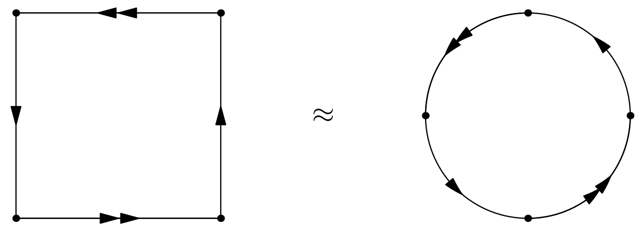
7 Simplicial Complexes

Definitions

Exercise 7.1.



Exercise 7.2. Consider some (nondegenerate) triangle with vertices P, x_0, y_0 in $\mathbb{R}^2$. Then define x_i to be the midpoint of P and x_{i-1}, and similarly define y_i. Then the union X of the triangle with all the line segments $x_i y_i$ is compact and connected.

We claim that it is not a polyhedron. Otherwise, there exists some simplicial complex K admitting a homeomorphism $h : |K| \to X$. But observe that K must have an infinite vertex set.

To see this, for each i, define s_i to be

$$s_i = \bigcap_{h^{-1}(x_i) \in s} s,$$

where s ranges over all simplices of K. Note that this intersection is over a nonempty set because $\bigcup s = |K|$, so there must exist some s containing $h^{-1}(x_i)$. Moreover, there are only finitely many simplices, so the intersection exists. Condition (ii) implies that s_i is a common face of s, and thus is a simplex. It must be 0-dimensional since the segment Px_i, $x_i y_i$, and $x_0 x_i$ cannot all be part of the same 1-simplex. In other words, x_i must be a common face of two 1-simplices, and so it must be a point.

Hence there are infinitely many vertices of K, a contradiction.

Exercise 7.3. Note that the upper right and lower right triangles are the same.

Exercise 7.4.

(i) The forwards direction is just the definition of the subspace topology. To see the backwards direction, suppose $F \cap s$ is closed in s for every $s \in K$. Each s is closed in $|K|$, so $F \cap s$ is closed in $|K|$. Since there are finitely many s and $\bigcup s = |K|$, it follows that we can take the union of all $F \cap s$. In particular, we have

$$F = \bigcup_{s \in K} (F \cap s)$$

is the finite union of closed sets, hence is itself closed in $|K|$.

(ii) This is obviously true if K has dimension 0.

If K (and hence s) has dimension > 1, then consider the complement of s°:

$$(s^\circ)^c = (|K| - s) \cup \dot{s}.$$

Then notice that

$$\left[(|K| - s) \cup \dot{s}\right] \cap s = \dot{s},$$

which is closed in s. Suppose $t \in K$ is not equal to s. Then consider

$$A_t = \left[(|K| - s) \cup \dot{s}\right] \cap t.$$

If $s \cap t = \emptyset$, then $A_t = \emptyset$ is closed in t. Otherwise, we know that $s \cap t$ is a face of t. Since s is of highest dimension, we know that either $s = t$, which we already took care of above, or $s \cap t$ is part of $\dot{s}$, in which case we know that

$$\dot{s} \cap t = s \cap t, \quad (|K| - s) \cap t = t - s \cap t.$$

Hence $A_t = t$, which is still closed in t.

The previous part proves the result.

Exercise 7.5. We begin by showing $s^\circ \cap t^\circ = \emptyset$ when $s \neq t$. Note that

$$s^\circ \cap t^\circ = (s - \dot{s}) \cap (t - \dot{t}) = s \cap t - \dot{s} \cap t - s \cap \dot{t}.$$

But $s \cap t$ is a face of both s and t. It can't be equal to both s and t since $s \neq t$. Thus $s \cap t$ is a *proper* face of at least one of s and t, say s. This means that $s \cap t$ is part of $\dot{s}$, and thus is in $\dot{s} \cap t$. This proves disjointness.

