Solutions to Rotman's algebraic topology

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0 Introduction

Notation

No exercises!

Brouwer Fixed Point Theorem

Exercise 0.1. As per the hint, observe that if $y \in G$, then we have $y = r(y) + (y - r(y))$. Obviously, we have $r(y) \in H$. Moreover, we know that

$$r(y - r(y)) = r(y) - r(r(y)) = 0,$$

and so $y - r(y) \in \ker r$. Thus $G \subseteq H \oplus \ker r$.

The reverse is obviously true, since H and $\ker r$ are both subgroups of G.

Exercise 0.2. Suppose instead that $f : D^1 \to D^1$ has no fixed point. Then consider the continuous map $g : D^1 \to S^0$ given by

$$g(x) = \begin{cases} 1 & \text{if } f(x) < x \\ -1 & \text{if } f(x) > x \end{cases}.$$

Notice that because $f(x) \neq x$ for all x, the function g is well-defined.

Moreover, we know that $f(-1) \neq -1$, since f has no fixed point, and so $f(-1) > -1$. Thus $g(-1) = -1$. Similarly, we have $g(1) = 1$.

Thus we have $g(D^1) = S^0$, which is disconnected. This is a contradiction, so f must have had a fixed point.

Exercise 0.3. Suppose that r is such a retract. Then we have the following commutative diagram:

$$\begin{tikzcd} & S^n \arrow[dr, "r"] & \\ S^{n-1} \arrow[ur, "i"] \arrow[rr, "1"] & & S^{n-1}. \end{tikzcd}$$

Applying H_{n-1}, we get another commutative diagram:

$$\begin{tikzcd} & H_{n-1}(S^n) \arrow[dr, "H_{n-1}(r)"] & \\ H_{n-1}(S^{n-1}) \arrow[ur, "H_{n-1}(i)"] \arrow[rr, "H_{n-1}(1)"] & & H_{n-1}(S^{n-1}). \end{tikzcd}$$

We know that $H_{n-1}(S^n) = 0$, however, implying that $H_{n-1}(1) = 0$. This contradicts the fact that $H_{n-1}(S^{n-1}) = \mathbb{Z} \neq 0$. Thus the retraction r could not have existed.

Exercise 0.4. Suppose $g : D^n \to X$ is a homeomorphism. Then we know that $g^{-1} \circ f \circ g$ is a continuous map from D^n to itself, and so it has a fixed point x. Then we know that $g^{-1}(f(g(x))) = x$, and so it follows that $f(g(x)) = g(x)$. Thus $g(x) \in X$ is a fixed point of f.

Exercise 0.5. Consider the function $h : \mathbb{I} \times \mathbb{I} \to \mathbb{I} \times \mathbb{I}$ given by

$$h(s,t) = f(s) - g(t) + (s,t).$$

This is the sum of continuous functions, and so it is itself continuous. Moreover, we know that $\mathbb{I} \times \mathbb{I}$ is homeomorphic to D^1, and so it follows that there is a fixed point (s,t) of h. But this means that $f(s) - g(t) = 0$, and so we are done.

Exercise 0.6. Observe that $x \in \Delta^{n-1}$ must contain some positive coordinate, because $\sum x_i = 1$ and $x_i \geq 0$ for all i. Since $a_{ij} > 0$ for every i, j, it follows that Ax contains only nonnegative coordinates and, moreover, contains at least one positive coordinate. Thus $\sigma(Ax) > 0$, and so $g(x)$ is well-defined.

Moreover, it is continuous because the linear map A, the map σ, and the division function are all continuous.

Because $\Delta^{n-1} \approx D^{n-1}$, it follows that there exists some x with

$$x = \frac{Ax}{\sigma(Ax)}.$$

Then $\lambda = \sigma(Ax) > 0$ is a positive eigenvalue for A and $x \in \Delta^{n-1}$ is a corresponding eigenvector.

We know that x contains only nonnegative coordinates. Suppose then that some coordinate, say x_1, is zero. Then obviously the first coordinate of λx is zero. However, the first coordinate of Ax is

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = a_{12}x_2 + \cdots + a_{1n}x_n.$$

Since $\sum x_i = 1$ and $x_1 = 0$, there exists some $k \neq 1$ such that $x_k > 0$. Then $a_{1k}x_k > 0$, and since each i already has $a_{1i}x_i \geq 0$, it follows that the first coordinate of Ax is strictly positive, contradicting that $Ax = \lambda x$.

Thus the eigenvector x has all positive coordinates.

Categories and Functors

Exercise 0.7. We know that

$$g \circ (f \circ h) = g \circ 1_b = g$$

and

$$(g \circ f) \circ h = 1_A \circ h = h,$$

and so associativity implies $g = h$.

Exercise 0.8.

(i) Notice that if 1_A and $1'_A$ are both identities, then we must have

$$1_A = 1_A \circ 1'_A = 1'_A,$$

which proves the desired result.

(ii) If $1'_A$ is the new identity in $\mathcal{C}'$, then we know that $1'_A \in \mathrm{Hom}_{\mathcal{C}'}(A, A) \subseteq \mathrm{Hom}_{\mathcal{C}}(A, A)$, and so $1_A \circ 1'_A$ is defined. But we know that

$$1'_A \circ 1_A = 1'_A = 1'_A \circ 1'_A,$$

and so Exercise 0.7 implies the result.

