

COMPLEX NUMBERS

PREFACE

- The real number system had limitations that were at first accepted and later overcome by a series of improvements in both concepts and mechanics.
- In connection with, quadratic, equations we encountered the situation where there is no real numbers satisfying specific quadratic equations that requires square root of negative numbers.
- as a result, a broader number system was devised so that such equations possess its solution in that number system, which was eventually named as **Complex Numbers** by Carl Friedrich Gauss.

INTRODUCTION

- A complex number is a number of the form $a + bi$ where a and b are real and $i^2 = -1$ or $i = \sqrt{-1}$. The letter 'a' is called the real part and 'b' is called the imaginary part of $a + bi$
- If $a = 0$, the number ib is said to be a purely imaginary number and if $b = 0$ the number a is real.
Hence, real numbers and pure imaginary numbers are special cases of complex numbers.
- The complex numbers are denoted by Z , i.e., $Z = a + bi$. In coordinate form, $Z = (a, b)$

Note : Every real number is a complex number with 0 as its imaginary part.

PROPERTIES OF COMPLEX NUMBERS

- The two complex numbers $a + bi$ and $c + di$ are equal if and only if $a = c$ and $b = d$
for example if.

$$(x-2) + 4yi = 3 + 12i$$

$$x-2 = 3, \quad y = 3$$

$$x = 5 \quad y = 3$$

- If any complex number vanishes then its real and imaginary parts will separately vanish
for example if

$$a + ib = 0$$

then

$$a = -i b$$

Squaring both sides, we have

$$a^2 = -b^2$$

$$a^2 + b^2 = 0$$

which hold true only when

$$a = 0, b = 0$$

CONJUGATE OF A COMPLEX NUMBER

Two complex numbers are called the conjugates of each other if their real parts are equal and their imaginary parts differ only in sign

- If $Z = a + bi$ the complex number $a - bi$ is called the conjugate of Z it is denoted by \bar{Z} i.e.,

$$\bar{Z} = \overline{a + bi} = a - bi$$

- Moreover $Z \cdot \bar{Z} = (a + bi) \cdot (a - bi) = a^2 + b^2$
- $Z + \bar{Z} = 2a$ and $Z - \bar{Z} = 2bi$

Result

If Z_1 and Z_2 are complex numbers, then

$$(1) \quad \overline{Z_1 + Z_2} = \overline{Z_1} + \overline{Z_2}$$

$$(2) \quad \overline{Z_1 \cdot Z_2} = \overline{Z_1} \cdot \overline{Z_2}$$

$$(3) \quad \overline{\left(\frac{Z_1}{Z_2}\right)} = \frac{\overline{Z_1}}{\overline{Z_2}}$$

POWER OF i

we know that

$$i^3 = i^2 \cdot i = (-1) \cdot i = -i$$

$$i^4 = (i^2)^2 = ((-1)^2) = 1$$

$$i^5 = (i^2)^2 \cdot i = ((-1)^2) \cdot i = i$$

$$i^6 = (i^4)^2 = ((-1)^3) = -1$$

Also, we have

$$i^{-1} = \frac{1}{i} \cdot \frac{i}{i} = \frac{i}{-1} = -i$$

$$i^{-2} = \frac{1}{i^2} = \frac{1}{-1} = -1$$

$$i^{-3} = \frac{1}{i^3} = \frac{1}{-i} = \frac{1}{-i} \cdot \frac{i}{i} = i$$

$$i^{-4} = \frac{1}{i^4} = \frac{1}{1} = 1$$

In general, for any integer k

$$i^{4k} = 1, i^{4k+1} = i, i^{4k+2} = -1, i^{4k+3} = -i$$

ALGEBRA OF COMPLEX NUMBERS

There are four algebraic operations of complex numbers

If $Z_1 = a_1 + b_1i$ and $Z_2 = a_2 + b_2i$ then

(1) Addition

$$\begin{aligned} Z_1 + Z_2 &= (a_1 + b_1i) + (a_2 + b_2i) \\ &= (a_1 + a_2) + i(b_1 + b_2) \end{aligned}$$

(2) Subtraction

$$\begin{aligned} Z_1 - Z_2 &= (a_1 + b_1i) - (a_2 + b_2i) \\ &= (a_1 - a_2) + i(b_1 - b_2) \end{aligned}$$

(3) Multiplication

$$\begin{aligned} Z_1 * Z_2 &= (a_1 + b_1i) \cdot (a_2 + b_2i) \\ &= a_1a_2 + b_1b_2i^2 + a_1b_2i + b_1a_2i \\ &= (a_1a_2 - b_1b_2) + i(a_1b_2 + b_1a_2) \end{aligned}$$

(4) Division where $Z_2 \neq 0$

$$\frac{z_1}{z_2} = \frac{a_1 + b_1i}{a_2 + b_2i}$$

Multiply Numerator and denominator by the number $a_2 - b_2i$ in order to make the denominator real.

$$\frac{z_1}{z_2} = \frac{(a_1 + b_1i)}{(a_2 + b_2i)} * \frac{a_2 - b_2i}{a_2 - b_2i}$$

$$= \frac{a_1 a_2 + b_1 b_2}{(a_2)^2 + (b_2)^2} + i \frac{b_1 a_2 - a_1 b_2}{(a_2)^2 + (b_2)^2}$$

Generally result will be expressed in the form $a + ib$

EXAMPLES Example 1

Prove that if the sum and product of two complex numbers are real , then the two numbers must be real or complex conjugates of each other

OR

Example 1

Show that $z_1 + z_2$ and $z_1 \cdot z_2$ are both real , then either z_1 and z_2 are both real or $z_2 = \overline{z_1}$

Solution

Let z_1 and z_2 be two complex numbers defined as $z_1 = a_1 + i b_1$ and $z_2 = a_2 + i b_2$

$$z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$$

Given $z_1 + z_2$ is real

$$b_1 + b_2 = 0$$

hence

$$b_1 = -b_2 \dots \dots (1) \text{ Also}$$

$$z_1 \cdot z_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$$

Given $z_1 \cdot z_2$ is real

$$a_1 b_2 + a_2 b_1 = 0 \dots \dots (2)$$

Using (1) in (2)

$$a_1 b_2 - a_2 b_1 = 0$$

$$b_2(a_1 - a_2) = 0$$

$$b_2 = 0 \text{ or } (a_1 - a_2) = 0$$

Case (i) $b_2 = 0$ hence $b_1 = 0$ (By (1))

$z_1 = a_1$ and $z_2 = a_2$

Two Complex Numbers are real

Case (ii) $(a_1 - a_2) = 0$

$a_1 = a_2$ and by (1) $b_1 = -b_2$

$z_1 = a_1 + i b_1$ and $z_2 = a_2 + i b_2 = a_1 - i b_1$

Two Complex Numbers are complex conjugates of each other

Hence Proved

ARGAND PLANE

- We know that corresponding to each ordered pair of real numbers (x, y) we get a unique point in the XY plane and vice-versa with reference to a set of mutually perpendicular lines known as the $x - axis$ and the $y - axis$
- The complex number $x + iy$ which corresponds to the ordered pair (x, y) can be represented geometrically as the unique point $P(x, y)$ in the $XY - plane$ and vice-versa

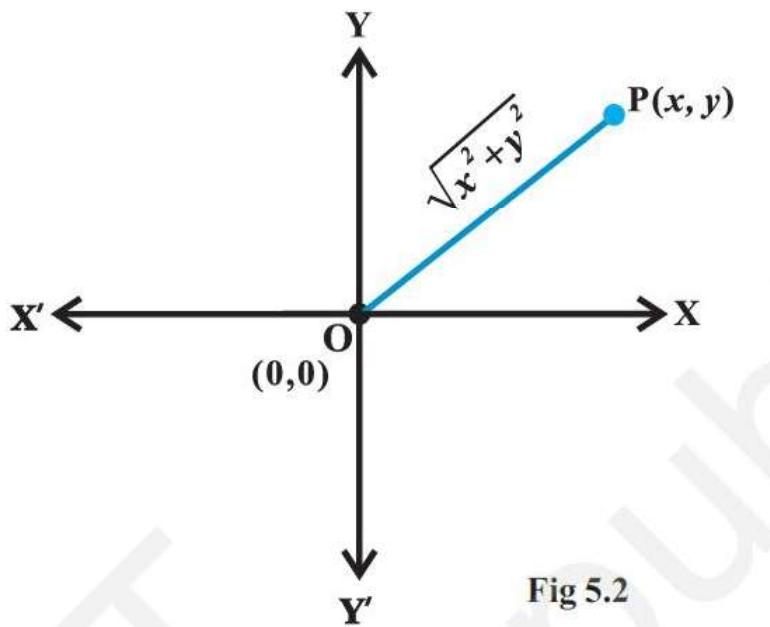


Fig 5.2

- The plane having complex number assigned to each of its points is called the ***Complex plane*** or ***Argand Plane***
- The $x - axis$ and $y - axis$ in the Argand plane are called ***Real axis*** and ***Imaginary axis*** respectively
- In the Argand Plane the ***Modulus*** of a complex number $x + iy = \sqrt{x^2 + y^2}$ is defined as the distance of a point $P(x, y)$ from

the origin $O(0, 0)$

- The points on real axis corresponds to complex number of the form $x + i0$ and points on imaginary axis corresponds to complex number of the form $0 + iy$