To see that $\bigcup s^\circ = |K|$, simply do this in the case of K as a simplex, and take unions. (To do this when K is a single simplex, use induction.)

Exercise 7.6. The backwards direction is obvious by the definition of st. For the forwards direction, suppose

$$x \in \mathrm{st}(p) = \bigcup_{p \in \mathrm{Vert}(t)} t^\circ.$$

Then we know that $x \in t^\circ$ for some t having p as a vertex. Uniqueness implies that $s = t$, so $p \in \mathrm{Vert}(s)$.

Exercise 7.7.

(i) Obviously the union is $|K|$ because every $s \in K$ has at least one vertex, hence is contained in at least one star. To see that $\mathrm{st}(p) \subseteq |K|$ is open, notice that

$$(\mathrm{st}(p))^c = \bigcup_{p \notin \mathrm{Vert}} s^\circ.$$

Intersect this with $t \in K$. If $p \notin \mathrm{Vert}(t)$, then this intersection is equal to t since no simplex of $\dot{t}$ can have p as a vertex. If $p \in \mathrm{Vert}(t)$, then write $t = [p, p_1, \ldots, p_k]$. The intersection can be seen to simply be $\{p_1, \ldots, p_k\}$, which is obviously closed. Thus Exercise 7.4 implies the result.

(ii) If $x \in \mathrm{st}(p)$, then $x \in s^\circ$ for some s with $p \in \mathrm{Vert}(s)$. Since $x, p \in s$ and s is convex, it follows that the line segment is also contained in $\mathrm{st}(p)$.

Exercise 7.8. The forwards direction is because $[p_0, \ldots, p_n]$ is in the intersection. The backwards direction is because there must exist some simplex $[p_0, \ldots, p_n, q_0, \ldots, q_m] \in K$. Since any face of a simplex in K is also in K, it follows that $[p_0, \ldots, p_n]$ is a simplex in K.

Simplicial Approximation

Exercise 7.9. In the forwards direction, suppose φ is a simplicial map. If $\bigcap \mathrm{st}(p_i) \neq \emptyset$, then there exists a simplex in K with vertices $[p_i]$. The definition implies that there must exist a simplex with vertices $[\varphi(p_i)]$, proving this direction. The backwards direction follows directly from Exercise 7.8.

Exercise 7.10. Suppose φ is a simplicial approximation to f, and suppose $x \in |K|$ with $f(x) \in s^\circ$. Write $x \in t^\circ$ for $t \in K$, and write $t = [p_1, \ldots, p_n]$. Then we know that $x \in \mathrm{st}(p_i)$ implies that $f(x) \in \mathrm{st}(\varphi(p_i))$, so that $s^\circ \subseteq \mathrm{st}(\varphi(p_i))$. Thus s has $\varphi(p_i)$ as a vertex for each $i = 1, \ldots, n$.

Hence $|\varphi|(x)$, which is determined by $\varphi(p_i)$, is in s by affineness.

Now suppose that $f(x) \in s^\circ$ implies $|\varphi|(x) \in s$. Let p be some vertex of K so that $x \in \mathrm{st}(p)$. Then $f(x) \in s^\circ$, so $|\varphi|(x) \in s$. Hence $\varphi(p)$ is a vertex of s by affineness and the definition of $|\varphi|$, from which it follows that

$$f(x) \in s^\circ \subseteq \mathrm{st}(\varphi(p)).$$

We can take the union over all $x \in \mathrm{st}(p)$:

$$\bigcup_{x \in \mathrm{st}(p)} f(x) \subseteq \mathrm{st}(\varphi(p)).$$

Of course, this left side is exactly $f(\mathrm{st}(p))$, and so we're done.

Exercise 7.11. Suppose $\varphi : K \to L$ is a simplicial approximation. Consider the obvious homotopy:

$$H(t, x) = (1 - t)|\varphi|(x) + tf(x).$$

We can do this because $|\varphi|(x)$ and $f(x)$ are, by Exercise 7.10, in the same simplex.

