9 Natural Transformations

Definitions and Examples

Exercise 9.1. This is obvious from Lemma 4.8.

Exercise 9.2. This is exactly the statement of **??**.

Exercise 9.3. The commutative diagram in **??** is exactly the statement that the map is natural.

Exercise 9.4. Commutativity of the diagram

$$\begin{tikzcd}
H_n(X,A) \arrow[r, "f_*"] \arrow[d, "\partial"'] & H_n(Y,B) \arrow[d, "\partial"] \\
H_{n-1}(A,\emptyset) \arrow[r, "(f|A)_*"'] & H_{n-1}(B,\emptyset)
\end{tikzcd}$$

follows from the exact sequence in Theorem 5.8, since $H_{n-1}(A,\emptyset) = H_{n-1}(A)$.

Exercise 9.5. This is again precisely the statement from **??**.

Exercise 9.6.

(i) Suppose $\sigma : F \to G$ and $\tau : G \to H$ are natural. Then we can "stack" the commutative diagrams:

$$\begin{tikzcd}
F(C) \arrow[r, "Ff"] \arrow[d, "\sigma_C"'] & F(D) \arrow[d, "\sigma_D"] \\
G(C) \arrow[r, "Gf"] \arrow[d, "\tau_C"'] & G(D) \arrow[d, "\tau_D"] \\
H(C) \arrow[r, "Hf"] & H(D)
\end{tikzcd}$$

Hence it follows that $\tau\sigma = (\tau_C \sigma_C)$ gives a natural transformation.

(ii) Reflexivity is due to the commutativity of the following diagram:

$$\begin{tikzcd}
F(C) \arrow[r, "Ff"] \arrow[d, "1_{F(C)}"'] & G(C) \arrow[d, "1_{G(C)}"] \\
F(C) \arrow[r, "Ff"] & G(C)
\end{tikzcd}$$

To see symmetry, simply choose τ_C^{-1} for each object C. This can be done because each τ_C is an equivalence. Finally, transitivity follows from the previous part and the fact that the composition of equivalences is an equivalence.

Exercise 9.7.

(i) If $\varphi \in \mathrm{Nat}(\mathrm{Hom}(\ ,A), F)$, then φ_A is a map from $\mathrm{Hom}(A,A)$ to $F(A)$. Since $1_A \in \mathrm{Hom}(A,A)$, it follows that $\varphi_A(1_A) \in F(A)$. Thus y is a well-defined function.

(ii) We must check that $\tau \in \mathrm{Nat}(\mathrm{Hom}(\ ,A), F)$ whenever $\mu \in F(A)$. First, observe that τ_X is indeed a morphism from $\mathrm{Hom}(X,A)$ to $F(X)$. After all, if $f : X \to A$, then $Ff : FA \to FX$. Hence $\tau_X(f) = (Ff)(\mu)$ is an element of $F(X)$.

To see that τ is natural, we must show that the following diagram commutes for all $f : X \to Y$.

$$\begin{tikzcd}
\mathrm{Hom}(X,A) \arrow[d, "\tau_X"'] & \mathrm{Hom}(Y,A) \arrow[l, "\mathrm{Hom}(f,A)"'] \arrow[d, "\tau_Y"] \\
F(X) & F(Y) \arrow[l, "Ff"]
\end{tikzcd}$$

But for each $g \in \mathrm{Hom}(X, A)$, we know that

$$(Ff) \circ \tau_Y(g) = (Ff)(Fg(\mu)) = F(g \circ f)\mu,$$

while we have $\mathrm{Hom}(f, A) = f^*$, so that

$$\tau_X \circ \mathrm{Hom}(f, A)(g) = \tau_X \circ f^*(g) = \tau_X(g \circ f) = F(g \circ f)\mu.$$

These are equal, so τ is a natural transformation.

(iii) First, we will show that $y \circ y' : F(A) \to F(A)$ is the identity. Let $\mu \in F(A)$. Then we know that

$$y'(\mu) = \{\tau_X : f \mapsto (Ff)(\mu)\},$$

and so we have that

$$y(y'(\mu)) = (y'(\mu))_A(1_A) = F(1_A)(\mu).$$

But F is a functor, so $F(1_A) = 1_{F(A)}$, and so this is exactly equal to $1_{F(A)}(\mu) = \mu$, which proves that $y \circ y'$ is the identity on $F(A)$.

