1] Find all complex roots of -16

Ans:- Let $x^3 = -16$

$x^3 = 16(-1)$

$x^3 = 16(-1 + 0i)$

$x^3 = 16(\cos\pi + i\sin\pi)$ [$\because \cos\pi = -1$ and $\sin\pi = 0$]

$\therefore x = 16^{1/3}(\cos\pi + i\sin\pi)^{1/3}$

Ingeneral, $x = 16^{1/3}\left[\cos(2k\pi+\pi) + i\sin(2k\pi+\pi)\right]^{1/3}$

$= 16^{1/3}\left[\cos(2k+1)\frac{\pi}{3} + i\sin(2k+1)\frac{\pi}{3}\right]$

- By DeMoivre's theorem

Where $k = 0, 1, 2$

i] $k=0$

$x_0 = 16^{1/3}\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right] = 16^{1/3}\left[\frac{1}{2} + \frac{\sqrt{3}i}{2}\right]$

ii] $k=1$

$x_1 = 16^{1/3}\left[\cos\pi + i\sin\pi\right] = 16^{1/3}(-1 + 0i) = -16^{1/3}$

iii] $k=2$

$x_2 = 16^{1/3}\left[\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right] = 16^{1/3}\left[\cos\left(\pi+\frac{2\pi}{3}\right) + i\sin\left(\pi+\frac{2\pi}{3}\right)\right] = 16^{1/3}\left(\frac{1}{2} - \frac{\sqrt{3}i}{2}\right)$

→ $\therefore$ Complex cube roots of -16 are $16^{1/3}\left(\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)$, $-16^{1/3}$ and $+16^{1/3}\left(\frac{1}{2}-\frac{\sqrt{3}i}{2}\right)$.

2] Find $\dfrac{(1+i)^6(1-i\sqrt{3})^4}{(1-i)^8(1+i\sqrt{3})^5}$ in $a+ib$ form using DeMoivre's theorem.

Ans:- Let $(1+i) = x_1 + iy_1 = r_1 e^{i\theta}$

then $x_1 = 1$, $y_1 = 1$

$r_1 = \sqrt{x_1^2 + y_1^2} = \sqrt{1^2+1^2} = \sqrt{2}$

$\theta_1 = \tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{4}$

$\therefore (1+i) = \sqrt{2}\, e^{i\pi/4}$ — ①

Replace $i \longrightarrow -i$

$(1-i) = \sqrt{2}\, e^{i\pi/4}$ — ②

Let $(1+i\sqrt{3}) = x_2 + iy_2 = r_2 e^{i\theta_2}$

then $x_2 = 1$, $y_2 = \sqrt{3}$

$r_2 = \sqrt{(1)^2+(\sqrt{3})^2} = 2$

$\theta_2 = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$

$\therefore (1+i\sqrt{3}) = 2e^{i\pi/3}$ — ③

Replace $i \longrightarrow (-i)$

$(1-i\sqrt{3}) = 2e^{-i\pi/3}$ — ④

From ①, ②, ③, ④,

$$\frac{(1+i)^6(1-i\sqrt{3})^4}{(1-i)^8(1+i\sqrt{3})^5} = \frac{(\sqrt{2})^6\, e^{i3\pi/2} \cdot 2^4 \cdot e^{-i4\pi/3}}{(\sqrt{2})^8\, e^{-i2\pi} \cdot 2^5 \cdot e^{i5\pi/3}}$$

$$= \frac{1}{4}\, e^{i\left(3\pi/2 - \frac{4\pi}{3} + 2\pi - \frac{5\pi}{3}\right)} = \frac{1}{4}\, e^{i\pi/2}$$

$$= \frac{1}{4}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = \frac{1}{4}\left(0 + 1i\right)$$

Hence, $\dfrac{(1+i)^6 \, (1-i\sqrt{3})^4}{(1-i)^8 \, (1+i\sqrt{3})^5} = \dfrac{1}{4}\left(0 + i\right).$

3] Find continued product of all the roots of $\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^{3/4}$

Ans :- $z = \left[\frac{1}{2} - \frac{i\sqrt{3}}{2}\right]^{3/4}$

$= \left[\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)^{3}\right]^{1/4}$