Exercise 7.12.

(i) This is true because it's true for simplices.

(ii) Order the vertices of K, and define $\varphi(b^s)$ to be the smallest vertex of s under this order. We claim that this gives a simplicial approximation to the identity. Consider a vertex b^s of $\mathrm{Sd}(K)$. Then we know that

$$f(\mathrm{st}(b^s)) = s^\circ \subseteq \mathrm{st}(\varphi(b^s))$$

by the definition of $\varphi(b^s)$, where f is the identity.

(iii) There exists a homeomorphism $g : |L| \to X$. If $g(v)$, then we are done. Otherwise, we know that $x \in g(s^\circ)$ for some unique $s \in L$. Consider the subdivision K of L obtained by drawing lines from s to every vertex of s. This gives a function $h : |K| \to X$ which is equal to g, and thus is a homeomorphism, as desired.

Exercise 7.13. Suppose that $\sum \lambda_i b^{s_i} = 0$. Since $s_0 < \cdots < s_q$, we knwo that there exists some vertex p_q which only appears in b^{s_q}, so $\lambda_q = 0$. But then there is a vertex p_{q-1} which only appears in $b^{s_{q-1}}$, so $\lambda_{q-1} = 0$, and so on. Thus $\lambda_i = 0$ for all i, proving independence.

Exercise 7.14. Every point of $\mathrm{Sd}\, K$ is contained in a unique open simplex of K, so it follows that an open simplex of $\mathrm{Sd}\, K$ can be contained in at most one open simplex of K. To see that there is at least one such simplex, note that $[b^{s_0}, \ldots, b^{s_q}]^\circ$ is contained in s_q°.

Exercise 7.15. This follows from the triangle inequality:

$$|x - y| \leq |x - p| + |p - y| \leq 2\mu,$$

because x and p are in one simplex, and y and p are in another.

Exercise 7.16. Write $s = [b^{s_0}, \ldots, b^{s_q}]$, where $s_0 < \cdots < s_q$. Then $\operatorname{diam} s = \sup ||b^{s_i} - b^{s_j}||$. If $i < j$, then we know that

$$||b^{s_i} - b^{s_j}|| \leq \frac{n_j}{n_j + 1} \operatorname{diam} s_j,$$

where $n_j = \dim s_j$. But $\operatorname{diam} s_j \leq \operatorname{mesh} K$ since $s_j \in K$, and $\frac{n_j}{n_j+1} \leq \frac{n}{n+1}$, since $n_j \leq n$. Hence it follows that

$$\operatorname{diam} s \leq \frac{n}{n+1} \operatorname{mesh} K,$$

and so $\operatorname{mesh} \mathrm{Sd}\, K \leq (n/n+1) \operatorname{mesh} K$. Induction implies the general result.

Exercise 7.17. If $s \in K^{(q)}$, then $s = [p_0, \ldots, p_r]$ for some $r \leq q$. Thus $\varphi(s) = [\varphi(p_0), \ldots, \varphi(p_r)] \in L^{(q)}$, as desired.

Exercise 7.18. Let b be the barycenter of the $(n+1)$-simplex, and consider

$$f(x) = \frac{x - b}{||x - b||} + b.$$

This is the desired homeomorphism.

Abstract Simplicial Complexes

No exercises!

Simplicial Homology

Exercise 7.19. In general, there are $\binom{n+2}{q+1}$ total q-simplices in an $(n+1)$-simplex. Since S^n is the n-skeleton of such a simplex, it follows that we must simply evaluate

$$\chi(S^n) = \sum_{q=0}^{n} \binom{n+2}{q+1}(-1)^q = \sum_{q=0}^{n+2} \binom{n+2}{q}(-1)^{q+1} + \binom{n+2}{0} + \binom{n+2}{n+2} = 2$$

when q is even. When q is odd, the last term is negative, and we find that $\chi(S^n) = 0$.

