3 The Fundamental Group

The Fundamental Groupoid

Exercise 3.1. The homotopy $H : X \times \mathbb{I} \to Z$ given by

$$H : (x,t) \mapsto \begin{cases} g_0(F(x,2t)) & \text{if } t \le \frac{1}{2}, \\ G(f_1(x), 2t-1) & \text{if } t \ge \frac{1}{2} \end{cases}$$

works. Continuity follows because $g_0(F(x,1)) = G(f_1(x), 0)$.

Moreover, this homotopy is indeed $\operatorname{rel} A$. For a detailed argument why this is so, simply suppose that $a \in A$ and $t \in I$. If $t \le \frac{1}{2}$, then $F(a,2t) = f_0(a)$ by definition of F. Hence $H(a,t) = g_0(f_0(a))$.

Similarly, we can show that if $t \ge \frac{1}{2}$, then $H(a,t) = g_1(f_1(a))$. This follows because $f_1(a) \in B$ and G is a homotopy $\operatorname{rel} B$.

It thus suffices to show that $g_0(f_0(a)) = g_1(f_1(a))$. But this is obvious because f_0 and f_1 agree on A, and g_0 and g_1 agree on $B \supseteq f_0(A)$.

Exercise 3.2.

(i) First, note that f' is well-defined because $f(0) = f(1)$. It is obvious by continuity of f and ln that f' is continuous.

Moreover, consider the map

$$H' : \left(e^{2\pi i\theta}, t\right) \mapsto H(\theta, t).$$

This is clearly continuous, for the same reasons that f' was continuous. If $t = 0$, clearly $H'(e^{2\pi i\theta}, t) = H(\theta, 0) = f(\theta) = f'(e^{2\pi i\theta})$, and similarly for $t = 1$. Thus H is indeed a homotopy from f' to g'.

To see that it is a homotopy $\operatorname{rel}\{1\}$, simply note that $e^{2\pi i\theta} = 1$ corresponds to $\theta = 0, 1$. Thus it follows that

$$H'(1,t) = H(1,t) = f(1)$$

for all t, proving the result.

(ii) Theorem 3.1 implies that $f * g \simeq f_1 * g_1 \operatorname{rel} \dot{\mathbb{I}}$. Using the previous part, we find that $(f*g)' \simeq (f_1 * g_1)' \operatorname{rel}\{1\}$. Now, using the observation that $(f*g)' = f' * g'$, we find that $f' * g' \simeq f_1' * g_1' \operatorname{rel}\{1\}$, as desired.

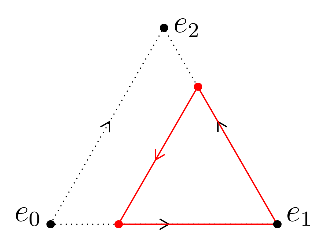
Exercise 3.3. The forward direction is trivial.

For the converse, note that g' is a constant map, and so f' is nullhomotopic. Then Theorem 1.6 implies that $f' \simeq g' \operatorname{rel}\{1\}$. In particular, note that $g' : S^1 \to X$ takes every element of S^1 to $g'(1) = g(0) = x_0$. Observe that $f'(1) = x_0$ as well, and so it follows that $f' \simeq g \operatorname{rel}\{1\}$, as desired.

Exercise 3.4.

(i) Instead of applying Theorem 1.6, I constructed an explicit homotopy. (If you are interested in a proof using Theorem 1.6, my guess would be that it relies on the fact that $\Delta^2 \approx D^2$. However, I have not gone through the details.)

The effective idea of the homotopy I constructed is to, at time $t \in [0,1]$, return the function which traverses the first t units of the face opposite e_0, then goes along a segment to the point t units away from e_1 on the fact opposite e_2, before returning back to e_1, as shown in the red path below.



