# 2  Simplexes[1]

## Affine Spaces

**Exercise 2.1.** Note that there is a maximal affine independent subset $S$ of $A$. This is directly implied by the fact that any set of greater than $n+1$ elements is not affine independent. Hence we can take an affine independent subset of $A$ with maximum size (because the empty set is affine independent).

Write $S = \{p_0, \ldots, p_m\}$. Then let $p_{m+1} \in A \setminus S$. By maximality of $S$, we know that $S \cup \{p\}$ is not affine independent. Hence there exist $s_i$ not all 0 such that

$$\sum_{i=0}^{m+1} s_i p_i = 0, \quad \sum_{i=0}^{m+1} s_i = 0.$$

Note that the second equation implies $\sum_{i=0}^{m} s_i \neq 0$ for some $i < m+1$. It follows then that

$$\sum_{i=0}^{m} \left( \frac{s_i}{\sum_{i=0}^{m} s_i} p_i \right) = p_{m+1}.$$

But we know that

$$\sum_{i=0}^{m} \frac{s_i}{\sum_{i=0}^{m} s_i} = 1,$$

and so it follows that $p_{m+1}$ is in fact in the affine span of $S$.

**Exercise 2.2.** Let $\varphi$ be the isomorphism from $\mathbb{R}^n$ to a subset of $\mathbb{R}^k$. Suppose $A \subseteq \mathbb{R}^n$ is an affine set containing $X$. Then $\varphi(X) \subseteq \varphi(A) \subseteq \mathbb{R}^k$.

Moreover, we claim that $\varphi(A)$ is affine. After all, for any $\varphi(x), \varphi(x') \in \varphi(A)$ and any $t \in \mathbb{R}$, the point $t\varphi(x) + (1-t)\varphi(x') = \varphi(tx + (1-t)x') \in \varphi(A)$ because $A$ is affine.

This implies that the intersection of all affine sets in $\mathbb{R}^n$ containing $X$ must contain the intersection of all affine sets in $\varphi(\mathbb{R}^n)$ containing $\varphi(X)$. Because $\varphi$ is an isomorphism, using $\varphi^{-1}$ gives the reverse inclusion. Thus the affine set spanned by $X$ in $\mathbb{R}^n$ is precisely the same as that spanned by $X$ in $\mathbb{R}^k$.

**Exercise 2.3.** This is evident in the case $n = 0$.

Suppose it is true for $n-1$ and consider the canonical injection $\iota : S^{n-1} \hookrightarrow S^n$ which takes $(x_0, \ldots, x_{n-1})$ to $(x_1, \ldots, x_{n-1}, 0)$. It is obvious that we can pick $n+1$ affine independent points $p_0, \ldots, p_n$ in this embedding.

Now consider the point $p_{n+1} = (0, \ldots, 0, 1) \in S^n$. Notice that the last coordinate of each $p_i$ for $i \neq n+1$ is zero. Thus suppose we have $s_i$ with $\sum s_i p_i = 0$ and $\sum s_i = 0$. Then $s_{n+1} = 0$, and so this reduces to the $n-1$ case. Affine independence of $\{p_0, \ldots, p_n\}$ proves the result.

**Exercise 2.4.** Consider the map $T'(x) = T(x) - T(0)$. We claim that $T'$ is a linear map.

Observe that $S = \{e_i\} \cup \{0\}$ spans $\mathbb{R}^n$. Thus we can write any point as the affine sum of elements of $S$. Note that the coefficient of the zero vector is flexible, and so we have effectively no restrictions on the sum of the coefficients.

Consider arbitrary elements $\sum r_i e_i + r \cdot 0$ and $\sum s_i e_i + s \cdot 0$ in $\mathbb{R}^n$, where $r = 1 - \sum r_i$ and similarly for $s$. Let $R, S \in \mathbb{R}$. Then note that

$$T' \left( R \sum r_i e_i + S \sum s_i e_i \right) = T' \left( \sum (Rr_i + Ss_i) e_i \right)$$
$$= T \left( \sum (Rr_i + Ss_i) e_i + \left( 1 - \sum (Rr_i + Ss_i) \right) \cdot 0 \right) - T(0)$$
$$= R \sum r_i T(e_i) + S \sum s_i T(e_i) - R \sum r_i T(0) - S \sum s_i T(0).$$

Considering the $R$-terms first, simply observe that we can add and subtract $RT(0)$ to give us that

$$R \sum r_i T(e_i) - R \sum r_i T(0) = R \left( T \left( \sum r_i T(e_i) + r \cdot 0 \right) - T(0) \right).$$

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[1] I usually use *simplices* as the plural of simplex, but Rotman doesn't; no matter.

