
Module 3: Homogeneous Functions

Lecture notes and Practice examples 2

By Dr. Keya Doshi

Homogeneous Functions

Definition

A function $f(x, y)$ of two independent variables x and y is said to be **Homogeneous** if degree of variables in each terms are equal

$$\begin{aligned}f(x, y) &= a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \dots + a_{n-1}xy^{n-1} + a_ny^n \\&= x^n \left[a_0 + a_1 \left(\frac{y}{x} \right) + a_2 \left(\frac{y}{x} \right)^2 + \dots + a_n \left(\frac{y}{x} \right)^n \right] \\f(x, y) &= x^n \phi \left(\frac{y}{x} \right)\end{aligned}$$

where n is an integer and it is also known as **degree of Homogeneous Function Condition**:

A two variable function $f(x, y)$ is called **Homogeneous function** if

$$f(\lambda x, \lambda y) = \lambda^n f(x, y)$$

Euler's Theorem (Two variables)

Statement

If $u = f(x, y)$ is a homogeneous function of two independent variables x and y of degree n then

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu$$

Proof

Given $u = f(x, y)$ is a homogeneous function of two independent variables x and y of degree n

\therefore by definition

$$u = f(x, y) = x^n \phi \left(\frac{y}{x} \right) \dots\dots\dots(1)$$

Partially differentiating (1) w.r.t x we have

$$\begin{aligned}
\frac{\partial u}{\partial x} &= x^n \frac{\partial}{\partial x} \left[\phi \left(\frac{y}{x} \right) \right] + \phi \left(\frac{y}{x} \right) \frac{\partial}{\partial x} (x^n) \\
&= x^n \left[\phi' \left(\frac{y}{x} \right) \right] \frac{\partial}{\partial x} \left(\frac{y}{x} \right) + \phi \left(\frac{y}{x} \right) (nx^{n-1}) \\
&= x^n \left[\phi' \left(\frac{y}{x} \right) \right] \left(\frac{-y}{x^2} \right) + \phi \left(\frac{y}{x} \right) (nx^{n-1}) \\
\frac{\partial u}{\partial x} &= nx^{n-1} \phi \left(\frac{y}{x} \right) - x^{n-2} y \phi' \left(\frac{y}{x} \right) \\
\Rightarrow x \frac{\partial u}{\partial x} &= nx^n \phi \left(\frac{y}{x} \right) - x^{n-1} y \phi' \left(\frac{y}{x} \right) \dots\dots\dots (2)
\end{aligned}$$

Again Partially differentiating (1) w.r.t y we have

$$\begin{aligned}
\frac{\partial u}{\partial y} &= x^n \frac{\partial}{\partial y} \left[\phi \left(\frac{y}{x} \right) \right] + \phi \left(\frac{y}{x} \right) \frac{\partial}{\partial y} (x^n) \\
&= x^n \left[\phi' \left(\frac{y}{x} \right) \right] \frac{\partial}{\partial y} \left(\frac{y}{x} \right) + 0 \\
&= x^n \left[\phi' \left(\frac{y}{x} \right) \right] \left(\frac{1}{x} \right) \\
\frac{\partial u}{\partial y} &= x^{n-1} \phi' \left(\frac{y}{x} \right) \\
\Rightarrow y \frac{\partial u}{\partial y} &= x^{n-1} y \phi' \left(\frac{y}{x} \right) \dots\dots\dots (3)
\end{aligned}$$

Adding (2) and (3) and using (1), we have

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nx^n \phi \left(\frac{y}{x} \right) = nu$$

Corollary 1 (Deduction 1)

Statement

If $u = f(x, y)$ is a homogeneous function of two independent variables x and y of degree n then

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

Proof

Given $u = f(x, y)$ is a homogeneous function of two independent variables x and y of degree n

\therefore by Euler's theorem

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nx^n \phi\left(\frac{y}{x}\right) = nu \dots \dots \dots (1)$$

Partially differentiating (1) w.r.t x we have

$$\begin{aligned} \frac{\partial}{\partial x} \left[x \frac{\partial u}{\partial x} \right] + \frac{\partial}{\partial x} \left[y \frac{\partial u}{\partial y} \right] &= \frac{\partial}{\partial x} [nu] \\ x \left[\frac{\partial^2 u}{\partial x^2} \right] + \frac{\partial u}{\partial x} + y \left[\frac{\partial^2 u}{\partial x \partial y} \right] &= n \frac{\partial u}{\partial x} \dots \dots \dots (2) \end{aligned}$$