Now to check $y'y$, suppose $\varphi \in \mathrm{Nat}(\mathrm{Hom}(\ , A), F)$. Then we know that

$$y'(y(\varphi)) = \{\tau_X : f \mapsto (Ff)(\varphi_A(1_A)).$$

We would like to show that

$$(Ff)(\varphi_A(1_A)) = \varphi_X f,$$

since this will imply that $y'(y(\varphi)) = \varphi$. But we know that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}(X, A) \arrow[d, "\varphi_X"'] & \mathrm{Hom}(A, A) \arrow[l, "f^*"'] \arrow[d, "\varphi_A"] \\
F(X) & F(A) \arrow[l, "Ff"]
\end{tikzcd}$$

Thus we know, in particular, that

$$Ff \circ \varphi_A(1_A) = \varphi_X f^*(1_A) = \varphi_X(1_A \circ f) = \varphi_x \circ f.$$

This is what we wanted.

(iv) If $\varphi : \mathrm{Hom}(\ , A) \to \mathrm{Hom}(\ , B)$ is natural, then we have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}(X, A) \arrow[d, "\varphi_X"'] & \mathrm{Hom}(Y, A) \arrow[l, "\mathrm{Hom}(f,A)"'] \arrow[d, "\varphi_Y"] \\
\mathrm{Hom}(X, B) & \mathrm{Hom}(Y, B) \arrow[l, "\mathrm{Hom}(f,B)"]
\end{tikzcd}$$

Let F be the functor $\mathrm{Hom}(\ , B)$. Then we know that

$$\begin{aligned}
\varphi_X(f) &= y'(y(\varphi))_X(f) \\
&= (Ff)(\varphi_A(1_A)) \\
&= \mathrm{Hom}(f, B)(\varphi_A(1_A)) \\
&= \varphi_A(1_A) \circ f,
\end{aligned}$$

where $f : X \to A$. Thus $\varphi_X(f) = \mu f$, as desired.

(v) Same proof.

Exercise 9.8. We must verify the properties of a category. To see that the family of $\operatorname{Hom}(F, G)$'s, where F and G are functors $\mathscr{C} \to \mathscr{A}$, is disjoint, notice that this means that there exists some $\tau = (\tau_C : F(C) \to G(C))$ and $\sigma = (\sigma_C : F'(C) \to G'(C))$ which are equal. Hence $F(C) = F'(C)$ and $G(C) = G'(C)$ for all C, since $\tau_C = \sigma_C$ is in both $\operatorname{Hom}(F(C), G(C))$ and $\operatorname{Hom}(F'(C), G'(C))$. Since this is true for all $C \in \mathscr{C}$, it follows that $F = F'$ and $G = G'$.

Composition of natural transformations reduces to composition of morphisms, which is associative.

Finally, note that $1_A \in \operatorname{Hom}(F, F)$ given by

$$1_A = \{(1_A)_C = 1_{F(C)}\}$$

works as an identity morphism.

Exercise 9.9.

(i) We shall verify the properties of a contravariant functor. The functor gives us a complex

$$\dots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \dots \longrightarrow C_0 \longrightarrow C_{-1} \longrightarrow \dots$$

Since C_n is abelian, we know that $n \in \mathbb{Z}$ implies that $C(n) = C_n \in \mathscr{A}$.

The only morphisms in $\mathbb{Z}$ are ι_y^x when $x \leq y$. Note that $C(\iota_y^x)$ is the composition $\partial_{x+1} \circ \dots \circ \partial y : C_y \to C_x$. We must verify that composition is reversed and identities are respected. But this is clear from the definition:

$$]iota_z^y \circ \iota_y^x = \iota_z^x = \partial_{x+1} \circ \dots \circ \partial_z$$

is exactly $C(\iota_y^x) \circ C(\iota_z^y)$, and $C(\iota_x^x)$ is the composition of an empty set of differentiation operators, and thus is the identity.

(ii) The chain map condition is exactly the condition of commutativity.

