Solutions to Rotman's algebraic topology

Jessica Zhang

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0 Introduction

Notation

No exercises!

Brouwer Fixed Point Theorem

Exercise 0.1. As per the hint, observe that if $y \in G$, then we have $y = r(y) + (y - r(y))$. Obviously, we have $r(y) \in H$. Moreover, we know that

$$r(y - r(y)) = r(y) - r(r(y)) = 0,$$

and so $y - r(y) \in \ker r$. Thus $G \subseteq H \oplus \ker r$.

The reverse is obviously true, since H and $\ker r$ are both subgroups of G.

Exercise 0.2. Suppose instead that $f : D^1 \to D^1$ has no fixed point. Then consider the continuous map $g : D^1 \to S^0$ given by

$$g(x) = \begin{cases} 1 & \text{if } f(x) < x \\ -1 & \text{if } f(x) > x \end{cases}.$$

Notice that because $f(x) \neq x$ for all x, the function g is well-defined.

Moreover, we know that $f(-1) \neq -1$, since f has no fixed point, and so $f(-1) > -1$. Thus $g(-1) = -1$. Similarly, we have $g(1) = 1$.

Thus we have $g(D^1) = S^0$, which is disconnected. This is a contradiction, so f must have had a fixed point.

Exercise 0.3. Suppose that r is such a retract. Then we have the following commutative diagram:

$$\begin{tikzcd} & S^n \arrow[dr, "r"] & \\ S^{n-1} \arrow[ur, "i"] \arrow[rr, "1"] & & S^{n-1}. \end{tikzcd}$$

Applying H_{n-1}, we get another commutative diagram:

$$\begin{tikzcd} & H_{n-1}(S^n) \arrow[dr, "H_{n-1}(r)"] & \\ H_{n-1}(S^{n-1}) \arrow[ur, "H_{n-1}(i)"] \arrow[rr, "H_{n-1}(1)"] & & H_{n-1}(S^{n-1}). \end{tikzcd}$$

We know that $H_{n-1}(S^n) = 0$, however, implying that $H_{n-1}(1) = 0$. This contradicts the fact that $H_{n-1}(S^{n-1}) = \mathbb{Z} \neq 0$. Thus the retraction r could not have existed.

Exercise 0.4. Suppose $g : D^n \to X$ is a homeomorphism. Then we know that $g^{-1} \circ f \circ g$ is a continuous map from D^n to itself, and so it has a fixed point x. Then we know that $g^{-1}(f(g(x))) = x$, and so it follows that $f(g(x)) = g(x)$. Thus $g(x) \in X$ is a fixed point of f.

Exercise 0.5. Consider the function $h : \mathbb{I} \times \mathbb{I} \to \mathbb{I} \times \mathbb{I}$ given by

$$h(s,t) = f(s) - g(t) + (s,t).$$

This is the sum of continuous functions, and so it is itself continuous. Moreover, we know that $\mathbb{I} \times \mathbb{I}$ is homeomorphic to D^1, and so it follows that there is a fixed point (s,t) of h. But this means that $f(s) - g(t) = 0$, and so we are done.

Exercise 0.6. Observe that $x \in \Delta^{n-1}$ must contain some positive coordinate, because $\sum x_i = 1$ and $x_i \geq 0$ for all i. Since $a_{ij} > 0$ for every i, j, it follows that Ax contains only nonnegative coordinates and, moreover, contains at least one positive coordinate. Thus $\sigma(Ax) > 0$, and so $g(x)$ is well-defined.

Moreover, it is continuous because the linear map A, the map σ, and the division function are all continuous.

Because $\Delta^{n-1} \approx D^{n-1}$, it follows that there exists some x with

$$x = \frac{Ax}{\sigma(Ax)}.$$

Then $\lambda = \sigma(Ax) > 0$ is a positive eigenvalue for A and $x \in \Delta^{n-1}$ is a corresponding eigenvector.

We know that x contains only nonnegative coordinates. Suppose then that some coordinate, say x_1, is zero. Then obviously the first coordinate of λx is zero. However, the first coordinate of Ax is

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = a_{12}x_2 + \cdots + a_{1n}x_n.$$

Since $\sum x_i = 1$ and $x_1 = 0$, there exists some $k \neq 1$ such that $x_k > 0$. Then $a_{1k}x_k > 0$, and since each i already has $a_{1i}x_i \geq 0$, it follows that the first coordinate of Ax is strictly positive, contradicting that $Ax = \lambda x$.

Thus the eigenvector x has all positive coordinates.

Categories and Functors

Exercise 0.7. We know that

$$g \circ (f \circ h) = g \circ 1_b = g$$

and

$$(g \circ f) \circ h = 1_A \circ h = h,$$

and so associativity implies $g = h$.

Exercise 0.8.

(i) Notice that if 1_A and $1'_A$ are both identities, then we must have

$$1_A = 1_A \circ 1'_A = 1'_A,$$

which proves the desired result.

(ii) If $1'_A$ is the new identity in $\mathcal{C}'$, then we know that $1'_A \in \mathrm{Hom}_{\mathcal{C}'}(A, A) \subseteq \mathrm{Hom}_{\mathcal{C}}(A, A)$, and so $1_A \circ 1'_A$ is defined. But we know that

$$1'_A \circ 1_A = 1'_A = 1'_A \circ 1'_A,$$

and so Exercise 0.7 implies the result.

Exercise 0.9. Clearly, the Hom-sets are pairwise disjoint, since each i_y^x appears at most once.

It is also obviously associative. In particular, if $a \leq b \leq c \leq d$, then we know that

$$i_d^c \circ \left(i_c^b \circ i_b^a\right) = i_d^c \circ i_c^a = i_d^a,$$

and similarly for $\left(i_d^c \circ i_c^b\right) \circ i_b^a$.

Finally, the map i_x^x is the identity on $x \in X$. To see that it is a left-identity, note that if $y \leq x$, then

$$i_x^x \circ i_x^y = i_x^y.$$

Similarly, we can show that this map is a right-identity as well, and so we are done.

Exercise 0.10. Disjointness is clear, since there is only one object. Because G is a monoid, it is associative and has an identity, proving that $\mathcal{C}$ is a category.

Exercise 0.11. It is pretty clear that obj(Top) $\subset$ obj(Top^2). Moreover, a continuous map $f : X \to Y$ between two topological spaces corresponds to the map $(f, \emptyset)$ in Top^2 from $(X, \emptyset)$ to $(Y, \emptyset)$, which then means that Top can be thought of as a subcategory of Top^2.

Exercise 0.12. It is worth noting that Rotman's definition here is incorrect. The morphisms in $\mathcal{M}$ should be the commutative squares, not merely the ordered pairs (h, k).

Indeed, consider the following counterexample to Rotman's definition. Let $\mathcal{C}$ be the category of sets. Furthermore, let A be a set with more than one element. Then the following diagrams are both commutative:

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "1_A"'] & A \arrow[d, "0"] \\ A \arrow[r, "0"] & \{0\} \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "0"] \arrow[d, "1_A"'] & A \arrow[d, "0"] \\ A \arrow[r, "0"] & \{0\}. \end{tikzcd}$$

This implies that the ordered pair $(1_A, 0)$, where 0 is considered to be the map that sends everything in A to the zero element, is both in $\mathrm{Hom}(1_A, 0)$ and in $\mathrm{Hom}(0, 0)$, contradicting disjointness.

If we instead consider morphisms of $\mathcal{M}$ to be the commutative squares, where composition is defined by "stacking" the squares on top of one another, disjointness is clear. After all, the squares contain f and g, and so Hom-sets of different objects must be disjoint.

Associativity is clear, as the morphisms of $\mathcal{C}$ are associative.

Finally, there is an identity 1_f for every $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, namely the one where $h = 1_A$ and $k = 1_B$.

Exercise 0.13. With the hint, this is clear. In particular, we consider Top^2 to be the subcategory of the arrow category of Top in which the objects are inclusions, and $\mathrm{Hom}_{\mathsf{Top}^2}(i, j) = \mathrm{Hom}_{\mathsf{Top}}(i, j)$.

Exercise 0.14. To see that it is a congruence at all, observe that Property (i) is satisfied because there is only one Hom-set. Moreover, if $x \sim x'$ and $y \sim y'$, then we know that $x(x')^{-1} = h_x$ and $y(y')^{-1} = h_y$ for some $h_x, h_y \in H$. But then we know that

$$(yx)(y'x')^{-1} = yx(x')^{-1}(y')^{-1} = yh_x(y')^{-1}.$$

However, since $(y')^{-1} = y^{-1}h_y$, we know that this is simply

$$(yx)(y'x')^{-1} = yh_xy^{-1}h_y.$$

Because H is normal, we know that $yh_xy^{-1} \in H$. Thus the product of this and h_y is in H as well, and so $xy \sim x'y'$, as desired.

To see that $[*, *] = G/H$ simply requires the observation that $x \sim y$ if and only if x and y are in the same coset of H.

Exercise 0.15. This follows from the fact that functors preserve (or, in the case of contravariant functors, reverse) the directions of the arrows. Thus the resulting diagram still commutes.

Exercise 0.16. Note that for (i)–(iv), we can simply use inverses. For instance, for Set, it suffices to note that if f is a bijection, then f^{-1} is a bijection, which is clearly true. Similarly, the inverse of a homeomorphism is a homeomorphism, and the inverse of a group or ring isomorphism is still an isomorphism.

For (v), note that i_x^y is defined and satisfies the requirements that $i_x^y \circ i_y^x = i_x^x$ and $i_y^x \circ i_x^y = i_y^y$.

For part (vi), notice that f^{-1} works because f is a homeomorphism. In particular, it is a bijection, and so $f^{-1}(A') = A$. Moreover, it is (bi)continuous since f is.

Finally, for the monoid G, if g has a two-sided inverse h, then $hg = gh = 1$, which is the identity element of $\mathrm{Hom}(G, G)$.

Exercise 0.17. To prove that T' is a functor, first observe that criterion (i) of a functor is satisfied because T does so. Moreover, if $[f] \in \mathrm{Hom}_{\mathscr{C}'}(A, B)$, then $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$, and so $T'([f]) = Tf$ is a morphism in $\mathscr{A}$. In particular, if $[g] \circ [f] = [g \circ f]$ is defined in $\mathscr{C}'$, then $g \circ f$ is defined in $\mathscr{C}$. This means, then, that

$$T'([g] \circ [f]) = T(g \circ f) = (Tg) \circ (Tf) = T'([g]) \circ T'([f]).$$

Finally, it remains to note that $T'([1_A]) = T_{1_A} = 1_{TA} = 1_{T'([A])}$ for every object A. Thus T' is a functor.

Exercise 0.18.

(i) It is clear that $tG \in \operatorname{obj} \mathsf{Ab}$ for every group G. Now suppose that we have a homomorphism $f : G \to H$. Then we know that $t(f)$ is a morphism $f|_{tG}$ from tG to tH. To see this, note that it is the restriction of a homomorphism, and thus is itself a homormophism. Moreover, if $x \in f(tG)$, then $x = f(y)$ for some $y \in G$ with finite order. But then there exists some n so that $y^n = 1$. Thus $x^n = f(y^n) = 1$, and so x has finite order. But $x \in f(G) \subseteq H$ implies that $x \in tH$.

Now we must check that t respects composition. Indeed, if $g \circ f$ is defined, then

$$t(g \circ f) = (g \circ f)_{tG} = g|_{f(tG)} \circ f|_{tG}.$$

But $f(tG) \subseteq tH$, and so this is simply

$$t(g \circ f) = g|_{tH} \circ f|_{tG} = t(g) \circ t(f),$$

which proves that composition is respected.

Finally, note simply that $t(1_G) = 1|_{tG}$, which is the identity on tG.

(ii) Suppose that f is an injective homomorphism from G to H. Then suppose that $t(f)(x) = t(f)(y)$. But $f(x) = f|_{tG}(x) = t(f)(x)$, and so it follows that $f(x) = f(y)$. Injectivity of f proves the result.

(iii) Let $G = \mathbb{Z}$ and $H = \mathbb{Z}/2\mathbb{Z}$ and let f take even integers to 0 and odd integers to 1. This is evidently surjective. But $tG = \{0\}$ while $tH = \{0, 1\}$, and so $t(f) : tG \to tH$ cannot be surjective.

Exercise 0.19.

(i) If f is a surjection, then consider an arbitrary coset $a + pH$ of H/pH. We know that there exists some $b \in G$ with $f(b) = a$, and so it follows that $F(f)$ takes $b + pG$ to $a + pH$, proving surjectivity of $F(f)$.

(ii) Consider the function $f : \mathbb{Z} \to \mathbb{Z}$ taking x to $2x$. Then, letting $p = 2$, we know that $F(f) : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ has $F(f)([0]) = F(f)([1])$.

Exercise 0.20.

(i) This is evident because $\mathbb{R}$ is a ring, and the operations are pointwise.

