

5 Long Exact Sequences

The Category Comp

Exercise 5.1. These results all follow directly from the definition of exactness.

- (i) Note that $\ker f = \operatorname{im} 0 = 0$, and so f is injective.
- (ii) In this case, we have $\operatorname{im} g = \ker 0 = C$.
- (iii) By the previous two parts, we know that f is bijective. Because f is a homomorphism as well, it follows that f is an isomorphism.
- (iv) Either observe that $0 \rightarrow A \rightarrow 0 \rightarrow 0$ is exact and apply the previous part, or note that $A \rightarrow 0$ is injective while $0 \rightarrow A$ is surjective, implying that $A \cong 0$, i.e., that $A = 0$.

Exercise 5.2. Note that f is surjective if and only if $\ker g = \operatorname{im} f = B$. But $\ker g = B$ if and only if g is the zero map, which is itself true exactly when $\ker h = \operatorname{im} g = 0$. Since $\ker h = 0$ if and only if h is injective, we are done.

Exercise 5.3. We know that $0 \rightarrow A \xrightarrow{i} B$ implies that i is an injection. But because i is a surjection onto its image, this implies that $iA \cong A$. Moreover, because $\ker p = \operatorname{im} i = iA$, we know that $B/iA = B/\ker p \cong \operatorname{im} p$. Because p is a surjection (see Exercise 5.1), the result follows.

Exercise 5.4. This amounts, effectively, to following the arrows and the equations given by exactness. In more detail, let $f_n : B_n \rightarrow C_n$ and $g_n : C_n \rightarrow A_{n-1}$. Now observe that $B_n = \operatorname{im} h_n = \ker f_n$. Thus f_n is the zero map. Moreover, because $\ker g_n = \operatorname{im} f_n$, we know that g_n is injective. Finally, we have $\operatorname{im} g_n = \ker h_{n-1}$. But h_{n-1} is an isomorphism, and so its kernel is trivial. Thus $\operatorname{im} g_n = 0$. Because g_n was injective, it follows that $C_n = 0$.

Exercise 5.5.

- (i) Let f be the map from A to B and g be the map from B to C . Let $\{a_\alpha\}$ and $\{c_\gamma\}$ be maximal independent sets of A and C , respectively. For every α , let $b_\alpha = f(a_\alpha)$. For every γ , pick some $b'_\gamma \in g^{-1}(c_\gamma)$, which is possible by surjectivity of g . If $\sum n_\alpha b_\alpha + \sum n'_\gamma b'_\gamma = 0$, then we know that

$$g\left(\sum n_\alpha b_\alpha + \sum n'_\gamma b'_\gamma\right) = 0.$$

But we also know that $\operatorname{im} f = \ker g$, and so $g(b_\alpha) = 0$. Thus this simply implies that $\sum n'_\gamma c_\gamma = 0$, implying that $n'_\gamma = 0$. But now we know that $\sum n_\alpha b_\alpha = 0$, and so injectivity of f implies that $\sum n_\alpha a_\alpha = 0$ as well. Thus $n_\alpha = 0$ for all α as well, and so $\{b_\alpha\} \cup \{b_\gamma\}$ is independent. Thus $\operatorname{rank} B \geq \operatorname{rank} A + \operatorname{rank} C$.

To show the opposite inequality, it suffices to show that $\{b_\alpha\} \cup \{b_\gamma\}$ is *maximally* independent. Note that $b \notin f(A)$. Otherwise, we could take f^{-1} on $\{b_\alpha\} \cup \{b\}$, which is not independent. Now consider $\{b_\gamma\} \cup \{b\}$. If $g(b) = g(b_\gamma)$ for any γ , then we know by Exercise 5.3 that $b - b_\gamma \in f(A)$. Obviously, we cannot have $b - b_\gamma = b_\alpha$ for any α , otherwise that would give us our linear dependence. Thus $\{b - b_\gamma\} \cup \{b_\alpha\}$ is a subset of $f(A)$ with $\operatorname{rank} A + 1$ elements. This is not independent, a contradiction.

