

3 The Fundamental Group

The Fundamental Groupoid

Exercise 3.1. The homotopy $H : X \times \mathbb{I} \rightarrow Z$ given by

$$H : (x, t) \mapsto \begin{cases} g_0(F(x, 2t)) & \text{if } t \leq \frac{1}{2}, \\ G(f_1(x), 2t - 1) & \text{if } t \geq \frac{1}{2} \end{cases}$$

works. Continuity follows because $g_0(F(x, 1)) = G(f_1(x), 0)$.

Moreover, this homotopy is indeed $\text{rel } A$. For a detailed argument why this is so, simply suppose that $a \in A$ and $t \in I$. If $t \leq \frac{1}{2}$, then $F(a, 2t) = f_0(a)$ by definition of F . Hence $H(a, t) = g_0(f_0(a))$.

Similarly, we can show that if $t \geq \frac{1}{2}$, then $H(a, t) = g_1(f_1(a))$. This follows because $f_1(a) \in B$ and G is a homotopy $\text{rel } B$.

It thus suffices to show that $g_0(f_0(a)) = g_1(f_1(a))$. But this is obvious because f_0 and f_1 agree on A , and g_0 and g_1 agree on $B \supseteq f_0(A)$.

Exercise 3.2.

- (i) First, note that f' is well-defined because $f(0) = f(1)$. It is obvious by continuity of f and \ln that f' is continuous.

Moreover, consider the map

$$H' : (e^{2\pi i \theta}, t) \mapsto H(\theta, t).$$

This is clearly continuous, for the same reasons that f' was continuous. If $t = 0$, clearly $H'(e^{2\pi i \theta}, t) = H(\theta, 0) = f(\theta) = f'(e^{2\pi i \theta})$, and similarly for $t = 1$. Thus H is indeed a homotopy from f' to g' .

To see that it is a homotopy $\text{rel}\{1\}$, simply note that $e^{2\pi i \theta} = 1$ corresponds to $\theta = 0, 1$. Thus it follows that

$$H'(1, t) = H(1, t) = f(1)$$

for all t , proving the result.

- (ii) Theorem 3.1 implies that $f * g \simeq f_1 * g_1 \text{ rel } \dot{\mathbb{I}}$. Using the previous part, we find that $(f * g)' \simeq (f_1 * g_1)' \text{ rel } \{1\}$. Now, using the observation that $(f * g)' = f' * g'$, we find that $f' * g' \simeq f'_1 * g'_1 \text{ rel } \{1\}$, as desired.

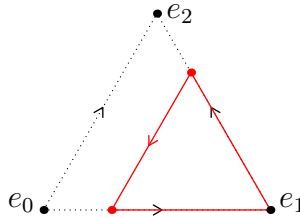
Exercise 3.3. The forward direction is trivial.

For the converse, note that g' is a constant map, and so f' is nullhomotopic. Then Theorem 1.6 implies that $f' \simeq g' \text{ rel } \{1\}$. In particular, note that $g' : S^1 \rightarrow X$ takes every element of S^1 to $g'(1) = g(0) = x_0$. Observe that $f'(1) = x_0$ as well, and so it follows that $f' \simeq g \text{ rel } \{1\}$, as desired.

Exercise 3.4.

- (i) Instead of applying Theorem 1.6, I constructed an explicit homotopy. (If you are interested in a proof using Theorem 1.6, my guess would be that it relies on the fact that $\Delta^2 \approx D^2$. However, I have not gone through the details.)

The effective idea of the homotopy I constructed is to, at time $t \in [0, 1]$, return the function which traverses the first t units of the face opposite e_0 , then goes along a segment to the point t units away from e_1 on the face opposite e_2 , before returning back to e_1 , as shown in the red path below.



