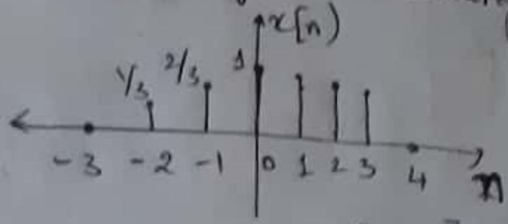
2.1) A discrete time signal $x(n)$ is defined as

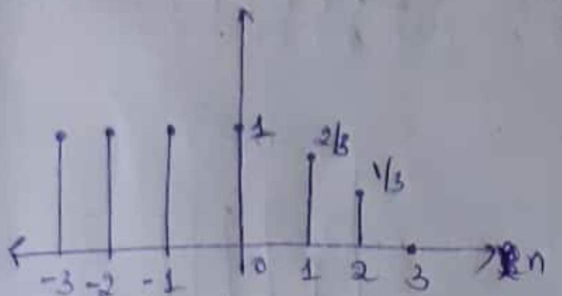
$$x(n) = \begin{cases} 1+\frac{n}{3} & -3 \le n \le -1 \\ 1 & 0 \le n \le 3 \\ 0 & \text{elsewhere} \end{cases}$$

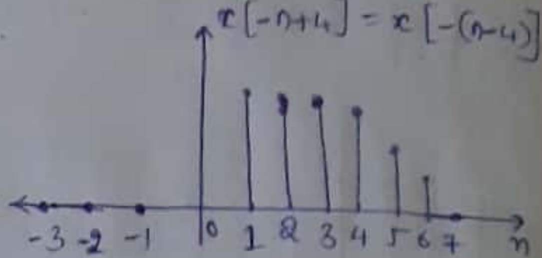
(a) Determine its value and sketch the signal $x(n)$

$x(n) = \{0, 0, \frac{1}{3}, \frac{2}{3}, 1, 1, 1, 1, 0, 0\}$

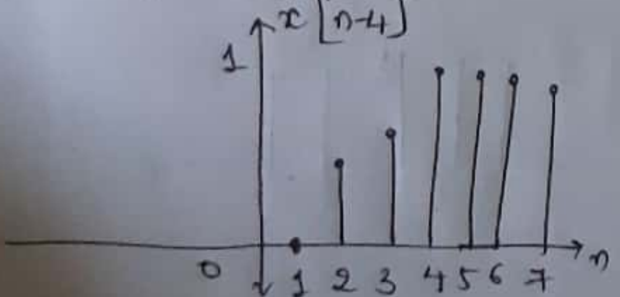
(b) (i) First fold the $x(n)$ and delay the resulting signal by four samples.

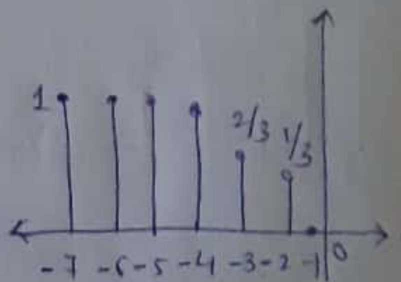




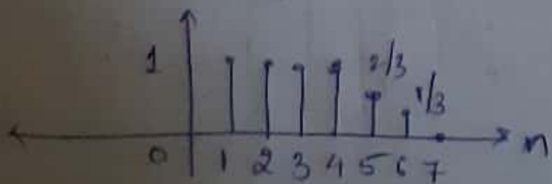


(ii) First delay $x[n]$ by four samples and then fold the resulting signal.





(c) Sketch the signal $x(-n+4)$



(d) Compare results in parts (b) and (c) derive a rule for obtaining the signal $x[-n+k]$ from $x(n)$.

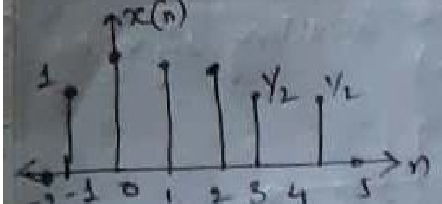
To obtain $x[-n+k)$ from $x(n)$. First we have to fold or reverse the signal and then delay it by K units.

(e) Can you express the signal in term of signals $\delta(n)$ and $u(n)$?

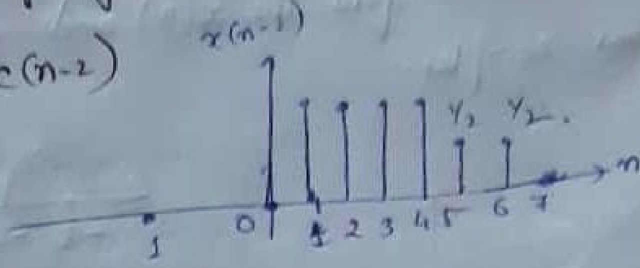
(i) $x(n) = \frac{1}{3}\delta(n+2) + \frac{2}{3}\delta(n+1) + \delta(n) + \delta(n-1) + \delta(n-2) + \delta(n-3)$

(ii) $x(n) = \frac{1}{3}\delta(n+2) + \frac{2}{3}\delta(n+1) - u(n-4) + u(n)$.

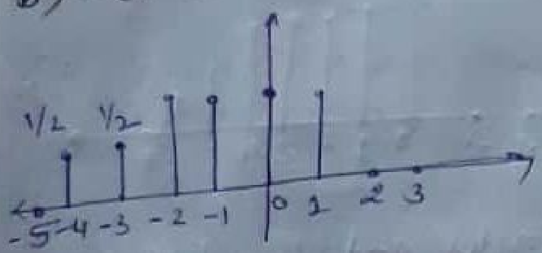
22) A discrete time signal $x(n)$ is shown. Sketch and label carefully each of the following signal

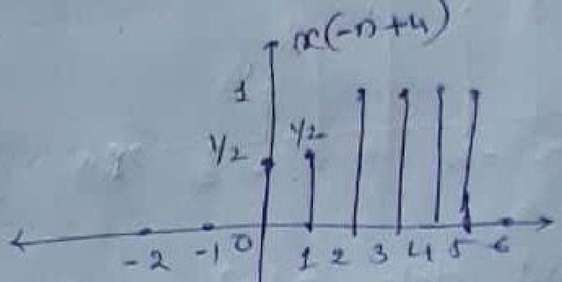


(a) $x(n-2)$

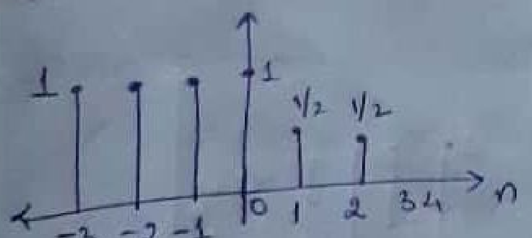


b) $x(4-n)$

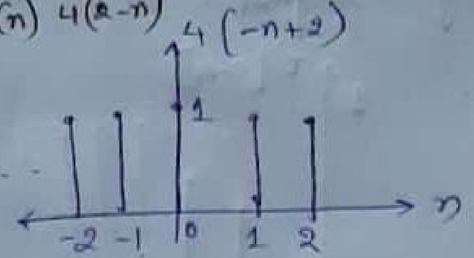




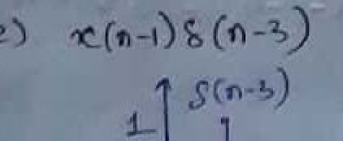
(c) $x(n+2)$

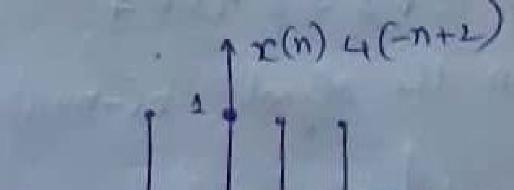


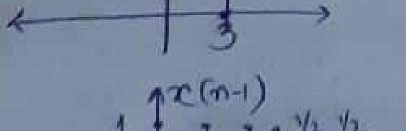
(d) $x(n)\,u(2-n)$

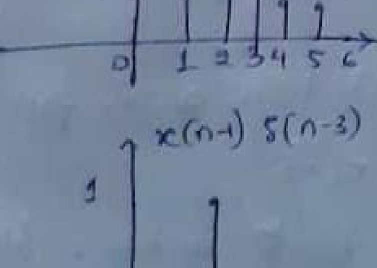


(e) $x(n-1)\,\delta(n-3)$

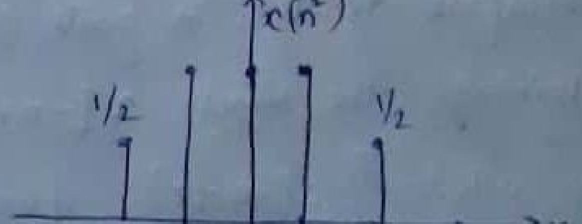




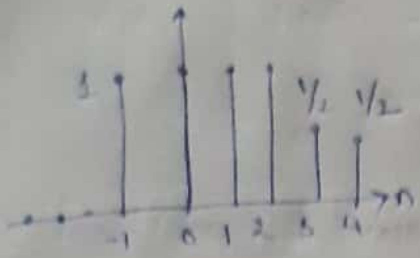


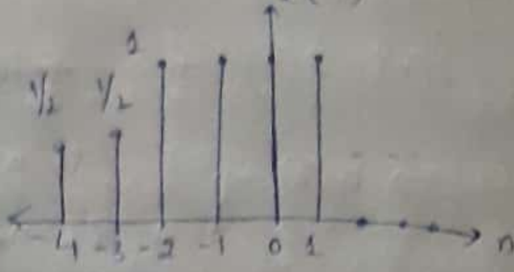


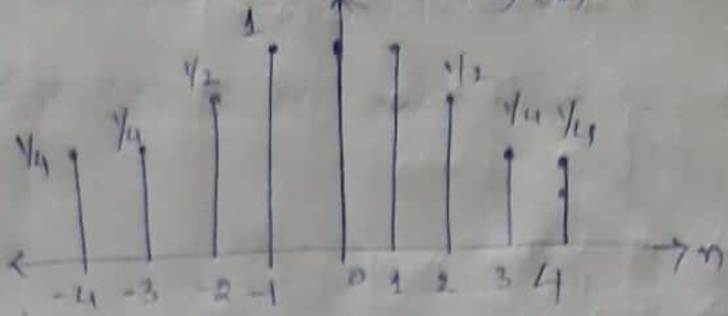
(f) $x(n^2)$



(g) even part of $x(n) = \frac{x(n) + x(-n)}{2}$

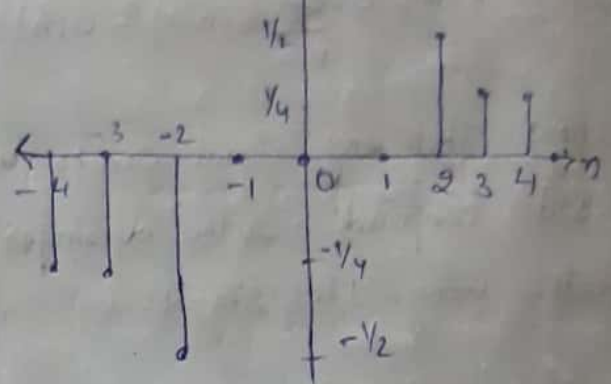






(h) odd part of $x(n)$.

$$x(n) = \frac{x(n) - x(-n)}{2}$$



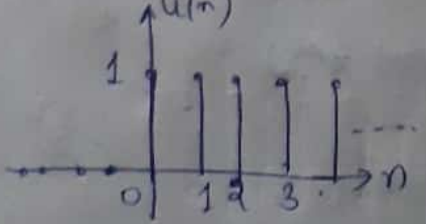
2.3 Show that

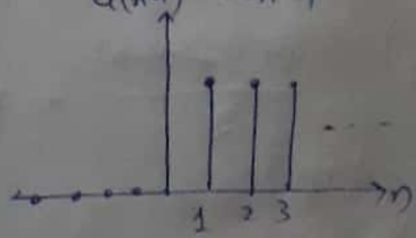
(a) $\delta(n) = u(n) - u(n-1)$

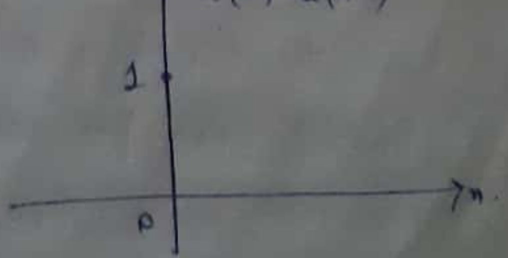
we know that $\delta(n)$ is defined as $\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0. \end{cases}$

and $u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$

$u(n-1) = \begin{cases} 1 & n \geq 1 \\ 0 & n < 1 \end{cases}$







(b) $u(n) = \sum_{k=-\infty}^{n} \delta(k) = \sum_{k=0}^{\infty} \delta(n-k)$

$\sum_{k=-\infty}^{n} \delta(k) = \delta(-\infty) + \dots + \delta(n)$

(1) $\longrightarrow u(n) = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases}$

$\sum_{k=0}^{\infty} \delta(n-k) = \delta(n) + \delta(n-1) + \dots \delta(n-\infty)$

(2) $\longrightarrow u(n) = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases}$

Reason: $0, \infty$ are all positive numbers.
if $n-k = 0 \rightarrow \delta(0) = 1$. Hence n must be a positive integer

Hence $\sum_{n=-\infty}^{n} \delta(k) = \sum_{k=0}^{\infty} \delta(n-k) = u(n)$.

2.4) Show that any signal can be decomposed into an even and an odd component. Is the decomposition unique? Illustrate your arguments using the signal.

Any signal can be written or formed by the combination even and odd components of the signal.
So decomposition is unique.

$x(n) = x_e(n) + x_o(n)$

We know that $x_e(n) = \frac{x(n) + x(-n)}{2}$ and $x_o(n) = \frac{x(n) - x(-n)}{2}$

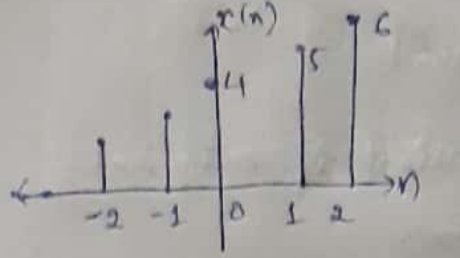
$\therefore x(n) = \frac{x(n) + x(-n)}{2} + \frac{x(n) - x(n) - \cancel{x(-n)}}{2} = \frac{x(n)}{2} + \frac{x(n)}{2}$

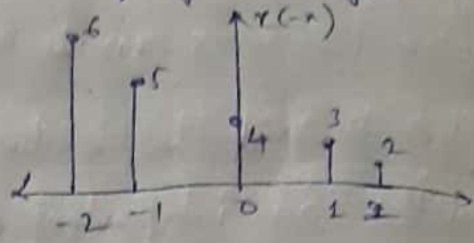
$= x(n)$

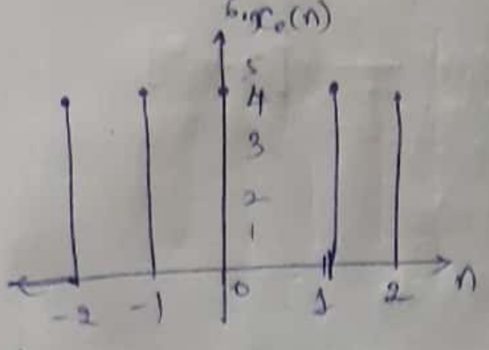
Hence proved.

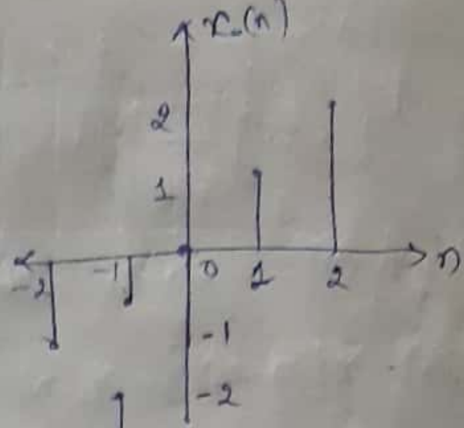
Given $x(n) = \{2, 3, \underset{\uparrow}{4}, 5, 6\}$

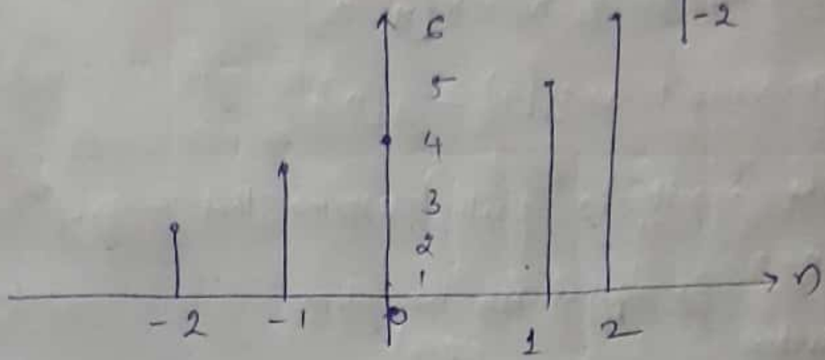
The odd and even component of signal is given as.











Hence proved

2.5) Show that the energy (power) of a real valued energy (power) signal is equal to the sum of the energies (power) of its even and odd component.

$$\sum_{n=-\infty}^{\infty} x^2(n) = \sum_{n=-\infty}^{\infty} [x_e(n) + x_o(n)]^2$$

$$= \sum_{n=-\infty}^{\infty} x_e^2[n] + \sum_{n=-\infty}^{\infty} x_o^2(n) + 2\sum_{n=-\infty}^{\infty} x_e(n)\, x_o(n)$$

$$= \sum_{n=-\infty}^{\infty} x_e^2(n) + \sum_{n=-\infty}^{\infty} x_o^2(n)$$

$$\sum_{x=-\infty}^{\infty} x_e(n)\, x_o(n) = \sum_{n=-\infty}^{\infty} = x_e(-m)\, x_o[-m]$$

$$= -\sum_{m=-\infty}^{\infty} x_e(m)\, x_o(m)$$

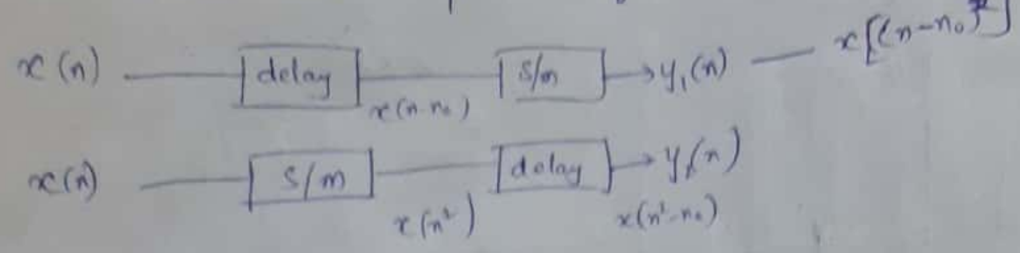
$$= -\sum_{n=-\infty}^{\infty} x_e(n)\, x_o(n) = \sum_{n=-\infty}^{\infty} x_e[n]\, x_o[n] = 0$$

2.6 Consider the system

$$y(n) = T(x(n)) = x(n^2)$$

(a) Determine if the system is time invariant.