- (ii) We prove this by induction. The previous part takes care of the base case. Consider the following commutative diagrams.

$$\begin{array}{ccccccc} 0 & \longrightarrow & A_n & \xrightarrow{f_n} & A_{n-1} & \xrightarrow{v} & A_{n-1}/\operatorname{im} f_n \longrightarrow 0 \\ & & & & & & \\ 0 & \longrightarrow & A_{n-1}/\ker f_{n-1} & \xrightarrow{\bar{f}_{n-1}} & A_{n-2} & \xrightarrow{f_{n-2}} \dots \xrightarrow{f_2} & A_1 \xrightarrow{f_1} A_0 \longrightarrow 0. \end{array}$$

Here v is the natural map and \bar{f}_{n-1} is the well-defined map taking $x + \ker f_{n-1}$ to $f_{n-1}(x)$.

Let r_i be the rank of A_i . Then the first diagram implies that $r_n - r_{n-1} + r = 0$, where r is the rank of $A_{n-1}/\operatorname{im} f_n$. Because $\operatorname{im} f_n = \ker f_{n-1}$, the second diagram implies by induction that $r - r_{n-2} + r_{n-3} + \dots = 0$. Thus we subtract the first from the second to find that $r_n - r_{n-1} + r_{n-2} - \dots = 0$, as desired.

Exercise 5.6. If $\partial_n = 0$ for all n , then we know that $H_n(S_*) = \ker \partial_n / \text{im } \partial_{n+1} = S_n / \{0\} = S_n$.

Exercise 5.7. If $f : S_* \rightarrow S'_*$ is an equivalence, then it has an inverse $g : S'_* \rightarrow S_*$. Thus at every n , there is a $g_n : S'_n \rightarrow S_n$ so that $g_n \circ f_n = \text{id}_{S_n}$ and $f_n \circ g_n = \text{id}_{S'_n}$. It follows that f_n is an isomorphism for every n .

Conversely, suppose f_n is an isomorphism for every n . Then let $g = \{g_n\}$, where $g_n = f_n^{-1}$. It is clear that f and g are inverses, and so f is indeed an equivalence in **Comp**.

Exercise 5.8. If the former sequence is exact in **Comp**, then we know that $\text{im } f = \ker g$. Then the terms of degree n of both $\text{im } f$ and $\ker g$ must be the same. In other words, we must have $\text{im } f_n = \ker g_n$, and so the latter sequence is exact in \mathcal{A} .

On the other hand, suppose that the latter sequence is exact for every integer n . Then we know that the degree n terms of $\ker f$ and $\text{im } g$ are the same. Moreover, we know that the differentiation operators are the same because they are defined, in both cases, simply as restrictions of the differentiation operator in S_* . Thus the two complexes are the same, as desired.

Exercise 5.9.

(i) We have the following diagram, where $\bar{\partial}_n$ represents the map taking $s_n + S'_n \mapsto \partial_n(s_n) + S'_{n-1}$.

$$\begin{array}{ccccccc} \dots & \longrightarrow & S_{n+1} & \xrightarrow{\partial_{n+1}} & S_n & \xrightarrow{\partial_n} & S_{n-1} \longrightarrow \dots \\ & & \downarrow v_{n+1} & & \downarrow v_n & & \downarrow v_{n-1} \\ \dots & \longrightarrow & S_{n+1}/S'_{n+1} & \xrightarrow{\bar{\partial}_{n+1}} & S_n/S'_n & \xrightarrow{\bar{\partial}_n} & S_{n-1}/S'_{n-1} \longrightarrow \dots \end{array}$$

To show that v is a chain map, we must show that $v_{n-1}\partial_n = \bar{\partial}_n v_n$ for every n . Pick a simplex $\sigma : \Delta^n \rightarrow X$. We know that $v_{n-1}(\partial_n \sigma) = \partial_n \sigma + S'_{n-1}$. However, we also have $\bar{\partial}_n v_n \sigma = \bar{\partial}_n(\sigma + S'_n) = \partial_n \sigma + S'_{n-1}$. Thus this is indeed a chain map.

Moreover, it is obvious that $\ker v_n = S'_n$ for every n . The definition of a subcomplex implies that the $\partial_n|_{\ker v_n}$ is the operation in S'_* . Thus $\ker v = S'_*$, as desired.

