6 Excision and Applications

Excision and Mayer–Vietoris

Exercise 6.1. Since A and B are both open, we know that $A^\circ = A$ and $B^\circ = B$. Thus $A^\circ \cup B^\circ = X$, and so we can use the Mayer–Vietoris sequence, along with the fact that $A \cap B = \emptyset$, to find an exact sequence

$$0 = H_n(\emptyset) \xrightarrow{(i_{1*}, i_{2*})} H_n(A) \oplus H_n(B) \xrightarrow{g_* - j_*} H_n(X) \xrightarrow{D} H_{n-1}(\emptyset) = 0 \ .$$

Thus the middle map $g_* - j_*$ is an isomorphism, which proves the result.

Exercise 6.2. We use excision directly. In particular, Excision II gives us an isomorphism $i_* : H_n(B, \emptyset) \to H_n(X, A)$. But $H_n(B, \emptyset) = H_n(B)$ for all $n \geq 0$, and so the conclusion follows.

Exercise 6.3. This is simply a diagram chase. Suppose $D_n([x]) = [x']$. Then by definition of D, we know that there exists some $[z] \in H_n(X_1, X_1 \cap X_2)$ such that $d_n([z]) = [x']$ and $h_n([z]) = q_n([x])$. It thus follows that

$$g_{n-1}(D_n([x])) = [f_{n-1}(x')] = f_{n-1}(d_n([z])).$$

Now set $y = f(x)$, so that $[y] = f_n([x])$. We would like to show that $D'_n([y]) = f_{n-1}(d_n([z]))$. To do this, set $z' = f_n(z)$. Then since d commutes with f by definition (cf. Theorem 5.7), we know that

$$d'_n([z']) = d'_n(f_n(z)) = f_{n-1}(d_n([z])) = g_{n-1}(D_n([x])).$$

Moreover, because h and q are just inclusions, we know that they commute with f. In particular, from the fact that $h_n([z]) = q_n([x])$, and so $f(h_n([z])) = f(q_n([x]))$, we find that

$$h'_n(f([z])) = q'_n(f_n([x])),$$

from which it follows by definition that $h'_n([z']) = q'_n([y])$. Thus it follows that

$$D'(f([x])) = D'([y]) = g_{n-1}(D_n([x])),$$

which proves that the desired diagram commutes.

Exercise 6.4. First note that the condition implies that, for all $n \geq 1$, we have

$$H_n(X_i) = H_n(X_i \cap X_j) = H_n(X_1 \cap X_2 \cap X_3) = 0.$$

Since each X_i is open, we can apply the Mayer–Vietoris sequence. Applying it on X_1 and X_2 gives an exact sequence

$$0 \longrightarrow H_n(X_1 \cup X_2) \longrightarrow 0 \ ,$$

and so we conclude that $H_n(X_1 \cup X_2) = 0$.

Now we can apply Mayer–Vietoris to $X_1 \cup X_2$ and X_3 to find an exact sequence

$$0 \longrightarrow H_n(X) \longrightarrow H_{n-1}((X_1 \cup X_2) \cap X_3). \ .$$

But the last term is exactly $H_{n-1}((X_1 \cap X_3) \cup (X_2 \cap X_3))$.

To see that this is 0, apply Mayer–Vietoris to $X_1 \cap X_3$ and $X_2 \cap X_3$. In particular, setting $H = H_{n-1}((X_1 \cap X_3) \cup (X_2 \cap X_3))$ as the desired homology group, we know that

$$H_{n-1}(X_1 \cap X_3) \oplus H_{n-1}(X_2 \cap X_3) \longrightarrow H \longrightarrow H_{n-2}((X_1 \cap X_3) \cap (X_2 \cap X_3))$$

is exact. If $n > 2$ or if $X_1 \cap X_2 \cap X_3 = \emptyset$, then the first and last terms are clearly 0, which proves that the middle homology group is indeed 0. If, on the other hand, we have $n = 2$ and $X_1 \cap X_2 \cap X_3$ is contractible, then the last term is $\mathbb{Z}$. However, the next map in the Mayer–Vietoris sequence is an injective map, since it is induced by inclusions. Thus we have an exact sequence

$$0 \longrightarrow H \longrightarrow \mathbb{Z} \longrightarrow A \ ,$$

where A is some homology group (in fact, it is $\mathbb{Z}^2$) and the map $\mathbb{Z} \to A$ is injective. Note that the image of the first map is 0, and so the map $H \to \mathbb{Z}$ is injective. But we also know that $\mathrm{im}(H \to \mathbb{Z}) = \ker(\mathbb{Z} \to A) = 0$. Thus $H = 0$, as desired.

Homology of Spheres and Some Applications

No exercises!