Exercise 7.20. Here, we have $\alpha_2 = 18$, $\alpha_1 = 27$, and $\alpha_0 = 9$. Thus $\chi(T) = 18 - 27 + 9 = 0$.

Exercise 7.21. Note that i is obviously an injection. Moreover, since the element $\sum b \in B_1 m_b b + \sum_{c \in B_2} m_c c \in F(b)$ is equal to

$$\sum b \in B_1 m_b b + \sum_{c \in B_2} m_c c \in F(b) = p\left(\sum m_b b, \sum -m_c c\right),$$

we see that p is surjective. Finally, note that

$$\begin{aligned} \ker p &= \left\{\left(\sum m_b b, \sum m_c c\right) : \sum m_b b = \sum m_c c\right\} \\ &= \{(x, x) : x \in F(B_1) \cap F(B_2)\} \\ &= \{(x, x) : x \in F(B_1 \cap B_2)\} = \operatorname{im} i, \end{aligned}$$

which completes the proof of exactness.

Comparison with Singular Homology

Exercise 7.22. For $q \geq 1$, the complexes are the same. If $q = 0$, we use the same argument as in Theorem 5.17, in particular, by restricting our attention to the ending:

$$0 \longrightarrow \ker \tilde{\partial}_0 \hookrightarrow C_0(K) \xrightarrow{\tilde{\partial}_0} C_{-1}(K) \longrightarrow 0 \ .$$

Exercise 7.23. This is simply because $\ker \tilde{\partial}_{-1} = C_{-1}(K)$.

Exercise 7.24.

(i) We can simply use the straight line homotopy between $\varphi(p)$ and $\psi(p)$ for all vertices p of K; the rest of the point follow by affineness. The reason this works is simply because $\varphi(p)$ and $\psi(p)$ belong to the same simplex, which is convex.

(ii) Since $|\varphi| \simeq |\psi$, we know that $|\varphi|_* = |\psi|_*$, which in turn implies that $\varphi_* = \psi_*$ by Theorem 7.22.

Exercise 7.25. Let L be a line segment, along with its endpoints and its midpoints. Thus it is composed of two 1-simplices, and three 0-simplices. Then let φ_1 map a 1-simplex to the left side of L, and φ_2 map it to the right side of L. Finally, if φ_3 maps the 1-simplex to the midpoint, it follows that $\varphi_1 \sim \varphi_3 \sim \varphi_2$, but obviously $\varphi_1 \not\sim \varphi_2$.

Exercise 7.26.

(i) This is clear by mapping the base points together, and mapping a given equivalence class to the corresponding equivalence class. For example, we have some point $x \in X$, then the homeomorphism would take $[[x]] \in (X \vee Y) \vee Z$ to $[x] \in X \vee (Y \vee Z)$. Similarly, it would take $[[y]] \mapsto [[y]]$ and $[z] \mapsto [[z]]$.

(ii) For $i = 1, 2$, there exists a simplicial complex L_i and a homeomorphism $h_i : |L_i| \to K_i$. Fix some vertex $x_i \in \operatorname{Vert}(L_i)$. Then let $L = L_1 \vee L_2$. Then, identifying each L_i with the natural corresponding set in L, we can apply Theorem 7.17 to find the exact senuence

$$\ldots \longrightarrow H_n(L_1 \cap L_2) \longrightarrow H_n(L_1) \oplus H_n(L_2) \longrightarrow H_n(L) \longrightarrow H_{n-1}(L_1 \cap L_2) \longrightarrow \ldots .$$

Of course, we have $L_1 \cap L_2$ is a singleton, so the homology groups are 0. Thus, if $n \geq 2$, then we know that $H_n(L_1 \cap L_2) = H_{n-1}(L_1 \cap L_2) = 0$, and so $H_n(L) \cong H_n(L_1) \oplus H_n(L_2)$, as desired. Otherwise, we can simply use the tail:

$$\ldots \longrightarrow H_1(L) \longrightarrow H_0(L_1 \cap L_2) \longrightarrow H_0(L_1) \oplus H_0(L_2) \longrightarrow H_0(L) \longrightarrow 0.$$

