3 The Fundamental Group

The Fundamental Groupoid

Exercise 3.1. The homotopy $H : X \times \mathbb{I} \to Z$ given by

$$H : (x,t) \mapsto \begin{cases} g_0(F(x,2t)) & \text{if } t \le \frac{1}{2}, \\ G(f_1(x), 2t-1) & \text{if } t \ge \frac{1}{2} \end{cases}$$

works. Continuity follows because $g_0(F(x,1)) = G(f_1(x),0)$.

Moreover, this homotopy is indeed $\operatorname{rel} A$. For a detailed argument why this is so, simply suppose that $a \in A$ and $t \in I$. If $t \le \frac{1}{2}$, then $F(a,2t) = f_0(a)$ by definition of F. Hence $H(a,t) = g_0(f_0(a))$.

Similarly, we can show that if $t \ge \frac{1}{2}$, then $H(a,t) = g_1(f_1(a))$. This follows because $f_1(a) \in B$ and G is a homotopy $\operatorname{rel} B$.

It thus suffices to show that $g_0(f_0(a)) = g_1(f_1(a))$. But this is obvious because f_0 and f_1 agree on A, and g_0 and g_1 agree on $B \supseteq f_0(A)$.

Exercise 3.2.

(i) First, note that f' is well-defined because $f(0) = f(1)$. It is obvious by continuity of f and ln that f' is continuous.

Moreover, consider the map
$$H' : \left(e^{2\pi i\theta}, t\right) \mapsto H(\theta, t).$$
This is clearly continuous, for the same reasons that f' was continuous. If $t = 0$, clearly $H'(e^{2\pi i\theta}, t) = H(\theta, 0) = f(\theta) = f'(e^{2\pi i\theta})$, and similarly for $t = 1$. Thus H is indeed a homotopy from f' to g'.

To see that it is a homotopy $\operatorname{rel}\{1\}$, simply note that $e^{2\pi i\theta} = 1$ corresponds to $\theta = 0, 1$. Thus it follows that
$$H'(1,t) = H(1,t) = f(1)$$
for all t, proving the result.

(ii) Theorem 3.1 implies that $f * g \simeq f_1 * g_1 \operatorname{rel} \dot{\mathbb{I}}$. Using the previous part, we find that $(f*g)' \simeq (f_1 * g_1)' \operatorname{rel}\{1\}$. Now, using the observation that $(f*g)' = f' * g'$, we find that $f' * g' \simeq f_1' * g_1' \operatorname{rel}\{1\}$, as desired.

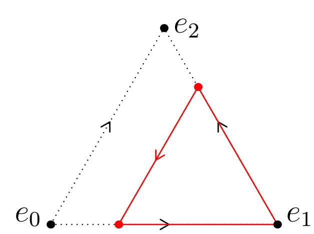
Exercise 3.3. The forward direction is trivial.

For the converse, note that g' is a constant map, and so f' is nullhomotopic. Then Theorem 1.6 implies that $f' \simeq g' \operatorname{rel}\{1\}$. In particular, note that $g' : S^1 \to X$ takes every element of S^1 to $g'(1) = g(0) = x_0$. Observe that $f'(1) = x_0$ as well, and so it follows that $f' \simeq g \operatorname{rel}\{1\}$, as desired.

Exercise 3.4.

(i) Instead of applying Theorem 1.6, I constructed an explicit homotopy. (If you are interested in a proof using Theorem 1.6, my guess would be that it relies on the fact that $\Delta^2 \approx D^2$. However, I have not gone through the details.)

The effective idea of the homotopy I constructed is to, at time $t \in [0,1]$, return the function which traverses the first t units of the face opposite e_0, then goes along a segment to the point t units away from e_1 on the fact opposite e_2, before returning back to e_1, as shown in the red path below.



The specific homotopy $H : \mathbb{I} \times \mathbb{I} \to X$ from $(\sigma_0 * \sigma_1^{-1}) * \sigma_2$ to the constant map at e_1 is as follows:

$$H(x,t) = \begin{cases} \sigma_0(4(1-t)x) & \text{if } x \leq \frac{1}{4}, \\ \sigma((1-x)\varepsilon_0(1-t) + x\varepsilon_2(t)) & \text{if } \frac{1}{4} \leq x \leq \frac{1}{2}, \\ \sigma(2tx - (2t-1)) & \text{if } x \geq \frac{1}{2}. \end{cases}$$

We leave it to the reader to check that this works.

(ii) One can generate a similar homotopy, which we do not do here.

(iii) This time, we use the homotopy which goes up along γ for t units, before going parallel to β and coming back down along δ^{-1}. The particular formula is as follows:

$$H(x,t) = \begin{cases} F(0, 4tx) & \text{if } x \leq \frac{1}{4}, \\ F(4x-1, t) & \text{if } \frac{1}{4} \leq x \leq \frac{1}{2}, \\ F(1, 2t(1-x)) & \text{if } \frac{1}{2} \leq x. \end{cases}$$

Once again, we leave the details to the reader to check.

