PROJECT 6: SIMPLE SOLUTION TO AN SECOND ORDER DIFFERENTIAL EQUATION USING A MEAN VALUE THEOREM APPROXIMATION

This project is about solving a second order differential equation using the mean value theorem (or as it was called in class, a "symmetric three point approximation method"). We will do this by taking this second order differential equation and writing it as two coupled first order differential equations. The second order differential equation (which happens to describe the motion of a driven pendulum) is:

$$\frac{d^2\theta}{dt^2} + \sin\theta = a\cos\omega t \tag{1}$$

Where t is time, θ is the angle the pendulum makes with the vertical, and ω is a frequency of the driving force. The right hand side of this expression describes the driving force. We can make the substitutions $y_1 = \theta$ and $y_2 = \frac{dy_1}{dt}$ so as to arrive at the two coupled first order differential equations:

$$\frac{dy_1}{dt} = y_2 \tag{2}$$

$$\frac{dy_2}{dt} = a\cos\omega t - \sin y_1 \tag{3}$$

The goal of this code is to plot three graphs: y_1 vs t (angle vs time), y_2 vs t (angular velocity vs time), and y_1 vs y_2 (angle vs angular velocity).

We will use the mean value theorem when considering the derivatives of y_1 and y_2. Although this code does not actually estimate $\frac{dy_1}{dt}$ or $\frac{dy_2}{dt}$ at any specific value, we use the derivative approximation formula (Eq 4) to solve for specific instances of y_1 and y_2. These values will give a recursion relation that allows us to plot the required graphs.

$$f\prime(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h} \tag{4}$$

Where x_0 is the middle point of interest and h is the distance between points.

In general, we can estimate the derivative of any function f at the value x_0 if we know one point on the function to either side of $f(x_0)$. The derivative $f\prime(x_0)$ is estimated by the slope of the secant line connecting the two symmetric points. If the function is continuous and differentiable on the interval then the mean value theorem, illustrated in Fig 1, guarantees that there exists a point somewhere between $x_0 - h$ and $x_0 + h$ that has a tangent line equal to the slope of the derivative. We are approximating the slope at x_0 as this secant slope.

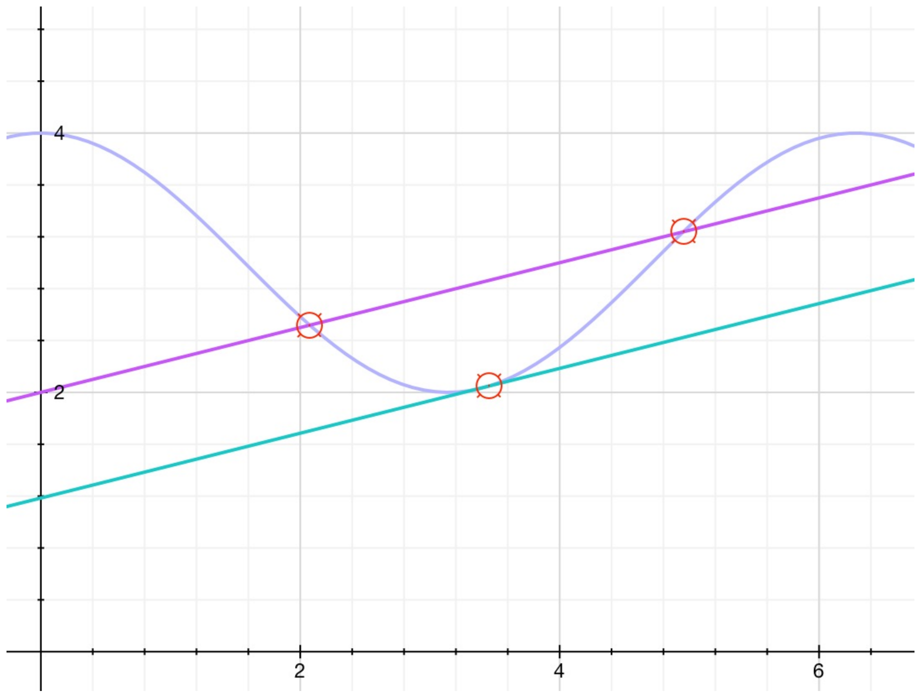

FIGURE 1. We use the mean value theorem to approximate the derivative of some function evaluated at the midpoint between two bounds.

Now, it is important to note that for this code we are never actually approximating a derivative. When we apply Equation 4 to y_1 and y_2 (defined in Equation 2 and 3) and rearrange we get:

$$y_1(t+h) = 2hy_2(t) + y_1(t-h) \tag{5}$$

$$y_2(t+h) = 2h\left[-\sin y_1(t) + a\cos\omega t\right] + y_2(t-h) \tag{6}$$

This is our recursion relation. We will start with some initial conditions and use these equations to step y_1 and y_2 forward so that we find these functions at sequential values of t. Then we will plot y_1 vs t, y_2 vs t, and y_1 vs y_2. The latter of which should be an ellipse.

Note, this is not an ideal method to numerically solve differential equations. Because of this we will see discretization error propagate across our results. If this error did not exist then our y_1 vs y_2 graph would be a periodic ellipse. It actuality this graph will not be periodic, but it should look something like an ellipse at the beginning.

The steps for the code include:

(1) Declaring the initial conditions $y_1(0) = 0$, $y_2(0) = 0$, $y_1(-h) = 0$, ,$y_2(-h) = 0$ and another other parameters such as ω or h.
(2) Calculating $y_1(t + h)$ and $y_2(t + h)$ via Equations 5 and 6. We will loop through this step to get out many sequential values of y_1 and y_2.
(3) Then we plot the graphs.