If L_i has c_i components, then notice that L has $c_1 + c_2 - 1$ components. Since the map $H_0(L_1) \oplus H_0(L_2) \to H_0(L)$ is surjective, it follows that its kernel is $\mathbb{Z}$ (or, more accurately, a free abelian group of rank 1). Hence the image of $H_0(L_1 \cap L_2) \to H_0(L_1) \oplus H_0(L_2)$ is $\mathbb{Z}$. The fact that $H_0(L_1 \cap L_2) = \mathbb{Z}$ implies that this map is an isomorphism, thus with empty kernel. Finally, we conclude that the image of $H_1(L) \to H_0(L_1 \cap L_2)$ is trivial, and so we again have the exact sequence

$$0 H_1(L_1) \oplus H_1(L_2) \longrightarrow H_1(L) \longrightarrow 0\ .$$

The result follows.

(iii) Use Corollary 7.19. In particular, let K_q consist of all proper faces of an oriented $(q+1)$-simplex. Then the corollary implies that $H_q(K_q) = \tilde{H}_q(K_q) = \mathbb{Z}$ and $H_r(K_q) = 0$ for any $r \neq q$. (Note that reduced homology matches the regular homology since $q \geq 1$.) Thus the previous part shows that the space

$$\bigvee_{q=1}^{n} \bigvee_{i=1}^{m_q} K_q,$$

where the wedge occurs at some identified vertex, satisfies the desired properties.

Exercise 7.27.

(i) This follows directly from the five lemma and Theorem 7.22, namely by looking at the following commutative diagram with exact rows:

$$\begin{array}{ccccccccccc} \ldots \longrightarrow & H_n(L) & \longrightarrow & H_n(K) & \longrightarrow & H_n(K, L) & \longrightarrow & H_{n-1}(L) & \longrightarrow & H_{n-1}(K) & \longrightarrow \ldots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \ldots \longrightarrow & H_n(|L|) & \longrightarrow & H_n(|K|) & \longrightarrow & H_n(|K|, |L|) & \longrightarrow & H_{n-1}(|L|) & \longrightarrow & H_{n-1}(|K|) & \longrightarrow \ldots \end{array}.$$

(ii) This follows from the previous part, Corollary 7.17, and Theorem 7.22.

Exercise 7.28. We can simply use the straight line homotopy to p. Exercise 7.7 implies that this is well-defined.

Exercise 7.29. In particular, we must show that

$$\left(\bigcap L_{\alpha_i}\right) \cap \left(\bigcap L_{\beta_i}\right) \neq \emptyset.$$

But notice that $\sigma_0 < \sigma_1 < \cdots < \sigma_q$ implies that $\sigma_0 \in L_{\beta_i}$ for each β_i. We also know that $\sigma_0 \in L_{\alpha_0} \cap \cdots \cap L_{\alpha_q}$, and so it follows that σ_0 is in the displayed intersection above. Hence g and f are contiguous.

Exercise 7.30. We have the following exact sequence:

$$H_q(M \cap L_1) \longrightarrow H_q(M) \oplus H_q(L_1) \longrightarrow H_q(M \cup L_1) \longrightarrow H_{q-1}(M \cap L_1).$$

The conditions imply that $H_q(M) \oplus H_q(L_1) = H_q(M)$ and the two outermost terms are both trivial. Thus $H_q(M) \cong H_q(M \cup L_1)$.