Exercise 0.9. Clearly, the Hom-sets are pairwise disjoint, since each i_y^x appears at most once.

It is also obviously associative. In particular, if $a \leq b \leq c \leq d$, then we know that

$$i_d^c \circ \left(i_c^b \circ i_b^a\right) = i_d^c \circ i_c^a = i_d^a,$$

and similarly for $\left(i_d^c \circ i_c^b\right) \circ i_b^a$.

Finally, the map i_x^x is the identity on $x \in X$. To see that it is a left-identity, note that if $y \leq x$, then

$$i_x^x \circ i_x^y = i_x^y.$$

Similarly, we can show that this map is a right-identity as well, and so we are done.

Exercise 0.10. Disjointness is clear, since there is only one object. Because G is a monoid, it is associative and has an identity, proving that $\mathcal{C}$ is a category.

Exercise 0.11. It is pretty clear that $\mathrm{obj}(\mathbf{Top}) \subset \mathrm{obj}(\mathbf{Top}^2)$. Moreover, a continuous map $f : X \to Y$ between two topological spaces corresponds to the map $(f, \emptyset)$ in $\mathbf{Top}^2$ from $(X, \emptyset)$ to $(Y, \emptyset)$, which then means that $\mathbf{Top}$ can be thought of as a subcategory of $\mathbf{Top}^2$.

Exercise 0.12. It is worth noting that Rotman's definition here is incorrect. The morphisms in $\mathcal{M}$ should be the commutative squares, not merely the ordered pairs (h, k).

Indeed, consider the following counterexample to Rotman's definition. Let $\mathcal{C}$ be the category of sets. Furthermore, let A be a set with more than one element. Then the following diagrams are both commutative:

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "1_A"] & A \arrow[d, "0"] \\ A \arrow[r, "0"] & \{0\} \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "0"] \arrow[d, "1_A"] & A \arrow[d, "0"] \\ A \arrow[r, "0"] & \{0\}. \end{tikzcd}$$

This implies that the ordered pair $(1_A, 0)$, where 0 is considered to be the map that sends everything in A to the zero element, is both in $\mathrm{Hom}(1_A, 0)$ and in $\mathrm{Hom}(0, 0)$, contradicting disjointness.

If we instead consider morphisms of $\mathcal{M}$ to be the commutative squares, where composition is defined by "stacking" the squares on top of one another, disjointness is clear. After all, the squares contain f and g, and so Hom-sets of different objects must be disjoint.

Associativity is clear, as the morphisms of $\mathcal{C}$ are associative.

Finally, there is an identity 1_f for every $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, namely the one where $h = 1_A$ and $k = 1_B$.

Exercise 0.13. With the hint, this is clear. In particular, we consider $\mathbf{Top}^2$ to be the subcategory of the arrow category of $\mathbf{Top}$ in which the objects are inclusions, and $\mathrm{Hom}_{\mathbf{Top}^2}(i, j) = \mathrm{Hom}_{\mathbf{Top}}(i, j)$.

Exercise 0.14. To see that it is a congruence at all, observe that Property (i) is satisfied because there is only one Hom-set. Moreover, if $x \sim x'$ and $y \sim y'$, then we know that $x(x')^{-1} = h_x$ and $y(y')^{-1} = h_y$ for some $h_x, h_y \in H$. But then we know that

$$(yx)(y'x')^{-1} = yx(x')^{-1}(y')^{-1} = yh_x(y')^{-1}.$$

However, since $(y')^{-1} = y^{-1}h_y$, we know that this is simply

$$(yx)(y'x')^{-1} = yh_xy^{-1}h_y.$$

Because H is normal, we know that $yh_xy^{-1} \in H$. Thus the product of this and h_y is in H as well, and so $xy \sim x'y'$, as desired.

To see that $[*, *] = G/H$ simply requires the observation that $x \sim y$ if and only if x and y are in the same coset of H.

Exercise 0.15. This follows from the fact that functors preserve (or, in the case of contravariant functors, reverse) the directions of the arrows. Thus the resulting diagram still commutes.

Exercise 0.16. Note that for (i)–(iv), we can simply use inverses. For instance, for **Set**, it suffices to note that if f is a bijection, then f^{-1} is a bijection, which is clearly true. Similarly, the inverse of a homeomorphism is a homeomorphism, and the inverse of a group or ring isomorphism is still an isomorphism.

For (v), note that i_x^y is defined and satisfies the requirements that $i_x^y \circ i_y^x = i_x^x$ and $i_y^x \circ i_x^y = i_y^y$.

For part (vi), notice that f^{-1} works because f is a homeomorphism. In particular, it is a bijection, and so $f^{-1}(A') = A$. Moreover, it is (bi)continuous since f is.

Finally, for the monoid G, if g has a two-sided inverse h, then $hg = gh = 1$, which is the identity element of $\mathrm{Hom}(G, G)$.