The specific homotopy $H : \mathbb{I}\times\mathbb{I}\to X$ from $(\sigma_0 * \sigma_1^{-1}) * \sigma_2$ to the constant map at e_1 is as follows:

$$H(x,t) = \begin{cases} \sigma_0(4(1-t)x) & \text{if } x \le \frac{1}{4}, \\ \sigma((1-x)\varepsilon_0(1-t) + x\varepsilon_2(t)) & \text{if } \frac{1}{4} \le x \le \frac{1}{2}, \\ \sigma(2tx - (2t-1)) & \text{if } x \ge \frac{1}{2}. \end{cases}$$

We leave it to the reader to check that this works.

(ii) One can generate a similar homotopy, which we do not do here.

(iii) This time, we use the homotopy which goes up along γ for t units, before going parallel to β and coming back down along δ^{-1}. The particular formula is as follows:

$$H(x,t) = \begin{cases} F(0,4tx) & \text{if } x \le \frac{1}{4}, \\ F(4x-1,t) & \text{if } \frac{1}{4} \le x \le \frac{1}{2}, \\ F(1,2t(1-x)) & \text{if } \frac{1}{2} \le x. \end{cases}$$

Once again, we leave the details to the reader to check.

Exercise 3.5. Simply use the homotopy $H : \mathbb{I}\times\mathbb{I}\to X\times Y$ which takes (s,t) to $(F(s,t),G(s,t))$. This is clearly a homotopy from (f_0,g_0) to (f_1,g_1). To see that it is still rel $\dot{\mathbb{I}}$, simply observe that $H(0,t) = (F(0,t),G(0,t))$. Because F and G are both rel $\dot{\mathbb{I}}$, it follows that $H(0,t)$ never changes. A similar argument shows that $H(1,t)$ is always the same, and so H is indeed a homotopy rel $\dot{\mathbb{I}}$.

Exercise 3.6.

(i) It is obvious that the homotopy $H' : (x,t)\mapsto H(x,1-t)$ works.

(ii) This is just some slightly annoying manipulation. In particular, note that

$$(f*g)(x) = \begin{cases} f(2x) & \text{if } x \le \frac{1}{2}, \\ g(2x-1) & \text{if } x \ge \frac{1}{2}. \end{cases}$$

By replacing x with $1-x$ to get the inverse, we find that

$$(f*g)^{-1}(x) = \begin{cases} f(2-2x) & \text{if } x \ge \frac{1}{2}, \\ g(1-2x) & \text{if } x \le \frac{1}{2}. \end{cases}$$

However, note that

$$\begin{aligned}(g^{-1}*f^{-1})(x) &= \begin{cases} g^{-1}(2x) & \text{if } x \ge \frac{1}{2}, \\ f^{-1}(2x-1) & \text{if } x \ge \frac{1}{2} \end{cases} \\ &= \begin{cases} g(1-2x) & \text{if } x \le \frac{1}{2}, \\ f(2-2x) & \text{if } x \ge \frac{1}{2}. \end{cases}\end{aligned}$$

Thus the two are indeed the same.

(iii) Take the closed path $f(t) = e^{2\pi i t}$ on S^1. Then note that $(f*f^{-1})(\frac{1}{8}) = f(\frac{1}{4}) = i$, while $(f^{-1}*f)(\frac{1}{8}) = f^{-1}(\frac{1}{4}) = -i$.

(iv) Suppose $i_p * f = f$ and f is not constant. Note that continuity implies that there must exist some $0 < t < 1$ so that $f(t) \ne p$. Thus there exists some $k\in\mathbb{N}$ so that $t < 1-2^{-k}$.

We claim, however, that f must be constant on $[0,1-2^{-n}]$ for every $n\in\mathbb{N}$. We prove this inductively. Clearly, it is true on $n=0$. If it is true on $n-1$, then we know that $i_p * f$ must be equal to p on $[0,\frac{1}{2}]$, as well as on $[\frac{1}{2},1-2^{-n}]$ (note that $1-2^{-n}$ comes from $2(1-2^{-n})-1$, which itself comes from the equation for the star operator). Thus f is constant on $[0,1-2^{-n}]$, as desired.

Thus it follows that $f(t) = p$, a contradiction. Thus f must have been constant in the first place.