The specific homotopy $H : \mathbb{I} \times \mathbb{I} \rightarrow X$ from $(\sigma_0 * \sigma_1^{-1}) * \sigma_2$ to the constant map at e_1 is as follows:

$$H(x, t) = \begin{cases} \sigma_0(4(1-t)x) & \text{if } x \leq \frac{1}{4}, \\ \sigma((1-x)\varepsilon_0(1-t) + x\varepsilon_2(t)) & \text{if } \frac{1}{4} \leq x \leq \frac{1}{2}, \\ \sigma(2tx - (2t-1)) & \text{if } x \geq \frac{1}{2}. \end{cases}$$

We leave it to the reader to check that this works.

- (ii) One can generate a similar homotopy, which we do not do here.
- (iii) This time, we use the homotopy which goes up along γ for t units, before going parallel to β and coming back down along δ^{-1} . The particular formula is as follows:

$$H(x, t) = \begin{cases} F(0, 4tx) & \text{if } x \leq \frac{1}{4}, \\ F(4x-1, t) & \text{if } \frac{1}{4} \leq x \leq \frac{1}{2}, \\ F(1, 2t(1-x)) & \text{if } \frac{1}{2} \leq x. \end{cases}$$

Once again, we leave the details to the reader to check.

Exercise 3.5. Simply use the homotopy $H : \mathbb{I} \times \mathbb{I} \rightarrow X \times Y$ which takes (s, t) to $(F(s, t), G(s, t))$. This is clearly a homotopy from (f_0, g_0) to (f_1, g_1) . To see that it is still $\text{rel } \mathbb{I}$, simply observe that $H(0, t) = (F(0, t), G(0, t))$. Because F and G are both $\text{rel } \mathbb{I}$, it follows that $H(0, t)$ never changes. A similar argument shows that $H(1, t)$ is always the same, and so H is indeed a homotopy $\text{rel } \mathbb{I}$.

Exercise 3.6.

- (i) It is obvious that the homotopy $H' : (x, t) \mapsto H(x, 1-t)$ works.
- (ii) This is just some slightly annoying manipulation. In particular, note that

$$(f * g)(x) = \begin{cases} f(2x) & \text{if } x \leq \frac{1}{2}, \\ g(2x-1) & \text{if } x \geq \frac{1}{2}. \end{cases}$$

By replacing x with $1-x$ to get the inverse, we find that

$$(f * g)^{-1}(x) = \begin{cases} f(2-2x) & \text{if } x \geq \frac{1}{2}, \\ g(1-2x) & \text{if } x \leq \frac{1}{2}. \end{cases}$$

However, note that

$$\begin{aligned} (g^{-1} * f^{-1})(x) &= \begin{cases} g^{-1}(2x) & \text{if } x \geq \frac{1}{2}, \\ f^{-1}(2x-1) & \text{if } x \leq \frac{1}{2} \end{cases} \\ &= \begin{cases} g(1-2x) & \text{if } x \leq \frac{1}{2}, \\ f(2-2x) & \text{if } x \geq \frac{1}{2}. \end{cases} \end{aligned}$$

Thus the two are indeed the same.

- (iii) Take the closed path $f(t) = e^{2\pi it}$ on S^1 . Then note that $(f * f^{-1})(\frac{1}{8}) = f(\frac{1}{4}) = i$, while $(f^{-1} * f)(\frac{1}{8}) = f^{-1}(\frac{1}{4}) = -i$.
- (iv) Suppose $i_p * f = f$ and f is not constant. Note that continuity implies that there must exist some $0 < t < 1$ so that $f(t) \neq p$. Thus there exists some $k \in \mathbb{N}$ so that $t < 1 - 2^{-k}$.

We claim, however, that f must be constant on $[0, 1 - 2^{-n}]$ for every $n \in \mathbb{N}$. We prove this inductively. Clearly, it is true on $n = 0$. If it is true on $n - 1$, then we know that $i_p * f$ must be equal to p on $[0, \frac{1}{2}]$, as well as on $[\frac{1}{2}, 1 - 2^{-n}]$ (note that $1 - 2^{-n}$ comes from $2(1 - 2^{-n}) - 1$, which itself comes from the equation for the star operator). Thus f is constant on $[0, 1 - 2^{-n}]$, as desired.