Exercise 0.17. To prove that T' is a functor, first observe that criterion (i) of a functor is satisfied because T does so. Moreover, if $[f] \in \mathrm{Hom}_{\mathcal{C}'}(A, B)$, then $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, and so $T'([f]) = Tf$ is a morphism in $\mathcal{A}$. In particular, if $[g] \circ [f] = [g \circ f]$ is defined in $\mathcal{C}'$, then $g \circ f$ is defined in $\mathcal{C}$. This means, then, that

$$T'([g] \circ [f]) = T(g \circ f) = (Tg) \circ (Tf) = T'([g]) \circ T'([f]).$$

Finally, it remains to note that $T'([1_A]) = T_{1_A} = 1_{TA} = 1_{T'([A])}$ for every object A. Thus T' is a functor.

Exercise 0.18.

(i) It is clear that $tG \in \text{obj}\,\mathbf{Ab}$ for every group G. Now suppose that we have a homomorphism $f : G \to H$. Then we know that $t(f)$ is a morphism $f|_{tG}$ from tG to tH. To see this, note that it is the restriction of a homomorphism, and thus is itself a homormophism. Moreover, if $x \in f(tG)$, then $x = f(y)$ for some $y \in G$ with finite order. But then there exists some n so that $y^n = 1$. Thus $x^n = f(y^n) = 1$, and so x has finite order. But $x \in f(G) \subseteq H$ implies that $x \in tH$.

Now we must check that t respects composition. Indeed, if $g \circ f$ is defined, then

$$t(g \circ f) = (g \circ f)_{tG} = g|_{f(tG)} \circ f|_{tG}.$$

But $f(tG) \subseteq tH$, and so this is simply

$$t(g \circ f) = g|_{tH} \circ f|_{tG} = t(g) \circ t(f),$$

which proves that composition is respected.

Finally, note simply that $t(1_G) = 1|_{tG}$, which is the identity on tG.

(ii) Suppose that f is an injective homomorphism from G to H. Then suppose that $t(f)(x) = t(f)(y)$. But $f(x) = f|_{tG}(x) = t(f)(x)$, and so it follows that $f(x) = f(y)$. Injectivity of f proves the result.

(iii) Let $G = \mathbb{Z}$ and $H = \mathbb{Z}/2\mathbb{Z}$ and let f take even integers to 0 and odd integers to 1. This is evidently surjective. But $tG = \{0\}$ while $tH = \{0, 1\}$, and so $t(f) : tG \to tH$ cannot be surjective.

Exercise 0.19.

(i) If f is a surjection, then consider an arbitrary coset $a + pH$ of H/pH. We know that there exists some $b \in G$ with $f(b) = a$, and so it follows that $F(f)$ takes $b + pG$ to $a + pH$, proving surjectivity of $F(f)$.

(ii) Consider the function $f : \mathbb{Z} \to \mathbb{Z}$ taking x to $2x$. Then, letting $p = 2$, we know that $F(f) : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ has $F(f)([0]) = F(f)([1])$.

Exercise 0.20.

(i) This is evident because $\mathbb{R}$ is a ring, and the operations are pointwise.

(ii) By the previous part, we know that if X is a topological space, then $C(X)$ is a ring. Now suppose that $f : X \to Y$ is a continuous map. Then define

$$\begin{aligned} C(f) : C(Y) &\to C(X) \\ g &\mapsto g \circ f \end{aligned}$$

and note that this is well-defined. Moreover, we know that $C(g \circ f)(h) = h \circ g \circ f$, while $C(f) \circ C(g)$ takes h to $C(f) \circ (h \circ g) = h \circ g \circ f$, which proves that C reverses composition. Finally, we know that $C(1_x)$ takes g to $g \circ 1_X = g$ and is therefore the identity on $C(Y)$. Thus C (or, rather, the map taking X to $C(X)$, to be precise) gives rise to a contravariant functor.

1 Some Basic Topological Notions

Homotopy

No exercises!

Convexity, Contractibility, and Cones

Exercise 1.1. Suppose $H : f_0 \simeq f_1$ is a homotopy. Then let $F(t) = H(x,t)$ for some fixed x. It is clear that $F(0) = x_0$ and $F(1) = 1$. Moreover, since H is continuous, it follows that so too is F. For the converse, simply let the homotopy $H : f_0 \simeq f_1$ take $(x,t) \in X \times \mathbb{I}$ to $F(t)$.

Exercise 1.2.

(i) There exist functions $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_Y$. Moreover, there is a homotopy $F : 1_X \simeq c$, where c denotes the constant map at some $x_0 \in X$. Then consider the map $G : Y \times \mathbb{I} \to Y$ which takes (y,t) to $f(F(g(y),t))$. In particular, we know that G is continuous and that it is thus a homotopy from $f \circ g$ to the constant map c' at $y_0 = f(x_0)$. But then we find that $1_Y \simeq f \circ g \simeq c'$, and so Y is contractible.

(ii) Consider, for example, the subsets $X, Y \subset \mathbb{R}^2$ where

$$\begin{aligned} X &= \{(x,0) : x \in [0,1]\}, \\ Y &= \left\{(x,x) : x \in \left[0, \frac{1}{2}\right]\right\} \cup \left\{(x, 1-x) : x \in \left[\frac{1}{2}, 1\right]\right\}. \end{aligned}$$

It is obvious that X is convex, but Y is not, even though there is an obvious homotopy equivalence from X to Y.

