence of curl of a vector is zero

Let $\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$

Del $\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

div curl $A = \vec{\nabla} \cdot \vec{\nabla} \times A$

$$= \vec{\nabla} \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left[\hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \hat{j}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$= \frac{\partial^2 A_z}{\partial x \partial y} - \frac{\partial^2 A_y}{\partial x \partial z} - \frac{\partial^2 A_z}{\partial y \partial x} + \frac{\partial^2 A_x}{\partial y \partial z} + \frac{\partial^2 A_y}{\partial z \partial x} - \frac{\partial^2 A_x}{\partial z \partial y}$$

As A_x, A_y, A_z are perfect differentials, the order of differentiation w.r.to x, y, z is immaterial.

i.e $\frac{\partial^2 A_z}{\partial x \partial y} = \frac{\partial^2 A_z}{\partial y \partial x}$ etc.

Hence we get div curl $A = \vec{\nabla} \cdot \vec{\nabla} \times \vec{A} = 0$

$\therefore$ Divergence of a curl vanishes.

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Curl of gradient of a scalar field

Let $\phi(x,y,z)$ be a scalar function

curl grad $\phi = \vec{\nabla} \times \vec{\nabla}\phi$

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right) - \hat{j}\left(\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial z\partial x}\right) + \hat{k}\left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right)$$

$$= \underline{\underline{0}}$$

Divergence of gradient of a scalar

div grad $\phi = \nabla\cdot\nabla\phi$

$$= \nabla^2\phi = \underline{\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}}$$

Note:

∇^2 is called Laplacian operator

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Three types of Integrals

1) Line Integral $\rightarrow$ Along path $\int_l dl$ or $\int_c dl$
2) Surface integral $\rightarrow$ surface $\int_s ds$ or $\iint ds$
3) Volume Integral $\rightarrow$ volume $\int_V dv$ or $\iiint dv$

Examples

1) **Line Integral**

Electric potential $\boxed{V = \int_l \vec{E} \cdot \vec{dl}}$

$V = E d$
$V = \int E \, dl$

2) **Surface Integral**

Current $\boxed{I = \int_s \vec{J} \cdot \vec{ds}}$

$J \rightarrow$ current density

$J = \frac{I}{Area}$
$I = J A$
$= \int J ds$

3) **Volume Integral**

Charge $\boxed{q = \int_V \rho \, dv}$

$\rho \rightarrow$ charge density

$\rho = \frac{charge}{Volume}$
$\rho = \frac{q}{V}$
$q = \rho V$
$= \int_V \rho dv$

Two Important Theorem used in Electrodynamics

1. DIVERGENCE THEOREM

Divergence theorem states that the volume integral of divergence of a vector field $\vec{A}$ over a volume is ~~is~~ equal to surface integral of vector $\vec{A}$ over surface enclosing the volume.

$$\boxed{\int_V \vec{\nabla}\cdot\vec{A}\, dV = \oint_S \vec{A}\cdot\vec{ds}}$$

2. STOKE'S THEOREM

Stoke's theorem states that the surface integral of curl of a vector field $\vec{A}$ over a surface s is equal to line integral of the vector field $\vec{A}$ over the boundary 'l' of surface.

$$\boxed{\int_S \vec{\nabla}\times\vec{A}\, ds = \oint_l \vec{A}\cdot\vec{dl}}$$

(i) Divergence theorem ⇒	$\int_V$ of divergence $\longrightarrow$ $\int_S$ of vector
(ii) Stokes theorem ⇒	$\int_S$ of curl $\longrightarrow$ $\int_l$ of vector

Equation of Continuity

It is the law of conservation of charge, which states that the current through the conductor is equal to the rate of loss of charge from it

$$I = -\frac{dq}{dt}$$

$$\int_S J \cdot ds = -\frac{d}{dt}\int_V \rho \, dv$$

$$\Downarrow \quad = -\int_V \frac{\partial \rho}{\partial t} dv$$

$J = \frac{I}{A}$

$I = JA$

$I = \int_S J \, ds$

$\rho = \frac{q}{V}$

$q = \rho \, dv$

$= \int \rho \, dv$

Using divergence theorem

$$\int_V (\nabla \cdot J) \, dv = -\int_V \frac{\partial \rho}{\partial t} dv$$

This equation holds good for any arbitrary volume we can put the integrand to be equal.

$$\boxed{\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}} \Rightarrow \text{Eqn. of Continuity}$$

note:

