# From Influence Kernels to Simulation Analytics: Modular Extensions and Robustness for the Living Influence Model (LIM)

**Abstract**
The Living Influence Model (LIM) unifies physics-based ball control, probabilistic passing, and off-ball scoring threat into a single spatio-temporal influence field for association football. While LIM is effective descriptively, deployment for *simulation analytics* requires probabilistic coherence, temporal stability, opponent adaptation, and uncertainty quantification. We propose a modular, theory-first framework that (i) enforces probability conservation and calibration, (ii) replaces brittle thresholding with smooth, value-consistent aggregation, (iii) introduces policy and opponent layers for decision modeling, (iv) propagates kinematic and observational uncertainty, (v) regularizes noise in space and time, and (vi) formalizes graph-theoretic health metrics with predictive semantics. We define a LIM-conditioned transition kernel for counterfactual rollouts and derive statistical guarantees via conformal prediction and doubly-robust off-policy evaluation. The result is a principled path from influence fields to a simulation-ready engine with interpretable error bars suitable for coaches and analysts and extensible to fan-safe visualizations.

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## 1. Introduction

Descriptive spatial models in football—e.g., pitch control and expected possession value—enable rich tactical analysis. LIM advances this line by **unifying** (a) physics-based time-to-intercept/control (TTI/TTC), (b) pass probabilities on the player graph, and (c) off-ball scoring opportunity into a scalar influence field $\Phi(z, t)$ over pitch coordinates $z$ and time $t$. However, coaching workflows and decision support demand more than descriptive maps: **simulation analytics** must answer *what-if* questions with coherent probabilities, calibrated uncertainty, and opponent response.

**Contributions.** We formalize a modular extension of LIM with six theoretical pillars:

1) **Probabilistic coherence** (Sec. 3): team-level conservation, multi-calibration, and a generative definition of $\Phi$.

2) **Stable fusion & value propagation** (Sec. 4): smooth aggregation and finite-horizon value iteration on the pass graph.

3) **Decision & opponent adaptation** (Sec. 5): behavior-cloned policies, pressing hazards, and collapse risk as a predictive object.

4) **Physics & uncertainty** (Sec. 6): stochastic TTI/TTC and ball kinematics with uncertainty propagation.

5) **Noise & regularization** (Sec. 7): temporal changepoints, anisotropic spatial smoothing, and physical feasibility.

6) **Network health** (Sec. 8): algebraic connectivity, effective resistance, and bottleneck indices with event-predictive semantics.

We then define a **LIM-conditioned transition kernel** $P(s_{t+\Delta} \mid s_t, a_t)$ for counterfactual rollouts (Sec. 8) and provide statistical guarantees (Sec. 9) via calibration, conformal coverage, and doubly-robust off-policy evaluation (OPE). Sections 10–13 specify an evaluation protocol, decision objects for coaches, limitations, and conclusions.

## 2. Probabilistic Coherence

### 2.1 Team-level conservation

Let $p_T(z,t)$ and $p_O(z,t)$ denote team-level control fields. Enforce

$$p_T(z,t) + p_O(z,t) = 1, \quad \forall z, t,$$

by coupling arrival times through a **soft difference** model:

$$p_T(z,t) = \sigma\Big(\beta\,(\min_{i\in T} \mathrm{TTI}_i - \min_{j\in O} \mathrm{TTI}_j)\Big), \quad \sigma(x) = \frac{1}{1+e^{-x}},$$

with temperature $\beta > 0$. This avoids non-conservation from hard minima or ad-hoc thresholds.

### 2.2 Generative influence definition

Let $Q \in \{\text{in-possession}, \text{out}\}$ and $D$ index action destinations (receivers, zones). Define

$$\Phi(z,t) \;=\; \mathbb{E}_{Q,D}[\, \pi(z,t)\, \mathbf{1}\{Q = \text{in}\} \cdot u(z,D,t)\,],$$

where $\pi(z,t)$ is local action propensity and $u$ is utility (e.g., short-horizon expected threat). This expectation binds $\Phi$ to observable events and prevents double counting.

### 2.3 Global calibration

For each probabilistic component (e.g., $P_{\text{pass}}, P_{\text{shot}}, p_T, \pi$), apply isotonic/Platt transforms on rolling windows with **multi-calibration** across zones, pressure, and phase. If $\hat{p}$ is a raw score and $g$ a calibrator, report $p^\star = g(\hat{p})$ alongside Brier/log loss and reliability curves per stratum.