Exercise 1.3. We know that $R(x) = e^{i\alpha}x$, and so the continuous map $F : S^1 \times \mathbb{I} \to S^1$ given by $F(x,t) = e^{i\alpha t}x$ is a homotopy $F : 1_S \simeq R$. Thus, if $g : S^1 \to S^1$ is continuous, then let θ be such that $g(1) = g(e^{i\cdot 0}) = e^{i\theta}$. Then we know that, letting R now be the rotation of $-\theta$ degrees, we must have $R \circ g \simeq 1_S \simeq g = g$ and $(R \circ g)(1) = 1$, as desired.

Exercise 1.4.

(i) Pick $(x_1, y_1), (x_2, y_2) \in X \times Y$. Then we know that, for any $t \in \mathbb{I}$, we have

$$t(x_1, y_1) + (1-t)(x_2, y_2) = (tx_1 + (1-t)x_2, ty_1 + (1-t)y_2).$$

The result follows from convexity of X and Y.

(ii) If $F_X : 1_X \simeq c_X$ and $F_Y : 1_Y \simeq c_Y$, where c_X and c_Y are constant maps at c_X and c_Y, respectively, then the map

$$\begin{aligned} F : (X \times Y) \times \mathbb{I} &\to X \times Y \\ (x, y, t) &\mapsto (F_X(x,t), F_Y(y,t)) \end{aligned}$$

is clearly a homotopy from $1_{X \times Y}$ to (c_X, c_Y).

Exercise 1.5. It is clear that X is compact. After all, any open cover of X must contain some set U containing 0, and thus containing cofinitely many elements of X.

If we have a map $h : X \to Y$, then because Y is discrete, we know that $\{h^{-1}(y) : y \in Y\}$ is an open covering of X and thus by compactness admits a finite subcovering. Thus there are only finitely many elements of y in the image of h.

Now suppose that $f : X \to Y$ is a homotopy equivalence. Then there exists some $g : Y \to X$ with a homotopy $H : f \circ g \simeq 1_Y$. But $H(\{y\} \times I)$ is the continuous image of a connected map and is therefore itself connected. Because Y is discrete, this means that $H(y,0) = H(y,1)$ for all y. But we know that f has finite image, and Y is infinite, so there exists some y such that $y \notin \operatorname{im} f$. In particular, we have $y \neq f(g(y))$, and so $H(y,0) = f(g(y)) \neq y = 1_Y(y)$, a contradiction. Thus X and Y are not of the same homotopy type.

Exercise 1.6. Suppose X is contractible, with $F : c \simeq 1_X$, where c is the constant map at p. Note that, for every $x \in X$, there is a path $F(x,t) : \{x\} \times \mathbb{I} \to X$ taking x to $p \in X$. In particular, this means that every x is in the same component as p, proving connectedness.

Exercise 1.7. The map $H : X \to \mathbb{I} \to X$ taking (x,t) to x and (y,t) to x if and only if $t > \frac{1}{2}$ works. Indeed, note that $H^{-1}(\{x\} \times \mathbb{I})$ is simply $\{x\} \times \mathbb{I} \cup \{y\} \times (\frac{1}{2}, 1]$, which is open in $X \times \mathbb{I}$.

Exercise 1.8.

(i) Consider the map taking the unit interval to S^1 given by $t \mapsto e^{2\pi it}$.

(ii) If $r : Y \to X$ is a retraction, then we know from $1_Y \simeq c$ that $r \circ 1_Y \circ i \simeq r \circ c \circ i$, where i is the injection $X \hookrightarrow Y$. But the left side is simply $r \circ i = 1_X$, while the left side is a constant map, proving the result.

Exercise 1.9. We know that there exists some constant map c with $f \simeq c$. But then $g \circ f \simeq g \circ c$, and the right side is a constant map. Thus $g \circ f$ is also nullhomotopic.

Exercise 1.10. First, suppose that g is an identification. Note that $(gf)^{-1}(U)$ open in X implies that $g^{-1}(U)$ is open in Y because f is an identification. But the hypothesis on g implies that U is open in Z. Since gf is clearly a continuous surjection, the result follows.

Now, suppose that gf is an identification. It suffices to prove that $g^{-1}(U) \subseteq Y$ open implies that $U \subseteq Z$ is open. But we know by continuity of f that $f^{-1}(g^{-1}(U))$ is open, and so gf being an identification implies the result.

Exercise 1.11. First, note that this is a well-defined function in the sense that $[x] = [y]$ in $X/\sim$ implies that $\overline{f}([x]) = \overline{f}([y])$.

This is evidently continuous. After all, suppose that $U \subseteq Y/\square$ is open. Then we know that

$$\overline{f}^{-1}(U) = \{[x] \in X/\sim : [f(x)] \in U\} = U'.$$

If we let $v : X \to X/\sim$ and $u : Y \to Y/\square$ be the natural maps, then we know that U' is open in $X/\sim$ because

$$v^{-1}(U') = \{x \in X : f(x) \in u^{-1}(U)\} = f^{-1}(u^{-1}(U))$$

is open.