Exercise 3.7. Recall that we defined the $\sin(1/x)$ space as the union of $A = \{(0, y) : -1 \le y \le 1\}$ and $G = \{(x, \sin(1/x)) : 0 < x \le 1/2\pi\}$. We also know that A and G are the path components of the $\sin(1/x)$ space. Moreover, both A and G are contractible, and so every path in either A or G is nullhomotopic. In particular, we conclude that the fundamental group at any basepoint is trivial.

Exercise 3.8. Let X be the $\sin(1/x)$ space. We know that CX is contractible. But consider an open ball around the point $x = ((0, 0), 0)$, that is, the point $(0, 0)$ on the "zeroth" level of the cone. Consider a small neighborhood (not including the points $(t, 1)$, in particular) around this point and pick some element $y = ((\varepsilon, \sin(1/\varepsilon)), 0)$ in the neighborhood. Now observe that any path between x and y can be projected down to a path between $(0, 0)$ and $(\varepsilon, \sin(1/\varepsilon))$ in X, which we know does not exist. Hence CX is contractible but not locally path connected.

Exercise 3.9. Note that composition is associative because $\circ$ is. Moreover, the path class of the trivial loop based at p is the identity on p. Thus this is a category.

To see that each morphism in $\mathscr{C}$, simply note that the inverse path, i.e., the path f^{-1} taking t to $f(1-t)$, gives a path class $[f^{-1}]$ which works as an inverse to $[f] \in \mathrm{Hom}(p, q)$.

Exercise 3.10. We simply let π_0 take $(X, x_0) \in \mathbf{Sets}_*$ to the set of all path components of X, with basepoint equal to the path component containing x_0. It takes a morphism $f \in \mathrm{Hom}((X, x_0), (Y, y_0))$ to the map $\pi_0(f)$ which takes each path component A of X to the path component B of Y which contains $f(A)$.

Note that this is possible because continuous images of path connected spaces are path connected and hence contained within a single path component of Y. Moreover, this is indeed a pointed map because the path component containing x_0 must be contained in the path component containing $f(x_0) = y_0$, which is the basepoint of $\pi_0((Y, y_0))$.

It is easy to check functoriality, completing the proof.

Exercise 3.11. Evidently the only possible path is the constant path at x_0. Hence $\pi_1(X, x_0)$ is the trivial group, i.e., $\{1\}$.

Exercise 3.12. Note that 1_S is a loop based at 1, i.e., an element of $\pi_1(S^1, 1)$. Thus if $\pi_1(S^1, 1)$ were trivial, then 1_S would be nullhomotopic. The hint gives the rest of the solution.

Exercise 3.13. We know that $\deg u = 1$. Since 1 is a generator for $\mathbb{Z}$, it follows that $[u]$ generates $\pi_1(S^1, 1)$.

Exercise 3.14. Let $\tilde{\gamma}(t) = m\tilde{f}(t)$, where $\tilde{f}$ is the lifting of f satisfying $\tilde{f}(0) = 0$. Now simply observe that

$$\exp \tilde{\gamma}(t) = \left(\exp \tilde{f}(t)\right)^m = f(t)^m$$

and $\tilde{\gamma}(0) = 0$. Thus $\tilde{\gamma}$ is indeed the lifting of f^m taking 0 to 0, and so we conclude that

$$\deg(f^m) = \tilde{\gamma}(1) = m\tilde{f}(1) = m \deg f.$$

Exercise 3.15. Note that **??** implies that there is a homotopy $F : R_f \circ f \simeq f$, where R_f is the rotation associated with f. Moreover, from the proof of that same exercise, it follows that F gives a closed path at every time t. Similarly, we have $G : g \simeq R_g \circ g$. Thus if $H : f \simeq g$ where H gives a closed path at every time t, then the homotopy which follows F, then H, and finally G is a homotopy between $R_f \circ f$ and $R_g \circ g$. Thus Corollary 3.18 implies that f and g have the same degree.

For the converse, simply use Corollary 3.18 to show that $\deg f = \deg g$ implies that there is a homotopy rel $\dot{\mathbb{I}}$ taking $R_f \circ f$ to $R_g \circ g$. Then using F and G defined above, it is clear that $g \simeq R_g \circ g \simeq R_f \circ f \simeq f$.

