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1. Find minimum value of x^2+y^2, subject to the constraint $ax+by=c$

Soln:-

To find the minimum value of $f(x,y)=x^2+y^2$ subject to the constraint $g(x,y)=ax+by-c=0$

By Langrange function

$$L(x,y,\lambda) = f(x,y) + \lambda g(x,y)$$

$$\therefore L = x^2+y^2+\lambda(ax+by-c)$$

For minimum value

$$\frac{\partial L}{\partial x} = 2x+\lambda a = 0 \quad ; \quad \frac{\partial L}{\partial y} = 2y+\lambda b = 0$$

$$\therefore 2x+\lambda a = 0 \qquad 2y+\lambda b = 0$$

$$\therefore \lambda = \frac{-2x}{a} \quad -① \qquad \lambda = \frac{-2y}{b} \quad -②$$

from ① and ②

$$\therefore \frac{y}{b} = \frac{b}{a} \cdot x$$

Let $\frac{y}{b} = \frac{x}{a} = k$

$\therefore x = ak$ and $y = bk$

But $ax+by=c$ according to constraint

$$\therefore a^2k + b^2k = c$$

$$\therefore k = \frac{c}{a^2+b^2}$$

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The minimum value of x^2+y^2 subject to constraint $ax+by=c$ is

$$x^2+y^2 = a^2k^2 + b^2k^2$$
$$= k^2(a^2+b^2)$$
$$= \frac{c^2}{(a^2+b^2)^2}(a^2+b^2)$$
$$= \frac{c^2}{a^2+b^2}$$

$\therefore$ Hence, the required minimum value of x^2+y^2 subject to constraint $ax+by=c$ is $\frac{c^2}{a^2+b^2}$.

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3] Find the maximum distance from the origin $(0,0)$ to the curve $3x^2+3y^2+4xy-2=0$ using Lagrange's Method.

Soln:- Let (x,y) be any point on the given surface $3x^2+3y^2+4xy-2=0$

To find (x,y) such that their distance $d=\sqrt{(x-0)^2+\dots}$

$\therefore d=\sqrt{x^2+y^2}$ from origin is maximum

subject to their constraint $g(x,y)=3x^2+3y^2+4xy\dots$

We maximize $f(x,y)=d^2=x^2+y^2$ — (1)

Subject to a constraint $g(x,y)=3x^2+3y^2+4xy-2$

Using Langrange Multiplier Method

For Langrange Function,

$L(x,y,\lambda)=f(x,y)+\lambda g(x,y)$

$\therefore L=x^2+y^2+\lambda(3x^2+3y^2+4xy-2)$

For maximum values,

$\frac{\partial L}{\partial x}=2x+\lambda(6x+4y)=0$; $\frac{\partial L}{\partial y}=2y+\lambda(6y+4x)=\dots$

subject to constraint is $3x^2+3y^2+4xy-2=\dots$

$\therefore$ we have

$2x+\lambda(6x+4y)=0$ and $2y+\lambda(6y+4x)=\dots$

$2x(1+3\lambda)=-4\lambda y$ $\qquad$ $2y(1+3\lambda)=-4\lambda x$ $\dots$

$\therefore y=\frac{-x(1+3\lambda)}{2\lambda}$ — (2)

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$\therefore (y+1)$ is a factor $f(y)$

Dividing $y^4 - 42y^2 + 64y + 105$ by $(y+1)$

$$
\begin{array}{r}
y^3 - y^2 - 41y + 105 \\
y+1 \,\big)\, y^4 - 0y^3 - 42y^2 + 64y + 105 \\
-(y^4 + y^3) \\
-y^3 - 42y^2 \\
-(-y^3 - y^2) \\
-41y^2 + 64y \\
-(-41y^2 - 41y) \\
105y + 105 \\
-(105y + 105) \\
0
\end{array}
$$

$\therefore y^4 - 42y^2 + 64y + 105 = (y+1)(y^3 - y^2 - 41y + 105)$ —(

Let $g(y) = y^3 - y^2 - 41y + 105$

$g(3) = (3)^2 - (3)^2 - 41(3) + 105 = 0$

$\therefore (y-3)$ is a factor of $y^3 - y^2 - 41y + 105$

Dividing $y^3 - y^2 - 41y + 105$ by $(y-3)$

$$
\begin{array}{r}
y^2 + 2y - 35 \\
y-3 \,\big)\, y^3 - y^2 - 41y + 105 \\
-(y^3 - 3y^2) \\
2y^2 - 41y \\
-(2y^2 - 6y) \\
-35y + 105 \\
-(-35y + 105) \\
0
\end{array}
$$

