10 Covering Spaces

Basic Properties

Exercise 10.1. Obviously $\mathbb{R}$ is path-connected and exp is continuous. Furthermore, for each $\exp(2\pi it) \in S^1$, consider the neighborhood $S^1 \setminus \{\exp(\pi + 2\pi it)\}$. Of course, we know that

$$\exp^{-1}(U) = \bigcup_{n\in\mathbb{Z}} \left(n + t - \frac{1}{2}, n + t + \frac{1}{2}\right),$$

so U is evenly covered.

Exercise 10.2. To see that $p_k : z \mapsto z^k$ is continuous, pick some open $U \subseteq S^1$. Pick $\exp(2\pi it) \in p_k^{-1}(U)$, so that $\exp(2\pi ikt) \in U$. Then there is some $\varepsilon > 0$ so that

$$\{\exp(2\pi ikx) : t - \varepsilon < x < t + \varepsilon\} \subseteq U.$$

Then it follows that the open set $\{\exp(2\pi ix) : t - \varepsilon < x < t + \varepsilon\}$ is contained in $p_k^{-1}(U)$, so that $p_k^{-1}(U)$ is open. Hence p_k is continuous.

To see that (S^1, p_k) is a covering space, let $e\exp(2\pi it) \in S^1$. Pick the open neighborhood

$$U = \{\exp(2\pi ix) : t - \tfrac{1}{2} < x < t + \tfrac{1}{2}\}.$$

Note then that

$$p_k^{-1}(U) = \bigcup_{n\in\mathbb{Z}} \left\{\exp\left(2\pi ix) : \frac{t - \frac{1}{2} + n}{k} < x < \frac{t + \frac{1}{2} + n}{k}\right)\right\},$$

proving that U is evenly covered.

Exercise 10.3. Informally: Note that a point of $\mathbb{R}P^n$ corresponds to a pair of antipodal points in S^n. Given some point in S^n, there is always a small open neighborhood which does not intersect its reflection (which is a neighborhood of the antipodal point). This neighborhood is evenly covered.

Exercise 10.4.

(i) Consider any (basic) open neighborhood of x_0. The preimage of this neighborhood under q looks like two disjoint intervals on S^1. The only possibility is if q restricted to a homeomorphism on each of these intervals. But this isn't the case (surjectivity fails), so no neighborhood of x_0 is evenly covered.

(ii) A non-tangency point obviously has an evenly covered neighborhood, while x_0 has an evenly covered neighborhood whose sheets correspond to small neighborhoods of the infinitely many tangency points of $\widetilde{X}$.

Exercise 10.5. Each element of $p^{-1}(x_0)$ belongs to a different sheet, since p is a homeomorphism on each sheet. Thus $p^{-1}(x_0)$ is discrete.

Exercise 10.6. Since a covering projection is a local homeomorphism, it follows that local topological properties are all inherited by picking a suitably small neighborhood of any given point.

Exercise 10.7. If $p^{-1}(U) = \bigcup S_i$ and $S_i' \subseteq S_i$ is $p^{-1}(V) \cap S_i$, then note that $p^{-1}(V) = \bigcup S_i'$. Furthermore, we know that $p : S_i' \to V$ is a homeomorphism since $p : S_i \to U$ is a homeomorphism and $p(S_i') = V$. Hence V is evenly covered.

Exercise 10.8.

(i) Pick $(x_1, x_2) \in (X_1, X_2)$. Suppose neighborhoods U_i of x_i are p_i-admissible for $i = 1, 2$. In particular, write

$$p_1^{-1}(U_1) = \bigcup S_i, \quad p_2^{-1}(U_2) = \bigcup T_j.$$

Then it is easy to check that

$$(p_1 \times p_2)^{-1}(U_1 \times U_2) = \bigcup S_i \times T_j.$$

Note that $\mathbb{R}$ is a covering space of S^1 (Exercise 10.1), and so it follows that $\mathbb{R} \times \mathbb{R}$ covers the torus $S^1 \times S^1$.

(ii) Either using Exercise 10.2 or by noting that $(X, 1_X)$ covers X for any path-connected space X, it is easy to see that S^1 is a covering space of S^1. Hence the conclusion follows from the previous part.

Exercise 10.9. Note that $q = \alpha^{-1} p \beta$ is continuous. Since $\widetilde{Y}$ and $\widetilde{X}$ are homeomorphic, we know that $\widetilde{Y}$ is path-connected.

Now let $y \in Y$ correspond to $x \in X$, i.e., have $\alpha(y) = x$. Note that x has a neighborhood U_x such that $p^{-1}(U_x) = \bigcup S_i$ for sheets S_i. Write $\beta^{-1}(S_i) = T_i$, where T_i and S_i are homeomorphic. Observe that

$$q(T_i) = \alpha^{-1} p \beta(T_i) = \alpha^{-1} p(S_i) = \alpha^{-1}(U_x).$$

Moreover, we know that $q|T_i$ is a homeomorphism, because p is a homeomorphism on S_i, making $q = \alpha^{-1} p \beta$ a composition of homeomorphisms. Hence $\alpha^{-1}(U_x)$ is a q-admissible neighborhood of y.