$= \left(\cos\pi - i\sin\pi\right)^{1/4}$ — By DeMoivre's theorem

$= \left(\cos\left(2k\pi+\pi\right) - i\sin\left(2k\pi+\pi\right)\right)^{1/4}$

$\therefore z = \cos\left(2k+1\right)\frac{\pi}{4} - i\sin\left(2k+1\right)\frac{\pi}{4}$ — By DeMoivre's Theorem

$= e^{-i(2k+1)\frac{\pi}{4}}$

where $k = 0, 1, 2, 3$

i] $k = 0$

$z_0 = e^{-i\pi/4}$

ii] $k = 1$

$z_1 = e^{-i3\pi/4}$

iii] $k = 2$

$z_2 = e^{-i5\pi/4}$

iv] $k = 3$

$z_3 = e^{-i7\pi/4}$

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$$Z_0 \cdot Z_1 \cdot Z_2 \cdot Z_3 = e^{-i\pi/4} \cdot e^{-i\frac{3\pi}{4}} \cdot e^{-i\frac{5\pi}{4}} \cdot e^{-i\frac{7\pi}{4}} = e^{-i(4\pi)}$$

$$= \cos 4\pi - i\sin 4\pi$$

$$= (-1)^4 - i(0)$$

$\therefore Z_0 \cdot Z_1 \cdot Z_2 \cdot Z_3 = 1$

$\therefore$ Continued product of $\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^{3/4}$ is $\underline{\underline{1}}$.

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4] Solve : $x^7 + x^4 + ix^3 + i = 0$

$$x^4(x^3+1) + i(x^3+1) = 0$$

$$(x^4+i)(x^3+1) = 0$$

$$x^4 + i = 0 \quad \text{or} \quad x^3+1 = 0$$

i] $x^4 = -i$

$$x^4 = \left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right)$$

$$= \left[\cos\left(2k\pi + \frac{\pi}{2}\right) - i\sin\left(2k\pi + \frac{\pi}{2}\right)\right]$$

$$x = \left[\cos(4k+1)\frac{\pi}{8} - i\sin(4k+1)\frac{\pi}{8}\right]$$

$$= \cos(4k+1)\frac{\pi}{8} - i\sin(4k+1)\frac{\pi}{8}$$ - By DeMoivre Theorem

$$x = e^{-i(4k+1)\frac{\pi}{8}}$$

where $k = 0, 1, 2, 3$

ii] $x^3 + 1 = 0$

$$x^3 = -1$$

$$= (\cos\pi + i\sin\pi)$$

$$x = [\cos(2k+1)\pi + i\sin(2k+1)\pi]^{1/3}$$

$$\therefore x = \cos(2k+1)\frac{\pi}{3} + i\sin(2k+1)\frac{\pi}{3}$$ - By DeMoivre's Theorem

$k = 0, 1, 2$

Hence,

$$x = e^{-i(4k+1)\frac{\pi}{8}} \quad \text{where } k = 0, 1, 2, 3$$

$$x = e^{i(2k+1)\frac{\pi}{3}} \quad \text{where } k = 0, 1, 2$$

5] Prove that $\frac{\sin 6\theta}{\sin 2\theta} = 16\cos^4\theta - 16\cos^2\theta + 3$

Ans:- By De Moivre's Theorem,

$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$

put $n=6$

$\therefore (\cos\theta + i\sin\theta)^6 = (\cos 6\theta + i\sin 6\theta)$

$\therefore \cos 6\theta + i\sin 6\theta = (\cos\theta + i\sin\theta)^6$

$= \cos^6\theta + 6\cos^5\theta(i\sin\theta) + 15\cos^4\theta\sin^2\theta(i)^2$
$+ 20\cos^3\theta(i\sin\theta)^3 + 15\cos^2\theta(i\sin\theta)^4$
$+ 6\cos\theta(i\sin\theta)^5 + (i\sin\theta)^6$

$= \cos^6\theta + 6(i\sin\theta)\cos^5\theta - 15\cos^4\theta(\sin^2\theta)$
$- i.20(\cos^3\theta)(\sin^3\theta) + 15\cos^2\theta(\sin^4\theta)$
$+ i.6(\cos\theta)(\sin^5\theta) - \sin^6\theta$

By comparing the real and imaginary parts on both sides,

$\sin 6\theta = 6\sin\theta\cos^5\theta - 20\sin^3\theta.\cos^3\theta + 6\cos\theta\sin^5\theta$

