

## 0 Introduction

### Notation

No exercises!

### Brouwer Fixed Point Theorem

**Exercise 0.1.** As per the hint, observe that if  $y \in G$ , then we have  $y = r(y) + (y - r(y))$ . Obviously, we have  $r(y) \in H$ . Moreover, we know that

$$r(y - r(y)) = r(y) - r(r(y)) = 0,$$

and so  $y - r(y) \in \ker r$ . Thus  $G \subseteq H \oplus \ker r$ .

The reverse is obviously true, since  $H$  and  $\ker r$  are both subgroups of  $G$ .

**Exercise 0.2.** Suppose instead that  $f : D^1 \rightarrow D^1$  has no fixed point. Then consider the continuous map  $g : D^1 \rightarrow S^0$  given by

$$g(x) = \begin{cases} 1 & \text{if } f(x) < x \\ -1 & \text{if } f(x) > x \end{cases}.$$

Notice that because  $f(x) \neq x$  for all  $x$ , the function  $g$  is well-defined.

Moreover, we know that  $f(-1) \neq -1$ , since  $f$  has no fixed point, and so  $f(-1) > -1$ . Thus  $g(-1) = -1$ . Similarly, we have  $g(1) = 1$ .

Thus we have  $g(D^1) = S^0$ , which is disconnected. This is a contradiction, so  $f$  must have had a fixed point.

**Exercise 0.3.** Suppose that  $r$  is such a retract. Then we have the following commutative diagram:

$$\begin{array}{ccc} & S^n & \\ i \nearrow & & \searrow r \\ S^{n-1} & \xrightarrow{1} & S^{n-1} \end{array}$$

Applying  $H_{n-1}$ , we get another commutative diagram:

$$\begin{array}{ccc} & H_{n-1}(S^n) & \\ H_{n-1}(i) \nearrow & & \searrow H_{n-1}(r) \\ H_{n-1}(S^{n-1}) & \xrightarrow{H_{n-1}(1)} & H_{n-1}(S^{n-1}) \end{array}$$

We know that  $H_{n-1}(S^n) = 0$ , however, implying that  $H_{n-1}(1) = 0$ . This contradicts the fact that  $H_{n-1}(S^{n-1}) = \mathbb{Z} \neq 0$ . Thus the retraction  $r$  could not have existed.

**Exercise 0.4.** Suppose  $g : D^n \rightarrow X$  is a homeomorphism. Then we know that  $g^{-1} \circ f \circ g$  is a continuous map from  $D^n$  to itself, and so it has a fixed point  $x$ . Then we know that  $g^{-1}(f(g(x))) = x$ , and so it follows that  $f(g(x)) = g(x)$ . Thus  $g(x) \in X$  is a fixed point of  $f$ .

**Exercise 0.5.** Consider the function  $h : \mathbb{I} \times \mathbb{I} \rightarrow \mathbb{I} \times \mathbb{I}$  given by

$$h(s, t) = f(s) - g(t) + (s, t).$$

This is the sum of continuous functions, and so it is itself continuous. Moreover, we know that  $\mathbb{I} \times \mathbb{I}$  is homeomorphic to  $D^1$ , and so it follows that there is a fixed point  $(s, t)$  of  $h$ . But this means that  $f(s) - g(t) = 0$ , and so we are done.

**Exercise 0.6.** Observe that  $x \in \Delta^{n-1}$  must contain some positive coordinate, because  $\sum x_i = 1$  and  $x_i \geq 0$  for all  $i$ . Since  $a_{ij} > 0$  for every  $i, j$ , it follows that  $Ax$  contains only nonnegative coordinates and, moreover, contains at least one positive coordinate. Thus  $\sigma(Ax) > 0$ , and so  $g(x)$  is well-defined.

Moreover, it is continuous because the linear map  $A$ , the map  $\sigma$ , and the division function are all continuous.