$\therefore y^3 - y^2 - 41y + 105 = (y+3)(y^2 + 2y - 35)$

$= (y+3)(y+7)(y-5)$ — (4)

∴ putting (4) in (3), we get

$$\therefore y^4 - 42y^2 + 64y + 105 = (y+1)(y+7)(y-3)(y-5)$$

$\therefore y = -7$, $y = -1$, $y = 3$, $y = 5$

(1) Putting $y = -7$ in eqn (3)

$$x = \frac{21 - (-7)^2}{4} = -7 \qquad \therefore x = -7$$

(2) Putting $y = -1$ in eqn (3)

$$x = \frac{21 - (-1)^2}{4} = 5 \qquad \therefore x = 5$$

(3) Putting $y = 3$ in eq (3)

$$x = \frac{21 - (3)^2}{4} = 3 \qquad \therefore x = 3$$

(4) Putting $y = 5$ in eqn (3)

$$x = \frac{21 - (5)^2}{4} = -1 \qquad \therefore x = -1$$

Hence, the critical points of this function are $(-7,-7)$, $(5,-1)$, $(3,3)$, $(-1,5)$.

Critical points	$(-7,-7)$	$(5,-1)$	$(3,3)$	$(-1,5)$
D	$=36(-7)(-7)-144$ $=1620>0$	$=36(5)(-1)-144$ $=-324<0$	$36(3)(3)-144$ $=180>0$	$36(-1)(5)-144$ $-324<0$
r	$=6(-7)$ $-42<0$	$6(5)$ $30>0$	$6(3)$ $18>0$	$6(-1)$ $-6<0$

Hence from the above table we can say that,

① $(5,-1)$ and $(-1,5)$ are saddle points.

② The maximum of this function is at $(-7,-7)$

Value at max. extreme $=(-7)^3+(-7)^3-63(-7-7)+12(-7)(-7)$
$=784$

③ The minimum of this function is at $(3,3)$

Value at min. extreme $=(3)^3+(3)^3-63(3+3)+12(3)(3)$
$=-216$

Ans. 1 The max. value of the function is 784.
2 The min. value of the function is -216.

3] Find the maximum distance from the origin $(0,0)$ to the curve $3x^2+3y^2+4xy-2=0$ using Lagrange's Method.

Soln:- Let (x,y) be any point on the given surface $3x^2+3y^2+4xy-2=0$

To find (x,y) such that their distance $d=\sqrt{(x-0)^2+\ldots}$
$\therefore d=\sqrt{x^2+y^2}$ from origin is maximum subject to their constraint $g(x,y)=3x^2+3y^2+4xy$

We maximize $f(x,y)=d^2=x^2+y^2$ — ①

Subject to a constraint $g(x,y)=3x^2+3y^2+4xy-2$
Using Langrange Multiplier Method

For Langrange Functions,

$$L(x,y,\lambda)=f(x,y)+\lambda g(x,y)$$

$$\therefore L=x^2+y^2+\lambda\left(3x^2+3y^2+4xy-2\right)$$

For maximum values,

$$\frac{\partial L}{\partial x}=2x+\lambda(6x+4y)=0 \quad ; \quad \frac{\partial L}{\partial y}=2y+\lambda(6y+4x)=0$$

subject to constraint is $3x^2+3y^2+4xy-2=0$

$\therefore$ we have

$$2x+\lambda(6x+4y)=0 \quad \text{and} \quad 2y+\lambda(6y+4x)=0$$

$$2x(1+3\lambda)=-4\lambda y \qquad 2y(1+3\lambda)=-4\lambda x \quad \text{③}$$

$$\therefore y=\frac{-x(1+3\lambda)}{2\lambda} \quad \text{— ②}$$

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Putting eqn (2) in (3)

$$2\left(\frac{-x(1+3s)}{2s}\right)(1+3s) + 4sx = 0$$

$$-9s^2x - 6sx - x + 4s^2x = 0$$

$$\therefore +5s^2 + 6s + 1 = 0$$

$$(5s+1)(s+1) = 0$$

$$\therefore s = \frac{-1}{5} \quad \text{or} \quad s = -1$$

(1) Putting $s = \frac{-1}{5}$ in eqn (2) and (3) we have

$$y = \frac{5x}{2}\left(\frac{2}{5}\right) \qquad 2y\left(\frac{2}{5}\right) = -4\left(\frac{-1}{5}\right)x$$