RESULT For complex numbers Z_1 and Z_2

$$(1) |Z_1 \cdot Z_2| = |Z_1| |Z_2|$$

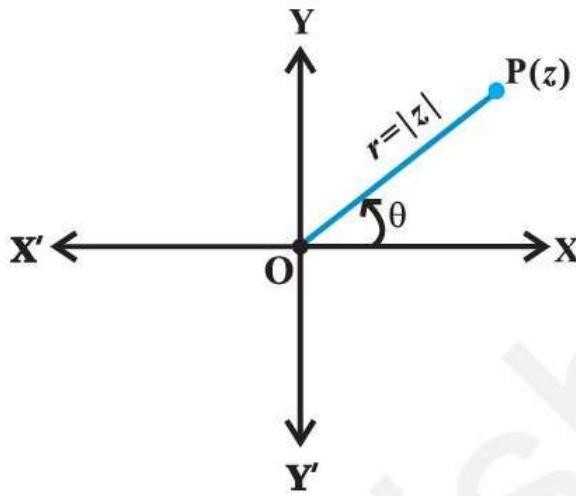
$$(2) \left| \frac{Z_1}{Z_2} \right| = \frac{|Z_1|}{|Z_2|}$$

$$(3) |Z_1 + Z_2| < |Z_1| + |Z_2|$$

$$(4) |Z_1 - Z_2| > |Z_1| - |Z_2|$$

POLAR REPRESENTATION

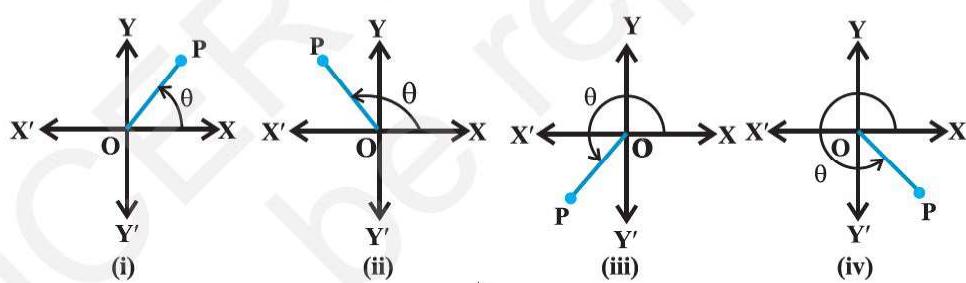
- Consider a point $P(x, y)$ representing a complex number $z = x + iy$
- The point P can be located in polar co-ordinate system by its distance from the pole say $r = |z|$ and its angular position with respect to +ve side of $X - axis$ as θ



- Considering polar coordinates for P
 $x = r \cos \theta$ and $y = r \sin \theta$
 $x^2 + y^2 = r^2$
- Hence for complex Number $z = x + iy$
 $z = x + iy = r \cos \theta + r \sin \theta = r(\cos \theta + i \sin \theta)$
- This expression is known as **Polar Form** of the complex Number z where
 - (1) $|z| = r$ is known as **modulus** of a complex number,defined as $r = \sqrt{x^2 + y^2}$

(2) θ is known as **argument or amplitude** of complex number defined as $\theta = \tan^{-1}|\frac{y}{x}|$

- The argument θ can have infinite number of values, the values lies in the interval $(-\pi, \pi]$ is known as **Principal value** of the argument
- For any complex number $z \neq 0$, there corresponds only one value of θ in $[0, 2\pi)$



However for any interval of length 2π , say $(-\pi, \pi]$, can be such interval where we have Principal value of the argument.

- In four quadrants, arguments of a complex numbers are defined as follows

Let $P(x, y)$ be a point representing a complex number $z = x + iy$ and let α be the angle that \overline{OP} makes with positive x axis then

- For $x > 0, y > 0$ (i.e. Point in first quadrant) $\arg(z) = \theta = \alpha = \tan^{-1}|\frac{y}{x}|$
fig(i)
- For $x < 0, y > 0$ (i.e. Point in second quadrant) $\arg(z) = \theta = \pi - \alpha = \pi - \tan^{-1}|\frac{y}{x}|$

fig(ii)

- For $x < 0, y < 0$ (i.e. Point in third quadrant) $\arg(z) = \theta = \pi + \alpha = \pi + \tan^{-1}|\frac{y}{x}|$

fig(iii)

- For $x > 0, y < 0$ (i.e. Point in fourth quadrant) $\arg(z) = \theta = 2\pi - \alpha = 2\pi - \tan^{-1}|\frac{y}{x}| = -\tan^{-1}|\frac{y}{x}| = -\alpha$

fig(iv)

Use of Polar Form in Multiplication and Division Let $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$ then

(1)

$$\arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2)$$

OR

$$z_1 \cdot z_2 = r_1 \cdot r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$$

(2)

$$\arg\left(\frac{z_1}{z_2}\right) = \frac{\arg(z_1)}{\arg(z_2)}$$

where

$$z_2 \neq 0$$

OR

$$\left(\frac{z_1}{z_2}\right) = \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]$$

EXPONENTIAL FORM

The polar form of complex number is given by

$$z = r(\cos \theta + i \sin \theta)$$

Also Euler's formula is given by

$$e^{i\theta} = \cos \theta + i \sin \theta$$

and

$$e^{-i\theta} = \cos \theta - i \sin \theta$$

Using Euler's formula in polar form of complex number, we have

$$z = r e^{i\theta}$$

which is known as **exponential form** of complex number

Example 2 Prove that, for any complex numbers z_1 and z_2

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$$

Solution

Let $z_1 = a + i b$ and $z_2 = c + i d$ be two complex numbers

$$|z_1 + z_2|^2 = (a + c)^2 + (b + d)^2 \text{ and } |z_1 - z_2|^2 = (a - c)^2 + (b - d)^2$$

$$\begin{aligned} |z_1 + z_2|^2 + |z_1 - z_2|^2 &= (a + c)^2 + (b + d)^2 + (a - c)^2 + (b - d)^2 \\ &= 2a^2 + 2b^2 + 2c^2 + 2d^2 \end{aligned}$$

$$= 2[a^2 + b^2 + c^2 + d^2] \dots (1)$$

Also $|z_1|^2 = a^2 + b^2$ and $|z_2|^2 = c^2 + d^2$

$$2[|z_1|^2 + |z_2|^2] = 2[a^2 + b^2 + c^2 + d^2] \dots (2)$$

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$$

Example 3

If $\arg(z + 1) = \frac{\pi}{6}$ and $\arg(z - 1) = \frac{2\pi}{3}$, then find Complex Number z