A system is said to be time invariant if the i/p and o/p characteristics are independent of time.



If $y_1(n) = y_2(n)$, system is called TIV s/m.

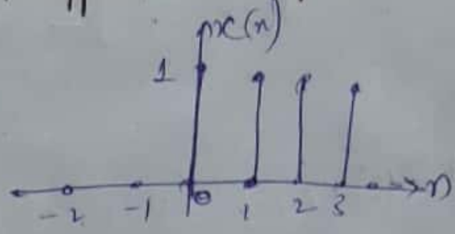
Here $y_1(n) = x[(n-n_0)^2]$ $y_2(n) = x[n^2-n_0]$

It is a Time variant as $y_1[n] \neq y_2[n]$

(b) To clarify the result in part(a) assume that the signal

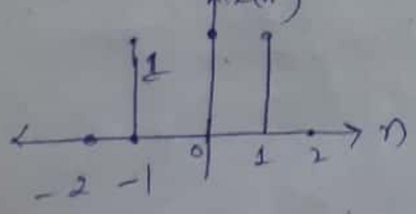
$x(n) = \begin{cases} 1 & 0 \le n \le 3 \\ 0 & \text{elsewhere} \end{cases}$ is applied into the system

(i) sketch the signal $x(n)$

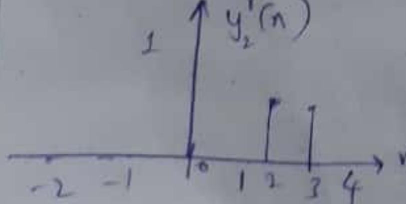


(ii) Determine and sketch the signals $y(n) = T(x(n)) = x[n^2]$

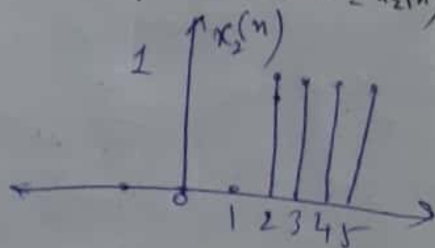
$n=0 \rightarrow x[0]$, $n=1 \rightarrow x[1]$, $n=2 \rightarrow x[4]$, $n=-1$, $x[1]$



(iii) Sketch the signal $y_2'(n) = y(n-2)$



(iv) Determine and sketch the signal $x(n-2) = x_2(n)$



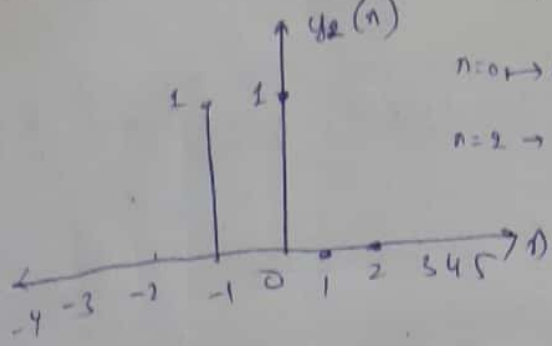
(v) Determine and sketch the signal $y_2[n] = T[x_2(n)]$

$= [x[(n-2)^2]]$

$= x[n^2 + 4 + 2n]$

$n=0 \to x[4],\ n=1 \to x[7]$

$n=2 \to x[12],\ n=-1 \to x[3]$



(6) Compare the signals $y_2(n)$ and $y(n-2)$. What is your conclusion.

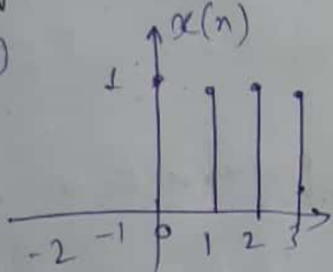
$y_2(n)$ and $y_2'(n)$ are not same

Therefore it is Time invariant system.

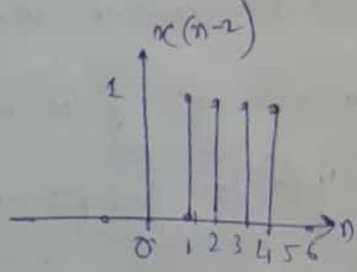
(c) Repeat part (b) for the system.

$y[n] = x(n) - x(n-1)$. Can you use this result to make any system about the TI of this system? why?

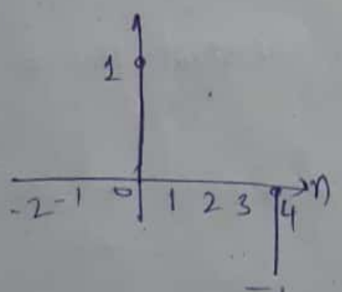
(i)



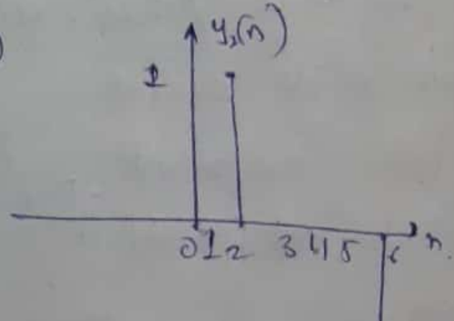
(iv)



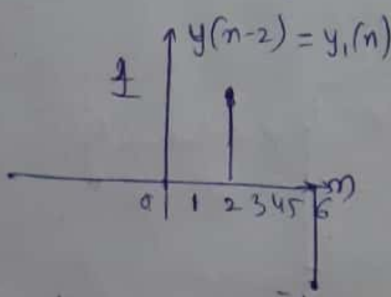
(ii)



(v)



(iii)



Since $y_1[n] = y_2[n]$ The system is Time Variant.

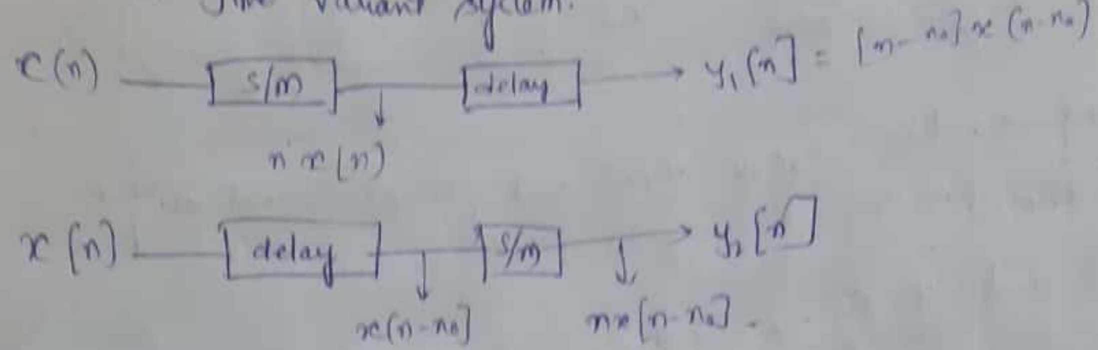
→ This way of finding Time invariancy is a long process. There are some points we have to remember i.e.,

(i) If i/p (or) o/p is a function of time then it is a T-V system.

(ii) Scaling operation makes system variable.

(d) Repeat parts (b) and (c) for the system $y(n) = T[x(n)] = n\,x(n)$

It is a Time variant system.



Hence proved.

2.7] A. DT System can be

1. Static (or) Dynamic
2. Lineal or non linear
3. Time Invariant or time varying
4. Casual or non casual
5. Stable (or) unstable.

Examine the following system w.r.t properties above

(a) $y(n) = \cos[x(n)]$.

(i) It is a static, non lineal, Time Invariant, Non casual and stable system.

$y[0] = \cos[x(0)]$
$y[1] = \cos[x(1)]$ } present values

$y_1[n] = \cos[x_1(n)]$
$y_2[n] = \cos[x_2(n)]$
$y(n) = \cos[x_1(n) + x_2(n)] \neq y_1(n) + y_2(n)$ } non linearity.

(ii) The value of cos ranges from -1 to 1. So for bounded input we are getting bounded output.

(b) $y[n] = \sum_{k=-\infty}^{n+1} x(k)$

Dynamic → it is depending on past values.

Lineal, Time Variant, Non causal and unstable system.

(c) $y[n] = x[n]\cos[\omega_0 n]$

Static, lineal, Time Variant, Causal and stable system.

(d) $y[n] = x[-n+2]$

Dynamic, lineal, Time invariant, Non causal and stable system.

(e) $y[n] = Trun[x(n)]$, where $Trun[x(n)]$ denotes the integer part of $x(n)$, obtained by truncation.

Static → $y[0] = Trun[x(0)]$, (ii) Non lineal, Time Invariant, causal, stable.

(f) $y(n) = Round[x(n)]$ where $Round[x(n)]$ denotes the integer part of $x(n)$ obtained by rounding.

Remark: The systems in parts (e) and (f) are quantizers that perform truncation and rounding, respectively.

Static, Non-lineal, Time invariant, causal and stable.

Let i/p ⇒ 1.6 −1.3

$$\underset{2}{\overset{1.6}{\downarrow}} + 2 = 4$$

i/p $(1.6+1.3) = 2.9$

(g) $y[n] = (x(n))$.

It, static, lineal, Time invariant, causal and stable system.

(h) $y[n] = x(n)u(n)$

Static, lineal, Time Variant, Non causal and stable system.

(i) $y(n) = x(n) + nx(n+1)$

Dynamic, lineal, Time Variable, Non Causal, and unstable system.

eg: $y[0] = 4[0] + 4[1] = 1$, $y[\infty] = \infty$ } Even though the input $x[n]$ is bounded at unbounded inputs.

(j) $y(n) = x(2n)$

Dynamic, linear, Time variant, Non causal and stable system.

(k) $y(n) = \begin{cases} x(n) & \text{if } x(n) \geq 0 \\ 0 & \text{if } x(n) < 0 \end{cases} = x(n)\,u(n).$

Static, linear, Time invariant, causal and stable system.

(l) $y(n) = x(-n)$

Dynamic, linear, Time invariant, Non causal and stable system.

(m) $y[n] = \text{sign}[x(n)] = -x(n)$

Static, linear, Time invariant, causal and stable.

(n) The ideal sampling system with input $x_a(t)$ and output $x(n) = x_a(nT)$, $-\infty < n < \infty$

Dynamic, linear, Time invariant, Non causal and stable system.

2.8) Two D.T systems T_1 and T_2 are connected in cascaded to form a new system T as shown in figure. Prove or disprove the following statements.

$$x(n) \longrightarrow \boxed{T_1} \longrightarrow \boxed{T_2} \longrightarrow y[n]$$

$T = T_1 T_2$

(a) If T_1 and T_2 are linear than T is linear (i.e.,) the cascaded of two linear is linear).

True

If $V_1(n) = T_1[x_1(n)]$ and $V_2(n) = T_1[x_2(n)]$.

$$\alpha_1 x_1(n) + \alpha_2 x_2(n) \xrightarrow{T_1} \alpha_1 V_1(n) + \alpha_2 V_2(n).$$

→ By the linearity property of T_1, similarity if

$$V_1[n] \longrightarrow y_1(n)$$
$$V_2(n) \longrightarrow y_2(n)$$

then $\beta_1 V_1(n) + \beta_2 V_2(n) \xrightarrow{T_2} \beta_1 y_1(n) + \beta_2 y_2(n)$

→ By linearity property of T_2

It follows that

$$A_1 x_1(n) + A_2 x_2(n) \xrightarrow{T} A_1 y_1(n) + A_2 y_2(n)$$

where $T = T_1 T_2$ and $A_1 = \alpha_1 \beta_1$, $A_2 = \alpha_2 \beta_2$.

Hence T is linear.

(b) If T_1 and T_2 are time Invariant, then T is time invariant.

True

For T_1 if $x(n) \xrightarrow{T_1} y(n)$

$$x(n-k) \xrightarrow{T_1} y(n-k)$$

For T_2 if $y(n) \xrightarrow{T_2} v(n)$

$$y(n-k) \xrightarrow{T_2} v(n-k)$$

Hence for $T_1 T_2$ if

$$x(n) \longrightarrow v(n)$$

$$x(n-k) \longrightarrow v(n-k)$$

$\therefore$ $T = T_1 T_2$ is time Invariant.

(c) If T_1 and T_2 are causal. Then T is causal.

True.

T_1 is causal $\Rightarrow$ $u(n)$ depends only on $x(k)$ for $k \le n$.

T_2 is causal $\Rightarrow$ $y(n)$ depends only on $u(k)$ for $k \le n$.

Therefore $y(n)$ depends only $x(k)$ for $k \le n$.

$\therefore$ $T = T_1 T_2$ is causal.

(d) If T_1 and T_2 are linear and time Invariant, the same holds for T

For T_1 $\alpha x(n) \xrightarrow{T_1} \alpha v(n)$

$$\alpha(x(n-k)) \xrightarrow[T_1]{} \alpha v(n-k)$$

For T_2 $\beta v(n) \xrightarrow{T_2} \alpha y(n)$

$$\beta v(n-k) \xrightarrow{T_2} \beta y(n-k)$$

For $T = T_1 T_2$ $\alpha x(n-k) \xrightarrow{T} \alpha y(n-k)$.

(e) If T_1 and T_2 are linear and time invariant. Then interchanging their order doesn't change the system T.

True. This follows form $h_1(n) + h_2(n) = h_2(n) + h_1(n)$

(f) As in part (e) except that T_1 and T_2 are now time varying.

False

(g) If T_1 and T_2 are nonlinear. Then T is nonlinear.

$T_1 : y(n) = x(n)+b \,,\quad T_2 : y(n) = x(n)-b \,,\ b \neq 0$

$T(x(n)) = T_2(T_1(x(n))] = T_2[x(n)+b)$

$= x(n) + b - b = x(n) \rightarrow$ linear

Hence false.

(h) If T_1 and T_2 are stable. Then T is stable.

True.

If T_1 is a stable system.

Then $u(n)$ is bounded if $x(n)$ is bounded.

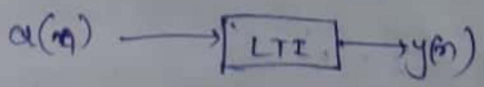
If T_2 is a stable system.

Then $y(n)$ is bounded if $u(n)$ is bounded.

Hence for $T = T_1 T_2$ $y(n)$ will be bounded if $x(n)$ is bounded

2.9) Let T be an LTI, relaxed, and BIBO stable system with input $x(n)$ and output $y(n)$. Show that.

(a) If $x(n)$ is periodic with period N [i.e, $x(n) = x(n+N)$] $\forall$ $n \geq 0$], the output $y(n)$ tends to a periodic signal with same period.



$$y(n) = \sum_{n=-\infty}^{n} h(k)\, x(n-k) = \sum_{k=-\infty}^{n} x(k)\, h(n-k)$$

Let $y(n) = \sum_{k=-\infty}^{n} h(k)\, x(n-k)$.

replacing 'n' by '$n+N$' we get

$$y(n+N) = \sum_{k=-\infty}^{n+N} h(k)\, x[n+N-k] = \sum_{k=-\infty}^{n+N} h(k)\, x(n-k)$$

where $x(n)$ is a periodic signal.

$$\Rightarrow y(n+N) = \sum_{k=-\infty}^{n+N} h(k)\, x(n-k)$$

$$= \sum_{k=-\infty}^{n} h(k)\, x(n-k) + \sum_{k=n+1}^{N+n} h(k)\, x(n-k)$$

$$= y(n) + \sum_{k=n+1}^{n+N} h(k)\, x(n-k)$$

Applying limit on both sides,

$$\lim_{n\to\infty} y(N+n) = y(n) + \lim_{n\to\infty} \sum_{k=n+1}^{n+N} h(k)\, x(n-k)$$

For a BIBO system, $\lim_{n\to\infty} |h(n)| = 0$.

$$\therefore \sum_{k=n+1}^{n+N} \lim_{n\to\infty} h(k)\, x(n-k) = 0.$$

$$\Rightarrow \lim_{n\to\infty} y(n+N) = y(n) \text{ is a periodic signal.}$$

(b) If $x(n)$ is bounded and tends to a constant, y the output will also tend to a constant.

Let $x(n) = x_0(n) + a\,u(n)$

where a is a constant and $x_0(n)$ is a bounded signal with $\lim_{n\to\infty} x_0(n) = 0$.

Then $y(n) = a \sum_{k=0}^{\infty} h(k)\, x(n-k)$

$$= a \sum_{k=0}^{\infty} h(k)\, a(n-k) + \sum_{k=0}^{\infty} h(k)\, x_0(n-k)$$

$$= a \sum_{k=0}^{n} h(k) + \sum_{k=0}^{\infty} h(k)\, x_0(n-k)$$

$$\lim_{n\to\infty} y(n) = a \sum_{k=0}^{n} h(k) + \lim_{n\to\infty} \sum_{k=0}^{\infty} h(k)\, x_0(n-k)$$

Clearly $\lim_{n\to\infty} \sum_{k=0}^{\infty} h(k)\, x_0(n-k) = 0$

$$\Rightarrow \lim_{n\to\infty} y(n) = a \sum_{k=0}^{n} h(k) = \text{Constant}.$$

2.10 The following input-output pairs have been observed during the operation of a time-invariant system.

$x_1(n) = \{\underset{\uparrow}{1}, 0, 2\} \xleftrightarrow{T} y_1(n) = \{0, \underset{\uparrow}{1}, 2\}$

$x_2(n) = \{\underset{\uparrow}{0}, 0, 3\} \xleftrightarrow{T} y_2(n) = \{\underset{\uparrow}{0}, 1, 0, 2\}$

$x_3(n) = \{\underset{\uparrow}{0}, 0, 0, 1\} \xleftrightarrow{T} y_3(n) = \{1, 2, 1\}$

Can you draw any conclusions regarding the linearity of the system. What is the impulse response of the system?

The given system is non linear.

Coming to $x_2(n)$ and $x_3(n)$

$x_2(n) = \{0, 0, 3\} \xrightarrow{(T)} \{\underset{\uparrow}{0}, 1, 0, 2\}$

$x_3(n-1) = \{\underset{\uparrow}{0}, 0, 0, 1\} \xrightarrow{T} \{1, \underset{\uparrow}{2}, 1\}$

$x_3(n+1) = \{0, 0, 1\} \xrightarrow{(T.I)} \{\underset{\uparrow}{1}, 2, 1\}$

Now if the system is linear then

$3x_3(n+1) \longleftrightarrow \{3, 6, 3\}$

But $\{3, 6, 3\} \neq \{0, 1, 0, 2\}$ Hence the system is non linear.

2.11 The following input-output pairs have been observed during the operation of a linear system.

$x_1(n) = \{-1, 2, 1\} \xleftrightarrow{T} y_1(n) = \{1, 2, -1, 0, 1\}$

$x_2(n) = \{1, -1, -1\} \xleftrightarrow{T} y_2(n) = \{-1, 1, 0, 2\}$

$x_3(n) = \{0, 1, 1\} \longleftrightarrow y_3(n) = \{1, 2, 1\}.$

Can you draw any conclusions about the time variance of this system?