(ii) By the previous part, we know that if X is a topological space, then $C(X)$ is a ring. Now suppose that $f : X \to Y$ is a continuous map. Then define

$$\begin{aligned} C(f) : C(Y) &\to C(X) \\ g &\mapsto g \circ f \end{aligned}$$

and note that this is well-defined. Moreover, we know that $C(g \circ f)(h) = h \circ g \circ f$, while $C(f) \circ C(g)$ takes h to $C(f) \circ (h \circ g) = h \circ g \circ f$, which proves that C reverses composition. Finally, we know that $C(1_x)$ takes g to $g \circ 1_X = g$ and is therefore the identity on $C(Y)$. Thus C (or, rather, the map taking X to $C(X)$, to be precise) gives rise to a contravariant functor.

1 Some Basic Topological Notions

Homotopy

No exercises!

Convexity, Contractibility, and Cones

Exercise 1.1. Suppose $H : f_0 \simeq f_1$ is a homotopy. Then let $F(t) = H(x,t)$ for some fixed x. It is clear that $F(0) = x_0$ and $F(1) = 1$. Moreover, since H is continuous, it follows that so too is F. For the converse, simply let the homotopy $H : f_0 \simeq f_1$ take $(x,t) \in X \times \mathbb{I}$ to $F(t)$.

Exercise 1.2.

(i) There exist functions $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_Y$. Moreover, there is a homotopy $F : 1_X \simeq c$, where c denotes the constant map at some $x_0 \in X$. Then consider the map $G : Y \times \mathbb{I} \to Y$ which takes (y,t) to $f(F(g(y),t))$. In particular, we know that G is continuous and that it is thus a homotopy from $f \circ g$ to the constant map c' at $y_0 = f(x_0)$. But then we find that $1_Y \simeq f \circ g \simeq c'$, and so Y is contractible.

(ii) Consider, for example, the subsets $X, Y \subset \mathbb{R}^2$ where

$$X = \{(x,0) : x \in [0,1]\},$$
$$Y = \left\{(x,x) : x \in \left[0, \frac{1}{2}\right]\right\} \cup \left\{(x, 1-x) : x \in \left[\frac{1}{2}, 1\right]\right\}.$$

It is obvious that X is convex, but Y is not, even though there is an obvious homotopy equivalence from X to Y.

Exercise 1.3. We know that $R(x) = e^{i\alpha}x$, and so the continuous map $F : S^1 \times \mathbb{I} \to S^1$ given by $F(x,t) = e^{i\alpha t}x$ is a homotopy $F : 1_S \simeq R$. Thus, if $g : S^1 \to S^1$ is continuous, then let θ be such that $g(1) = g(e^{i \cdot 0}) = e^{i\theta}$. Then we know that, letting R now be the rotation of $-\theta$ degrees, we must have $R \circ g \simeq 1_S \simeq g = g$ and $(R \circ g)(1) = 1$, as desired.

Exercise 1.4.

(i) Pick $(x_1, y_1), (x_2, y_2) \in X \times Y$. Then we know that, for any $t \in \mathbb{I}$, we have

$$t(x_1, y_1) + (1-t)(x_2, y_2) = (tx_1 + (1-t)x_2, ty_1 + (1-t)y_2).$$

The result follows from convexity of X and Y.

(ii) If $F_X : 1_X \simeq c_X$ and $F_Y : 1_Y \simeq c_Y$, where c_X and c_Y are constant maps at c_X and c_Y, respectively, then the map

$$\begin{aligned} F : (X \times Y) \times \mathbb{I} &\to X \times Y \\ (x, y, t) &\mapsto (F_X(x,t), F_Y(y,t)) \end{aligned}$$

is clearly a homotopy from $1_{X \times Y}$ to (c_X, c_Y).

Exercise 1.5. It is clear that X is compact. After all, any open cover of X must contain some set U containing 0, and thus containing cofinitely many elements of X.

If we have a map $h : X \to Y$, then because Y is discrete, we know that $\{h^{-1}(y) : y \in Y\}$ is an open covering of X and thus by compactness admits a finite subcovering. Thus there are only finitely many elements of y in the image of h.

Now suppose that $f : X \to Y$ is a homotopy equivalence. Then there exists some $g : Y \to X$ with a homotopy $H : f \circ g \simeq 1_Y$. But $H(\{y\} \times I)$ is the continuous image of a connected map and is therefore itself connected. Because Y is discrete, this means that $H(y,0) = H(y,1)$ for all y. But we know that f has finite image, and Y is infinite, so there exists some y such that $y \notin \operatorname{im} f$. In particular, we have $y \neq f(g(y))$, and so $H(y,0) = f(g(y)) \neq y = 1_Y(y)$, a contradiction. Thus X and Y are not of the same homotopy type.

Exercise 1.6. Suppose X is contractible, with $F : c \simeq 1_X$, where c is the constant map at p. Note that, for every $x \in X$, there is a path $F(x,t) : \{x\} \times \mathbb{I} \to X$ taking x to $p \in X$. In particular, this means that every x is in the same component as p, proving connectedness.

Exercise 1.7. The map $H : X \to \mathbb{I} \to X$ taking (x,t) to x and (y,t) to x if and only if $t > \frac{1}{2}$ works. Indeed, note that $H^{-1}(\{x\} \times \mathbb{I})$ is simply $\{x\} \times \mathbb{I} \cup \{y\} \times (\frac{1}{2}, 1]$, which is open in $X \times \mathbb{I}$.

Exercise 1.8.

(i) Consider the map taking the unit interval to S^1 given by $t \mapsto e^{2\pi i t}$.

(ii) If $r : Y \to X$ is a retraction, then we know from $1_Y \simeq c$ that $r \circ 1_Y \circ i \simeq r \circ c \circ i$, where i is the injection $X \hookrightarrow Y$. But the left side is simply $r \circ i = 1_X$, while the left side is a constant map, proving the result.

Exercise 1.9. We know that there exists some constant map c with $f \simeq c$. But then $g \circ f \simeq g \circ c$, and the right side is a constant map. Thus $g \circ f$ is also nullhomotopic.

Exercise 1.10. First, suppose that g is an identification. Note that $(gf)^{-1}(U)$ open in X implies that $g^{-1}(U)$ is open in Y because f is an identification. But the hypothesis on g implies that U is open in Z. Since gf is clearly a continuous surjection, the result follows.

Now, suppose that gf is an identification. It suffices to prove that $g^{-1}(U) \subseteq Y$ open implies that $U \subseteq Z$ is open. But we know by continuity of f that $f^{-1}(g^{-1}(U))$ is open, and so gf being an identification implies the result.

Exercise 1.11. First, note that this is a well-defined function in the sense that $[x] = [y]$ in $X/\sim$ implies that $\overline{f}([x]) = \overline{f}([y])$.

This is evidently continuous. After all, suppose that $U \subseteq Y/\Box$ is open. Then we know that

$$\overline{f}^{-1}(U) = \{[x] \in X/\sim : [f(x)] \in U\} = U'.$$

If we let $v : X \to X/\sim$ and $u : Y \to Y/\Box$ be the natural maps, then we know that U' is open in $X/\sim$ because

$$v^{-1}(U') = \{x \in X : f(x) \in u^{-1}(U)\} = f^{-1}(u^{-1}(U))$$

is open.

Finally, we will show that $\overline{f}$ is an identification. It is obviously surjective. Moreover, if $U' = \overline{f}^{-1}(U)$ is open in $X/\sim$, then we simply note that a similar argument as above gives us that $v^{-1}(U') = f^{-1}(u^{-1}(U))$ is open. Since f and u are identifications, it follows that U was an open set in the first place, proving the result.

Exercise 1.12. Note that if $K \subseteq Z$ is closed, then it is compact and so $h(K)$ is compact in Z, hence itself closed. Thus h is a closed map, and hence an identification.

Now because $v : X \to X/\ker h$ is an identification, Corollary 1.9 applies. Indeed, Corollary 1.9 implies that $hv^{-1} = \varphi$ is a closed map. Thus it is an identification, i.e., a continuous surjection.

But the same corollary also implies that $\varphi^{-1} = vh^{-1}$ is continuous. This, combined with Example 1.3, in which it was shown that φ is injective, proves the result, as φ is now a bicontinuous bijection, i.e., a homeomorphism.

Exercise 1.13. First observe that $f(x) = f(y)$ implies that $[x,t] = [y,t]$ and so $t = 1$. Thus f is injective and hence bijective onto its image $CX_t = \{[x,t] \in CX : x \in X\}$. Then open sets in CX_t are precisely of the form $U \cap CX_t$ for an open set $U \subseteq CX$. But clearly we can assume that $[x,1] \notin U$ because $[x,1] \notin CX_t$, and thus we wind up with $X \times [0,1)$, where $CX_t = X \times \{t\}$. This is obviously homeomorphic to X.

Exercise 1.14. The functor takes a map $f : X \to Y$ to $Cf : CX \to CY$ given by $C([x,t]) = [f(x), t]$. Note that this is well-defined. Moreover, it is obvious that this is satisfies the properties of a functor. Indeed, if $g : Y \to Z$, then

$$C(g \circ f)([x,t]) = [g(f(x)), t] = ((Cg) \circ (Cf))([x,t])$$

and clearly $C(1_X)$ is the identity on CX.

Paths and Path Connectedness

Exercise 1.15. Using the hint, suppose that $g : \mathbb{I} \to X$ is a path with $g(0) = (0, a) \in A$ and with $g(t) \in G$ for all $t > 0$. Then note that $\pi_i \circ g$ is continuous for $i = 1, 2$, where π_i are the projections to the x- and y-axes. This implies the existence of an $\varepsilon > 0$ such that $t \in (0, \varepsilon)$ implies that $g(t) = (x(t), \sin(1/x(t)))$ has $x(t), |\sin(1/x(t)) - a| < \delta$. But this is obviously impossible, as $\sin(1/x(t))$ will oscillate wildly between -1 and 1.

Exercise 1.16. Let (a_i) and (b_i) be points in S^n. We will construct n paths which, when joined together in the customary fashion (i.e., by traversing each of the $n-1$ subpaths in $1/(n-1)$ time), will give us a path from (a_i) to (b_i).

The first path f_1 is defined as

$$f_1(t) = ((1-t)a_1 + tb_1, c_2, a_3, a_4, \ldots, a_n),$$

where c_2 is chosen to be of the same sign as a_2 and in such a way that $f(t) \in S^n$. Note that such a c_2 always exists.

In general, for $1 \le i \le n-1$, the path f_i will fix every coordinate except for the i-th, which it will take to b_i, and the $(i+1)$-th, which we use as a "free" coordinate to allow for such adjusting. Moreover, observe that if the first $n-1$ coordinates of two points on S^1 are the same, then the n-th coordinates either will be the same or will be negatives.

If joining the paths $f_1, f_2, \ldots, f_{n-1}$ together gives a path from (a_i) to (b_i), then we are done. Note that this occurs if a_n and b_n have the same sign.

Otherwise, construct a path g which adjusts the n-th coordinate and uses the $(n-1)$-th coordinate as a "free" one, preserving the sign. This effectively allows us to switch the sign of the n-th coordinate so that the n-th coordinate is just b_n. Moreover, because we preserved the sign of the $(n-1)$-th coordinate, it is still equal to b_{n-1}.

Exercise 1.17. It suffices to show the forward direction, so suppose that U is not path connected. Then there are at least two path components.

We will show that each path component is open, which will prove that U is not connected. But because U is open, we know that open sets in U (as a subspace) or also open in $\mathbb{R}^n$. Thus, for every $x \in U$, there is a ball B_x centered at x and contained in U. This ball is obviously path-connected. As such, if x is in the path component A, it must follow that $B_x \subseteq A$, proving that A is open.

Exercise 1.18. We know that if X is contractible then there exists a point $c \in X$ such that 1_X is homotopic to the constant map at c from X to itself. Now consider the map $c : \mathbb{I} \to X$ satisfying $c(t) = c$ for all t. In the proof of Theorem 1.13, we saw that any path is homotopic to c. In particular, the constant maps $x : \mathbb{I} \to X$ and $y : \mathbb{I} \to X$ at x and y, respectively, are both homotopic to c. Note that these give rise to paths from x to c and from c to y, respectively, which in turn give rise to a path from x to y. This proves path connectedness.

Exercise 1.19.

(i) If X is path connected, then let c and c' be constant maps. Let f be a path from (the point) c to (the point) c' and define $H : X \times \mathbb{I} \to X$ as $H(x, t) = f(t)$. Then H is a homotopy from c to c'.

For the reverse direction, let H be a homotopy from c to c' and define the path $f : \mathbb{I} \to X$ as $f(t) = H(c, t)$.

(ii) Let $f : X \to Y$ be a continuous function. Fix some $y_0 \in Y$ and consider the map

$$\begin{aligned} H : X \times \mathbb{I} &\to Y \\ (x, t) &\mapsto p_x(t), \end{aligned}$$

where p_x is a path from $f(x)$ to y_0. This is a homotopy from f to the constant map mapping X to y_0.