(ii) At each n , we know from the previous part that we have the following commutative diagram in \mathcal{A} .

$$\begin{array}{ccc} S_n & \xrightarrow{\partial_n} & S_{n-1} \\ \downarrow v_n & & \downarrow v_{n-1} \\ S_n / \ker f_n & \xrightarrow{\bar{\partial}_n} & S_{n-1} / \ker f_{n-1} \end{array}$$

By the first isomorphism theorem for groups, however, we know that there is an isomorphism θ_n from $S_n / \ker f_n \rightarrow \text{im } f_n$ such that $\theta_n v_n = f_n$.

We claim that $\theta = \{\theta_n\}$ is the desired chain map. To see this, observe that

$$\theta_{n-1} \bar{\partial}_n(\sigma + \ker f_n) = \theta_{n-1}(\partial_n \sigma + \ker f_{n-1}) = f_{n-1} \partial_n \sigma.$$

On the other hand, because $\sigma + \ker f_n = v_n(\sigma)$, we know that

$$\partial'_n \theta_n(\sigma + \ker f_n) = \partial'_n(f_n(\sigma)) = \partial_n f_n \sigma.$$

The two final expressions in the above equations are equal, moreover, because $\{f_n\}$ is itself a chain map.

Exercise 5.10. First, note that both $S'_*/(S'_* \cap S''_*)$ and $(S'_* + S''_*)/S''_*$ are well-defined because everything is abelian. Now consider the map

$$\begin{aligned} \varphi : S'_* &\rightarrow \frac{S'_* + S''_*}{S''_*} \\ S'_n &\mapsto S'_n + S''_*. \end{aligned}$$

Note that we have boundary maps

$$\partial'_n : S'_n \rightarrow S'_{n-1}$$

and

$$\bar{\partial}_n : \frac{S'_n + S''_n}{S''_n} \rightarrow \frac{S'_{n-1} + S''_{n-1}}{S''_{n-1}},$$

where $\bar{\partial}_n$ takes $(s'_n + s''_n) + S''_n$ to $(\partial'_n s'_n + \partial''_n s''_n) + S''_{n-1}$.

We claim that φ is a chain map. To see this, it suffices to show that $\varphi_{n-1} \partial'_n = \bar{\partial}_n \varphi_n$. But for any $\sigma' \in S'_n$, we know that

$$\varphi_{n-1} \partial'_n(\sigma) = \partial'_n \sigma + S''_{n-1} = \bar{\partial}_n(\sigma + S''_n) = \bar{\partial}_n \varphi_n(\sigma),$$

as desired. Moreover, the second isomorphism theorem for groups implies that φ_n is a homomorphism with kernel $S'_n \cap S''_n$, from which it follows that φ is a chain map with $\ker \varphi = S'_* \cap S''_*$. The first isomorphism theorem (Exercise 5.9) implies the result.

Exercise 5.11. Consider the sequence in the problem, namely

$$0 \longrightarrow T_*/U_* \xrightarrow{i} S_*/U_* \xrightarrow{p} S_*/T_* \longrightarrow 0.$$

Clearly, we have

$$\text{im } i_n = \{t_n + U_n : t_n \in T_n\}.$$

Moreover, we know that $p_n(s_n + U_n) = s_n + T_n$, so

$$\ker p_n = \{s_n + U_n : s_n \in T_n\}.$$

Clearly these are equal.

It now suffices to prove that $\ker i_n = 0$ and $\text{im } p_n = S_n/T_n$. But note that $i_n(t_n + U_n) = 0$ implies that $t_n \in U_n$. Hence $t_n + U_n = 0$ as an element of T_n/U_n as well. Moreover, consider an arbitrary element $s_n + T_n \in S_n/T_n$. It is equal to $p_n(s_n + U_n)$, which proves that p is surjective. Hence the sequence of complexes is exact.

