Module 2:COMPLEX NUMBERS

CIRCULAR FUNCTIONS OF COMPLEX NUMBERS

- By Euler's Formula $e^{i\theta} = cos\ \theta + i\ sin\ \theta ...(1)$ and $e^{-i\theta} = cos\ \theta - i\ sin\ \theta ....(2)$

- Adding and subtracting (1) and (2), we have

$$cos\ \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

and

$$sin\ \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

respectively , known as **Euler's exponential forms of circular functions**, where θ is a real number.

- If z is complex number,then

$$cos\ z = \frac{e^{iz} + e^{-iz}}{2}$$

and

$$sin\ z = \frac{e^{iz} - e^{-iz}}{2i}$$

, known as **circular functions of complex numbers**

HYPERBOLIC FUNCTIONS

- The hyperbolic functions, a new class of transcendental functions which appear in some scientific and mathematical applications

- In mathematics, hyperbolic functions are analogues of the ordinary trigonometric functions, but defined using the hyperbola rather than the circle

- Just as the points $(cos\ t, sin\ t)$ form a circle with a unit radius, the points $(cosh\ t, sinh\ t)$ form the right half of the unit hyperbola

- In complex analysis, the hyperbolic functions arise as the imaginary parts of sine and cosine.

- If x is real or Complex number then

$$\frac{e^x + e^{-x}}{2}$$

  is called **Hyperbolic cosine of** x or **Cosine hyperbolic of** x denoted by $cosh\ (x)$

- Also if x is real or Complex number then

$$\frac{e^x - e^{-x}}{2}$$

  is called **Hyperbolic sine of** x or **Sine hyperbolic of** x denoted by $sinh\ (x)$

- Thus we have

$$cosh\ (x) = \frac{e^x + e^{-x}}{2}$$

and

$$sinh\ (x) = \frac{e^x - e^{-x}}{2}$$

- Using above definitions we can also define

$$tanh\ (x) = \frac{sinh\ x}{cosh\ x} = \frac{e^x - e^{-x}/2}{e^x + e^{-x}/2}$$

Hence

$$tanh\ (x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

- Similarly

$$coth\ (x) = \frac{1}{tanh\ (x)} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$sech\ (x) = \frac{1}{cosh\ (x)} = \frac{2}{e^x + e^{-x}}$$

$$cosech\ (x) = \frac{1}{sinh\ (x)} = \frac{2}{e^x - e^{-x}}$$

Relation between Circular and Hyperbolic Functions

- There are twelve relationships between circular and hyperbolic functions

- First six are conversion of circular to hyperbolic functions and other six are conversion of hyperbolic to circular functions

Conversion of Circular to Hyperbolic Functions

(1) $sin\ (ix) = i\ sinh\ (x)$

Proof:We know that

$$sin\ (x) = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\therefore sin\ (ix) = \frac{e^{i(ix)} - e^{-i(ix)}}{2i}$$

$$= \frac{e^{i^2x} - e^{-i^2x}}{2i}$$

$$= \frac{e^{-x} - e^{x}}{2i}$$

$$= \frac{-1}{i}\left(\frac{e^{x} - e^{-x}}{2}\right)$$

$$\therefore sin\ (ix) = i\ sinh\ (x)$$

(2) $cos\ (ix) = cosh\ (x)$

Proof:We know that

$$\begin{aligned} cos\ (x) &= \frac{e^{ix} + e^{-ix}}{2} \\ \therefore cos\ (ix) &= \frac{e^{i(ix)} + e^{-i(ix)}}{2} \\ &= \frac{e^{i^2 x} + e^{-i^2 x}}{2} \\ &= \frac{e^{-x} + e^{x}}{2} \\ &= \frac{e^{x} + e^{-x}}{2} \\ \therefore cos\ (ix) &= cosh\ (x) \end{aligned}$$

Similarly we can prove

- $tan\ (ix) = i\ tanh\ (x)$

- $cot\ (ix) = -i\ coth\ (x)$

- $sec\ (ix) = sech\ (x)$

- $cosec\ (ix) = -i\ cosech\ (x)$

Conversion of Hyperbolic to Circular Functions

(1) $sinh\ (ix) = i\ sin\ (x)$

Proof:We know that

$$sinh\ (x) = \frac{e^x - e^{-x}}{2}$$
$$\therefore sinh\ (ix) = \frac{e^{(ix)} - e^{(-ix)}}{2}$$
$$= i\ \left(\frac{e^{i2x} - e^{-ix}}{2i}\right)$$
$$\therefore sinh\ (ix) = i\ sin\ (x)$$

(2) $cosh\ (ix) = cos\ (x)$

Proof:We know that

$$cosh\ (x) = \frac{e^x + e^{-x}}{2}$$
$$\therefore cosh\ (ix) = \frac{e^{(ix)} + e^{(-ix)}}{2}$$
$$\therefore cosh\ (ix) = cos\ (x)$$

Similarly we can prove

- $tanh\ (ix) = i\ tan\ (x)$

- $coth\ (ix) = -i\ cot\ (x)$

- $sech\ (ix) = sec\ (x)$

- $cosech\ (ix) = -i\ cosec\ (x)$

HYPERBOLIC IDENTITIES

Hyperbolic identities can be obtained from circular identities by replacing x by ix and using relation between circular and hyperbolic functions

(A) Square hyperbolic identities

(1) $cosh^2 x - sinh^2 x = 1$

Proof:We know that

$$cos^2\ x + sin^2\ x = 1$$
$$\therefore [cos\ (ix)]^2 + [sin\ (ix)]^2 = 1....(replacing\ x\ by\ ix)$$
$$\therefore [cosh\ (x)]^2 + [i\ sinh\ (x)]^2 = 1(using\ relation)$$
$$\therefore cosh^2\ x - sinh^2\ x = 1$$