Again Partially differentiating (1) w.r.t y we have

$$\begin{aligned} \frac{\partial}{\partial y} \left[x \frac{\partial u}{\partial x} \right] + \frac{\partial}{\partial y} \left[y \frac{\partial u}{\partial y} \right] &= \frac{\partial}{\partial y} [nu] \\ x \left[\frac{\partial^2 u}{\partial y \partial x} \right] + y \left[\frac{\partial^2 u}{\partial y^2} \right] + \frac{\partial u}{\partial y} &= n \frac{\partial u}{\partial y} \dots \dots \dots (3) \end{aligned}$$

Adding (2) and (3) and using (1), we have

$$\begin{aligned} x^2 \left[\frac{\partial^2 u}{\partial x^2} \right] + xy \left[\frac{\partial^2 u}{\partial x \partial y} \right] + xy \left[\frac{\partial^2 u}{\partial y \partial x} \right] + y^2 \left[\frac{\partial^2 u}{\partial y^2} \right] &= (n-1) \\ &\quad \left[x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right] \end{aligned}$$

For Homogeneous functions mixed partial derivatives are equal hence

$$\begin{aligned} x^2 \left[\frac{\partial^2 u}{\partial x^2} \right] + xy \left[\frac{\partial^2 u}{\partial x \partial y} \right] + xy \left[\frac{\partial^2 u}{\partial y \partial x} \right] + y^2 \left[\frac{\partial^2 u}{\partial y^2} \right] &= (n-1) \\ &\quad \left[x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right] \end{aligned}$$

by(1)

$$x^2 \left[\frac{\partial^2 u}{\partial x^2} \right] + 2xy \left[\frac{\partial^2 u}{\partial x \partial y} \right] + y^2 \left[\frac{\partial^2 u}{\partial y^2} \right] = n(n-1)u$$

Hence Proved.

Corollary 2 (Deduction 2)

Statement

If $z(x, y) = f(u)$ is a homogeneous function of two independent variables x and y of degree n then

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n \frac{f(u)}{f'(u)}$$

Proof

Given $z(x, y) = f(u)$ is a homogeneous function of two independent variables x and y of degree n

\therefore by Euler's theorem

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = nz \dots \dots \dots (1)$$

Given $z = f(u)$

Partially differentiating (1) w.r.t x and y we have

$$\frac{\partial z}{\partial x} = f'(u) \frac{\partial u}{\partial x} \dots \dots (2)$$

and

$$\frac{\partial z}{\partial y} = f'(u) \frac{\partial u}{\partial y} \dots \dots (3)$$

Using (2) and (3) in (1), we have

$$\begin{aligned}
x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} &= nz \\
\therefore x f'(u) \frac{\partial u}{\partial x} + y f'(u) \frac{\partial u}{\partial y} &= n f(u) \\
\therefore f'(u) \left[x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right] &= n f(u) \\
\therefore x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} &= n \frac{f(u)}{f'(u)}
\end{aligned}$$

Corollary 3 (Deduction 3)

Statement

If $z(x, y) = f(u)$ is a homogeneous function of two independent variables x and y of degree n then

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1]$$

where $g(u) = n \frac{f(u)}{f'(u)}$ **Proof**

Given $z(x, y) = f(u)$ is a homogeneous function of two independent variables x and y of degree n

\therefore by Deduction to Euler's theorem

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n \frac{f(u)}{f'(u)} \dots \dots \dots (1)$$

Let $g(u) = n \frac{f(u)}{f'(u)}$

Substituting in (1) we have

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n g(u) \dots \dots \dots (2)$$

Partially differentiation (2), w.r.t. x , we have

$$\begin{aligned}\frac{\partial}{\partial x} \left[x \frac{\partial u}{\partial x} \right] + \frac{\partial}{\partial x} \left[y \frac{\partial u}{\partial y} \right] &= \frac{\partial}{\partial x} [g(u)] \\ x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y \frac{\partial^2 u}{\partial x \partial y} &= g'(u) \frac{\partial u}{\partial x} \dots \dots \dots (3)\end{aligned}$$