Eilenberg–Steenrod Axioms

No exercises!

Chain Equivalences

Exercise 9.10. To prove (i) implies (ii), note that $ps = 1_C$ implies s is injective. Then the same argument as in Corollary 9.2 implies that $B = \ker p \oplus \operatorname{im} s$. Of course, we have $\ker p = \operatorname{im} i$ and $C' = \operatorname{im} s = s(C) \cong C$. Since $p(C') = C$, this proves the first implication.

The second implication is clear. In particular, consider $q : B \to A$ defined by $(i(x), c) \mapsto x$. Then $qi(a) = q(i(a)) = a$.

Finally, to show (iii) implies (i), define $s(c)$ as

$$s(c) = p^{-1}(c) - iqp^{-1}(c).$$

To see that this is well-defined, pick $b \in \ker p = \operatorname{im} i$, so $b = i(a)$. Thus $b - iq(b) = i(a) - iqi(a) = 0$. Hence $p(b) = p(b')$ means that $b - iqb = b' - iqb'$, proving well-definedness. To see that this choice of s gives a split exact sequence, simply verify that

$$ps(c) = p(p^{-1}(c) - iqp^{-1}(c)) = c - piqp^{-1}(c).$$

Since $pi = 0$, this is equal to c.

Acyclic Models

Exercise 9.11. First we show that the diagram given by Rotman commutes, i.e., that

$$\partial_n(t_n - t;_n - s_{n-1}d_n) = 0.$$

We know that

$$\partial_n t_n - \partial_n t'_n - \partial_n s_{n-1} d_n = t_{n-1} d_n - t'_{n-1} d_n - \partial_n s_{n-1} d_n.$$

The inductive hypothesis implies that

$$\partial_n s_{n-1} = t_{n-1} - t'_{n-1} - s_{n-2} d_{n-1}.$$

Plugging this value in and canceling gives us

$$\partial_n t_n - \partial_n t'_n - \partial_n s_{n-1} d_n = s_{n-2} d_{n-1} d_n = 0,$$

because $dd = 0$.

Thus the diagram commutes. In particular, we know that

$$\operatorname{im}(t_n - t'_n - s_{n-1} d_n) \subseteq \ker \partial_n = \operatorname{im} \partial_{n+1},$$

where the final equality comes from the fact that E_* is an acyclic complex. This means that we can rewrite the diagram as follows:

$$\begin{tikzcd} & F_n \arrow[d, "t_n - t'_n - s_{n-1} d_n"'] & \\ E_{n+1} \arrow[r, "\partial_{n+1}"'] & \begin{array}{c}\operatorname{im}(t_n - t'_n - s_{n-1} d_n) \\ = \operatorname{im} \partial_{n+1} = \ker \partial_n\end{array} \arrow[r, "\partial_n = 0"'] & 0 \end{tikzcd}$$

Thus Theorem 9.1 implies that we can find s_n with the desired properties.

Exercise 9.12. We have $F(g) = F(0 + g) = F(0) + F(g)$, so $F(0)$ acts as the 0 element. If A is the zero group, then its identity is the zero homomorphism. Hence $1_{F(A)} = F(1_A)$ is the zero homomorphism, so $F(A) = 0$.

Exercise 9.13.

(i) We'll prove the covariant case. By Exercise 9.10, we have a morphism $q : A \to B$ with $qi = 1_A$. Note that $(Fp) \circ (Fs) = F(p \circ s) = F(1_C) = 1_{F(C)}$, and similarly for q and i, so that we still have a split sequence, as long as it is exact. Moreover, these imply that Fp is surjective and Fi is injective.

It now suffices to check that $\operatorname{im} Fi = \ker Fp$. But notice that $B \cong iq(B) \oplus sp(B)$ implies that $F(B)$ is equal to the functored version of the right side, thus making the center of the short functored sequence exact.