(2) $1 - tanh^2\ (x) = sech^2\ (x)$

Proof:We know that

$$1 + tan^2\ (x) = sec^2\ (x)$$

$$\therefore 1 + [tan\ (ix)]^2 = [sec\ (ix)]^2 .... (replacing\ x\ by\ ix)$$

$$\therefore 1 + [i\ tanh\ (x)]^2 = [sech\ (x)]^2 (using\ relation)$$

$$\therefore 1 - tanh^2\ x = sech^2\ x$$

(3) $1 - coth^2\ (x) = -cosech^2\ (x)$

(B) Sum and diference hyperbolic formulas

(1) $sinh\ (x + y) = sinh\ (x)\ cosh\ (y) + cosh\ (x)\ sinh\ (y)$

Proof:We know that

$$sin\ (x + y) = sin\ (x)\ cos\ (y) + cos\ (x)\ sin\ (y)$$

$$\therefore sin\ (ix+iy) = sin\ (ix)\ cos\ (iy) + cos\ (ix)\ sin\ (iy) (replacing\ x\ by\ ix\ and$$

$$\therefore sin\ [i(x+y)] = i\ sinh\ (x)\ cosh\ (y) + cosh\ (x)\ i\ sinh\ (y) (using\ relation)$$

$$\therefore i\ sinh\ (x + y) = i\ [sinh\ (x)\ cosh\ (y) + cosh\ (x)\ sinh\ (y)]$$

$$\therefore sinh\ (x + y) = sinh\ (x)\ cosh\ (y) + cosh\ (x)\ sinh\ (y)$$

Similarly we can prove that

(2) $sinh\ (x - y) = sinh\ (x)\ cosh\ (y) - cosh\ (x)\ sinh\ (y)$

(3) $cosh\ (x + y) = cosh\ (x)\ cosh\ (y) + sinh\ (x)\ sinh\ (y)$

(4) $cosh\ (x-y) = cosh\ (x)\ cosh\ (y) - sinh\ (x)\ sinh\ (y)$

(5) $tanh\ (x+y) = \frac{tanh\ (x)+tanh\ (y)}{1+tanh\ (x)\ \ tanh\ (y)}$

(5) $tanh\ (x-y) = \frac{tanh\ (x)-tanh\ (y)}{1-tanh\ (x)\ \ tanh\ (y)}$

(C) Multiple angle hyperbolic formulas

(1) $sinh\ (2x) = 2\ sinh\ (x)\ cosh\ (x)$

(2) $sinh\ (2x) = \frac{2\ tanh\ (x)}{1-tanh^2\ (x)}$

(3) $cosh\ (2x) = cosh^2\ (x) + sinh^2\ (x)$

(4) $cosh\ (2x) = 2\ cosh^2\ (x) - 1$

(5) $cosh\ (2x) = 1 + 2\ sinh^2\ (x)$

(6) $cosh\ (2x) = \frac{1+\ tanh^2\ (x)}{1-tanh^2\ (x)}$

(7) $tanh\ (2x) = \frac{2\ tanh\ (x)}{1+tanh^2\ (x)}$

(C) Multiple angle hyperbolic formulas

(8) $sinh\ (3x) = 3\ sinh\ (x) + 4\ sinh^3\ (x)$

(9) $cosh\ (3x) = 4\ cosh^3\ (x) - 3\ cosh\ (x)$

(10) $tanh\ (3x) = \frac{3\ tanh\ (x)+tanh^2\ (x)}{1+3\ tanh^2\ (x)}$

(D) Product hyperbolic formulas

(1) $2\ sinh\ (x)\ \ cosh\ (y) = sinh\ (x+y) + sinh\ (x-y)$

(2) $2\ cosh\ (x)\ \ sinh\ (y) = sinh\ (x+y) - sinh\ (x-y)$

(1) $2\ cosh\ (x)\ \ cosh\ (y) = cosh\ (x+y) + cosh\ (x-y)$

(2) $2\ sinh\ (x)\ \ sinh\ (y) = cosh\ (x+y) - cosh\ (x-y)$

(E) Defactorization hyperbolic formulas

(1) $sinh\ (x) + sinh\ (y) = 2\ sinh\left(\frac{x+y}{2}\right)\ cosh\left(\frac{x-y}{2}\right)$

(2) $sinh\ (x) - sinh\ (y) = 2\ cosh\left(\frac{x+y}{2}\right)\ sinh\left(\frac{x-y}{2}\right)$

(3) $cosh\ (x) + cosh\ (y) = 2\ cosh\left(\frac{x+y}{2}\right)\ cosh\left(\frac{x-y}{2}\right)$

(4) $cosh\ (x) - cosh\ (y) = 2\ sinh\left(\frac{x+y}{2}\right)\ sinh\left(\frac{x-y}{2}\right)$

Examples

Example 1

If $x = \sqrt{3}$ find value of $tanh\ (log\ x)$

Solution

By definition of $tanh\ (x)$, we have

$$
\begin{aligned}
tanh\ (x) &= \frac{e^x - e^{-x}}{e^x + e^{-x}} \\
\therefore tanh\ (log\ x) &= \frac{e^{log\ x} - e^{-log\ x}}{e^{log\ x} + e^{-log\ x}} \\
&= \frac{e^{log\ x} - e^{log\ x^{-1}}}{e^{log\ x} + e^{log\ x^{-1}}}
\end{aligned}
$$

$$= \frac{x - x^{-1}}{x + x^{-1}}$$
$$= \frac{x^2 - 1}{x^2 + 1}$$
$$= \frac{(\sqrt{3})^2 - 1}{(\sqrt{3})^2 + 1}$$
$$= \frac{1}{2}$$

Example 2

Solve the equation for real values of x

$$17\ cosh\ (x) + 18\ sinh\ (x) = 1$$

Solution

Given

$$17\ cosh\ (x) + 18\ sinh\ (x) = 1$$
$$\therefore 17\ \left(\frac{e^x + e^{-x}}{2}\right) + 18\ \left(\frac{e^x - e^{-x}}{2}\right) = 1$$
$$\therefore \frac{17\ e^x + 17\ e^{-x} + 18\ e^x - 18\ e^{-x}}{2} = 1$$
$$\therefore 35\ e^x - e^{-x} = 2$$
$$\therefore 35\ e^x - \frac{1}{e^x} = 2$$

$$\implies 35\ (e^x)^2 - 2\ e^x - 1 = 0$$

which is quadratic equation in e^x

$$\therefore e^x = \frac{-(-2) \pm \sqrt{4 - 4(35)(-1)}}{2(35)}$$

$$= \frac{2 \pm\ 2\ \sqrt{36}}{2(35)}$$

$$\therefore e^x = \frac{1}{5} or \frac{-1}{7}$$

$$\therefore x = log\left(\frac{1}{5}\right) orlog\left(\frac{-1}{7}\right)$$

$$x = log\left(\frac{-1}{7}\right)$$

is not possible since x is real.