Partially differentiation (2), w.r.t. y , we have

$$\begin{aligned}\frac{\partial}{\partial y} \left[x \frac{\partial u}{\partial x} \right] + \frac{\partial}{\partial y} \left[y \frac{\partial u}{\partial y} \right] &= \frac{\partial}{\partial y} [g(u)] \\ x \frac{\partial^2 u}{\partial y \partial x} + \frac{\partial u}{\partial y} + y \frac{\partial^2 u}{\partial y^2} &= g'(u) \frac{\partial u}{\partial y} \dots \dots \dots (4)\end{aligned}$$

$x(3) + y(4)$ we have

$$\begin{aligned}x^2 \frac{\partial^2 u}{\partial x^2} + x \frac{\partial u}{\partial x} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y \frac{\partial u}{\partial y} + y^2 \frac{\partial^2 u}{\partial y^2} &= xg'(u) \frac{\partial u}{\partial x} + yg'(u) \frac{\partial u}{\partial y} \\ \therefore x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} &= (g'(u) - 1) \left[x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right] \\ \therefore x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} &= g(u)(g'(u) - 1)\end{aligned}$$

Hence Proved

Corollary 4 (Deduction 4)

Statement

If $u = f(v)$ where v is a homogeneous function of two independent variables x and y of degree n then

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nv f'(v)$$

Proof

Given $u = f(v)$ where v is a homogeneous function of two independent variables x and y of degree n

\therefore by Euler's theorem

$$x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = nv \dots (1)$$

Also

$$u = f(v)$$

$$\therefore \frac{\partial u}{\partial x} = f'(v) \frac{\partial v}{\partial x} \dots (2)$$

and

$$\frac{\partial u}{\partial y} = f'(v) \frac{\partial v}{\partial y} \dots (3)$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = x f'(v) \frac{\partial v}{\partial x} + y f'(v) \frac{\partial v}{\partial y}$$

$$= f'(v) \left[x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} \right]$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nv f'(v)$$

Examples

Example (Type 1)

Find values of

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}$$

and

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}$$

if

$$(1) \quad u = \frac{\sqrt{x} + \sqrt{y}}{x+y}$$

$$(2) \quad u = \tan^{-1} \left(\frac{x^3 + y^3}{x-y} \right)$$

Solution

(1) Given

$$\begin{aligned} u(x, y) &= \frac{\sqrt{x} + \sqrt{y}}{x + y} \\ \therefore u(\lambda x, \lambda y) &= \frac{\sqrt{\lambda x} + \sqrt{\lambda y}}{\lambda x + \lambda y} \\ &= \frac{(\lambda)^{\frac{1}{2}} (\sqrt{x} + \sqrt{y})}{\lambda(x + y)} \\ &= \lambda^{-\frac{1}{2}} \frac{(\sqrt{x} + \sqrt{y})}{(x + y)} \\ \therefore u(\lambda x, \lambda y) &= \lambda^{-\frac{1}{2}} u(x, y) \end{aligned}$$

$\therefore u$ is a homogeneous function of degree $n = -\frac{1}{2}$

\therefore by Euler's theorem

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \left(-\frac{1}{2}\right) u$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \left(-\frac{1}{2}\right) \frac{\sqrt{x} + \sqrt{y}}{x + y} \dots\dots\dots(1)$$

Also by deduction to Euler's theorem

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \left(-\frac{1}{2}\right) \left(-\frac{1}{2} - 1\right) u$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \left(\frac{3}{4}\right) \frac{\sqrt{x} + \sqrt{y}}{x + y} \dots\dots\dots(2)$$

(1) and (2) are required results

(2)

$$\begin{aligned} u &= \tan^{-1} \left(\frac{x^3 + y^3}{x - y} \right) \\ \implies \tan u &= \frac{x^3 + y^3}{x - y} \end{aligned}$$

Let $z = f(u) = \tan u$ and

$$z(x, y) = \frac{x^3 + y^3}{x - y}$$

$$\therefore z(\lambda x, \lambda y) = \frac{(\lambda x)^3 + (\lambda y)^3}{\lambda x - \lambda y}$$

$$= \frac{(\lambda)^3 (x^3 + y^3)}{\lambda(x - y)}$$

$$= \lambda^2 \frac{(x^3 + y^3)}{(x - y)}$$

$$\therefore z(\lambda x, \lambda y) = \lambda^2 z(x, y)$$

$\therefore z$ is a homogeneous function in x and y of degree $n = 2$ and $z = f(u)$