Exercise 10.10. In this case, we have $p_* \pi_1(\widetilde{X}, \tilde{x}_0)$ trivial, so that

$$m = [\pi_1(X, x_0) : p_* \pi_1(\widetilde{X}, \tilde{x}_0)] = |\pi_1(X, x_0)|.$$

If m is prime, then the only group of order m is $\mathbb{Z}/m\mathbb{Z}$.

Exercise 10.11. This is the same logic as Exercise 10.10.

Exercise 10.12. Pick $u \in U$. Then $p^{-1}(U)$ has cardinality m. But no two elements of $p^{-1}(U)$ can be in the same sheet, else $p|S_i$ would not be injective, while there is at least one element of $p^{-1}(U)$ in each S_i, else $p|S_i$ would not be surjective onto U. Since no element can be in multiple sheets, as the sheets are disjoint, it follows that the elements of $p^{-1}(U)$ can be put into 1-1 correspondence with the sheets S_i. Hence $|I| = m$.

Exercise 10.13.

(i) Note that $[f] \in \ker \theta$ iff $\tilde{x}[f] = \tilde{x}$ for all $\tilde{x} \in p^{-1}(x_0)$. By definition, we know that $\tilde{x}[f] = \tilde{f}(1)$. Hence the following are all equivalent:

- $[f] \in \ker \theta$
- $\tilde{f}(1) = \tilde{x}[f] = \tilde{x}$ for all $\tilde{x} \in p^{-1}(x_0)$
- $\tilde{f}$ is a closed loop at $\tilde{x}$ for all $\tilde{x}$
- $p[\tilde{f}] \in p_* \pi_1(\widetilde{X}, \tilde{x})$
- $[f] \in \bigcap_{\tilde{x} \in Y} p_* \pi_1(\widetilde{X}, \tilde{x})$

The equivalence of the first and last conditions is what we originally wanted.

(ii) Note that $\widetilde{X}$ being simply connected implies that $\bigcap p_* \pi_1(\widetilde{X}, \tilde{x})$ is trivial. Hence $\ker \theta$ is trivial, so θ is an injection.

Exercise 10.14. We know that G is a covering space for G/H with $p : g \mapsto gH$. Let $\theta : \pi(G/H, 1) \to S_H$ be as defined in Exercise 10.13. Since G is simply connected, we know that θ is injective. Hence $\pi_1(G/H, 1) \cong \operatorname{im} \theta$. Note that θ takes $\tilde{x} \mapsto \tilde{f}(1)$, where $\tilde{f}(0) = \tilde{x}$. But these path lifts in $\operatorname{im} \theta$ correspond precisely to elements of H, since $\tilde{f}(1) \in \ker p = H$.

Exercise 10.15. Every subgroup of an abelian group is normal. Hence $p_* \pi_1(\widetilde{X}, \tilde{x}_0)$ is a normal subgroup of $\pi_1(X, x_0)$. This is the definition of a regular covering space.

Exercise 10.16. Pick $x \in X$. Let $\tilde{y} \in q^{-1}(x)$ and $\tilde{x} = h(\tilde{y})$. We have the following diagram:

$$\begin{tikzcd} (\widetilde{Y}, \tilde{y}) \arrow[rr, "h"] \arrow[dr, "q"'] & & (\widetilde{X}, \tilde{x}) \arrow[dl, "p"] \\ & (X, x) & \end{tikzcd}$$

Pick some path $\tilde{f}_X : \mathbb{I} \to \widetilde{X}$ with $\tilde{f}_X(0) = \tilde{x}$. Define $f = p\tilde{f}_X$. There is a unique $\tilde{f}_Y$ lifting f to $(\widetilde{Y}, \tilde{y})$, i.e., with $\tilde{f}_Y(0) = \tilde{y}$. But now notice that $h\tilde{f}_Y$ is a path lifting f into $\widetilde{X}$ such that $(h\tilde{f}_Y)(1) = h(\tilde{y}) = \tilde{x}$. Uniqueness implies that $h\tilde{f}_Y = \tilde{f}_X$.

Now $\operatorname{im} h$ contains $\operatorname{im} \tilde{f}_x$, which contains x. Thus $x \in \operatorname{im} h$. Since x was arbitrary, this proves that h is surjective.