In case of stationary currents, i.e when the charge density at any point with in the region remains constant, $\frac{\partial \rho}{\partial t} = 0$

$$\therefore \quad \nabla \cdot J = 0$$

Four Important Laws

1. Gauss's Law for Electrostatics
2. Gauss's Law for magnetostatics
3. Faraday's Law
4. Ampere's Circuital Law

MAXWELL'S FOUR EQUATIONS

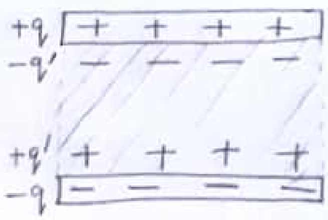
First Equation – Gauss's Law

Gauss's Law states that the total electric flux over a closed surface is $\frac{1}{\varepsilon_0}$ times the net charge enclosed within surface

$$\phi_E = \frac{1}{\varepsilon_0} q$$

$$\oint_S E \cdot ds = \frac{1}{\varepsilon_0} \int_V [\text{charge density} \times dV]$$

Consider a dielectric medium which is kept in $\vec{E}$ field. Then the external field $\vec{E}$ polarises the dielectric medium and charges are induced, called bound charges or polarisation charges. Then total charge density will be the sum of free charge density (ρ_f) and polarisation charge density (ρ_P)



Polarisation charge density $\rho_P = -\nabla \cdot \vec{P}$

where p is called <u>polarisation</u> (Induced dipolement per unit volume)

$\therefore$ Total charge density $= \rho_f + \rho_P$

$$\oint_S E \cdot ds = \frac{1}{\varepsilon_0} \int_V (\rho_f + \rho_P)\, dV$$

Let $\rho_f = \rho$

$$= \frac{1}{\varepsilon_0} \int_V (\rho_f - \nabla \cdot \vec{P})\, dV$$

Using Divergence Theorem

$$\int_V (\nabla \cdot E)\, dV = \frac{1}{\varepsilon_0} \int_V (\rho_f - \nabla \cdot P)\, dV$$

$$\varepsilon_0 \int_V (\nabla \cdot E)\, dV + \int_V \nabla \cdot P\, dV = \int_V \rho_f\, dV$$

$$\int_V \nabla \cdot (\varepsilon_0 E + P)\, dV = \int_V \rho_f\, dV$$

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The quantity $(\varepsilon_0\vec{E}+\vec{P})$ is denoted by a quantity called <u>electric displacement $\vec{D}$</u>

$$\vec{D} = \varepsilon_0\vec{E} + \vec{P}$$

$$\therefore \int_V (\nabla\cdot D)\,dv = \int_V \rho_f\,dv$$

Since this eqn is true for all arbitrary volume, the integrands in this equation must be equal

$$\boxed{\nabla\cdot D = \rho_f}$$ ⇒ Maxwell's Ist equation in Differential form.

note :⇒ If we are not considering polarisation $(P=0)$ $D=\varepsilon_0 E$

Then $\nabla\cdot\varepsilon_0 E = \rho$ or

$$\boxed{\nabla\cdot E = \frac{1}{\varepsilon_0}\rho}$$

<u>Integral Form</u>

Differential form of Ist eqn is

$$\nabla\cdot D = \rho$$

On integrating ~~ab~~ over a volume V, we have

$$\int_V \nabla\cdot D\,dv = \int_V \rho\,dv$$

⇓ Using Divergence theorem

$$\oint_S \vec{D}\cdot\vec{ds} = \int_V \rho\,dv = q$$

or $$\boxed{\oint_S \vec{D}\cdot\vec{ds} = q}$$

Second Equation — Gauss's law for magnetism

Gauss's law in magnetostatics states that the total magnetic flux over any closed surface is zero. i.e

$$\phi_B = 0$$

$$\oint_S \vec{B} \cdot \vec{ds} = 0$$

$\Downarrow$ Using Divergence theorem

$$\int_V (\nabla \cdot B)\, dv = 0$$

$\therefore$ The integrand in the above equation should vanish

$$\boxed{\vec{\nabla} \cdot \vec{B} = 0} \Rightarrow \text{2nd eqn in differential form.}$$

Integral form

Differential form is $\vec{\nabla} \cdot \vec{B} = 0$

Take volume integral

$$\int_V (\nabla \cdot B)\, dV = 0$$

$\Downarrow$ using divergence theorem

$$\boxed{\oint_S (\vec{B} \cdot \vec{ds}) = 0}$$

which signifies that the total outward flux of magnetic induction B through any closed surface is equal to zero.

note: This is due to the fact that unlike electric charges, isolated magnetic poles donot exist

(14) Third Equation — Faraday's law

According to Faraday's law of electromagnetic induction, the induced emf in any closed loop of wire is equal to rate of change of magnetic flux linked with it, i.e

Induced emf $e = -\dfrac{d\phi_B}{dt}$

Also this induced emf $e = \oint_l E \cdot dl$

Here $\vec{E}$ is intensity of field associated with induced emf

On equating both equation

$$\oint_l E \cdot dl = -\frac{d\phi_B}{dt}$$

$$= -\frac{d}{dt}\int_s \vec{B} \cdot \vec{ds}$$

$\Downarrow$ Using Stoke's theorem

$$\int_s (\nabla \times E) \cdot \vec{ds} = -\int_s \frac{\partial B}{\partial t}\, ds$$