Finally, we will show that $\overline{f}$ is an identification. It is obviously surjective. Moreover, if $U' = \overline{f}^{-1}(U)$ is open in $X/\sim$, then we simply note that a similar argument as above gives us that $v^{-1}(U') = f^{-1}(u^{-1}(U))$ is open. Since f and u are identifications, it follows that U was an open set in the first place, proving the result.

Exercise 1.12. Note that if $K \subseteq Z$ is closed, then it is compact and so $h(K)$ is compact in Z, hence itself closed. Thus h is a closed map, and hence an identification.

Now because $v : X \to X/\ker h$ is an identification, Corollary 1.9 applies. Indeed, Corollary 1.9 implies that $hv^{-1} = \varphi$ is a closed map. Thus it is an identification, i.e., a continuous surjection.

But the same corollary also implies that $\varphi^{-1} = vh^{-1}$ is continuous. This, combined with Example 1.3, in which it was shown that φ is injective, proves the result, as φ is now a bicontinuous bijection, i.e., a homeomorphism.

Exercise 1.13. First observe that $f(x) = f(y)$ implies that $[x,t] = [y,t]$ and so $t = 1$. Thus f is injective and hence bijective onto its image $CX_t = \{[x,t] \in CX : x \in X\}$. Then open sets in CX_t are precisely of the form $U \cap CX_t$ for an open set $U \subseteq CX$. But clearly we can assume that $[x,1] \notin U$ because $[x,1] \notin CX_t$, and thus we wind up with $X \times [0,1)$, where $CX_t = X \times \{t\}$. This is obviously homeomorphic to X.

Exercise 1.14. The functor takes a map $f : X \to Y$ to $Cf : CX \to CY$ given by $C([x,t]) = [f(x),t]$. Note that this is well-defined. Moreover, it is obvious that this is satisfies the properties of a functor. Indeed, if $g : Y \to Z$, then

$$C(g \circ f)([x,t]) = [g(f(x)),t] = ((Cg) \circ (Cf))([x,t])$$

and clearly $C(1_X)$ is the identity on CX.

Paths and Path Connectedness

Exercise 1.15. Using the hint, suppose that $g : \mathbb{I} \to X$ is a path with $g(0) = (0, a) \in A$ and with $g(t) \in G$ for all $t > 0$. Then note that $\pi_i \circ g$ is continuous for $i = 1, 2$, where π_i are the projections to the x- and y-axes. This implies the existence of an $\epsilon > 0$ such that $t \in (0, \epsilon)$ implies that $g(t) = (x(t), \sin(1/x(t)))$ has $x(t), |\sin(1/x(t)) - a| < \delta$. But this is obviously impossible, as $\sin(1/x(t))$ will oscillate wildly between -1 and 1.

Exercise 1.16. Let (a_i) and (b_i) be points in S^n. We will construct n paths which, when joined together in the customary fashion (i.e., by traversing each of the $n-1$ subpaths in $1/(n-1)$ time), will give us a path from (a_i) to (b_i).

The first path f_1 is defined as

$$f_1(t) = ((1-t)a_1 + tb_1, c_2, a_3, a_4, \ldots, a_n),$$

where c_2 is chosen to be of the same sign as a_2 and in such a way that $f(t) \in S^n$. Note that such a c_2 always exists.

In general, for $1 \leq i \leq n-1$, the path f_i will fix every coordinate except for the i-th, which it will take to b_i, and the $(i+1)$-th, which we use as a "free" coordinate to allow for such adjusting. Moreover, observe that if the first $n-1$ coordinates of two points on S^1 are the same, then the n-th coordinates either will be the same or will be negatives.

If joining the paths $f_1, f_2, \ldots, f_{n-1}$ together gives a path from (a_i) to (b_i), then we are done. Note that this occurs if a_n and b_n have the same sign.

Otherwise, construct a path g which adjusts the n-th coordinate and uses the $(n-1)$-th coordinate as a "free" one, preserving the sign. This effectively allows us to switch the sign of the n-th coordinate so that the n-th coordinate is just b_n. Moreover, because we preserved the sign of the $(n-1)$-th coordinate, it is still equal to b_{n-1}.

Exercise 1.17. It suffices to show the forward direction, so suppose that U is not path connected. Then there are at least two path components.

We will show that each path component is open, which will prove that U is not connected. But because U is open, we know that open sets in U (as a subspace) or also open in $\mathbb{R}^n$. Thus, for every $x \in U$, there is a ball B_x centered at x and contained in U. This ball is obviously path-connected. As such, if x is in the path component A, it must follow that $B_x \subseteq A$, proving that A is open.

Exercise 1.18. We know that if X is contractible then there exists a point $c \in X$ such that 1_X is homotopic to the constant map at c from X to itself. Now consider the map $c : \mathbb{I} \to X$ satisfying $c(t) = c$ for all t. In the proof of Theorem 1.13, we saw that any path is homotopic to c. In particular, the constant maps $x : \mathbb{I} \to X$ and $y : \mathbb{I} \to X$ at x and y, respectively, are both homotopic to c. Note that these give rise to paths from x to c and from c to y, respectively, which in turn give rise to a path from x to y. This proves path connectedness.