(ii) This simply uses induction on $|I| = n + 1$ and the following short exact sequence:

$$0 \longrightarrow \textstyle\sum_{i=1}^{n} A_i \xrightarrow{\ i\ } \sum_{i=1}^{n+1} A_i \xrightarrow{\ p\ } A_{n+1} \longrightarrow 0.$$

Note that this is split exact with $s : a_{n+1} \mapsto (0, \ldots, 0, a_{n+1})$. Thus the previous part applies, and Exercise 9.10 implies that

$$F\left(\sum_{i=1}^{n+1} A_i\right) \cong F\left(\sum_{i=1}^{n} A_i\right) \oplus F(A_{n+1}) = \sum_{i=1}^{n+1} F(A_i),$$

where the last equality follows from the inductive hypothesis.

Exercise 9.14.

(i) If $\partial_n \partial_{n+1} = 0$, then $F(\partial_n \partial_{n+1}) = 0$ thanks to additivity. This proves that the functored complex is a chain complex too.

(ii) Note that $f_{n-1}\partial_n = \partial'_n f_n$ implies that $F(f_{n-1}\partial_n) = F(\partial'_n f_n)$. Since functors respect composition, this proves the result.

(iii) Note that additive functors respect homotopy because they rerspect both composition and addition. Hence if $g : B_* \to A_*$ makes f an equivalence, i.e., if $g \circ f \simeq 1_{A_*}$ and $f \circ g \simeq 1_{B_*}$, then it follows that

$$Fg \circ Ff \simeq F1_{A_*} = 1_{FA_*},$$

and similarly for B. Hence Fg is an inverse for Ff, so Ff is a chain equivalence.

Exercise 9.15.

(i) This simply involves applying Corollary 9.13(ii). In particulars, we know that F_p and S_p arer both free with basis in $\mathcal{M} = \{\Delta^p\}$. We want to show that Δ^p is totally S- and F-acyclic. But notice that $\tilde{H}_n(S_*(\Delta^k)) = 0$ because Δ^k is contractible, and similarly for F_p, since it coincides with C_p ono Δ. This proves acyclicity, and so the two are naturally chain equivalent.

(ii) Theorem 9.8 implies that singular and large simplicial homology are the same, while Theorem 7.22 implies that normal simplicial and singular homology are the same.

Lefschetz Fixed Point Theorem

Exercise 9.16. Notice that 1_G induces $1_{G/tG} : x + tG \mapsto x + tG$. Hence, with any basis $\{x_1, \ldots, x_n\}$ of G/tG, we have $1_{G/tG}$ equal to the identity matrix whose dimension is $\operatorname{rank} G/tG$.

Exercise 9.17. A basis $\{x_1, \ldots, x_k\}$ of G'/tG' can be extended to $\{x_1, \ldots, x_n\}$ of G/tG. Since G'' is just G/G' and $f''(g + G') = pf(g) = f(g) + G'$. Thus $f''(x_i + G') = f(x_i) + G'$ for $i = k+1, \ldots, n$. Thus the matrix of $\bar{f}$ is diagonal, of the form shown on p. 259 of the textbook, which implies the result.

Exercise 9.18. If $f : S^n \to S^n$, then f_{0*} and f_{n*} are maps $\mathbb{Z} \to \mathbb{Z}$. Note that f_{0*} is the identity, and thus has trace 1. If $\operatorname{tr} f_{n*} = 1$ as well, then the whole map is homotopic to either the identity or the antipodal map, implying that f is a homotopy equivalence. Thus $\operatorname{tr} f_{n*} = 0$, and so $\lambda(f) = 1 \neq 0$. The Lefschetz fixed point theorem implies the result.

Tensor Products

Exercise 9.19. Note that

$$\begin{aligned} a \otimes 0 + a' \otimes b' &= a \otimes 0 + (a \otimes b' + (a' - a) \otimes b') \\ &= a \otimes b' + (a' - a) \otimes b' \\ &= a' \otimes b', \end{aligned}$$

and similarly for $0 \otimes b$.

Exercise 9.20. We would like to show that $m(a, b) \sim (ma, b)$ for $m \in \mathbb{Z}$. It is true for $m > 0$ by induction, true for $m = 0$ by Exercise 9.19, and true for $m < 0$ by inverses.

Exercise 9.21. The hint gives the full solution. If $a \in A$ then there exists some $m > 0$ so that $ma = 0$. Hence $a \otimes q = ma \otimes (q/m) = 0$. Since this is true fora all generators of $A \otimes \mathbb{Q}$, the result follows.