\therefore by Deduction to Euler's theorem

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n \frac{f(u)}{f'(u)}$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2 \left(\frac{\tan u}{\sec^2 u} \right)$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2 (\sin u \cos u)$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \sin 2u$$

Let $g(u) = \sin 2u$

Then by deduction to Euler's theorem

$$\begin{aligned}
 x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} &= g(u) [g'(u) - 1] \\
 &= (\sin 2u) [2 \cos 2u - 1] \\
 &= 2 \sin u \cos u [2 (2 \cos^2 u - 1) - 1] \\
 &= 2 \sin u \cos u [4 \cos^2 u - 3] \\
 &= 2 \sin u [4 \cos^3 u - 3 \cos u] \\
 x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} &= 2 \sin u \cos 3u \dots \dots (2)
 \end{aligned}$$

(1) and (2) are required results

Example (Type 2)

Find values of

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}$$

if

$$u = e^{x+y} + \log(x^3 + y^3 - x^2y - xy^2)$$

Solution

Given

$$u = e^{x+y} + \log(x^3 + y^3 - x^2y - xy^2)$$

Let

$$u(x, y) = v + w$$

where $v(x, y) = e^{x+y}$ and $w(x, y) = \log(x^3 + y^3 - x^2y - xy^2)$

Now

$$v(x, y) = e^{x+y}$$

$$\therefore \log v = x + y$$

Let $f(v) = \log v$ and

$$\begin{aligned} z(x, y) &= x + y \\ \therefore z(\lambda x, \lambda y) &= \lambda x + \lambda y \\ \therefore z(\lambda x, \lambda y) &= \lambda(x + y) \\ \therefore z(\lambda x, \lambda y) &= \lambda z(x, y) \end{aligned}$$

$\therefore z$ is a homogeneous function of degree $n = 1$ and $z = f(v) = \log v \therefore$ by deduction to Euler's theorem

$$x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = n \frac{f(v)}{f'(v)}$$

$$x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = 1 \frac{\log v}{\frac{1}{v}}$$

$$x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = v \log v \dots\dots(1)$$

and

for $g(v) = v \log v$

$$\begin{aligned} x^2 \frac{\partial^2 v}{\partial x^2} + 2xy \frac{\partial^2 v}{\partial x \partial y} + y^2 \frac{\partial^2 v}{\partial y^2} &= g(v) [g'(v) - 1] \\ &= v \log v [\log v + 1 - 1] \\ &= v (\log v)^2 \dots\dots(2) \end{aligned}$$

Also

$$\begin{aligned} w(x, y) &= \log(x^3 + y^3 - x^2y - xy^2) \\ \therefore e^w &= x^3 + y^3 - x^2y - xy^2 \end{aligned}$$

Let $g(w) = e^w$ and

$$\begin{aligned} p(x, y) &= x^3 + y^3 - x^2y - xy^2 \\ \therefore p(\lambda x, \lambda y) &= (\lambda x)^3 + (\lambda y)^3 + (\lambda x)^2 \lambda y - (\lambda y)^2 \lambda x \\ \therefore p(\lambda x, \lambda y) &= (\lambda)^3 (x^3 + y^3 - x^2y - xy^2) \\ \therefore p(\lambda x, \lambda y) &= \lambda^3 p(x, y) \end{aligned}$$

$\therefore p$ is a homogeneous function of degree $n = 3$ and $p = f(w) = e^w \therefore$ by deduction to Euler's theorem

$$x \frac{\partial w}{\partial x} + y \frac{\partial w}{\partial y} = n \frac{f(w)}{f'(w)}$$

$$x \frac{\partial w}{\partial x} + y \frac{\partial w}{\partial y} = 3 \frac{e^w}{e^w}$$

$$x \frac{\partial w}{\partial x} + y \frac{\partial w}{\partial y} = 3 \dots \dots \dots (3)$$

and
for $g(w) = 3$

$$\begin{aligned} x^2 \frac{\partial^2 w}{\partial x^2} + 2xy \frac{\partial^2 w}{\partial x \partial y} + y^2 \frac{\partial^2 w}{\partial y^2} &= g(w) [g'(w) - 1] \\ &= 3 [0 - 1] \\ &= -3 \dots \dots \dots (4) \end{aligned}$$