Solution

$$\text{Given } \arg(z + 1) = \frac{\pi}{6}$$

$$\text{Let } z = a + i b$$

Then

$$z + 1 = (a + i b) + 1 = (a + 1) + i b$$

$$\arg(z + 1) = \tan^{-1}\left(\frac{b}{a+1}\right) = \frac{\pi}{6}$$

$$\frac{b}{a+1} = \frac{1}{\sqrt{3}}$$

$$\sqrt{3}b = a + 1$$

$$a - \sqrt{3}b = -1 \dots (1)$$

Also, $\arg(z - 1) = \frac{2\pi}{3}$ Let $z = a + i b$

Then $z - 1 = (a + i b) - 1 = (a - 1) + i b$

$$\arg(z - 1) = \tan^{-1}\left(\frac{b}{a-1}\right) = \frac{2\pi}{3}$$

$$\frac{b}{a-1} = \tan\left(\frac{2\pi}{3}\right)$$

$$\frac{b}{a-1} = -\sqrt{3}$$

$$b = -\sqrt{3}a + \sqrt{3} \dots\dots\dots(2)$$

Using (2) in (1)

$$a - \sqrt{3}(-\sqrt{3}a + \sqrt{3}) = -1$$

$$a + 3a - 3 = -1$$

$$4a = -2$$

$$a = \frac{1}{2}$$

$$b = \frac{\sqrt{3}}{2} \dots\dots\dots(By(2))$$

Hence required complex number is

$$z = \frac{1}{2} + i \frac{\sqrt{3}}{2}$$

DE MOIVRE'S THEOREM

For any two complex numbers with $r_1 = r_2 = 1$

$$(\cos \theta_1 + i\sin \theta_1) \cdot (\cos \theta_2 + i\sin \theta_2)$$

$$= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$$

In particular, if $\theta = \theta_1 = \theta_2$ we have

$$(\cos \theta + i\sin \theta)^2 = \cos 2\theta + i\sin 2\theta$$

For any positive integer n , by induction on n , the result may be generalized as

$$(\cos \theta + i\sin \theta)^n = \cos n\theta + i\sin n\theta$$

and this is known as the **DeMoivre's Theorem for integer index**

For any negative integer $n = -m$, where m is positive

$$(\cos \theta + i\sin \theta)^n = (\cos \theta + i\sin \theta)^{-m}$$

$$= \frac{1}{(\cos \theta + i\sin \theta)^m}$$

$$= \frac{1}{\cos m\theta + i\sin m\theta}$$

$$= \frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta}$$

$$\begin{aligned}
&= \cos m\theta - i \sin m\theta \\
&= \cos (-m)\theta + i \sin (-m)\theta \\
&= \cos n\theta + i \sin n\theta
\end{aligned}$$

Hence result also holds for negative integers n

For any rational number $n = \frac{p}{q}$, where $p, q \in \mathbb{Z}$

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$$

Hence result also holds for rational number n

In general, for any real number θ , rational number n , one of the

solution of $(\cos \theta + i \sin \theta)^n$ is $\cos n\theta + i \sin n\theta$

APPLICATION OF DE MOIVRE'S THEOREM

- Power of complex Numbers
- Expansion of $\cos^n \theta$ and $\sin^n \theta$ in terms of sines and cosines multiples of θ
- Expansion of $\cos (n\theta)$ and $\sin (n\theta)$ in terms of powers of $\sin \theta$ and $\cos \theta$
- Roots of complex Numbers

Powers of complex Numbers Properties

- $z^m \cdot z^n = z^{m+n}$

- $\frac{z^m}{z^n} = z^{m-n}$

- $(z^m)^n = z^{mn}$

Example 1

Prove that $(1 + \cos\theta + i\sin\theta)^n + (1 + \cos\theta - i\sin\theta)^n = 2^{n+1} \cos^n(\frac{\theta}{2}) \cos(\frac{n\theta}{2})$

Solution

$$\begin{aligned}
 & (1 + \cos\theta + i\sin\theta)^n + (1 + \cos\theta - i\sin\theta)^n \\
 &= [2\cos^2(\frac{\theta}{2}) + 2i\cos(\frac{\theta}{2})\sin(\frac{\theta}{2})]^n + [2\cos^2(\frac{\theta}{2}) - 2i\cos(\frac{\theta}{2})\sin(\frac{\theta}{2})]^n \\
 &= [2\cos(\frac{\theta}{2})(\cos(\frac{\theta}{2}) + i\sin(\frac{\theta}{2}))]^n + [2\cos(\frac{\theta}{2})(\cos(\frac{\theta}{2}) - i\sin(\frac{\theta}{2}))]^n \\
 &= 2^n \cos^n(\frac{\theta}{2}) [[\cos(\frac{\theta}{2}) + i\sin(\frac{\theta}{2})]^n + [\cos(\frac{\theta}{2}) - i\sin(\frac{\theta}{2})]^n] \\
 &= 2^n \cos^n(\frac{\theta}{2}) [\cos(\frac{n\theta}{2}) + i\sin(\frac{n\theta}{2})] + [\cos(\frac{n\theta}{2}) - i\sin(\frac{n\theta}{2})]
 \end{aligned}$$

(By De Moivre's Theorem)

$$= 2^n \cos^n(\frac{\theta}{2}) (2\cos(\frac{n\theta}{2}))$$

$$= 2^{n+1} \cos^n(\frac{\theta}{2}) (2\cos(\frac{n\theta}{2}))$$

Example 2

If $2 \cos \theta = x + \frac{1}{x}$; Prove that $x^r + \frac{1}{x^r} = 2 \cos r\theta$

Solution

Given

$$2 \cos \theta = x + \frac{1}{x}$$

$$\frac{x^2 + 1}{x} = 2 \cos \theta$$

$$x^2 - (2 \cos \theta)x + 1 = 0$$

which is quadratic equation in x whose roots are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \cos \theta \pm i \sin \theta$$

Case(i)

$$x = \cos \theta + i \sin \theta$$

$$x^r = (\cos \theta + i \sin \theta)^r$$

$$x^r = \cos r\theta + i \sin r\theta \dots (\text{By De Moivre's theorem})$$

Also

$$x^{-r} = (\cos \theta + i \sin \theta)^{-r}$$

$$x^{-r} = \cos (-r)\theta + i \sin (-r)\theta \dots (\text{By De Moivre's theorem})$$

$$x^{-r} = \cos r\theta - i \sin r\theta$$

Hence

$$x^r + \frac{1}{x^r} = \cos r\theta + i \sin r\theta + \cos r\theta - i \sin r\theta$$

$$x^r + \frac{1}{x^r} = 2 \cos r\theta$$

Hence result holds true in this case. **Case(ii)**

$$x = \cos \theta - i \sin \theta$$

$$x^r = (\cos \theta - i \sin \theta)^r$$

$$x^r = \cos r\theta - i \sin r\theta \dots (\text{By De Moivre's theorem})$$

Also

$$x^{-r} = (\cos \theta - i \sin \theta)^{-r}$$

$$x^{-r} = \cos (-r)\theta - i \sin (-r)\theta \dots (\text{By De Moivre's theorem})$$

$$x^{-r} = \cos r\theta + i \sin r\theta$$

Hence

$$x^r + \frac{1}{x^r} = \cos r\theta - i \sin r\theta + \cos r\theta + i \sin r\theta$$

$$x^r + \frac{1}{x^r} = 2 \cos r\theta$$

Hence result holds true in this case.

Example 3

If $\alpha = 1 + i$; $\beta = 1 - i$ and $\cot \theta = x + 1$, Prove that

$$(x + \alpha)^n - (x + \beta)^n = (\alpha - \beta) \sin n\theta \cosec^n \theta$$

Solution

Given

$$\alpha = 1 + i; \beta = 1 - i \text{ and } \cot \theta = x + 1 \text{ hence } x = \cot \theta - 1$$

$$x + \alpha$$

$$= \cot \theta - 1 + 1 + i$$

$$= \cot \theta + i$$

$$= \frac{\cos \theta + i \sin \theta}{\sin \theta}$$

$$\text{Also } x + \beta$$

$$= \cot \theta - 1 + 1 - i$$

$$= \cot \theta - i$$

$$= \frac{\cos \theta - i \sin \theta}{\sin \theta}$$

$$\text{Hence } (x + \alpha)^n - (x + \beta)^n$$

$$= \left[\frac{\cos \theta + i \sin \theta}{\sin \theta} \right]^n - \left[\frac{\cos \theta - i \sin \theta}{\sin \theta} \right]^n$$

$$= \cosec^n \theta [(\cos \theta + i \sin \theta)^n - (\cos \theta - i \sin \theta)^n]$$

$$= \cosec^n \theta [\cos n\theta + i \sin n\theta - \cos n\theta + i \sin n\theta]$$

(By De Moivre's theorem)

$$= \cosec^n \theta [2i \sin n\theta]$$

$$= 2i \sin n\theta \cosec^n \theta$$

$$= (\alpha - \beta) \sin n\theta \cosec^n \theta \text{ (By given data)}$$

Hence proved.