Exercise 9.22. Let m be the order of $a \in A$ and n the order of $b \in B$. Then we know that $\gcd(m, n) = 1$, so that there exist integers x, y with $mx + ny = 1$. Hence we have that

$$\begin{aligned} a \otimes b &= a \otimes (mx + ny)b \\ &= (mx + ny)(a \otimes b) \\ &= mx(a \otimes b) + ny(a \otimes b) \\ &= (mxa \otimes b) + (a \otimes nyb) = 0. \end{aligned}$$

Exercise 9.23. This is the exact same argument as Corollary 9.27.

Exercise 9.24.

(i) Use Theorem 9.25(ii) with the fact that $A \times B \cong B \times A$.

(ii) To see this, simply consider the following commutative diagram:

$$\begin{tikzcd}
A \otimes B \arrow[r, "1_A \otimes f"] \arrow[d] & A \otimes C \arrow[d] \\
B \otimes A \arrow[r, "f \otimes 1_A"'] & C \otimes A
\end{tikzcd}$$

Exercise 9.25. Note that $T_A(f+g) = 1_A \otimes f + 1_A \otimes g$, since both maps complete the diagram

$$\begin{tikzcd}
A \times B \arrow[rr] \arrow[dr, "\varphi"'] & & A \otimes B \arrow[dl, dashed] \\
& A \otimes B &
\end{tikzcd}$$

where $\varphi(a, b) = (a, (f+g)(b))$. But note that $1_A \otimes f + 1_A \otimes g$ is just $T_A(f) + T_A(g)$, proving additivity.

Exercise 9.26. This is clear, since we have

$$\begin{aligned}
1_A \otimes f : A \otimes B &\to A \times B \\
a \otimes b &\mapsto a \otimes fb = a \otimes mb = m(a \otimes b).
\end{aligned}$$

Exercise 9.27.

(i) This is easy to show directly. In particular, we show that $a \mapsto 1 \otimes a$ is an isomorphism. We would like to show that $1 \otimes a = n \otimes b$ if $nb = a$. But $n \otimes b = n(1 \otimes b) = 1 \otimes (nb) = 1 \otimes a$, as desired. Hence this map is surjective. It is injective because, otherwise, every $1 \otimes a$ would be 0, which would violate the universal property of tensor products given by Theorem 9.25. Hence this is an isomorphism.

(ii) We must show that the following commutes:

$$\begin{tikzcd}
\mathbb{Z} \otimes A \arrow[r, "1_A \otimes f"] \arrow[d, "\tau_A"'] & \mathbb{Z} \otimes B \arrow[d, "\tau_B"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

This commutes because

$$\tau_B \circ (1_A \otimes f) : (n, a) \mapsto (n, f(a)) \mapsto nf(a),$$

while

$$f \circ \tau_A : (n, a) \mapsto na \mapsto f(na),$$

and $nf(a) = f(na)$ since f is a homomorphism.

Universal Coefficients

Exercise 9.28.

(i) We can write $F = \sum A_j$ where $A_j = \mathbb{Z}x_j$, and $F' = \sum A'_k$ where $A'_k = \mathbb{Z}x'_k$. Then $F \otimes F'$ is just

$$\begin{aligned}
F \otimes F' &= F \otimes \sum A'_k = \sum (F \otimes A'_k) \\
&= \sum \left(\sum A_j \otimes A'_k \right) \\
&= \sum_{j,k} A_j \otimes A'_k.
\end{aligned}$$

But it is easy to verify that $A_j \otimes A'_k = \mathbb{Z}(x_j \otimes x'_k) \cong \mathbb{Z}$, which proves the result.

(ii) This is obvious from the previous part since

$$\operatorname{rank} F \otimes F' = |J \times K| = |J||K| = \operatorname{rank} F \operatorname{rank} F'$$

Exercise 9.29. This is simply an application of Theorem 9.28 and Corollary 9.30, along with Exercise 9.27. We end up with

$$A \otimes B = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}.$$

Exercise 9.30.

(i) Using coordinate-wise addition and scalar multiplication of the form

$$p \sum (q_i, g_i) = \sum (pq_i, g_i)$$

shows that $\mathbb{Q} \otimes G$ is a $\mathbb{Q}$-vector space. Hence $\dim \mathbb{Q} \otimes G$ is defined.