Adding(1),(3)

$$\begin{aligned} x \left[\frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \right] + y \left[\frac{\partial v}{\partial y} + \frac{\partial w}{\partial y} \right] &= v \log v + 3 \\ \left[x \frac{\partial}{\partial x} \right] (v + w) + \left[y \frac{\partial}{\partial y} \right] (v + w) &= e^{(x+y)}(x + y) + 3 \\ \left[x \frac{\partial}{\partial x} \right] (u) + \left[y \frac{\partial}{\partial y} \right] (u) &= e^{(x+y)}(x + y) + 3 \\ x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} &= e^{(x+y)}(x + y) + 3 \dots \dots \dots (5) \end{aligned}$$

Adding(2),(4)

$$\begin{aligned}
x^2 \left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 w}{\partial x^2} \right] + 2xy \left[\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial y} \right] + y^2 \left[\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial y^2} \right] &= v (\log v)^2 - 3 \\
\left[x^2 \frac{\partial^2}{\partial x^2} \right] (v + w) + \left[2xy \frac{\partial^2}{\partial x \partial y} \right] (v + w) + \left[y^2 \frac{\partial^2}{\partial y^2} \right] (v + w) &= e^{(x+y)} (x + y)^2 - 3 \\
\left[x^2 \frac{\partial^2}{\partial x^2} \right] (u) + \left[2xy \frac{\partial^2}{\partial x \partial y} \right] (u) + \left[y^2 \frac{\partial^2}{\partial y^2} \right] (u) &= e^{(x+y)} (x + y)^2 - 3 \\
x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} &= e^{(x+y)} (x + y)^2 - 3 \dots (6)
\end{aligned}$$

Adding(5) and (6)

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = e^{(x+y)} (x + y) + 3 + e^{(x+y)} (x + y)^2 - 3$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = e^{(x+y)} (x + y) + e^{(x+y)} (x + y)^2$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = e^{(x+y)} (x + y) [1 + x + y]$$

EXAMPLES

(1) Find values of

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}$$

and

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}$$

if

$$(1) \quad u = \frac{x^3 + y^3}{y\sqrt{x}}$$

$$(2) \quad u = \sin^{-1}(\sqrt{x} + \sqrt{y})$$

$$(3) \quad u = e^{\left(\frac{x^2 y^2}{x^2 + y^2}\right)}$$

$$(4) \quad u = \tan^{-1} \left(\frac{\sqrt{x^3 + y^3}}{\sqrt{x} + \sqrt{y}} \right)$$

(2) Find value of

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}$$

at $x = 1$ and $y = 1$ if

$$(1) \quad u = x^3 \sin^{-1} \left(\frac{y}{x} \right) + x^4 \tan^{-1} \left(\frac{y}{x} \right)$$

$$(2) \quad u = x^3 \left[\tan^{-1} \left(\frac{y}{x} \right) + \frac{y}{x} e^{-\frac{y}{x}} \right] + y^{-3} \left[\sin^{-1} \left(\frac{x}{y} \right) + \frac{x}{y} \log \left(\frac{x}{y} \right) \right]$$

(3) If

$$u = \log r$$

and

$$r^2 = x^2 + y^2$$

Prove that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + 1 = 0$$

(4) Find value of

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}$$

if

(1)

$$u = \operatorname{cosec}^{-1} \sqrt{\left[\frac{x^{\frac{1}{2}} + y^{\frac{1}{2}}}{x^{\frac{1}{3}} + y^{\frac{1}{3}}} \right]}$$

(2)

$$u = \log \left[\frac{x+y}{\sqrt{x^2+y^2}} \right] + \sin^{-1} \left[\frac{x+y}{\sqrt{x} + \sqrt{y}} \right]$$

(5) Verify Euler's theorem for

(1)

$$u = \cos^{-1} \left(\frac{x}{y} \right) + \cot^{-1} \left(\frac{y}{x} \right)$$

(2)

$$u = \left[\frac{x^{\frac{1}{3}} + y^{\frac{1}{3}}}{x^{\frac{1}{4}} - y^{\frac{1}{4}}} \right]$$