PROJECT 4: ESTIMATING THE VOLUME OF AN N DIMENSIONAL SPHERE USING MONTE CARLO METHODS

In this project we are to consider a series of higher and higher dimensional spheres. We will fill in these volumes by using a Monte Carlo process in which we generate coordinates (ordered pairs) via a random number generator. We will then count the number of coordinates that fall within the volume and the number of coordinates that fall outside of the volume. For two dimensions these counts can be used to estimate the value of π. This process is illustrated pictorially for the 2 dimensional case at `http://en.wikipedia.org/wiki/File:Pi_30K.gif`.

At first pass on this project I thought our task what to estimate *pi* for the N dimensional case. It is not. **The general goal of this program is to obtain the volumes of the N dimensional sphere.**

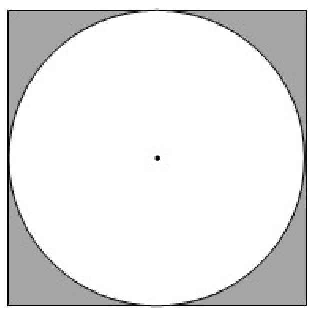

FIGURE 1. A sphere of dimension 2

Lets start with the familiar circle. We need to accomplish the following:

(1) Draw (or imagine drawing) a circle inscribed in a square. Lets say this circle has radius 1 and center at the origin. Thus the areas are $A_{circle} = \pi$ and $A_{square} = 4$. We want to calculate the A_{circle} using the number of hits and total shots. In essence, for this 2 dimensional case, we are estimating π. NOTE this estimating π task is only for the two dimensional case.
(2) Initiate counters N_{hits} and $N_{totalshots}$ at 0. Draw two random numbers x and y from a uniform distribution $P(x) = 1$ where $x \in [0, 1]$.
(3) Rescale these values so that $\mathbf{r} = (2x - 1, 2y - 1)$. This ensures that they appear somewhere within the area of the square.
(4) Increment the counters so that $N_{totalshots}$ increases on each step and $N_{hits} = N_{hits} + 1$ if $r \leq 1$.

(5) We stop this loop at some arbitrary value and analyze the estimator $A_{circle} = \hat{\pi} = 4 \cdot \frac{N_{hits}}{N_{totalshots}}$
(6) Finally we analyze the error in our estimate via the variance:
$\sigma^2 = 4 \cdot \frac{N_{hits}}{N_{totalshots}} \left(1 - \frac{N_{hits}}{N_{totalshots}}\right)$

So now that we know how to do this in 2 dimensions, we need to write a code that will do this for $N = 1...12$ dimensions. What does a 12 dimensional sphere look like, you ask? The answer is I have no idea. But fortunately it doesn't matter. All we need to know are N_{hits} and $N_{totalshots}$ to arrive at the volume for the N sphere. Analytically, this volume is given by

$$V_{Nsphere} = \frac{\pi^{N/2}}{\Gamma(N/2+1)} \tag{1}$$

Where Γ is the gamma function and is defined by $\Gamma(n) = (n-1)!$ for positive integers n. We can use this forum to check our answers for the estimated $\hat{V}_{Nsphere}$ that our code will find.

The formula for an N dimensional cube is

$$V_{Ncube} = 2^N \tag{2}$$

Where the cube's length is 2. We will use this formula in our code because it is well known. We are to repeat this process for spheres up to dimension 12. The general process of estimating $V_{Nsphere}$ is

$$\hat{V}_{Nsphere} = V_{Ncube} \cdot \frac{N_{hits}}{N_{totalshots}} \tag{3}$$

The variance for the estimate $\hat{V}_{Nsphere}$ is

$$\sigma^2 = V_{Ncube} \cdot \frac{N_{hits}}{N_{totalshots}} \left(1 - \frac{N_{hits}}{N_{totalshots}}\right) \tag{4}$$