Exercise 3.16. Theorem 3.7 implies that $\pi_1(T, t_0) = \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$.

Exercise 3.17. Because D^2 is contractible, its fundamental group is trivial. Thus if there were to exist a retraction $r : D^2 \to S^1$, then $r_* : \pi_1(D) \to \pi_1(S^1)$ would be a constant. But then, letting $i : S^1 \to D^2$ be the canonical injection, we would have that $(r \circ i)_* = r_* \circ i_*$ is a constant. However, we also know that $r \circ i$ is the identity on S^1, and so $(r \circ i)_*$ is the identity on $\pi_1(S^1)$, which is *not* a constant. This is a contradiction, from which we conclude that S^1 is not a retract of D^2, as desired.

Exercise 3.18. This was proved in Theorem 0.3, which required only the fact proved in the above problem, namely that S^1 is not a retract of D^2.

Exercise 3.19.

(i) Let $\tilde{f}$ be the unique lifting of f with $\tilde{f}(0) = 0$. Then if $\tilde{f}(1) \geq 1$, the intermediate value theorem implies that every point in the interval $[0, 1] \subset \mathbb{R}$ is in the image of $\tilde{f}$. But this implies that $f = \exp \circ \tilde{f}$ must be surjective, a contradiction.

(ii) Consider the map which traverses the circle once counterclockwise, reaching the point 1 at time $t = \frac{1}{2}$, before looping back and making a clockwise rotation. Clearly it is surjective. However, it is composed of two loops, one of which has degree 1 and one of which has degree -1. Because $\deg(f * g) = \deg f + \deg g$, it follows that this map has degree 0.

Exercise 3.20. As per the hint, consider an arbitrary closed path f in X and let λ be a Lebesgue number of the open cover $\{f^{-1}(U_j) : j \in J\}$ of $\mathbb{I}$. Note that λ exists by the Lebesgue number lemma and compactness of the unit interval. Picking $N \in \mathbb{N}$ with $N > 1/\lambda$, it follows that if we subdivide I into N equal intervals $I_k = [\frac{k}{N}, \frac{k+1}{N}]$, then $f(I_k) \subseteq U_{j_k}$ for some $j_k \in J$.

Define f_k as the path in U_{j_k} obtained by restricting f to I_k and then stretching suitably so that the domain is all of $\mathbb{I}$. With notation, define $f_k(t) = f\left(\frac{k+t}{N}\right) \in U_{j_k}$. Because f_k is a path in U_{j_k}, it follows that $[f_k'] = [i_{j_k} \circ f_k] \in \operatorname{im}(i_j)_*$. But now simply observe that $[f_0' * \cdots * f_{N-1}'] = [f]$, implying that $[f]$ is contained in the group generated by the subsets $\operatorname{im}(i_j)_*$. This proves the result.

Exercise 3.21. Let U_1 and U_2 be defined as in the hint, and let i_k be the injection from U_k to S^n for $k = 1, 2$. Observe that, by the previous exercise, it suffices to show that $\operatorname{im}(i_k)_*$ is trivial for $k = 1, 2$.

Without loss of generality, let $k = 1$. But we know that $(i_1)_*$ takes a closed path $f : \mathbb{I} \to U_1$ to the path class $[i_1 \circ f]$. (Note that the basepoint doesn't really matter for us as long as it is neither the north nor the south pole.Thus we omit it.) Because $U_1 \approx D^n$ and is therefore contractible, it follows that f is nullhomotopic. In particular, we know that $i_1 \circ f$ is nullhomotopic, and so $[i_1 \circ f] = [1]$ for every f. Thus $\operatorname{im}(i_1)_*$ is trivial, and similarly for $k = 2$, proving the result.

Exercise 3.22. Corollary 3.11 implies that path connected spaces of the same homotopy type must have isomorphic fundamental groups. But obviously $\mathbb{Z} \not\cong \{1\}$, and so S^1 and S^n do not have the same homotopy type for $n > 1$.