$$\therefore y = x \quad -(4) \qquad \therefore y = x$$

(2) Putting $s = -1$ in eqn (2) and (3) we have

$$y = \frac{+x}{2}(-2) \qquad 2y(-2) = -4(-1)x$$

$$\therefore y = -x \quad -(5) \qquad \therefore y = -x$$

from (4) and (5) we get that $y = \pm x$

Putting $y = \pm x$ in the constraint

(i) $y = x$

$$3(x)^2 + 3(x)^2 + 4(x)(x) - 2 = 0$$

$$10x^2 - 2 = 0$$

$$\therefore x^2 = \frac{1}{5}$$

$$\therefore x = \pm\frac{1}{\sqrt{5}}$$

(ii) $y = -x$

$$3x^2 + 3(-x)^2 + 4(x)(-x) - 2 = 0$$

$$2x^2 - 2 = 0$$

$$\therefore (x-1)(x+1) = 0$$

$$\therefore x = \pm 1$$

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and $y = x = \pm \frac{1}{\sqrt{5}}$ $y = -x = \mp 1$

$\therefore f(x, y) = d^2 = x^2 + y^2$ takes on extreme values on curve at 4 points $\left(\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$, $\left(\frac{-1}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right)$, $(1, -1)$, $(-1, 1)$. f_{max} at $(-1, 1)$ and $(1, -1)$

Hence the maximum distance from the origin to the given curve is $d = \sqrt{x^2 + y^2}$

$$= \sqrt{1^2 + (-1)^2}$$

$$= \sqrt{2}$$

$\therefore$ Maximum distance from origin to curve is $\sqrt{2}$ unit

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4] A soldier is placed at a point $(3,4)$ wants to shoot the fighter plane of an enemy which is flying along the curve $y=x^2+4$ when it is nearest to him, find such distance.

Solⁿ:- Let $f(x,y)$ be the nearest point on the curve $y=x^2+4$ from $(3,4)$ and distance between them be D.

$$\therefore D^2 = (x-3)^2 + (y-4)^2 \quad \text{—— ①}$$

Putting $y = x^2+4$ in eqⁿ ①

$$\therefore D^2 = (x-3)^2 + (x^2+4-4)^2$$

$$\therefore D = \sqrt{(x-3)^2 + x^4}$$

For obtaining the nearest point

$$\frac{\partial D}{\partial x} = 0$$

$$\therefore \frac{\partial D}{\partial x} = \frac{2(x-3)+4x^3}{2\sqrt{(x-3)^2+x^4}} = 0$$

$$\therefore 4x^3+2x-6 = 0$$

Let $g(x) = 4x^3+2x-6$

$g(1) = 4(1)^3+2(1)-6 = 0$

$\therefore (x-1)$ is a factor of $4x^3+2x-6$

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Dividing $4x^3+2x-6$ by $(x-1)$

$$
\begin{array}{r}
4x^2+4x+6 \\
x-1 \,\big)\, 4x^3+0x^2+2x-6 \\
-(4x^3-4x^2) \\
\hline
4x^2+2x \\
-(4x^2-4x) \\
\hline
6x-6 \\
-(6x-6) \\
\hline
0 \\
\hline
\end{array}
$$

$\therefore 4x^3+2x-6 = (x-1)(4x^2+4x+6) = 0$

$\therefore x = 1$ or $4x^2+4x+6 = 0$

But $4x^2+4x+6 > 0$ for all x

As $\Delta < 0$

$\therefore x = 1$

$\therefore y = (1)^2+4 = 5$

$\therefore f(x,y) = (1,5)$ are the nearest points

$\therefore D = \sqrt{(1-3)^2+(1)^4} = \sqrt{5} = 2.236$ units

Ans:- The nearest distance according to the given condition is 2.236 units.

