1 Some Basic Topological Notions

Homotopy

No exercises!

Convexity, Contractibility, and Cones

Exercise 1.1. Suppose $H : f_0 \simeq f_1$ is a homotopy. Then let $F(t) = H(x,t)$ for some fixed x. It is clear that $F(0) = x_0$ and $F(1) = 1$. Moreover, since H is continuous, it follows that so too is F. For the converse, simply let the homotopy $H : f_0 \simeq f_1$ take $(x,t) \in X \times \mathbb{I}$ to $F(t)$.

Exercise 1.2.

(i) There exist functions $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_Y$. Moreover, there is a homotopy $F : 1_X \simeq c$, where c denotes the constant map at some $x_0 \in X$. Then consider the map $G : Y \times \mathbb{I} \to Y$ which takes (y,t) to $f(F(g(y),t))$. In particular, we know that G is continuous and that it is thus a homotopy from $f \circ g$ to the constant map c' at $y_0 = f(x_0)$. But then we find that $1_Y \simeq f \circ g \simeq c'$, and so Y is contractible.

(ii) Consider, for example, the subsets $X, Y \subset \mathbb{R}^2$ where

$$X = \{(x,0) : x \in [0,1]\},$$
$$Y = \left\{(x,x) : x \in \left[0, \frac{1}{2}\right]\right\} \cup \left\{(x, 1-x) : x \in \left[\frac{1}{2}, 1\right]\right\}.$$

It is obvious that X is convex, but Y is not, even though there is an obvious homotopy equivalence from X to Y.

Exercise 1.3. We know that $R(x) = e^{i\alpha}x$, and so the continuous map $F : S^1 \times \mathbb{I} \to S^1$ given by $F(x,t) = e^{i\alpha t}x$ is a homotopy $F : 1_S \simeq R$. Thus, if $g : S^1 \to S^1$ is continuous, then let θ be such that $g(1) = g(e^{i \cdot 0}) = e^{i\theta}$. Then we know that, letting R now be the rotation of $-\theta$ degrees, we must have $R \circ g \simeq 1_S \simeq g = g$ and $(R \circ g)(1) = 1$, as desired.

Exercise 1.4.

(i) Pick $(x_1, y_1), (x_2, y_2) \in X \times Y$. Then we know that, for any $t \in \mathbb{I}$, we have

$$t(x_1, y_1) + (1-t)(x_2, y_2) = (tx_1 + (1-t)x_2, ty_1 + (1-t)y_2).$$

The result follows from convexity of X and Y.

(ii) If $F_X : 1_X \simeq c_X$ and $F_Y : 1_Y \simeq c_Y$, where c_X and c_Y are constant maps at c_X and c_Y, respectively, then the map

$$\begin{aligned} F : (X \times Y) \times \mathbb{I} &\to X \times Y \\ (x,y,t) &\mapsto (F_X(x,t), F_Y(y,t)) \end{aligned}$$

is clearly a homotopy from $1_{X \times Y}$ to (c_X, c_Y).

Exercise 1.5. It is clear that X is compact. After all, any open cover of X must contain some set U containing 0, and thus containing cofinitely many elements of X.

If we have a map $h : X \to Y$, then because Y is discrete, we know that $\{h^{-1}(y) : y \in Y\}$ is an open covering of X and thus by compactness admits a finite subcovering. Thus there are only finitely many elements of y in the image of h.

Now suppose that $f : X \to Y$ is a homotopy equivalence. Then there exists some $g : Y \to X$ with a homotopy $H : f \circ g \simeq 1_Y$. But $H(\{y\} \times I)$ is the continuous image of a connected map and is therefore itself connected. Because Y is discrete, this means that $H(y,0) = H(y,1)$ for all y. But we know that f has finite image, and Y is infinite, so there exists some y such that $y \notin \operatorname{im} f$. In particular, we have $y \neq f(g(y))$, and so $H(y,0) = f(g(y)) \neq y = 1_Y(y)$, a contradiction. Thus X and Y are not of the same homotopy type.

Exercise 1.6. Suppose X is contractible, with $F : c \simeq 1_X$, where c is the constant map at p. Note that, for every $x \in X$, there is a path $F(x,t) : \{x\} \times \mathbb{I} \to X$ taking x to $p \in X$. In particular, this means that every x is in the same component as p, proving connectedness.

Exercise 1.7. The map $H : X \to \mathbb{I} \to X$ taking (x,t) to x and (y,t) to x if and only if $t > \frac{1}{2}$ works. Indeed, note that $H^{-1}(\{x\} \times \mathbb{I})$ is simply $\{x\} \times \mathbb{I} \cup \{y\} \times (\frac{1}{2}, 1]$, which is open in $X \times \mathbb{I}$.

Exercise 1.8.