Exercise 10.17. Let $\tilde{x}_0 \in p^{-1}(x_0)$. Now consider the following statements, all of which are equivalent:

- $[f] \in p_*\pi_1(\widetilde{X}, \tilde{x}_0)$
- $[f]$ stabilizes $\tilde{x}_0$
- $\tilde{x}_0[f] = \tilde{x}_0$
- $\tilde{f}(1) = \tilde{x}_0$ where $\tilde{f}$ is the lifting with $0 \mapsto \tilde{x}_0$
- $\tilde{f}$ is closed at $\tilde{x}_0$

Covering Transformations

Exercise 10.18. The isomorphism is the composition of two isomorphisms. The first is $\mathrm{Cov}(\widetilde{X}/X) \to \mathrm{Aut}(p^{-1}(x_0))$ which takes h to $h|p^{-1}(x_0)$. The second is $\mathrm{Aut}(p^{-1}(x_0)) \to \pi_1(X, x_0)$ which takes φ to $[f^{-1}]$ where f is the well-defined path so that $\varphi(\tilde{x}_0) = \tilde{x}_0[f]$ for $\tilde{x}_0$ in the fiber over x_0.

Hence the isomorphism $\mathrm{Cov}(\tilde{X}/X)$ takes h to the map $[f^{-1}]$ defined by $h(\tilde{x}_0) = \tilde{x}_0[f]$ for $\tilde{x}_0 \in p^{-1}(x_0)$.

Exercise 10.19. No. Any neighborhood of $p \in S^n$ is necessarily going to include some $q\tilde{p}$, thus making an even covering of any neighborhood impossible. To see why any neighborhood of p intersects $\{q : q \sim p\}$ at a point that isn't p, simply note that the equivalence class of p is connected.

Exercise 10.20. This is exactly the same argument as Example 10.2.

Exercise 10.21. Suppose that, for each closed path $f : \mathbb{I} \to X$, either every lifting $\tilde{f}$ of f is closed, or no lifting $\tilde{f}$ is closed. Exercise 10.17 implies that $p_*\pi_1(\widetilde{X}, \tilde{x}_0) = p_*\pi_1(\widetilde{X}, \tilde{x}_1)$ for all $\tilde{x}_0, \tilde{x}_1 \in p^{-1}(x_0)$. Now Corollary 10.12 says that, if $\tilde{x}_0 \in p^{-1}(x_0)$, then $gp_*\pi_1(\widetilde{X}, \tilde{x}_0)g^{-1} = p_*\pi_1(\widetilde{X}, \tilde{x}_1)$ for some $\tilde{x}_1 \in p^{-1}(x_0)$. Hence the conjugate of $p_*\pi_1(\widetilde{X}, \tilde{x}_0)$ is itself, making it a normal subgroup. This is true for every x_0, so $(\widetilde{X}, p)$ is regular.

Now suppose $(\widetilde{X}, p)$ is regular. Then $p_*\pi_1(\widetilde{X}, \tilde{x}_0)$ is normal for each $\tilde{x}_0$. Corollary 10.12(i) implies that $p_*\pi_1(\widetilde{X}, \tilde{x}_0)$ and $p_*\pi_1(\widetilde{X}, \tilde{x}_1)$ are conjugate for all $\tilde{x}_0, \tilde{x}_1$ in the fiber over x_0. Hence they are equal. Now use Exercise 10.17:

$$p_*\pi_1(\widetilde{X}, \tilde{x}_0) = \{[f] : \tilde{f} \text{ closed at } \tilde{x}_0\}.$$

Hence if the lifting $\tilde{f}$ to $\tilde{x}_0$ is closed, then so too is the lifting to $\tilde{x}_1$.

Exercise 10.22. The monodromy group is $\pi_1(X, x_0)/\ker\theta$ where $\ker\theta = \bigcap_{\tilde{x} \in p^{-1}(x_0)} p_*\pi_1(\widetilde{X}, \tilde{x})$. But the p_*'s are all equal by Corollary 10.12(i). Hence $\ker\theta = p_*\pi_1(\widetilde{X}, \tilde{x}_0)$ for some $\tilde{x}_0 \in p^{-1}(x_0)$. Corollary 10.28 implies the result.

Exercise 10.23. If X is an H-space, then π_1 is abelian. Hence every subgroup is normal, so every covering space X is regular.

Existence

Exercise 10.24. It suffices to show that $[\bar{f}] = [f^{-1}]$. After all, if this is true, then $\bar{f} * f$ is nullhomotopic, hence $[\bar{f} * f] \in G$, hence $\langle f \rangle_G = \langle f^{-1} \rangle_G$. But note that $e \simeq \tilde{f} \circ f \simeq \tilde{f} * f \operatorname{rel} \dot{\mathbb{I}}$. Hence $[\bar{f}] = [f^{-1}]$ as desired.

Exercise 10.25. A similar argument, on multiplication only, holds.