But if $g : X \to Y$ is another continuous function, then the same argument shows that $g \simeq y_0$, and so $f \simeq g$, as desired.

Exercise 1.20. It suffices to show that if $a \in A$ and $b \in B$, then there is a path from a to b. But fix some point $x \in A \cap B$. Then there is a path from a to x, and a path from x to b. Joining the two paths gives a path from a to b.

Exercise 1.21. This is simply done by noting that for any $(x, y), (x', y') \in X \times Y$, we can join the paths $f(t) = ((1-t)x + tx', y)$ and $g(t) = (x', (1-t)y + ty')$.

Exercise 1.22. Suppose $f(a), f(b) \in Y$. Then let p be a path from a to b in X. Now simply note that $q(t) = f(p(t))$ is a path from $f(a)$ to $f(b)$, proving the result.

Exercise 1.23.

(i) We already know that there are at least two path components because the entire space is not path connected. Moreover, both A and G are path connected, and so it follows that they must themselves be the path components.

(ii) Simply note that the sequence $\left\{\left(\frac{1}{n\pi}, \sin(n\pi)\right)\right\} \subset G$ approaches $(0,0) \in A$.

(iii) As per the hint, consider U to be the open disk with center $(0, \frac{1}{2})$ and radius $\frac{1}{4}$. Then $X \cap U$ is open in X. But note that $v(X \cap U)$ is not open in $X/A \approx [0, \frac{1}{2\pi}]$. After all, note that any ball B_ε around the point 0 (which is the image of A under the natural map in this case) must contain some point $\frac{1}{n\pi} < \varepsilon$. But $\frac{1}{n\pi}$, which corresponds to the point $\left(\frac{1}{n\pi}, 0\right) \in X \setminus U$, is not contained in $v(X \cap U)$.

Exercise 1.24. By definition, path components are path connected. Moreover, if C is a path component and there exists some point $x \in X$ and $c \in C$ so that there is a path between x and c, then the definition of path components implies that $x \in C$. Thus path components are maximally path connected.

Finally, suppose that A is path connected and pick $a \in A$. There exists a unique path component C such that $a \in C$. Then for all $b \in A$, we know that there is a path between a and b, and so $b \in C$. Thus $A \subseteq C$, as desired.

Exercise 1.25. Simply use Exercise 1.22 and observe that I is path connected.

Exercise 1.26. Note that, if X is locally path connected, then for all $x \in X$, there exists some open path connected, hence connected, neighborhood V of x. Alternatively, note that if $U \subseteq X$ is open, then its components are unions of its path components and thus open.

Exercise 1.27. Given any open subset U of $X \times Y$ containing a given point $(x, y) \in X \times Y$, there must exist a basic open neighborhood $U_x \times U_y \subseteq U$ of (x, y). Then we know that there exists some path connected V_x with $x \in V_x \subseteq U_x$, and similarly for y. Then $V_x \times V_y$ is path connected by Exercise 1.21. The result follows.

Exercise 1.28. Note that open subsets of open subsets are open in the main space. In particular, let $A \subseteq X$ be open. Given any $x \in A$, let U be an open neighborhood of x in A. Note that this is also an open neighborhood in X, and so there exists an open path connected V in X (and hence open in A as well) such that $x \in V \subseteq U$.

Exercise 1.29. Consider the map $F : (\mathbb{R}^{n+1} \setminus \{0\}) \times \mathbb{I} \to \mathbb{R}^{n+1} \setminus \{0\}$ given by

$$F((x_i), t) = \left[(1-t) + \frac{t}{\sqrt{\sum x_i^2}}\right](x_i).$$

This is evidently a homotopy which makes S^n a deformation retract.

Exercise 1.30. The exact same map as in Exercise 1.29 works for this case.

Exercise 1.31. It is easy to see that the deformation retract of a deformation retract is a deformation retract, either by a direct argument or by applying Theorem 1.22. Thus the previous exercise implies that it suffices to show that $D^n \setminus \{0\}$ is a deformation retract of $S^n \setminus \{a, b\}$. But the map $(x_i) \mapsto (x_1, \ldots, x_{n-1}, 0)$ is exactly the map needed, and so we are done.

Exercise 1.32. If $H : f_0 \simeq f_1$, then the map $H' : (y, t) \mapsto H(r(y), t)$ is a homotopy from $\tilde{f}_0$ to $\tilde{f}_1$.

Exercise 1.33. Let $Y = \{y\}$ and observe that $(x, 1) \sim y$ for all $x \in X$. Thus $(x, 1) \sim (x', 1)$ for alL $x, x' \in X$. Moreover, this is the only equivalence. Thus M_f is precisely the quotient space $(X \times \mathbb{I})/(X \times \{1\}) = CX$.

Exercise 1.34.

(i) We first tackle i. It is obvious that i is injective, and thus a bijection onto its image $i(X) = \{[x, 0] : x \in X\}$. Moreover, the open sets in $i(X)$ are precisely of the $U \cap i(X)$ for open sets U in M_f.

Note that we can suppose without loss of generality that U is contained in $v(X \times [0, 1))$, where v is the natural map. Thus U simply looks like the Cartesian product of an open interval with an open set of X. This proves that i is a homeomorphism, for the open sets of $i(X)$ map exactly to the open sets of X.

We can show that j is a homeomorphism onto $j(Y)$ in a similar manner. The main idea is simply that $y \not\sim y'$ for any $y, y' \in j(Y)$.

(ii) It is obvious that $(rj)(y) = r[y] = y = 1_Y(y)$ for any $y \in Y$. It is also clearly continuous by the gluing lemma. Thus r is indeed a retraction.

(iii) Define $F : M_f \times \mathbb{I} \to M_f$ as suggested in the hint. It is evident that F is continuous. Moreover, for any $[x, t] \in M_f$, we know that

$$F([x, t], 0) = [x, t]$$
$$F([x, t], 1) = [x, 1] = [f(x)] \in Y.$$

Similarly, if $[y] \in Y$, then the definition implies that the remaining criteria for this homotopy to induce a deformation retraction $r(x) = F(x, 1)$ are satisfied.

(iv) Note that Rotman writes that f is homotopic to $r \circ i$; in fact, we can and do prove the stronger statement that f coincides with $r \circ i$.

Let $f : X \to Y$ be continuous. Then it is clear that the map $f = r \circ i$, where $i : X \to M_f$ is an injection and $r : M_f \to Y$ is the retraction taking $[x, t]$ to $[f(x)]$ and taking $[y]$ to itself, proving the result.

2 Simplexes[1]

Affine Spaces

Exercise 2.1. Note that there is a maximal affine independent subset S of A. This is directly implied by the fact that any set of greater than $n+1$ elements is not affine independent. Hence we can take an affine independent subset of A with maximum size (because the empty set is affine independent).

Write $S = \{p_0, \ldots, p_m\}$. Then let $p_{m+1} \in A \setminus S$. By maximality of S, we know that $S \cup \{p\}$ is not affine independent. Hence there exist s_i not all 0 such that

$$\sum_{i=0}^{m+1} s_i p_i = 0, \quad \sum_{i=0}^{m+1} s_i = 0.$$

Note that the second equation implies $\sum_{i=0}^{m} s_i \neq 0$ for some $i < m+1$. It follows then that

$$\sum_{i=0}^{m} \left(\frac{s_i}{\sum_{i=0}^{m} s_i} p_i \right) = p_{m+1}.$$

But we know that

$$\sum_{i=0}^{m} \frac{s_i}{\sum_{i=0}^{m} s_i} = 1,$$

and so it follows that p_{m+1} is in fact in the affine span of S.

Exercise 2.2. Let φ be the isomorphism from $\mathbb{R}^n$ to a subset of $\mathbb{R}^k$. Suppose $A \subseteq \mathbb{R}^n$ is an affine set containing X. Then $\varphi(X) \subseteq \varphi(A) \subseteq \mathbb{R}^k$.

Moreover, we claim that $\varphi(A)$ is affine. After all, for any $\varphi(x), \varphi(x') \in \varphi(A)$ and any $t \in \mathbb{R}$, the point $t\varphi(x) + (1-t)\varphi(x') = \varphi(tx + (1-t)x') \in \varphi(A)$ because A is affine.

This implies that the intersection of all affine sets in $\mathbb{R}^n$ containing X must contain the intersection of all affine sets in $\varphi(\mathbb{R}^n)$ containing $\varphi(X)$. Because φ is an isomorphism, using φ^{-1} gives the reverse inclusion. Thus the affine set spanned by X in $\mathbb{R}^n$ is precisely the same as that spanned by X in $\mathbb{R}^k$.

Exercise 2.3. This is evident in the case $n = 0$.

Suppose it is true for $n-1$ and consider the canonical injection $\iota : S^{n-1} \hookrightarrow S^n$ which takes $(x_0, \ldots, x_{n-1})$ to $(x_1, \ldots, x_{n-1}, 0)$. It is obvious that we can pick $n+1$ affine independent points $p_0, \ldots, p_n$ in this embedding.

Now consider the point $p_{n+1} = (0, \ldots, 0, 1) \in S^n$. Notice that the last coordinate of each p_i for $i \neq n+1$ is zero. Thus suppose we have s_i with $\sum s_i p_i = 0$ and $\sum s_i = 0$. Then $s_{n+1} = 0$, and so this reduces to the $n-1$ case. Affine independence of $\{p_0, \ldots, p_n\}$ proves the result.

Exercise 2.4. Consider the map $T'(x) = T(x) - T(0)$. We claim that T' is a linear map.

Observe that $S = \{e_i\} \cup \{0\}$ spans $\mathbb{R}^n$. Thus we can write any point as the affine sum of elements of S. Note that the coefficient of the zero vector is flexible, and so we have effectively no restrictions on the sum of the coefficients.

Consider arbitrary elements $\sum r_i e_i + r \cdot 0$ and $\sum s_i e_i + s \cdot 0$ in $\mathbb{R}^n$, where $r = 1 - \sum r_i$ and similarly for s. Let $R, S \in \mathbb{R}$. Then note that

$$\begin{aligned} T'\left(R\sum r_i e_i + S\sum s_i e_i\right) &= T'\left(\sum (Rr_i + Ss_i) e_i\right) \\ &= T\left(\sum (Rr_i + Ss_i) e_i + \left(1 - \sum (Rr_i + Ss_i)\right) \cdot 0\right) - T(0) \\ &= R\sum r_i T(e_i) + S\sum s_i T(e_i) - R\sum r_i T(0) - S\sum s_i T(0). \end{aligned}$$

Considering the R-terms first, simply observe that we can add and subtract $RT(0)$ to give us that

$$R\sum r_i T(e_i) - R\sum r_i T(0) = R\left(T\left(\sum r_i T(e_i) + r \cdot 0\right) - T(0)\right).$$

[1] I usually use *simplices* as the plural of simplex, but Rotman doesn't; no matter.

This is simply $RT'\left(\sum r_i e_i\right)$. A similar result holds for the S-terms, from which we conclude that

$$T'\left(R\sum r_i e_i + S\sum s_i e_i\right) = RT'\left(\sum r_i e_i\right) + ST'\left(\sum s_i e_i\right),$$

proving linearity.

Exercise 2.5. This is obvious from the previous exercise and continuity of linear maps.

Exercise 2.6. Given two m-simplexes $[p_0, \ldots, p_m]$ and $[q_0, \ldots, q_m]$, the map f taking p_i to q_i for every i is a homeomorphism. Bijectivity is obvious by the definition. Continuity is clear by how we extend f from $\{p_i\}$ to $[p_i]$. Finally, the inverse is of the same form as f, only with the q_i's taking the place of the p_i's and vice versa; thus f^{-1} is also continuous.

Exercise 2.7. The following map works:

$$f : x \mapsto \frac{t_2 - t_1}{s_2 - s_1}(x - s_1) + t_1.$$

Exercise 2.8. Pick arbitrary $T(x), T(x') \in T(X)$ and observe that

$$tT(x) + (1-t)T(x') = T(tx + (1-t)x') \in T(X).$$

Thus $T(X)$ is affine if X is affine, and convex if X is convex. The second statement of the exercise follows by noting that ℓ is convex.

Exercise 2.9. Without loss of generality, we delete p_0. Now suppose that

$$\sum_{i=1}^{m} s_i p_i + sb = 0, \quad \sum_{i=1}^{m} s_i + s = 0.$$

Then we know by definition of the barycenter b that

$$\sum_{i=1}^{m} s_i p_i + \frac{s}{m+1}\sum_{i=0}^{m} p_i = 0.$$

Moreover, letting s'_i be the coefficient of p_i in the above equation, it is obvious that $\sum_{i=0}^{m} s'_i = s + \sum_{i=1}^{m} s_i = 0$. Thus $s'_i = 0$ for all i because $\{p_0, \ldots, p_m\}$ was affine independent. But then we conclude that $0 = s'_0 = \frac{s}{m+1}$, and so $s = 0$. For every $i \in \{1, \ldots, m\}$, we have $0 = s'_i = \frac{s}{m+1} + s_i$. Thus $s = 0$ implies $s_i = 0$ for every i,a nd so it follows that $\{b, p_1, \ldots, p_m\}$ is affine independent, as desired.