(i) Consider the map taking the unit interval to S^1 given by $t \mapsto e^{2\pi i t}$.

(ii) If $r : Y \to X$ is a retraction, then we know from $1_Y \simeq c$ that $r \circ 1_Y \circ i \simeq r \circ c \circ i$, where i is the injection $X \hookrightarrow Y$. But the left side is simply $r \circ i = 1_X$, while the left side is a constant map, proving the result.

Exercise 1.9. We know that there exists some constant map c with $f \simeq c$. But then $g \circ f \simeq g \circ c$, and the right side is a constant map. Thus $g \circ f$ is also nullhomotopic.

Exercise 1.10. First, suppose that g is an identification. Note that $(gf)^{-1}(U)$ open in X implies that $g^{-1}(U)$ is open in Y because f is an identification. But the hypothesis on g implies that U is open in Z. Since gf is clearly a continuous surjection, the result follows.

Now, suppose that gf is an identification. It suffices to prove that $g^{-1}(U) \subseteq Y$ open implies that $U \subseteq Z$ is open. But we know by continuity of f that $f^{-1}(g^{-1}(U))$ is open, and so gf being an identification implies the result.

Exercise 1.11. First, note that this is a well-defined function in the sense that $[x] = [y]$ in $X/\sim$ implies that $\overline{f}([x]) = \overline{f}([y])$.

This is evidently continuous. After all, suppose that $U \subseteq Y/\square$ is open. Then we know that

$$\overline{f}^{-1}(U) = \{[x] \in X/\sim : [f(x)] \in U\} = U'.$$

If we let $v : X \to X/\sim$ and $u : Y \to Y/\square$ be the natural maps, then we know that U' is open in $X/\sim$ because

$$v^{-1}(U') = \{x \in X : f(x) \in u^{-1}(U)\} = f^{-1}(u^{-1}(U))$$

is open.

Finally, we will show that $\overline{f}$ is an identification. It is obviously surjective. Moreover, if $U' = \overline{f}^{-1}(U)$ is open in $X/\sim$, then we simply note that a similar argument as above gives us that $v^{-1}(U') = f^{-1}(u^{-1}(U))$ is open. Since f and u are identifications, it follows that U was an open set in the first place, proving the result.

Exercise 1.12. Note that if $K \subseteq Z$ is closed, then it is compact and so $h(K)$ is compact in Z, hence itself closed. Thus h is a closed map, and hence an identification.

Now because $v : X \to X/\ker h$ is an identification, Corollary 1.9 applies. Indeed, Corollary 1.9 implies that $hv^{-1} = \varphi$ is a closed map. Thus it is an identification, i.e., a continuous surjection.

But the same corollary also implies that $\varphi^{-1} = vh^{-1}$ is continuous. This, combined with Example 1.3, in which it was shown that φ is injective, proves the result, as φ is now a bicontinuous bijection, i.e., a homeomorphism.

Exercise 1.13. First observe that $f(x) = f(y)$ implies that $[x,t] = [y,t]$ and so $t = 1$. Thus f is injective and hence bijective onto its image $CX_t = \{[x,t] \in CX : x \in X\}$. Then open sets in CX_t are precisely of the form $U \cap CX_t$ for an open set $U \subseteq CX$. But clearly we can assume that $[x,1] \notin U$ because $[x,1] \notin CX_t$, and thus we wind up with $X \times [0,1)$, where $CX_t = X \times \{t\}$. This is obviously homeomorphic to X.

Exercise 1.14. The functor takes a map $f : X \to Y$ to $Cf : CX \to CY$ given by $C([x,t]) = [f(x),t]$. Note that this is well-defined. Moreover, it is obvious that this is satisfies the properties of a functor. Indeed, if $g : Y \to Z$, then

$$C(g \circ f)([x,t]) = [g(f(x)),t] = ((Cg) \circ (Cf))([x,t])$$

and clearly $C(1_X)$ is the identity on CX.

Paths and Path Connectedness

Exercise 1.15. Using the hint, suppose that $g : \mathbb{I} \to X$ is a path with $g(0) = (0, a) \in A$ and with $g(t) \in G$ for all $t > 0$. Then note that $\pi_i \circ g$ is continuous for $i = 1, 2$, where π_i are the projections to the x- and y-axes. This implies the existence of an $\varepsilon > 0$ such that $t \in (0, \varepsilon)$ implies that $g(t) = (x(t), \sin(1/x(t)))$ has $x(t), |\sin(1/x(t)) - a| < \delta$. But this is obviously impossible, as $\sin(1/x(t))$ will oscillate wildly between -1 and 1.

Exercise 1.16. Let (a_i) and (b_i) be points in S^n. We will construct n paths which, when joined together in the customary fashion (i.e., by traversing each of the $n - 1$ subpaths in $1/(n - 1)$ time), will give us a path from (a_i) to (b_i).