$\because$ $x_1(n) + x_2(n) = \delta(n)$ and the system is linear.

The impulse response of the system is given by

$y_1(n) + y_2(n) = \{0, 3, -1, 2, 1\}.$

If the system is further time-invariant, then it would be an LIT system.

Then $x_3(n) + h(n) = \{0,1,1\} + \{0,3,-1,2,1\}$.

$y(n) = \{3,2,1,3,1\}$.

But the given o/p $y_3(n) = \{1,2,1\}$. Hence system is TV.

~~(12) The only available information about a system consists of N input-output, using the information above, if the system is known to be linear? Pairs.~~

~~Any weighted linear combination of the signals $x_i(n)$, $i=1,2,\dots N$~~

~~(b) The pair of signals.~~

2.12) The only available information about a system consists of N input-output pairs of signals $y_i(n) = T[x_i(n)]$ $i=1,2,\dots N$.

(a) What is the class of input signals for which we can determine the output, using the information above, if the system is known to be linear?

Any weighted linear combination of the signals $x_i(n)$ $i=1,2,\dots N$

(b) The same as above; if the system is known to be time invariant.

Any $x_i(n-k)$ where k is any integer and $i=1,2,\dots N$.

2.13) Show that the necessary and sufficient condition for closed LIT system to be BIBO stable is

$$\sum_{n=\infty}^{\infty} |h(n)| \leq M_h \leq \infty \text{ for some constant } N_n.$$

A system is stable if and only if for a banded input the output should also be banded.

We know that for an LTI system the input and output relation is $y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$

Let the input to the LTI system is bounded then

$|x(n)| < M_x < \infty$

$\Rightarrow$

$$|y(n)| = \sum_{k=-\infty}^{\infty} |h[k]|\,|x(n-k)|$$

$$\leq M_x \sum_{k=-\infty}^{\infty} |h(k)|$$

Where $|x(n-k)| \leq M_x$. Therefore $|y(n)| < \infty \;\forall\, n$ if and only if $\sum_{k=\infty}^{\infty} |h(k)| < \infty$.

2.14 Show that

(a) A relaxed linear system is causal if and only if for any input $x(n)$ such that $x(n)=0 \;\forall\, n < n_0$

$\rightarrow y(n) = 0$ for $n < n_0$.

A system is said to be causal if and only if the output depends only on the present and past values of the input.

Given $x(n) = 0$ for $n < n_0$

$\therefore$ The o/p depends only on the present values of the system

$\therefore$ The system is causal as the output becomes nonzero when the input becomes non-zero.

Hence $x(n) = 0$ for $n < n_0 \Rightarrow y(n) = 0$ for $n < n_0$.

(b) A relaxed LTI system is causal if and only if $h(n) = 0$ for $n < 0$.

For an LTI system i/p - o/p relation is given as

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\,x(n-k), \quad x(n) = 0 \text{ for } n < 0$$

If $h(k) = 0$ for $k < 0$ then

$$y[n] = \sum_{0}^{n} h(k)\,x(n-k)$$ and hence $y[n] = 0$ for $n < 0$

on the other hand, if $y(n)=0$ for $n<0$, then

$$\sum_{-\infty}^{n} h(k)\, x(n-k) \Rightarrow h(k) = 0,\ k<0.$$

2.15) (a) Show that for any real or complex constant a, and any finite integer numbers M and N, we have

$$\sum_{n=M}^{N} a^n = \begin{cases} \dfrac{a^M - a^{N+1}}{1-a} & \text{if } a \neq 1 \\ N-M+1 & \text{if } a=1. \end{cases}$$

For $a=1$, $\displaystyle\sum_{n=M}^{N} a^n = N-M+1$

for $a \neq 1$, $\displaystyle\sum_{n=M}^{N} a^n = a^M + a^{M+1} + \cdots + a^N$.

$$(1-a)\sum_{n=M}^{N} a^n = a^M + a^{M+1} - a^{M+1} + \cdots + a^N - a^N - a^{N+1}$$

$$= a^M - a^{N+1}.$$

(b) Show that if $|a|<1$, then $\displaystyle\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$.

For $M=0$, $|a|<1$, and $N \to \infty$

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a},\ |a|<1.$$

2.16) (a) If $y(n) = x(n) * h(n)$, show that $\Sigma_y = \Sigma_x \Sigma_h$,

where $\Sigma_x = \displaystyle\sum_{n=-\infty}^{\infty} x(n)$

$$y[n] = \sum_{k=-\infty}^{\infty} h(k)\, x[n-k]$$

Applying summation on both sides we get.

$$\sum_{n=-\infty}^{\infty} y[n] = \sum_{n=-\infty}^{\infty} h(k)\, x[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h(k) \sum_{n=-\infty}^{\infty} x(n-k)$$

$$\sum_{n=-\infty}^{\infty} y(n) = \left[\sum_{k} h(k)\right]\left[\sum_{n} x(n)\right].$$

(b) Compute the convolution $y(n) = x(n) * h(n)$ of the following signals and check the correctness of the results by using the test in a

(i) $x(n) = \{1, \underset{\uparrow}{2}, 4\}$ $h(n) = \{\underset{\uparrow}{1}, 1, 1, 1, 1\}$

	↓ 1	1	1	1	1
→ 1	1	1	1	1	1
2	2	2	2	2	2
4	4	4	4	4	4

$y(n) = x(n) * h(n)$

$y(n) = \{1, 3, 7, 7, 7, 6, 4\}$

$\sum_n y(n) = 35$

Verification

$\sum_n y(n) = \left[\sum_k h(k)\right]\left[\sum_n x(n)\right] = (7)(5) = 35$

(ii) $x(n) = \{\underset{\uparrow}{1}, 2, -1\}$, $h(n) = x(n)$

$\sum_n x(n) = 2 = \sum_k h(k)$

$\sum_n y(n) = 2(2) = 4$

$y(n) = \{1, 4, 2, -4, 1\}$

$\rightarrow \sum_n y(n) = 4$

	↓ 1	2	-1
→ 1	1	2	-1
2	2	4	-2
-1	-1	-2	1

(iii) $x(n) = \{0, 1, -2, 3, -4\}$, $h(n) = \{\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}\}$

	↓ 0	1	-2	3	-4
→ 1/2	0	1/2	-1	3/2	-2
1/2	0	1/2	-1	3/2	-2
1	0	1	-2	3	-4
-1/2	0	1/2	-1	3/2	-2

$y(n) = x(n) * h(n)$

$y(n) = \{0, \frac{1}{2}, -\frac{1}{2}, \frac{3}{2}, -2, 0, \frac{-5}{2}, -2\}$

$\sum_n y(n) = -5$

$\sum_k h(k) = \frac{5}{2}$, $\sum_n x(n) = -2$

$\Rightarrow \sum_n y(n) = (\frac{5}{2})(-2) = -5.$

(iv) $x(n) = \{1, 2, 3, 4, 5\}$, $h(n) = \{1\}$

	1	2	3	4	5
1	1	2	3	4	5

$y(n) = x(n) \times h(n)$

$= \{1, 2, 3, 4, 5\}$

$\sum_k h(k) = 1$, $\sum_n x(n) = 15$

$\sum_n y(n) = (1)(15) = 15$.

(v) $x(n) = \{\underset{\uparrow}{1}, -2, 3\}$, $h(n) = \{\underset{\uparrow}{0}, 0, 1, 1, 1, 1\}$.

	0	0	1	1	1	1
→ 1	0	0	1	1	1	1
−2	0	0	−2	−2	−2	−2
3	0	0	3	3	3	3

$y(n) = \{0, 0, 1, -1, 2, 2, 1, 3\}$

$\sum_n y(n) = 8$

$\sum_n x(n) = 2$, $\sum_k h(k) = 4 \Rightarrow \sum_n y(n) = (2)(4) = 8$.

(vi) $x(n) = \{0, 0, 1, 1, 1, 1\}$, $h(n) = \{1, -2, 3\}$.

$x[n] \times y[n]$

	0	0	1	1	1	1
1	0	0	1	1	1	1
→ −2	0	0	−2	−2	−2	−2
3	0	0	3	3	3	3

$y[n] = \{0, \underset{\uparrow}{0}, 1, -1, 2, 2, 1, 3\}$

$\sum_n y(n) = \left[\sum_n h(k)\right]\left[\sum_n x(n)\right]$

$= (2)(4) = 8$.

(vii) $x(n) = \{\underset{\uparrow}{0}, 1, 4, -3\}$, $h(n) = \{\underset{\uparrow}{1}, 0, -1, -1\}$.

	1	0	−1	−1
→ 0	0	0	0	0
1	1	0	−1	−1
4	4	0	−4	−4
−3	−3	0	3	3

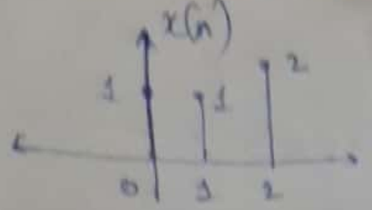
$y(n) = \{\underset{\uparrow}{0}, 1, 4, -4, -5, -1, 3\}$

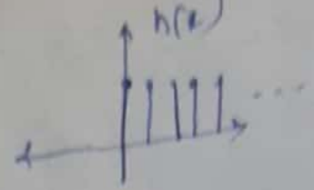
$\sum_n x(n) = 2$, $\sum_k h(k) = -1$

$\therefore \sum_n y(n) = (2)(-1) = -2$.

(viii) $x(n) = \{\underset{\uparrow}{1}, 1, 2\}$, $h(n) = u(n)$.

$\sum_n y(n) = \infty$, $\sum_n x(n) = 4$ $\sum_k h(k) = \infty$





$y(0) = h(n)x[-k]$ or $x[k]h[-k]$ $\left[\because y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)\right]$

(IX) $x(n) = \{1, 1, \underset{\uparrow}{0}, 1, 1\}$, $h(n) = \{1, -2, -3, \underset{\uparrow}{4}\}$.

	1	1	0	1	1
1	1	1	0	1	1
-2	-2	-2	0	-2	-2
-3	-3	-3	0	-3	-3
4	4	4	0	4	4

$y[n] = \{1, -1, -5, 2, 3, -5, 1, 4\}$

$\sum_n y[n] = 0$

$\sum_n x[n] = 4$, $\sum_k h(k) = 0$

$\sum_n y(n) = (4)(0) = 0$.

(X) $x[n] = \{1, 2, \underset{\uparrow}{0}, 2, 1\}$, $h(n) = x(n)$

	1	2	0	2	1
1	1	2	0	2	1
2	2	4	0	4	2
0	0	0	0	0	0
2	2	4	0	4	2
1	1	2	0	2	1

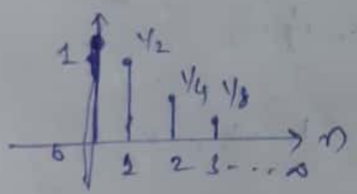
$y[n] = \{1, 4, 4, 4, \underset{\uparrow}{10}, 4, 4, 4, 1\}$

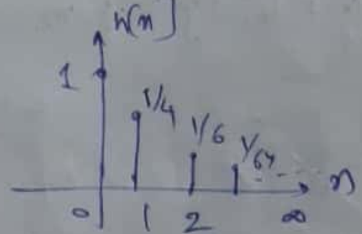
$\sum_n x[n] = 6$, $\sum_k h(k) = 6$

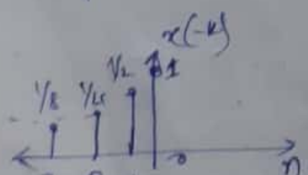
$\sum_n y[n] = \sum_n x[n] \cdot \sum_k h(k)$

$= (6)(6) = 36$.

(XI) $x[n] = \left(\frac{1}{2}\right)^n u[n]$, $h(n) = \left(\frac{1}{4}\right)^n u[n]$







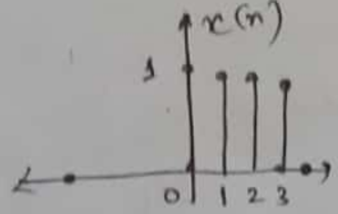
$\sum_n x(n) = \sum_n \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2$

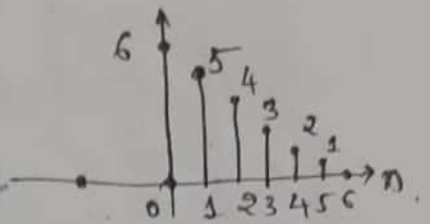
$\sum_k h(k) = \sum_k \left(\frac{1}{4}\right)^n = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$

$\Rightarrow \sum_n y[n] = \left(\sum_n x(n)\right)\left(\sum_k h(k)\right) = (2)\left(\frac{4}{3}\right) = \frac{8}{3}$

2-17) Compute and plot the convolutions $x(n) \times h(n)$ and $h(n) \times x(n)$ for the pair of signals shown in figure.

(i)





$x(n) \times h(n) = y_1[n]$

	6	5	4	3	2	1
→ 1	6	5	4	3	2	1
1	6	5	4	3	2	1
1	6	5	4	3	2	1
1	6	5	4	3	2	1

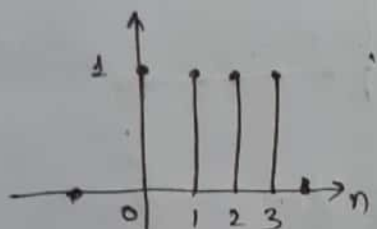
$y_1[n] = \{6, 11, 15, 18, 14, 10, 6, 3, 1\}$

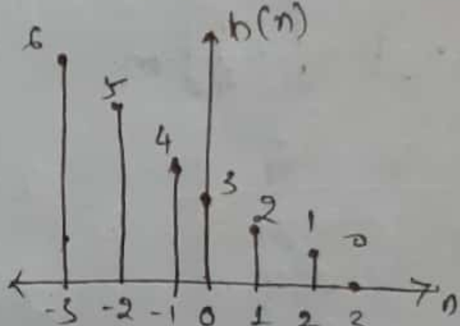
$y_2(n] = h(n) \times x(n)$

	1	1	1	1
→ 6	6	6	6	6
5	5	5	5	5
4	4	4	4	4
3	3	3	3	3
2	2	2	2	2
1	1	1	1	1

$y_2(n) = \{\underset{\uparrow}{6}, 11, 15, 18, 14, 10, 6, 3, 1\}$

(ii)





	6	5	4	3	2	1
→ 1	6	5	4	3	2	1
1	6	5	4	3	2	1
1	6	5	4	3	2	1
1	6	5	4	3	2	1

$y_1[n] = \{6, 11, 15, \underset{\uparrow}{18}, 14, 10, 6, 3, 1\}$

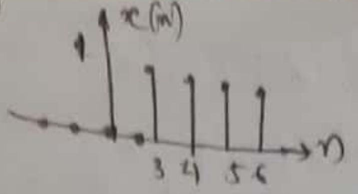
$= x(n) * h(n)$.

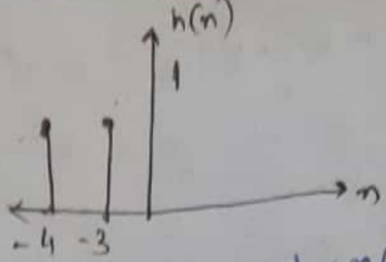
$y_2(n) = h(n) \times x(n)$

	1	1	1	1
6	6	6	6	6
5	5	5	5	5
4	4	4	4	4
→ 3	3	3	3	3
2	2	2	2	2
1	1	1	1	1

$y_2[n] = \{6, 11, 15, 18, 14, 10, 6, 3, 1$

(iii)





$y_1(n) = x(n) \times h(n)$

	1	1	0	0	↓ 0
→ 0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
1	1	1	0	0	0
1	1	1	0	0	0
1	1	1	0	0	0
1	1	1	0	0	0

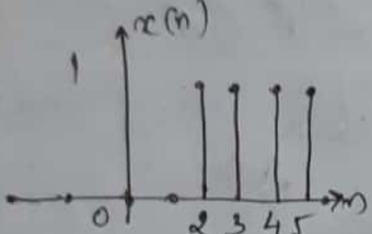
$y_1(n) = \{0,0,0,1,2,2,2,1,0,0,0\}$

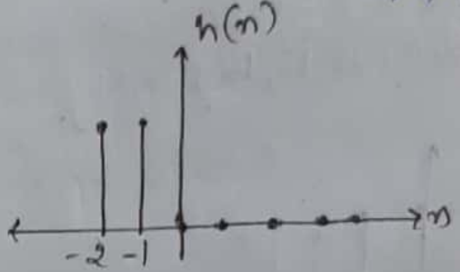
$y_2(n) = h(n) \times x(n)$

	↓ 0	0	0	1	1	1	1
1	0	0	0	1	1	1	1
1	0	0	0	1	1	1	1
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
→ 0	0	0	0	0	0	0	0

$y_2(n) = \{0,0,0,1,2,2,2,1,0,0,0\}$.

(iv)





	1	1	↓ 0
→ 0	0	0	0
0	0	0	0
1	1	1	0
1	1	1	0
1	1	1	0
1	1	1	0

$y_1(n) = x(n) \times h(n)$

$= \{0,0,1,2,2,2,1,0\}$.

	↓ 0	0	1	1	1	1
1	0	0	1	1	1	1
1	0	0	1	1	1	1
→ 0	0	0	0	0	0	0

$y_2(n) = h(n) \times x(n)$

$= \{0,0,1,2,2,2,1,0\}$.

2.18) Determine and sketch the convolution y(n) of the signals.

$$x(n) = \begin{cases} \frac{n}{3} & 0 \le n \le 6 \\ 0 & \text{elsewhere} \end{cases} \qquad h(n) = \begin{cases} 1 & -2 \le n \le 2 \\ 0, & \text{elsewhere} \end{cases}$$

both Graphically and Analytically.