Exercise 1.19.

(i) If X is path connected, then let c and c' be constant maps. Let f be a path from (the point) c to (the point) c' and define $H : X \times \mathbb{I} \to X$ as $H(x,t) = f(t)$. Then H is a homotopy from c to c'.

For the reverse direction, let H be a homotopy from c to c' and define the path $f : \mathbb{I} \to X$ as $f(t) = H(c, t)$.

(ii) Let $f : X \to Y$ be a continuous function. Fix some $y_0 \in Y$ and consider the map

$$\begin{aligned} H : X \times \mathbb{I} &\to Y \\ (x,t) &\mapsto p_x(t), \end{aligned}$$

where p_x is a path from $f(x)$ to y_0. This is a homotopy from f to the constant map mapping X to y_0.

But if $g : X \to Y$ is another continuous function, then the same argument shows that $g \simeq y_0$, and so $f \simeq g$, as desired.

Exercise 1.20. It suffices to show that if $a \in A$ and $b \in B$, then there is a path from a to b. But fix some point $x \in A \cap B$. Then there is a path from a to x, and a path from x to b. Joining the two paths gives a path from a to b.

Exercise 1.21. This is simply done by noting that for any $(x, y), (x', y') \in X \times Y$, we can join the paths $f(t) = ((1-t)x + tx', y)$ and $g(t) = (x', (1-t)y + ty')$.

Exercise 1.22. Suppose $f(a), f(b) \in Y$. Then let p be a path from a to b in X. Now simply note that $q(t) = f(p(t))$ is a path from $f(a)$ to $f(b)$, proving the result.

Exercise 1.23.

(i) We already know that there are at least two path components because the entire space is not path connected. Moreover, both A and G are path connected, and so it follows that they must themselves be the path components.

(ii) Simply note that the sequence $\left\{\left(\frac{1}{n\pi}, \sin(n\pi)\right)\right\} \subset G$ approaches $(0,0) \in A$.

(iii) As per the hint, consider U to be the open disk with center $(0, \frac{1}{2})$ and radius $\frac{1}{4}$. Then $X \cap U$ is open in X. But note that $v(X \cap U)$ is not open in $X/A \approx [0, \frac{1}{2\pi}]$. After all, note that any ball B_ϵ around the point 0 (which is the image of A under the natural map in this case) must contain some point $\frac{1}{n\pi} < \epsilon$. But $\frac{1}{n\pi}$, which corresponds to the point $\left(\frac{1}{n\pi}, 0\right) \in X \setminus U$, is not contained in $v(X \cap U)$.

Exercise 1.24. By definition, path components are path connected. Moreover, if C is a path component and there exists some point $x \in X$ and $c \in C$ so that there is a path between x and c, then the definition of path components implies that $x \in C$. Thus path components are maximally path connected.

Finally, suppose that A is path connected and pick $a \in A$. There exists a unique path component C such that $a \in C$. Then for all $b \in A$, we know that there is a path between a and b, and so $b \in C$. Thus $A \subseteq C$, as desired.

Exercise 1.25. Simply use Exercise 1.22 and observe that I is path connected.

Exercise 1.26. Note that, if X is locally path connected, then for all $x \in X$, there exists some open path connected, hence connected, neighborhood V of x. Alternatively, note that if $U \subseteq X$ is open, then its components are unions of its path components and thus open.

Exercise 1.27. Given any open subset U of $X \times Y$ containing a given point $(x, y) \in X \times Y$, there must exist a basic open neighborhood $U_x \times U_y \subseteq U$ of (x, y). Then we know that there exists some path connected V_x with $x \in V_x \subseteq U_x$, and similarly for y. Then $V_x \times V_y$ is path connected by Exercise 1.21. The result follows.

Exercise 1.28. Note that open subsets of open subsets are open in the main space. In particular, let $A \subseteq X$ be open. Given any $x \in A$, let U be an open neighborhood of x in A. Note that this is also an open neighborhood in X, and so there exists an open path connected V in X (and hence open in A as well) such that $x \in V \subseteq U$.

Exercise 1.29. Consider the map $F : (\mathbb{R}^{n+1} \setminus \{0\}) \times \mathbb{I} \to \mathbb{R}^{n+1} \setminus \{0\}$ given by

$$F((x_i), t) = \left[(1-t) + \frac{t}{\sqrt{\sum x_i^2}}\right](x_i).$$

This is evidently a homotopy which makes S^n a deformation retract.

Exercise 1.30. The exact same map as in Exercise 1.29 works for this case.

Exercise 1.31. It is easy to see that the deformation retract of a deformation retract is a deformation retract, either by a direct argument or by applying Theorem 1.22. Thus the previous exercise implies that it suffices to show that $D^n \setminus \{0\}$ is a deformation retract of $S^n \setminus \{a, b\}$. But the map $(x_i) \mapsto (x_1, \ldots, x_{n-1}, 0)$ is exactly the map needed, and so we are done.

Exercise 1.32. If $H : f_0 \simeq f_1$, then the map $H' : (y, t) \mapsto H(r(y), t)$ is a homotopy from $\tilde{f}_0$ to $\tilde{f}_1$.