$$\therefore x = log\left(\frac{1}{5}\right) = -log(5)$$

Example 3

If $log(tan\ x) = y$ Prove that

(1) $cosh\ (ny) = \frac{1}{2}\ (tan^n\ (x) + cot^n\ (x)$

(2) $sinh\ [(n+1)y] + sinh\ [(n-1)y] = 2\ sinh\ (ny)cosec\ (2x)$

Solution

Given

$$log(tan\ x) = y$$
$$\therefore tan\ x = e^y cot\ x = e^{-y}$$

(1)

$$cosh(ny) = \frac{e^{ny} + e^{-ny}}{2}$$
$$= \frac{(tanx)^n + (cotx)^n}{2}$$

(2)Using $sinh\ (A) + sinh\ (B) = 2\ sinh\left(\frac{A+B}{2}\right)\ cosh\left(\frac{A-B}{2}\right)$ we have

$$sinh\ [(n+1)y] + sinh\ [(n-1)y]$$
$$= 2\ sinh\left(\frac{(n+1)y + (n-1)y}{2}\right) cosh\left(\frac{(n+1)y - (n-1)y}{2}\right)$$
$$= 2\ sinh\ (ny)\ cosh\ (y)$$
$$= 2\ sinh\ (ny)\ \left(\frac{e^y + e^{-y}}{2}\right)$$
$$= 2\ sinh\ (ny)\ \left(\frac{tan\ x + cot\ x}{2}\right)$$
$$= 2\ sinh\ (ny)\ \left(\frac{sin^2\ x + cos^2\ x}{2\ sin\ x\ cos\ x}\right)$$
$$= 2\ sinh\ (ny)\ \left(\frac{1}{sin\ 2x}\right)$$
$$\therefore sinh\ [(n+1)y] + sinh\ [(n-1)y] = 2\ sinh\ (ny)\ \ (cosech\ (2x))$$

Example 4

If $u = log\left[tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right]$ Prove that

(1) $cosh\ (u) = sec\ \theta$

(2) $sinh\ (u) = tan\ \theta$

(3) $tanh\ (u) = sin\ \theta$

(4) $tanh\ (\frac{u}{2}) = tan\ \frac{\theta}{2}$

Solution

Given

$$u = log\left[tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right]$$

$$\therefore e^u = tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$$

$$= \frac{tan\ \frac{\pi}{4} + tan\ \frac{\theta}{2}}{1 - tan\ \frac{\pi}{4}\ tan\ \frac{\theta}{2}}$$

$$= \frac{1 + tan\ \frac{\theta}{2}}{1 - tan\ \frac{\theta}{2}}$$

$$= \frac{cos\ \frac{\theta}{2} + sin\ \frac{\theta}{2}}{cos\ \frac{\theta}{2} - sin\ \frac{\theta}{2}}$$

$$e^u = \left(\frac{cos\ \frac{\theta}{2} + sin\ \frac{\theta}{2}}{cos\ \frac{\theta}{2} - sin\ \frac{\theta}{2}}\right)\ \left(\frac{cos\ \frac{\theta}{2} + sin\ \frac{\theta}{2}}{cos\ \frac{\theta}{2} + sin\ \frac{\theta}{2}}\right)$$

$$= \frac{\left(cos\ \frac{\theta}{2} + sin\ \frac{\theta}{2}\right)^2}{cos^2\ \frac{\theta}{2} - sin^2\ \frac{\theta}{2}}$$

$$= \frac{cos^2\ \frac{\theta}{2} + sin^2\ \frac{\theta}{2} + 2\ sin\ \frac{\theta}{2}\ \ cos\ \frac{\theta}{2}}{cos\ \theta}$$

$$= \frac{1 + sin\ \theta}{cos\ \theta}$$

$$e^u = sec\ \theta + tan\ \theta$$

$$\therefore e^{-u} = sec\ \theta - tan\ \theta$$

(1)

$$cosh\ u = \frac{e^u + e^{-u}}{2}$$

$$= \frac{sec\ \theta + tan\ \theta + sec\ \theta - tan\ \theta}{2}$$

$$\therefore cosh\ u = sec\ \theta$$

(2)

$$sinh\ u = \frac{e^u - e^{-u}}{2}$$

$$= \frac{sec\ \theta + tan\ \theta - sec\ \theta + tan\ \theta}{2}$$

$$\therefore sinh\ u = tan\ \theta$$

(3)

$$tanh\ u = \frac{sinh\ u}{cosh\ u}$$

$$= \frac{tan\ \theta}{sec\ \theta}$$

$$\therefore tanh\ u = sin\ \theta$$

(4)

$$tanh\ (\frac{u}{2}) = \frac{sinh\ (\frac{u}{2})}{cosh\ (\frac{u}{2})}$$

$$= \frac{2\ sinh\ (\frac{u}{2}) cosh\ (\frac{u}{2})}{2\ cosh^2\ (\frac{u}{2})}$$

$$= \frac{sinh\ (u)}{cosh\ (u) + 1}$$

$$= \frac{tan\ (\theta)}{sec\ (\theta) + 1}$$

$$= \frac{(tan\ (\theta))\ (sec\ (\theta) - 1)}{sec^2\ (\theta) - 1}$$

$$= \frac{sec\ (\theta) - 1}{tan\ (\theta)}$$

$$= \frac{1 - cos\ (\theta)}{sin\ (\theta)}$$

$$= \frac{2\ sin^2(\frac{\theta}{2})}{2\ sin(\frac{\theta}{2})\ cos(\frac{\theta}{2})}$$

$$tanh\ (\frac{u}{2}) = tan\ (\frac{\theta}{2})$$

Inverse Hyperbolic Functions

- If $sinh\ (x) = y$ then $x = sinh^{-1}\ (y)$ is called inverse hyperbolic sine of y
- If $cosh\ (x) = y$ then $x = cosh^{-1}\ (y)$ is called inverse hyperbolic cosine of y
- If $tanh\ (x) = y$ then $x = tanh^{-1}\ (y)$ is called inverse hyperbolic tangent of y

