4 Singular Homology

Holes and Green's Theorem

No exercises!

Free Abelian Groups

Exercise 4.1. If $\gamma \in F$, then we can write $\gamma = \sum_{b \in B} m_b b$, where $m_b \in \mathbb{Z}$ is zero for almost all b. Now, writing $B = \cup B_\lambda$ for disjoint B_λ, we can define for each λ the value $\gamma_\lambda = \sum_{b \in B_\lambda} m_b b \in F_\lambda$. Then obviously $\gamma = \sum \gamma_\lambda$.

To see that this expression is unique, simply observe that if $\gamma = \sum \gamma'_\lambda$, then because the sums are formal sums only, it follows that $\gamma_\lambda = \gamma'_\lambda$ for every λ. But then it follows that the coefficient for each $b \in B_\lambda$ must be the same in γ_λ and in γ'_λ, and so the two expressions are the same. Moreover, it is clear that almost every γ_λ is zero. After all, only finitely many m_b's are nonzero, and so only finitely many γ_λ contain a nonzero coefficient.

Finally, the converse is clear. In particular, if $\gamma = \sum \gamma_\lambda$ and $\gamma_\lambda = \sum_{b \in B_\lambda} m_b b$, then $\gamma = \sum_{b \in B} m_b b$.

Exercise 4.2. To see the forward direction (isomorphic implies same rank), simply restrict to the basis. In particular, if $\varphi : F \to F'$ is an isomorphism between two free abelian groups, and if B is a basis for F, then $\varphi(B)$ is a basis for F'. But clearly B and $\varphi(B)$ have the same cardinality because φ is injective. Thus F and F' have the same rank.

To see the converse, consider bases B and B' for F and F', respectively. Because B and B' have the same cardinality, there is a bijection $\varphi|_B$ between them. Pick such a bijection and extend it to all of F linearly. Theorem 4.1 tells us that this is a homomorphism; indeed, it is an isomorphism because $\varphi|_B$ was a bijection.

Exercise 4.3.

(i) An arbitrary element of $S_1(X)$ looks like $\sum m_\sigma \sigma$, where σ ranges over paths in X. Then we know that ∂_1 takes $\sum m_\sigma \sigma + \sum n_\sigma \sigma$ to

$$\sum_\sigma m_\sigma \sigma(1) + \sum_\sigma n_\sigma \sigma(1) - \sum_\sigma m_\sigma \sigma(0) - \sum_\sigma n_\sigma \sigma(0) = \partial_1(m) + \partial_1(n),$$

where $m = \sum m_\sigma \sigma$ and similarly for n. Thus this is a homomorphism.

(ii) If x_0 and x_1 lie in the same path component of X, then there is a path σ between them. This path is an element of X (indeed, it is a *basis* element of X), and satisfies $\partial_1(\sigma) = x_1 - x_0$.

The converse is slightly trickier, however. Suppose that x_0 and x_1 belong to different path components, say X_0 and X_1, respectively. Then consider the map $\varphi : S_0(X) \to \mathbb{Z}$ which takes $x \in X$ to 1 if $x \in X_0$ and to 0 otherwise. This defines φ on the basis of $S_0(X)$, so we can linearly extend it to a group homomorphism (Theorem 4.1).

Any element in the image of ∂_1 can be written as $(\sum m_\sigma \sigma)(1) - (\sum m_\sigma \sigma)(0)$. Then we know that

$$\varphi\left((\sum m_\sigma \sigma)(1) - (\sum m_\sigma \sigma)(0)\right) = \sum m_\sigma \varphi(\sigma(1) - \sigma(0)).$$

But because σ is a path, obviously $\sigma(1)$ and $\sigma(0)$ are in the same path component. In particular, we have $\varphi(\sigma(1) - \sigma(0)) = 0$, and so $\operatorname{im} \partial_1 \subset \ker \varphi$. Now observe that $\varphi(x_1 - x_0) = -1$. Thus $x_1 - x_0 \notin \operatorname{im} \partial_1$, proving the converse.