Exercise 2.10. Once again, suppose without loss of generality that $i = 0$. Then the map taking $\sum t_i p_i \in [p_0, p_1, \ldots, p_m]$ to $(\sum_{i=1}^{m} t_i p_i, t_0)$ works. Note that this actually requires the affine independence of the p_i's, as well as the fact that the coefficients t_i are all between 0 and 1.

Exercise 2.11. Notice that $[0, e_1, \ldots, e_n]$, where e_i are the standard basis vectors in $\mathbb{R}^n$, is an n-simplex. Thus there is a homeomorphism $[p_0, \ldots, p_n] \to [0, e_1, \ldots, e_n]$. If we translate the image by $\mathbf{v} = (-\frac{1}{4}, \ldots, -\frac{1}{4})$, then we can map the result to D^n by taking a radial mapping. In particular, this map will take

$$\begin{aligned} p_0 &\mapsto \frac{\mathbf{v}}{\|\mathbf{v}\|} \\ p_i &\mapsto \frac{e_i + \mathbf{v}}{\|e_i + \mathbf{v}\|} \text{ for } i \neq 0. \end{aligned}$$

Note that this extends to a homeomorphism.

3 The Fundamental Group

The Fundamental Groupoid

Exercise 3.1. The homotopy $H : X \times \mathbb{I} \to Z$ given by

$$H : (x,t) \mapsto \begin{cases} g_0(F(x,2t)) & \text{if } t \leq \frac{1}{2}, \\ G(f_1(x), 2t-1) & \text{if } t \geq \frac{1}{2} \end{cases}$$

works. Continuity follows because $g_0(F(x,1)) = G(f_1(x), 0)$.

Moreover, this homotopy is indeed $\operatorname{rel} A$. For a detailed argument why this is so, simply suppose that $a \in A$ and $t \in I$. If $t \leq \frac{1}{2}$, then $F(a, 2t) = f_0(a)$ by definition of F. Hence $H(a,t) = g_0(f_0(a))$.

Similarly, we can show that if $t \geq \frac{1}{2}$, then $H(a,t) = g_1(f_1(a))$. This follows because $f_1(a) \in B$ and G is a homotopy $\operatorname{rel} B$.

It thus suffices to show that $g_0(f_0(a)) = g_1(f_1(a))$. But this is obvious because f_0 and f_1 agree on A, and g_0 and g_1 agree on $B \supseteq f_0(A)$.

Exercise 3.2.

(i) First, note that f' is well-defined because $f(0) = f(1)$. It is obvious by continuity of f and ln that f' is continuous.

Moreover, consider the map

$$H' : \left(e^{2\pi i\theta}, t\right) \mapsto H(\theta, t).$$

This is clearly continuous, for the same reasons that f' was continuous. If $t = 0$, clearly $H'(e^{2\pi i\theta}, t) = H(\theta, 0) = f(\theta) = f'(e^{2\pi i\theta})$, and similarly for $t = 1$. Thus H is indeed a homotopy from f' to g'.

To see that it is a homotopy $\operatorname{rel}\{1\}$, simply note that $e^{2\pi i\theta} = 1$ corresponds to $\theta = 0, 1$. Thus it follows that

$$H'(1,t) = H(1,t) = f(1)$$

for all t, proving the result.

(ii) Theorem 3.1 implies that $f * g \simeq f_1 * g_1 \operatorname{rel} \dot{\mathbb{I}}$. Using the previous part, we find that $(f * g)' \simeq (f_1 * g_1)' \operatorname{rel}\{1\}$. Now, using the observation that $(f * g)' = f' * g'$, we find that $f' * g' \simeq f_1' * g_1' \operatorname{rel}\{1\}$, as desired.

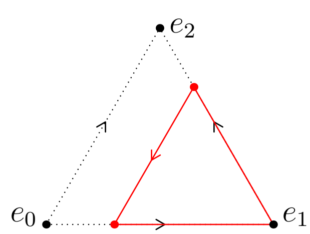
Exercise 3.3. The forward direction is trivial.

For the converse, note that g' is a constant map, and so f' is nullhomotopic. Then Theorem 1.6 implies that $f' \simeq g' \operatorname{rel}\{1\}$. In particular, note that $g' : S^1 \to X$ takes every element of S^1 to $g'(1) = g(0) = x_0$. Observe that $f'(1) = x_0$ as well, and so it follows that $f' \simeq g \operatorname{rel}\{1\}$, as desired.

Exercise 3.4.

(i) Instead of applying Theorem 1.6, I constructed an explicit homotopy. (If you are interested in a proof using Theorem 1.6, my guess would be that it relies on the fact that $\Delta^2 \approx D^2$. However, I have not gone through the details.)

The effective idea of the homotopy I constructed is to, at time $t \in [0,1]$, return the function which traverses the first t units of the face opposite e_0, then goes along a segment to the point t units away from e_1 on the fact opposite e_2, before returning back to e_1, as shown in the red path below.



The specific homotopy $H : \mathbb{I}\times\mathbb{I} \to X$ from $(\sigma_0 * \sigma_1^{-1}) * \sigma_2$ to the constant map at e_1 is as follows:

$$H(x,t) = \begin{cases} \sigma_0(4(1-t)x) & \text{if } x \le \frac{1}{4}, \\ \sigma((1-x)\varepsilon_0(1-t) + x\varepsilon_2(t)) & \text{if } \frac{1}{4} \le x \le \frac{1}{2}, \\ \sigma(2tx - (2t-1)) & \text{if } x \ge \frac{1}{2}. \end{cases}$$

We leave it to the reader to check that this works.

(ii) One can generate a similar homotopy, which we do not do here.

(iii) This time, we use the homotopy which goes up along γ for t units, before going parallel to β and coming back down along δ^{-1}. The particular formula is as follows:

$$H(x,t) = \begin{cases} F(0,4tx) & \text{if } x \le \frac{1}{4}, \\ F(4x-1,t) & \text{if } \frac{1}{4} \le x \le \frac{1}{2}, \\ F(1,2t(1-x)) & \text{if } \frac{1}{2} \le x. \end{cases}$$

Once again, we leave the details to the reader to check.

Exercise 3.5. Simply use the homotopy $H : \mathbb{I}\times\mathbb{I} \to X \times Y$ which takes (s,t) to $(F(s,t), G(s,t))$. This is clearly a homotopy from (f_0, g_0) to (f_1, g_1). To see that it is still $\operatorname{rel} \dot{\mathbb{I}}$, simply observe that $H(0,t) = (F(0,t), G(0,t))$. Because F and G are both $\operatorname{rel} \dot{\mathbb{I}}$, it follows that $H(0,t)$ never changes. A similar argument shows that $H(1,t)$ is always the same, and so H is indeed a homotopy $\operatorname{rel} \dot{\mathbb{I}}$.

Exercise 3.6.

(i) It is obvious that the homotopy $H' : (x,t) \mapsto H(x, 1-t)$ works.

(ii) This is just some slightly annoying manipulation. In particular, note that

$$(f * g)(x) = \begin{cases} f(2x) & \text{if } x \le \frac{1}{2}, \\ g(2x-1) & \text{if } x \ge \frac{1}{2}. \end{cases}$$

By replacing x with $1-x$ to get the inverse, we find that

$$(f * g)^{-1}(x) = \begin{cases} f(2-2x) & \text{if } x \ge \frac{1}{2}, \\ g(1-2x) & \text{if } x \le \frac{1}{2}. \end{cases}$$

However, note that

$$\begin{aligned}(g^{-1} * f^{-1})(x) &= \begin{cases} g^{-1}(2x) & \text{if } x \ge \frac{1}{2}, \\ f^{-1}(2x-1) & \text{if } x \ge \frac{1}{2} \end{cases} \\ &= \begin{cases} g(1-2x) & \text{if } x \le \frac{1}{2}, \\ f(2-2x) & \text{if } x \ge \frac{1}{2}. \end{cases}\end{aligned}$$

Thus the two are indeed the same.

(iii) Take the closed path $f(t) = e^{2\pi i t}$ on S^1. Then note that $(f * f^{-1})(\frac{1}{8}) = f(\frac{1}{4}) = i$, while $(f^{-1} * f)(\frac{1}{8}) = f^{-1}(\frac{1}{4}) = -i$.

(iv) Suppose $i_p * f = f$ and f is not constant. Note that continuity implies that there must exist some $0 < t < 1$ so that $f(t) \ne p$. Thus there exists some $k \in \mathbb{N}$ so that $t < 1 - 2^{-k}$.

We claim, however, that f must be constant on $[0, 1-2^{-n}]$ for every $n \in \mathbb{N}$. We prove this inductively. Clearly, it is true on $n = 0$. If it is true on $n-1$, then we know that $i_p * f$ must be equal to p on $[0, \frac{1}{2}]$, as well as on $[\frac{1}{2}, 1-2^{-n}]$ (note that $1 - 2^{-n}$ comes from $2(1-2^{-n}) - 1$, which itself comes from the equation for the star operator). Thus f is constant on $[0, 1-2^{-n}]$, as desired.

Thus it follows that $f(t) = p$, a contradiction. Thus f must have been constant in the first place.

Exercise 3.7. Recall that we defined the $\sin(1/x)$ space as the union of $A = \{(0, y) : -1 \leq y \leq 1\}$ and $G = \{(x, \sin(1/x)) : 0 < x \leq 1/2\pi\}$. We also know that A and G are the path components of the $\sin(1/x)$ space. Moreover, both A and G are contractible, and so every path in either A or G is nullhomotopic. In particular, we conclude that the fundamental group at any basepoint is trivial.

Exercise 3.8. Let X be the $\sin(1/x)$ space. We know that CX is contractible. But consider an open ball around the point $x = ((0, 0), 0)$, that is, the point $(0, 0)$ on the "zeroth" level of the cone. Consider a small neighborhood (not including the points $(t, 1)$, in particular) around this point and pick some element $y = ((\varepsilon, \sin(1/\varepsilon)), 0)$ in the neighborhood. Now observe that any path between x and y can be projected down to a path between $(0, 0)$ and $(\varepsilon, \sin(1/\varepsilon))$ in X, which we know does not exist. Hence CX is contractible but not locally path connected.

Exercise 3.9. Note that composition is associative because $\circ$ is. Moreover, the path class of the trivial loop based at p is the identity on p. Thus this is a category.

To see that each morphism in $\mathscr{C}$, simply note that the inverse path, i.e., the path f^{-1} taking t to $f(1-t)$, gives a path class $[f^{-1}]$ which works as an inverse to $[f] \in \mathrm{Hom}(p, q)$.

Exercise 3.10. We simply let π_0 take $(X, x_0) \in \mathsf{Sets}_*$ to the set of all path components of X, with basepoint equal to the path component containing x_0. It takes a morphism $f \in \mathrm{Hom}((X, x_0), (Y, y_0))$ to the map $\pi_0(f)$ which takes each path component A of X to the path component B of Y which contains $f(A)$.

Note that this is possible because continuous images of path connected spaces are path connected and hence contained within a single path component of Y. Moreover, this is indeed a pointed map because the path component containing x_0 must be contained in the path component containing $f(x_0) = y_0$, which is the basepoint of $\pi_0((Y, y_0))$.

It is easy to check functoriality, completing the proof.

Exercise 3.11. Evidently the only possible path is the constant path at x_0. Hence $\pi_1(X, x_0)$ is the trivial group, i.e., $\{1\}$.

Exercise 3.12. Note that 1_S is a loop based at 1, i.e., an element of $\pi_1(S^1, 1)$. Thus if $\pi_1(S^1, 1)$ were trivial, then 1_S would be nullhomotopic. The hint gives the rest of the solution.

Exercise 3.13. We know that $\deg u = 1$. Since 1 is a generator for $\mathbb{Z}$, it follows that $[u]$ generates $\pi_1(S^1, 1)$.

Exercise 3.14. Let $\tilde{\gamma}(t) = m\tilde{f}(t)$, where $\tilde{f}$ is the lifting of f satisfying $\tilde{f}(0) = 0$. Now simply observe that

$$\exp \tilde{\gamma}(t) = \left(\exp \tilde{f}(t)\right)^m = f(t)^m$$

and $\tilde{\gamma}(0) = 0$. Thus $\tilde{\gamma}$ is indeed the lifting of f^m taking 0 to 0, and so we conclude that

$$\deg(f^m) = \tilde{\gamma}(1) = m\tilde{f}(1) = m \deg f.$$

Exercise 3.15. Note that Exercise 1.3 implies that there is a homotopy $F : R_f \circ f \simeq f$, where R_f is the rotation associated with f. Moreover, from the proof of that same exercise, it follows that F gives a closed path at every time t. Similarly, we have $G : g \simeq R_g \circ g$. Thus if $H : f \simeq g$ where H gives a closed path at every time t, then the homotopy which follows F, then H, and finally G is a homotopy between $R_f \circ f$ and $R_g \circ g$. Thus Corollary 3.18 implies that f and g have the same degree.