Examples

Example 1

Prove that for real values of x,

$$sinh^{-1}\ (x) = log\left[x + \sqrt{x^2+1}\right]$$

Solution 1

$$\begin{aligned}
sinh^{-1}(x) &= y \\
\therefore x &= sinh(y) \\
\therefore x + \sqrt{x^2+1} &= sinh(y) + \sqrt{sinh^2(y)+1} \\
&= sinh(y) + cosh(y) \\
&= \left(\frac{e^y - e^{-y}}{2}\right) + \left(\frac{e^y + e^{-y}}{2}\right) \\
\therefore x + \sqrt{x^2+1} &= e^y \\
\therefore y &= log\left[x + \sqrt{x^2+1}\right]
\end{aligned}$$

Hence

$$sinh^{-1}(x) = log\left[x + \sqrt{x^2+1}\right]$$

Solution 2

Let

$$\begin{aligned}
sinh^{-1}(x) &= y \\
\therefore x &= sinh(y) \\
\therefore x &= \frac{e^y - e^{-y}}{2} \\
\implies 2x &= e^y - e^{-y} \\
\implies e^{2y} - 2xe^y - 1 &= 0
\end{aligned}$$

which is quadratic equation in e^y Hence

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2}$$
$$= x \pm \sqrt{x^2+1}$$

considering positive sign only

$$sinh^{-1}\ (x) = log\left[x + \sqrt{x^2+1}\right]$$

Examples (HW)

Example 2

Prove that for real values of x,

$$cosh^{-1}\ (x) = log\left[x + \sqrt{x^2-1}\right]$$

Example 3

Prove that for real values of x,

$$tanh^{-1}\ (x) = \frac{1}{2}\ log\left(\frac{1+x}{1-x}\right)$$

Example 4 Prove that

$$sech^{-1}\ (sin\ \theta) = log\ cot\ (\frac{\theta}{2})$$

Solution

Let

$$
\begin{aligned}
sech^{-1}\ (sin\ \theta) &= y \\
\therefore sin\ \theta &= sech\ (y) \\
\therefore sin\ \theta &= \sqrt{1 - tanh^2\ (y)} \\
\therefore tanh^2\ (y) &= 1 - sin^2\ \theta = cos^2\ \theta \\
\therefore tanh\ (y) &= cos\ \theta \\
\therefore y &= tanh^{-1}(cos\ \theta)
\end{aligned}
$$

Now ,

$$tanh^{-1}\ (x) = \frac{1}{2}\ log \left(\frac{1+x}{1-x}\right)$$

$$\therefore y = tanh^{-1}(cos\ \theta)$$

$$\implies y = \frac{1}{2}\ log \left(\frac{1+cos\ \theta}{1-cos\ \theta}\right)$$

$$= \frac{1}{2}\ log \left(\frac{2\ cos^2\ \frac{\theta}{2}}{2\ sin^2\ \frac{\theta}{2}}\right)$$

$$= \frac{1}{2}\ log\ cot^2\ \left(\frac{\theta}{2}\right)$$

$$= log\ \sqrt{cot^2\ \left(\frac{\theta}{2}\right)}$$

$$\therefore y = tanh^{-1} cot\ \left(\frac{\theta}{2}\right)$$

$$\therefore sech^{-1}\ (sin\ \theta) = log\ cot\ \left(\frac{\theta}{2}\right)$$

Example 5

Prove that

$$tan^{-1}\ \left[i\ \left(\frac{x-a}{x+a}\right)\right] = \frac{-i}{2}\ log\ (\frac{a}{x})$$

Solution 1

Let

$$\begin{aligned}
tan^{-1}\ \left[i\ \left(\frac{x-a}{x+a}\right)\right] &= tan^{-1}\ \left[\frac{i\ \left(\frac{x}{a}-1\right)a}{\left(\frac{x}{a}+1\right)a}\right] \\
&= tan^{-1}\ \left[\frac{i\ \left(\frac{x}{a}-1\right)}{\left(\frac{x}{a}+1\right)}\right]
\end{aligned}$$

Put $\frac{x}{a} = e^y \implies log\ \left(\frac{x}{a}\right) = y$

$$tan^{-1}\ \left[i\ \left(\frac{x-a}{x+a}\right)\right] = tan^{-1}\ \left[i\ \left(\frac{e^y-1}{e^y+1}\right)\right]$$

Let

$$\begin{aligned}
tan^{-1}\ \left[i\ \left(\frac{x-a}{x+a}\right)\right] &= tan^{-1}\ \left[i\ \left(\frac{e^{\frac{y}{2}}-e^{\frac{-y}{2}}}{e^{\frac{y}{2}}+e^{\frac{-y}{2}}}\right)\right] \\
&= tan^{-1}\ \left[i\ tanh\ \left(\frac{y}{2}\right)\right] \\
&= tan^{-1}\ tan\left(\frac{iy}{2}\right) \\
&= \frac{i}{2}\ y \\
&= \frac{i}{2}\ log\ \left(\frac{x}{a}\right) \\
&= \frac{i}{2}\ log\ \left(\frac{a}{x}\right)^{-1} \\
\therefore tan^{-1}\ \left[i\ \left(\frac{x-a}{x+a}\right)\right] &= \frac{-i}{2}\ log\ \left(\frac{a}{x}\right)
\end{aligned}$$

Solution 2

Let

$$\begin{aligned}
tan^{-1}\ \left[i\ \left(\frac{x-a}{x+a}\right)\right] &= \alpha + i\beta \\
\therefore tan(\alpha + i\beta) &= i\ \left(\frac{x-a}{x+a}\right) \\
\therefore tan(\alpha + i\beta) &= -i\ \left(\frac{x-a}{x+a}\right) \\
\therefore tan(2\alpha) &= tan\left[(\alpha + i\beta) + (\alpha - i\beta)\right] \\
&= \frac{tan(\alpha + i\beta) + tan(\alpha - i\beta)}{1 - tan(\alpha + i\beta)tan(\alpha - i\beta)} \\
&= \frac{i\ \left(\frac{x-a}{x+a}\right) - i\ \left(\frac{x-a}{x+a}\right)}{1 - \left(i\ \left(\frac{x-a}{x+a}\right)\right)\left(-i\ \left(\frac{x-a}{x+a}\right)\right)} \\
\therefore tan(2\alpha) &= 0 \\
\therefore \alpha &= 0
\end{aligned}$$