(i) $x(n) = \{\underset{\uparrow}{0}, \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, 2\}$

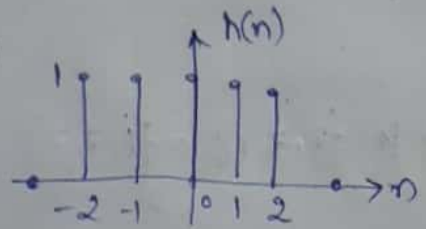
$h(n) = \{1, 1, \underset{\uparrow}{1}, 1, 1\}$

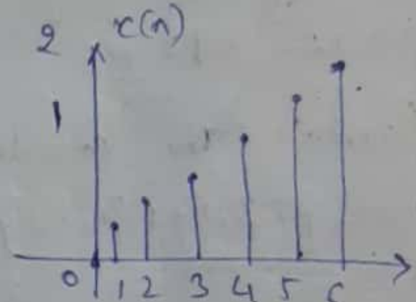
$y(n) = x(n) \times h(n)$

$y(n) = \{0, \frac{1}{3}, \frac{1}{2}, 2, \frac{10}{3}, 4, \frac{17}{3}, 5, 4, \frac{11}{3}, 2\}$

	1	1	1	1	1
→ 0	0	0	0	0	0
1/3	1/3	1/3	1/3	1/3	1/3
2/3	2/3	2/3	2/3	2/3	2/3
1	1	1	1	1	1
4/3	4/3	4/3	4/3	4/3	4/3
5/3	5/3	5/3	5/3	5/3	5/3
2	2	2	2	2	2

(ii)



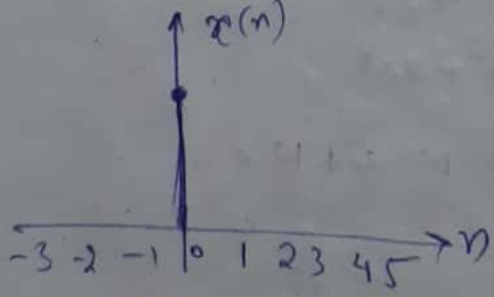


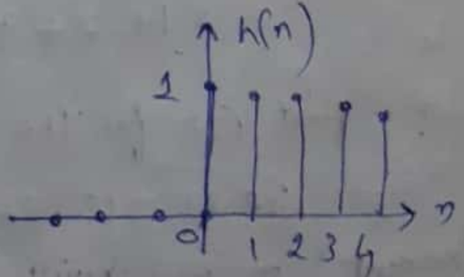
$$y[n] = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$$y(0) = x(k)\, h(-k)$$

2.19) Compute the convolution y(n) of the signals.

$$x(n) = \begin{cases} \alpha^n & -3 \le n \le 5 \\ 0 & \text{elsewhere} \end{cases} \qquad h(n) = \begin{cases} 1 & 0 \le n \le 4 \\ 0 & \text{elsewhere} \end{cases}$$





$x(n) = \{\alpha^{-3}, \alpha^{-2}, \alpha^{-1}, \underset{\uparrow}{1}, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5\}$

$h(n) = \{\underset{\uparrow}{1}, 1, 1, 1, 1\}$

	α^{-3}	α^{-2}	α^{-1}	$\downarrow$	α	α^{2}	α^{3}	α^{4}	α^{5}
$\rightarrow$ 1	α^{-3}	α^{-2}	α^{-1}		α	α^{2}	α^{3}	α^{4}	α^{5}
1	α^{-3}	α^{-2}	α^{-1}	1	α	α^{2}	α^{3}	α^{4}	α^{5}
1	α^{-3}	α^{-2}	α^{-1}	1	α	α^{2}	α^{3}	α^{4}	α^{5}
1	α^{-3}	α^{-2}	α^{-1}	1	α	α^{2}	α^{3}	α^{4}	α^{5}
1	α^{-3}	α^{-2}	α^{-1}	1	α	α^{2}	α^{3}	α^{4}	α^{5}

$y(n) = \{\alpha^{-3}, \alpha^{-3}+\alpha^{-2}, \alpha^{-1}+\alpha^{-2}+\alpha^{-3}, \underset{\uparrow}{1+\alpha^{-1}+\alpha^{-2}+\alpha^{-3}}, \alpha^{-3}+\alpha^{-2}+\alpha^{-1}+1+\alpha,$
$\alpha^{-2}+\alpha^{-1}+1+\alpha+\alpha^{2}, \alpha^{-1}+1+\alpha+\alpha^{2}+\alpha^{3}, 1+\alpha+\alpha^{2}+\alpha^{3}+\alpha^{4},$
$\alpha+\alpha^{2}+\alpha^{3}+\alpha^{4}+\alpha^{5}, \alpha^{2}+\alpha^{3}+\alpha^{4}+\alpha^{5}, \alpha^{3}+\alpha^{4}+\alpha^{5}, \alpha^{4}+\alpha^{5}, \alpha^{5}$

2.20] Consider the following three operations

(a) Multiply the integer numbers 131 and 122.

$131 \times 122 = 15982$

(b) Compute the convolution of signals $\{1,3,1\} \times \{1,2,2\}$

$y(n) = \{1,3,1\} * \{\underset{\uparrow}{1},2,2\}$

	1	2	2
1	1	2	2
3	3	6	6
1	1	2	2

$y(n) = \{1, 5, 9, 8, 2\}$

(c) Multiply the polynomials $1+3z+z^2$ and $1+2z+2z^2$.

$(1+3z+z^2)(1+2z+2z^2) = 1+2z+2z^2+3z+6z^2+6z^3+z^2+2z^3+2z^4.$

$= 1+5z+9z^2+8z^3+2z^4.$

(d) Repeat part (a) for the numbers 1.31 and 12.2.

$(1.31)(12.2) = 15.982$

(e) Comment on your results.

There are different ways to perform convolution.

2.21) Compute the convolution $y(n) = x(n) * h(n)$ of the following pair of signals.

(a) $x(n) = \alpha^n u[n]$, $h(n) = \beta^n u[n]$ where when $\alpha = \beta$ and $\alpha \neq \beta$

$$y[n] = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$$= \sum_{k=-\infty}^{\infty} \alpha^k u[k]\, \beta^{n-k} u(n-k) = \sum_{k=0}^{n} (\alpha\beta^{-1})^k \beta^n .$$

$$\Rightarrow y(n) = \beta^n \sum_{k=0}^{n} (\alpha\beta^{-1})^k$$

$$y(n) = \begin{cases} \dfrac{b^{n+1} - a^{n+1}}{b-a}\, u(n) & a \neq b \\ b^n (n+1)\, u(n) & a = b . \end{cases}$$

(b) $x(n) = \begin{cases} 1 & n = -2, 0, 1 \\ 2 & n = -1 \\ 0 & \text{elsewhere} \end{cases}$, $h(n) = \delta(n) - \delta(n-1) + \delta(n-4) + \delta(n-5)$

$x(n) = \{1, 2, \underset{\uparrow}{1}, 1\}$, $h(n) = \{\underset{\uparrow}{1}, -1, 0, 0, 1, 1\}$

	1	-1	0	0	1	1
1	1	-1	0	0	1	1
2	2	-2	0	0	2	2
→ 1	1	-1	0	0	1	1
1	1	-1	0	0	1	1

$y(n) = \{1, 1, -1, 0, 0, 3, 3, 2, 1\}$

(c) $x(n) = u(n+1) - u(n-4] - \delta(n-5)$

$h(n) = [u(n+2) - u(n-3)]\,(3 - |n|)$

$x_1(n) = \{1, 1, 1, 1, 1, 0, -1\}$ $h(n) = \{1, 2, \underset{\uparrow}{3}, 2, 1\}$

	1	1	1	1	1	0	-1
1	1	1	1	1	1	0	-1
2	2	2	2	2	2	0	-2
→ 3	3	3	3	3	3	0	-3
2	2	2	2	2	2	0	-2
1	1	1	1	1	1	0	-1

$y(n) = \{1, 3, 6, \underset{\uparrow}{8}, 9, 8, 5, 1, -2, -1, -1\}$

$x(n) = u(n) - u(n-5)$, $h(n) = u(n-2) - u(n-8) + u(n-11)$
$- u(n-17)$

$x(n) = \{\underset{\uparrow}{1}, 1, 1, 1, 1\}$

$h(n) = \{\overset{\uparrow}{0}, 0, 1, 1, 1, 1, 1, 1, 0, 0, 0, 1, 1, 1, 1, 1, 1\}$

	0	0	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1
1	0	0	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1
1	0	0	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1
1	0	0	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1
1	0	0	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1
1	0	0	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1

$y(n) = \{\overset{\uparrow}{0}, 0, 1, 2, 3, 4, 5, 5, 4, 3, 2, 2, 2, 3, 4, 5, 5, 4, 3, 2, 1\}$

2.22) Let $x(n)$ be input signal to a discrete time filter with impulse response $h_i(n)$ and $y_i(n)$ be the corresponding output

(a) Compute and sketch $x(n)$ and $y_i(n)$ using the same scale in all figures.

$x(n) = \{1, 4, 2, 3, 5, 3, 3, 4, 5, 7, 6, 9\}$

$h_1(n) = \{1, 1\}$, $h_2(n) = \{1, 2, 1\}$, $h_3(n) = \{\frac{1}{2}, \frac{1}{2}\}$.

$h_4(n) = \{\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\}$ $h_5(n) = \{\frac{1}{4}, -\frac{1}{2}, \frac{1}{4}\}$

Sketch $x(n)$, $y_1(n)$, $y_2(n)$ on one graph and $x(n)$, $y_3(n)$, $y_4(n)$, $y_5(n)$ on another graph.

$y_1(n) = x(n) * h_1(n)$

	1	1
1	1	1
4	4	4
2	2	2
3	3	3
5	5	5
3	3	3
3	3	3
4	4	4
5	5	5
7	7	7
6	6	6
9	9	9

$y_2(n) = x(n) * h_2(n)$

	1	2	1
1	1	2	1
4	4	8	4
2	2	4	2
3	3	6	3
5	5	10	5
3	3	6	3
3	3	6	3
4	4	8	4
5	5	10	5
7	7	14	7
6	6	12	6
9	9	18	9

$y_1(n) = \{1, 5, 6, 5, 8, 8, 6, 7, 9, 12, 13, 15, 9\}$

$y_2(n) = \{1, 6, 11, 11, 13, 16, 14, 13, 16, 21, 25, 28, 24, 9\}$

	1/2	1/2
1	1/2	1/2
4	2	2
2	1	1
3	3/2	3/2
5	5/2	5/2
3	3/2	3/2
3	3/2	3/2
4	2	2
5	5/2	5/2
7	7/2	7/2
6	3	3
9	9/2	9/2

	1/4	1/2	1/4
1	1/4	1/2	1/4
4	1	2	1
2	1/2	1	1/2
3	3/4	3/2	3/4
5	5/4	5/2	5/4
3	3/4	3/2	3/4
3	3/4	3/2	3/4
4	1	2	1
5	5/4	5/2	5/4
7	7/4	7/2	7/4
6	6/4	3	6/4
9	9/4	9/2	9/4

$y_3[n] = \{\frac{1}{2}, 5/2, 3, 5/2, 4, 3, 7/2, 9/2, 6, \frac{13}{2}, 15/2, 9/2\}$

$y_4(n) = \{\frac{1}{4}, 3/2, 11/4, 11/4, 13/4, 8/2, 14/4, 13/4, 8/2, \frac{21}{4}, \frac{25}{4}, 7, 6, 9/4\}$

	-1/4	1/2	-1/4
1	1/4		
4	1		
2	1/2		
3	3/4		
5	5/4		
3			
3			
4			
5			
7			
6			
9			

	-1/4	-1/2	1/4
1	1/4	-1/2	1/4
4	1	-2	1
2	1/2	-1	1/2
3	3/4	-3/2	3/4
5	5/4	-5/2	5/4
3	3/4	-3/2	3/4
3	3/4	-3/2	3/4
4	1	-2	1
5	5/4	-5/2	5/4
7	7/4	-7/2	7/4
6	6/4	-3	6/4
9	9/4	-9/2	9/4

$y_5(n) = \{\frac{1}{4}, \frac{1}{2}, -\frac{5}{4}, \frac{3}{4}, \frac{1}{4}, -1, \frac{1}{2}, \frac{1}{4}, 0, \frac{1}{4}, -\frac{3}{4}, 1, -3, 9/4\}$

(b) What is the difference b/w $y_1(n)$ and $y_2(n)$ and b/w $y_3(n)$, $y_4(n)$?

$y_3(n) = \frac{1}{2} y_1(n)$ because $h_3(n) = \frac{1}{2} h_1(n)$

$y_4(n) = \frac{1}{4} y_2(n)$ because $h_4(n) = \frac{1}{4} h_2(n)$

(c) Comment on the smoothness of $y_1(n)$ and $y_2(n)$ which factors affect the smoothness?

$y_2(n)$ and $y_1(n)$ are smoother than $y_1(n)$, but $y_4(n)$ will appear even smoother because of smaller scale factor.

(d) Compare $y_4(n)$ with $y_5(n)$ what is difference? Can you explain it?

System 4 results in a smoother output. The negative half $h_5(n)$ is responsible for the non-smooth characteristics of $y_5(n)$

(e) Let $h_6(n) = \{\frac{1}{2}, \frac{1}{2}\}$ Compare $y_6(n)$. Sketch $x(n)$, $y_2(n)$ and $y_6(n)$ on the same figure and comment on the results?

	1/2	-1/2
1	1/2	-1/2
4	2	-2
2	1	-1
3	3/2	-3/2
5	5/2	-5/2
3	3/2	-3/2
3	3/2	-3/2
4	2	-2
5	5/2	-5/2
7	7/2	-7/2
6	3	-3
7	9/2	-9/2

$$y_6[n] = \{\overset{\uparrow}{1/2}, 3/2, 1/2, 1, -1, 0, 1/2, \frac{1}{2}, 1, \frac{-1}{2}, \frac{3}{2}, -9/2\}$$

$\Rightarrow y_2(n)$ is smoother than $y_6(n)$

2.23) The discrete time system

$$y(n) = n\,y[n-1) + x(n) \qquad n \geq 0$$ is at rest [i.e., $y(-1) = 0$]. Check if the system is linear time invariant and BIBO stable.

$$x_1(n) \longrightarrow y_1(n) = n y_1(n-1) + x_1(n)$$

$$x_2(n) \longrightarrow y_2(n) = n y_2(n-1) + x_2(n)$$

$a x_1(n) + b x_2(n) \longrightarrow a y_1(n) + b y_2(n)$

$= a\left(n y_1(n-1) + x_1(n)\right) + b\left(n y_2(n-1) + x_2(n)\right)$

$= n\left[a y_1(n-1) + b y_2(n-1)\right] + \left(a x_1(n) + b x_2(n)\right)$

Hence the system is linear

$x(n) \longrightarrow y(n) = n(y(n-1)) + x(n)$

$x(n-1) \longrightarrow y_1(n) = (n-1)\, y(n-2) + x(n-1)$

But $y_2(n) = y(n-1) = n y(n-2) + x(n-1)$

$\therefore\ y_1(n) \neq y_2(n)$

Hence the system is Time variant.

If $x(n) = u(n)$, then $|x(n)| \leq 1$. But for this bounded input, the output is,

$y(0) = 1$, $y(1) = 1+1 = 2$, $y(2) = 2\times 2+1 = 5$

$y(3) = 3\times 5+1 = 16$, $y(4) = 4\times 16+1 = 65$, ...

which is unbounded. Hence, the system is unstable.

2.24) Consider the signal $r(n) = \alpha^n u(n)$, $0 < \alpha < 1$

(a) show that any sequence $x(n)$ can be decomposed as $x(n) = \sum_{n=-\infty}^{\infty} c_k r(n-k)$ and express c_k in terms of $x(n)$.

$\delta(n) = r[n] - a r(n-1)$

$\Rightarrow \delta(n-k) = r(n-k) - a r(n-k-1)$

we know that $x(n)$ can be written as $x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$

$x(n) = \sum_{k=-\infty}^{\infty} x(k)\left[r(n-k) - a r[n-k-1)\right]$

$= \sum_{k=-\infty}^{\infty} x[k]\, r(n-k) - a \sum_{k=-\infty}^{\infty} x(k)\, r(n-k-1)$

$= \sum_{k} x(k)\, r(n-k) - a \sum_{k} x(k-1)\, r(n-k)$

$\Rightarrow x(n) = \sum_{k} [x(k) - ax(k-1)]\, \gamma(n-k)$

$\therefore \; c_k = x(k) - ax(k-1)$

(b) Use the properties of linearity and time invariance to express output $y(n) = T(x(n))$ in terms of input $x(n)$ and the signal $g(n) = T(y(n))$. where $T(.)$ is an LTI system.

$$y(n) = T(x(n)) = T\left[\sum_{k} c_k\, \gamma(n-k)\right]$$

$$= \sum_{k=-\infty}^{\infty} c_k T[\gamma(n-k)]$$

$$= \sum_{k=-\infty}^{\infty} c_k\, g(n-k)$$

(c) Express the impulse response $h(n) = T[\delta(n)]$ in terms of $g(n)$.

$$h(n) = T[\delta(n)]$$

$$= T[\gamma(n) - a\gamma(n-1)]$$

$$= g(n) - ag(n-1)$$

2.25) Determine the zero-input response of the system described by the second-order difference equation.

$$x(n) - 3y(n-1) - 4y(n-2) = 0$$

Zero-input response is the response of the system when no input is given.

$$\Rightarrow x(n) = 0 \;\Rightarrow\; -3y(n-1) - 4y(n-2) = 0$$

$$y(n-1) + \frac{4}{3} y(n-2) = 0$$

$$\Rightarrow \; y(n-1) = -\frac{4}{3} y(n-2)$$

when $n=0$, $\; y(-1) = \frac{-4}{3} y(-2)$

$n=1$, $\; y(0) = -\frac{4}{3} y(-1) = \left(\frac{4}{3}\right)^2 y(-2)$

$n=2$, $y(1) = \frac{-4}{3} y(0) = -\left(\frac{4}{3}\right)^2 y(-2)$

$\therefore n = k+1$, $y(k) = \left(\frac{-4}{3}\right)^{k+2} y(-2) \longrightarrow$ Zero Input response

2.26] Determine the particular solution of the difference equation $y(n) = \frac{5}{6} y(n-1) - \frac{1}{6} y[n-2] + x(n)$ when the forcing function is $x(n) = 2^n u(n)$?

Consider the homogeneous equation

$$y(n) - \frac{5}{6} y(n-1) + \frac{1}{6} y(n-2) = 0$$

Characteristic equation is $\lambda^2 - \frac{5}{6}\lambda + \frac{1}{6} = 0$

$\Rightarrow 6\lambda^2 - 5\lambda + 1 = 0 \quad \Rightarrow \lambda = \frac{1}{2}, \frac{1}{3}$

$\therefore \quad y_h(n) = C_1 \left(\frac{1}{2}\right)^n + C_2 \left(\frac{1}{3}\right)^n$

The particular solution to $x(n) = 2^n u[n]$ is

$$y_p(n) = k(2^n)\, u(n)$$

Substitute the solution into the difference equation.

$$k(2^n) u(n) - k\left(\frac{5}{6}\right)(2^{n-1}) u(n-1) + k\left(\frac{1}{6}\right) 2^{n-2} u(n-2) = 2^n u(n)$$

For $n=2$,

$$4k - \frac{5k}{3} + \frac{k}{6} = 4 \quad \Rightarrow \quad 24k - 10k + k = 24$$

$$15k = 24$$

$$\Rightarrow k = \frac{24}{15} = 8/5$$

Therefore

$y(n) = y_p(n) + y_h(n)$

$= \frac{8}{5} 2^n u(n) + C_1 \left(\frac{1}{2}\right)^n u(n) + C_2 \left(\frac{1}{3}\right)^n u(n)$

To determine C_1 and C_2, assume that $y(-2) = y(-1) = 0$. Then,

$y(0) = 1$ and $y(1) = \frac{5}{6} y(0) + 2 = \frac{17}{6}$.