For the converse, simply use Corollary 3.18 to show that $\deg f = \deg g$ implies that there is a homotopy rel $\dot{\mathbb{I}}$ taking $R_f \circ f$ to $R_g \circ g$. Then using F and G defined above, it is clear that $g \simeq R_g \circ g \simeq R_f \circ f \simeq f$.

Exercise 3.16. Theorem 3.7 implies that $\pi_1(T, t_0) = \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$.

Exercise 3.17. Because D^2 is contractible, its fundamental group is trivial. Thus if there were to exist a retraction $r : D^2 \to S^1$, then $r_* : \pi_1(D) \to \pi_1(S^1)$ would be a constant. But then, letting $i : S^1 \to D^2$ be the canonical injection, we would have that $(r \circ i)_* = r_* \circ i_*$ is a constant. However, we also know that $r \circ i$ is the identity on S^1, and so $(r \circ i)_*$ is the identity on $\pi_1(S^1)$, which is *not* a constant. This is a contradiction, from which we conclude that S^1 is not a retract of D^2, as desired.

Exercise 3.18. This was proved in Theorem 0.3, which required only the fact proved in the above problem, namely that S^1 is not a retract of D^2.

Exercise 3.19.

(i) Let $\tilde{f}$ be the unique lifting of f with $\tilde{f}(0) = 0$. Then if $\tilde{f}(1) \geq 1$, the intermediate value theorem implies that every point in the interval $[0, 1] \subset \mathbb{R}$ is in the image of $\tilde{f}$. But this implies that $f = \exp \circ \tilde{f}$ must be surjective, a contradiction.

(ii) Consider the map which traverses the circle once counterclockwise, reaching the point 1 at time $t = \frac{1}{2}$, before looping back and making a clockwise rotation. Clearly it is surjective. However, it is composed of two loops, one of which has degree 1 and one of which has degree -1. Because $\deg(f * g) = \deg f + \deg g$, it follows that this map has degree 0.

Exercise 3.20. As per the hint, consider an arbitrary closed path f in X and let λ be a Lebesgue number of the open cover $\{f^{-1}(U_j) : j \in J\}$ of $\mathbb{I}$. Note that λ exists by the Lebesgue number lemma and compactness of the unit interval. Picking $N \in \mathbb{N}$ with $N > 1/\lambda$, it follows that if we subdivide I into N equal intervals $I_k = [\frac{k}{N}, \frac{k+1}{N}]$, then $f(I_k) \subseteq U_{j_k}$ for some $j_k \in J$.

Define f_k as the path in U_{j_k} obtained by restricting f to I_k and then stretching suitably so that the domain is all of $\mathbb{I}$. With notation, define $f_k(t) = f\left(\frac{k+t}{N}\right) \in U_{j_k}$. Because f_k is a path in U_{j_k}, it follows that $[f_k'] = [i_{j_k} \circ f_k] \in \operatorname{im}(i_j)_*$. But now simply observe that $[f_0' * \cdots * f_{N-1}'] = [f]$, implying that $[f]$ is contained in the group generated by the subsets $\operatorname{im}(i_j)_*$. This proves the result.

Exercise 3.21. Let U_1 and U_2 be defined as in the hint, and let i_k be the injection from U_k to S^n for $k = 1, 2$. Observe that, by the previous exercise, it suffices to show that $\operatorname{im}(i_k)_*$ is trivial for $k = 1, 2$.

Without loss of generality, let $k = 1$. But we know that $(i_1)_*$ takes a closed path $f : \mathbb{I} \to U_1$ to the path class $[i_1 \circ f]$. (Note that the basepoint doesn't really matter for us as long as it is neither the north nor the south pole.Thus we omit it.) Because $U_1 \approx D^n$ and is therefore contractible, it follows that f is nullhomotopic. In particular, we know that $i_1 \circ f$ is nullhomotopic, and so $[i_1 \circ f] = [1]$ for every f. Thus $\operatorname{im}(i_1)_*$ is trivial, and similarly for $k = 2$, proving the result.

Exercise 3.22. Corollary 3.11 implies that path connected spaces of the same homotopy type must have isomorphic fundamental groups. But obviously $\mathbb{Z} \ncong \{1\}$, and so S^1 and S^n do not have the same homotopy type for $n > 1$.

Exercise 3.23. The multiplication map μ on G/H is continuous. After all, if we let v be the natural map, then for any open set $U \subseteq G/H$, we have

$$\mu^{-1}(U) = \{([x], [y]) : xy \in v^{-1}(U)\}.$$

But this set is open in $G/H \times G/H$ because the set consisting of elements $(v^{-1}([x]), v^{-1}([y]))$ for each $([x], [y]) \in \mu^{-1}(U)$ is just $\mu^{-1}(v^{-1}(U))$, which is clearly open.

For the inversion map $i : G/H \to G/H$, a very similar argument holds. In particular, for any open set $U \subseteq G/H$, we have

$$v^{-1}(i^{-1}(U)) = \{x^{-1} : x \in v^{-1}(U)\}.$$

Thus $v^{-1}(i^{-1}(U))$ is open, and so $i^{-1}(U)$ is open, proving continuity.

Exercise 3.24. First, we will show that we can lift a loop $f : (\mathbb{I}, 0) \to (G/H, 1)$ to a unique continuous map $\tilde{f} : (\mathbb{I}, 0) \to (G, h)$ for any $h_0 \in H$, as shown below.

$$\begin{tikzcd} & (G, h_0) \arrow[d, "v"] \\ (\mathbb{I}, 0) \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & (G/H, 1) \end{tikzcd}$$

In the above diagram, the map v is the natural map taking g to the coset $gH \in G/H$.

First, we will find a suitable neighborhood U of 1 such that the family $\{hU : h \in H\}$ is pairwise disjoint. Discreteness of H implies that there exists an open neighborhood V of 1 with $V \cap H = \{1\}$. It is clear that the map $\varphi : (x, y) \mapsto xy^{-1}$ is the composition $\mu \circ (\mathrm{id} \times i)$ and is therefore continuous. Thus $\varphi^{-1}(V) \subseteq G \times G$ is an open neighborhood of $(1, 1)$. This implies that we can find an open neighborhood U of 1 such that $U \times U \subseteq f^{-1}(V)$.

Now suppose that there are $h_1, h_2 \in H$ and $x, y \in U$ with $h_1x = h_2y$. But this would require that $xy^{-1} = h_1^{-1}h_2$. It is clear that $xy^{-1} \in \varphi(U) \subseteq V$. Moreover, because H is a subgroup, we know that $h_1^{-1}h_2 \in H$, and so $xy^{-1} \in V \cap H$. Thus $x = y$ and $h_1 = h_2$, proving that the sets hU are disjoint, as desired. Note that any translate $U_g = gU$ of U is a neighborhood of $g \in G$ and has $\{hU_g : h \in H\}$ disjoint.

Note that v is an open map, and so the set $W = v(U) \subseteq G/H$ is open. Moreover, because $v|_U$ is the restriction of a continuous open map to an open set, it follows that $v|_U$ is itself continuous and open. It is also a bijection onto W, and so $v|_U : U \to V$ is a homeomorphism.

Note that the collection of sets $V[g]$ for $[g] \in G/H$ forms an open cover of G/H. Thus, if we are given some $f : (\mathbb{I}, 0) \to (G/H, 1)$, then we can consider the open cover

$$\{f^{-1}(V[g]) : [g] \in G/H\}$$

of $\mathbb{I}$. Note that we can find a finite subcover of this open cover. This means that we can take subsets of the sets in this open cover, given us a finite collection open overlapping subintervals which are, in order of their smaller coordinate, labeled $I_1, \dots, I_k$. Let the group elements $g_1, \dots, g_k$ be such that $I_j \subseteq f^{-1}(V[g_j])$. This is simply because $\mathbb{I} = [0, 1]$ is connected compact.

Now we can lift f to each interval $f^{-1}(V[g])$ in this finite subcover. Note that $0 = t_1 \in I_1 \subseteq f^{-1}(V[g_1])$. Moreover, we know that $v^{-1}(V[g_1])$ consists of disjoint unions of U, and so we can pick the one containing h_0. Now, for each $t \in I_1$, we let $\tilde{f}(t)$ to be the unique element in this copy of U such that $v(\tilde{f}(t)) = f(t)$. Because the intervals overlap, we know that there is some $t_2 \in I_2 \cap I_1$, and so we can do the same thing, all the way to t_k. This lets us define $\tilde{f}(t)$ for all $t \in \mathbb{I}$, and it is easy to show that our construction is indeed a lifting satisfying the commutative diagram above.

Now consider the map $d : \pi_1(G/H, 1) \to H$ taking a loop $[f]$ to $d([f]) = \tilde{f}(1)$, where $\tilde{f}$ is the unique lifting of f with $\tilde{f}(0) = 1$. It is obvious that $\operatorname{im} d \subseteq H$ because $v(\tilde{f}(1)) = f(1) = [1]$ implies that $d([f]) = \tilde{f}(1) \in H$. Moreover, the reverse inclusion holds, showing surjectivity. In particular, if $h \in H$, then path connectedness of G implies that there is a path $\tilde{f}$ from 1 to h. Taking its projection $f = v \circ \tilde{f}$, note that f is a loop because $v(\tilde{f}(1)) = v(h) = [1]$. Thus $d([f])$ is defined and equal to h. To show injectivity, simply note that $\tilde{f}(1) = 1$ implies that $\tilde{f}$ is a loop in G. Because G is simply connected, however, it follows that $\tilde{f}$, and hence f, is nullhomotopic. Thus $f \in \ker d$ implies that $[f] = [1]$. Finally, we must show that d is indeed a homomorphism, i.e., that $d(f * g) = d(f)d(g)$. But this is clear if we lift f to $\tilde{f}$ with $\tilde{f}(0) = 1$, and if we lift g to $\tilde{g}$ with $\tilde{g}(0) = d(f)$. This follows the same proof layout as Theorem 3.16, and proves the result.

Exercise 3.25. If $S \subseteq GL(n, \mathbb{R})$ is a subgroup of $GL(n, \mathbb{R})$, then note that $\mu : S \times S \to S$ is continuous. After all, the product, entrywise, is simply a polynomial, and so μ is a polynomial in each of its n^2 entries. Since polynomials are continuous in $\mathbb{R}^2$, it follows that each of the n^2 components of μ is continuous. Hence μ is continuous.

To see that the inversion i is continuous, observe that the determinant $\det A$ is a continuous function, since it too is a polynomial (and is never zero, by definition of GL). It thus suffices to show that the function $A \mapsto \operatorname{adj} A$ is continuous. But it is easy to see that the adjugate matrix, which is the transpose of the cofactor matrix, is also a polynomial in the entries of A, and so $\operatorname{adj} A$ is continuous too. Thus i is continuous, and so S is a topological group, as desired.

Exercise 3.26. As hinted in the exercise, fix $h_0 \in H$ and let $\varphi : G \to H$ be the map taking x to $xh_0x^{-1}h_0^{-1}$. Note that $xh_0x^{-1} \in H$ because H is normal, and so $\varphi(x)$ is indeed an element of H. Moreover, we know that φ is continuous and $\{h\} \subseteq H$ is open for each $h \in H$. Thus $\{\varphi^{-1}(h) : h \in H\}$ is an open cover of G consisting of disjoint open sets.

In particular, if there are two elements $h_1, h_2 \in H$ such that $\varphi^{-1}(h_i) \neq \emptyset$ for $i = 1, 2$, then setting $A = \varphi^{-1}(h_1)$ and $B = \bigcup_{h \neq h_1} \varphi^{-1}(h)$ will give us two disjoint open sets A and B that cover G. This implies that G is disconnected, a contradiction. Thus for all but one element of H, we must have $\varphi^{-1}(h) = \emptyset$, proving that φ is constant. But obviously, setting $x = h_0$, we find that $\varphi(x) = 1$. Thus $xh_0x^{-1}h_0^{-1} = 1$ for all $x \in G$, and so $xh_0 = h_0x$ for each $h_0 \in H$. This proves the result.

4 Singular Homology

Holes and Green's Theorem

No exercises!

Free Abelian Groups

Exercise 4.1. If $\gamma \in F$, then we can write $\gamma = \sum_{b \in B} m_b b$, where $m_b \in \mathbb{Z}$ is zero for almost all b. Now, writing $B = \cup B_\lambda$ for disjoint B_λ, we can define for each λ the value $\gamma_\lambda = \sum_{b \in B_\lambda} m_b b \in F_\lambda$. Then obviously $\gamma = \sum \gamma_\lambda$.