Also

$$
\begin{aligned}
tan(2i\beta) &= tan\left[(\alpha + i\beta) - (\alpha - i\beta)\right] \\
&= \frac{tan(\alpha + i\beta) - tan(\alpha - i\beta)}{1 + tan(\alpha + i\beta)tan(\alpha - i\beta)} \\
&= \frac{2i\ \left(\frac{x-a}{x+a}\right)}{1 + \left(\frac{x-a}{x+a}\right)^2} \\
&= \frac{2i\ (x-a)(x+a)}{(x+a)^2 + (x-a)^2} \\
&= \frac{2i\ (x^2 - a^2)}{2(x^2 + a^2)} \\
\therefore itanh2\beta &= i\frac{x^2 - a^2}{x^2 + a^2} \\
\therefore 2\beta &= tanh^{-1}\left[\frac{x^2 - a^2}{x^2 + a^2}\right] \\
&= \frac{1}{2}log\left[\frac{1 + \frac{x^2-a^2}{x^2+a^2}}{1 - \frac{x^2-a^2}{x^2+a^2}}\right] \\
&= \frac{1}{2}log\left[\frac{x}{a}\right]
\end{aligned}
$$

$$
\therefore tan^{-1}\ \left[i\ \left(\frac{x-a}{x+a}\right)\right] = \alpha + i\beta = \frac{i}{2}log\left[\frac{x}{a}\right]
$$

Separation into real and imaginary parts
Example 1

Separate into real and imaginary parts

(a) $sin\ (x+iy)$

(b) $cos\ (x+iy)$...(HW)

(c) $tan\ (x+iy)$

(d) $sinh\ (x+iy)$

(e) $cosh\ (x+iy)$...(HW)

(f) $tanh\ (x+iy)$...(HW)

Solution

(a) Using $sin\ (A+B) = sin\ A\ cos\ B + cos\ A\ sin\ B$ we have

$$\begin{aligned} sin\ (x+iy) &= sin\ (x)\ cos\ (iy) + cos\ (x)\ sin\ (iy) \\ &= sin\ (x)\ cosh\ (y) + cos\ (x)\ i\ sinh\ (y) \\ &= sin\ (x)\ cosh\ (y) + i\ cos\ (x)\ sinh\ (y) \end{aligned}$$

$\therefore$ Real Part=$sin\ (x)\ cosh\ (y)$
Imaginary Part=$cos\ (x)\ sinh\ (y)$

(c)

$$tan\ (x+iy) = \frac{sin\ (x+iy)}{cos\ (x+iy)}$$

$$= \frac{2\ sin\ (x+iy)\ cos\ (x-iy)}{2\ cos\ (x+iy)\ cos\ (x-iy)}$$

$$= \frac{sin\ (2x) + sin\ (2iy)}{cos\ (2x) + cos\ (2iy)}$$

$$= \frac{sin\ (2x) + i\ sinh\ (2y)}{cos\ (2x) + cosh\ (2y)}$$

$\therefore$ Real Part=$\frac{sin\ (2x)}{cos\ (2x)+cosh\ (2y)}$
Imaginary Part=$\frac{sinh\ (2y)}{cos\ (2x)+cosh\ (2y)}$

(d)

$$sinh\ (x+iy) = -i\ sin\ [i(x+iy)] ... (\because sin(i\theta) = i\ sinh\ (\theta))$$

$$= -i\ sin\ [ix + i^2 y]$$

$$= -i\ sin\ [-y + ix]$$

$$= -i\ [sin\ (-y).\ cos\ (ix) + cos\ (-y)\ sin\ (ix)]$$

$$= -i\ [-sin\ (y).\ cosh\ (x) + cos\ (y)\ i\ sinh\ (x)]$$

$$= i\ sin\ (y).\ cosh\ (x) - i^2\ cos\ (y)\ sinh\ (x)$$

$$= cos\ (y)\ sinh\ (x) + i\ sin\ (y).\ cosh\ (x)$$

$\therefore$ Real Part=$cos\ (y)\ sinh\ (x)$
Imaginary Part=$sin\ (y).\ cosh\ (x)$

Example 2

Separate into real and imaginary parts

(a) $sin^{-1}\ (e^{i\theta})$

(b) $cos^{-1}\ (e^{i\theta})$...(HW)

(c) $tan^{-1}\ (e^{i\theta})$

(d) $tanh^{-1}\ (x+iy)$...(HW)

(e) $sinh^{-1}\ (ix)$...(HW)

Solution

(a) Let

$$sin^{-1}(e^{i\theta}) = a + ib$$
$$\implies e^{i\theta} = sin\ (a+ib)$$
$$\implies e^{i\theta} = sin\ (a)\ cos\ (ib) + cos\ (a)\ sin\ (ib)$$
$$\implies cos\ (\theta) + i\ sin\ (\theta) = sin\ (a)\ cosh\ (b) + i\ cos\ (a)\ sinh\ (b)$$

Equation real and imaginary parts on both sides

$$cos\,(\theta) = sin\,(a)\,cosh\,(b) \implies cosh\,(b) = \frac{cos\ (\theta)}{sin\ (a)} \text{...}\mathbf{(1)} sin\,(\theta) = cos\,(a)\,sin$$

Now

$$cosh^2\ (b) - sinh^2\ (b) = 1$$

$$\therefore \left[\frac{cos\ (\theta)}{sin\ (a)}\right]^2 - \left[\frac{sin\ (\theta)}{cos\ (a)}\right]^2 = 1 \text{....}\mathbf{By\ (1)\ and\ (2)}$$

$$\therefore cos^2\ (\theta)\ cos^2\ (a) - sin^2\ (\theta)\ sin^2\ (a) = sin^2\ (a)\ cos^2\ (a)$$

$$\therefore cos^2\ (\theta)\ cos^2\ (a) - (1 - cos^2\ (\theta))\ (1 - cos^2\ (a)) = (1 - cos^2\ (a))\ cos^2\ (a)$$

$$\therefore cos^2\ (\theta)\ cos^2\ (a) - 1 + cos^2\ (\theta) + cos^2\ (a) - cos^2\ (\theta)\ cos^2\ (a)$$

$$= cos^2\ (a) - cos^4\ (a)$$

$$\therefore -1 + cos^2\ (\theta) = -cos^4\ (a)$$

$$\therefore 1 - cos^2\ (\theta) = cos^4\ (a)$$

$$\therefore sin^2\ (\theta) = cos^4\ (a)$$

$$\therefore cos^2\ (a) = sin\ (\theta)$$

$$\therefore cos\ (a) = \pm\ \sqrt{sin\ (\theta)}$$

$$\therefore a = cos^{-1}\ \left(\pm\ \sqrt{sin\ (\theta)}\right) \text{....}\mathbf{(3)}$$