Barycentric Subdivision and Proof of Excision

Exercise 6.5. We use induction. In particularly, for Σ^0 we know we have $(0+1)! = 1$ 0-simplexes. Now suppose the statement is true for $n-1$. Using the notation in the definition, note that each n-simplex in Σ^n is spanned by b and an $(n-1)$-simplex in $\operatorname{Sd}\varphi_i$. Since there are $n+1$ total possible φ_i's, and since there are $n!$ total $(n-1)$-simplexes in $\operatorname{Sd}\varphi_i$, it follows that $\operatorname{Sd}\Sigma^n$ has $(n+1)!$ total n-simplexes, as desired,

Exercise 6.6.

(i) By construction, every point is the barycenter of at least one face. Moreover, writing $\Sigma^n = [p_0, \ldots, p_{n+1}]$, suppose that b is the barycenter of $[p_{i_1}, \ldots, p_{i_k}]$, as well as of $[p_{j_1}, \ldots, p_{j_\ell}]$. Then

$$\frac{1}{k+1}(p_{i_1} + \cdots + p_{i_k}) = \frac{1}{\ell+1}(p_{j_1} + \cdots + p_{j_\ell}).$$

This implies linear dependence, unless the two subsimplexes of Σ^n are actually the same simplex.

(ii) This is clear for $n = 0$. Now suppose that the statement is true for $n-1$ Note that, by definition, every n-simplex of Σ^n is spanned by the barycenter b of Σ^n and an $(n-1)$-simplex of $\operatorname{Sd}\varphi_i$. But every face of Σ^n is a subset of Σ^n. Thus we can write an n-simplex of Σ^n as $[b^{\sigma_0}, \ldots, b^{\sigma_n}]$ with $\sigma_0 < \cdots < \sigma_{n-1} < \sigma_n = \Sigma^n$.

Exercise 6.7.

(i) Note that $\operatorname{Sd}_1(\delta^1)$ is exactly $b_1.\operatorname{Sd}_0(\partial\delta^1)$. Since Sd_0 is the identity, we know that this is $b_1.(\partial\delta^1)$. But $\partial\delta^1 = e_1 - e_0$, while $b_1 = \frac{1}{2}(e_0 + e_1)$, and so it follows that

$$\begin{aligned}\operatorname{Sd}_1(\delta^1)(t) &= b_1.e_1(t) - b_1.e_0(t) \\ &= \left(\frac{t}{2}(e_0+e_1) + (1-t)e_1\right) - \left(\frac{t}{2}(e_0+e_1) + (1-t)e_0\right).\end{aligned}$$

Note that both terms within the large parentheses are 1-simplexes, and so we cannot "cancel" the $\frac{t}{2}(e_0 + e_1)$ terms.

For $n = 2$, we would like to evaluate $b_2.\operatorname{Sd}_1(\partial\delta^2)$. Note that $\partial\delta^2 = [e_1, e_2] - [e_0, e_2] + [e_0, e_1]$. Thus we know, either by using the same argument as before or by appealing to the case $n = 1$ in part (ii) below, that

$$\begin{aligned}\operatorname{Sd}_1(\partial\delta^2)(t) = &\left(\frac{t}{2}(e_1+e_2) + (1-t)e_2\right) - \left(\frac{t}{2}(e_1+e_2) + (1-t)e_1\right) \\ &- \left(\frac{t}{2}(e_0+e_2) + (1-t)e_2\right) + \left(\frac{t}{2}(e_0+e_2) + (1-t)e_0\right) \\ &+ \left(\frac{t}{2}(e_0+e_1) + (1-t)e_1\right) - \left(\frac{t}{2}(e_0+e_1) + (1-t)e_0\right).\end{aligned}$$

Thus we may evaluate $\operatorname{Sd}_2(\delta^2)$ on a term-by-term basis. For example, the first term of $\operatorname{Sd}_2(\delta^2)(t_1, t_2)$ is

$$t_1 b_2 + (1-t_1)\left(\frac{t_2/(1-t_1)}{2}(e_1+e_2) + (1 - t_2/(1-t_1))e_2\right) = t_1 b_2 + \left(\frac{t_2}{2}(e_1+e_2) + (1-t_1-t_2)e_2\right).$$

We can do this with each term to find $\operatorname{Sd}_2(\delta^2)$.

(ii) We can simply evaluate this using the previous part. In particular, we have

$$\begin{aligned}\operatorname{Sd}_1(\sigma) &= \sigma_\# \operatorname{Sd}_1(\delta^1) \\ &= \left(\frac{t}{2}(\sigma(e_0)+\sigma(e_1)) + (1-t)\sigma(e_1)\right) - \left(\frac{t}{2}(\sigma(e_0)+\sigma(e_1)) + (1-t)\sigma(e_0)\right).\end{aligned}$$

Similarly, by replacing each e_i in $\operatorname{Sd}_2(\delta^2)$ with $\sigma(e_i)$, we have $\operatorname{Sd}_2(\sigma)$.

