## Chapter 0: Introduction

Jessica Zhang

June 4, 2020

**Exercise 0.1.** As per the hint, observe that if  $y \in G$ , then we have y = r(y) + (y - r(y)). Obviously, we have  $r(y) \in H$ . Moreover, we know that

$$r(y - r(y)) = r(y) - r(r(y)) = 0,$$

and so  $y - r(y) \in \ker r$ . Thus  $G \subseteq H \oplus \ker r$ .

The reverse is obviously true, since H and  $\ker r$  are both subgroups of G.

**Exercise 0.2.** Suppose instead that  $f: D^1 \to D^1$  has no fixed point. Then consider the continuous map  $g: D^1 \to S^0$  given by

$$g(x) = \begin{cases} 1 & \text{if } f(x) < x \\ -1 & \text{if } f(x) > x \end{cases}.$$

Notice that because  $f(x) \neq x$  for all x, the function q is well-defined.

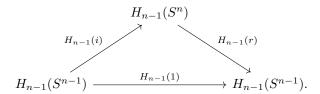
Moreover, we know that  $f(-1) \neq -1$ , since f has no fixed point, and so f(-1) > -1. Thus g(-1) = -1. Similarly, we have g(1) = 1.

Thus we have  $g(D^1)=S^0$ , which is disconnected. This is a contradiction, so f must have had a fixed point.

**Exercise 0.3.** Suppose that r is such a retract. Then we have the following commutative diagram:

$$S^{n-1} \xrightarrow{1} S^{n-1}$$

Applying  $H_{n-1}$ , we get another commutative diagram:



We know that  $H_{n-1}(S^n) = 0$ , however, implying that  $H_{n-1}(1) = 0$ . This contradicts the fact that  $H_{n-1}(S^{n-1}) = \mathbb{Z} \neq 0$ . Thus the retraction r could not have existed.

**Exercise 0.4.** Suppose  $g: D^n \to X$  is a homeomorphism. Then we know that  $g^{-1} \circ f \circ g$  is a continuous map from  $D^n$  to itself, and so it has a fixed point x. Then we know that  $g^{-1}(f(g(x))) = x$ , and so it follows that f(g(x)) = g(x). Thus  $g(x) \in X$  is a fixed point of f.