Exercise 3.5. Simply use the homotopy $H : \mathbb{I} \times \mathbb{I} \to X \times Y$ which takes (s,t) to $(F(s,t), G(s,t))$. This is clearly a homotopy from (f_0, g_0) to (f_1, g_1). To see that it is still $\operatorname{rel} \dot{\mathbb{I}}$, simply observe that $H(0,t) = (F(0,t), G(0,t))$. Because F and G are both $\operatorname{rel} \dot{\mathbb{I}}$, it follows that $H(0,t)$ never changes. A similar argument shows that $H(1,t)$ is always the same, and so H is indeed a homotopy $\operatorname{rel} \dot{\mathbb{I}}$.

Exercise 3.6.

(i) It is obvious that the homotopy $H' : (x,t) \mapsto H(x, 1-t)$ works.

(ii) This is just some slightly annoying manipulation. In particular, note that

$$(f * g)(x) = \begin{cases} f(2x) & \text{if } x \leq \frac{1}{2}, \\ g(2x-1) & \text{if } x \geq \frac{1}{2}. \end{cases}$$

By replacing x with $1-x$ to get the inverse, we find that

$$(f * g)^{-1}(x) = \begin{cases} f(2-2x) & \text{if } x \geq \frac{1}{2}, \\ g(1-2x) & \text{if } x \leq \frac{1}{2}. \end{cases}$$

However, note that

$$\begin{aligned} (g^{-1} * f^{-1})(x) &= \begin{cases} g^{-1}(2x) & \text{if } x \geq \frac{1}{2}, \\ f^{-1}(2x-1) & \text{if } x \geq \frac{1}{2} \end{cases} \\ &= \begin{cases} g(1-2x) & \text{if } x \leq \frac{1}{2}, \\ f(2-2x) & \text{if } x \geq \frac{1}{2}. \end{cases} \end{aligned}$$

Thus the two are indeed the same.

(iii) Take the closed path $f(t) = e^{2\pi i t}$ on S^1. Then note that $(f * f^{-1})(\frac{1}{8}) = f(\frac{1}{4}) = i$, while $(f^{-1} * f)(\frac{1}{8}) = f^{-1}(\frac{1}{4}) = -i$.

(iv) Suppose $i_p * f = f$ and f is not constant. Note that continuity implies that there must exist some $0 < t < 1$ so that $f(t) \neq p$. Thus there exists some $k \in \mathbb{N}$ so that $t < 1 - 2^{-k}$.

We claim, however, that f must be constant on $[0, 1-2^{-n}]$ for every $n \in \mathbb{N}$. We prove this inductively. Clearly, it is true on $n = 0$. If it is true on $n-1$, then we know that $i_p * f$ must be equal to p on $[0, \frac{1}{2}]$, as well as on $[\frac{1}{2}, 1-2^{-n}]$ (note that $1 - 2^{-n}$ comes from $2(1-2^{-n}) - 1$, which itself comes from the equation for the star operator). Thus f is constant on $[0, 1-2^{-n}]$, as desired.

Thus it follows that $f(t) = p$, a contradiction. Thus f must have been constant in the first place.

Exercise 3.7. Recall that we defined the $\sin(1/x)$ space as the union of $A = \{(0, y) : -1 \leq y \leq 1\}$ and $G = \{(x, \sin(1/x)) : 0 < x \leq 1/2\pi\}$. We also know that A and G are the path components of the $\sin(1/x)$ space. Moreover, both A and G are contractible, and so every path in either A or G is nullhomotopic. In particular, we conclude that the fundamental group at any basepoint is trivial.

Exercise 3.8. Let X be the $\sin(1/x)$ space. We know that CX is contractible. But consider an open ball around the point $x = ((0, 0), 0)$, that is, the point $(0, 0)$ on the "zeroth" level of the cone. Consider a small neighborhood (not including the points $(t, 1)$, in particular) around this point and pick some element $y = ((\varepsilon, \sin(1/\varepsilon)), 0)$ in the neighborhood. Now observe that any path between x and y can be projected down to a path between $(0, 0)$ and $(\varepsilon, \sin(1/\varepsilon))$ in X, which we know does not exist. Hence CX is contractible but not locally path connected.

Exercise 3.9. Note that composition is associative because $\circ$ is. Moreover, the path class of the trivial loop based at p is the identity on p. Thus this is a category.

To see that each morphism in $\mathscr{C}$, simply note that the inverse path, i.e., the path f^{-1} taking t to $f(1-t)$, gives a path class $[f^{-1}]$ which works as an inverse to $[f] \in \mathrm{Hom}(p, q)$.

Exercise 3.10. We simply let π_0 take $(X, x_0) \in \mathbf{Sets}_*$ to the set of all path components of X, with basepoint equal to the path component containing x_0. It takes a morphism $f \in \mathrm{Hom}((X, x_0), (Y, y_0))$ to the map $\pi_0(f)$ which takes each path component A of X to the path component B of Y which contains $f(A)$.

Note that this is possible because continuous images of path connected spaces are path connected and hence contained within a single path component of Y. Moreover, this is indeed a pointed map because the path component containing x_0 must be contained in the path component containing $f(x_0) = y_0$, which is the basepoing of $\pi_0((Y, y_0))$.

It is easy to check functoriality, completing the proof.

Exercise 3.11. Evidently the only possible path is the constant path at x_0. Hence $\pi_1(X, x_0)$ is the trivial group, i.e., $\{1\}$.