Thus it follows that $f(t) = p$, a contradiction. Thus f must have been constant in the first place.

The Functor π_1

Exercise 3.7. Recall that we defined the $\sin(1/x)$ space as the union of $A = \{(0, y) : -1 \leq y \leq 1\}$ and $G = \{(x, \sin(1/x)) : 0 < x \leq 1/2\pi\}$. We also know that A and G are the path components of the $\sin(1/x)$ space. Moreover, both A and G are contractible, and so every path in either A or G is nullhomotopic. In particular, we conclude that the fundamental group at any basepoint is trivial.

Exercise 3.8. Let X be the $\sin(1/x)$ space. We know that CX is contractible. But consider an open ball around the point $x = ((0, 0), 0)$, that is, the point $(0, 0)$ on the “zeroth” level of the cone. Consider a small neighborhood (not including the points $(t, 1)$, in particular) around this point and pick some element $y = ((\varepsilon, \sin(1/\varepsilon)), 0)$ in the neighborhood. Now observe that any path between x and y can be projected down to a path between $(0, 0)$ and $(\varepsilon, \sin(1/\varepsilon))$ in X , which we know does not exist. Hence CX is contractible but not locally path connected.

Exercise 3.9. Note that composition is associative because \circ is. Moreover, the path class of the trivial loop based at p is the identity on p . Thus this is a category.

To see that each morphism in \mathcal{C} , simply note that the inverse path, i.e., the path f^{-1} taking t to $f(1-t)$, gives a path class $[f^{-1}]$ which works as an inverse to $[f] \in \text{Hom}(p, q)$.

Exercise 3.10. We simply let π_0 take $(X, x_0) \in \text{Sets}_*$ to the set of all path components of X , with basepoint equal to the path component containing x_0 . It takes a morphism $f \in \text{Hom}((X, x_0), (Y, y_0))$ to the map $\pi_0(f)$ which takes each path component A of X to the path component B of Y which contains $f(A)$.

Note that this is possible because continuous images of path connected spaces are path connected and hence contained within a single path component of Y . Moreover, this is indeed a pointed map because the path component containing x_0 must be contained in the path component containing $f(x_0) = y_0$, which is the basepoint of $\pi_0((Y, y_0))$.

It is easy to check functoriality, completing the proof.

Exercise 3.11. Evidently the only possible path is the constant path at x_0 . Hence $\pi_1(X, x_0)$ is the trivial group, i.e., $\{1\}$.

$$\pi_1(S^1)$$

Exercise 3.12. Note that 1_S is a loop based at 1, i.e., an element of $\pi_1(S^1, 1)$. Thus if $\pi_1(S^1, 1)$ were trivial, then 1_S would be nullhomotopic. The hint gives the rest of the solution.

Exercise 3.13. We know that $\deg u = 1$. Since 1 is a generator for \mathbb{Z} , it follows that $[u]$ generates $\pi_1(S^1, 1)$.

Exercise 3.14. Let $\tilde{\gamma}(t) = m\tilde{f}(t)$, where \tilde{f} is the lifting of f satisfying $\tilde{f}(0) = 0$. Now simply observe that

$$\exp \tilde{\gamma}(t) = \left(\exp \tilde{f}(t) \right)^m = f(t)^m$$

and $\tilde{\gamma}(0) = 0$. Thus $\tilde{\gamma}$ is indeed the lifting of f^m taking 0 to 0, and so we conclude that

$$\deg(f^m) = \tilde{\gamma}(1) = m\tilde{f}(1) = m \deg f.$$

Exercise 3.15. Note that $??$ implies that there is a homotopy $F : R_f \circ f \simeq f$, where R_f is the rotation associated with f . Moreover, from the proof of that same exercise, it follows that F gives a closed path at every time t . Similarly, we have $G : g \simeq R_g \circ g$. Thus if $H : f \simeq g$ where H gives a closed path at every time t , then the homotopy which follows F , then H , and finally G is a homotopy between $R_f \circ f$ and $R_g \circ g$. Thus Corollary 3.18 implies that f and g have the same degree.