The first path f_1 is defined as

$$f_1(t) = ((1 - t)a_1 + tb_1, c_2, a_3, a_4, \dots, a_n),$$

where c_2 is chosen to be of the same sign as a_2 and in such a way that $f(t) \in S^n$. Note that such a c_2 always exists.

In general, for $1 \leq i \leq n - 1$, the path f_i will fix every coordinate except for the i-th, which it will take to b_i, and the $(i + 1)$-th, which we use as a "free" coordinate to allow for such adjusting. Moreover, observe that if the first $n - 1$ coordinates of two points on S^1 are the same, then the n-th coordinates either will be the same or will be negatives.

If joining the paths $f_1, f_2, \dots, f_{n-1}$ together gives a path from (a_i) to (b_i), then we are done. Note that this occurs if a_n and b_n have the same sign.

Otherwise, construct a path g which adjusts the n-th coordinate and uses the $(n - 1)$-th coordinate as a "free" one, preserving the sign. This effectively allows us to switch the sign of the n-th coordinate so that the n-th coordinate is just b_n. Moreover, because we preserved the sign of the $(n - 1)$-th coordinate, it is still equal to b_{n-1}.

Exercise 1.17. It suffices to show the forward direction, so suppose that U is not path connected. Then there are at least two path components.

We will show that each path component is open, which will prove that U is not connected. But because U is open, we know that open sets in U (as a subspace) or also open in $\mathbb{R}^n$. Thus, for every $x \in U$, there is a ball B_x centered at x and contained in U. This ball is obviously path-connected. As such, if x is in the path component A, it must follow that $B_x \subseteq A$, proving that A is open.

Exercise 1.18. We know that if X is contractible then there exists a point $c \in X$ such that 1_X is homotopic to the constant map at c from X to itself. Now consider the map $c : \mathbb{I} \to X$ satisfying $c(t) = c$ for all t. In the proof of Theorem 1.13, we saw that any path is homotopic to c. In particular, the constant maps $x : \mathbb{I} \to X$ and $y : \mathbb{I} \to X$ at x and y, respectively, are both homotopic to c. Note that these give rise to paths from x to c and from c to y, respectively, which in turn give rise to a path from x to y. This proves path connectedness.

Exercise 1.19.

(i) If X is path connected, then let c and c' be constant maps. Let f be a path from (the point) c to (the point) c' and define $H : X \times \mathbb{I} \to X$ as $H(x, t) = f(t)$. Then H is a homotopy from c to c'.

For the reverse direction, let H be a homotopy from c to c' and define the path $f : \mathbb{I} \to X$ as $f(t) = H(c, t)$.

(ii) Let $f : X \to Y$ be a continuous function. Fix some $y_0 \in Y$ and consider the map

$$\begin{aligned} H : X \times \mathbb{I} &\to Y \\ (x, t) &\mapsto p_x(t), \end{aligned}$$

where p_x is a path from $f(x)$ to y_0. This is a homotopy from f to the constant map mapping X to y_0.

But if $g : X \to Y$ is another continuous function, then the same argument shows that $g \simeq y_0$, and so $f \simeq g$, as desired.

Exercise 1.20. It suffices to show that if $a \in A$ and $b \in B$, then there is a path from a to b. But fix some point $x \in A \cap B$. Then there is a path from a to x, and a path from x to b. Joining the two paths gives a path from a to b.

Exercise 1.21. This is simply done by noting that for any $(x, y), (x', y') \in X \times Y$, we can join the paths $f(t) = ((1-t)x + tx', y)$ and $g(t) = (x', (1-t)y + ty')$.

Exercise 1.22. Suppose $f(a), f(b) \in Y$. Then let p be a path from a to b in X. Now simply note that $q(t) = f(p(t))$ is a path from $f(a)$ to $f(b)$, proving the result.

Exercise 1.23.

(i) We already know that there are at least two path components because the entire space is not path connected. Moreover, both A and G are path connected, and so it follows that they must themselves be the path components.

(ii) Simply note that the sequence $\left\{\left(\frac{1}{n\pi}, \sin(n\pi)\right)\right\} \subset G$ approaches $(0,0) \in A$.

(iii) As per the hint, consider U to be the open disk with center $(0, \frac{1}{2})$ and radius $\frac{1}{4}$. Then $X \cap U$ is open in X. But note that $v(X \cap U)$ is not open in $X/A \approx [0, \frac{1}{2\pi}]$. After all, note that any ball B_ε around the point 0 (which is the image of A under the natural map in this case) must contain some point $\frac{1}{n\pi} < \varepsilon$. But $\frac{1}{n\pi}$, which corresponds to the point $\left(\frac{1}{n\pi}, 0\right) \in X \setminus U$, is not contained in $v(X \cap U)$.

