5 The Category Comp

Exercise 5.1. These results all follow directly from the definition of exactness.

(i) Note that $\ker f = \operatorname{im} 0 = 0$, and so f is injective.

(ii) In this case, we have $\operatorname{im} g = \ker 0 = C$.

(iii) By the previous two parts, we know that f is bijective. Because f is a homomorphism as well, it follows that f is an isomorphism.

(iv) Either observe that $0 \to A \to 0 \to 0$ is exact and apply the previous part, or note that $A \to 0$ is injective while $0 \to A$ is surjective, implying that $A \cong 0$, i.e., that $A = 0$.

Exercise 5.2. Note that f is surjective if and only if $\ker g = \operatorname{im} f = B$. But $\ker g = B$ if and only if g is the zero map, which is itself true exactly when $\ker h = \operatorname{im} g = 0$. Since $\ker h = 0$ if and only if h is injective, we are done.

Exercise 5.3. We know that $0 \to A \xrightarrow{i} B$ implies that i is an injection. But because i is a surjection onto its image, this implies that $iA \cong A$. Moreover, because $\ker p = \operatorname{im} i = iA$, we know that $B/iA = B/\ker p \cong \operatorname{im} p$. Because p is a surjection (see Exercise 5.1), the result follows.

Exercise 5.4. This amounts, effectively, to following the arrows and the equations given by exactness. In more detail, let $f_n : B_n \to C_n$ and $g_n : C_n \to A_{n-1}$. Now observe that $B_n = \operatorname{im} h_n = \ker f_n$. Thus f_n is the zero map. Moreover, because $\ker g_n = \operatorname{im} f_n$, we know that g_n is injective. Finally, we have $\operatorname{im} g_n = \ker h_{n-1}$. But h_{n-1} is an isomorphism, and so its kernel is trivial. Thus $\operatorname{im} g_n = 0$. Because g_n was injective, it follows that $C_n = 0$.