## 3. Stable Fusion & Value Propagation

### 3.1 Smooth aggregation

Replace hard max and binary thresholds with **log-sum-exp (LSE)**:

$$\mathrm{LSE}_\tau(x_1,\ldots,x_n) = \tau \log \sum_i e^{x_i/\tau},$$

which is Lipschitz for $\tau > 0$ and approaches max as $\tau \to 0$. For team aggregation, use temperature-controlled softmax/Shapley allocations to preserve synergy and stability.

### 3.2 Finite-horizon value iteration on the pass graph

Let $G_t = (V, E_t)$ with edge success $w_{ij}(t) = P_{\text{pass}}(i \to j \mid s_t)$. Define

$$V_i^{(0)} = u_i, \quad V_i^{(h+1)} = (1-\gamma)\, u_i + \gamma \sum_j w_{ij}(t)\, V_j^{(h)},$$

with discount $\gamma \in (0,1)$ and horizon $h$. This yields **multi-hop** value propagation consistent with $\Phi$.

## 4. Decision & Opponent Adaptation

### 4.1 Player policy layer

State $s_t$ includes positions, velocities, orientation, possession, and context tags. A behavior-cloned policy $\pi(a \mid s_t)$ covers $a \in \{\mathrm{pass}_j, \mathrm{dribble}_\theta, \mathrm{shoot}\}$, with mixed effects capturing player/team heterogeneity. Self-edges encode dribbles; a risk parameter $\rho$ modulates exploration.

### 4.2 Opponent response

Define pressing hazard $h(z,t)$ for interception/turnover intensity given spacing and angles. Model level-$k$ adaptation: acting team rolls out under $\pi$; opponent responds with $\pi_{\mathrm{opp}}^{(k)}$ best-replying to first-order forecasts (or via fictitious play). This enables **two-player** rollouts with calibrated probabilities.

### 4.3 Collapse risk as a predictive object

Let $\lambda_2(t)$ be algebraic connectivity of $G_t$. Define collapse at time $\tau$ if $\lambda_2(\tau) < \theta$. Fit a time-to-event model for $T_c = \inf\{t : \lambda_2(t) < \theta\}$ using covariates $\{p_T, \Phi, V^{(h)}, h\}$:

$$\Pr(T_c \le t + \Delta \mid s_t) = 1 - \exp\Big(-\int_t^{t+\Delta} \alpha(u)\, du\Big),$$

linking spectral fragility to imminent turnovers/press breaks.

## 5. Physics & Uncertainty

### 5.1 Stochastic interception and control

Replace point TTI/TTC with **time-to-reach** distributions $\mathrm{TTR}_i(z,t) \sim \mathcal{D}_i(z,t)$ (tracking noise, anisotropic speed/orientation). Team control integrates over $\mathcal{D}$:

$$p_T(z,t) = \Pr\left(\min_{i \in T} \mathrm{TTR}_i < \min_{j \in O} \mathrm{TTR}_j\right).$$

### 5.2 Ball kinematics

Use piecewise dynamics with drag $\dot{v} = -\kappa v$ and event-conditioned impulses. Propagate uncertainty in $\kappa$ and launch parameters via Monte Carlo to obtain calibrated arrival time distributions at candidate receivers.

## 6. Noise & Regularization

- **Temporal:** exponentially-weighted moving averages (EWMA) for latent fields; Bayesian online changepoint detection to preserve genuine tactical shifts; hysteresis to prevent flicker.

- **Spatial:** anisotropic diffusion or total-variation regularization aligned to motion fields to suppress speckle while keeping fronts sharp.

- **Physical feasibility:** caps on velocity/acceleration; feasible dribble sets; minimum reach times consistent with biomechanics.

## 7. Network Health Metrics

Beyond $\lambda_2$, formalize: - **Effective resistance** $R_{\text{eff}}$: global redundancy; high values imply vulnerability to edge removals.
- **Edge betweenness** $b_e$: identifies funnel passes whose removal disrupts value flow.

Hypothesis: increases in $R_{\text{eff}}$ and max $b_e$ elevate collapse hazard $\alpha(t)$, giving **predictive semantics** to “fragility”.

## 8. LIM-Conditioned Simulation Kernel

Let $s_t$ be state, $a_t$ an action, and $u$ utilities from Sec. 3–4.

**Definition (Transition kernel).**

$$P(s_{t+\Delta} \mid s_t, a_t) \;=\; \underbrace{P_{\text{kin}}(\cdot \mid s_t, a_t)}_{\text{physics}} * \underbrace{P_{\text{evt}}(\cdot \mid s_t, a_t; \Phi, p_T, P_{\text{pass}}, P_{\text{shot}})}_{\text{events}},$$

a convolution of kinematic propagation and calibrated event realizations.