(ii) This follows immediately from the Tor exact sequence, along with the fact that $\operatorname{Tor}(\mathbb{Q}, B) = 0$ for all B.

Exercise 9.31. This is simply a calculation using the properties of Tor. We end up with

$$\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}.$$

Exercise 9.32. Using Exercise 9.30 with the short exact sequence

$$0 \to F \to G \to G/F \to 0$$

gives us

$$\dim \mathbb{Q} \otimes G = \dim \mathbb{Q} \otimes F + \dim \mathbb{Q} \otimes G/F.$$

But $\dim \mathbb{Q} \otimes G/F = 0$ by Exercise 9.21 and the fact that G/F is torsion. Moreover, we know that $\mathbb{Q} \otimes F$ has basis $(1, x_i)$, where x_i is a generator of F, so $\dim \mathbb{Q} \otimes F = \operatorname{rank} F = \operatorname{rank} G$, which proves the result.

Exercise 9.33. Note that [Tor 1] and [Tor 5] imply that there is an exact sequence

$$0 \to \operatorname{Tor}(B', A) \to \operatorname{Tor}(B, A) \to \operatorname{Tor}(B'', A) \to B' \otimes A \to B \otimes A \to B'' \otimes A \to 0,$$

since $B \otimes A \cong A \otimes B$ by Exercise 9.24. But if A is torsion-free, then $\operatorname{Tor}(B'', A) = 0$ by [Tor 2], which gives us the desired exact sequence.

Exercise 9.34. This is false! Consider, for example, when $F = \mathbb{Z}$ and $H = \mathbb{Z}/2\mathbb{Z}$, and $a = 2$, $h = 1$. In general, we need the condition that if $a = \sum m_j x_j$, where $\{x_j\}$ is a basis for F, then $m_j h \neq 0$ for at least some j. After all, we need that

$$a \otimes h = (m_j x_j \otimes h)_j = (m_j h) \neq 0.$$

Exercise 9.35. Let α be the map $(\operatorname{cls} z) \otimes g \mapsto \operatorname{cls}(z \otimes g)$. Then the Universal Coefficients Theorem implies that

$$0 \longrightarrow H_n(X) \otimes G \xrightarrow{\alpha} H_n(X; G) \longrightarrow \operatorname{Tor}(H_{n-1}(X), G) \longrightarrow 0$$

is exact. Of course, since G is torsion-free, we know that $\operatorname{Tor}(H_{n-1}(X), G) = 0$. Hence α is an isomorphism.

Exercise 9.36. Use the second part of the Universal Coefficients Theorem. In particular, it gives us that

$$H_n(X; \mathbb{Z}/m\mathbb{Z}) \cong (H_n(X) \otimes \mathbb{Z}/m\mathbb{Z}) \oplus H_{n-1}(X)[m],$$

since

$$\operatorname{Tor}(H_{n-1}(X), \mathbb{Z}/m\mathbb{Z}) = H_{n-1}(X)[m]$$

by [Tor 4]. If $H_{n-1}(X)$ is torsion-free, the second term is zero, which gives the conclusion.

Eilenberg–Zilber Theorem and the Künneth Formula

Exercise 9.37. This is a straightforward calculation. In particular, we find that

$$\begin{aligned}(\lambda\otimes\mu)_{n-1}D_n(c_i\otimes e_j) &= (\lambda\otimes\mu)_{n-1}(dc_i\otimes e_j+(-1)^i c_i\otimes\partial e_j)\\ &= (\lambda_{i-1}\otimes\mu_j)(dc_i\otimes e_j)+(\lambda_i\otimes\mu_{j-1})((-1)^i c_i\otimes\partial e_j)\\ &= \lambda_{i-1}dc_i\otimes\mu_j e_j+(-1)^i\lambda_i c_i\otimes\mu_{j-1}\partial e_j.\end{aligned}$$

A similar calculation gives

$$\begin{aligned}D'_n(\lambda\otimes\mu)_n(c_i\otimes e_j) &= D'_n(\lambda_i\otimes\mu_j)(c_i\otimes e_j)\\ &= D'_n(\lambda_i c_i\otimes\mu_j e_j)\\ &= d\lambda_i c_i\otimes\mu_j e_j+(-1)^i\lambda_i c_i\otimes\partial(\mu_j e_j).\end{aligned}$$

Of course, we know that $d\lambda=\lambda d$ and $\partial\mu=\mu\partial$, which implies the result.