Substituting (3) in (2), we have

$$sin^2\ (\theta) = cos^2\ (a)\ sinh^2\ (b)$$

$$sin^2\ (\theta) = sin\ (\theta)\ sinh^2\ (b)$$

$$sin\ (\theta) = sinh^2\ (b)$$

$$sinh\ (b) = \pm\ \sqrt{sin\ (\theta)}$$

$$b = sinh^{-1}\left[\pm\ \sqrt{sin\ (\theta)}\right] \text{....}\mathbf{(4)}$$

By (3) and (4)

$$sin^{-1}(e^{i\theta}) = cos^{-1}\ \left(\pm\ \sqrt{sin\ (\theta)}\right) + i\ sinh^{-1}\left[\pm\ \sqrt{sin\ (\theta)}\right]$$

$$sin^{-1}(e^{i\theta}) = cos^{-1}\ \left(\pm\ \sqrt{sin\ (\theta)}\right) + i\ log\left(\pm\ \sqrt{sin\ (\theta)} + \sqrt{sin\ (\theta) + 1}\right)$$

(c) Let

$$tan^{-1}(e^{i\theta}) = a + ib$$
$$\implies tan^{-1}(e^{-i\theta}) = a - ib$$

Hence

$$e^{i\theta} = tan\ (a + ib)$$
$$e^{-i\theta} = tan\ (a - ib)$$

Now

$$\begin{aligned} tan\ (2a) &= tan\ [(a + ib) + (a - ib)] \\ &= \frac{tan\ (a + ib) + tan\ (a - ib)}{1 - tan\ (a + ib)\ tan\ (a - ib)} \\ &= \frac{e^{i\theta} + e^{-i\theta}}{1 - e^{i\theta}\ e^{-i\theta}} \\ &= \frac{2\ cos\ (\theta)}{0} \\ &= \infty \end{aligned}$$

$$\therefore tan\ (2a) = \infty$$
$$\therefore 2a = tan^{-1}\ (\infty)$$

$$\therefore 2a = (2n+1)\frac{\pi}{2}$$
$$\therefore a = (2n+1)\frac{\pi}{4}$$

Also

$$tan\ (2ib) = tan\ [(a+ib)-(a-ib)]$$
$$= \frac{tan\ (a+ib) - tan\ (a-ib)}{1 + tan\ (a+ib)\ tan\ (a-ib)}$$
$$= \frac{e^{i\theta} - e^{-i\theta}}{1 + e^{i\theta}\ e^{-i\theta}}$$
$$= \frac{2i\ sin\ (\theta)}{2}$$
$$= i\ sin\ (\theta)$$
$$\therefore tan\ (2ib) = i\ sin\ (\theta)$$
$$\therefore i\ tanh\ (2b) = i\ sin\ (\theta)$$
$$\therefore tanh\ (2b) = sin\ (\theta)$$
$$\therefore 2b = tanh^{-1}\ (sin\ (\theta))$$
$$\therefore 2b = \frac{1}{2}log\ \left(\frac{1 + sin\ (\theta)}{1 - sin\ (\theta)}\right)$$
$$\therefore b = \frac{1}{4}log\ \left(\frac{1 + sin\ (\theta)}{1 - sin\ (\theta)}\right)$$

Hence

$$tan^{-1}(e^{i\theta}) = (2n+1)\frac{\pi}{4} + i\ \frac{1}{4}log\ \left(\frac{1 + sin\ (\theta)}{1 - sin\ (\theta)}\right)$$

$$tan^{-1}(e^{i\theta}) = \frac{1}{4}\ \left[(2n+1)\pi + i\ log\ \left(\frac{1+sin\ (\theta)}{1-sin\ (\theta)}\right)\right]$$

Example 3

If $cos\ (x+iy) = \alpha + i\ \beta$ Prove that

(a)

$$\frac{\alpha^2}{cosh^2\ y} + \frac{\beta^2}{sinh^2\ y} = 1$$

(b)

$$\frac{\alpha^2}{cos^2\ x} - \frac{\beta^2}{sin^2\ x} = 1$$

Solution

Given

$$cos\ (x+iy) = \alpha + i\ \beta$$

$$\therefore cos\ (x)\ cos\ (iy) - sin\ (x)\ sin\ (iy) = \alpha + i\ \beta$$

$$\therefore cos\ (x)\ cosh\ (y) - i\ sin\ (x)\ sinh\ (y) = \alpha + i\ \beta$$

Equation real and imaginary parts on both sides

$$cos\ (x)\ cosh\ (y) = \alpha.....(1)$$

.and

$$-sin\ (x)\ sinh\ (y) = \beta.......(2)$$

From (1) and (2),

$$cos\ (x) = \frac{\alpha}{cosh\ (y))}$$

and

$$sin\ (x) = \frac{-\beta}{sinh\ (y)}$$

Eliminating x using

$$cos^2\ (x) + sin^2\ (x) = 1$$

$$\left[\frac{\alpha}{cosh\ (y))}\right]^2 - \left[\frac{-\beta}{sinh\ (y)}\right]^2 = 1$$

$$\frac{\alpha^2}{cosh^2\ y} + \frac{\beta^2}{sinh^2\ y} = 1$$

Again from (1) and (2),

$$cosh\ (y) = \frac{\alpha}{cos\ (x)}$$

and

$$sinh\ (y) = \frac{-\beta}{sin\ (x)}$$

Eliminating y using

$$cosh^2\ (y) - sinh^2\ (y) = 1$$

$$\left[\frac{\alpha}{cos\ (x)}\right]^2 - \left[\frac{-\beta}{sin\ (x)}\right]^2 = 1$$

$$\frac{\alpha^2}{cos^2\ x} - \frac{\beta^2}{sin^2\ x} = 1$$

Logarithm of a Complex Numbers

- If z and w are two complex numbers and $z = e^w$ then $w = log\ (z)$ is called logarithm of complex number z