(iii) By definition, we have that $\sigma \in \ker \partial_1$ if and only if $\sigma(1) - \sigma(0) = 0$. Because σ is a path, however, this condition is equivalent to saying that σ is a closed path.

To see that the path condition on σ is necessary, note that the sum of two closed paths is in $\ker \partial_1$ but is not itself a closed path.

Exercise 4.4. Note that $S_n(X) = \emptyset$ for all n, because there is no function $\Delta^n \to X = \emptyset$. Thus $\ker \partial = \operatorname{im} \partial = \emptyset$, and so $H_n(X)$ is trivial.

Exercise 4.5. We know that ∂_0 is the zero map, and so $\ker \partial_0 = S_0(X)$. Moreover, the proof of the dimension axiom shows that ∂_1 is the zero map as well. In particular, we find that $Z_0(X)/B_0(X) \cong S_0(X)$. But we know, once again from the proof of the dimension axiom, that $S_0(X)$ is infinite cyclic and hence $H_0(X) \cong \mathbb{Z}$.

Exercise 4.6. We already know how S_n acts on objects of **Top**. Defining $S_n(f) = f_\#$ on morphisms, it is easy to see that S_n satisfies the functorial properties $S_n(1_X) = 1_{S_n(X)}$ and $S_n(g \circ f) = S_n(g) \circ S_n(f)$.

Exercise 4.7. We know that S^0 is the disjoint union of two points, and so $H_n(S^0) = H_n(\{0\}) \oplus H_n(\{1\})$. But the dimension axiom and Exercise 4.5 imply that

$$H_n(S^0) = \begin{cases} \mathbb{Z}^2 & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}.$$

Exercise 4.8. Because the Cantor set is the disjoint union of countably many points, it follows that $H_0(X) = \mathbb{Z}^\omega$ and $H_n(X) = 0$ for all $n > 0$.

Exercise 4.9.

(i) For $n = 0$, note that $\beta_1 = [a_0, b_0]$, and so $\partial_1\beta_1$ is the constant map taking $e_0 \in \Delta^0$ to $b_0 - a_0 = (e_0, 1) - (e_1, 0)$. On the other hand, we know that P^Δ_{-1} is the zero map, and $\lambda^\Delta_{i\,\#}(\delta) = \lambda^\Delta_i$. Thus the right-hand side of the equation is simply

$$\lambda^\Delta_1 - \lambda^\Delta_0,$$

which is the map taking $e_0 \in \Delta^0$ to $(e_0, 1) - (e_1, 0)$. The two sides are therefore the same.

For $n = 1$, we first consider the left-hand side. Note that

$$\begin{aligned} \partial_2\beta_2 &= [b_0, b_1] - [a_0, b_1] + [a_0, b_0] - [a_1, b_1] + [a_0, b_1] - [a_0, a_1] \\ &= [b_0, b_1] + [a_0, b_0] - [a_1, b_1] - [a_0, a_1], \end{aligned}$$

and so it is simply the constant map $\Delta^1 \to \Delta^1 \times \mathbb{I}$ taking everything to $b_0 - a_1 = (e_0, 1) - (e_1, 0)$. For the right-hand side, on the other hand, we already know that

$$\lambda^\Delta_{1\,\#}(\delta) - \lambda^\Delta_{0\,\#}(\delta) = \lambda^\Delta_1 - \lambda^\Delta_0 : t \mapsto (t, 1) - (t, 0).$$

Moreover, because $\partial_1\Delta^1 = e_1 - e_0$, we know that

$$P^\Delta_0 \partial\delta : t \mapsto ((e_1 - e_0)(e_0), t) = (e_1, t) - (e_0, t).$$

Thus the right-hand side takes e_0 to

$$(e_0, 1) - (e_0, 0) - (e_1, 0) + (e_0, 0) = (e_0, 1) - (e_1, 0)$$

and takes e_1 to

$$(e_1, 1) - (e_1, 0) - (e_1, 1) + (e_0, 1) = (e_0, 1) - (e_1, 0).$$

hus the two sides agree on e_0 and e_1, from which we conclude the result.