Dividing both sides by $\sin 2\theta$

$\frac{\sin 6\theta}{\sin 2\theta} = \frac{6\sin\theta\cos^5\theta - 20\sin^3\theta\cos^3\theta + 6\cos\theta\sin^5\theta}{\sin 2\theta}$

$= \frac{2\sin\theta\cos\theta(3\cos^4\theta - 10\sin^2\theta\cos^2\theta + 3\sin^4\theta)}{2\sin\theta\cos\theta}$

$= 3\cos^4\theta - 10(1-\cos^2\theta)\cos^2\theta + 3(1-\cos^2\theta)^2$

$[\because \sin^2\theta + \cos^2\theta = 1]$

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$$= 3\cos^4\theta - (10 + 10\cos^2\theta)\cos^2\theta + 3[1 + \cos^4\theta - 2\cos^2\theta]$$

$$= 16\cos^4\theta - 16\cos^2\theta + 3$$

Hence,

$$\frac{\sin 6\theta}{\sin 2\theta} = 16\cos^4\theta - 16\cos^2\theta + 3$$

Hence proved

6] Prove that $2^7 \sin^6\theta \cdot \cos^2\theta = 5 - 4\cos 2\theta - 4\cos 4\theta + 4\cos 6\theta - \cos 8\theta$

Ans :- Let $x = \cos\theta + i\sin\theta = e^{i\theta}$

$\frac{1}{x} = \cos\theta - i\sin\theta = e^{-i\theta}$

$x + \frac{1}{x} = 2\cos\theta$ — ① $\quad x - \frac{1}{x} = 2i\sin\theta$ — ②

$x^n + \frac{1}{x^n} = \left[\cos\theta + i\sin\theta\right]^n + \left(\cos\theta - i\sin\theta\right)^n$

$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

[By DeMoivre's Theorem]

$\therefore x^n + \frac{1}{x^n} = 2\cos n\theta$ — ③

$x^n - \frac{1}{x^n} = 2i\sin n\theta$ — ④

Given $= 2^7 \sin^6\theta \cdot \cos^2\theta$

$2^7 \sin^6\theta \cos^2\theta = 2^7 \left[\frac{1}{2i}\left(\frac{x-1}{x}\right)\right]^6 \cdot \left[\frac{1}{2}\left(\frac{x+1}{x}\right)\right]^2$ from ①, ②

$= \frac{2^7}{2^8 \cdot i^6}\left[\left(x - \frac{1}{x}\right)^6 \left(x + \frac{1}{x}\right)\right]$

$= \frac{1}{-2}\left[\left(x - \frac{1}{x}\right)^4 \left(x^2 - \frac{1}{x^2}\right)^2\right]$

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$$= \frac{-1}{2}\left[\left(x^4 - 4x^2 + 6 - 4\frac{1}{x^2} + \frac{1}{x^4}\right)\left(x^4 + \frac{1}{x^4} - 2\right)\right]$$

$$= \frac{-1}{2}\left[x^8 + 1 - 4x^6 - 2x^4 - 4\left(\frac{1}{x^2}\right) + 8x^2 + 6x^4 + \frac{6}{x^4} - 12 - 4x^2 - 4\left(\frac{1}{x^6}\right) + \frac{8}{x^2} + 1 + \frac{1}{x^8} - \frac{2}{x^4}\right]$$

$$= \frac{-1}{2}\left[4\left(x^2 + \frac{1}{x^2}\right) + 4\left(x^4 + \frac{1}{x^4}\right) - 4\left(x^6 + \frac{1}{x^6}\right) + \left(x^8 + \frac{1}{x^8}\right)\right]$$

$$= \frac{-1}{2}\left[4(2\cos 2\theta) + 4(2\cos 4\theta) - 4(2\cos 6\theta) + 2\cos 8\theta - 10\right]$$

$$= -4\cos 2\theta - 4\cos 4\theta + 4\cos 6\theta - \cos 8\theta + 5$$

$$= 5 - 4\cos 2\theta - 4\cos 4\theta + 4\cos 6\theta - \cos 8\theta$$

$= RHS$

$\therefore\ 2^7 \sin^6\theta \cdot \cos^2\theta = 5 - 4\cos 2\theta - 4\cos 4\theta + 4\cos 6\theta - \cos 8\theta$

Hence proved.

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