Exercise 1.24. By definition, path components are path connected. Moreover, if C is a path component and there exists some point $x \in X$ and $c \in C$ so that there is a path between x and c, then the definition of path components implies that $x \in C$. Thus path components are maximally path connected.

Finally, suppose that A is path connected and pick $a \in A$. There exists a unique path component C such that $a \in C$. Then for all $b \in A$, we know that there is a path between a and b, and so $b \in C$. Thus $A \subseteq C$, as desired.

Exercise 1.25. Simply use **??** and observe that I is path connected.

Exercise 1.26. Note that, if X is locally path connected, then for all $x \in X$, there exists some open path connected, hence connected, neighborhood V of x. Alternatively, note that if $U \subseteq X$ is open, then its components are unions of its path components and thus open.

Exercise 1.27. Given any open subset U of $X \times Y$ containing a given point $(x, y) \in X \times Y$, there must exist a basic open neighborhood $U_x \times U_y \subseteq U$ of (x, y). Then we know that there exists some path connected V_x with $x \in V_x \subseteq U_x$, and similarly for y. Then $V_x \times V_y$ is path connected by **??**. The result follows.

Exercise 1.28. Note that open subsets of open subsets are open in the main space. In particular, let $A \subseteq X$ be open. Given any $x \in A$, let U be an open neighborhood of x in A. Note that this is also an open neighborhood in X, and so there exists an open path connected V in X (and hence open in A as well) such that $x \in V \subseteq U$.

Exercise 1.29. Consider the map $F : (\mathbb{R}^{n+1} \setminus \{0\}) \times \mathbb{I} \to \mathbb{R}^{n+1} \setminus \{0\}$ given by

$$F((x_i), t) = \left[(1-t) + \frac{t}{\sqrt{\sum x_i^2}}\right](x_i).$$

This is evidently a homotopy which makes S^n a deformation retract.

Exercise 1.30. The exact same map as in **??** works for this case.

Exercise 1.31. It is easy to see that the deformation retract of a deformation retract is a deformation retract, either by a direct argument or by applying Theorem 1.22. Thus the previous exercise implies that it suffices to show that $D^n \setminus \{0\}$ is a deformation retract of $S^n \setminus \{a, b\}$. But the map $(x_i) \mapsto (x_1, \ldots, x_{n-1}, 0)$ is exactly the map needed, and so we are done.

Exercise 1.32. If $H : f_0 \simeq f_1$, then the map $H' : (y, t) \mapsto H(r(y), t)$ is a homotopy from $\tilde{f}_0$ to $\tilde{f}_1$.

Exercise 1.33. Let $Y = \{y\}$ and observe that $(x, 1) \sim y$ for all $x \in X$. Thus $(x, 1) \sim (x', 1)$ for alL $x, x' \in X$. Moreover, this is the only equivalence. Thus M_f is precisely the quotient space $(X \times \mathbb{I})/(X \times \{1\}) = CX$.

Exercise 1.34.

(i) We first tackle i. It is obvious that i is injective, and thus a bijection onto its image $i(X) = \{[x, 0] : x \in X\}$. Moreover, the open sets in $i(X)$ are precisely of the $U \cap i(X)$ for open sets U in M_f.

Note that we can suppose without loss of generality that U is contained in $v(X \times [0, 1))$, where v is the natural map. Thus U simply looks like the Cartesian product of an open interval with an open set of X. This proves that i is a homeomorphism, for the open sets of $i(X)$ map exactly to the open sets of X.

We can show that j is a homeomorphism onto $j(Y)$ in a similar manner. The main idea is simply that $y \not\sim y'$ for any $y, y' \in j(Y)$.

(ii) It is obvious that $(rj)(y) = r[y] = y = 1_Y(y)$ for any $y \in Y$. It is also clearly continuous by the gluing lemma. Thus r is indeed a retraction.

(iii) Define $F : M_f \times \mathbb{I} \to M_f$ as suggested in the hint. It is evident that F is continuous. Moreover, for any $[x, t] \in M_f$, we know that

$$F([x, t], 0) = [x, t]$$
$$F([x, t], 1) = [x, 1] = [f(x)] \in Y.$$

Similarly, if $[y] \in Y$, then the definition implies that the remaining criteria for this homotopy to induce a deformation retraction $r(x) = F(x, 1)$ are satisfied.

(iv) Note that Rotman writes that f is homotopic to $r \circ i$; in fact, we can and do prove the stronger statement that f coincides with $r \circ i$.

Let $f : X \to Y$ be continuous. Then it is clear that the map $f = r \circ i$, where $i : X \to M_f$ is an injection and $r : M_f \to Y$ is the retraction taking $[x, t]$ to $[f(x)]$ and taking $[y]$ to itself, proving the result.