Exercise 10.26. Let $\mathcal{U}$ be an open cover of $\widetilde{X}$. For $x \in X$, consider an admissible neighborhood V_x of x:

$$p^{-1}(V_x) = \bigcup_{i=1}^{j} S_i.$$

Let W_x be admissible with $\overline{W_x} \subseteq V_x$. Then we can write

$$p^{-1}(W_x) = \bigcup_{i=1}^{j} T_i,$$

where $\overline{T_i} \subseteq S_i$. Note that $T_i \approx W_x$ and $S_i \approx V_x$ for each $i = 1, \ldots, j$. Hence $\overline{T_i}$ is compact, since it is homeomorphic to a closed subset of the compact set $\overline{W_x}$.

Thus there are, for each $i = 1, \ldots, j$, finitely many sets of $\mathcal{U}$ which together cover $\overline{T_i}$. Take all of them to obtain a (finite) cover of

$$\bigcup \overline{T_i} \supseteq \bigcup T_i = p^{-1}(W_x).$$

Call this finite cover to be $\mathcal{W}_x$.

Note that the family of all W_x's covers X. Since X is compact, it follows that finitely many W_x's cover X, say $W_{x_1}, \ldots, W_{x_n}$. Then

$$\bigcup_{i=1}^{n} W_{x_i} = \widetilde{X},$$

which gives a finite subcover of $\mathcal{U}$.

Exercise 10.27. The j-sheeted covering spaces is exactly the number of $\widetilde{X}_G$ where $\widetilde{X}_G$ is j-sheeted. By Theorem 10.9(iii), this is exactly the number rof G with $[\pi_1(X, x_0) : p_*\pi_1(\widetilde{X}_G, \tilde{x}_0)] = j$. Of course, this p_*-group is exactly G, and so this is exactly the number of subgroups having index j.

Exercise 10.28. The result follows as long as any finite CW complex has a finitely generated π_1. But this can be seen to be true by Van Kampen.

Orbit Spaces

Exercise 10.29. We know by Theorem 10.54 that $\mathrm{Cov}(\widetilde{X}/(\widetilde{X}/H)) = H$, and so we can think of G as a subgroup of $\mathrm{Cov}(\widetilde{X}/(\widetilde{X}/H))$. Now use Theorem 10.52, with $X = \widetilde{X}/H$. We know, in particular, that G is a subgroup of $\mathrm{Cov}(\widetilde{X}/X)$, and thus is exactly a covering space $(\widetilde{X}/G, v)$ of $X = \widetilde{X}/H$, as desired.

Exercise 10.30.

(i) Suppose $gx = x$ and consider a proper neighborhood V of x. Then we know that $gV \cap V = \emptyset$, but $x = gx \in gV \cap V$, contradiction.

(ii) If $G = \{e, g_1, \ldots, g_n\}$ and $x \in X$, then, since X is Hausdorff and since $g_i x \neq x$, there exists a neighborhood V of X which does not contain any $g_i x$. Obviously, this V is a proper neighborhood.

Exercise 10.31. This is exactly the argument in the proof of Theorem 10.2, namely in the first full paragraph on p. 276.

Exercise 10.32.

(i) The group $\mathbb{Z}/p\mathbb{Z}$ acts on S^3 via $m \bullet (z_0, z_1) = (\zeta^m z_0, \zeta^{mq} z_1)$. This action is proper because part (ii) of Exercise 10.30 obviously applies.

(ii) Note that $S^3/(\mathbb{Z}/p\mathbb{Z})$ is exactly $L(p, q)$. Thanks to the previous part, Theorem 10.54(ii) applies, which implies that

$$\pi_1(L(p, q)) = \pi_1(S^3/(\mathbb{Z}/p\mathbb{Z})) = \mathbb{Z}/p\mathbb{Z}.$$

(iii) We know that $L(p, q)$ inherits the local properties of S^3, since there is a local homeomorphism between them. Thus $L(p, q)$ is a 3-manifold.

If $\mathcal{U}$ is an open cover of $L(p, q)$, then $p^{-1}(\mathcal{U})$ is an open cover of S^3. Hence finitely many elements of $p^{-1}(\mathcal{U})$, say $p^{-1}(U_i)$ for $i = 1, \ldots, n$, cover S^3. But then $\{U_1, \ldots, U_n\}$ is a finite subcover of $\mathcal{U}$ which covers $L(p, q)$, proving compactness.

Finally, note that $A \subseteq L(p, q)$ clopen implies that $p^{-1}(A)$ is clopen in S^3. Hence $p^{-1}(A) = \emptyset, S^3$, and so $A = \emptyset, L(p, q)$. Thus $L(p, q)$ is connected too.

Exercise 10.33. Notice that $T \to T/G$ is a universal covering space since T is simply connected. Moreover, since T/G is a connected 1-complex, we know by Corollary 7.35 that $\pi_1(T/G)$ is free. But Theorem 10.54(iii) implies that $\pi_1(T/G) \cong G$, and so G is free.