This is simply $RT'\left(\sum r_i e_i\right)$. A similar result holds for the $S$-terms, from which we conclude that

$$T'\left(R\sum r_i e_i + S\sum s_i e_i\right) = RT'\left(\sum r_i e_i\right) + ST'\left(\sum s_i e_i\right),$$

proving linearity.

**Exercise 2.5.** This is obvious from the previous exercise and continuity of linear maps.

**Exercise 2.6.** Given two $m$-simplexes $[p_0, \ldots, p_m]$ and $[q_0, \ldots, q_m]$, the map $f$ taking $p_i$ to $q_i$ for every $i$ is a homeomorphism. Bijectivity is obvious by the definition. Continuity is clear by how we extend $f$ from $\{p_i\}$ to $[p_i]$. Finally, the inverse is of the same form as $f$, only with the $q_i$'s taking the place of the $p_i$'s and vice versa; thus $f^{-1}$ is also continuous.

**Exercise 2.7.** The following map works:

$$f : x \mapsto \frac{t_2 - t_1}{s_2 - s_1}(x - s_1) + t_1.$$

**Exercise 2.8.** Pick arbitrary $T(x), T(x') \in T(X)$ and observe that

$$tT(x) + (1 - t)T(x') = T(tx + (1 - t)x') \in T(X).$$

Thus $T(X)$ is affine if $X$ is affine, and convex if $X$ is convex. The second statement of the exercise follows by noting that $\ell$ is convex.

**Exercise 2.9.** Without loss of generality, we delete $p_0$. Now suppose that

$$\sum_{i=1}^{m} s_i p_i + sb = 0, \quad \sum_{i=1}^{m} s_i + s = 0.$$

Then we know by definition of the barycenter $b$ that

$$\sum_{i=1}^{m} s_i p_i + \frac{s}{m+1} \sum_{i=0}^{m} p_i = 0.$$

Moreover, letting $s_i'$ be the coefficient of $p_i$ in the above equation, it is obvious that $\sum_{i=0}^{m} s_i' = s + \sum_{i=1}^{m} s_i = 0$. Thus $s_i' = 0$ for all $i$ because $\{p_0, \ldots, p_m\}$ was affine independent. But then we conclude that $0 = s_0' = \frac{s}{m+1}$, and so $s = 0$. For every $i \in \{1, \ldots, m\}$, we have $0 = s_i' = \frac{s}{m+1} + s_i$. Thus $s = 0$ implies $s_i = 0$ for every $i$,and so it follows that $\{b, p_1, \ldots, p_m\}$ is affine independent, as desired.

**Exercise 2.10.** Once again, suppose without loss of generality that $i = 0$. Then the map taking $\sum t_i p_i \in [p_0, p_1, \ldots, p_m]$ to $(\sum_{i=1}^{m} t_i p_i, t_0)$ works. Note that this actually requires the affine independence of the $p_i$'s, as well as the fact that the coefficients $t_i$ are all between 0 and 1.

**Exercise 2.11.** Notice that $[0, e_1, \ldots, e_n]$, where $e_i$ are the standard basis vectors in $\mathbb{R}^n$, is an $n$-simplex. Thus there is a homeomorphism $[p_0, \ldots, p_n] \to [0, e_1, \ldots, e_n]$. If we translate the image by $\mathbf{v} = (-\frac{1}{4}, \ldots, -\frac{1}{4})$, then we can map the result to $D^n$ by taking a radial mapping. In particular, this map will take

$$p_0 \mapsto \frac{\mathbf{v}}{\|\mathbf{v}\|}$$
$$p_i \mapsto \frac{e_i + \mathbf{v}}{\|e_i + \mathbf{v}\|} \text{ for } i \neq 0.$$

Note that this extends to a homeomorphism.