Thus $\frac{8}{5} + C_1 + C_2 = 1 \Rightarrow C_1 + C_2 = -3/5$

$$\frac{16}{5} + \frac{C_1}{2} + \frac{C_2}{3} = \frac{17}{6} \quad \Rightarrow 3C_1 + 2C_2 = -11/5$$

$\therefore\ C_1 = 1,\ C_2 = 2/5$

The total solution is $y(n) = \left[\frac{8}{5}(2)^n - \left(\frac{1}{2}\right)^n + \frac{2}{5}\left(\frac{1}{3}\right)^n\right]u(n)$

2.2.7) Determine the response $y(n)$, $n \geq 0$ of the system described by second-order difference equation.

$$y(n) - 3y(n-1) - 4y(n-2) = x(n) + 2x(n-1)$$ to the input

$x(n) = 4^n u(n)$.

Sol: $y(n) - 3y(n-1) - 4y(n-2) = x(n) + 2x(n-1)$

Characteristic equation is

$$\lambda^2 - 3\lambda - 4 = 0$$

$\Rightarrow \lambda = 4, -1$

$$y_h(n) = C_1 4^n + C_2(-1)^n = C_1 4^n + C_2(-1)^n.$$

Given the input is $x(n) = 4^n u(n)$

we assume a particular solution of the form

$y_p(n) = Kn4^n u(n)$

Then $Kn4^n u(n) - 3K(n-1)4^{n-1}u(n-1) - 4K(n-2)4^{n-2}u(n-2)$

$$= 4^n u(n) + 2(4^{n-1})u(n-1)$$

For $n = 2$

$$K(32-12) = 4^2 + 8 = 24 \Rightarrow K = 6/5$$

The total solution is

$$y(n) = y_p(n) + y_h(n) = \left[\frac{6}{5}n4^n + C_1 4^n + C_2(-1)^n\right]u(n)$$

To solve for C_1 and C_2 we assume that $y(-1) = y(-2) = 0$

Then $y(0) = 1$ and

$$y(1) = 3y(0) + 4 + 2 = 9$$

Hence $C_1 + C_2 = 1$ and $\frac{24}{5} + 4C_1 - C_2 = 9 \Rightarrow 4C_1 - C_2 = \frac{21}{5}$

$\therefore\ C_1 = \frac{26}{25}$ and $C_2 = -1/25$

$\therefore\ y(n) = \left[\frac{6}{5} n 4^n + \frac{26}{25} 4^n - \frac{1}{25}(-1)^n\right] u(n)$

2.28) Determine the impulse response of the following causal system. $y(n) - 3y(n-1) - 4y(n-2) = x(n) + 2x(n-1)$

$y(n) - 3y(n-1) - 4y(n-2) = x(n) + 2x(n-1)$

The characteristic equation is $\lambda^2 - 3\lambda - 4 = 0$

$\Rightarrow \lambda = 4, -1$

$\therefore\ y_h(n) = C_1 4^n + C_2 (-1)^n$

when $x(n) = \delta(n)$, we find out

$y(0) = 1$ and $y(1) - 3y(0) = 2$ or $y(1) = 5$

Hence $C_1 + C_2 = 1$ and $4C_1 - C_2 = 5$

This yields $C_1 = 5/3$ and $C_2 = -1/5$

$\therefore\ h(n) = \left[\frac{6}{5} 4^n - \frac{1}{5}(-1)^n\right] u(n).$

2.29) Let $x(n)$ $N_1 \le n \le N_2$ and $h(n)$, $M_1 \le n \le M_2$ be two finite duration signals.

(a) Determine the range $L_1 \le n \le L_2$ of their convolution, in terms of N_1, N_2, M_1 and M_2.

$L_1 = N_1 + M_1$, $L_2 : N_2 + M_2$

(b) Determine the limit of the cases of partial overlap from the left, full overlap, and partial overlap from the right, assume that $h(n)$ has shorter duration than $x(n)$.

Partial overlap from right

low $N_2 + M_1 + 1$ high $N_2 + M_2$

Full overlap

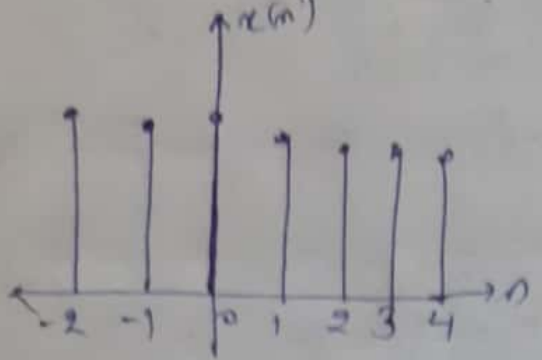
low $N_1 + M_2$ high $N_2 + M_1$

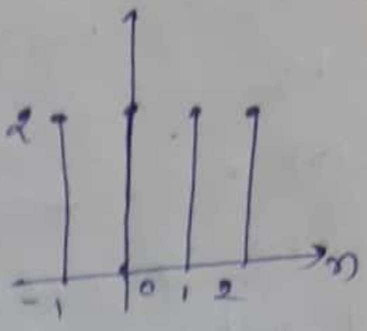
Partial overlap from left:

low $N_1 + M_1$ high $N_1 + M_2$

(c) Illustrate the validity of your results by computing the convolution of the signals-

$$x(n) = \begin{cases} 1 & -2 \le n \le 4 \\ 0 & \text{elsewhere} \end{cases} \qquad h(n) = \begin{cases} 2 & -1 \le n \le 2 \\ 0 & \text{elsewhere} \end{cases}$$





$y(n) = x(n) * h(n)$, $L_1 = 7$, $L_1 = 4$

	2	↓ 2	2	2
1	2	2	2	2
1	2	2	2	2
→ 1	2	2	2	2
1	2	2	2	2
1	2	2	2	2
1	2	2	2	2
1	2	2	2	2

$y(n) = \{2, 4, 6, \underset{\uparrow}{8}, 8, 8, 8, 6, 4, 2\}$

$L = 10$

$N_1 = -2 \qquad M_1 = -1$

$N_2 = 4 \qquad M_2 = 2$

Partial overlap from left : $n = -3$ $n = -1$ $L_1 = -3$

Full overlap : $n = 0$, $n = 3$

Partial overlap from right : $n = 4$ $n = 6$ $L_2 = 6$

2.30) Determine the impulse response and unit step response of the system described by the difference equation

(a) $y(n) = 0.6\,y(n-1) - 0.08\,y(n-2) + x(n)$

$y(n) - 0.6\,y(n-1) + 0.08\,y(n-2) = x(n)$

The characteristic equation is $\lambda^2 - 0.6\lambda + 0.08 = 0$

$\Rightarrow \lambda = \frac{2}{5}, \frac{1}{5}$.

$\Rightarrow y_h(n) = c_1\left(\frac{1}{5}\right)^n + c_2\left(\frac{2}{5}\right)^n$

(i) Given $x(n) = \delta(n) = \begin{cases} 1 & n=0 \\ 0 & n\neq 0 \end{cases}$

when $n=0 \Rightarrow y(0) = x(0) = 1$

$n=1 \Rightarrow y(1) - 0.6y(0) = 0 \Rightarrow y(1) = 0.6.$

$\therefore$ when $n=0 \Rightarrow c_1 + c_2 = 1 \Rightarrow c_2 = 1 - c_1$

$n=1 \Rightarrow c_1\left(\frac{1}{5}\right) + c_2\left(\frac{2}{5}\right) = 0.6$

$\Rightarrow c_1\left(\frac{1}{5}\right) + (1-c_1)\left(\frac{2}{5}\right) = 0.6$

$\left(\frac{1}{5} - \frac{2}{5}\right)c_1 = 0.2$

$-\frac{1}{5}c_1 = 0.2 \Rightarrow c_1 = -1$

$\therefore c_2 = 2.$

$\therefore h(n) = \left[\left(-\frac{1}{5}\right)^n + 2\left(\frac{2}{5}\right)^n\right] u(n) = \left[\left(-\frac{1}{5}\right)^n + 2\left(\frac{2}{5}\right)^n\right] u(n)$

(ii) If Given $x(n) = u(n)$.

$s[n] = \sum_{k=-\infty}^{\infty} u(k)\,h(n-k) = \sum_{k=0}^{n} h(n-k)$

$= \sum_{k=0}^{n}\left[2\left(\frac{2}{5}\right)^{n-k} - \left(\frac{1}{5}\right)^{n-k}\right].$

(b)

$y(n) = 0.7y(n-1) - 0.1y(n-2) + 2x(n) - x(n-2)$

$y(n) - 0.7y(n-1) + 0.1y(n-2) = 2x(n) - x(n-2)$

The characteristic equation is

$\lambda^2 - 0.7\lambda + 0.1 = 0 \Rightarrow \lambda = \frac{1}{2}, \frac{1}{5}$

$\Rightarrow y_h(n) = c_1\left(\frac{1}{2}\right)^n + c_2\left(\frac{1}{5}\right)^n$

(i) Given $x(n) = \delta(n) = \begin{cases} 1 & n=0 \\ 0 & n \neq 0 \end{cases}$

When $n=0 \Rightarrow y(0) = 2x(0) = 2$

$n=1 \Rightarrow y(1) - 0.7y(0) = 0 \Rightarrow y(1) = 1.4$

$\therefore c_1 + c_2 = 2 \Rightarrow c_2 = 2 - c_1$

$c_1\left(\frac{1}{2}\right) + c_2\left(\frac{1}{5}\right) = 1.4 \Rightarrow c_1\left(\frac{1}{2}\right) + (2-c_1)\left(\frac{1}{5}\right) = 1.4$

$\therefore \frac{3}{10}c_1 = \frac{-1}{1} \Rightarrow c_1 = \frac{-10}{3}$ and $c_2 = \frac{4}{3}$

$\therefore h[n] = \left[\left(\frac{10}{3}\right)\left(\frac{1}{2}\right)^n + \left(\frac{-4}{3}\right)\left(\frac{1}{5}\right)^n\right]u(n)$

$\therefore h(n) = \left[\frac{10}{3}\left(\frac{1}{2}\right)^n + \left(-\frac{4}{3}\right)\left(\frac{1}{5}\right)^n\right]u(n)$

(ii) Given $x(n) = u(n)$

$s(n) = \sum_{k=0}^{n} h(n-k)$

$= \sum_{k=0}^{n}\left[\left(\frac{10}{3}\right)\left(\frac{1}{2}\right)^{n-k} + \left(-\frac{4}{3}\right)\left(\frac{1}{5}\right)^n \sum_{k=0}^{n} 5^k\right.$

$= \left(\frac{10}{3}\right)\left(\frac{1}{2}\right)^n \sum_{k=0}^{n} 2^k - \left(\frac{4}{3}\right)\left(\frac{1}{5}\right)^n \sum_{k=0}^{n} 5^k$

$= \frac{10}{3}\left(\frac{1}{2}\right)^n (2^{n+1} - 1)u(n) - \left(\frac{4}{3}\right)\left(\frac{1}{5}\right)^n (5^{n+1} - 1)u(n).$

2.31) Consider a system with Impulse response

$$h(n) = \begin{cases} \left(\frac{1}{2}\right)^n & 0 \le n \le 4 \\ 0 & \text{elsewhere} \end{cases}$$

Determine the input $x(n)$ for $0 \le n \le 8$ that will generate the output sequence $y(n) = \{\underset{\uparrow}{1}, 2, 2, 5, 3, 3, 3, 2, 1, 0, \ldots\}$

$$h(n) = \left\{1, \left(\tfrac{1}{2}\right), \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{16}\right\}$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
1	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
1/2	$x_0/2$	$\frac{x_1}{2}$	$\frac{x_2}{2}$	$\frac{x_3}{2}$	$\frac{x_4}{2}$	$\frac{x_5}{2}$	$\frac{x_6}{2}$	$\frac{x_7}{2}$	$\frac{x_8}{2}$
1/4	$\frac{x_0}{4}$	$\frac{x_1}{4}$	$\frac{x_2}{4}$	$\frac{x_3}{4}$	$\frac{x_4}{4}$	$\frac{x_5}{4}$	$\frac{x_6}{4}$	$\frac{x_7}{4}$	$\frac{x_8}{4}$
1/8	$\frac{x_0}{8}$	$\frac{x_1}{8}$	$\frac{x_2}{8}$	$\frac{x_3}{8}$	$\frac{x_4}{8}$	$\frac{x_5}{8}$	$\frac{x_6}{8}$	$\frac{x_7}{8}$	$\frac{x_8}{8}$
1/16	$\frac{x_0}{16}$	$\frac{x_1}{16}$	$\frac{x_2}{16}$	$\frac{x_3}{16}$	$\frac{x_4}{16}$	$\frac{x_5}{16}$	$\frac{x_6}{16}$	$\frac{x_7}{16}$	$\frac{x_8}{16}$

$\therefore y(n) = \Big\{ x_0,\ \frac{x_0}{2} + x_1,\ \frac{x_0}{4} + \frac{x_1}{2} + x_2,\ \frac{x_0}{8} + \frac{x_1}{4} + \frac{x_2}{2} + x_3,$

$\frac{x_0}{16} + \frac{x_1}{8} + \frac{x_2}{4} + \frac{x_3}{2} + x_4,\ \frac{x_1}{16} + \frac{x_2}{8} + \frac{x_3}{4} + \frac{x_4}{4} + \frac{x_4}{2} + x_5,$

$\frac{x_2}{16} + \frac{x_3}{8} + \frac{x_4}{4} + \frac{x_5}{2} + x_6,\ \frac{x_3}{16} + \frac{x_4}{8} + \frac{x_5}{4} + \frac{x_6}{2} + x_7,$

$\frac{x_4}{16} + \frac{x_5}{8} + \frac{x_6}{4} + \frac{x_7}{2} + x_8,\ \frac{x_5}{16},\ \frac{x_6}{8} + \frac{x_7}{4} + \frac{x_8}{2},\ \frac{x_6}{16} + \frac{x_7}{8} + \frac{x_8}{4},$

$\frac{x_7}{16} + \frac{x_8}{8},\ \frac{x_8}{16} \Big\}$

Comparing we get

$x_0 = 1$, $\frac{x_0}{2} + x_1 = 2 \Rightarrow x_1 = 2 - \frac{1}{2} = \frac{3}{2}$.

$\Rightarrow \frac{x_0}{4} + \frac{x_1}{2} + x_2 = 2.5 \Rightarrow x_2 = 2 - \frac{1}{4} - \frac{3}{4} = 2.5 - 1 = 1.5$

$x_3 + \frac{x_2}{2} + \frac{x_1}{4} + \frac{x_0}{8} = 3 \Rightarrow x_3 = 3 - \frac{5}{4} - \frac{3}{8} - \frac{1}{8} = 7/4$

$x_4 + \frac{x_3}{2} + \frac{x_2}{4} + \frac{x_1}{8} + \frac{x_0}{16} = 3$

$\Rightarrow x_4 = 3 - \frac{x_3}{2} - \frac{x_4}{2} - \frac{x_1}{8} - \frac{x_0}{16} = 3 - \frac{7}{8} - \frac{3}{8} - \frac{3}{16} - \frac{1}{16} = 3/2$

$x_5 + \frac{x_4}{2} + \frac{x_3}{4} + \frac{x_2}{8} + \frac{x_1}{16} = 3 \Rightarrow x_5 = 3 - \frac{3}{8} - \frac{7}{16} - \frac{3}{16} - \frac{3}{32}$

$\therefore x_5 = 49/32$

$x_6 + \frac{x_5}{2} + \frac{x_4}{4} + \frac{x_3}{8} + \frac{x_2}{16} = 2 \rightarrow x_6 = 2 - \frac{49}{64} - \frac{3}{8} - \frac{7}{32} - \frac{3}{32}$

$= 35/64$

$x_7 + \frac{x_6}{2} + \frac{x_5}{4} + \frac{x_4}{8} + \frac{x_3}{16} = 1$

$\Rightarrow x_7 = 1 - \frac{x_6}{2} - \frac{x_5}{4} - \frac{x_4}{8} - \frac{x_3}{16} = 1 - \frac{35}{128} - \frac{49}{128} - \frac{3}{16} - \frac{7}{64}$

$\therefore x_7 = 3/64$

$x_8 + \frac{x_7}{2} + \frac{x_6}{4} + \frac{x_5}{8} + \frac{x_4}{16} = 0 \Rightarrow x_8 = -\frac{3}{128} - \frac{35}{256} - \frac{49}{256} - \frac{3}{32}$

$x_8 = 51/64$

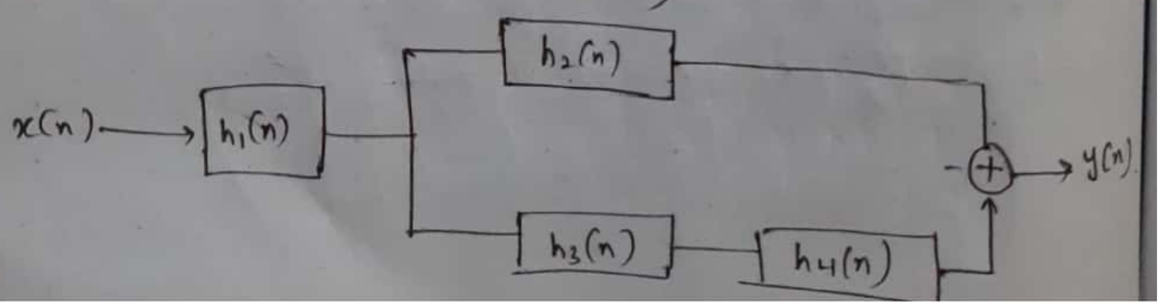
2.32) Consider the interconnection of LTI system as shown in figure.

(a) Express the overall impulse response in terms of $h_1(n)$, $h_2(n)$, $h_3(n)$ and $h_4(n)$.