To see that this expression is unique, simply observe that if $\gamma = \sum \gamma'_\lambda$, then because the sums are formal sums only, it follows that $\gamma_\lambda = \gamma'_\lambda$ for every λ. But then it follows that the coefficient for each $b \in B_\lambda$ must be the same in γ_λ and in γ'_λ, and so the two expressions are the same. Moreover, it is clear that almost every γ_λ is zero. After all, only finitely many m_b's are nonzero, and so only finitely many γ_λ contain a nonzero coefficient.

Finally, the converse is clear. In particular, if $\gamma = \sum \gamma_\lambda$ and $\gamma_\lambda = \sum_{b \in B_\lambda} m_b b$, then $\gamma = \sum_{b \in B} m_b b$.

Exercise 4.2. To see the forward direction (isomorphic implies same rank), simply restrict to the basis. In particular, if $\varphi : F \to F'$ is an isomorphism between two free abelian groups, and if B is a basis for F, then $\varphi(B)$ is a basis for F'. But clearly B and $\varphi(B)$ have the same cardinality because φ is injective. Thus F and F' have the same rank.

To see the converse, consider bases B and B' for F and F', respectively. Because B and B' have the same cardinality, there is a bijection $\varphi|_B$ between them. Pick such a bijection and extend it to all of F linearly. Theorem 4.1 tells us that this is a homomorphism; indeed, it is an isomorphism because $\varphi|_B$ was a bijection.

Exercise 4.3.

(i) An arbitrary element of $S_1(X)$ looks like $\sum m_\sigma \sigma$, where σ ranges over paths in X. Then we know that ∂_1 takes $\sum m_\sigma \sigma + \sum n_\sigma \sigma$ to

$$\sum_\sigma m_\sigma \sigma(1) + \sum_\sigma n_\sigma \sigma(1) - \sum_\sigma m_\sigma \sigma(0) - \sum_\sigma n_\sigma \sigma(0) = \partial_1(m) + \partial_1(n),$$

where $m = \sum m_\sigma \sigma$ and similarly for n. Thus this is a homomorphism.

(ii) If x_0 and x_1 lie in the same path component of X, then there is a path σ between them. This path is an element of X (indeed, it is a *basis* element of X), and satisfies $\partial_1(\sigma) = x_1 - x_0$.

The converse is slightly trickier, however. Suppose that x_0 and x_1 belong to different path components, say X_0 and X_1, respectively. Then consider the map $\varphi : S_0(X) \to \mathbb{Z}$ which takes $x \in X$ to 1 if $x \in X_0$ and to 0 otherwise. This defines φ on the basis of $S_0(X)$, so we can linearly extend it to a group homomorphism (Theorem 4.1).

Any element in the image of ∂_1 can be written as $(\sum m_\sigma \sigma)(1) - (\sum m_\sigma \sigma)(0)$. Then we know that

$$\varphi\Big((\sum m_\sigma \sigma)(1) - (\sum m_\sigma \sigma)(0)\Big) = \sum m_\sigma \varphi(\sigma(1) - \sigma(0)).$$

But because σ is a path, obviously $\sigma(1)$ and $\sigma(0)$ are in the same path component. In particular, we have $\varphi(\sigma(1) - \sigma(0)) = 0$, and so $\operatorname{im} \partial_1 \subset \ker \varphi$. Now observe that $\varphi(x_1 - x_0) = -1$. Thus $x_1 - x_0 \notin \operatorname{im} \partial_1$, proving the converse.

(iii) By definition, we have that $\sigma \in \ker \partial_1$ if and only if $\sigma(1) - \sigma(0) = 0$. Because σ is a path, however, this condition is equivalent to saying that σ is a closed path.

To see that the path condition on σ is necessary, note that the sum of two closed paths is in $\ker \partial_1$ but is not itself a closed path.

Exercise 4.4. Note that $S_n(X) = \emptyset$ for all n, because there is no function $\Delta^n \to X = \emptyset$. Thus $\ker \partial = \operatorname{im} \partial = \emptyset$, and so $H_n(X)$ is trivial.

Exercise 4.5. We know that ∂_0 is the zero map, and so $\ker \partial_0 = S_0(X)$. Moreover, the proof of the dimension axiom shows that ∂_1 is the zero map as well. In particular, we find that $Z_0(X)/B_0(X) \cong S_0(X)$. But we know, once again from the proof of the dimension axiom, that $S_0(X)$ is infinite cyclic and hence $H_0(X) \cong \mathbb{Z}$.

Exercise 4.6. We already know how S_n acts on objects of Top. Defining $S_n(f) = f_\#$ on morphisms, it is easy to see that S_n satisfies the functorial properties $S_n(1_X) = 1_{S_n(X)}$ and $S_n(g \circ f) = S_n(g) \circ S_n(f)$.

Exercise 4.7. We know that S^0 is the disjoint union of two points, and so $H_n(S^0) = H_n(\{0\}) \oplus H_n(\{1\})$. But the dimension axiom and Exercise 4.5 imply that

$$H_n(S^0) = \begin{cases} \mathbb{Z}^2 & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}.$$

Exercise 4.8. Because the Cantor set is the disjoint union of countably many points, it follows that $H_0(X) = \mathbb{Z}^\omega$ and $H_n(X) = 0$ for all $n > 0$.

Exercise 4.9.

(i) For $n = 0$, note that $\beta_1 = [a_0, b_0]$, and so $\partial_1 \beta_1$ is the constant map taking $e_0 \in \Delta^0$ to $b_0 - a_0 = (e_0, 1) - (e_1, 0)$. On the other hand, we know that P_{-1}^Δ is the zero map, and $\lambda_{i\,\#}^\Delta(\delta) = \lambda_i^\Delta$. Thus the right-hand side of the equation is simply
$$\lambda_1^\Delta - \lambda_0^\Delta,$$
which is the map taking $e_0 \in \Delta^0$ to $(e_0, 1) - (e_1, 0)$. The two sides are therefore the same.

For $n = 1$, we first consider the left-hand side. Note that

$$\begin{aligned} \partial_2 \beta_2 &= [b_0, b_1] - [a_0, b_1] + [a_0, b_0] - [a_1, b_1] + [a_0, b_1] - [a_0, a_1] \\ &= [b_0, b_1] + [a_0, b_0] - [a_1, b_1] - [a_0, a_1], \end{aligned}$$

and so it is simply the constant map $\Delta^1 \to \Delta^1 \times \mathbb{I}$ taking everything to $b_0 - a_1 = (e_0, 1) - (e_1, 0)$. For the right-hand side, on the other hand, we already know that

$$\lambda_{1\,\#}^\Delta(\delta) - \lambda_{0\,\#}^\Delta(\delta) = \lambda_1^\Delta - \lambda_0^\Delta : t \mapsto (t, 1) - (t, 0).$$

Moreover, because $\partial_1 \Delta^1 = e_1 - e_0$, we know that

$$P_0^\Delta \partial \delta : t \mapsto ((e_1 - e_0)(e_0), t) = (e_1, t) - (e_0, t).$$

Thus the right-hand side takes e_0 to

$$(e_0, 1) - (e_0, 0) - (e_1, 0) + (e_0, 0) = (e_0, 1) - (e_1, 0)$$

and takes e_1 to
$$(e_1, 1) - (e_1, 0) - (e_1, 1) + (e_0, 1) = (e_0, 1) - (e_1, 0).$$

hus the two sides agree on e_0 and e_1, from which we conclude the result.

(ii) We know that

$$\begin{aligned} P_1^X(\sigma) &= (\sigma \times 1)_\#(\beta_2) \\ &= (\sigma \times 1) \circ [a_0, b_0, b_1] - (\sigma \times 1) \circ [a_0, a_1, b_1]. \end{aligned}$$

The first term takes an arbitrary element $(t_0, t_1, t_2) \in \Delta^2$, where we use barycentric coordinates, to the point $(\sigma((t_0 + t_1)e_0 + t_2 e_1), t_1 + t_2)$. By corresponding a point $(1-t)e_0 + te_1 \in \Delta^1$ to t, we find that the first term takes (t_i) to $(\sigma(t_2), t_1 + t_2)$. Similarly, the second term takes (t_i) to $(\sigma(t_1 + t_2), t_2)$. Thus we find the following explicit formula:

$$P_1^X(\sigma) : (t_0, t_1, t_2) \mapsto (\sigma(t_2), t_1 + t_2) + (\sigma(t_1 + t_2), t_2).$$

Exercise 4.10. Let $\sigma : \Delta^n \to X$ be a simplex. Then note that $P_n^X(\sigma) = (\sigma \times 1)_\#(\beta_{n+1})$. Thus

$$(f \times 1)_\# P_n^X(\sigma) = (f\sigma \times 1)_\#(\beta_{n+1}).$$

On the other hand, we know that

$$P_n^Y f_\#(\sigma) = (f_\#\sigma \times 1)\#(\beta_{n+1}),$$

which is the same as the previous expression because σ is a simplex and so $f_\#\sigma = f\sigma$.

Exercise 4.11. The inclusion i is a homotopy equivalence, and so Corollary 4.24 implies that i_* is an isomorphism.

Exercise 4.12. Note that the $\sin(1/x)$ space has two path components, both of which are contractible. Thus $H_0(X) = \mathbb{Z}^2$ and $H_n(X) = 0$ for $n > 0$.

Exercise 4.13. We know that $\varphi \circ h_\#$ takes the path class $[f]$ to $\varphi[h \circ f] = \operatorname{cls} hf\eta$. On the flip side, we know that $h_* \circ \varphi$ takes φ to $h_* \operatorname{cls} f\eta$. But because $f\eta$ is a simplex, this is simply $\operatorname{cls} hf\eta$ as well.

Exercise 4.14. We know that

$$f * f^{-1} * (f * f^{-1})^{-1} \simeq c$$

for some constant map c. But note that $(f * f^{-1})^{-1} = f * f^{-1}$. Thus we can apply the Hurewicz map to find that

$$2\operatorname{cls}((f + f^{-1})\eta) = [0].$$

It follows that $f + f^{-1} \in B_1(X)$, where f and f^{-1} are considered as 1-chains. Thus f and $-f^{-1}$ are homologous, as desired.

Exercise 4.15. Note that the boundary of the second triangle is $\alpha * \beta + \gamma - (\alpha * \beta) * \gamma$. Thus $\operatorname{cls}(\alpha * \beta * \gamma) = \operatorname{cls}(\alpha * \beta + \gamma)$. Repeating this procedure on the first triangle, we find that $\operatorname{cls}(\alpha * \beta * \gamma) = \operatorname{cls}(\alpha + \beta + \gamma)$. Note that, in the text, there is a second equality, namely that these expressions equal $\operatorname{cls} \alpha + \operatorname{cls} \beta + \operatorname{cls} \gamma$. However, homology classes are not actually defined for paths which are not closed, so this seems to be an error.

Exercise 4.16. This is proved in Theorem 6.20.

5 The Category Comp

Exercise 5.1. These results all follow directly from the definition of exactness.

(i) Note that $\ker f = \operatorname{im} 0 = 0$, and so f is injective.

(ii) In this case, we have $\operatorname{im} g = \ker 0 = C$.

(iii) By the previous two parts, we know that f is bijective. Because f is a homomorphism as well, it follows that f is an isomorphism.

(iv) Either observe that $0 \to A \to 0 \to 0$ is exact and apply the previous part, or note that $A \to 0$ is injective while $0 \to A$ is surjective, implying that $A \cong 0$, i.e., that $A = 0$.

Exercise 5.2. Note that f is surjective if and only if $\ker g = \operatorname{im} f = B$. But $\ker g = B$ if and only if g is the zero map, which is itself true exactly when $\ker h = \operatorname{im} g = 0$. Since $\ker h = 0$ if and only if h is injective, we are done.

Exercise 5.3. We know that $0 \to A \xrightarrow{i} B$ implies that i is an injection. But because i is a surjection onto its image, this implies that $iA \cong A$. Moreover, because $\ker p = \operatorname{im} i = iA$, we know that $B/iA = B/\ker p \cong \operatorname{im} p$. Because p is a surjection (see Exercise 5.1), the result follows.

Exercise 5.4. This amounts, effectively, to following the arrows and the equations given by exactness. In more detail, let $f_n : B_n \to C_n$ and $g_n : C_n \to A_{n-1}$. Now observe that $B_n = \operatorname{im} h_n = \ker f_n$. Thus f_n is the zero map. Moreover, because $\ker g_n = \operatorname{im} f_n$, we know that g_n is injective. Finally, we have $\operatorname{im} g_n = \ker h_{n-1}$. But h_{n-1} is an isomorphism, and so its kernel is trivial. Thus $\operatorname{im} g_n = 0$. Because g_n was injective, it follows that $C_n = 0$.

Exercise 5.5.