Now consider the following exact sequence:

$$H_q((M \cup L_1) \cap L_2) \longrightarrow H_q(M \cup L_1) \oplus H_q(L_2) \longrightarrow H_q(M \cup L_1 \cup L_2) \longrightarrow H_{q-1}((M \cup L_1) \cap L_2).$$

But notice that

$$(M \cup L_1) \cap L_2 = (M \cap L_2) \cup (L_1 \cap L_2) = M \cap L_2$$

since $L_1 \cap L_2 \subseteq M$. Hence the flanking terms of the exact sequence displayed above are again 0. Since L_2 is acyclic, it follows that $H_q(M \cup L_1) \cong H_q(M \cup L_1 \cup L_2)$. Repeating this proves the result.

Calculations

Exercise 7.31. Consider the Klein bottle, as in Figure 1. Let P be the entire square. Then we can define

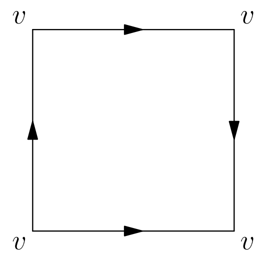


Figure 1: The Klein bottle

the adequate subcomplex with chains

$$E_2 = \langle P \rangle, \quad E_1 = \langle a \rangle \oplus \langle b \rangle, \quad E_2 = \langle v \rangle.$$

We have

$$\begin{aligned} \partial P &= a + b + a - b \\ \partial a = \partial b &= 0 \\ \partial v &= 0. \end{aligned}$$

Hence it follows that we have

$$\begin{array}{lll} Z_2 = 0, & Z_1 = \langle a \rangle \oplus \langle b \rangle, & Z_0 = \langle v \rangle, \\ B_2 = 0, & B_1 = \langle 2a \rangle, & B_0 = 0. \end{array}$$

The results are obvious.

Exercise 7.32. This time, if we let a denote each edge and v denote each vertex, we have

$$\partial P = ka, \quad \partial a = 0, \quad \partial v = 0.$$

Thus we now have

$$\begin{array}{lll} Z_2 = 0, & Z_1 = \langle a \rangle \oplus \langle b \rangle, & Z_0 = \langle v \rangle, \\ B_2 = 0, & B_1 = \langle ka \rangle, & B_0 = 0. \end{array}$$

This gives the desired homology groups.

Fundamental Groups of Polyhedra

Exercise 7.33. This is true because equality is an equivalence relation.

Exercise 7.34.

(i) It suffices to show that $o(()\alpha)$ cannot be changed in a single move. But this is clear. In particular, using the definition, note that $o(()\alpha)$ is $o(()\beta)$ if $\beta \neq \emptyset$, and is p if β is empty. The same holds for $o(()\alpha')$, so $o(()\alpha)$ is preserved. Similarly, $\mathrm{e}(()\alpha) = \mathrm{e}(()\alpha')$.

(ii) Again, it suffices to show this for a single elementary move. We can further assume that $\beta = \beta'$. Write $\alpha = \gamma(p,q)(q,r)\delta$ and $\alpha' = \gamma(p,r)\delta$. Then

$$\alpha\beta = \gamma(p,q)(q,r)\delta\beta = \gamma(p,r)\delta\beta = \alpha'\beta'.$$

(Recall $\beta = \beta'$.)

Exercise 7.35. An edge path, by definition, only goes along the 1-skeleton. Thus K being connected automatically implies that $K^{(1)}$ is.

If $K^{(1)}$ is connected, then let $x, y \in |K|$. There are unique open simplices s°, t° with $x \in s^\circ$ and $y \in t^\circ$. Pick vertices v and w of s and t, respectively. Then consider the path taken by going straight line from x to v, then along the edges to w, then along a straight to y. Hence $|K|$ is connected (indeed, path-connected).

If $|K|$ is connected, then $|K|$ is clearly path-connected.

Finally, if $|K|$ is path-connected, then we can find edge paths between any two vertices of K in the following manner: Each time the path crosses the 1-skeleton, say along the edge between v and w, pick either v and w and append that vertex (or, rather, the edge between that vertex and the previous one) to the edge path. That this works is clear.

Exercise 7.36. This is exactly the proof of Theorem 3.6, with γ as the edge path from p_0 to p_1.