This equation must be true for any surface

$\therefore$ Integrands must be equal

$$\boxed{\vec{\nabla} \times \vec{E} = -\frac{\partial B}{\partial t}}$$

$\Rightarrow$ Maxwell's third eqn Differential form

Integral form

Differential form is $\vec{\nabla} \times \vec{E} = -\dfrac{\partial B}{\partial t}$

On integrating over a surface s bounded by closed path

$$\int_s (\nabla \times E)\, ds = -\int_s \frac{\partial B}{\partial t}\, ds$$

$\Downarrow$ Using Stoke's theorem

$$\boxed{\oint_l \vec{E} \cdot \vec{dl} = -\frac{\partial}{\partial t}\int_s \vec{B} \cdot \vec{ds}}$$

FOURTH EQUATION — Ampere's law

Ampere's Law states that the line integral of the magnetic field around any closed path is equal to μ_0 times the current enclosed with in the path

$$\oint_l B \cdot dl = \mu_0 I$$

$$= \mu_0 \int_s J \cdot ds$$

$\Downarrow$ Using Stoke's theorem

$$\int_s (\nabla \times B)\, ds = \mu_0 \int_s J \cdot ds$$

$$\int_s (\nabla \times B)\, ds = \int_s \mu_0 J \cdot ds$$

$$\therefore \boxed{\nabla \times B = \mu_0 J}$$

$B = \mu_0 H$

$$\therefore \boxed{\nabla \times H = J}$$

This relation holds good only for steady current. However, for the changing electric fields, the current density J should be modified. Because

Taking divergence of $\nabla \times B = \mu_0 J$

$$\nabla \cdot (\nabla \times B) = \mu_0 (\nabla \cdot J)$$

$\Downarrow$ Divergence of curl is always zero.

$$0 = \mu_0 (\nabla \cdot J)$$

or $\quad \nabla \cdot J = 0 \quad$ (since $\mu_0 \neq 0$)

But from eqn of continuity $\nabla \cdot J = -\frac{\partial \rho}{\partial t}$

($\nabla \cdot J = 0$ is true only in the static case)

$\therefore$ Modify the current density to get corrected Ampere's law.

$\therefore$ Total current density is

$$J + J_D$$

where J_D is known as displacement current density given by

$$\boxed{J_D = \varepsilon_0 \frac{\partial E}{\partial t} = \frac{\partial D}{\partial t}}$$

$$\nabla \cdot J = -\frac{\partial \rho}{\partial t}$$

Maxwell's 1^{st} eqn

$$\nabla \cdot E = \frac{1}{\varepsilon_0}\rho$$

$$\rho = \varepsilon_0 \nabla \cdot E$$

$$\nabla \cdot J = -\frac{\partial}{\partial t}(\varepsilon_0 \nabla \cdot E)$$

$$\nabla \cdot \underbrace{\left(J + \varepsilon_0 \frac{\partial E}{\partial t}\right)}_{\text{Total current densit}} = 0$$

$\therefore$ Maxwell's fourth eqn can be written as

$$\nabla \times B = \mu_0 \left(J + \varepsilon_0 \frac{\partial E}{\partial t}\right)$$

$$\nabla \times B = \mu_0 \left(J + \frac{\partial D}{\partial t}\right)$$

But $B = \mu_0 H$

$$\therefore \boxed{\nabla \times H = J + \frac{\partial D}{\partial t}}$$

$\Rightarrow$ 4th eqn or modified Ampere's law Differential form.

Integral form

Differential form $\nabla \times H = J + \frac{\partial D}{\partial t}$

On integrating over surface.

$$\int_S (\nabla \times H)\, ds = \int_S \left(J + \frac{\partial D}{\partial t}\right) ds.$$

$\Downarrow$ using stoke's theorem

$$\boxed{\oint_l H \cdot dl = \int_S \left(J + \frac{\partial D}{\partial t}\right) ds}$$

Physical Significance of Maxwell's Eqns

Maxwell's eqns represent the fundamentals of electricity and magnetism in a concise way.

1) Maxwell's first eqn shows that total electric flux density D through the surface enclosing volume is equal to charge density ρ within volume. It means that a charge distribution generates a steady electric field.

2) Maxwell's 2nd eqn tells that net magnetic flux through a closed surface is zero. It implies that magnetic monopoles do not exist as electric charges do

3) The third eqn shows that emf around a closed path is equal to time derivative of magnetic flux density through the surface bounded by path. It implies that an electric field can also be generated by time varying magnetic fields.

4) Maxwell's 4th eqn gives time derivative of electric flux density $\frac{\partial D}{\partial t}$ as displacement current. This gives the meaning that magnetic field is generated by time varying electric fields

Maxwell's equations are very important in understanding working of antennas, waveguide and satellite communication