11 Homotopy Groups

Function Spaces

No exercises!

Group Objects and Cogroup Objects

Exercise 11.1.

(i) By definition of a product, there is a unique morphism $\theta : (X, q_1, q_2) \to (C_1 \times C_2, p_1, p_2)$ in $\mathscr{C}$ making the diagram commute, namely $\theta = (q_1, q_2)$.

(ii) The objects are ordered triples (X, k_1, k_2) where X is a set and $k_i : C_i \to X$ are functions. Morphisms $\theta : (X, k_1, k_2) \to (Y, \ell_1, \ell_2)$ are functions $\theta : X \to Y$ making the following commute:

$$\begin{tikzcd}
& X \arrow[dd, "\theta"] & \\
C_1 \arrow[ur, "k_1"] \arrow[dr, "\ell_1"'] & & C_1 \arrow[ul, "k_2"'] \arrow[dl, "\ell_2"] \\
& Y &
\end{tikzcd}$$

Exercise 11.2. We first tackle **Ab**.

The map $\theta : X \to G_1 \oplus G_2$ in the product diagram is given by $\theta(g) = (q_1 g, q_2 g)$. Commutativity follows from the fact that $p_i(\theta(g)) = q_i(g)$. Uniqueness of θ follows from the fact that any other θ' must satisfy $\theta'(g) = (g_1, g_2)$ where $g_i = p_i(g_1, g_2) = q_i(g)$. Hence $\theta' = \theta$.

The map $\eta : G_1 \oplus G_2 \to X$ in the coproduct diagram is given by $(g, h) \mapsto k_1(g) + k_2(h)$, where $+$ denotes the operation in the abelian group X. We can easily check commutativity and uniqueness using the fact that η must be a group homomorphism.

Now, for **Grp**, note that the free product property on p. 173 is exactly the coproduct property. The same argument as in the abelian case shows that direct product is the product in **Grp**.

Exercise 11.3.

(i) We will show this for $\mathbf{Top}_*$. Suppose we have $((X, x), k_1, k_2)$. It is obvious that the map $\theta : (A_1 \vee A_2, *) \to (X, x)$, if it exists, must take $*$ to x, and $* \neq a_i \in j_i(A_i)$ to $k_i(a_i)$. We need only show that this map θ is continuous. (In contrast, the proof has already been completed for $\mathbf{Set}_*$; commutativity of the relevant diagram is obvious from the definition of θ.)

Suppose $U \subseteq X$ is closed. Note that $\theta^{-1}(U) \cap A_i = k_i^{-1}(U)$. (This statement is clear if $* \notin U$. If $* \in U$, then

$$\theta^{-1}(U) \cap A_i = (\theta^{-1}(U \setminus \{x\}) \cap A_i) \cup \{*\} = k_i^{-1}(U \setminus \{x\}) \cup \{a_i\} = k_i^{-1}(U),$$

which proves the statement anyway.) The definition of the topology of the wedge (see Example 8.9) implies that $\theta^{-1}(U)$ is closed. Hence θ is continuous, completing the proof.

(ii) Call this subset S. The map $f : A_1 \vee A_2 \to S$ which takes $a \in A_i$ (or, more accurately, $a \in j_i(A_i)$) to (a, a_2) if $i = 1$ and to (a_1, a) if $i = 2$ is continuous by the previous argument. It is clearly bijective and closed, since a closed set F in $A_1 \vee A_2$ is still closed in $A_1 \times A_2$. Thus it is a homeomorphism.

Exercise 11.4. Commutativity follows from the interchanging of C_1 and C_2 in the definition. To see associativity, consider the following diagram:

$$\begin{tikzcd}
 & & (C_1 \times C_2) \times C_3 \arrow[dl, bend right] \arrow[ddr, bend left] & \\
 & C_1 \times C_2 \arrow[dl] \arrow[dr] & & \\
C_1 & & C_2 & C_3 \\
 & & X \arrow[ull, bend left] \arrow[uul, bend left] \arrow[u] \arrow[ur, bend right] &
\end{tikzcd}$$

There is a unique map $X \to (C_1 \times C_2) \times C_3$ making this diagram commute.

Now define $p_1 : (C_1 \times C_2) \times C_3 \to C_1$ to be the composition of the red arrows below. Furthermore, the product property of $C_2 \times C_3$ implies the existence of the following blue and green arrows:

$$\begin{tikzcd}
 & & (C_1 \times C_2) \times C_3 \arrow[dl, bend right, red] \arrow[dr, blue] \arrow[ddrr, bend left] & & \\
 & C_1 \times C_2 \arrow[dl, red] \arrow[dr] & & C_2 \times C_3 \arrow[dl, green] \arrow[dr, green] & \\
C_1 & & C_2 & & C_3 \\
 & & X \arrow[ull, bend left] \arrow[uul, bend left] \arrow[u] \arrow[urr, bend right] & &
\end{tikzcd}$$

Let p_2 be the blue arrow. The fact that there is still the same unique map $X \to (C_1 \times C_2) \times C_2$ making this commute, then, implies that $(C_1 \times C_2) \times C_3$ is the product of C_1 and $C_2 \times C_3$, thus proving associativity.

Exercise 11.5.

(i) We would like to find $f_1 \times f_2$ making the following commute:

$$\begin{tikzcd}
C_2 \arrow[r, "f_2"] & D_2 \\
C_1 \times C_2 \arrow[u] \arrow[d] \arrow[r, "f_1 \times f_2"] & D_1 \times D_2 \arrow[u] \arrow[d] \\
C_1 \arrow[r, "f_1"] & D_1
\end{tikzcd}$$

But the existence of maps $C_1 \times C_2 \to C_i \to D_i$ implies, by the product property of $D_1 \times D_2$, a unique map $f_1 \times f_2$ into $D_1 \times D_2$ making the diagram commute.

(ii) Same idea.

Exercise 11.6.

(i) Note that Δ_X is the unique map making the red part of the diagram commute, while $q_1 \times q_2$ is the unique map making the blue part commute:

$$
\begin{tikzcd}
 & D_1 \times D_2 \arrow[dl] \arrow[dr] & \\
D_1 & & D_2 \\
 & X \times X \arrow[uu, dashed, "q_1 \times q_2"'] \arrow[dl] \arrow[dr] & \\
X \arrow[uu] & & X \arrow[uu] \\
 & X \arrow[ul] \arrow[ur] \arrow[uu, dashed, "\Delta_X"'] &
\end{tikzcd}
$$

But of course, since the maps $q_i \circ 1_X = X \to X \to D_i$ are equal to simply the maps $q_i : X \to D_i$, we know that the unique map $X \mapsto D_1 \times D_2$ making this entire diagram commute is (q_1, q_2). Uniqueness implies that (q_1, q_2) must be equal to $(q_1 \times q_2)\Delta_X$.

(ii) This is the same idea.

(iii) We already showed the first statement. For the second, notice that $\nabla_B(f \times g) = (f, g)$. But $(f, g)\Delta_A(a) = (f(a), g(a)) = (f + g)(a)$ because $A \oplus B = A \amalg B$.

Exercise 11.7.

(i) Everything follows from the hint, except that we must verify that $1_{X\times Z}$ and $\theta\lambda$ complete the given diagram. Commutativity of the left triangle is obvious in both cases. To see that $q1_{X\times Z} = t$, note that Z being terminal implies that $q = t$. To show that $q\theta\lambda = t$, note that $q\theta\lambda : X \times Z \to X \to X \times Z \to Z$. Thus Z being terminal again implies the result.

Now θ and λ are inverses, and so $X \times Z$ and X are equivalent.

(ii) This is the dualized version of the previous part.