Exercise 7.37. Since an elementary move only moves across a 2-simplex, it follows that the edge path group is only dependent on the 2-skeleton.

Exercise 7.38.

(i) This is clear.

(ii) If v and w are in the same component as some point x, then by taking an edge path from v to x, then from x to w, we have an edge path between v and w. This proves that components are connected.

Obviously the union of the components is K. To see that the unions are disjoint, suppose $v \in [x] \cap [y]$ and $w \in [x]$. Then the path $w \to x \to v \to y$ implies that $w \in [y]$. Since w was arbitrary, and since $w \in [y]$ would similarly imply $w \in [x]$, it follows that $[x] = [y]$. This proves disjointness.

(iii) Suppose $[\alpha] \in \pi(K, x)$. Then we claim that $[\alpha] \in \pi(L, x)$. But this is simply because any vertex along α is necessarily connected to p via an edge path, hence belongs to L.

Exercise 7.39.

(i) Write $\alpha = e_1 \dots e_m$ and $\beta = e_{m+1} \dots e_{m+n}$. Then $(\alpha\beta)^\circ : I_{m+n} \to K$ takes v_i to p_i, where $p_i = \alpha^\circ(v_i)$ for $0 \leq i \leq m$ and $p_i = \beta^\circ(v_{i-m-1})$ otherwise. This is exactly γ.

(ii) It suffices to show this if α and β are separated by one step. But, writing $\alpha = \gamma(p,q)(q,r)\delta = \gamma(p,r)\delta = \beta$, simply note that we can use the straight line homotopy from the center of (p,r) to go to q. Resizing intervals as necessary, as in the previous part, gives the result.

Exercise 7.40. It suffices to show that trees are contractible. This is true for zero or one 1-simplices. For $(n+1)$ total 1-simplices, simply pick an edge one of whose endpoints is a leaf. Then we can contract that edge to the other vertex, which is connected to the rest of the tree. Induction implies the result.

Exercise 7.41. Suppose $e_1 \dots e_n$ were a circuit in $T_1 \cup T_2$. Suppose without loss of generality that $e_1 \in T_1$. Let i and j be the first and last indices, respectively, such that $e_i, e_j \in T_1 \cap T_2$. There is a path α which starts with e_i and ends with e_j contained in $T_1 \cap T_2$. Now notice that $e_1 \dots e_{i-1}\alpha e_{j+1} \dots e_n$ is a circuit contained entirely within T_1, contradicting that T_1 is a tree.

Exercise 7.42. Let G be any abelian group, and let $\varphi : \{xF' : x \to X\} \to G$. Our goal is to show that there is a unique homomorphism $\psi : F/F' \to G$ with $\psi(xF') = \varphi(xF')$ for all $xF' \in F/F'$. (See Theorem 4.1(i).)

As in the definition of a free group, let $\tilde{\varphi}$ be the unique homomorphism from F to G with $\tilde{\varphi}(x) = \varphi(xF')$ for all $x \in X$. Now define

$$\begin{aligned} \psi : F/F' &\to G \\ fF' &\mapsto \tilde{\varphi}(f). \end{aligned}$$

To see that this is well-defined, notice that $f \in F'$ implies that $f = g^{-1}h^{-1}gh$ for some $g, h \in F$. Thus

$$\tilde{\varphi}(f) = \tilde{\varphi}(g)^{-1}\tilde{\varphi}(h)^{-1}\tilde{\varphi}(g)\tilde{\varphi}(h).$$

But G is abelian, so this is exactly 1, which proves well-definedness.

To see that ψ does indeed satisfy that $\psi(xF') = \varphi(xF')$, simply notice that $\psi(xF') = \tilde{\varphi}(x)$, which is defined to be $\varphi(xF')$.

Finally, to see that ψ is the *unique* homomorphism with this property, note that any other function ψ' would have to have $\psi'(fF') = \tilde{\varphi}(f)$, and thus be exactly equal to ψ.