Example 4

Prove that

$$(1 - e^{i\theta})^{-\frac{1}{2}} + (1 - e^{-i\theta})^{-\frac{1}{2}} = [1 + \operatorname{cosec}(\frac{\theta}{2})]^{\frac{1}{2}}$$

Solution

By Euler's Formula

$$e^{i\theta} = \cos \theta + i \sin \theta$$

and

$$e^{-i\theta} = \cos \theta - i \sin \theta$$

$$\begin{aligned} & \text{Hence } (1 - e^{i\theta})^{-\frac{1}{2}} + (1 - e^{-i\theta})^{-\frac{1}{2}} \\ &= (\cos \theta + i \sin \theta)^{-\frac{1}{2}} + (\cos \theta - i \sin \theta)^{-\frac{1}{2}} \\ &= [2\sin^2(\frac{\theta}{2}) - 2i \sin(\frac{\theta}{2})\cos(\frac{\theta}{2})]^{-\frac{1}{2}} + [2\sin^2(\frac{\theta}{2}) + 2i \sin(\frac{\theta}{2})\cos(\frac{\theta}{2})]^{-\frac{1}{2}} \\ &= [2\sin(\frac{\theta}{2})(\sin(\frac{\theta}{2}) - i \cos(\frac{\theta}{2}))]^{-\frac{1}{2}} + [2\sin(\frac{\theta}{2})(\sin(\frac{\theta}{2}) + i \cos(\frac{\theta}{2}))]^{-\frac{1}{2}} \\ &= [2\sin(\frac{\theta}{2})(\cos(\frac{\pi}{2} - \frac{\theta}{2}) - i \sin(\frac{\pi}{2} - \frac{\theta}{2}))]^{-\frac{1}{2}} + \\ & \quad [2\sin(\frac{\theta}{2})(\cos(\frac{\pi}{2} - \frac{\theta}{2}) + i \sin(\frac{\pi}{2} - \frac{\theta}{2}))]^{-\frac{1}{2}} \end{aligned}$$

Let $\frac{\pi}{2} - \frac{\theta}{2} = \alpha$ then above result can be written as

$$\begin{aligned}
& (1 - e^{i\theta})^{-\frac{1}{2}} + (1 - e^{-i\theta})^{-\frac{1}{2}} \\
&= [2\sin(\frac{\theta}{2})(\cos(\alpha) - i \sin(\alpha))]^{-\frac{1}{2}} + [2\sin(\frac{\theta}{2})(\cos(\alpha) + i \sin(\alpha))]^{-\frac{1}{2}} \\
&= [2\sin(\frac{\theta}{2})]^{-\frac{1}{2}} [\cos(\frac{\alpha}{2}) + i \sin(\frac{\alpha}{2}) + \cos(\frac{\alpha}{2}) - i \sin(\frac{\alpha}{2})] \\
&= \frac{1}{2\sin(\frac{\theta}{2})^{\frac{1}{2}}} [2 \cos(\frac{\alpha}{2})] \\
&= \frac{[4 \cos^2(\frac{\alpha}{2})]^{\frac{1}{2}}}{[2\sin(\frac{\theta}{2})]^{\frac{1}{2}}} \\
&= \left[\frac{1+\cos \alpha}{\sin(\frac{\theta}{2})} \right]^{\frac{1}{2}} \\
&= \left[\frac{1+\cos (\frac{\pi}{2}-\frac{\theta}{2})}{\sin(\frac{\theta}{2})} \right]^{\frac{1}{2}} \\
&= \left[\frac{1+\sin(\frac{\theta}{2})}{\sin(\frac{\theta}{2})} \right]^{\frac{1}{2}} \\
&= [1 + \operatorname{cosec}(\frac{\theta}{2})]^{\frac{1}{2}}
\end{aligned}$$

Hence proved

Example 5

Find 10th Power of $\sqrt{3} + i$ Using De Moivre's Theorem and express

your answer in standard form

Solution

Given $z = \sqrt{3} + i$

So point (x, y) lies in first quadrant

$$r = \sqrt{3+1} = 2 \text{ and } \theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

Hence Polar form : $z = r(\cos\theta + i \sin\theta) = 2\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$

Taking 10^{th} power on both the sides, by De Moivre's theorem

$$z^{10} = [2\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)]^{10}$$

$$= 2^{10}\left[\cos\left(\frac{10\pi}{6}\right) + i\sin\left(\frac{10\pi}{6}\right)\right]$$

$$= 2^{10}\left[\cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right)\right]$$

$$= 2^{10}\left[\cos\left(2\pi - \frac{\pi}{3}\right) + i\sin\left(2\pi - \frac{\pi}{3}\right)\right]$$

$$= 2^{10}\left[\cos\left(\frac{\pi}{3}\right) - i\sin\left(\frac{\pi}{3}\right)\right]$$

$$= 2^{10}\left[\frac{1}{2} - i\frac{\sqrt{3}}{2}\right]$$

$$= 2^9[1 - \sqrt{3}i]$$

$$\text{Hence } z^{10} = 512[1 - \sqrt{3}i]$$

Example 6

Express $z = (1 + 7i)(2 - i)^{-2}$ in polar form and prove that second

power of z is an imaginary number and fourth power is a negative real number.

Solution

Let

$$z = (1 + 7i)(2 - i)^{-2}$$

$$= \frac{1+7i}{(2-i)^2}$$

$$= \frac{1+7i}{3-4i}$$

$$= \frac{1+7i}{3-4i} \cdot \frac{3+4i}{3+4i}$$

$$= \frac{-25+25i}{25}$$

$$z = -1 + i$$

Hence Point lies in second quadrant

$$\text{so } r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$

$$\text{and } \arg(z) = \theta = \pi - \tan^{-1}\left|\frac{1}{1}\right| = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

Hence polar form of z is given by

$$\begin{aligned} z &= r(\cos \theta + i \sin \theta) \\ &= \sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i \sin\left(\frac{3\pi}{4}\right)\right) \end{aligned}$$

$$= \sqrt{2}\left(\cos\left(\pi - \frac{\pi}{4}\right) + i \sin\left(\pi - \frac{\pi}{4}\right)\right)$$

$$z = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right)\right)$$

By De Moivre's Theorem

$$z^2 = \left[\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right)\right)\right]^2$$

$$\begin{aligned} z^2 &= (\sqrt{2})^2 \left[\left(\cos\left(\frac{2\pi}{4}\right) + i \sin\left(\frac{2\pi}{4}\right)\right)\right] \\ z^2 &= 2 \left[\left(\cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right)\right)\right] \end{aligned}$$

$$z^2 = 2(0 + i)$$

$z^2 = 2i$ which is purely imaginary number

Also

$$z^4 = \left[\sqrt{2}(\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4})) \right]^4$$

$$z^4 = (\sqrt{2})^4 \left[(\cos(\frac{4\pi}{4}) + i\sin(\frac{4\pi}{4})) \right]$$

$$z^4 = 4[(\cos(\pi) + i\sin(\pi))]$$

$$z^4 = 4(-1 + 0)$$

$z^4 = -4$ which is negative real number

Expansion of $\cos^n\theta$ and $\sin^n\theta$ in terms of sine and Cosine multiples of θ (Concept)

- By Euler's Formula

$$z = e^{i\theta} = \cos \theta + i \sin \theta$$

$$\frac{1}{z} = e^{-i\theta} = \cos \theta - i \sin \theta$$

- Also by De Moivre's theorem

$$z^m = e^{im\theta} = \cos m\theta + i \sin m\theta$$

$$\frac{1}{z^m} = e^{-im\theta} = \cos m\theta - i \sin m\theta$$

- Then

$$z + \frac{1}{z} = 2 \cos \theta \dots (1) \quad z - \frac{1}{z} = 2i \sin \theta \dots (2)$$

$$z^m + \frac{1}{z^m} = 2 \cos m\theta \dots (3) \quad z^m - \frac{1}{z^m} = 2i \sin m\theta \dots (4)$$

- Expressing $(z + \frac{1}{z})^n$ or $(z - \frac{1}{z})^n$ using (1) or (2) and expanding by Binomial expansion, we can merge the result in $z^m + \frac{1}{z^m}$ or $z^m - \frac{1}{z^m}$ form
- Using (3) and (4) we can get expansion of $\cos^n \theta$ and $\sin^n \theta$ in terms of sine and Cosine multiples of θ

Expansion of $\cos n\theta$ and $\sin n\theta$ in terms of powers of $\sin \theta$ and $\cos \theta$ (Concept)

- To find expansion of $\cos n\theta$ and $\sin n\theta$ in terms of powers of $\sin \theta$ and $\cos \theta$, we write De Moivre's theorem in reverse order i.e $\cos n\theta + i \sin n\theta = (\cos \theta + i \sin \theta)^n$
- We expand $(\cos \theta + i \sin \theta)^n$ by Binomial expansion and write the expression in standard form $A + iB$

- Equating real and imaginary parts on both sides we get expansion of $\cos n\theta$ and $\sin n\theta$ respectively.