**LIM Simulation Rollout (theory, high-level)**

```
Input: s0, horizon H, rollouts N, policies (pi, pi_opp)
for n = 1..N:
  s <- s0
  for t = 0..H-1:
    a ~ pi(. | s); a_opp ~ pi_opp(. | s)
    s ~ P(. | s, a)            # LIM-conditioned kernel
Aggregate outcomes: Delta_xT, TTA, EIG, CollapseRisk
Report medians with coverage-guaranteed intervals
```

## 9. Statistical Guarantees

### 9.1 Calibration

Apply isotonic/Platt calibration to each probabilistic submodule on rolling windows. Under standard regularity, isotonic regression yields vanishing calibration error $o_p(1)$ with sample growth. Report ECE/reliability per stratum.

### 9.2 Distribution-free coverage (conformal)

For scenario outputs $Y$ (e.g., $\Delta$xT, TTA), use **Mondrian conformal** by phase/zone to construct intervals $[L, U]$ satisfying

$$\Pr\{Y \in [L, U]\} \geq 1 - \alpha$$

without distributional assumptions, assuming exchangeability within strata.

### 9.3 Off-policy evaluation (OPE)

Interventions induce policies $\pi'$ off the behavior distribution. Use **doubly-robust** estimators

$$\hat{V}_{\text{DR}} = \frac{1}{n} \sum_{i=1}^{n} \Big[\hat{Q}(s_i, a_i) + w_i\{r_i - \hat{Q}(s_i, a_i)\}\Big],$$

with stabilized, clipped importance weights $w_i$ and learned $\hat{Q}$. Consistency holds if either $w$ or $\hat{Q}$ is correct; efficiency improves when both are strong.

### 9.4 Stability to perturbations

With LSE aggregation and Lipschitz kinematics, the mapping $s_t \mapsto \Phi(\cdot, t)$ is **Lipschitz-stable** under bounded tracking noise (proof sketch: bound gradient of LSE; propagate through calibrated maps; apply Grönwall inequality to the flow).

## 10. Evaluation Protocol (theory template)

- **Rolling-origin backtests** at possession and frame scales; report log-loss/Brier, CRPS for spatial fields, and ECE by stratum.
- **Counterfactual consistency:** nearest-neighbor "twin states" to compare simulated top-$k$ actions with realized choices (Kendall $\tau$, top-$k$ recall).
- **Ablations:** remove off-ball, pass-graph propagation, or opponent response to quantify causal contribution to $\Delta$xT and TTA.
- **Robustness:** timestamp jitter, positional noise, missing frames; invariances to rotation/translation and player relabeling.
- **Pre-registration:** fix horizons (e.g., 10s/20s), metrics, and $\alpha$ for conformal coverage.

## 11. Coach-Facing Decision Objects

Define **scenario cards** as tuples

$$\mathcal{C} = \langle \text{intervention},\ \widetilde{\Delta\text{xT}},\ \text{TTA},\ \text{EIG},\ \Delta\text{CollapseRisk},\ [L, U] \rangle,$$

with median $\widetilde{\Delta\text{xT}}$ and conformal interval $[L, U]$. Each card includes a concise rationale derived from marginal effects (lane opening, overload shift) to ensure interpretability.

## 12. Discussion & Limitations

The modules leave the **core LIM idea** intact while making it **probabilistically valid** and **simulation-ready**. Limitations include (i) exchangeability assumptions for conformal intervals, (ii) behavioral policy misspecification in rare states, and (iii) computational budgets for near-real-time usage. Future work: human-in-the-loop scenario selection, inverse-RL opponent models, and multi-objective utilities blending tactical and player-health costs.

## 13. Conclusion

We present a theory-first path from unified influence fields to **rigorous simulation analytics**. By enforcing conservation, calibration, stability, adaptation, uncertainty propagation, and spectral health semantics, LIM becomes a principled engine for *what-if* reasoning with interpretable uncertainty—suitable for coaches and analysts and extensible to fan-safe visualizations.

## Appendix A. Notation (abbrev.)

- $z$: pitch coordinate; $t$: time; $s_t$: state; $a_t$: action.

- TTI/TTC/TTR: time-to-intercept/control/reach (stochastic).
- $p_T, p_O$: team control fields; $\Phi$: influence field; $u$: local utility.
- $G_t$: pass graph; $\lambda_2$: algebraic connectivity; $R_{\text{eff}}$: effective resistance.
- $\pi$: policy; $h$: pressing hazard; $T_c$: collapse time.
- $\Delta$xT: change in expected threat; **TTA**: time-to-advantage; **EIG**: expected influence gain.