Exercise 5.12. We claim that

$$\ker \left(\sum \partial_n^\lambda \right) = \sum \ker \partial_n^\lambda.$$

If $\sum s_n^\lambda \in \ker \left(\sum \partial_n^\lambda \right)$, then by definition we must have $\partial_n^\lambda(s_n^\lambda) = 0$ for each λ . The converse is also clearly true. Similarly, we find that

$$\text{im} \left(\sum \partial_{n+1}^\lambda \right) = \left\{ \sum \partial_{n+1}^\lambda(s_{n+1}^\lambda) \right\} = \sum \text{im } \partial_{n+1}^\lambda.$$

Thus we conclude that

$$H_n \left(\sum S_*^\lambda \right) = \frac{\ker \partial_n}{\text{im } \partial_{n+1}} = \frac{\sum \ker \partial_n^\lambda}{\sum \text{im } \partial_{n+1}^\lambda} = \sum \frac{\ker \partial_n^\lambda}{\text{im } \partial_{n+1}^\lambda} = \sum H_n(S_*^\lambda).$$

Exact Homology Sequences

Exercise 5.13. Suppose $\sum m_\sigma \sigma + S_n(A) = \sum m'_\sigma \sigma + S_n(A)$. This implies that $m_\sigma - m'_\sigma = 0$ for every σ with $\text{im } \sigma \not\subseteq A$. Hence we can pick the unique representative

$$\sum_{\text{im } \sigma \not\subseteq A} m_\sigma \sigma = \sum_{\text{im } \sigma \not\subseteq A} m'_\sigma \sigma,$$

thus showing that this is indeed the free abelian group generated by σ with $\text{im } \sigma \not\subseteq A$.

Exercise 5.14.

- (i) It suffices to prove that p_n is surjective and that $\text{im } i_n = \ker p_n$. To see that p_n is surjective, consider the following segment of the long exact sequence:

$$B_n \xrightarrow{p_n} C_n \longrightarrow A_{n-1} \xrightarrow{i_{n-1}} B_{n-1}$$

Note that the map $f_n : C_n \rightarrow A_{n-1}$ has $\text{im } f_n = \ker i_{n-1} = 0$, and so $\ker f_n = C_n$. Thus $\text{im } f_n = C_n$, proving surjectivity.

To see that $\text{im } i_n = \ker p_n$, simply consider the following segment:

$$A_n \xrightarrow{i_n} B_n \xrightarrow{p_n} C_n$$

The result immediately follows.

- (ii) There exists a map r with $r \circ i = \text{id}_A$. Thus $r_* \circ i_* = \text{id}_{H_n(A)}$, from which it follows that i_* is injective. Theorem 5.8 gives an exact sequence

$$\dots \longrightarrow H_n(A) \xrightarrow{i_n} H_n(X) \xrightarrow{p_n} H_n(X, A) \xrightarrow{d} \dots$$

where p_* is induced by the quotient map $S_*(X) \rightarrow S_*(X)/S_*(A)$. Since i_* injective implies that i_n is injective, we can apply the previous part to find an exact sequence

$$0 \longrightarrow H_n(A) \xrightarrow{i_n} H_n(X) \xrightarrow{p_n} H_n(X, A) \longrightarrow 0$$

Then Exercise 5.3 implies that $H_n \oplus H_n(X, A) = H_n(X)$, as desired.

- (iii) We now have $i \circ r \simeq \text{id}_X$ as well. In particular, since A and X have the same homotopy type, we must have $H_n(A) \cong H_n(X)$ by Corollary 4.24. Thus i_n in the exact sequence given in the previous part must be the identity, and so $\ker p_n = \text{im } i_n = H_n(X)$. But since p_n is surjective, it follows that $H_n(X, A) = 0$, as desired.

Exercise 5.15. We prove this in cases. We will use the follow commutative diagram, where the columns are exact:

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ \dots & \longrightarrow & S'_{n+1} & \xrightarrow{\partial'_{n+1}} & S'_n & \xrightarrow{\partial'_n} & S'_{n-1} \longrightarrow \dots \\ & & \downarrow i_{n+1} & & \downarrow i_n & & \downarrow i_{n-1} \\ \dots & \longrightarrow & S_{n+1} & \xrightarrow{\partial_{n+1}} & S_n & \xrightarrow{\partial_n} & S_{n-1} \longrightarrow \dots \\ & & \downarrow p_{n+1} & & \downarrow p_n & & \downarrow p_n \\ \dots & \longrightarrow & S''_{n+1} & \xrightarrow{\partial''_{n+1}} & S''_n & \xrightarrow{\partial''_n} & S''_{n-1} \longrightarrow \dots \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

Case 1. S_* and S'_* are acyclic.