Hence F/F' is indeed free abelian, with the desired basis.

Exercise 7.43. Exercise 7.42 shows that the rank of the free group F is the rank of the free abelian group F/F'. But this latter rank is invariant with respect to X.

Exercise 7.44.

(i) By picking a maximal tree T, and setting some edge not in the tree to be x, we can see that every other edge becomes either x, x^{-1}, or 1. Hence $G_{\mathbb{R}P^2,T} \cong \mathbb{Z}/2\mathbb{Z}$, and Corollary 7.37 implies the result.

(ii) Hurewicz's theorem applies since $\mathbb{R}P^2$ is obviously path-connected. Moreover, since $\mathbb{Z}/2\mathbb{Z}$ is abelian, its commutator subgroup is trivial. Thus $H_1(\mathbb{R}P^2) \cong \mathbb{Z}/2\mathbb{Z}$, as desired.

Exercise 7.45.

(i) Pick points $x, y \in X$. Then consider vertices p and q of the simplices containing x and y, respectively. Consider the following path: Take the straight line from x to p, then take the path mapped out by $F(p,t)$ as $t \in \mathbb{I}$, then the path mapped out by $F(q, 1-t)$, and finally the straight line from q to y.

(ii) Let $F : X \times \mathbb{I} \to X$ have $F(v, 0)$ for all $v \in X^{(1)}$ and $F(\ ,1)$ a constant function. Then by taking the homotopy along F, we can go from (p,q) to the constant point, then back to some arbitrary edge of T, where T is a maximal tree of X. Hence $(p,q) = 1$, implying a trivial edge path group. Thus the fundamental group is trivial too.

Exercise 7.46. Since there are n vertices, we know that there are $n-1$ edges of a maximal tree. The result follows from Corollary 7.35.

Exercise 7.47. If X has m edges and n vertices, then $\chi(X) = -m + n$. Thus $1 - \chi(X) = m - n + 1$. Now use Hurewicz's theorem, Exercise 7.35, Exercise 7.42, and Corollary 7.35 to find the result for H_1. Note that $H_0(X) = \mathbb{Z}$ because X is connected, and $H_q(X) = 0$ for $q \geq 2$ because X has dimension 1.

Exercise 7.48. We know that S^m is the boundary of an $(m+1)$-simplex. Thus there is an edge between any two vertices, so we can fix one vertex p and let T be the star consisting of all edges (p,q). Now consider any other edge (q,r). Note that $\{p,q,r\}$ forms a simplex, so $(p,q)(q,r) = (p,r)$. But in $G_{K,T}$, we know that $(p,q) = (p,r) = 1$, so $(q,r) = 1$ as well. Thus $\pi(K,p) \cong G_{K,T} = 1$, and so $\pi_1(S^m) = 1$. Hence S^m is simply connected.

Exercise 7.49.

(i) Since every vertex is contained in $K^{(q)}$, we can pick any simplex of maximal dimension. Its vertices are contained in $\text{Vert}(K^{(q)})$, but it does not itself belong in the q-skeleton.

(ii) If a full subcomplex L exists, we know that it would need to include every simplex of K with vertices in A. Moreover, adding any other simplex would introduce new vertices. Thus such a subcomplex would be unique. Note that the set thus described is indeed a subcomplex, since any faces of $s \in L$ would have to have vertices in A as well.

The second part of the statement follows from the description of L.

Exercise 7.50. Consider some element $[\alpha] = \in \pi(K, v_0)$. Then there is some path $\alpha' \simeq \alpha$ with $\alpha' \in \pi(L, v_0)$. Thus $i[\alpha'] = [\alpha]$, proving surjectivity.

If K is the 2-simplex and L is its boundary, then obviously any closed edge path in K is also in L (and, in particular, is homotopic to a closed edge path in L). But the fact that K is simply connected while L is not implies that there cannot be an isomorphism.

The Seifert–van Kampen Theorem

No exercises!