EXAMPLES

Example 1

Prove that $\cos^6\theta - \sin^6\theta = \frac{1}{16}[\cos 6\theta + 15\cos 2\theta]$

Solution

Using Euler's Formula , define

$$z = e^{i\theta} = \cos \theta + i \sin \theta$$

$$\frac{1}{z} = e^{-i\theta} = \cos \theta - i \sin \theta$$

Also by De Moivre's theorem

$$z^m = e^{im\theta} = \cos m\theta + i \sin m\theta$$

$$\frac{1}{z^m} = e^{-im\theta} = \cos m\theta - i \sin m\theta$$

Then

$$z + \frac{1}{z} = 2 \cos \theta \dots (1) \quad z - \frac{1}{z} = 2i \sin \theta \dots (2)$$

$$z^m + \frac{1}{z^m} = 2 \cos m\theta \dots (3) \quad z^m - \frac{1}{z^m} = 2i \sin m\theta \dots (4)$$

Consider

$$(2 \cos \theta)^6 = \left(z + \frac{1}{z}\right)^6 \dots \dots [\text{by (1)}]$$

$$= z^6 + {}^6 C_1 z^5 \left(\frac{1}{z}\right) + {}^6 C_2 z^4 \left(\frac{1}{z^2}\right) + {}^6 C_3 z^3 \left(\frac{1}{z^3}\right) + {}^6 C_4 z^2 \left(\frac{1}{z^4}\right)$$

$$+^6C_5 z \left(\frac{1}{z^5}\right) +^6 C_6 \left(\frac{1}{z^6}\right)$$

$$\begin{aligned} \therefore 2^6 \cos^6 \theta &= z^6 + 6 z^5 \left(\frac{1}{z}\right) + 15 z^4 \left(\frac{1}{z^2}\right) + 20 z^3 \left(\frac{1}{z^3}\right) + 15 z^2 \left(\frac{1}{z^4}\right) \\ &+ 6 z \left(\frac{1}{z^5}\right) + \left(\frac{1}{z^6}\right) \\ \therefore 2^6 \cos^6 \theta &= \left(z^6 + \frac{1}{z^6}\right) + 6 \left(z^4 + \frac{1}{z^4}\right) + 15 \left(z^2 + \frac{1}{z^2}\right) + 20 \end{aligned}$$

$$\therefore \cos^6 \theta = \frac{1}{2^6} [2 \cos 6\theta + 6 * 2 \cos 4\theta + 15 * 2 \cos 2\theta + 20] \dots [\text{By (3)}]$$

$$\cos^6 \theta = \frac{1}{2^5} [\cos 6\theta + 6 \cos 4\theta + 15 \cos 2\theta + 10] \dots (5)$$

Also Consider

$$(2i \sin \theta)^6 = \left(z - \frac{1}{z}\right)^6 \dots [\text{by (2)}]$$

$$\begin{aligned} &= z^6 - ^6C_1 z^5 \left(\frac{1}{z}\right) + ^6C_2 z^4 \left(\frac{1}{z^2}\right) \\ &- ^6C_3 z^3 \left(\frac{1}{z^3}\right) + ^6C_4 z^2 \left(\frac{1}{z^4}\right) \\ &- ^6C_5 z \left(\frac{1}{z^5}\right) + ^6C_6 \left(\frac{1}{z^6}\right) \end{aligned}$$

$$\begin{aligned} \therefore (2i)^6 \sin^6 \theta &= z^6 - 6 z^5 \left(\frac{1}{z}\right) + 15 z^4 \left(\frac{1}{z^2}\right) - 20 z^3 \left(\frac{1}{z^3}\right) + 15 z^2 \left(\frac{1}{z^4}\right) \\ &- 6 z \left(\frac{1}{z^5}\right) + \left(\frac{1}{z^6}\right) \end{aligned}$$

$$\begin{aligned} \therefore 2^6 i^6 \sin^6 \theta &= \left(z^6 + \frac{1}{z^6}\right) - 6 \left(z^4 + \frac{1}{z^4}\right) + 15 \left(z^2 + \frac{1}{z^2}\right) - 20 \\ \therefore 2^6 (-1) \sin^6 \theta &= [2 \cos 6\theta - 6 * 2 \cos 4\theta + 15 * 2 \cos 2\theta - 20] \dots [\text{By (3)}] \end{aligned}$$

$$\therefore \sin^6 \theta = \frac{-1}{2^5} [\cos 6\theta - 6 \cos 4\theta + 15 \cos 2\theta - 10] \dots (6)$$

Hence (5)-(6) gives

$$\begin{aligned}
& \cos^6 \theta - \sin^6 \theta \\
&= \frac{1}{2^5} [\cos 6\theta + \cos 6\theta + 6 \cos 4\theta - 6 \cos 4\theta + 15 \cos 2\theta + 15 \cos 2\theta + \\
&\quad 10 - 10] \\
&= \frac{1}{32} [2 \cos 6\theta + 30 \cos 2\theta]
\end{aligned}$$

$$\cos^6 \theta - \sin^6 \theta = \frac{1}{16} [\cos 6\theta + 15 \cos 2\theta]$$

Hence Proved.

Example 2

Prove that

$$2^6 \sin^4 \theta \cos^3 \theta = [\cos 7\theta - \cos 5\theta - \cos 3\theta + \cos \theta]$$

Solution

Using Euler's Formula , define

$$\begin{aligned}
z &= e^{i\theta} = \cos \theta + i \sin \theta \\
\frac{1}{z} &= e^{-i\theta} = \cos \theta - i \sin \theta
\end{aligned}$$

Also by De Moivre's theorem

$$\begin{aligned}
z^m &= e^{im\theta} = \cos m\theta + i \sin m\theta \\
\frac{1}{z^m} &= e^{-im\theta} = \cos m\theta - i \sin m\theta
\end{aligned}$$

Then

$$z + \frac{1}{z} = 2 \cos \theta \dots (1) \quad z - \frac{1}{z} = 2i \sin \theta \dots (2)$$

$$z^m + \frac{1}{z^m} = 2 \cos m\theta \dots (3) \quad z^m - \frac{1}{z^m} = 2i \sin m\theta \dots (4)$$

Consider $(2i \sin \theta)^4 (2 \cos \theta)^3 = (z - \frac{1}{z})^4 (z + \frac{1}{z})^3 \dots \dots \text{[by (1) and (2)]}$

$$\therefore 2^4 i^4 \sin^4 \theta 2^3 \cos^3 \theta = (z - \frac{1}{z}) \left[(z - \frac{1}{z})^3 (z + \frac{1}{z})^3 \right]$$

$$\begin{aligned} & \therefore 2^7 \sin^4 \theta \cos^3 \theta \\ &= (z - \frac{1}{z}) (z^2 - \frac{1}{z^2})^3 \end{aligned}$$

$$= (z - \frac{1}{z}) \left[(z^2)^3 - 3(z^2)^2 \frac{1}{z^2} + 3z^2 \frac{1}{(z^2)^2} - (\frac{1}{z^2})^3 \right]$$

$$= (z - \frac{1}{z}) \left[z^6 - 3z^2 + 3\frac{1}{z^2} - \frac{1}{z^6} \right]$$

$$= z^7 - 3z^3 + 3\frac{1}{z} - \frac{1}{z^5} - z^5 - 3z - 3\frac{1}{z^3} + \frac{1}{z^7}$$

$$= (z^7 + \frac{1}{z^7}) - 3(z^3 + \frac{1}{z^3}) + 3(z + \frac{1}{z}) - (z^5 + \frac{1}{z^5})$$

$$= 2 \cos 7\theta - 3 * 2 \cos 3\theta + 3 * 2 \cos \theta - 2 \cos 5\theta$$

$$\therefore 2^6 \sin^4 \theta \cos^3 \theta = \cos 7\theta - \cos 5\theta - 3 \cos 3\theta + 3 \cos \theta$$

Hence Proved.