(b) Determine $h(n)$ when $h_1(n) = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{2}\}$, $h_2(n) = h_3(n) = (n+1)u(n)$

$h_4(n) = \delta(n-2)$

(c) Determine the response of the system in part (b) if $x(n) = \delta(n+2) + 3\delta(n-1) - 4\delta(n-3)$

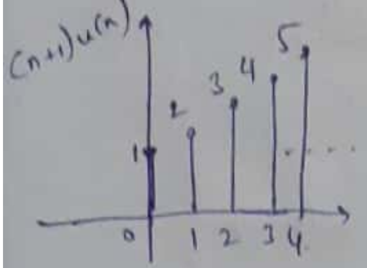


(a) $h(n) = h_1(n) * [h_2(n) - h_3(n) \times h_4(n)]$

(b) $h_3(n) \times h_4(n) = (n+1)\,u(n) \times \delta(n-2)$

$$= (n-2+1)\,u(n-2) = (n-1)\,u(n-2)$$

$h_2(n) - h_3(n) + h_4(n) = (n+1)\,u(n) - (n-1)\,u(n-2)$



$2u(n) - \delta(n)$

$\Rightarrow h(n) = \left(\frac{1}{2}\delta(n) + \frac{1}{4}\delta(n-1) + \frac{1}{2}\delta(n-2)\right] * (2u(n) - \delta(n))$

$$= u(n) + \frac{1}{2}u(n-1) + u(n-2) - \frac{1}{2}\delta(n) - \frac{1}{4}\delta(n-1) - \frac{1}{2}\delta(n-2)$$

$$= \frac{1}{2}\delta(n) + \frac{5}{4}\delta(n-1) + 2\delta(n-2) + \frac{5}{4}u(n-3)$$

(c) $y(n) = x(n) * h(n)$

$= (\delta(n+2) + 3\delta(n-1) - 4\delta(n-3)) * h(n)$

$= h(n+2) + 3h(n-1) - 4h(n-3)$

$= \frac{1}{2}\delta(n+2) + \frac{5}{4}\delta(n+1) + 2\delta(n) + \frac{5}{4}u(n-1) + \frac{3}{2}\delta(n-1) +$

$\frac{15}{4}\delta(n-2) + 6\delta(n-3) + \frac{15}{2}u(n-4) - 2\delta(n-3) - 5\delta(n-4)$

$-8\delta(n-5) - 10\delta(n-6)$

$n=-2 \rightarrow y(n) = \frac{1}{2}$, $n=-1 \rightarrow \frac{5}{4}$, $n=0 \rightarrow 2$; $n=1 \rightarrow \frac{5}{2}+\frac{3}{2}=4$

$n=2 \rightarrow \frac{15}{4} + \frac{5}{2} = \frac{25}{2}$; $n=3 \rightarrow \frac{5}{2} + 6 - 2 = 13/2$

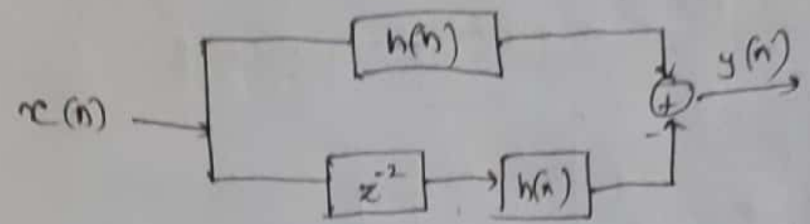
$n=4 \rightarrow \frac{5}{2} + \frac{15}{2} - 5 = 5$

$n=5 \rightarrow \frac{5}{2} + \frac{15}{2} - 8 = 2$; $n=6 \rightarrow -10 + \frac{15}{2} + \frac{5}{2} = 0$

$\therefore\ y(n) = \left\{\frac{1}{2}, \frac{5}{4}, \underset{\uparrow}{2}, \frac{25}{4}, \frac{13}{2}, 5, 2, 0, 0 \cdots\right\}$

2.37] Consider the system with $h[n] = a^n u(n), -1 < a < 1$

Determine the response $y(n)$ of the system to the excitation $x(n) = u(n+5) - u(n-10)$



$h'(n) = h(n) - h(n-2) = a^n u(n) - a^{n-2} u(n-2)$

$y(n) = x(n) * h'(n)$

$= [u(n+5) - u(n-10)] * [a^n u(n) - a^{n-2} u(n-2)]$

$= a^n u(n) * u(n+5) - a^{n-2} u(n-2) * u(n+5) - a^n u(n) * u(n-10)$

$+ a^{n-2} u(n-2) * u(n-10)$

$\therefore a^n u(n) * u(n+5) = \sum_{k=-\infty}^{\infty} u(k+5)\, a^{n-k} u(n-k) = \sum_{k=-5}^{n} a^{n-k}$

$= 1 + a + \cdots + a^{n+5} = \dfrac{a^{n+6} - 1}{a-1} u(n+5)$

$a^{n-2} u(n-2) * u(n+5) = \sum_{k=\infty} u(k+5)\, a^{n-k-2} u(n-k-2)$

$= \sum_{k=-5}^{n-2} a^{n-k-2} = \sum_{k=-3}^{n} a^{n-k}$ [replacing k by k-2]

$= 1 + a + \cdots + a^{n+3} = \dfrac{a^{n+4} - 1}{a-1} u(n+3)$

$a^n u(n) * u(n-10) = \sum_{k=-\infty}^{\infty} u(k-10)\, a^{n-k} u(n-k) = \sum_{k=10}^{n} a^{n-k}$

$= 1 + a + \cdots + a^{n-10} = \dfrac{a^{n+10+1} - 1}{a-1} u(n-10)$

$= \dfrac{a^{n-9} - 1}{a-1} u(n-10)$

$a^{n-2} u(n-2) * u(n-10) = \sum_{k=\infty} u(k-10)\, a^{n-k-2} u(n-k-2)$

$= \sum_{k=10}^{n-2} a^{n-k-2} = \sum_{k=12}^{n} a^{n-k} = 1 + a + \cdots + a^{k-12}$

$$= \frac{a^{n-11}-1}{a-1} u(n-12)$$

$$\therefore y(n) = \frac{a^{n+6}-1}{a-1} u(n+1) - \frac{a^{n+4}-1}{a-1} u(n+3) - \frac{a^{n-9}-1}{a-1} u(n-10)$$

$$+ \frac{a^{n-4}-1}{a-1} u(n-12)$$

2.34) Compute and sketch the step response of the system

$$y(n) = \frac{1}{M} \sum_{k=0}^{M-1} x(n-k)$$

$$h(n) = \frac{1}{M} [u(n) - u(n-M)]$$

$$s(n) = u(n) * h(n) = u(n) + \frac{1}{M} [u(n) - u(n-M)]$$

$$= \frac{1}{M} (u[n] * u(n)] - \frac{1}{M} [u(n) \times u(n-M)]$$

$$= \frac{1}{M} \sum_{k=\infty}^{\infty} u(n-k)\, u(k) - \frac{1}{M} \sum_{k=\infty}^{\infty} u(k)\, u(n-k-M)$$

$$= \frac{1}{M} \sum_{k=0}^{n} (1) - \frac{1}{M} \sum_{k=0}^{n-M} (1) = \frac{1}{M} \sum_{k=0}^{n} (1) - \frac{1}{n} \sum_{k=M}^{n} (1)$$

If $n \geq M = \frac{1}{M} [n+1 - (n-M+1)] = 1$

$n < M = \frac{1}{M} [n+1-0] = \frac{n+1}{M}$

$$\therefore s(n) = \begin{cases} \frac{n+1}{M} & ; n < M \\ 1 & ; n \geq M. \end{cases}$$

2.35) Determine the range of values of the parameter 'a' for which the LIT system with impulse response

$$h(n) = \begin{cases} a^n & n \geq 0,\ n \text{ even} \\ 0 & \text{otherwise} \end{cases}$$ is stable.

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=0}^{\infty} |a^n| = \sum_{n=0}^{\infty} |a|^{2n}$$

$$= \frac{1}{1-|a|^2}$$

stable if $|a| < 1$.

2.36] Determine the response of the system with the impulse response $h(n) = a^n u(n)$ to the input signal $x(n) = u(n) - u(n-10)$

$h(n) = a^n u(n)$

$y_1(n) = \sum_{k=0}^{\infty} u(k) h(n-k) = \sum_{k=0}^{n} a^{n-k} = a^n \sum_{k=0}^{n} a^{-k}$

$= \frac{1-a^{n+1}}{1-a} u(n)$

$\therefore y(n) = y_1(n) - y_1(n-10)$

$= \frac{1-a^{n+1}}{1-a} u(n) - \frac{1-a^{n-9}}{1-a} u(n-10)$

2.37] Determine the response of the (relaxed) system characterized by the impulse response to the input signal: $h(n) = \left(\frac{1}{2}\right)^n u(n)$ $x(n) = \begin{cases} 1 & 0 \le n \le 10 \\ 0, & \text{otherwise.} \end{cases}$

$h(n) = \left(\frac{1}{2}\right)^n u(n) = a^n u(n)$ with $a = 1/2$

and $x(n)$ can be written as $x(n) = u(n) - u(n-10)$

$\therefore y(n) = \frac{1-a^{n+1}}{1-a} u(n) - \frac{1-a^{n-9}}{1-a} u(n-10)$ with $a = 1/2$

$= 2\left(1-\left(\frac{1}{2}\right)^{n+1} u(n) - \left(1-\left(\frac{1}{2}\right)^{n-9}\right) u(n-10)\right]$.

2.38] Determine the response of the (relaxed) system characterized by the impulse response $h(n) = \left(\frac{1}{2}\right)^n u(n)$. to the input signals (a) $x(n) = 2^n u(n)$ (b) $x(n) = u(-n)$

(a) $y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k) = \sum_{k=-\infty}^{\infty} h(k) x(n-k)$

$= \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^k u(k) 2^{n-k} u(n-k) = \sum_{k=0}^{n} 2^{-k} 2^{n-k} = 2^n \sum_{k=0}^{n} 2^{-2k}$

$= 2^n \sum_{k=0}^{n} \left(\frac{1}{2}\right)^{2k} = 2^n \sum_{k=0}^{n} \left(\frac{1}{4}\right)^k = 2^n \left[\frac{1-(1/4)^{n+1}}{1-1/4}\right]$

$= \left(\frac{4}{3}\right) 2^n \left[1 - \frac{1}{4^{n+1}}\right] = \left(\frac{4}{3}\right) 2^n \left[\frac{4^{n+1}-1}{4^{n+1}}\right] = \frac{2^n}{3}\left[\frac{4^{n+1}-1}{4^n}\right]$

(b) $y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^k u(k)\, u(-(n-k))$

$= \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^k u(k)\, u(-n+k)$

If $n<0$ then $y(n) = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = \frac{1}{1-1/2} = 2$

If $n \geq 0$ then $y(n) = \sum_{k=n}^{\infty} \left(\frac{1}{2}\right)^k = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k - \sum_{k=0}^{n-1} \left(\frac{1}{2}\right)^k$

$= 2 - \left(\frac{1-(1/2)^n}{1/2}\right) = 2\left(\frac{1}{2}\right)^n, \; n \geq 0.$

2.39) Three systems with impulse responses $h_1(n) = \delta(n) - \delta(n-1)$, $h_2(n) = h(n)$ and $h_3(n) = u(n)$ are connected in cascade.

(a) What is the impulse response $h_c(n)$ of the overall system?

$h_c(n) = h_1(n) * h_2(n) * h_3(n) = [\delta(n) - \delta(n-1)] * h_2(n) * h_3(n)$

$= (h_1(n) * h_3(n) * h_2(n)) \longrightarrow$ commutative property

$= (\delta(n) - \delta(n-1)) * u(n) * h(n)$

$= (u(n) - u(n-1)) * h(n)$

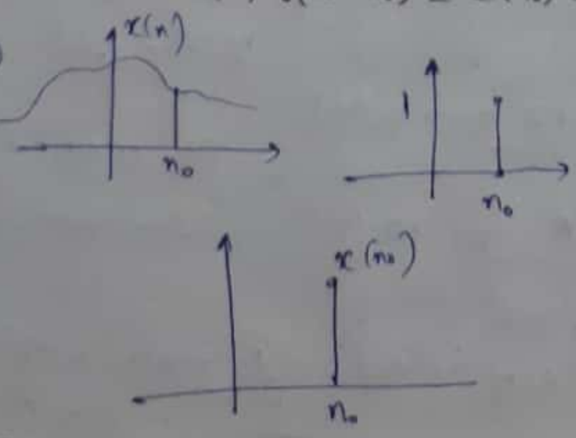
$= \delta(n) * h(n) = h(n)$

(b) Does the order of the interconnection affect the overall system?

No.

2.40) (a) Prove and Explain graphically the difference b/w the relations $x(n)\,\delta(n-n_0) = x(n_0)\,\delta(n-n_0)$ and $x(n) * \delta(n-n_0)$

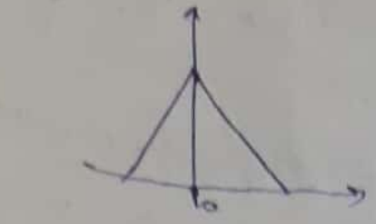
(a)



$x(n) * \delta(n-n_0) = x(n-n_0)$

$x(n) \cdot \delta(n-n_0) = x(n_0)\delta(n-n_0)$

(b)



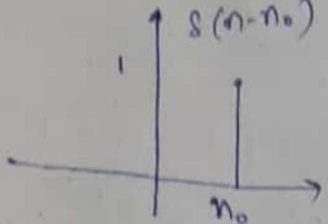
$$x(n) * \delta(n-n_0) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k-n_0)$$

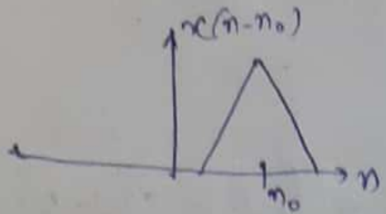
$$= \sum_{k=-\infty}^{\infty} x(n-n_0)\,\delta(k-n+n_0)$$

$$= x(n-n_0)$$

As the area of delta function is one

Hence proved.





(b) Show that the discrete time system; which is described by a convolution summation is LTI and relaxed.

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\,x(n-k) = h(n) \times x(n)$$

Linearity property

$$x_1(n) \longrightarrow y_1(n)$$
$$x_2(n) \longrightarrow y_2(n)$$

$$\alpha_1 x_1(n) + \alpha_2 x_2(n) \longrightarrow \alpha_1 y_1(n) + \alpha_2 y_2(n)$$

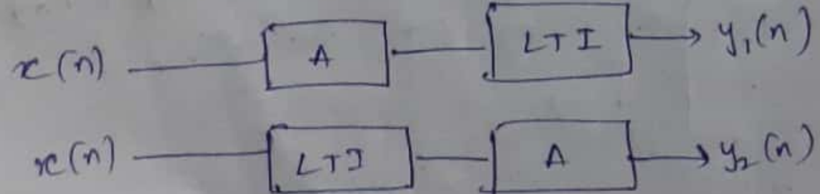
$$[x(n) = \alpha x_1(n) + \beta x_2(n) \longrightarrow y(n) = h(n) * x(n)]$$

$$= h(n) * (\alpha x_1(n) + \beta x_2(n))$$

$$= \alpha h(n) * x_1(n) + \beta h(n) * x_2(n)$$

$$= \alpha y_1(n) + \beta y_2(n)$$

Time invariance.



$y_1(n) = y_2(n)$

s/m is said to be Time Invariant

$$x(n) \longrightarrow y(n) = x(n) * h(n)$$

$$x(n-n_0) \longrightarrow y_1(n) = h(n) * x(n-n_0) = \sum_{k=-\infty}^{\infty} h(k)\,x(n-n_0-k)$$

$$= y(n-n_0)$$

(c) What is the impulse response of the system describe by $y(n) = x(n-n_0)$?

$h(n) = \delta(n-n_0)$ is the impulse response of the system

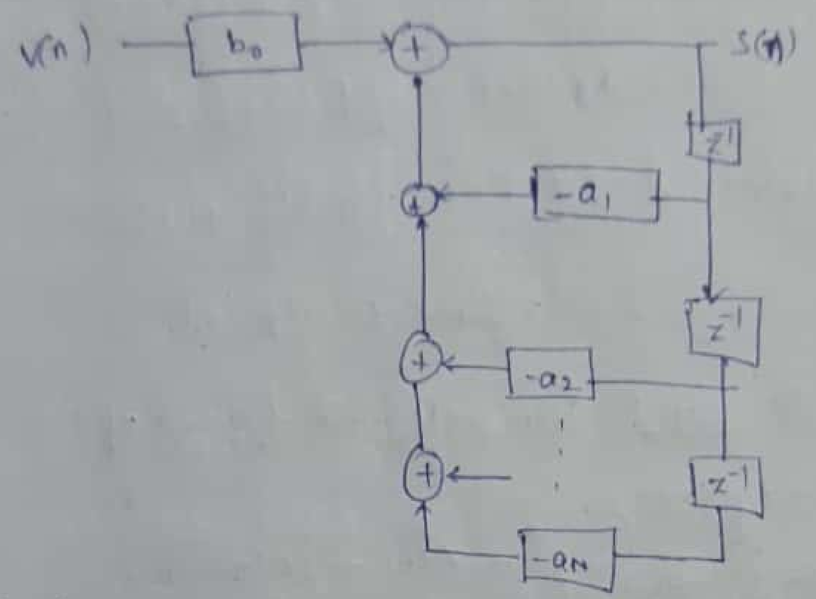
2.41) Two signals $s(n)$ and $v(n)$ are related through the following difference equations

$$s(n) + a_1 s(n-1) + \cdots + a_N s(n-N) = b_0 v(n)$$

Design the block diagram realization of:

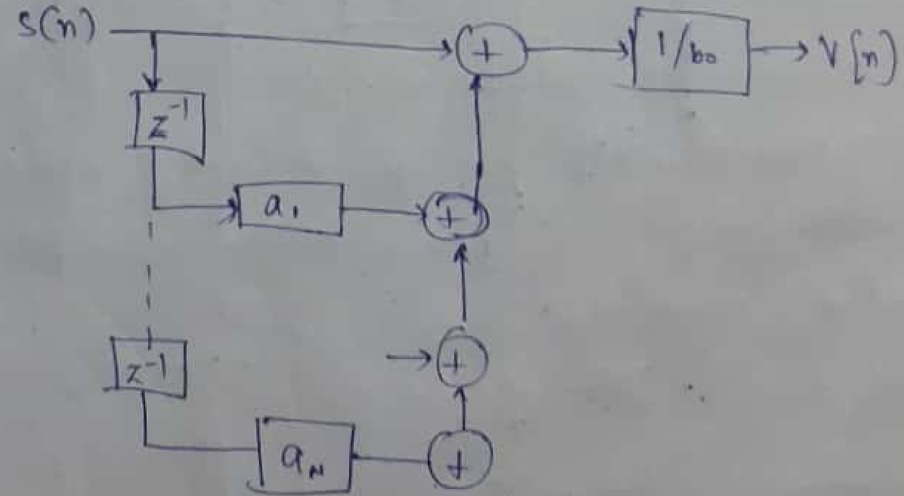
(a) The system that generates $s(n)$ when excited by $v(n)$

$$s(n) = +b_0 v(n) - a_1 s(n-1) - a_2 s(n-2) \cdots a_N s(n-N)$$



(b) The system that generates $v(n)$ when excited by $s(n)$

$$v(n) = \frac{1}{b_0}\left[s(n) + a_1 s(n-1) + a_2 s(n-2) + \cdots + a_N s(n-N)\right]$$



2.42] Compute the zero-state response of the system described by the difference equation

$$y[n] + \frac{1}{2}y(n-1) = x(n) + 2x(n-2)$$ to the input $x(n) = \{1, 2, \underset{\uparrow}{3}, 4, 2, 1\}$ by solving the difference equation recursively.

$$y(n) = -\frac{1}{2}y(n-1) + x(n) + 2x(n-2)$$

At $n=2$ $\quad y(-2) = -\frac{1}{2}y(-3) + x(-2) + 2x(-4)$

$$= 0 + 1 + 0 = 1$$

$$y(-1) = -\frac{1}{2}y(-2) + x(-1) + 2x(-3) = -\frac{1}{2}(1) + 2 + 0 = 2 - \frac{1}{2} = \frac{3}{2}$$

$$y(0) = -\frac{1}{2}y(-1) + x(0) + 2x(-2) = -\frac{1}{2}(3/2) + 3 + 2 = 5 - \frac{3}{4} = \frac{17}{4}$$