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5] Find nth derivative of

$e^{2x} \sin\left(\frac{x}{2}\right) \cos\left(\frac{x}{2}\right) \sin(3x)$

Let $y = e^{2x} \sin\left(\frac{x}{2}\right) \cos\left(\frac{x}{2}\right) \cdot \sin 3x$

$= \frac{e^{2x}}{2} \cdot 2\sin\left(\frac{x}{2}\right) \cos\left(\frac{x}{2}\right) \sin 3x$

$= \frac{e^{2x}}{2} \sin(x) \sin(3x)$

$= \frac{e^{2x}}{4} 2\sin(3x) \sin(x)$ $\left[\because 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) = \sin\right.$

$= \frac{e^{2x}}{4} \left[\cos 2x - \cos 4x\right]$

$[\because 2\sin a \sin b = \cos(a-b) - \cos(a+$

$\therefore y = \frac{e^{2x}}{4} \left[\cos 2x - \cos 4x\right]$

Using successive differentiation,

Given expression is of type $y = e^{ax} \cos(bx+c)$

$\therefore y_n = \left[\left(\sqrt{a^2+b}\right)^n e^{ax} \cos bx + c + n\tan^{-1}\left(\frac{b}{a}\right)\right]$

Here

$y_n = \frac{1}{4}\left[\left(2\sqrt{2}\right)^n e^{ax} \cos\left(2x + n\tan^{-1}(1)\right) + \left(2\sqrt{5}\right)^n e^{2x} \cos\left(4x + n\tan^{-1} 2\right)\right]$

$\therefore y_n = \frac{e^{2x}}{4}\left[\left(2\sqrt{2}\right)^n \cos\left(2x + \frac{n\pi}{4}\right) + 2\sqrt{5} \cos\left(4x + n\tan^{-1}(2\right.\right.$

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6] If $x = \sin\theta$ and $y = \cos m\theta$, then prove that
$(1-x^2)y_{n+2} - (2n+1)x y_{n+1} + (m^2-n^2)y_n = 0$

Solⁿ:
$$\frac{dx}{d\theta} = \cos\theta = \sqrt{1-x^2}$$

$$\frac{dy}{\partial\theta} = -m\sin m\theta = -m\sqrt{1-y^2}$$

$$\frac{dy}{dx} = \frac{-m\sqrt{1-y^2}}{\sqrt{1-x^2}} = y_1$$

$$y_1^2(1-x^2) = m^2(1-y^2)$$

Differentiating w.r.t x

$$(1-x^2)2y_1y_2 - 2xy_1^2 = -2yy_1m^2$$

$$(1-x^2)y_2 - xy_1 = -ym^2$$

$$y_2(1-x^2) - xy_1 = -m^2y$$

$$\therefore y_2(1-x^2) - xy_1 + m^2y = 0$$

By Leibnitz Rule, successive differentiation upto n

$$(1-x^2)y_{n+2} + (-2xn)y_{n+1} - xy_{n+1} - n(n-1)y_n + m^2y_n = 0$$

$$\therefore (1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$$

Hence Proved

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7] If $y = \sec^{-1}x$

Prove that $x(x^2-1)y_{n+2} + [(2+3n)x^2 - (n+1)]y_{n+1} + n(3n+1)xy_n + n^2(n-1)y_{n-1} = 0$

Soln:- $y = \sec^{-1}x$ — ①

$x = \sec y$ — ②

Differentiating w.r.t x

$1 = \sec y \tan y \; y_1$

$\therefore \tan y \,(y_1) = \frac{1}{\sec y}$

$\frac{1}{x} = \sqrt{x^2-1}\; y_1$ from ② and $(\sec^2 y - 1 = \tan^2 y)$

$\frac{1}{x^2} = (x^2-1)(y_1)^2$ — ③

Differentiating w.r.t x

$-\frac{2}{x^3} = 2(x^2-1)y_1 y_2 + 2xy_1^2$

$-\frac{1}{x}\left(\frac{1}{x^2}\right) = (x^2-1)\, y_1 y_2 + xy_1^2$

$-\frac{1}{x}(x^2-1)y_1^2 = (x^2-1)\, y_2 y_1 + xy_1^2$ from ③

$\therefore -(x^2-1)y_1^2 = x(x^2-1)y_2 y_1 + x^2 y_1^2$

$\therefore y_1(2x^2-1) + y_2 x(x^2-1) = 0$

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Using Leibnitz theorem,

For $x(x^2-1)y_2 + y_1(2x^2-1) = 0$

$$x(x^2-1)\,y_{n+2} + \left[(3n+2)x^2-(n+1)\right]y_{n+1} + \left[\frac{6n(n+1)}{2}+4n\right]xy_n$$

$$+ \left(6n(n-1)(n-2) + \frac{4n}{2}(n-1)\,y_{n-1}\right)y_{n-1} = 0$$

$$\therefore\ x(x^2-1)y_{n+2} + \left[(2+3n)x^2-(n+1)\right]y_{n+1} + n(3n+1)xy_n$$

$$+\ n^2(n-1)y_{n-1} = 0$$

Hence proved.