Exercise 9.38. Note that it suffices to prove the hint, since transitivity will finish the proof. The proof of the hint is a routine, if long, computation.

Exercise 9.39. Suppose $\lambda : C_*\to C'_*$ and $\lambda' : C'_*\to C_*$ with $\lambda\circ\lambda'\simeq 1_{C'_*}$ and $\lambda'\circ\lambda\simeq 1_{C_*}$. Similarly define μ and μ'. Then Exercise 9.38 implies that

$$\lambda\otimes\mu : C_*\otimes E_*\to C'_*\otimes E'_*,$$

and similarly for $\lambda'\otimes\mu'$). But

$$(\lambda\otimes\mu)\circ(\lambda'\otimes\mu')=(\lambda\lambda')\otimes(\mu\mu')\simeq 1_{C'_*}\otimes 1_{E'_*}=1_{C'_*\otimes E'_*}.$$

The same calculation holds for the other composition, which proves chain equivalence.

Exercise 9.40. Each n (i.e., each $0\to S'_n\to S_n\to S''_n\to 0$) works because E_* is a chain complex, hence E_n is free.

Exercise 9.41. For $n\geq 1$, we know that $H_n(X)=0=H_n(Y)$. Hence the Künneth formula implies that

$$H_n(X\times Y)\cong\sum_{i+j=n}H_i(X)\otimes H_j(Y)\oplus\sum_{p+q=n-1}\mathrm{Tor}(H_p(X),H_q(Y)).$$

But the first term is 0 since one of i,j is at least 1, and thus one of $H_i(X),H_j(Y)$ is 0. The second term is zero since the only way for $H_p(X)$ and $H_q(Y)$ to both be nonzero is if $p=q=0$, in which case both homology groups are free. Hence the torsion $\mathrm{Tor}(H_0(X),H_0(Y))$ is zero in that case too.

Exercise 9.42. For path-connected X and Y, we have

$$H_1(X\times Y)=H_0(X)\otimes H_1(Y)\oplus H_1(X)\otimes H_0(Y)\oplus\mathrm{Tor}(H_0(X),H_0(Y)).$$

But $H_0(X)=H_0(Y)=\mathbb{Z}$, and so using Exercise 9.27 gives us that the first two terms are $H_1(Y)$ and $H_1(X)$, respectively, while [Tor 2] implies that the last term is 0. This gives the first equation.

For H_2, notice that the Tor terms have either $H_0(X)$ or $H_0(Y)$, so [Tor 2] implies that they are 0. Hence

$$H_2(X\times Y)=[H_0(X)\otimes H_2(Y)]\oplus[H_1(X)\otimes H_1(Y)]\oplus[H_2(X)\otimes H_0(Y)].$$

Using $H_0(X)=H_0(Y)=\mathbb{Z}$ again gives the result.

Exercise 9.43. This splits into multiple cases and is slightly annoying. We end up with the following:

$$H_p(K\times\mathbb{R}P^n)=\begin{cases}0 & p\geq n+2\\ \mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z} & p=n+1, n\text{ odd}\\ \mathbb{Z}/2\mathbb{Z} & p=n+1, n\text{ even}\\ \mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z} & p=n, n\text{ odd}\\ \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z} & p=n, n\text{ even}\\ \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z} & 1<p<n\\ \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z} & p=1, p\neq n\\ \mathbb{Z} & p=0\end{cases}$$

Exercise 9.44. We once again have many, many cases.

$$H_p(\mathbb{R}P^n \times S^m) = \begin{cases} \mathbb{Z} & p = 0, m \neq 0 \\ \mathbb{Z} \oplus \mathbb{Z} & p = m = 0 \\ \mathbb{Z}/2\mathbb{Z} & p \text{ odd}, p < \min(m, n) \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & m \text{ odd}, p = m \\ \mathbb{Z}/2\mathbb{Z} & m \text{ odd}, p \text{ odd between } m \text{ and } n \\ \mathbb{Z} & m \text{ odd}, p \text{ even}, p \leq m+n \\ \mathbb{Z} & m \text{ even}, p = m \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & m \text{ even}, p \text{ odd between } m \text{ and } n \\ \mathbb{Z} & m \text{ even}, p \text{ odd}, p \leq m+n \\ 0 & \text{otherwise} \end{cases}$$

(Something like that, I can't quite read my work anymore.)