(ii) We know that

$$\begin{aligned} P^X_1(\sigma) &= (\sigma \times 1)_\#(\beta_2) \\ &= (\sigma \times 1) \circ [a_0, b_0, b_1] - (\sigma \times 1) \circ [a_0, a_1, b_1]. \end{aligned}$$

The first term takes an arbitrary element $(t_0, t_1, t_2) \in \Delta^2$, where we use barycentric coordinates, to the point $(\sigma((t_0 + t_1)e_0 + t_2e_1), t_1 + t_2)$. By corresponding a point $(1 - t)e_0 + te_1 \in \Delta^1$ to t, we find that the first term takes (t_i) to $(\sigma(t_2), t_1 + t_2)$. Similarly, the second term takes (t_i) to $(\sigma(t_1 + t_2), t_2)$. Thus we find the following explicit formula:

$$P^X_1(\sigma) : (t_0, t_1, t_2) \mapsto (\sigma(t_2), t_1 + t_2) + (\sigma(t_1 + t_2), t_2).$$

Exercise 4.10. Let $\sigma : \Delta^n \to X$ be a simplex. Then note that $P_n^X(\sigma) = (\sigma \times 1)_\#(\beta_{n+1})$. Thus

$$(f \times 1)_\# P_n^X(\sigma) = (f\sigma \times 1)_\#(\beta_{n+1}).$$

On the other hand, we know that

$$P_n^Y f_\#(\sigma) = (f_\#\sigma \times 1)\#(\beta_{n+1}),$$

which is the same as the previous expression because σ is a simplex and so $f_\#\sigma = f\sigma$.

Exercise 4.11. The inclusion i is a homotopy equivalence, and so Corollary 4.24 implies that i_* is an isomorphism.

Exercise 4.12. Note that the $\sin(1/x)$ space has two path components, both of which are contractible. Thus $H_0(X) = \mathbb{Z}^2$ and $H_n(X) = 0$ for $n > 0$.

Exercise 4.13. We know that $\varphi \circ h_\#$ takes the path class $[f]$ to $\varphi[h \circ f] = \operatorname{cls} hf\eta$. On the flip side, we know that $h_* \circ \varphi$ takes φ to $h_* \operatorname{cls} f\eta$. But because $f\eta$ is a simplex, this is simply $\operatorname{cls} hf\eta$ as well.

Exercise 4.14. We know that

$$f * f^{-1} * (f * f^{-1})^{-1} \simeq c$$

for some constant map c. But note that $(f * f^{-1})^{-1} = f * f^{-1}$. Thus we can apply the Hurewicz map to find that

$$2 \operatorname{cls}((f + f^{-1})\eta) = [0].$$

It follows that $f + f^{-1} \in B_1(X)$, where f and f^{-1} are considered as 1-chains. Thus f and $-f^{-1}$ are homologous, as desired.

Exercise 4.15. Note that the boundary of the second triangle is $\alpha * \beta + \gamma - (\alpha * \beta) * \gamma$. Thus $\operatorname{cls}(\alpha * \beta * \gamma) = \operatorname{cls}(\alpha * \beta + \gamma)$. Repeating this procedure on the first triangle, we find that $\operatorname{cls}(\alpha * \beta * \gamma) = \operatorname{cls}(\alpha + \beta + \gamma)$. Note that, in the text, there is a second equality, namely that these expressions equal $\operatorname{cls} \alpha + \operatorname{cls} \beta + \operatorname{cls} \gamma$. However, homology classes are not actually defined for paths which are not closed, so this seems to be an error.

Exercise 4.16. This is proved in Theorem 6.20.