Exercise 1.33. Let $Y = \{y\}$ and observe that $(x,1) \sim y$ for all $x \in X$. Thus $(x,1) \sim (x',1)$ for alL $x, x' \in X$. Moreover, this is the only equivalence. Thus M_f is precisely the quotient space $(X \times \mathbb{I})/(X \times \{1\}) = CX$.

Exercise 1.34.

(i) We first tackle i. It is obvious that i is injective, and thus a bijection onto its image $i(X) = \{[x,0] : x \in X\}$. Moreover, the open sets in $i(X)$ are precisely of the $U \cap i(X)$ for open sets U in M_f.

Note that we can suppose without loss of generality that U is contained in $v(X \times [0,1))$, where v is the natural map. Thus U simply looks like the Cartesian product of an open interval with an open set of X. This proves that i is a homeomorphism, for the open sets of $i(X)$ map exactly to the open sets of X.

We can show that j is a homeomorphism onto $j(Y)$ in a similar manner. The main idea is simply that $y \not\sim y'$ for any $y, y' \in j(Y)$.

(ii) It is obvious that $(rj)(y) = r[y] = y = 1_Y(y)$ for any $y \in Y$. It is also clearly continuous by the gluing lemma. Thus r is indeed a retraction.

(iii) Define $F : M_f \times \mathbb{I} \to M_f$ as suggested in the hint. It is evident that F is continuous. Moreover, for any $[x,t] \in M_f$, we know that

$$F([x,t],0) = [x,t]$$
$$F([x,t],1) = [x,1] = [f(x)] \in Y.$$

Similarly, if $[y] \in Y$, then the definition implies that the remaining criteria for this homotopy to induce a deformation retraction $r(x) = F(x,1)$ are satisfied.

(iv) Note that Rotman writes that f is homotopic to $r \circ i$; in fact, we can and do prove the stronger statement that f coincides with $r \circ i$.

Let $f : X \to Y$ be continuous. Then it is clear that the map $f = r \circ i$, where $i : X \to M_f$ is an injection and $r : M_f \to Y$ is the retraction taking $[x,t]$ to $[f(x)]$ and taking $[y]$ to itself, proving the result.

2 Simplexes[1]

Affine Spaces

Exercise 2.1. Note that there is a maximal affine independent subset S of A. This is directly implied by the fact that any set of greater than $n+1$ elements is not affine independent. Hence we can take an affine independent subset of A with maximum size (because the empty set is affine independent).

Wrrite $S = \{p_0, \ldots, p_m\}$. Then let $p_{m+1} \in A \setminus S$. By maximality of S, we know that $S \cup \{p\}$ is not affine independent. Hence there exist s_i not all 0 such that

$$\sum_{i=0}^{m+1} s_i p_i = 0, \quad \sum_{i=0}^{m+1} s_i = 0.$$

Note that the second equation implies $\sum_{i=0}^m s_i \neq 0$ for some $i < m+1$. It follows then that

$$\sum_{i=0}^{m} \left(\frac{s_i}{\sum_{i=0}^m s_i} p_i \right) = p_{m+1}.$$

But we know that

$$\sum_{i=0}^{m} \frac{s_i}{\sum_{i=0}^m s_i} = 1,$$

and so it follows that p_{m+1} is in fact in the affine span of S.

Exercise 2.2. Let φ be the isomorphism from $\mathbb{R}^n$ to a subset of $\mathbb{R}^k$. Suppose $A \subseteq \mathbb{R}^n$ is an affine set containing X. Then $\varphi(X) \subseteq \varphi(A) \subseteq \mathbb{R}^k$.

Moreover, we claim that $\varphi(A)$ is affine. After all, for any $\varphi(x), \varphi(x') \in \varphi(A)$ and any $t \in \mathbb{R}$, the point $t\varphi(x) + (1-t)\varphi(x') = \varphi(tx + (1-t)x') \in \varphi(A)$ because A is affine.

This implies that the intersection of all affine sets in $\mathbb{R}^n$ containing X must contain the intersection of all affine sets in $\varphi(\mathbb{R}^n)$ containing $\varphi(X)$. Because φ is an isomorphism, using φ^{-1} gives the reverse inclusion. Thus the affine set spanned by X in $\mathbb{R}^n$ is precisely the same as that spanned by X in $\mathbb{R}^k$.

Exercise 2.3. This is evident in the case $n = 0$.

Suppose it is true for $n-1$ and consider the canonical injection $\iota : S^{n-1} \hookrightarrow S^n$ which takes $(x_0, \ldots, x_{n-1})$ to $(x_1, \ldots, x_{n-1}, 0)$. It is obvious that we can pick $n+1$ affine independent points $p_0, \ldots, p_n$ in this embedding.

Now consider the point $p_{n+1} = (0, \ldots, 0, 1) \in S^n$. Notice that the last coordinate of each p_i for $i \neq n+1$ is zero. Thus suppose we have s_i with $\sum s_i p_i = 0$ and $\sum s_i = 0$. Then $s_{n+1} = 0$, and so this reduces to the $n-1$ case. Affine independence of $\{p_0, \ldots, p_n\}$ proves the result.