Exercise 9.45. This is the exact same idea, but I'll admit I didn't work it all out.

Exercise 9.46. It turns out that the machinery we have (i.e., fundamental groups) isn't sufficient to distinguish $S^1 \vee S^2 \vee S^3$ from $S^1 \times S^2$, as they both have fundamental group $\mathbb{Z}$. In fact, this seems to require cohomology (see **??**).

Exercise 9.47. We use the Künneth formula here. Note that the homology groups of S^1 are all cyclic or zero, so the Tor terms are zero. Hence

$$H_n(S^1 \times S^1) = \sum_{i+j=n} H_i(S^1) \otimes H_j(S^1).$$

If $n > 2$, then one of $H_i(S^1)$ and $H_j(S^1)$ is zero, so

$$H_n(S^1 \times S^1) = 0 \quad n > 2.$$

When $n = 0$, then we have $i = 0, j = 0$, so

$$H_0(S^1 \times S^1) = \mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z}$$

If $n = 1$, then we have $(i, j) = (0, 1), (1, 0)$, and so

$$H_1(S^1 \times S^1) = \mathbb{Z} \otimes \mathbb{Z} \oplus \mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z} \oplus \mathbb{Z}.$$

Finally, if $n = 2$, then we only have to consider $(i, j) = (1, 1)$, so that

$$H_2(S^1 \times S^1) = \mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z}.$$

Now recall that

$$H_n(K_1 \vee K_2) \cong H_n(K_1) \oplus H_n(K_2),$$

so we know that

$$H_n(S^2 \vee S^1 \vee S^1) \cong H_n(S^2) \oplus H_n(S^1) \oplus H_n(S^1).$$

We know the homology groups of S^1 and S^2, and working them out gives the same homology groups as those of $S^1 \times S^1$.

(Interestingly, the fundamental groups of these two spaces are different from one another, which one can show using Seifert–Van Kampen. I wonder if Rotman mixed up this problem with Exercise 9.46.)

Exercise 9.48.

(i) This is straightforward using the fact that the homology of wedges is the direct sum of homology groups. Hence both homology groups are $\mathbb{Z}$ when $n = 0, 3$, $\mathbb{Z}/2\mathbb{Z}$ when $n = 1$, and 0 otherwise.

(ii) According to a cursory search online, this requires universal coverings.

(iii) This seems to be another mistake on Rotman's part, as he seems to have thought that $\mathbb{R}P^3$ and $\mathbb{R}P^2 \vee S^3$ had different fundamental groups. In fact, they both have $\mathbb{Z}/2\mathbb{Z}$ as their fundamental group, and so it is obvious that $\mathbb{R}P^3 \times \mathbb{R}P^2$ and $(\mathbb{R}P^2 \vee S^3) \times \mathbb{R}P^2$ have the same homology groups and fundamental group.

Exercise 9.49. Since the homology groups of S^1 are all cyclic or zero, the Tor terms in the Künneth formula don't count. Suppose that

$$H_n(T^{r-1}) = \mathbb{Z}^{\binom{r-1}{n}}.$$

Note that this is true for $r = 1$. Then we have that

$$H_n(S^1 \times T^{r-1}) \cong \sum_{i+j=n} H_i(S^1) \otimes H_j(T^{r-1}) = H_n(T^{r-1}) \oplus H_{n-1}(T^{r-1}).$$

But of course we know that

$$\binom{r-1}{n} + \binom{r-1}{n-1} = \binom{r}{n},$$

and so it follows that

$$H_n(T^r) = H_n(S^1 \times T^{r-1}) = \mathbb{Z}^{\binom{r}{n}}.$$