(i) Let f be the map from A to B and g be the map from B to C. Let $\{a_\alpha\}$ and $\{c_\gamma\}$ be maximal independent sets of A and C, respectively. For every α, let $b_\alpha = f(a_\alpha)$. For every γ, pick some $b_\gamma \in g^{-1}(c_\gamma)$, which is possible by surjectivity of g. If $\sum n_\alpha b_\alpha + \sum n'_\gamma b'_\gamma = 0$, then we know that

$$g\left(\sum n_\alpha b_\alpha + \sum n'_\gamma b'_\gamma\right) = 0.$$

But we also know that $\operatorname{im} f = \ker g$, and so $g(b_\alpha) = 0$. Thus this simply implies that $\sum n'_\gamma c_\gamma = 0$, implying that $n'_\gamma = 0$. But now we know that $\sum n_\alpha b_\alpha = 0$, and so injectivity of f implies that $\sum n_\alpha a_\alpha = 0$ as well. Thus $n_\alpha = 0$ for all α as well, and so $\{b_\alpha\} \cup \{b_\gamma\}$ is independent. Thus $\operatorname{rank} B \geq \operatorname{rank} A + \operatorname{rank} C$.

To show the opposite inequality, it suffices to show that $\{b_\alpha\} \cup \{b_\gamma\}$ is *maximally* independent. Note that $b \notin f(A)$. Otherwise, we could take f^{-1} on $\{b_\alpha\} \cup \{b\}$, which is not independent. Now consider $\{b_\gamma\} \cup \{b\}$. If $g(b) = g(b_\gamma)$ for any γ, then we know by Exercise 5.3 that $b - b_\gamma \in f(A)$. Obviously, we cannot have $b - b_\gamma = b_\alpha$ for any α, otherwise that would give us our linear dependence. Thus $\{b - b_\gamma\} \cup \{b_\alpha\}$ is a subset of $f(A)$ with $\operatorname{rank} A + 1$ elements. This is not independent, a contradiction.

(ii) We prove this by induction. The previous part takes care of the base case. Consider the following commutative diagrams.

$$0 \longrightarrow A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{v} A_{n-1}/\operatorname{im} f_n \longrightarrow 0$$

$$0 \longrightarrow A_{n-1}/\ker f_{n-1} \xrightarrow{\bar{f}_{n-1}} A_{n-2} \xrightarrow{f_{n-2}} \dots \xrightarrow{f_2} A_1 \xrightarrow{f_1} A_0 \longrightarrow 0.$$

Here v is the natural map and $\bar{f}_{n-1}$ is the well-defined map taking $x + \ker f_{n-1}$ to $f_{n-1}(x)$.

Let r_i be the rank of A_i. Then the first diagram implies that $r_n - r_{n-1} + r = 0$, where r is the rank of $A_{n-1}/\operatorname{im} f_n$. Because $\operatorname{im} f_n = \ker f_{n-1}$, the second diagram implies by induction that $r - r_{n-2} + r_{n-3} + \cdots = 0$. Thus we subtract the first from the second to find that $r_n - r_{n-1} + r_{n-2} - \cdots = 0$, as desired.

Exercise 5.6. If $\partial_n = 0$ for all n, then we know that $H_n(S_*) = \ker \partial_n / \operatorname{im} \partial_{n+1} = S_n/\{0\} = S_n$.

Exercise 5.7. If $f : S_* \to S'_*$ is an equivalence, then it has an inverse $g : S'_* \to S_*$. Thus at every n, there is a $g_n : S'_n \to S_n$ so that $g_n \circ f_n = \mathrm{id}_{S_n}$ and $f_n \circ g_n = \mathrm{id}_{S'_n}$. It follows that f_n is an isomorphism for every n.

Conversely, suppose f_n is an isomorphism for every n. Then let $g = \{g_n\}$, where $g_n = f_n^{-1}$. It is clear that f and g are inverses, and so f is indeed an equivalence in Comp.

Exercise 5.8. If the former sequence is exact in Comp, then we know that $\operatorname{im} f = \ker g$. Then the terms of degree n of both $\operatorname{im} f$ and $\ker g$ must be the same. In other words, we must have $\operatorname{im} f_n = \ker g_n$, and so the latter sequence is exact in Ab.

On the other hand, suppose that the latter sequence is exact for every integer n. Then we know that the degree n terms of $\ker f$ and $\operatorname{im} g$ are the same. Moreover, we know that the differentiation operators are the same because they are defined, in both cases, simply as restrictions of the differentiation operator in S_*. Thus the two complexes are the same, as desired.

Exercise 5.9.

(i) We have the following diagram, where $\bar{\partial}_n$ represents the map taking $s_n + S'_n \mapsto \partial_n(s_n) + S'_{n-1}$.

$$\begin{tikzcd}
\dots \arrow[r] & S_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "v_{n+1}"] & S_n \arrow[r, "\partial_n"] \arrow[d, "v_n"] & S_{n-1} \arrow[r] \arrow[d, "v_{n-1}"] & \dots \\
\dots \arrow[r] & S_{n+1}/S'_{n+1} \arrow[r, "\bar{\partial}_{n+1}"] & S_n/S'_n \arrow[r, "\bar{\partial}_n"] & S_{n-1}/S'_{n-1} \arrow[r] & \dots
\end{tikzcd}$$

To show that v is a chain map, we must show that $v_{n-1}\partial_n = \bar{\partial}_n v_n$ for every n. Pick a simplex $\sigma : \Delta^n \to X$. We know that $v_{n-1}(\partial n\sigma) = \partial_n\sigma + S'_{n-1}$. However, we also have $\bar{\partial}_n v_n \sigma = \bar{\partial}_n(\sigma + S'_n) = \partial_n\sigma + S'_{n-1}$. Thus this is indeed a chain map.

Moreover, it is obvious that $\ker v_n = S'_n$ for every n. The definition of a subcomplex implies that the $\partial_n | \ker v_n$ is the operation in S'_*. Thus $\ker v = S'_*$, as desired.

(ii) At each n, we know from the previous part that we have the following commutative diagram in Ab.

$$\begin{tikzcd}
S_n \arrow[r, "\partial_n"] \arrow[d, "v_n"] & S_{n-1} \arrow[d, "v_{n-1}"] \\
S_n/\ker f_n \arrow[r, "\bar{\partial}_n"] & S_{n-1}/\ker f_{n-1}
\end{tikzcd}$$

By the first isomorphism theorem for groups, however, we know that there is an isomorphism θ_n from $S_n/\ker f_n \to \operatorname{im} f_n$ such that $\theta_n v_n = f_n$.

We claim that $\theta = \{\theta_n\}$ is the desired chain map. To see this, observe that

$$\theta_{n-1}\bar{\partial}_n(\sigma + \ker f_n) = \theta_{n-1}(\partial_n\sigma + \ker f_{n-1}) = f_{n-1}\partial_n\sigma.$$

On the other hand, because $\sigma + \ker f_n = v_n(\sigma)$, we know that

$$\partial'_n\theta_n(\sigma + \ker f_n) = \partial'_n(f_n(\sigma)) = \partial_n f_n\sigma.$$

The two final expressions in the above equations are equal, moreover, because $\{f_n\}$ is itself a chain map.

Exercise 5.10. First, note that both $S'_*/(S'_* \cap S''_*)$ and $(S'_* + S''_*)/S''_*$ are well-defined because everything is abelian. Now consider the map

$$\begin{aligned} \varphi : S'_* &\to \frac{S'_* + S''_*}{S''_*} \\ S'_n &\mapsto S'_n + S''_*. \end{aligned}$$

Note that we have boundary maps

$$\partial'_n : S'_n \to S'_{n-1}$$

and

$$\overline{\partial}_n : \frac{S'_n + S''_n}{S''_n} \to \frac{S'_{n-1} + S''_{n-1}}{S''_{n-1}},$$

where $\overline{\partial}_n$ takes $(s'_n + s''_n) + S''_n$ to $(\partial'_n s'_n + \partial''_n s''_n) + S''_{n-1}$.

We claim that φ is a chain map. To see this, it suffices to show that $\varphi_{n-1}\partial'_n = \overline{\partial}_n \varphi_n$. But for any $\sigma' \in S'_n$, we know that

$$\varphi_{n-1}\partial'_n(\sigma) = \partial'_n\sigma + S''_{n-1} = \overline{\partial}_n(\sigma + S''_n) = \overline{\partial}_n\varphi_n(\sigma),$$

as desired. Moreover, the second isomorphism theorem for groups implies that φ_n is a homomorphism with kernel $S'_n \cap S''_n$, from which it follows that φ is a chain map with $\ker\varphi = S'_* \cap S''_*$. The first isomorphism theorem (Exercise 5.9) implies the result.

Exercise 5.11. Consider the sequence in the problem, namely

$$0 \longrightarrow T_*/U_* \xrightarrow{i} S_*/U_* \xrightarrow{p} S_*/T_* \longrightarrow 0\ .$$

Clearly, we have

$$\operatorname{im} i_n = \{t_n + U_n : t_n \in T_n\}.$$

Moreover, we know that $p_n(s_n + U_n) = s_n + T_n$, so

$$\ker p_n = \{s_n + U_n : s_n \in T_n\}.$$

Clearly these are equal.

It now suffices to prove that $\ker i_n = 0$ and $\operatorname{im} p_n = S_n/T_n$. But note that $i_n(t_n + U_n) = 0$ implies that $t_n \in U_n$. Hence $t_n + U_n = 0$ as an element of T_n/U_n as well. Moreover, consider an arbitrary element $s_n + T_n \in S_n/T_n$. It is equal to $p_n(s_n + U_n)$, which proves that p is surjective. Hence the sequence of complexes is exact.

Exercise 5.12. We claim that

$$\ker\left(\sum \partial_n^\lambda\right) = \sum \ker \partial_n^\lambda.$$

If $\sum s_n^\lambda \in \ker\left(\sum \partial_n^\lambda\right)$, then by definition we must have $\partial_n^\lambda(s_n^\lambda) = 0$ for each λ. The converse is also clearly true. Similarly, we find that

$$\operatorname{im}\left(\sum \partial_{n+1}^\lambda\right) = \left\{\sum \partial_{n+1}^\lambda(s_{n+1}^\lambda)\right\} = \sum \operatorname{im}\partial_{n+1}^\lambda.$$

Thus we conclude that

$$H_n\left(\sum S_*^\lambda\right) = \frac{\ker\partial_n}{\operatorname{im}\partial_{n+1}} = \frac{\sum \ker\partial_n^\lambda}{\sum \operatorname{im}\partial_{n+1}^\lambda} = \sum \frac{\ker\partial_n^\lambda}{\operatorname{im}\partial_{n+1}^\lambda} = \sum H_n(S_*^\lambda).$$

Exercise 5.13. Suppose $\sum m_\sigma\sigma + S_n(A) = \sum m'_\sigma\sigma + S_n(A)$. This implies that $m_\sigma - m'_\sigma = 0$ for every σ with $\operatorname{im}\sigma \not\subseteq A$. Hence we can pick the unique representative

$$\sum_{\operatorname{im}\sigma\not\subseteq A} m_\sigma\sigma = \sum_{\operatorname{im}\sigma\not\subseteq A} m'_\sigma\sigma,$$

thus showing that this is indeed the free abelian group generated by σ with $\operatorname{im}\sigma \not\subseteq A$.

Exercise 5.14.

(i) It suffices to prove that p_n is surjective and that $\operatorname{im} i_n = \ker p_n$. To see that p_n is surjective, consider the following segment of the long exact sequence:

$$B_n \xrightarrow{p_n} C_n \longrightarrow A_{n-1} \xrightarrow{i_{n-1}} B_{n-1}$$

Note that the map $f_n : C_n \to A_{n-1}$ has $\operatorname{im} f_n = \ker i_{n-1} = 0$, and so $\ker f_n = C_n$. Thus $\operatorname{im} f_n = C_n$, proving surjectivity.

To see that $\operatorname{im} i_n = \ker p_n$, simply consider the following segment:

$$A_n \xrightarrow{i_n} B_n \xrightarrow{p_n} C_n$$

The result immediately follows.

(ii) There exists a map r with $r \circ i = \mathrm{id}_A$. Thus $r_* \circ i_* = \mathrm{id}_{H_n(A)}$, from which it follows that i_* is injective. Theorem 5.8 gives an exact sequence

$$\dots \longrightarrow H_n(A) \xrightarrow{i_n} H_n(X) \xrightarrow{p_n} H_n(X, A) \xrightarrow{d} \dots$$

where p_* is induced by the quotient map $S_*(X) \to S_*(X)/S_*(A)$. Since i_* injective implies that i_n is injective, we can apply the previous part to find an exact sequence

$$0 \longrightarrow H_n(A) \xrightarrow{i_n} H_n(X) \xrightarrow{p_n} H_n(X, A) \longrightarrow 0$$

Then Exercise 5.3 implies that $H_n \oplus H_n(X, A) = H_n(X)$, as desired.

(iii) We now have $i \circ r \simeq \mathrm{id}_X$ as well. In particular, since A and X have the same homotopy type, we must have $H_n(A) \cong H_n(X)$ by Corollary 4.24. Thus i_n in the exact sequence given in the previous part must be the identity, and so $\ker p_n = \operatorname{im} i_n = H_n(X)$. But since p_n is surjective, it follows that $H_n(X, A) = 0$, as desired.