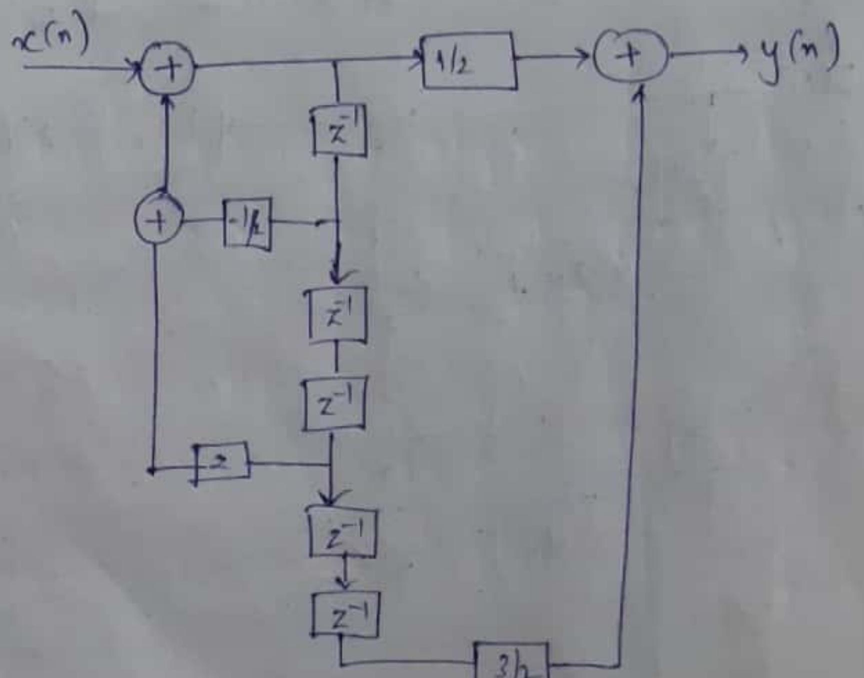
$$y(1) = -\frac{1}{2}y(0) + x(1) + 2x(-1) = -\frac{1}{2}\left(\frac{17}{4}\right) + 4 + 4$$

$$= -\frac{17}{8} + 8 = 47/8 \text{ etc..}$$

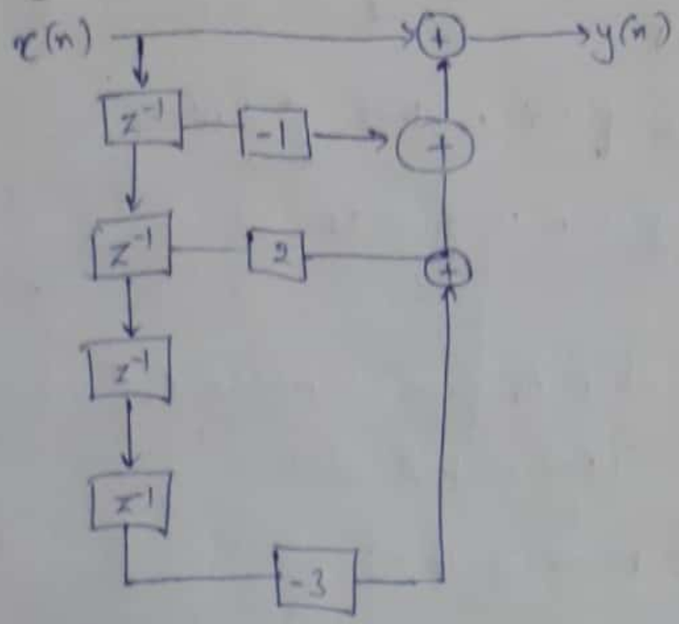
2.43] Determine the direct form realization for each of the following LTI systems.

(a) $2y(n) + y(n-1) - 4y(n-3) = x(n) + 3x(n-5)$

$$y(n) = \frac{1}{2}\left[x(n) + 3x(n-5) - y(n-1) + 4y(n-3)\right]$$

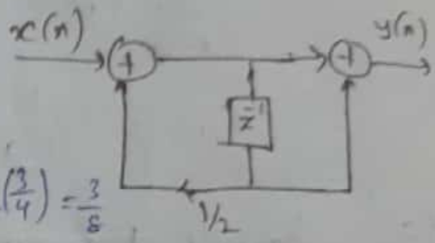


(b) $y(n) = x(n) - x(n-1) + 2x(n-2) - 3x(n-4)$



2.44) Consider the discrete time system shown below.

(a) Compute the 10 first samples of its impulse response



$x(n) = \delta(n)$

$y(n) = x(n) + x(n-1) + \frac{1}{2} y(n-1)$

$y(0) = x(0) = 1$

$y(1) = \frac{1}{2}(1) + 1 = 3/2$

$y(2) = 0 + 0 + \frac{1}{2}(3/2) = \frac{3}{4}$; $y(3) = 0 + 0 + \frac{1}{2}\left(\frac{3}{4}\right) = \frac{3}{8}$

$\therefore y(n) = \{1, 3/2, 3/4, 3/8, 3/16, 3/32, 3/64, \dots\}$.

(b) Find Input - output relation.

$y(n) = x(n) + x(n-1) + \frac{1}{2} y(n-1)$

$y(n) - \frac{1}{2} y(n-1) = x(n) + x(n-1)$

(c) Apply the input $x(n) = \{1, 1, 1, \dots\}$ and compute the first 10 samples of the output.

$y(0) = 1 + 0 + 0 = 1$

$y(1) = 1 + 1 + \frac{1}{2} = 2 + \frac{1}{2} = 5/2$

$y(2) = 1 + 1 + \frac{1}{2}\left(\frac{5}{2}\right) = 2 + 5/4 = 13/4$

$y(3) = 1 + 1 + \frac{1}{2}(13/4) = 2 + \frac{13}{8} = \frac{29}{8}$

$\therefore y(n) = \{1, 5/2, 13/4, 29/8, \dots\}$

2.45] Consider the system described by the difference equation

$y(n) = ay(n-1) + bx(n)$.

(a) Determine b in terms of 'a' so that $\sum_{n=-\infty}^{\infty} h(n) = 1$

The characteristic equation is given as

$y_h(n) = a\lambda + b = 0$

$\therefore$ The homogeneous solution is given as $h(n) = C_1 a^n u(n)$

$y(0) = b \rightarrow C_1 = b \quad \therefore h(n) = b a^n u(n)$

$\Rightarrow \sum_{n=-\infty}^{\infty} h(n) = \frac{b}{1-a} = 1 \Rightarrow b = 1-a.$

(b) Compute the zero-state response $s(n)$ of the system and choose b so that $s(\infty) = 1$

$$S(n) = \sum_{k=0}^{n} h(n-k) = b\left[\frac{1-a^{n+1}}{1-a}\right]u(n)$$

$$S(\infty) = \frac{b}{1-a} = 1 \Rightarrow b = 1-a.$$

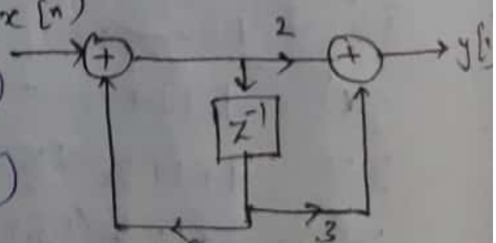
(c) Compute the zero-state response. Compute the value of b obtained in part (a) and (b) what did you notice?

$b = 1-a$; In both cases they are equal.

2.46) A discrete time system is realized by the structure shown in figure.

(a) Determine the impulse response.

(b) Determine a realization for its inverse system, that is the system which produces $x(n)$ as an output when $y(n)$ is used as an input.



$y(n) = 0.8\,y(n-1) + 2x(n) + 3x(n-1)$

$y(n) - 0.8\,y(n-1) = 2x(n) + 3x(n-1)$

The characteristic equation is

$\lambda - 0.8 = 0 \rightarrow \lambda = 0.8$

$\Rightarrow y_b(n) = C_1 (0.8)^n$.

Let us consider the response of the system.

$y[n] - 0.8y[n-1] = x[n]$ when input $x[n] = \delta(n)$

$y(0) = 1 \Rightarrow C_1 = 1$

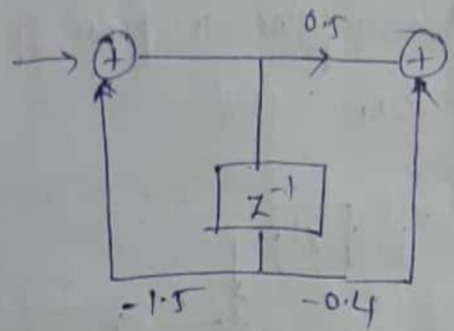
Then the impulse response of the original system is

$h(n) = 2(0.8)^n u(n) + 3(0.8)^{n-1} u(n-1)$

$= 2\delta(n) + 4.6(0.8)^{n-1} u(n-1)$

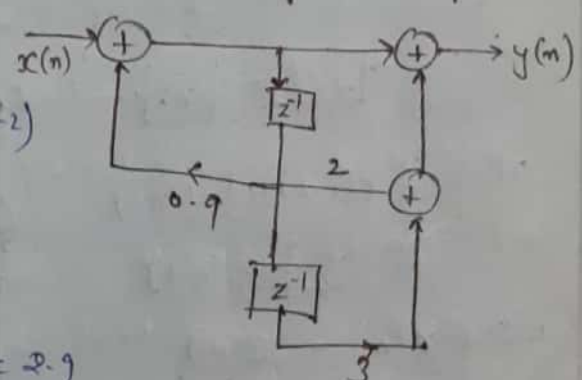
(b) The inverse system is characterised by the difference equation.

$x[n] = -1.5x[n-1] + \frac{1}{2} y(n) - 0.4y(n-1)$



2.47) Consider the discrete time system as shown in figure.

(a) Compute the first six values of the impulse response of the system



$y(n) = 0.9y(n-1) + 2x(n-1) + 3x(n-2)$

for $x(n) = \delta(n)$

$y(0) = 0 + 0 + 0 = 1+1$

$y(1) = 0.9(1) + 2(1) + 0 + 0 = 2.9$

$y(2) = 0.9(2.9) + 0 + 0 + 3 = 2.61 + 3 = 5.61$

$y(3) = 0.9(5.61) + 0 + 0 + 0 = 5.049$

$y(4) = 0.9(5.049) = 4.5441$

$y(5) = 4.08969 \dots\dots$

(b) Compute the first six values of zero state step response of the system.

$y(n) = 0.9y[n-1] + x(n) + 2x(n-1) + 3x(n-2)$

when $x(n) = u(n)$

$y(0) = 1$

$y(1) = 0.9(1) + 1 + 2 = 3.9$

$y(2) = (0.9)(3.9) + 1 + 2 + 3 = 9.51$

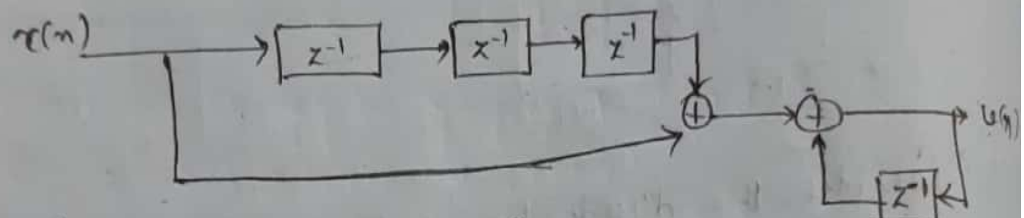
$y(3) = (0.9)(9.51) + 6 = 14.56$

$y(4) = (0.9)(14.56) + 6 = 19.10$

$y(5) = 23.19.$

2.48) Determine and sketch the impulse response of the following systems for $n = 0, 1, \ldots 9$.

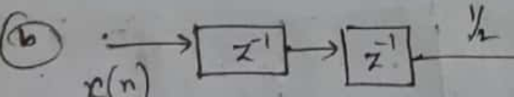
(a)

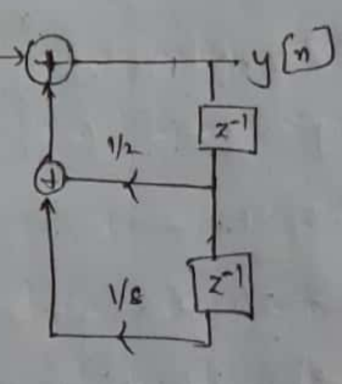


$y(n) = \frac{1}{3}x(n-3) + \frac{1}{3}x(n) + y(n-1)$

for $x(n) = \delta(n)$ we have $h(n) = \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \ldots\}$

(b)



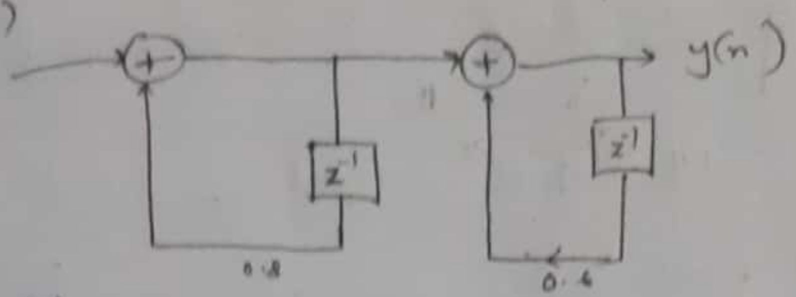


$y(n) = \frac{1}{2}x(n-2) + \frac{1}{2}y(n-1) + \frac{1}{8}y(n-2)$

with $x(n) = \delta(n)$

$y(-1)$ and $y(-2) = 0$

$h(n) = \{0, 0, \frac{1}{2}, \frac{1}{4}, \frac{3}{16}, \frac{1}{8}, \frac{11}{128}, \frac{15}{256}, \frac{41}{1024}, \ldots\}$

(c) 

$y(n) = x(n) + 0.48\,y(n-2) + 4y(n-1) = x(n) - 0.48\,y(n-2) + 1.4y(n-1)$

with $x(n) = \delta(n)$

$h(n) = \{\underset{\uparrow}{1},\ 1.4,\ 1.48,\ 1.4,\ 1.2496,\ 1.0774,\ 0.9086, \cdots\}$

(d) Classify above as FIR or IIR.

All the systems are IIR

(e) Find an explicit expression for the impulse response of the system in part (c)

$y(n) = 1.4y(n-1) - 0.48\,y(n-2) + x(n)$

$\lambda^2 - 1.4\lambda + 0.48 = 0 \Rightarrow \lambda = 0.8, 0.6$

$\therefore\ y_h(n) = C_1(0.8)^n + C_2(0.6)^n$

when $x(n) = \delta(n) \Rightarrow y(0) = C_1 + C_2 = 1$

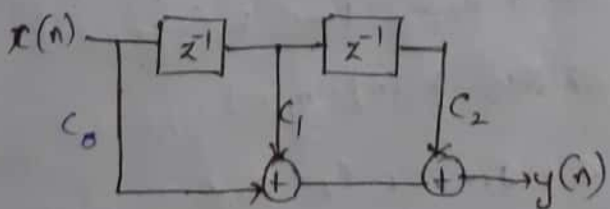
$y(1) = 0.8C_1 + 0.6C_2 = 1.4$

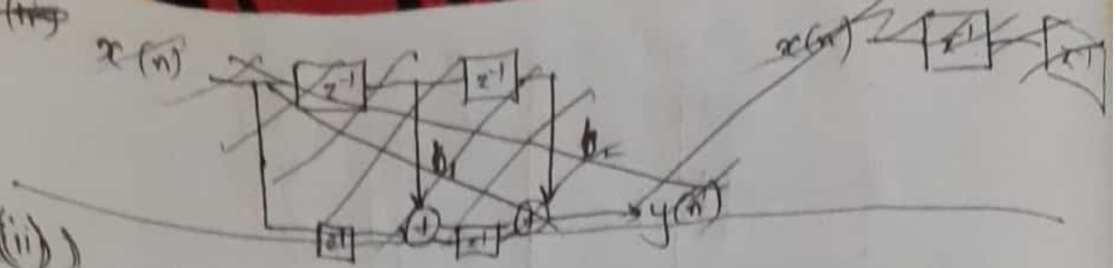
$\Rightarrow C_2 = 1 - C_1$

$\therefore\ 0.8C_1 + 0.6(1 - C_1) = 1.4 \Rightarrow C_1 = 4 \Rightarrow C_2 = 1 - 4 = -3$.

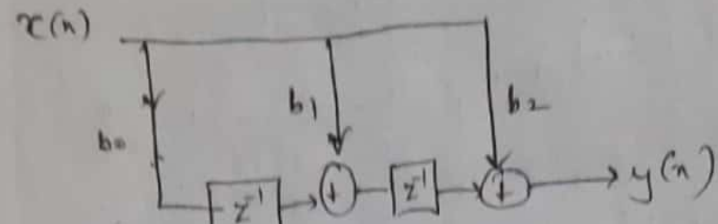
$\therefore\ h(n) = \left[4(0.8)^n - 3(0.6)^n\right] u(n)$

2.49) Consider the system shown in figure.

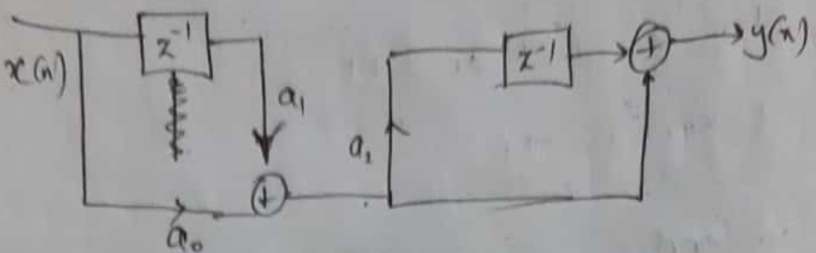
(i) 



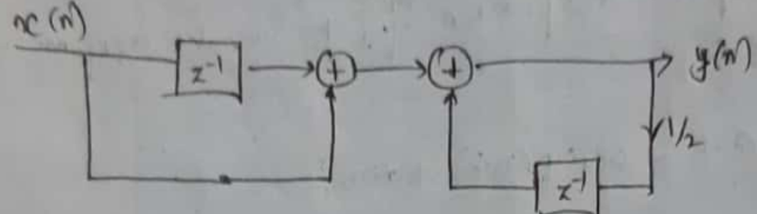
(ii)



(iii)



(iv)



(a) Determine its impulse response $h_1(n)$, $h_2(n)$ and $h_3(n)$

$$y_1(n) = c_0 x(n) + c_1 x(n-1) + c_2 x(n-2)$$

$$h_1(n) = c_0 \delta(n) + c_1 \delta(n-1) + c_2 \delta(n-2)$$

$$h_2(n) = b_2 \delta(n) + b_1 \delta(n-1) + b_0 \delta(n-2)$$

$$y_2(n) = b_2 x(n) + b_1 x(n-1) + b_0 x(n-2)$$

$$y_3(n) = a_0 x(n) + (a_1 + a_0 a_2) x(n-1) + a_1 a_2 x(n-2)$$

$$h_3(n) = a_0 \delta(n) + (a_1 + a_0 a_2) \delta(n-1) + a_1 a_2 \delta(n-2)$$

(b) Is it possible to choose the coefficient of these systems in such a way that $h_1(n) = h_2(n) = h_3(n)$

If $c_1 = b_1$, $c_0 = b_2$ and $b_0 = c_2$ then $h_1(n) = h_2(n)$

The only question is whether $h_3(n) = h_2(n)$

Let $c_0 = a_0$, $c_1 = a_1 + a_0 a_2$, $c_2 = a_1 a_2$

$\Rightarrow a_1 + a_0 a_2 = c_1$. $\quad c_2 = a_1 a_2 \Rightarrow \frac{c_2}{a_2} = a_1$

$a_1 + c_0 c_2 = c_1 \Rightarrow \frac{c_2}{a_2} = c_1 - c_0 c_2$

$\Rightarrow C_0 a_2^2 - C_1 a_2 + C_2 = 0$ for $C_0 \neq 0$

This quadratic has a real solution if and only if $C_1^2 - 4C_0C_2 \geq 0$

2.50] Consider the system in figure.