Example 3

Expand $\frac{\sin 7\theta}{\sin \theta}$ in powers of $\sin \theta$

Solution

By De Moivre's theorem

$$\cos 7\theta + i \sin 7\theta = (\cos \theta + i \sin \theta)^7$$

$$= (\cos \theta)^7 + {}^7C_1 (\cos \theta)^6 (i \sin \theta) + {}^7C_2 (\cos \theta)^5 (i \sin \theta)^2 + {}^7C_3 (\cos \theta)^4 (i \sin \theta)^3 + {}^7C_4 (\cos \theta)^3 (i \sin \theta)^4 + {}^7C_5 (\cos \theta)^2 (i \sin \theta)^5 + {}^7C_6 (\cos \theta) (i \sin \theta)^6 + {}^7C_7 (i \sin \theta)^7$$

$$= \cos^7\theta + 7i \cos^6\theta \sin\theta - 21 \cos^5\theta \sin^2\theta - 35i \cos^4\theta \sin^3\theta + 35 \cos^3\theta \sin^4\theta + 21i \cos^2\theta \sin^5\theta - 7 \cos \theta \sin^6\theta - i \sin^7\theta$$

$$= (\cos^7\theta - 21 \cos^5\theta \sin^2\theta + 35 \cos^3\theta \sin^4\theta - 7 \cos \theta \sin^6\theta) + i (7 \cos^6\theta \sin\theta - 35 \cos^4\theta \sin^3\theta + 21 \cos^2\theta \sin^5\theta - \sin^7\theta)$$

$$\therefore \frac{\sin 7\theta}{\sin \theta} = 7 \cos^6\theta - 35 \cos^4\theta \sin^2\theta + 21 \cos^2\theta \sin^4\theta - \sin^6\theta$$

$$= 7(1 - \sin^2\theta)^3 - 35(1 - \sin^2\theta)^2 \sin^2\theta + 21(1 - \sin^2\theta) \sin^4\theta - \sin^6\theta$$

$$\therefore \frac{\sin 7\theta}{\sin \theta} = 7(1 - 3 \sin^2\theta + 3 \sin^4\theta - \sin^6\theta) - 35(1 - 2 \sin^2\theta +$$

$$\sin^4\theta) \sin^2\theta + 21(\sin^4\theta - \sin^6\theta) - \sin^6\theta$$

$$= 7 - 21 \sin^2\theta + 21 \sin^4\theta - 7 \sin^6\theta - 35 \sin^2\theta + 70 \sin^4\theta -$$

$$35 \sin^6\theta + 21 \sin^4\theta - 21 \sin^6\theta - \sin^6\theta$$

$$\therefore \frac{\sin 7\theta}{\sin \theta} = 7 - 56 \sin^2\theta + 112 \sin^4\theta - 64 \sin^6\theta$$

which is required expansion.

Example 4

Using De Moivre's theorem show that

$$2(1 + \cos 8\theta) = (x^4 - 4x^2 - 2)^2$$

where $x = 2 \cos \theta$

Solution

Consider

$$2(1 + \cos 8\theta) = 2 * 2 \cos^2 (4\theta) = (2 \cos 4\theta)^2 \dots (1)$$

By De Moivre's theorem

$$\cos 4\theta + i \sin 4\theta$$

$$= (\cos \theta + i \sin \theta)^4$$

$$= (\cos \theta)^4 + {}^4C_1 (\cos \theta)^3(i \sin \theta) + {}^4C_2 (\cos \theta)^2(i \sin \theta)^2 + {}^4C_3 (\cos \theta)(i \sin \theta)^3 + \sin^4 \theta$$

$$= \cos^4 \theta + 4i \cos^3 \theta \sin \theta - 6 \cos^2 \theta \sin^2 \theta - 4i \cos \theta \sin^3 \theta + \sin^4 \theta \\ \therefore \cos 4\theta + i \sin 4\theta$$

$$= (\cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta)$$

$$i (4 \cos^3 \theta \sin \theta - 4 \cos \theta \sin^3 \theta)$$

Equating real part on both sides, we have

$$\cos 4\theta = \cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta$$

$$\therefore 2 \cos 4\theta = 2 \cos^4 \theta - 12 \cos^2 \theta \sin^2 \theta + 2 \sin^4 \theta \\ = 2 \cos^4 \theta - 12 \cos^2 \theta (1 - \cos^2 \theta) + 2 (1 - \cos^2 \theta)^2$$

$$\begin{aligned}
&= 2 \cos^4 \theta - 12 \cos^2 \theta + 12 \cos^4 \theta + 2 (1 - 2 \cos^2 \theta + \cos^4 \theta) \\
&= 2 \cos^4 \theta - 12 \cos^2 \theta + 12 \cos^4 \theta + 2 - 4 \cos^2 \theta \\
&= 16 \cos^4 \theta - 16 \cos^2 \theta + 2
\end{aligned}$$

By (1) $2(1 + \cos 8\theta) = [16 \cos^4 \theta - 16 \cos^2 \theta + 2]^2$

$$= [(2 \cos \theta)^4 - 4(2 \cos \theta)^2 + 2]^2$$

$$\therefore 2(1 + \cos 8\theta) = (x^4 - 4x^2 - 2)^2$$

where $x = 2 \cos \theta$

Application of De Moirve's theorem to find Roots of Complex Numbers(Concept)

- The n th roots of a complex number z are the n values of w which satisfy the equation $w^n = z$.
- write $z = \cos \theta + i \sin \theta$ and assuming that the equation is satisfied by $w = \cos \phi + i \sin \phi$, then

$$w^n = z \implies (\cos \phi + i \sin \phi)^n = \cos n\phi + i \sin n\phi = \cos \theta + i \sin \theta$$

Equating real and imaginary parts on both sides, we have

$$\begin{aligned} \cos \theta &= \cos n\phi & \sin \theta &= \sin n\phi \\ \therefore n\phi &= 2k\pi + \theta \implies \phi &= \frac{2k\pi + \theta}{n} & k \in \mathbb{Z} \end{aligned}$$

- We obtain n distinct complex roots of z with the values of ϕ obtained above for $k = 0, 1, 2, \dots, n-1$ as for $k < 0$ and $k > n-1$ the root obtained is one of the root, mentioned above.
- Hence, the equation $w^n = z = r(\cos \theta + i \sin \theta)$ has n distinct complex roots, given by

$$w = r^{\frac{1}{n}} \left[\cos \left(\frac{2k\pi + \theta}{n} \right) + i \sin \left(\frac{2k\pi + \theta}{n} \right) \right] \quad k = 0, 1, 2, \dots, n-1$$

where θ is the principal argument.

Working Rule

- To find n distinct roots of Complex number by De Moirve's theorem, equation should be in one of the following form

$$x^n = \text{complex number} \dots (1)$$

OR

$$[f(x)]^n = \text{complex Number} \dots (2)$$

- Write the complex number in polar form using its modulus and principal argument

- write the generalized polar form of complex number as discussed before

- By (1) , n distinct roots of complex number is given by

$$x = r^{\frac{1}{n}} [\cos(2k\pi + \theta) + i \sin(2k\pi + \theta)]^{\frac{1}{n}} \quad k = 0, 1, 2 \dots n - 1$$

$$\therefore x = r^{\frac{1}{n}} \left[\cos \left(\frac{2k\pi + \theta}{n} \right) + i \sin \left(\frac{2k\pi + \theta}{n} \right) \right] \quad k = 0, 1, 2 \dots n - 1$$

(By De Moivre's theorem)

- substituting values of k we, get required n distinct roots

EXAMPLES

Example 1

Find the cube roots of unity (or 1) Show that if ω is complex cube root of unity, then $a) 1 + \omega + \omega^2 = 0 \quad b) (1 - \omega)^6 = -27$

Solution

Consider

$$x = \sqrt[3]{1}$$

$$\implies x^3 = 1$$

$$= \cos 0 + i \sin 0$$

$$= \cos (0 + 2k\pi) + i \sin (0 + 2k\pi)$$

$$\therefore x^3 = \cos (2k\pi) + i \sin (2k\pi)$$

$$\implies x_r = [\cos (2k\pi) + i \sin (2k\pi)]^{\frac{1}{3}}$$

$$= [\cos (\frac{2k\pi}{3}) + i \sin (\frac{2k\pi}{3})] \dots \text{where } k = 0, 1, 2$$

for $k = 0$ and $r = 0$

$$x_0 = \cos 0 + i \sin 0$$

$$x_0 = 1$$

for $k = 1$ and $r = 1$

$$x_1 = \cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right) \dots\dots (1)$$

$$= \cos\left(\pi - \frac{\pi}{3}\right) + i \sin\left(\pi - \frac{\pi}{3}\right)$$

$$= \left(-\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right)\right)$$

$$x_1 = -\frac{1}{2} + i \frac{\sqrt{3}}{2}$$

for $k = 2$ and $r = 2$

$$x_2 = \cos\left(\frac{4\pi}{3}\right) + i \sin\left(\frac{4\pi}{3}\right) \dots\dots (2)$$

$$= \cos\left(\pi + \frac{\pi}{3}\right) + i \sin\left(\pi + \frac{\pi}{3}\right)$$

$$= \left(-\cos\left(\frac{\pi}{3}\right) - i \sin\left(\frac{\pi}{3}\right)\right)$$

$$x_2 = -\frac{1}{2} - i \frac{\sqrt{3}}{2}$$

Hence required cube roots of unity are 1 , $-\frac{1}{2} + i \frac{\sqrt{3}}{2}$ and $-\frac{1}{2} - i \frac{\sqrt{3}}{2}$