- Let
$$z = x + iy = r\ e^{i\theta}$$
where $r = \sqrt{x^2 + y^2}$ and $\theta = tan^{-1}\frac{y}{x}$
$$\therefore log\ (z) = log(re^{i\theta}) = log\ r + log\ e^{i\theta}$$

$$\therefore log\ (z) = log(r) + i\theta = log\ \sqrt{x^2 + y^2} + i\ tan^{-1}\frac{y}{x}$$

- Hence
$$log\ (x + iy) = log(r) + i\theta = \frac{1}{2}\ log\ (x^2 + y^2) + i\ tan^{-1}\left(\frac{y}{x}\right)$$
which is known as **Principal value of Logarithm of complex Number**

- The **General Value of Logarithm of complex Number** is denoted by $Log\ (z)$ and is defimed as
$$Log\ (x + iy) = log(r) + i\ (\theta + 2\ n\pi)$$
$$Log\ (x + iy) = \frac{1}{2}\ log\ (x^2 + y^2) + i\ (tan^{-1}\left(\frac{y}{x}\right) + 2\ n\pi)$$
$$Log\ (z) = log\ (z) + i\ (2\ n\pi)$$

EXAMPLES

Example 1

Find the value of

(a) $log\ (-3)$

(b) $log_{(-2)}\ (-3)$...(HW)

(c) $log_2\ (-5)$

(d) $log\ (i)$...(HW)

(e) $log\ (i^i)$

(f) $sin[log\ (i^i)]$...(HW)

(g) $cos[log\ (i^i)]$

(h) $log\ (1+i)$...(HW)

(i) $Log_i\ (i)$

(j) $Log\ (1+i) + Log\ (1-i)$...(HW)

Solution

(a)

$$log(-3) = log(-3 + i\ 0)$$

$$= log\ \sqrt{(-3)^2 + 0^2} + i\ tan^{-1}\ (\frac{0}{-3})....(By Definition)$$

$$\therefore log(-3) = log\ 3 + i\ \pi$$

(c)

$$log_{(2)}(-5) = \frac{log(-5)}{log\ 2}.....(\because\ log_n m = \frac{log\ m}{log\ n})$$

$$log_{(2)}(-5) = \frac{log(-5 + i\ 0)}{log\ 2}$$

$$= \frac{log\ \sqrt{(-5)^2 + 0^2} + i\ tan^{-1}\ (\frac{0}{-5})}{log\ 2}$$

$$= \frac{log\ 5 + i\ \pi}{log\ 2}$$

(e)

$$log(i^i) = i\ log(i) = i\ log(0 + i\ 1)$$

$$= i \left(\ log\ \sqrt{0^2 + 1^2} + i\ tan^{-1}\ (\frac{1}{0})\right) ...(By Definition)$$

$$= i\ \left(\ log\ 1 + i\ tan^{-1}\ \infty\right)$$

$$= i\ \left(0 + i\ \frac{\pi}{2}\right)$$

$$= i^2 \left(\frac{\pi}{2}\right)$$

$$= -\frac{\pi}{2}$$

$$log(i^i) = -\frac{\pi}{2}$$

(g)

$$cos[log(i^i)] = cos[i\ log(i)] = cos[i\ log(0 + i\ 1)]$$

$$= cos[i\left(\ log\ \sqrt{0^2+1^2} + i\ tan^{-1}\ (\frac{1}{0})\right)]...(By Definition)$$

$$= cos[i\ \left(\ log\ 1 + i\ tan^{-1}\ \infty\right)]$$

$$= cos[i\ \left(0 + i\ \frac{\pi}{2}\right)]$$

$$= cos[i^2\left(\frac{\pi}{2}\right)]$$

$$= cos[-\frac{\pi}{2}]$$

$$cos[log(i^i)] = cos[\frac{\pi}{2}] = 0$$

(i)

$$Log_i(i) = \frac{Log\ i}{Log\ i}$$

$$= \frac{log\ i + i\ 2\ n\pi}{log\ i + i\ 2\ m\pi}$$

$$= \frac{log(0+1\ i) + i\ 2\ n\pi}{log(0+1\ i) + i\ 2\ m\pi}$$

$$= \frac{log\sqrt{0^2+1^2} + i\ tan^{-1}\frac{1}{0} + i\ 2\ n\pi}{log\sqrt{0^2+1^2} +\ \ tan^{-1}\frac{1}{0} + i\ 2\ m\pi}$$

$$= \frac{log1 + i\ tan^{-1}\infty + i\ 2\ n\pi}{log1 + i\ tan^{-1}\infty + i\ 2\ m\pi} = \frac{\frac{i\ \pi}{2}(1+4n)}{\frac{i\ \pi}{2}(1+4m)}$$

$$= \frac{(1+4n)}{(1+4m)}$$

Example 2 Simplify

(a) $log\left(e^{(i\theta)} + e^{(i\phi)}\right)$

(b) $i\ log\left(\frac{x-i}{x+i}\right)$...(HW)

(a)

$$log\left(e^{(i\theta)} + e^{(i\phi)}\right)$$
$$= log\left[(cos\ \theta + i\ sin\ \theta) + (cos\ \phi + i\ sin\ \phi)\right]$$
$$= log\left[(cos\ \theta + cos\ \phi) + i\ (sin\ \theta + \ sin\ \phi)\right]$$
$$= log\left[2\ cos\left(\frac{\theta+\phi}{2}\right)\ cos\left(\frac{\theta-\phi}{2}\right) + i\ 2\ sin\left(\frac{\theta+\phi}{2}\right) cos\left(\frac{\theta-\phi}{2}\right)\right]$$
$$= log\left[2\ cos\ \left(\frac{\theta-\phi}{2}\right)\ \left(cos\ \left(\frac{\theta+\phi}{2}\right) + i\ sin\ \left(\frac{\theta+\phi}{2}\right)\right)\right]$$
$$= log\left[2\ cos\ \left(\frac{\theta-\phi}{2}\right)\ \left(e^{i\ \left(\frac{\theta+\phi}{2}\right)}\right)\right]$$
$$= log\left[2\ cos\ \left(\frac{\theta-\phi}{2}\right)\right] + log\left(e^{i\ \left(\frac{\theta+\phi}{2}\right)}\right)$$
$$= log\left[2\ cos\ \left(\frac{\theta-\phi}{2}\right)\right] + i\ \left(\frac{\theta+\phi}{2}\right)$$