We would like to show that $Z''_n = \ker \partial''_n$ is equal to $B''_n = \text{im } \partial''_{n+1}$. We already know that $B''_n \subseteq Z''_n$. Surjectivity of p implies that we can rewrite Z''_n as

$$Z''_n = p_n(\ker(\partial''_n p_n)) = p_n(\ker(p_{n-1} \partial_n)).$$

This, in turn, can be written as

$$Z''_n = \{p_n s_n : p_{n-1} \partial_n s_n = 0\}.$$

On the other hand, we can rewrite B_n'' as

$$B_n'' = \text{im}(\partial_{n+1}'' p_{n+1} = \text{im}(p_n \partial_{n+1})) = p_n \text{im } \partial_{n+1}.$$

Since S_* is acyclic, we know that $\text{im } \partial_{n+1} = \ker \partial_n$, and so we find that

$$B_n'' = \{p_n z_n : \partial_n z_n = 0\}.$$

Now consider an arbitrary element $p_n s_n \in Z_n''$. Since $p_{n-1} \partial_n s_n = 0$, we know that $\partial_n s_n \in \ker p_{n-1} = \text{im } i_{n-1}$, where again we use the fact that S_* is acyclic. Injectivity of i_{n-1} implies the existence of a unique $s'_{n-1} \in S'_{n-1}$ with $i_{n-1} s'_{n-1} = \partial_n s_n$. We know, however, that $\partial_{n-1} \partial_n = 0$, and so

$$0 = \partial_{n-1} \partial_n s = \partial_{n-1} i_{n-1} s'_{n-1} = i_{n-2} \partial'_{n-1} s'_{n-1}.$$

Since i_{n-2} is injective, it follows that $\partial'_{n-1} s'_{n-1} = 0$, and so acyclicity of S'_* implies that $s'_{n-1} \in \ker \partial'_{n-1} = \text{im } \partial'_n$. In particular, we can write $s'_{n-1} = \partial'_n s'_n$.

Now notice that

$$\partial_n i_n s'_n = i_{n-1} \partial'_n s'_n = i_{n-1} s'_{n-1} = \partial_n s_n,$$

where the last equality follows from the definition of s'_{n-1} . We know that $z_n = s_n - i_n s'_n \in Z_n$ since ∂'_n is a homomorphism. But we also know that

$$p_n z_n = p_n (s_n - i_n s'_n) = p_n s_n - p_n i_n s'_n = p_n s_n,$$

where we use exactness of the columns. In other words, we have a z_n with $\partial_n z_n = 0$, such that $p_n z_n = p_n s_n$. Thus $p_n s_n \in B_n''$, proving that $Z_n'' = B_n''$. To be even more explicit, this implies that $H_n'' = Z_n''/B_n'' = 0$ for all n , proving that S_*'' is an acyclic complex as well.

Case 2. S'_* and S_*'' are acyclic.

Suppose $s_n \in Z_n$, i.e., that $\partial_n s_n = 0$. Then $\partial_n'' p_n = p_{n-1} \partial_n$ implies that $p_n s_n \in \ker \partial_n'' = \text{im } \partial_{n+1}''$. Hence write $p_n s_n = \partial_{n+1}'' s''_{n+1}$. Since p_{n+1} is surjective, we can find s_{n+1} with $p_{n+1} s_{n+1} = s''_{n+1}$, and so

$$p_n \partial_{n+1} s_{n+1} = \partial_{n+1}'' p_{n+1} s_{n+1} = p_n s_n.$$

But then we know that $\partial_{n+1} s_{n+1} - s_n \in \ker p_n = \text{im } i_n$. Thus there exists a unique s'_n with $i_n s'_n = \partial_{n+1} s_{n+1} - s_n$. We can take ∂_n of both sides to find that

$$0 = \partial_n \partial_{n+1} s_{n+1} - \partial_n s_n = \partial_n i_n s'_n = i_{n-1} \partial'_n s'_n,$$