Now by (1)

$$x_1 = \cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right)$$

$$x_1 = \omega \quad (\text{say})$$

Then by (2)

$$x_2 = \cos\left(\frac{4\pi}{3}\right) + i \sin\left(\frac{4\pi}{3}\right)$$

$$x_2 = \left[\cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right)\right]^2$$

$$x_2 = \omega^2 \quad (\text{say})$$

i.e. $x_0 = 1; x_1 = \omega; x_2 = \omega^2$

$$(1) 1 + \omega + \omega^2$$

$$= 1 + \cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right) + \cos\left(\frac{4\pi}{3}\right) + i \sin\left(\frac{4\pi}{3}\right)$$

$$= 1 - \cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right) - \cos\left(\frac{\pi}{3}\right) - i \sin\left(\frac{\pi}{3}\right)$$

$$= 1 - 1$$

Hence $1 + \omega + \omega^2 = 0$

$$(2)(1 - \omega)^6$$

$$= [(1 - \omega)^2]^3$$

$$= (1 - 2\omega + \omega^2)^3$$

$$= (-3\omega)^3 \dots (\text{By above Result})$$

Hence $(1 - \omega)^6 = -27$

Example 2

Find the roots, $\alpha, \alpha^2, \alpha^3, \alpha^4$ of equation $x^5 = 1$. Hence prove that

$$(1) (1 - \alpha)(1 - \alpha^2)(1 - \alpha^3)(1 - \alpha^4) = 5$$

$$(2) x^5 - 1 = (x - 1)(x^2 + 2x \cos(\frac{\pi}{5}) + 1)(x^2 + 2x \cos(\frac{3\pi}{5}) + 1) = 0$$

Solution

Consider

$$x^5 = 1$$

$$= \cos 0 + i \sin 0$$

$$= \cos(0 + 2k\pi) + i \sin(0 + 2k\pi)$$

$$\therefore x^5 = \cos(2k\pi) + i \sin(2k\pi)$$

$$\implies x_r = [\cos(2k\pi) + i \sin(2k\pi)]^{\frac{1}{5}}$$

$$= [\cos(\frac{2k\pi}{5}) + i \sin(\frac{2k\pi}{5})]$$

....where $k = 0, 1, 2, 3, 4$

for $k = 0$ and $r = 0$

$$x_0 = \cos 0 + i \sin 0$$

$$x_0 = 1$$

for $k = 1$ and $r = 1$

$$x_1 = \cos\left(\frac{2\pi}{5}\right) + i \sin\left(\frac{2\pi}{5}\right) = \alpha \quad (\text{say}) \dots \dots (1)$$

for $k = 2$ and $r = 2$

$$x_2 = \cos\left(\frac{4\pi}{5}\right) + i \sin\left(\frac{4\pi}{5}\right)$$

$$x_2 = [\cos\left(\frac{2\pi}{5}\right) + i \sin\left(\frac{2\pi}{5}\right)]^2 = \alpha^2 \quad (\text{say}) \dots \dots (2)$$

for $k = 3$ and $r = 3$

$$x_3 = \cos\left(\frac{6\pi}{5}\right) + i \sin\left(\frac{6\pi}{5}\right)$$

$$x_3 = [\cos\left(\frac{2\pi}{5}\right) + i \sin\left(\frac{2\pi}{5}\right)]^3 = \alpha^3 \quad (\text{say}) \dots \dots (3)$$

for $k = 4$ and $r = 4$

$$x_4 = \cos\left(\frac{8\pi}{5}\right) + i \sin\left(\frac{8\pi}{5}\right)$$

$$x_4 = [\cos\left(\frac{2\pi}{5}\right) + i \sin\left(\frac{2\pi}{5}\right)]^4 = \alpha^4 \quad (\text{say}) \dots \dots (4)$$

Hence (1),(2),(3) and (4) gives required roots of $x^5 - 1 = 0$

Now

$$x^5 - 1 = (x - 1)(x - \alpha)(x - \alpha^2)(x - \alpha^3)(x - \alpha^4)$$

$$\frac{x^5 - 1}{x - 1} = (x - \alpha)(x - \alpha^2)(x - \alpha^3)(x - \alpha^4)$$

$$1 + x + x^2 + x^3 + x^4 = (x - \alpha)(x - \alpha^2)(x - \alpha^3)(x - \alpha^4)$$

Since $x = 1$ is one of the root, let $x = 1$ in above expression

$$(1 - \alpha)(1 - \alpha^2)(1 - \alpha^3)(1 - \alpha^4) = 5 \dots\dots\dots (\mathbf{A})$$

Again by (2),(3)and (4)

$$\alpha^2 = \cos\left(\frac{4\pi}{5}\right) + i \sin\left(\frac{4\pi}{5}\right)$$

$$\alpha^2 = \cos\left(\pi - \frac{\pi}{5}\right) + i \sin\left(\pi - \frac{\pi}{5}\right)$$

$$\alpha^2 = -\cos\left(\frac{\pi}{5}\right) + i \sin\left(\frac{\pi}{5}\right) \dots\dots (\mathbf{5})$$

$$\alpha^3 = \cos\left(\frac{6\pi}{5}\right) + i \sin\left(\frac{6\pi}{5}\right)$$

$$\alpha^3 = \cos\left(\pi + \frac{\pi}{5}\right) + i \sin\left(\pi + \frac{\pi}{5}\right)$$

$$\alpha^3 = -\cos\left(\frac{\pi}{5}\right) - i \sin\left(\frac{\pi}{5}\right) \dots\dots (\mathbf{6})$$

$$\alpha^4 = \cos\left(\frac{8\pi}{5}\right) + i \sin\left(\frac{8\pi}{5}\right)$$

$$\alpha^4 = \cos\left(2\pi - \frac{2\pi}{5}\right) + i \sin\left(2\pi + \frac{2\pi}{5}\right)$$

$$\alpha^4 = \cos\left(\frac{2\pi}{5}\right) - i \sin\left(\frac{2\pi}{5}\right) \dots\dots (7)$$

Now by (2),(5),(6)and (7) , roots of the equation $x^5 - 1 = 0$ are of the form

$$x = -\cos\left(\frac{\pi}{5}\right) \pm i \sin\left(\frac{\pi}{5}\right) \quad \text{and}$$

$$x = \cos\left(\frac{2\pi}{5}\right) \pm i \sin\left(\frac{2\pi}{5}\right)$$

Let Case(1)

$$x = -\cos\left(\frac{\pi}{5}\right) \pm i \sin\left(\frac{\pi}{5}\right)$$

$$x + \cos\left(\frac{\pi}{5}\right) = \pm i \sin\left(\frac{\pi}{5}\right)$$

$$(x + \cos\left(\frac{\pi}{5}\right))^2 = i^2 \sin^2\left(\frac{\pi}{5}\right)$$

$$x^2 + 2 x \cos\left(\frac{\pi}{5}\right) + \cos^2\left(\frac{\pi}{5}\right) = -\sin^2\left(\frac{\pi}{5}\right)$$

$$x^2 + 2 x \cos\left(\frac{\pi}{5}\right) + \cos^2\left(\frac{\pi}{5}\right) + \sin^2\left(\frac{\pi}{5}\right) = 0$$

$$x^2 + 2 x \cos\left(\frac{\pi}{5}\right) + 1 = 0 \dots (8)$$

Also Case(2)

$$x = \cos\left(\frac{2\pi}{5}\right) \pm i \sin\left(\frac{2\pi}{5}\right)$$

$$x - \cos\left(\frac{2\pi}{5}\right) = \pm i \sin\left(\frac{2\pi}{5}\right)$$

$$(x - \cos\left(\frac{2\pi}{5}\right))^2 = i^2 \sin^2\left(\frac{2\pi}{5}\right)$$

$$x^2 - 2x \cos\left(\frac{2\pi}{5}\right) + \cos^2\left(\frac{2\pi}{5}\right) = -\sin^2\left(\frac{2\pi}{5}\right)$$

$$x^2 - 2x \cos\left(\frac{2\pi}{5}\right) + \cos^2\left(\frac{2\pi}{5}\right) + \sin^2\left(\frac{2\pi}{5}\right) = 0$$

$$x^2 - 2x \cos\left(\frac{2\pi}{5}\right) + 1 = 0$$

$$x^2 - 2x \cos\left(\pi - \frac{3\pi}{5}\right) + 1 = 0$$

$$x^2 + 2x \cos\left(\frac{3\pi}{5}\right) + 1 = 0 \dots \text{(9)}$$

substituting (8) and (9) in

$$x^5 - 1 = (x - 1)(x - \alpha)(x - \alpha^2)(x - \alpha^3)(x - \alpha^4) = 0$$

$$x^5 - 1 = (x - 1)(x^2 + 2x \cos\left(\frac{\pi}{5}\right) + 1)(x^2 + 2x \cos\left(\frac{3\pi}{5}\right) + 1) = 0 \dots \text{(B)}$$