Exercise 6.8. It is sufficient to show commutativity for generators $\sigma : \Delta^n \to X$. But note that $f_\# \operatorname{Sd}_n(\sigma) = f_\# \sigma_\# \operatorname{Sd}_n(\delta^n)$. However, since $f_\# \sigma_\# = (f \circ \sigma)_\#$, it follows that this is in turn equal to $(f \circ \sigma)_\# \operatorname{Sd}_n(\delta^n) = \operatorname{Sd}_n(f \circ \sigma) = \operatorname{Sd}_n f_\# \sigma$. This proves commutativity, as desired.

Exercise 6.9. Recall that the j-th face of σ is $\sigma\varepsilon_j : [e_0, \ldots, e_{n-1}] \to [e_0, \ldots, \hat{e}_i, \ldots, e_n]$. Now observe that ε_j is clearly affine. After all, we know that

$$\varepsilon_j \left(\sum_i t_i e_i \right) = \sum_{i<j} t_i e_i + \sum_{i \geq j} t_i e_{i+1} = \sum_i t_i \varepsilon_j(e_i).$$

Thus we know that

$$\sigma\varepsilon_j \left(\sum t_i e_i \right) = \sigma \left(\sum t_i \varepsilon_j(e_i) \right) = \sum t_i \sigma(\varepsilon_j(e_i)).$$

Thus $\sigma\varepsilon_j$ is affine. Since $\sigma\varepsilon_j(e_i)$ is either e_i (if $i < j$) or e_{i+1} (if $i \geq j$), it follows that the vertex set of $\sigma\varepsilon_j$ is a (proper) subset of the vertex set of σ. Since $\partial\sigma$ is just an alternating sum of faces of σ, it follows that $\partial\sigma$ is affine whenever σ is.

Exercise 6.10. Recall the definition of a cone:

$$b.\sigma(t_0, \ldots, t_{n+1}) = \begin{cases} b & \text{if } t_0 = 1, \\ t_0 b + (1 - t_0)\sigma \left(\frac{t_1}{1-t_0}, \ldots, \frac{t_{n+1}}{1-t_0} \right) & \text{if } t_0 < 1. \end{cases}$$

It is clear that b is affine. Since the case $t_0 < 1$ results in the sum of affine maps, it follows that this is also affine.

Now note that $b.\sigma(e_0) = b$ and $b.\sigma(e_i) = \sigma(e_i)$ for $i \neq 0$. Thus the vertex set of $b.\sigma$ is the union of $\{b\}$ and the vertex set of σ. Note that $\operatorname{Sd}_0 \sigma$ is affine whenever $\sigma : \Delta^0 \to E$ is affine. If Sd_{n-1} preserves affineness, then note that Sd_n must as well. After all, we know that $\operatorname{Sd}_n \sigma = b_n . \operatorname{Sd}_{n-1}(\partial\sigma)$ is the cone of some point $b_n \in E$ and the affine function $\operatorname{Sd}_{n-1}(\partial\sigma)$. (Note that this last function is affine because $\partial\sigma$ is, by Exercise 6.9, affine.) The result follows.

More Applications to Euclidean Space

Exercise 6.11. Writing $(1 + a^n_\#)\gamma$ as $\gamma + a^n_\# \gamma$, note that

$$\begin{aligned} \partial(\gamma + a^n_\# \gamma) &= \partial\gamma + a^{n-1}_\# \partial\gamma \\ &= \gamma(e_1) - \gamma(e_0) + (-\gamma(e_1)) - (-\gamma(e_0)). \end{aligned}$$

But recall that $-\gamma(e_1) = \gamma(e_0)$ and $-\gamma(e_0) = \gamma(e_1)$, and so we know that the terms cancel out to 0. Thus $(1 + a^n_\#)\gamma$ is a 1-cycle.

Exercise 6.12. We can simply compute evaluate $(1 + a^n_\#)(1 - a^n_\#)$ on a simplex σ. In particular, we find that

$$\begin{aligned} (1 + a^n_\#)(1 - a^n_\#)\sigma &= (1 + a^n_\#)(\sigma - a^n\sigma) \\ &= \sigma + a^n_\#\sigma - a^n\sigma - a^n_\#(a^n\sigma) \\ &= \sigma + a^n\sigma - a^n\sigma - a^n a^n \sigma. \end{aligned}$$

But of course we have $\sigma = a^n a^n \sigma$, and so the terms all cancel out. We can simply extend to any 1-chain γ.