and so it follows that $\partial'_n s'_n = 0$. In particular, we know that $s'_n \in \text{im } \partial'_{n+1}$, so we can find s'_{n+1} whose boundary is s'_n . Recall that we had

$$s_n = \partial_{n+1} s_{n+1} - i_n s'_n.$$

But the last term is equal to $i_n \partial'_{n+1} s'_{n+1} = \partial_{n+1} i_{n+1} s'_{n+1}$, and so this is in turn equal to

$$s_n = \partial_{n+1} (s_{n+1} - i_{n+1} s'_{n+1}).$$

This proves that $s_n \in B_n$, and so $Z_n = B_n$.

Case 3. S_* and S_*'' are acyclic.

This final case is handled similarly to the first two, but we lay out the details below. Let $s'_n \in \ker \partial'_n = Z'_n$ be arbitrary. Then $i_n s'_n \in \ker \partial_n = \text{im } \partial_{n+1}$, and so

$$i_n s'_n = \partial_{n+1} s_{n+1}$$

for some s_{n+1} . We know that $p_{n+1} s_{n+1} \in \ker \partial_{n+1}'' = \text{im } \partial_{n+2}''$ because $p_n i_n s'_n = 0$. Hence there exists s''_{n+2} with $\partial_{n+2}'' s''_{n+2} = p_{n+1} s_{n+1}$. But then it follows that

$$p_{n+1} \partial_{n+2} s_{n+2} = \partial_{n+2}'' p_{n+2} s_{n+2} = p_{n+1} s_{n+1},$$

from which it follows that $s_{n+1} - \partial_{n+2} s_{n+2} \in \ker p_{n+1} = \text{im } i_{n+1}$. Thus there exists s'_{n+1} with $i_{n+1} s'_{n+1} = s_{n+1} - \partial_{n+2} s_{n+2}$. We then find that

$$i_n \partial'_{n+1} s'_{n+1} = \partial_{n+1} i_{n+1} s'_{n+1} = \partial_{n+1} s_{n+1} = i_n s'_n.$$

Injectivity implies $s'_n = \partial'_{n+1} s'_{n+1} \in B_n$, thus proving the final case.

Exercise 5.16. To show that $f_{\#}(Z_n(X, A)) \subseteq Z_n(X', A')$, consider $\gamma \in Z_n(X, A)$. Note that $\gamma \in S_n(X)$, and so Lemma 4.8 implies that

$$\partial'_n f_{\#} \gamma = f_{\#} \partial_n \gamma.$$

We know, moreover, that $\partial_n \gamma = \sum m_{\sigma} \sigma$, where the sum ranges over all σ with $\text{im } \sigma \subseteq A$. Thus it follows that

$$\partial'_n f_{\#} \gamma = f_{\#} \left(\sum m_{\sigma} \sigma \right) = \sum_{\text{im}(f\sigma) \subseteq f(A)} m_{\sigma} f\sigma.$$

Since $f\sigma$ is a simplex into X' with image contained in A' , it follows that this is an element of $S_{n-1}(A')$. Hence we conclude that $f_{\#} \gamma \in Z_n(X', A')$, as desired.

The proof for boundaries is similar.

Exercise 5.17. As defined, we have that $f_{\#} : H_n(X, A) \rightarrow H_n(X', A')$ is given by

$$f_{\#} : \bar{\gamma} + \text{im } \bar{\partial}_{n+1} \mapsto f_{\#}(\bar{\gamma}) + \text{im } \bar{\partial}'_{n+1},$$

where $\bar{\partial}$ and $\bar{\partial}'$ denote the boundary maps of the quotient complexes $S_*(X)/S_*(A)$ and $S_*(X')/S_*(A')$, respectively, and where

$$\bar{\gamma} \in \ker \bar{\partial}_n = Z_n(X, A)/S_n(A).$$

The third isomorphism theorem gives an isomorphism

$$H_n(X, A) = \frac{Z_n(X, A)/S_n(A)}{B_n(X, A)/S_n(A)} \rightarrow \frac{Z_n(X, A)}{B_n(X, A)}$$

which takes

$$\bar{\gamma} + B_n(X, A)/S_n(A) \mapsto \gamma + B_n(X, A).$$

Since $f_{\#}(\bar{\gamma}) = \overline{f_{\#}(\gamma)}$, we find that, thinking of $f_{\#}$ as a map from $Z_n(X, A)/B_n(X, A)$ to a map from $Z_n(X', A')/B_n(X', A')$, it takes

$$\gamma + B_n(X, A) \mapsto f_{\#}(\gamma) + B_n(X', A'),$$

as desired.