(A) and (B) are required results

Example 3 Find continued product of the roots of $\left(\frac{1}{2} - i \frac{\sqrt{3}}{2}\right)^{\frac{3}{4}}$

Solution

Consider

$$x = \left(\frac{1}{2} - i \frac{\sqrt{3}}{2}\right)^{\frac{3}{4}}$$

$$\implies x^4 = \left(\frac{1}{2} - i \frac{\sqrt{3}}{2}\right)^3$$

$$= \left[\left(\cos \frac{\pi}{3} - i \sin \frac{\pi}{3}\right)^3\right]$$

$$= \left[\left(\cos \frac{3\pi}{3} - i \sin \frac{3\pi}{3}\right)\right]$$

$$x^4 = [\cos \pi - i \sin \pi]$$

$$\therefore x_r = [\cos (\pi + 2k\pi) - i \sin (\pi + 2k\pi)]^{\frac{1}{4}}$$

$$\therefore x_r = \left[\cos \left(\frac{\pi+2k\pi}{4}\right) - i \sin \left(\frac{\pi+2k\pi}{4}\right)\right]$$

...where $k = 0, 1, 2, 3$

$$x_r = e^{-i \left(\frac{\pi+2k\pi}{4}\right)} \dots \text{where } k = 0, 1, 2, 3$$

for $k = 0$ and $r = 0$

$$x_0 = e^{-i \frac{\pi}{4}}$$

for $k = 1$ and $r = 1$

$$x_1 = e^{-i \frac{3\pi}{4}}$$

for $k = 2$ and $r = 2$ $x_2 = e^{-i \frac{5\pi}{4}}$

for $k = 3$ and $r = 3$ $x_3 = e^{-i \frac{7\pi}{4}}$

\therefore Continued product of roots is defined as

$$x_0.x_1.x_2.x_3 = e^{-i(\frac{\pi}{4})}.e^{-i(\frac{3\pi}{4})}.e^{-i(\frac{5\pi}{4})}.e^{-i(\frac{7\pi}{4})}$$

$$= e^{-i(\frac{\pi}{4} + \frac{3\pi}{4} + \frac{5\pi}{4} + \frac{7\pi}{4})}$$

$$= e^{-i(4\pi)}$$

$$= \cos 4\pi - i \sin 4\pi$$

$$= 1$$

Example 4

Show that roots of $(x + 1)^7 = (x - 1)^7$ are given by

$$\pm i \cot \frac{r\pi}{7} \dots r = 1, 2, 3, 4, 5, 6$$

Solution

Given

$$(x + 1)^7 = (x - 1)^7$$

$$\therefore \frac{(x + 1)^7}{(x - 1)^7} = 1$$

$$\therefore \left(\frac{x + 1}{x - 1} \right)^7 = \cos 0 + i \sin 0$$

$$\therefore \left(\frac{x + 1}{x - 1} \right)^7 = \cos (2k\pi) + i \sin (2k\pi)$$

$$\therefore \frac{x + 1}{x - 1} = [\cos (2k\pi) + i \sin (2k\pi)]^{\frac{1}{7}}$$

$$\therefore \frac{x+1}{x-1} = [\cos\left(\frac{2k\pi}{7}\right) + i \sin\left(\frac{2k\pi}{7}\right)] \text{.. where } k = 0, 1, 2, 3, 4, 5, 6$$

$$\frac{x+1}{x-1} = \frac{\cos\left(\frac{2k\pi}{7}\right) + i \sin\left(\frac{2k\pi}{7}\right)}{1} \text{.. where } k = 0, 1, 2, 3, 4, 5, 6$$

Using Componendo and dividendo

$$\frac{x-1+x+1}{x-1-x-1} = \frac{1 + \cos\left(\frac{2k\pi}{7}\right) + i \sin\left(\frac{2k\pi}{7}\right)}{1 - \cos\left(\frac{2k\pi}{7}\right) - i \sin\left(\frac{2k\pi}{7}\right)}$$

$$\frac{2x}{-2} = \frac{2 \cos^2\left(\frac{k\pi}{7}\right) + i 2 \sin\left(\frac{k\pi}{7}\right) \cos\left(\frac{k\pi}{7}\right)}{2 \sin^2\left(\frac{k\pi}{7}\right) - i 2 \sin\left(\frac{k\pi}{7}\right) \cos\left(\frac{k\pi}{7}\right)}$$

$$\frac{x}{1} = \frac{2 \cos\left(\frac{k\pi}{7}\right) [\cos\left(\frac{k\pi}{7}\right) + i \sin\left(\frac{k\pi}{7}\right)]}{-2 \sin^2\left(\frac{k\pi}{7}\right) + i 2 \sin\left(\frac{k\pi}{7}\right) \cos\left(\frac{k\pi}{7}\right)}$$

$$x = \frac{2 \cos\left(\frac{k\pi}{7}\right) [\cos\left(\frac{k\pi}{7}\right) + i \sin\left(\frac{k\pi}{7}\right)]}{i^2 2 \sin^2\left(\frac{k\pi}{7}\right) + i 2 \sin\left(\frac{k\pi}{7}\right) \cos\left(\frac{k\pi}{7}\right)}$$

$$x = \frac{1}{i} \frac{\cos\left(\frac{k\pi}{7}\right) [\cos\left(\frac{k\pi}{7}\right) + i \sin\left(\frac{k\pi}{7}\right)]}{\sin\left(\frac{k\pi}{7}\right) [\cos\left(\frac{k\pi}{7}\right) + i \sin\left(\frac{k\pi}{7}\right)]}$$

$$x = \frac{1}{i} \cot\left(\frac{k\pi}{7}\right)$$

$$x = -i \cot\left(\frac{k\pi}{7}\right) \text{ where } k = 1, 2, 3, 4, 5, 6$$

For $k = 0$ x is undefined

For $k = 1$

$$x_1 = -i \cot\left(\frac{\pi}{7}\right)$$

For $k = 2$

$$x_1 = -i \cot\left(\frac{2\pi}{7}\right)$$

For $k = 3$

$$x_1 = -i \cot\left(\frac{3\pi}{7}\right)$$

For $k = 4$

$$x_1 = -i \cot\left(\frac{4\pi}{7}\right)$$

$$x_1 = i \cot\left(\frac{3\pi}{7}\right)$$

For $k = 5$

$$x_1 = -i \cot\left(\frac{5\pi}{7}\right)$$

$$x_1 = i \cot\left(\frac{2\pi}{7}\right)$$

For $k = 6$

$$x_1 = -i \cot\left(\frac{6\pi}{7}\right)$$

$$x_1 = i \cot\left(\frac{\pi}{7}\right)$$

Hence roots of the equation $(x + 1)^7 = (x - 1)^7$ are of the form

$$\pm i \cot\left(\frac{r\pi}{7}\right) \dots \text{where } r = 1, 2, 3$$

CIRCULAR FUNCTIONS OF COMPLEX NUMBERS

- By Euler's Formula $e^{i\theta} = \cos \theta + i \sin \theta \dots (1)$ and $e^{-i\theta} = \cos \theta - i \sin \theta \dots (2)$
- Adding and subtracting (1) and (2), we have

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

and

$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

respectively , known as **Euler's exponential forms of circular functions**, where θ is a real number.

- If z is complex number,then

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

and

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

, known as **circular functions of complex numbers**