(a) Determine the impulse response $h(n)$.

$y(n) = \frac{1}{2} y(n-1) + x(n) + x(n-1)$

$y(n) - \frac{1}{2} y(n-1) = x(n) + x(n-1)$

For $y(n) - \frac{1}{2} y(n-1) = \delta(n)$; The solution is

$$h(n) = \left(\tfrac{1}{2}\right)^n u(n) + \left(\tfrac{1}{2}\right)^{n-1} u(n-1)$$

(b) Show that $h(n)$ is equal to the convolution of the following signals. $h_1(n) = \delta(n) + \delta(n-1)$; $h_2(n) = \left(\frac{1}{2}\right)^n u(n)$.

$h_1(n) + h_2(n) = (\delta(n) + \delta(n-1)) * \left(\tfrac{1}{2}\right)^n u(n)$

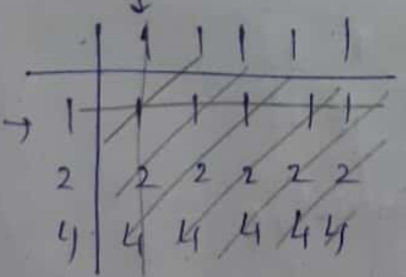
$= \left(\tfrac{1}{2}\right)^n u(n) + \left(\tfrac{1}{2}\right)^{n-1} u(n-1)$

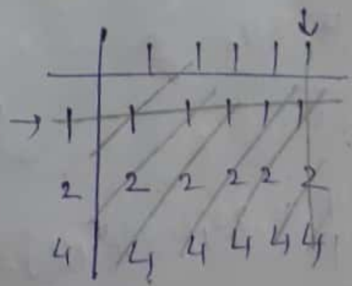
Hence proved.

2.51) Compute the sketch the convolution $y_1(n)$ and the correlation $r_1(n)$ sequence for the following pairs of signals and comment on the result obtained.

(a) $x_1(n) = \{\underset{\uparrow}{1}, 2, 4\}$, $h(n) = \{\underset{\uparrow}{1}, 1, 1, 1, 1\}$

$y_1(n) = x_1(n) * h_1(n)$ $\qquad$ $y_2(n) = x_1(n) * h_2(-n)$





$y_1(n) = \{\underset{\uparrow}{1}, 3, 7, 7, 7, 6, 4\}$ $\qquad$ $y_2(n) = \{1, 3, 7, 7, \underset{\uparrow}{7}, 6, 4\}$

(b) $x_2(n) = \{\underset{\uparrow}{0}, 1, -2, 3, -4\}$ $\quad h_2(n) = \{\frac{1}{2}, 1, \underset{\uparrow}{2}, 1, \frac{1}{2}\}$

	1/2	1	2	1	1/2
→ 0	0	0	0	0	0
1	1/2	1	2	1	1/2
−2	−1	−2	−4	−2	−1
3	3/2	3	6	3	3/2
−4	−2	−4	−8	−4	−2

$y_1(n) = x(n) * h(n)$

$= \{0, \frac{1}{2}, \underset{\uparrow}{0}, \frac{3}{2}, -2, \frac{1}{2}, -6, -\frac{5}{2}, -2\}$

$y_2(n) = x_2(n) * h_2(-n)$

	1/2	1	2	1	1/2
→ 0	0	0	0	0	0
1	1/2	1	2	1	1/2
~~2~~	~~1/2~~	~~1~~	~~2~~	~~1~~	~~1/2~~
−2	−1	−2	−4	−2	−1
3	3/2	3	6	3	3/2
−4	−4/2	−4	−8	−4	−4/2

Correlation $y_2(n)$

$= \{0, \frac{1}{2}, 0, \frac{3}{2}, -2, -\frac{1}{2}, -6, -\frac{5}{2}, -2\}$

$y_1(n) = y_2(n)$ because $h_2(n) = h_2(-n)$

is an even signal.

(c) $x_3(n) = \{\underset{\uparrow}{1}, 2, 3, 4\}$ $\quad h_3(n) = \{\underset{\uparrow}{4}, 3, 2, 1\}$

	4	3	2	1
→ 1	4	3	2	1
2	8	6	4	2
3	12	9	6	3
4	16	12	8	4

	1	2	3	4
→ 1	1	2	3	4
2	2	4	6	8
3	3	6	9	12
4	4	8	12	16

Convolution

$y_1(n) = \{\underset{\uparrow}{4}, 11, 20, 30, 20, 4, 4\}$

Correlation

$y_2(n) = \{1, 4, 10, \underset{\uparrow}{20}, 25, 24, 16\}$

(d) $x_4(n) = \{\underset{\uparrow}{1}, 2, 3, 4\}$ $\qquad$ $h_4(n) = \{\underset{\uparrow}{1}, 2, 3, 4\}$

	1	2	3	4
1	1	2	3	4
2	2	4	6	8
3	3	6	9	12
4	4	8	12	16

Convolution

$y_1(n) = \{\underset{\uparrow}{1}, 4, 10, 20, 25, 24, 16\}$

	4	3	2	1
1	4	3	2	1
2	8	6	4	2
3	12	9	6	3
4	16	12	8	4

Correlation

$y_2(n) = \{4, 11, 20, \underset{\uparrow}{30}, 20, 11, 4\}$

2.52) The zero state response of a ~~casual~~ causal LTI system to the input $x[n] = \{\underset{\uparrow}{1}, 3, 3, 1\}$ is $y(n) = \{\underset{\uparrow}{1}, 4, 6, 4, 1\}$. Determine its impulse response

$\therefore$ The length of $h(n)$ is 2

	h_0	h_1
1	h_0	h_1
3	$3h_0$	$3h_1$
3	$3h_0$	$3h_1$
1	h_0	h_1

$\rightarrow y(n) = \{h_0, 3h_0 + h_1 + 3h_0 + h_1$
$+ h_0 + 3h_1, + h_1\}.$

$\therefore h_0 = 1, h_1 = 1$

$\therefore h(n) = \{1, 1\}$

2.53) Prove by direct substitution the equivalence of equations (2.5.9) and (2.5.10) which describe the direct form II structure to the relation (2.5.6) which describes the direct form I structure.

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k) \longrightarrow 2.5.6$$

$$w(n) = -\sum_{k=1}^{N} a_k (w(n-k)) + x(n) \longrightarrow 2.5.9$$

$$y(n) = \sum_{k=0}^{M} b_k\, w(n-k) \rightarrow 2.5.10$$

from 2.5.9 obtain

$$x(n) = w[n] + \sum_{k=1}^{N} a_k w(n-k) \rightarrow ①$$

By substituting (2.5.10) for $y(n)$ and (①) into 2.5.6 we obtain LHS = RHS

$$\sum_{k=0}^{M} b_k\, w(n-k)$$

2.54) Determine the response $y(n)$, $n \geq 0$ of the system des by the second order difference equation $y(n) - 4y(n-1) + 4y(n-2)$ $= x(n) - x(n-1)$ when the input is $x(n) = (-1)^n u(n)$ and the initial conditions are $y(-1) = y(-2) = 0$

$$y(n) + 4y(n-2) - 4y(n-1) = x(n) - x(n-1)$$

The characteristic equation is

$$\lambda^2 - 4\lambda + 4 = 0 \Rightarrow \lambda = 2, 2$$

$$y_h(n) = C_1 2^n + C_1 n(2^n)$$

The particular solution is $y_p(n) = k(-1)^n u(n)$

Substitute the solution in the difference equation, we obtain

$$k(-1)^n u(n) - 4k(-1)^{n-1} u(n-1) + 4k(-1)^{n-2} u(n-2) = (-1)^n u(n) - (-1)^{n-1} u(n-1)$$

$$[k(-1)^2 - 4k(-1) + 4k] = 1+1$$

$$9k = 2 \Rightarrow k = 2/9$$

Hence the total solution is $y(n) = \left[C_1 2^n + C_2 \frac{2^n}{n} + \frac{2}{9}(-1)^n\right]$

From the initial conditions, we obtain,

$y(0) = 1$

$y(1) = 2$

$$y(0) = C_1 + \frac{2}{9} = 1 \Rightarrow C_1 = 1 - \frac{2}{9} = 7/9$$

$$2C_1 + 2C_2 - 2/9 = 2 \Rightarrow 2C_2 = 2 + \frac{2}{9} - \frac{14}{9} = 2 - \frac{12}{9}$$

$2c_2 = \frac{+6}{9} \Rightarrow c_2 = \frac{+6}{18} = \frac{+1}{3}$.

$\therefore y(n) = \left[\frac{7}{9}(2^n) + \frac{n}{3}(2^n) + \frac{2}{9}(-1)^n\right] u(n)$

2.55] Determine the impulse response $h(n)$ for the system described by the second order difference equation
$y(n) - 4y(n-1) + 4y(n-2) = x(n) - x(n-1)$

The characteristic equation is $\lambda^2 - 4\lambda + 4 = 0$

$\Rightarrow \lambda = 2, 2$.

$\Rightarrow \quad y_h(n) = h(n) = \left[c_1 2^n + c_2(n) 2^n\right] u(n)$

$y(0) = 1, \; y(1) = 3$

$\therefore \; c_1 = 1, \; 2c_1 + 2c_2 = 3 \Rightarrow c_2 = 1/2$

$\therefore \; h(n) = \left[2^n + \left(\frac{1}{2}\right)^n (2^n)\right] u(n)$

2.56] Show that any discrete time signal $x(n)$ can be expressed as $x(n) = \sum_{k=-\infty}^{\infty} [x(k) - x(k-1)]\, u(n-k)$ where $u(n-k)$ is a unit step delayed by k units in time that is

$$u(n-k) = \begin{cases} 1 & n \geq k \\ 0 & \text{otherwise} \end{cases}$$

$$x(n) = x(n) * \delta(n) = x(n) * \{u(n) - u(n-1)\}$$

$$= (x(n) - x(n-1)) * u(n)$$

$$= \sum_{k=-\infty}^{\infty} [x(k) - x(k-1)]\, u(n-k)$$

(or)

$$x(n) = x(n) * \delta(n) = x(n) * \frac{d}{dn} u(n) = \frac{d}{dn} x(n) * u(n)$$

$$= \sum_{k=-\infty}^{\infty} [x(k) - x(k-1)]\, u(n-k)$$

2.57] Show that the output of an LTI system can be expressed in terms of its unit step response $s(n)$ as follows

$$y(n) = \sum_{k=-\infty}^{\infty} [s(k) - s(k-1)]\, x(n-k)$$

$$= \sum_{k=-\infty}^{\infty} [x(k) - x(k-1)]\, s(n-k)$$

Let $h(m)$ be the impulse response of the system

$$s(k) = \sum_{m=-\infty}^{k} h(m)$$

$$\Rightarrow h(k) = s(k) - s(k-1)$$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

$$= \sum_{k=-\infty}^{\infty} (s(k) - s(k-1))\, x(n-k)$$

(ii) $s(n) = h(n) * u(n)$; $s(n-1) = h(n) * u(n-1)$

$$s(n) - s(n-1) = [h(n) - h(n-1)] * u(n)$$

$$s(n) - s(n-1) = h(n) * [u(n) - u(n-1)] = h(n) * \delta(n) = h(n)$$

$$y(n) = x(n) * h(n) = x(n) * [s(n) - s(n-1)]$$

$$= \sum_{k=-\infty}^{\infty} [s(k) - s(k-1)]\, x(n-k)$$

$$= \sum_{k=-\infty}^{\infty} [x(k) - x(k-1)]\, s(n-k)$$

2.58] Compute the correlation sequence $r_{xx}(l)$ and $r_{xy}(l)$ for the following sequence

$$x(n) = \begin{cases} 1 ; & n_0 - N \le n \le n_0 + N \\ 0 ; & \text{otherwise} \end{cases}$$

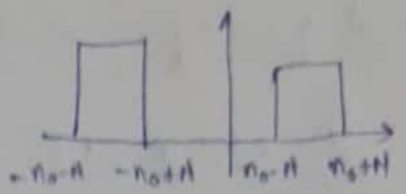
$$y(n) = \begin{cases} 1 ; & -N \le n \le N \\ 0 ; & \text{otherwise.} \end{cases}$$

$$r_{xx}(l) = \sum_{k=-\infty}^{\infty} x(-k)\, x(l-k) = \sum_{k=-\infty}^{\infty} x(k)\, x(k-l) = x(n) * x(-n)$$

The range of non-zero values of $r_{xx}(l)$ is determined by

$n_0 - N \le n \le n_0 + N$

$n_0 - N \le n - l \le n_0 + N$



which implies

$-2N \le l \le 2N$ range from $-2N \le l \le 2N$

$n_0 - N + n_0 - N = -2N$

$n_0 + N - n_0 + N = 2N$

for a given shift 'l' the no. of terms in the summation for which both $x(n)$ and $x(n-l)$ are non zero is $2N+1-|l|$ and the value of each term $|x|$,

Hence $r_{xx}(l) = \begin{cases} 2N+1-|l| & -2N \le l \le 2N \\ 0 & \text{otherwise} \end{cases}$

For $r_{xy}(l)$ we have

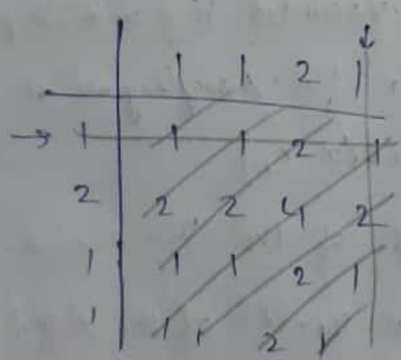
$$r_{xy}(l) = \begin{cases} N+1-|l-n_0|, & n_0 - 2N \le l \le n_0 + 2N \\ 0 & \text{otherwise.} \end{cases}$$

2.59) Determine the autocorrelation of the following sequence.

(a) $x(n) = \{1, 2, 1, 1\}$

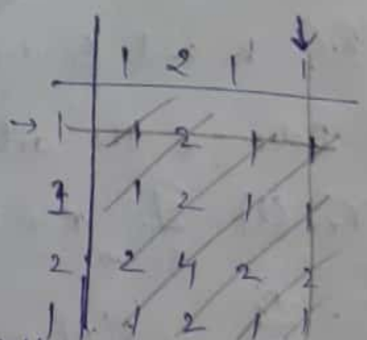
(b) $y(n) = \{1, 1, 2, 1\}$

$r_{xx}(l) = x(n) * x(-n)$



$r_{yy}(l) = y(n) * y(-n)$

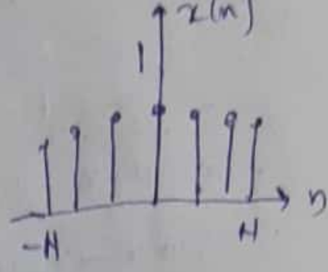
$= \{1, 3, 5, 7, 5, 3, 1\}$



We observe that $y(n) = x(-n+3)$ which is equivalent to reversing the sequence $x(n)$. This has not changed the auto correlation sequence.

2.60) What is the normalized auto correlation sequence of a signal $x(n)$ given by $x(n) = \begin{cases} 1 & , -N \le n \le N \\ 0 & \text{otherwise} \end{cases}$

$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n)\, x(n-l)$$



$$\sum_{-N}^{n+N} (1 \cdot 1) = 2N+1+n$$

$$\sum_{n-N}^{N} (1 \cdot 1) = 2N+1-n$$

$$r_{xx}(n) = \begin{cases} 2N+1+n & -2N \le n < 0 \\ 2N+1-n & 0 \le n \le 2N \end{cases}$$

Hence $r_{xx}(l) = \begin{cases} 2N+1-|l| & ; -2N \le l \le 2N \\ 0 & ; \text{otherwise.} \end{cases}$

$r_{xx}(0) = 2N+1$

Therefore the normalized autocorrelation is

$$r_{xx}(l) = \begin{cases} \frac{1}{2N+1}(2N+1-|l|) & ; -2N \le l \le 2N \\ 0 & ; \text{otherwise.} \end{cases}$$

2.61] An audio signal $s(t)$ generated by a loudspeaker is reflected two different walls with reflection coefficients r_1 and r_2. The signal $x(t)$ recorded by a microphone close to the signal loudspeaker, after sampling is

$$x(n) = s(n) + r_1 s(n-k_1) + r_2 s(n-k_2)$$

where k_1 and k_2 are the delays of the two echoes.

(b) Determine the auto correlation $r_{xx}(l)$ of the signal $x(n)$

$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n)\, x(n-l)$$

$$= \sum_{n=-\infty}^{\infty} [s(n) + r_1 s(n-k_1) + r_2 s(n-k_2)] *$$

$$[s(n-l) + r_1 s(n-l-k_1) + r_2 s(n-l-k_2)]$$

$= (1+r_1^2+r_2^2)\, r_{ss}(l) + r_1\left[r_{ss}(l+k_1) + r_{ss}(l-k_1)\right]$

$+ r_2\left[r_{ss}(l+k_2) + r_{ss}(l-k_2)\right] + r_1 r_2 \left[r_{ss}(l+k_1-k_2)\right.$

$\left.+ r_{ss}(l+k_2-k_1)\right]$.

(b) Can we obtain r_1, r_2, k_1 and k_2 by observing $r_{xx}(l)$?

$r_{xx}(l)$ has peaks at $l=0$ $\pm k_1$, $\pm k_2$ and $\pm(k_1 \pm k_2)$

suppose that $k_1 < k_2$. Then we can determine r_1 and k_1.

The problem is to determine r_2 and k_2 from other peaks.

(c) what happens if $r_2=0$?

If $r_2=0$; the peaks occur at $l=0$ and $l=\pm k_1$. Then

it is easy to obtain r_1 and k_1