Because  $\Delta^{n-1} \approx D^{n-1}$ , it follows that there exists some  $x$  with

$$x = \frac{Ax}{\sigma(Ax)}.$$

Then  $\lambda = \sigma(Ax) > 0$  is a positive eigenvalue for  $A$  and  $x \in \Delta^{n-1}$  is a corresponding eigenvector.

We know that  $x$  contains only nonnegative coordinates. Suppose then that some coordinate, say  $x_1$ , is zero. Then obviously the first coordinate of  $\lambda x$  is zero. However, the first coordinate of  $Ax$  is

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = a_{12}x_2 + \cdots + a_{1n}x_n.$$

Since  $\sum x_i = 1$  and  $x_1 = 0$ , there exists some  $k \neq 1$  such that  $x_k > 0$ . Then  $a_{1k}x_k > 0$ , and since each  $i$  already has  $a_{1i}x_i \geq 0$ , it follows that the first coordinate of  $Ax$  is strictly positive, contradicting that  $Ax = \lambda x$ .

Thus the eigenvector  $x$  has all positive coordinates.

## Categories and Functors

**Exercise 0.7.** We know that

$$g \circ (f \circ h) = g \circ 1_b = g$$

and

$$(g \circ f) \circ h = 1_A \circ h = h,$$

and so associativity implies  $g = h$ .

**Exercise 0.8.**

(i) Notice that if  $1_A$  and  $1'_A$  are both identities, then we must have

$$1_A = 1_A \circ 1'_A = 1'_A,$$

which proves the desired result.

(ii) If  $1'_A$  is the new identity in  $\mathcal{C}'$ , then we know that  $1'_A \in \text{Hom}_{\mathcal{C}'}(A, A) \subseteq \text{Hom}_{\mathcal{C}}(A, A)$ , and so  $1_A \circ 1'_A$  is defined. But we know that

$$1'_A \circ 1_A = 1'_A = 1'_A \circ 1'_A,$$

and so Exercise 0.7 implies the result.

**Exercise 0.9.** Clearly, the Hom-sets are pairwise disjoint, since each  $i_y^x$  appears at most once.

It is also obviously associative. In particular, if  $a \leq b \leq c \leq d$ , then we know that

$$i_d^c \circ (i_c^b \circ i_b^a) = i_d^c \circ i_c^a = i_d^a,$$

and similarly for  $(i_d^c \circ i_c^b) \circ i_b^a$ .

Finally, the map  $i_x^x$  is the identity on  $x \in X$ . To see that it is a left-identity, note that if  $y \leq x$ , then

$$i_x^x \circ i_x^y = i_x^y.$$

Similarly, we can show that this map is a right-identity as well, and so we are done.

**Exercise 0.10.** Disjointness is clear, since there is only one object. Because  $G$  is a monoid, it is associative and has an identity, proving that  $\mathcal{C}$  is a category.

**Exercise 0.11.** It is pretty clear that  $\text{obj}(\mathbf{Top}) \subset \text{obj}(\mathbf{Top}^2)$ . Moreover, a continuous map  $f : X \rightarrow Y$  between two topological spaces corresponds to the map  $(f, \emptyset)$  in  $\mathbf{Top}^2$  from  $(X, \emptyset)$  to  $(Y, \emptyset)$ , which then means that  $\mathbf{Top}$  can be thought of as a subcategory of  $\mathbf{Top}^2$ .

**Exercise 0.12.** It is worth noting that Rotman's definition here is incorrect. The morphisms in  $\mathcal{M}$  should be the commutative squares, not merely the ordered pairs  $(h, k)$ .