Exercise 3.23. The multiplication map μ on G/H is continuous. After all, if we let v be the natural map, then for any open set $U \subseteq G/H$, we have

$$\mu^{-1}(U) = \{([x], [y]) : xy \in v^{-1}(U)\}.$$

But this set is open in $G/H \times G/H$ because the set consisting of elements $(v^{-1}([x]), v^{-1}([y]))$ for each $([x], [y]) \in \mu^{-1}(U)$ is just $\mu^{-1}(v^{-1}(U))$, which is clearly open.

For the inversion map $i : G/H \to G/H$, a very similar argument holds. In particular, for any open set $U \subseteq G/H$, we have

$$v^{-1}(i^{-1}(U)) = \{x^{-1} : x \in v^{-1}(U)\}.$$

Thus $v^{-1}(i^{-1}(U))$ is open, and so $i^{-1}(U)$ is open, proving continuity.

Exercise 3.24. First, we will show that we can lift a loop $f : (\mathbb{I}, 0) \to (G/H, 1)$ to a unique continuous map $\tilde{f} : (\mathbb{I}, 0) \to (G, h)$ for any $h_0 \in H$, as shown below.

$$\begin{tikzcd} & (G, h_0) \arrow[d, "v"] \\ (\mathbb{I}, 0) \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & (G/H, 1) \end{tikzcd}$$

In the above diagram, the map v is the natural map taking g to the coset $gH \in G/H$.

First, we will find a suitable neighborhood U of 1 such that the family $\{hU : h \in H\}$ is pairwise disjoint. Discreteness of H implies that there exists an open neighborhood V of 1 with $V \cap H = \{1\}$. It is clear that the map $\varphi : (x, y) \mapsto xy^{-1}$ is the composition $\mu \circ (\mathrm{id} \times i)$ and is therefore continuous. Thus $\varphi^{-1}(V) \subseteq G \times G$ is an open neighborhood of $(1, 1)$. This implies that we can find an open neighborhood U of 1 such that $U \times U \subseteq f^{-1}(V)$.

Now suppose that there are $h_1, h_2 \in H$ and $x, y \in U$ with $h_1 x = h_2 y$. But this would require that $xy^{-1} = h_1^{-1} h_2$. It is clear that $xy^{-1} \in \varphi(U) \subseteq V$. Moreover, because H is a subgroup, we know that $h_1^{-1} h_2 \in H$, and so $xy^{-1} \in V \cap H$. Thus $x = y$ and $h_1 = h_2$, proving that the sets hU are disjoint, as desired. Note that any translate $U_g = gU$ of U is a neighborhood of $g \in G$ and has $\{hU_g : h \in H\}$ disjoint.

Note that v is an open map, and so the set $W = v(U) \subseteq G/H$ is open. Moreover, because $v|_U$ is the restriction of a continuous open map to an open set, it follows that $v|_U$ is itself continuous and open. It is also a bijection onto W, and so $v|_U : U \to V$ is a homeomorphism.

Note that the collection of sets $V[g]$ for $[g] \in G/H$ forms an open cover of G/H. Thus, if we are given some $f : (\mathbb{I}, 0) \to (G/H, 1)$, then we can consider the open cover

$$\{f^{-1}(V[g]) : [g] \in G/H\}$$

of $\mathbb{I}$. Note that we can find a finite subcover of this open cover. This means that we can take subsets of the sets in this open cover, given us a finite collection open overlapping subintervals which are, in order of their smaller coordinate, labeled $I_1, \ldots, I_k$. Let the group elements $g_1, \ldots, g_k$ be such that $I_j \subseteq f^{-1}(V[g_j])$. This is simply because $\mathbb{I} = [0, 1]$ is connected compact.