Exercise 5.18. Recall the definition of $\varepsilon_i : \Delta^{n-1} \rightarrow \Delta^n$ as taking $\{e_0, \dots, e_{n-1}\}$ to $\{e_0, \dots, \hat{e}_i, \dots, e_{n-1}\}$. Thus we have

$$\partial_n \sigma = \sum_{i=0}^n (-1)^i \sigma \varepsilon_i.$$

Since $\sigma \varepsilon_i : \Delta^{n-1} \rightarrow X$ has image in A by hypothesis, this is in $S_{n-1}(A)$, as desired.

Reduced Homology

Exercise 5.19. By Theorem 5.6, it is sufficient to show that we have a short exact sequence

$$0 \longrightarrow \tilde{S}_*(A) \longrightarrow \tilde{S}_*(X) \longrightarrow S_*(X, A) \longrightarrow 0.$$

When $n \geq 1$, this is clear by Theorem 5.8. When $n = 0$, we have the sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow 0 \rightarrow 0,$$

which is easily verified to be exact.

Exercise 5.20. Consider the exact sequence

$$H_1(CX) \longrightarrow H_1(CX, X) \longrightarrow \tilde{H}_0(X) \longrightarrow \tilde{H}_0(CX).$$

Note that $H_1(CX) = 0$ because CX is contractible. On the other hand, Corollary 5.18 implies that $\tilde{H}_0(X) \cong \mathbb{Z}^4$, while $\tilde{H}_0(CX) \cong 0$. Thus the map $H_1(CX, X) \rightarrow \tilde{H}_0(X)$ is surjective. Moreover, its kernel is equal to the image of the map $H_1(CX) \rightarrow H_1(CX, X)$, which is simply 0 since $H_1(CX) = 0$. Thus the map is also injective, from which it immediately follows that $H_1(CX, X) \cong \mathbb{Z}^4$.

Exercise 5.21. Consider the exact sequence

$$\tilde{H}_1(S^0) \longrightarrow \tilde{H}_1(S^1) \longrightarrow H_1(S^1, S^0) \longrightarrow \tilde{H}_0(S^0) \longrightarrow \tilde{H}_0(S^1),$$

which is simply equal to

$$0 \longrightarrow \mathbb{Z} \longrightarrow \boxed{} \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Now note that the first map has $\text{im} = 0$, so the second map has $\ker = 0$. Thus the second map has $\text{im} \cong \mathbb{Z}$, and so the third map has $\ker \cong \mathbb{Z}$. Yet we also know that the last map is the zero map, and so the third map has $\text{im} = \mathbb{Z}$, from which it follows that the $H_1(S^1, S^0) = \mathbb{Z} \times \mathbb{Z}$.

Exercise 5.22. When $n = 0$, this follows from the exact sequence

$$\tilde{H}_0(X) \longrightarrow \tilde{H}_0(X) \longrightarrow H_0(X, X) \longrightarrow 0.$$

After all, the first map is the identity, and so the second map is the zero map. But the second map is surjective, and so $H_0(X, X) = 0$.

For $n > 0$, we have the exact sequence

$$\tilde{H}_n(X) \longrightarrow \tilde{H}_n(X) \longrightarrow H_n(X, X) \longrightarrow \tilde{H}_{n-1}(X) \longrightarrow \tilde{H}_{n-1}(X).$$

The first map is the identity, and so the second map is everywhere zero. Thus the image of the second map, which is the kernel of the third map too, is equal to 0. Since the kernel of the last map, which is 0 (the map is the identity), is equal to the image of the third map, it follows that the third map is everywhere zero. Hence the third map is injective, but also everywhere zero, and so $H_n(X, X)$ must have been 0 in the first place.