Example 3 Separate into real and imaginary Parts

(a) $Log(3+4i)$...(HW)

(b) $\frac{(a+ib)^{x+iy}}{(a-ib)^{x-iy}}$

(c) $(1+i\sqrt{3})^{1+i\sqrt{3}}$

Solution (b)

Let

$$\frac{(a+ib)^{x+iy}}{(a-ib)^{x-iy}} = A+iB$$

Taking *log* on both sides

$$log\left[\frac{(a+ib)^{x+iy}}{(a-ib)^{x-iy}}\right] = log(A+iB)$$

$$\therefore log\left[(a+ib)^{x+iy}\right] - log\left[(a-ib)^{x-iy}\right] = log(A+iB)$$

$$\therefore (x+iy)log\left[(a+ib)\right] - (x-iy)log\left[(a-ib)\right] = log(A+iB)$$

$$\therefore (x+iy)log\left[\sqrt{a^2+b^2}+itan^{-1}\left(\frac{b}{a}\right)\right]$$

$$-(x-iy)log\left[\sqrt{a^2+b^2}-itan^{-1}\left(\frac{b}{a}\right)\right] = log(A+iB)$$

$$\therefore xlog\left[\sqrt{a^2+b^2}\right]+xitan^{-1}\left(\frac{b}{a}\right)+iylog\left[\sqrt{a^2+b^2}\right]+iyitan^{-1}\left(\frac{b}{a}\right)$$

$$-xlog\left[\sqrt{a^2+b^2}+xitan^{-1}\left(\frac{b}{a}\right)+iylog\left[\sqrt{a^2+b^2}\right]-iyitan^{-1}\left(\frac{b}{a}\right)\right) = lo$$

$$\therefore 2xitan^{-1}\left(\frac{b}{a}\right)+2iylog\left[\sqrt{a^2+b^2}\right] = log(A+iB)$$

$$\therefore e^{i\left(2xtan^{-1}\left(\frac{b}{a}\right)+2ylog\left[\sqrt{a^2+b^2}\right]\right)} = A+iB$$

$$\therefore cos\left(2xtan^{-1}\left(\frac{b}{a}\right)+2ylog\left[\sqrt{a^2+b^2}\right]\right)$$

$$+isin\left(2xtan^{-1}\left(\frac{b}{a}\right)+2ylog\left[\sqrt{a^2+b^2}\right]\right) = A+iB$$

Equating real and imaginary parts on both sides

Real Part=

$$cos\left(2xtan^{-1}\left(\frac{b}{a}\right)+2ylog\left[\sqrt{a^2+b^2}\right]\right)$$

Imaginary Part=

$$sin\left(2xtan^{-1}\left(\frac{b}{a}\right)+2ylog\left[\sqrt{a^2+b^2}\right]\right)$$

Practice examples:

(1) Find $tanh\ x$ if

$$5\ sinh\ (x) - cosh\ (x) = 5$$

(2) Prove that

$$\left(\frac{1 + tanh\ x}{1 - tanh\ x}\right)^3 = cosh\ 6x + sinh\ 6x$$

(3) Prove that

$$cosh^5\ (x) = \frac{1}{16}\left[cosh\ (5x) + 5\ cosh\ (3x) + 10\ cosh\ (x)\right]$$

(4) Prove that

$$(cosh\ (x) + sinh\ (x))^n = cosh\ (nx) + sinh\ (nx)$$

(5) If $cos\ (\alpha).cosh\ (\beta) = \frac{x}{2}$ and $sin\ (\alpha).sinh\ (\beta) = \frac{y}{2}$ then show that

(a)

$$sec(\alpha - i\ \beta) + sec(\alpha + i\ \beta) = \frac{4x}{(x^2 + y^2)}$$

(b)

$$sec(\alpha - i\ \beta) - sec(\alpha + i\ \beta) = -\frac{4iy}{(x^2+y^2)}$$

(6) If $log(tan\ x) = y$ Prove that

(a)

$$sinh\ (ny) = \frac{1}{2}\ [tan^n\ (x) - cot^n\ (x)]$$

(b)

$$cosh\,[(n{+}1)y]{+}cosh\,[(n{-}1)y] = 2\,cosh\,(ny)\,cosec\,(2x)$$

(7) If $u = log\left[tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right]$ Prove that $cosh\ u.cos\ \theta = 1$

(8) Prove that

$$tanh^{-1}\ (sin\ \theta) = cosh^{-1}\ (sec\ \theta)$$

(9) Prove that

$$cosh^{-1}\ (\sqrt{1+x^2}) = tanh^{-1}\ (\frac{x}{\sqrt{1+x^2}})$$

(10) Separate into real and imaginary parts

$$sin^{-1}(cosec\theta)$$

(11) If $sinh\ (x+iy) = e^{i\theta}$ Prove that

(a)

$$sinh^4\ (x) = cos^2\ \theta$$

(b)

$$cos^2\ (y) = \pm cos\theta$$

(12) If $u+iv = cosh[\alpha+\frac{\pi}{4}]$ Find u^2-v^2

(13) If $sin\ (x+iy) = \alpha + i\ \beta$ Prove that

(a)

$$\frac{\alpha^2}{cosh^2\ y} + \frac{\beta^2}{sinh^2\ y} = 1$$

(b)

$$\frac{\alpha^2}{sin^2\ x} - \frac{\beta^2}{cos^2\ x} = 1$$

(14) Prove that

(a)

$$log\left(\frac{1}{1-e^{i\alpha}}\right) = log\left(\frac{1}{2}cosec\frac{\alpha}{2}\right) + i\ \left(\frac{\pi}{2} - \frac{\alpha}{2}\right)$$

(b)

$$log\left[\frac{sin(x+iy)}{sin(x-iy)}\right] = 2i\ tan^{-1}(cot\ xtanh\ y)$$

(15) If $e^{i\alpha} = i^{\beta}$, Prove that $\frac{\alpha}{\beta} = 2\ n\pi + \frac{\pi}{2}$

(16) If $i^{log(1+i)} = A + iB$, Prove that one value of A is,$e^{-\frac{\pi^2}{8}}cos\left(\frac{\pi}{4}\ log\ 2\right)$