Exercise 5.15. We prove this in cases. We will use the follow commutative diagram, where the columns are exact:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\dots \arrow[r] & S'_{n+1} \arrow[r, "\partial'_{n+1}"] \arrow[d, "i_{n+1}"] & S'_n \arrow[r, "\partial'_n"] \arrow[d, "i_n"] & S'_{n-1} \arrow[r] \arrow[d, "i_{n-1}"] & \dots \\
\dots \arrow[r] & S_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "p_{n+1}"] & S_n \arrow[r, "\partial_n"] \arrow[d, "p_n"] & S_{n-1} \arrow[r] \arrow[d, "p_n"] & \dots \\
\dots \arrow[r] & S''_{n+1} \arrow[r, "\partial''_{n+1}"] \arrow[d] & S''_n \arrow[r, "\partial''_n"] \arrow[d] & S''_{n-1} \arrow[r] \arrow[d] & \dots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Case 1. S_* and S'_* are acyclic.

We would like to show than $Z''_n = \ker \partial''_n$ is equal to $B''_n = \operatorname{im} \partial''_{n+1}$. We already know that $B''_n \subseteq Z''_n$. Surjectivity of p implies that we can rewrite Z''_n as

$$Z''_n = p_n(\ker(\partial''_n p_n)) = p_n(\ker(p_{n-1}\partial_n)).$$

This, in turn, can be written as

$$Z''_n = \{p_n s_n : p_{n-1}\partial_n s_n = 0\}.$$

On the other hand, we can rewrite B''_n as

$$B''_n = \operatorname{im}(\partial''_{n+1} p_{n+1} = \operatorname{im}(p_n \partial_{n+1})) = p_n \operatorname{im} \partial_{n+1}.$$

Since S_* is acyclic, we know that $\operatorname{im}\partial_{n+1} = \ker\partial_n$, and so we find that

$$B''_n = \{p_n z_n : \partial_n z_n = 0\}.$$

Now consider an arbitrary element $p_n s_n \in Z''_n$. Since $p_{n-1}\partial_n s_n = 0$, we know that $\partial_n s_n \in \ker p_{n-1} = \operatorname{im} i_{n-1}$, where again we use the fact that S_* is acyclic. Injectivity of i_{n-1} implies the existence of a unique $s'_{n-1} \in S'_{n-1}$ with $i_{n-1}s'_{n-1} = \partial_n s_n$. We know, however, that $\partial_{n-1}\partial_n = 0$, and so

$$0 = \partial_{n-1}\partial_n s = \partial_{n-1} i_{n-1} s'_{n-1} = i_{n-2}\partial'_{n-1}s'_{n-1}.$$

Since i_{n-2} is injective, it follows that $\partial'_{n-1}s_{n-1} = 0$, and so acyclicity of S'_* implies that $s_{n-1} \in \ker\partial'_{n-1} = \operatorname{im}\partial'_n$. In particular, we can write $s'_{n-1} = \partial'_n s'_n$.

Now notice that

$$\partial_n i_n s'_n = i_{n-1}\partial'_n s'_n = i_{n-1}s'_{n-1} = \partial_n s_n,$$

where the last equality follows from the definition of s'_{n-1}. We know that $z_n = s_n - i_n s'_n \in Z_n$ since ∂'_n is a homomorphism. But we also know that

$$p_n z_n = p_n(s_n - i_n s'_n) = p_n s_n - p_n i_n s'_n = p_n s_n,$$

where we use exactness of the columns. In other words, we have a z_n with $\partial_n z_n = 0$, such that $p_n z_n = p_n s_n$. Thus $p_n s_n \in B''_n$, proving that $Z''_n = B''_n$. To be even more explicit, this implies that $H''_n = Z''_n/B''_n = 0$ for all n, proving that S''_* is an acyclic complex as well.

Case 2. S'_* and S''_* are acyclic.

Suppose $s_n \in Z_n$, i.e., that $\partial_n s_n = 0$. Then $\partial''_n p_n = p_{n-1}\partial_n$ implies that $p_n s_n \in \ker\partial''_n = \operatorname{im}\partial''_{n+1}$. Hence write $p_n s_n = \partial''_{n+1}s''_{n+1}$. Since p_{n+1} is surjective, we can find s_{n+1} with $p_{n+1}s_{n+1} = s''_{n+1}$, and so

$$p_n\partial_{n+1}s_{n+1} = \partial''_{n+1}p_{n+1}s_{n+1} = p_n s_n.$$

But then we know that $\partial_{n+1}s_{n+1} - s_n \in \ker p_n = \operatorname{im} i_n$. Thus there exists a unique s'_n with $i_n s'_n = \partial_{n+1}s_{n+1} - s_n$. We can take ∂_n of both sides to find that

$$0 = \partial_n\partial_{n+1}s_{n+1} - \partial_n s_n = \partial_n i_n s'_n = i_{n-1}\partial'_n s'_n,$$

and so it follows that $\partial'_n s'_n = 0$. In particular, we know that $s'_n \in \operatorname{im}\partial'_{n+1}$, so we can find s'_{n+1} whose boundary is s'_n. Recall that we had

$$s_n = \partial_{n+1}s_{n+1} - i_n s'_n.$$

But the last term is equal to $i_n\partial'_{n+1}s'_{n+1} = \partial_{n+1}i_{n+1}s'_{n+1}$, and so this is in turn equal to

$$s_n = \partial_{n+1}(s_{n+1} - i_{n+1}s'_{n+1}).$$

This proves that $s_n \in B_n$, and so $Z_n = B_n$.

Case 3. S_* and S''_* are acyclic.

This final case is handled similarly to the first two, but we lay out the details below. Let $s'_n \in \ker\partial'_n = Z'_n$ be arbitrary. Then $i_n s'_n \in \ker\partial_n = \operatorname{im}\partial_{n+1}$, and so

$$i_n s'_n = \partial_{n+1}s_{n+1}$$

for some s_{n+1}. We know that $p_{n+1}s_{n+1} \in \ker\partial''_{n+1} = \operatorname{im}\partial''_{n+2}$ because $p_n i_n s'_n = 0$. Hence there exists s''_{n+2} with $\partial''_{n+2}s''_{n+2} = p_{n+1}s_{n+1}$. But then it follows that

$$p_{n+1}\partial_{n+2}s_{n+2} = \partial''_{n+2}p_{n+2}s_{n+2} = p_{n+1}s_{n+1},$$

from which it follows that $s_{n+1} - \partial_{n+2}s_{n+2} \in \ker p_{n+1} = \operatorname{im} i_{n+1}$. Thus there exists s'_{n+1} with $i_{n+1}s'_{n+1} = s_{n+1} - \partial_{n+2}s_{n+2}$. We then find that

$$i_n\partial'_{n+1}s'_{n+1} = \partial_{n+1}i_{n+1}s'_{n+1} = \partial_{n+1}s_{n+1} = i_n s'_n.$$

Injectivity implies $s'_n = \partial'_{n+1}s'_{n+1} \in B_n$, thus proving the final case.

Exercise 5.16. To show that $f_{\#}(Z_n(X,A)) \subseteq Z_n(X',A')$, consider $\gamma \in Z_n(X,A)$. Note that $\gamma \in S_n(X)$, and so Lemma 4.8 implies that

$$\partial'_n f_{\#}\gamma = f_{\#}\partial_n \gamma.$$

We know, moreover, that $\partial_n \gamma = \sum m_\sigma \sigma$, where the sum ranges over all σ with $\operatorname{im}\sigma \subseteq A$. Thus it follows that

$$\partial'_n f_{\#}\gamma = f_{\#}\left(\sum m_\sigma \sigma\right) = \sum_{\operatorname{im}(f\sigma)\subseteq f(A)} m_\sigma f\sigma.$$

Since $f\sigma$ is a simplex into X' with image contained in A', it follows that this is an element of $S_{n-1}(A')$. Hence we conclude that $f_{\#}\gamma \in Z_n(X',A')$, as desired.

The proof for boundaries is similar.

Exercise 5.17. As defined, we have that $f_{\#} : H_n(X,A) \to H_n(X',A')$ is given by

$$f_{\#} : \overline{\gamma} + \operatorname{im}\overline{\partial}_{n+1} \mapsto f_{\#}\left(\overline{\gamma}\right) + \operatorname{im}\overline{\partial'}_{n+1},$$

where $\overline{\partial}$ and $\overline{\partial'}$ denote the boundary maps of the quotient complexes $S_*(X)/S_*(A)$ and $S_*(X')/S_*(A')$, respectively, and where

$$\overline{\gamma} \in \ker \overline{\partial}_n = Z_n(X,A)/S_n(A).$$

The third isomorphism theorem gives an isomorphism

$$H_n(X,A) = \frac{Z_n(X,A)/S_n(A)}{B_n(X,A)/S_n(A)} \to \frac{Z_n(X,A)}{B_n(X,A)}$$

which takes

$$\overline{\gamma} + B_n(X,A)/S_n(A) \mapsto \gamma + B_n(X,A).$$

Since $f_{\#}(\overline{\gamma}) = \overline{f_{\#}(\gamma)}$, we find that, thinking of $f_{\#}$ as a map from $Z_n(X,A)/B_n(X,A)$ to a map from $Z_n(X',A')/B_n(X',A')$, it takes

$$\gamma + B_n(X,A) \mapsto f_{\#}(\gamma) + B_n(X',A'),$$

as desired.

Exercise 5.18. Recall the definition of $\varepsilon_i : \Delta^{n-1} \to \Delta^n$ as taking $\{e_0, \ldots, e_{n-1}\}$ to $\{e_0, \ldots, \hat{e}_i, \ldots, e_{n-1}\}$. Thus we have

$$\partial_n \sigma = \sum_{i=0}^{n} (-1)^i \sigma\varepsilon_i.$$

Since $\sigma\varepsilon_i : \Delta^{n-1} \to X$ has image in A by hypothesis, this is in $S_{n-1}(A)$, as desired.

Exercise 5.19. By Theorem 5.6, it is sufficient to show that we have a short exact sequence

$$0 \longrightarrow \widetilde{S}_*(A) \longrightarrow \widetilde{S}_*(X) \longrightarrow S_*(X,A) \longrightarrow 0.$$

When $n \geq 1$, this is clear by Theorem 5.8. When $n = 0$, we have the sequence

$$0 \to \mathbb{Z} \to \mathbb{Z} \to 0 \to 0,$$

which is easily verified to be exact.

Exercise 5.20. Consider the exact sequence

$$H_1(CX) \longrightarrow H_1(CX,X) \longrightarrow \widetilde{H}_0(X) \longrightarrow \widetilde{H}_0(CX).$$

Note that $H_1(CX) = 0$ because CX is contractible. On the other hand, Corollary 5.18 implies that $\widetilde{H}_0(X) \cong \mathbb{Z}^4$, while $\widetilde{H}_0(CX) \cong 0$. Thus the map $H_1(CX,X) \to \widetilde{H}_0(X)$ is surjective. Moreover, its kernel is equal to the image of the map $H_1(CX) \to H_1(CX,X)$, which is simply 0 since $H_1(CX) = 0$. Thus the map is also injective, from which it immediately follows that $H_1(CX,X) \cong \mathbb{Z}^4$.

Exercise 5.21. Consider the exact sequence

$$\widetilde{H}_1(S^0) \longrightarrow \widetilde{H}_1(S^1) \longrightarrow H_1(S^1, S^0) \longrightarrow \widetilde{H}_0(S^0) \longrightarrow \widetilde{H}_0(S^1),$$

which is simply equal to

$$0 \longrightarrow \mathbb{Z} \longrightarrow \boxed{} \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Now note that the first map has im $= 0$, so the second map has ker $= 0$. Thus the second map has im $\cong \mathbb{Z}$, and so the third map has ker $\cong \mathbb{Z}$. Yet we also know that the last map is the zero map, and so the third map has im $= \mathbb{Z}$, from which it follows that the $H_1(S^1, S^0) = \mathbb{Z} \times \mathbb{Z}$.

Exercise 5.22. When $n = 0$, this follows from the exact sequence

$$\widetilde{H}_0(X) \longrightarrow \widetilde{H}_0(X) \longrightarrow H_0(X, X) \longrightarrow 0\ .$$

After all, the first map is the identity, and so the second map is the zero map. But the second map is surjective, and so $H_0(X, X) = 0$.

For $n > 0$, we have the exact sequence

$$\widetilde{H}_n(X) \longrightarrow \widetilde{H}_n(X) \longrightarrow H_n(X, X) \longrightarrow \widetilde{H}_{n-1}(X) \longrightarrow \widetilde{H}_{n-1}(X).$$

The first map is the identity, and so the second map is everywhere zero. Thus the image of the second map, which is the kernel of the third map too, is equal to 0. Since the kernel of the last map, which is 0 (the map is the identity), is equal to the image of the third map, it follows that the third map is everywhere zero. Hence the third map is injective, but also everywhere zero, and so $H_n(X, X)$ must have been 0 in the first place.