10 Covering Spaces

Basic Properties

Exercise 10.1.

Exercise 10.2.

Exercise 10.3.

Exercise 10.4.

Exercise 10.5.

Exercise 10.6.

Exercise 10.7.

Exercise 10.8.

Exercise 10.9.

Exercise 10.10.

Exercise 10.11.

Exercise 10.12.

Exercise 10.13.

Exercise 10.14.

Exercise 10.15.

Exercise 10.16.

Exercise 10.17.

Covering Transformations

Exercise 10.18.

Exercise 10.19.

Exercise 10.20.

Exercise 10.21.

Exercise 10.22.

Exercise 10.23.

Existence

Exercise 10.24.

Exercise 10.25.

Exercise 10.26.

Exercise 10.27.

Exercise 10.28.

Orbit Spaces

Exercise 10.29. We know by Theorem 10.54 that $\mathrm{Cov}(\tilde{X}/(\tilde{X}/H)) = H$, and so we can think of G as a subgroup of $\mathrm{Cov}(\tilde{X}/(\tilde{X}/H))$. Now use Theorem 10.52, with $X = \tilde{X}/H$. We know, in particular, that G is a subgroup of $\mathrm{Cov}(\tilde{X}/X)$, and thus is exactly a covering space $(\tilde{X}/G, v)$ of $X = \tilde{X}/H$, as desired.

Exercise 10.30.

(i) Suppose $gx = x$ and consider a proper neighborhood V of x. Then we know that $gV \cap V = \emptyset$, but $x = gx \in gV \cap V$, contradiction.

(ii) If $G = \{e, g_1, \ldots, g_n\}$ and $x \in X$, then, since X is Hausdorff and since $g_i x \neq x$, there exists a neighborhood V of X which does not contain any $g_i x$. Obviously, this V is a proper neighborhood.

Exercise 10.31. This is exactly the argument in the proof of Theorem 10.2, namely in the first full paragraph on p. 276.

Exercise 10.32.

(i) The group $\mathbb{Z}/p\mathbb{Z}$ acts on S^3 via $m \bullet (z_0, z_1) = (\zeta^m z_0, \zeta^{mq} z_1)$. This action is proper because part (ii) of Exercise 10.30 obviously applies.

(ii) Note that $S^3/(\mathbb{Z}/p\mathbb{Z})$ is exactly $L(p,q)$. Thanks to the previous part, Theorem 10.54(ii) applies, which implies that
$$\pi_1(L(p,q)) = \pi_1(S^3/(\mathbb{Z}/p\mathbb{Z})) = \mathbb{Z}/p\mathbb{Z}.$$

(iii) We know that $L(p,q)$ inherits the local properties of S^3, since there is a local homeomorphism between them. Thus $L(p,q)$ is a 3-manifold.

If $\mathcal{U}$ is an open cover of $L(p,q)$, then $p^{-1}(\mathcal{U})$ is an open cover of S^3. Hence finitely many elements of $p^{-1}(\mathcal{U})$, say $p^{-1}(U_i)$ for $i = 1, \ldots, n$, cover S^3. But then $\{U_1, \ldots, U_n\}$ is a finite subcover of $\mathcal{U}$ which covers $L(p,q)$, proving compactness.

Finally, note that $A \subseteq L(p,q)$ clopen implies that $p^{-1}(A)$ is clopen in S^3. Hence $p^{-1}(A) = \emptyset, S^3$, and so $A = \emptyset, L(p,q)$. Thus $L(p,q)$ is connected too.

Exercise 10.33. Notice that $T \to T/G$ is a universal covering space since T is simply connected. Moreover, since T/G is a connected 1-complex, we know by Corollary 7.35 that $\pi_1(T/G)$ is free. But Theorem 10.54(iii) implies that $\pi_1(T/G) \cong G$, and so G is free.