Chapter 0: Introduction

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Exercise 0.1. As per the hint, observe that if $y \in G$, then we have $y = r(y) + (y - r(y))$. Obviously, we have $r(y) \in H$. Moreover, we know that

$$r(y - r(y)) = r(y) - r(r(y)) = 0,$$

and so $y - r(y) \in \ker r$. Thus $G \subseteq H \oplus \ker r$.

The reverse is obviously true, since H and $\ker r$ are both subgroups of G.

Exercise 0.2. Suppose instead that $f : D^1 \to D^1$ has no fixed point. Then consider the continuous map $g : D^1 \to S^0$ given by

$$g(x) = \begin{cases} 1 & \text{if } f(x) < x \\ -1 & \text{if } f(x) > x \end{cases}.$$

Notice that because $f(x) \neq x$ for all x, the function g is well-defined.

Moreover, we know that $f(-1) \neq -1$, since f has no fixed point, and so $f(-1) > -1$. Thus $g(-1) = -1$. Similarly, we have $g(1) = 1$.

Thus we have $g(D^1) = S^0$, which is disconnected. This is a contradiction, so f must have had a fixed point.

Exercise 0.3. Suppose that r is such a retract. Then we have the following commutative diagram:

$$\begin{tikzcd} & S^n \arrow[dr, "r"] & \\ S^{n-1} \arrow[ur, "i"] \arrow[rr, "1"] & & S^{n-1}. \end{tikzcd}$$

Applying H_{n-1}, we get another commutative diagram:

$$\begin{tikzcd} & H_{n-1}(S^n) \arrow[dr, "H_{n-1}(r)"] & \\ H_{n-1}(S^{n-1}) \arrow[ur, "H_{n-1}(i)"] \arrow[rr, "H_{n-1}(1)"] & & H_{n-1}(S^{n-1}). \end{tikzcd}$$

We know that $H_{n-1}(S^n) = 0$, however, implying that $H_{n-1}(1) = 0$. This contradicts the fact that $H_{n-1}(S^{n-1}) = \mathbb{Z} \neq 0$. Thus the retraction r could not have existed.

Exercise 0.4. Suppose $g : D^n \to X$ is a homeomorphism. Then we know that $g^{-1} \circ f \circ g$ is a continuous map from D^n to itself, and so it has a fixed point x. Then we know that $g^{-1}(f(g(x))) = x$, and so it follows that $f(g(x)) = g(x)$. Thus $g(x) \in X$ is a fixed point of f.