Now we can lift f to each interval $f^{-1}(V[g])$ in this finite subcover. Note that $0 = t_1 \in I_1 \subseteq f^{-1}(V[g_1])$. Moreover, we know that $v^{-1}(V[g_1])$ consists of disjoint unions of U, and so we can pick the one containing h_0. Now, for each $t \in I_1$, we let $\tilde{f}(t)$ to be the unique element in this copy of U such that $v(\tilde{f}(t)) = f(t)$. Because the intervals overlap, we know that there is some $t_2 \in I_2 \cap I_1$, and so we can do the same thing, all the way to t_k. This lets us define $\tilde{f}(t)$ for all $t \in \mathbb{I}$, and it is easy to show that our construction is indeed a lifting satisfying the commutative diagram above.

Now consider the map $d : \pi_1(G/H, 1) \to H$ taking a loop $[f]$ to $d([f]) = \tilde{f}(1)$, where $\tilde{f}$ is the unique lifting of f with $\tilde{f}(0) = 1$. It is obvious that $\operatorname{im} d \subseteq H$ because $v(\tilde{f}(1)) = f(1) = [1]$ implies that $d([f]) = \tilde{f}(1) \in H$. Moreover, the reverse inclusion holds, showing surjectivity. In particular, if $h \in H$, then path connectedness of G implies that there is a path $\tilde{f}$ from 1 to h. Taking its projection $f = v \circ \tilde{f}$, note that f is a loop because $v(\tilde{f}(1)) = v(h) = [1]$. Thus $d([f])$ is defined and equal to h. To show injectivity, simply note that $\tilde{f}(1) = 1$ implies that $\tilde{f}$ is a loop in G. Because G is simply connected, however, it follows that $\tilde{f}$, and hence f, is nullhomotopic. Thus $f \in \ker d$ implies that $[f] = [1]$. Finally, we must show that d is indeed a homomorphism, i.e., that $d(f * g) = d(f)d(g)$. But this is clear if we lift f to $\tilde{f}$ with $\tilde{f}(0) = 1$, and if we lift g to $\tilde{g}$ with $\tilde{g}(0) = d(f)$. This follows the same proof layout as Theorem 3.16, and proves the result.

Exercise 3.25. If $S \subseteq GL(n, \mathbb{R})$ is a subgroup of $GL(n, \mathbb{R})$, then note that $\mu : S \times S \to S$ is continuous. After all, the product, entrywise, is simply a polynomial, and so μ is a polynomial in each of its n^2 entries. Since polynomials are continuous in $\mathbb{R}^2$, it follows that each of the n^2 components of μ is continuous. Hence μ is continuous.

To see that the inversion i is continuous, observe that the determinant $\det A$ is a continuous function, since it too is a polynomial (and is never zero, by definition of GL). It thus suffices to show that the function $A \mapsto \operatorname{adj} A$ is continuous. But it is easy to see that the adjugate matrix, which is the transpose of the cofactor matrix, is also a polynomial in the entries of A, and so $\operatorname{adj} A$ is continuous too. Thus i is continuous, and so S is a topological group, as desired.

Exercise 3.26. As hinted in the exercise, fix $h_0 \in H$ and let $\varphi : G \to H$ be the map taking x to $xh_0x^{-1}h_0^{-1}$. Note that $xh_0x^{-1} \in H$ because H is normal, and so $\varphi(x)$ is indeed an element of H. Moreover, we know that φ is continuous and $\{h\} \subseteq H$ is open for each $h \in H$. Thus $\{\varphi^{-1}(h) : h \in H\}$ is an open cover of G consisting of disjoint open sets.

In particular, if there are two elements $h_1, h_2 \in H$ such that $\varphi^{-1}(h_i) \neq \emptyset$ for $i = 1, 2$, then setting $A = \varphi^{-1}(h_1)$ and $B = \bigcup_{h \neq h_1} \varphi^{-1}(h)$ will give us two disjoint open sets A and B that cover G. This implies that G is disconnected, a contradiction. Thus for all but one element of H, we must have $\varphi^{-1}(h) = \emptyset$, proving that φ is constant. But obviously, setting $x = h_0$, we find that $\varphi(x) = 1$. Thus $xh_0x^{-1}h_0^{-1} = 1$ for all $x \in G$, and so $xh_0 = h_0x$ for each $h_0 \in H$. This proves the result.