Indeed, consider the following counterexample to Rotman's definition. Let  $\mathcal{C}$  be the category of sets. Furthermore, let  $A$  be a set with more than one element. Then the following diagrams are both commutative:

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ \downarrow 1_A & & \downarrow 0 \\ A & \xrightarrow{0} & \{0\} \end{array} \quad \begin{array}{ccc} A & \xrightarrow{0} & A \\ \downarrow 1_A & & \downarrow 0 \\ A & \xrightarrow{0} & \{0\}. \end{array}$$

This implies that the ordered pair  $(1_A, 0)$ , where  $0$  is considered to be the map that sends everything in  $A$  to the zero element, is both in  $\text{Hom}(1_A, 0)$  and in  $\text{Hom}(0, 0)$ , contradicting disjointness.

If we instead consider morphisms of  $\mathcal{M}$  to be the commutative squares, where composition is defined by “stacking” the squares on top of one another, disjointness is clear. After all, the squares contain  $f$  and  $g$ , and so Hom-sets of different objects must be disjoint.

Associativity is clear, as the morphisms of  $\mathcal{C}$  are associative.

Finally, there is an identity  $1_f$  for every  $f \in \text{Hom}_{\mathcal{C}}(A, B)$ , namely the one where  $h = 1_A$  and  $k = 1_B$ .

**Exercise 0.13.** With the hint, this is clear. In particular, we consider  $\mathbf{Top}^2$  to be the subcategory of the arrow category of  $\mathbf{Top}$  in which the objects are inclusions, and  $\text{Hom}_{\mathbf{Top}^2}(i, j) = \text{Hom}_{\mathbf{Top}}(i, j)$ .

**Exercise 0.14.** To see that it is a congruence at all, observe that Property (i) is satisfied because there is only one Hom-set. Moreover, if  $x \sim x'$  and  $y \sim y'$ , then we know that  $x(x')^{-1} = h_x$  and  $y(y')^{-1} = h_y$  for some  $h_x, h_y \in H$ . But then we know that

$$(yx)(y'x')^{-1} = yx(x')^{-1}(y')^{-1} = yh_x(y')^{-1}.$$

However, since  $(y')^{-1} = y^{-1}h_y$ , we know that this is simply

$$(yx)(y'x')^{-1} = yh_xy^{-1}h_y.$$

Because  $H$  is normal, we know that  $yh_xy^{-1} \in H$ . Thus the product of this and  $h_y$  is in  $H$  as well, and so  $xy \sim x'y'$ , as desired.

To see that  $[\ast, \ast] = G/H$  simply requires the observation that  $x \sim y$  if and only if  $x$  and  $y$  are in the same coset of  $H$ .

**Exercise 0.15.** This follows from the fact that functors preserve (or, in the case of contravariant functors, reverse) the directions of the arrows. Thus the resulting diagram still commutes.

**Exercise 0.16.** Note that for (i)–(iv), we can simply use inverses. For instance, for **Set**, it suffices to note that if  $f$  is a bijection, then  $f^{-1}$  is a bijection, which is clearly true. Similarly, the inverse of a homeomorphism is a homeomorphism, and the inverse of a group or ring isomorphism is still an isomorphism.

For (v), note that  $i_x^y$  is defined and satisfies the requirements that  $i_x^y \circ i_y^x = i_x^x$  and  $i_y^x \circ i_x^y = i_y^y$ .

For part (vi), notice that  $f^{-1}$  works because  $f$  is a homeomorphism. In particular, it is a bijection, and so  $f^{-1}(A') = A$ . Moreover, it is (bi)continuous since  $f$  is.

Finally, for the monoid  $G$ , if  $g$  has a two-sided inverse  $h$ , then  $hg = gh = 1$ , which is the identity element of  $\text{Hom}(G, G)$ .

**Exercise 0.17.** To prove that  $T'$  is a functor, first observe that criterion (i) of a functor is satisfied because  $T$  does so. Moreover, if  $[f] \in \text{Hom}_{\mathcal{C}'}(A, B)$ , then  $f \in \text{Hom}_{\mathcal{C}}(A, B)$ , and so  $T'([f]) = Tf$  is a morphism in  $\mathcal{A}$ . In particular, if  $[g] \circ [f] = [g \circ f]$  is defined in  $\mathcal{C}'$ , then  $g \circ f$  is defined in  $\mathcal{C}$ . This means, then, that

$$T'([g] \circ [f]) = T(g \circ f) = (Tg) \circ (Tf) = T'([g]) \circ T'([f]).$$

Finally, it remains to note that  $T'([1_A]) = T_{1_A} = 1_{TA} = 1_{T'([A])}$  for every object  $A$ . Thus  $T'$  is a functor.