Exercise 2.4. Consider the map $T'(x) = T(x) - T(0)$. We claim that T' is a linear map.

Observe that $S = \{e_i\} \cup \{0\}$ spans $\mathbb{R}^n$. Thus we can write any point as the affine sum of elements of S. Note that the coefficient of the zero vector is flexible, and so we have effectively no restrictions on the sum of the coefficients.

Consider arbitrary elements $\sum r_i e_i + r \cdot 0$ and $\sum s_i e_i + s \cdot 0$ in $\mathbb{R}^n$, where $r = 1 - \sum r_i$ and similarly for s. Let $R, S \in \mathbb{R}$. Then note that

$$\begin{aligned}
T'\left(R\sum r_i e_i + S\sum s_i e_i\right) &= T'\left(\sum (Rr_i + Ss_i)e_i\right) \\
&= T\left(\sum (Rr_i + Ss_i)e_i + \left(1 - \sum (Rr_i + Ss_i)\right) \cdot 0\right) - T(0) \\
&= R\sum r_i T(e_i) + S\sum s_i T(e_i) - R\sum r_i T(0) - S\sum s_i T(0).
\end{aligned}$$

Considering the R-terms first, simply observe that we can add and subtract $RT(0)$ to give us that

$$R\sum r_i T(e_i) - R\sum r_i T(0) = R\left(T\left(\sum r_i T(e_i) + r \cdot 0\right) - T(0)\right).$$

[1] I usually use *simplices* as the plural of simplex, but Rotman doesn't; no matter.

This is simply $RT'\left(\sum r_i e_i\right)$. A similar result holds for the S-terms, from which we conclude that

$$T'\left(R\sum r_i e_i + S\sum s_i e_i\right) = RT'\left(\sum r_i e_i\right) + ST'\left(\sum s_i e_i\right),$$

proving linearity.

Exercise 2.5. This is obvious from the previous exercise and continuity of linear maps.

Exercise 2.6. Given two m-simplexes $[p_0, \ldots, p_m]$ and $[q_0, \ldots, q_m]$, the map f taking p_i to q_i for every i is a homeomorphism. Bijectivity is obvious by the definition. Continuity is clear by how we extend f from $\{p_i\}$ to $[p_i]$. Finally, the inverse is of the same form as f, only with the q_i's taking the place of the p_i's and vice versa; thus f^{-1} is also continuous.

Exercise 2.7. The following map works:

$$f : x \mapsto \frac{t_2 - t_1}{s_2 - s_1}(x - s_1) + t_1.$$

Exercise 2.8. Pick arbitrary $T(x), T(x') \in T(X)$ and observe that

$$tT(x) + (1-t)T(x') = T(tx + (1-t)x') \in T(X).$$

Thus $T(X)$ is affine if X is affine, and convex if X is convex. The second statement of the exercise follows by noting that ℓ is convex.

Exercise 2.9. Without loss of generality, we delete p_0. Now suppose that

$$\sum_{i=1}^{m} s_i p_i + sb = 0, \quad \sum_{i=1}^{m} s_i + s = 0.$$

Then we know by definition of the barycenter b that

$$\sum_{i=1}^{m} s_i p_i + \frac{s}{m+1}\sum_{i=0}^{m} p_i = 0.$$

Moreover, letting s_i' be the coefficient of p_i in the above equation, it is obvious that $\sum_{i=0}^{m} s_i' = s + \sum_{i=1}^{m} s_i = 0$. Thus $s_i' = 0$ for all i because $\{p_0, \ldots, p_m\}$ was affine independent. But then we conclude that $0 = s_0' = \frac{s}{m+1}$, and so $s = 0$. For every $i \in \{1, \ldots, m\}$, we have $0 = s_i' = \frac{s}{m+1} + s_i$. Thus $s = 0$ implies $s_i = 0$ for every i,a nd so it follows that $\{b, p_1, \ldots, p_m\}$ is affine independent, as desired.

Exercise 2.10. Once again, suppose without loss of generality that $i = 0$. Then the map taking $\sum t_i p_i \in [p_0, p_1, \ldots, p_m]$ to $(\sum_{i=1}^{m} t_i p_i, t_0)$ works. Note that this actually requires the affine independence of the p_i's, as well as the fact that the coefficients t_i are all between 0 and 1.

Exercise 2.11. Notice that $[0, e_1, \ldots, e_n]$, where e_i are the standard basis vectors in $\mathbb{R}^n$, is an n-simplex. Thus there is a homeomorphism $[p_0, \ldots, p_n] \to [0, e_1, \ldots, e_n]$. If we translate the image by $\mathbf{v} = (-\frac{1}{4}, -\frac{1}{4}, \ldots, -\frac{1}{4})$, then we can map the result to D^n by taking a radial mapping. In particular, this map will take

$$p_0 \mapsto \frac{\mathbf{v}}{\|\mathbf{v}\|}$$
$$p_i \mapsto \frac{e_i + \mathbf{v}}{\|e_i + \mathbf{v}\|} \text{ for } i \neq 0.$$

Note that this extends to a homeomorphism.