Exercise 6.13. Again, we evaluate the expression on a simplex σ and find that

$$\begin{aligned} (1 + a^n_\#)(1 + a^n_\#)\sigma &= (1 + a^n_\#)(\sigma + a^n\sigma) \\ &= \sigma + a^n\sigma + a^n\sigma + a^n a^n \sigma \\ &= 2\sigma + 2a^n\sigma = 2(1 + a^n_\#)\sigma. \end{aligned}$$

As before, we can extend.

Exercise 6.14. Letting τ be, as in Theorem 6.22, the southerly path in S^1 from $a^1(y)$ to y, recall that the homology class of the cycle $\sigma + \tau$ generates all of $H_1(S^1)$. But notice that $[(1 + a^1_{\#})\sigma] = [\sigma + a^1\sigma] = [\sigma + \tau]$, which proves the result.

Exercise 6.15. Suppose $f : S^1 \to \mathbb{R}$ is continuous. Note that such a function is effectively a continuous function $f : [0, 1] \to \mathbb{R}$ with $f(0) = f(1)$, and so the intermediate value theorem implies the result.

Exercise 6.16. Suppose $S \subseteq \mathbb{R}^2$ is homeomorphic to S^2. Then there is a function $\varphi : S^2 \to S \hookrightarrow \mathbb{R}^2$, where the $S^2 \to S$ part is a homeomorphism. The Borsuk-Ulam theorem, however, implies that there exists some point $x \in S^2$ with $\varphi(x) = \varphi(-x)$. But φ is an injection, a contradiction.

Exercise 6.17. This is obvious from Borsuk-Ulam. Since there exists an x with $f(x) = f(-x)$, but $f(-x) = -f(x)$ for all x, it follows that there exists an x with $f(x) = -f(x)$, i.e., with $f(x) = 0$.

Exercise 6.18. This follows the same proof as that of Borsuk-Ulam. In particular, we use the function

$$\begin{aligned} g(x) : S^n &\to S^{n-1} \\ x &\mapsto \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}. \end{aligned}$$

This would be an antipodal map, a contradiction.

Exercise 6.19. Suppose $a^n(F_i) \cap F_i = \emptyset$ for $i = 1, \ldots, n$. There exist functions $g_i : S^n \to I$ with $g_i(F_i) = 0$ and $g_i(a^n F_i) = 1$. Then define $f : S^n \to \mathbb{R}^n$ by $f(x) = (g_1(x), \ldots, g_n(x))$. Note that Exercise 6.18 implies that there exists some $x_0 \in S^n$ with $f(x_0) = f(-x_0)$. Thus it follows that

$$g_i(x_0) = g_i(-x_0) = g_i(a^n x_0)$$

for all i. Hence if $x_0 \in F_i$ then the left side of the equation is 0 while the right side is 1, and if $x_0 \in a^n F_i$ then the left side is 1 while the right side is 0. Either way, this is a contradiction, and so it follows that $x_0, a^n x_0 \notin F_i$ for all i. Thus $x_0 a^n x_0 \in F_{n+1}$.

I have not come up with a counterexample in the $n + 2$ case, unfortunately.

Exercise 6.20. Suppose $A \subseteq S^n$ is a subspace, and suppose that $h : S^n \to A$ is a homeomorphism. Invariance of domain implies that A is open in S^n. But we also know by compactness of S^n that A must be compact, and hence closed in its ambient space. Thus A is clopen. Since A is obviously nonempty, it follows by connectedness that $A = S^n$.

Exercise 6.21. This follows because we can just write $S^n = \mathbb{R}^n \cup \{\infty\}$. Hence any open set in $\mathbb{R}^n$, including $\mathbb{R}^n$ itself, is just an open set in S^n.

Walking through this in more detail, suppose $U, V \subseteq \mathbb{R}^n$ with a homeomorphism $h : U \to V$ and with U open. Then U is an open subset of S^n because $\mathbb{R}^n$ is open in S^n. Hence invariance of domain on S^n implies that V is open in S^n, and so $V \cap \mathbb{R}^n = V$ is open in $\mathbb{R}^n$ as well.

Exercise 6.22. Suppose $\varphi : X \to Y$ is a homeomorphism. Then we can simply pass to a homeomorphism between U and V in X to a homeomorphism between $\varphi(U)$ and $\varphi(V)$ in Y. Since every open set in Y is of the form $\varphi(U)$ for some open U in X, the result follows immediately.

Exercise 6.23. Consider the map $h : D^n \to \overline{D_{\frac{1}{2}}(0)}$ defined by $h(x) = \frac{x}{2}$. It effectively shrinks $D^n n$ down to the closed ball with radius $\frac{1}{2}$. Obviously the two disks are homeomorphic. But D^n is open while $\overline{D_{\frac{1}{2}}(0)}$ is not.