**Exercise 0.18.**

- (i) It is clear that  $tG \in \text{obj } \mathbf{Ab}$  for every group  $G$ . Now suppose that we have a homomorphism  $f : G \rightarrow H$ . Then we know that  $t(f)$  is a morphism  $f|_{tG}$  from  $tG$  to  $tH$ . To see this, note that it is the restriction of a homomorphism, and thus is itself a homomorphism. Moreover, if  $x \in f(tG)$ , then  $x = f(y)$  for some  $y \in G$  with finite order. But then there exists some  $n$  so that  $y^n = 1$ . Thus  $x^n = f(y^n) = 1$ , and so  $x$  has finite order. But  $x \in f(G) \subseteq H$  implies that  $x \in tH$ .

Now we must check that  $t$  respects composition. Indeed, if  $g \circ f$  is defined, then

$$t(g \circ f) = (g \circ f)_{tG} = g|_{f(tG)} \circ f|_{tG}.$$

But  $f(tG) \subseteq tH$ , and so this is simply

$$t(g \circ f) = g|_{tH} \circ f|_{tG} = t(g) \circ t(f),$$

which proves that composition is respected.

Finally, note simply that  $t(1_G) = 1|_{tG}$ , which is the identity on  $tG$ .

- (ii) Suppose that  $f$  is an injective homomorphism from  $G$  to  $H$ . Then suppose that  $t(f)(x) = t(f)(y)$ . But  $f(x) = f|_{tG}(x) = t(f)(x)$ , and so it follows that  $f(x) = f(y)$ . Injectivity of  $f$  proves the result.
- (iii) Let  $G = \mathbb{Z}$  and  $H = \mathbb{Z}/2\mathbb{Z}$  and let  $f$  take even integers to 0 and odd integers to 1. This is evidently surjective. But  $tG = \{0\}$  while  $tH = \{0, 1\}$ , and so  $t(f) : tG \rightarrow tH$  cannot be surjective.

**Exercise 0.19.**

- (i) If  $f$  is a surjection, then consider an arbitrary coset  $a + pH$  of  $H/pH$ . We know that there exists some  $b \in G$  with  $f(b) = a$ , and so it follows that  $F(f)$  takes  $b + pG$  to  $a + pH$ , proving surjectivity of  $F(f)$ .
- (ii) Consider the function  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  taking  $x$  to  $2x$ . Then, letting  $p = 2$ , we know that  $F(f) : \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$  has  $F(f)([0]) = F(f)([1])$ .

**Exercise 0.20.**

- (i) This is evident because  $\mathbb{R}$  is a ring, and the operations are pointwise.
- (ii) By the previous part, we know that if  $X$  is a topological space, then  $C(X)$  is a ring. Now suppose that  $f : X \rightarrow Y$  is a continuous map. Then define

$$\begin{aligned} C(f) : C(Y) &\rightarrow C(X) \\ g &\mapsto g \circ f \end{aligned}$$

and note that this is well-defined. Moreover, we know that  $C(g \circ f)(h) = h \circ g \circ f$ , while  $C(f) \circ C(g)$  takes  $h$  to  $C(f) \circ (h \circ g) = h \circ g \circ f$ , which proves that  $C$  reverses composition. Finally, we know that  $C(1_x)$  takes  $g$  to  $g \circ 1_X = g$  and is therefore the identity on  $C(Y)$ . Thus  $C$  (or, rather, the map taking  $X$  to  $C(X)$ , to be precise) gives rise to a contravariant functor.