For the converse, simply use Corollary 3.18 to show that $\deg f = \deg g$ implies that there is a homotopy $\text{rel } \mathbb{I}$ taking $R_f \circ f$ to $R_g \circ g$. Then using F and G defined above, it is clear that $g \simeq R_g \circ g \simeq R_f \circ f \simeq f$.

Exercise 3.16. Theorem 3.7 implies that $\pi_1(T, t_0) = \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$.

Exercise 3.17. Because D^2 is contractible, its fundamental group is trivial. Thus if there were to exist a retraction $r : D^2 \rightarrow S^1$, then $r_* : \pi_1(D) \rightarrow \pi_1(S^1)$ would be a constant. But then, letting $i : S^1 \rightarrow D^2$ be the canonical injection, we would have that $(r \circ i)_* = r_* \circ i_*$ is a constant. However, we also know that $r \circ i$ is the identity on S^1 , and so $(r \circ i)_*$ is the identity on $\pi_1(S^1)$, which is *not* a constant. This is a contradiction, from which we conclude that S^1 is not a retract of D^2 , as desired.

Exercise 3.18. This was proved in Theorem 0.3, which required only the fact proved in the above problem, namely that S^1 is not a retract of D^2 .

Exercise 3.19.

- (i) Let \tilde{f} be the unique lifting of f with $\tilde{f}(0) = 0$. Then if $\tilde{f}(1) \geq 1$, the intermediate value theorem implies that every point in the interval $[0, 1] \subset \mathbb{R}$ is in the image of \tilde{f} . But this implies that $f = \exp \circ \tilde{f}$ must be surjective, a contradiction.
- (ii) Consider the map which traverses the circle once counterclockwise, reaching the point 1 at time $t = \frac{1}{2}$, before looping back and making a clockwise rotation. Clearly it is surjective. However, it is composed of two loops, one of which has degree 1 and one of which has degree -1 . Because $\deg(f * g) = \deg f + \deg g$, it follows that this map has degree 0.

Exercise 3.20. As per the hint, consider an arbitrary closed path f in X and let λ be a Lebesgue number of the open cover $\{f^{-1}(U_j) : j \in J\}$ of \mathbb{I} . Note that λ exists by the Lebesgue number lemma and compactness of the unit interval. Picking $N \in \mathbb{N}$ with $N > 1/\lambda$, it follows that if we subdivide I into N equal intervals $I_k = [\frac{k}{N}, \frac{k+1}{N}]$, then $f(I_k) \subseteq U_{j_k}$ for some $j_k \in J$.

Define f_k as the path in U_{j_k} obtained by restricting f to I_k and then stretching suitably so that the domain is all of \mathbb{I} . With notation, define $f_k(t) = f(\frac{k+t}{N}) \in U_{j_k}$. Because f_k is a path in U_{j_k} , it follows that $[f'_k] = [i_{j_k} \circ f_k] \in \text{im}(i_{j_k})_*$. But now simply observe that $[f'_0 * \cdots * f'_{N-1}] = [f]$, implying that $[f]$ is contained in the group generated by the subsets $\text{im}(i_j)_*$. This proves the result.

Exercise 3.21. Let U_1 and U_2 be defined as in the hint, and let i_k be the injection from U_k to S^n for $k = 1, 2$. Observe that, by the previous exercise, it suffices to show that $\text{im}(i_k)_*$ is trivial for $k = 1, 2$.

Without loss of generality, let $k = 1$. But we know that $(i_1)_*$ takes a closed path $f : \mathbb{I} \rightarrow U_1$ to the path class $[i_1 \circ f]$. (Note that the basepoint doesn't really matter for us as long as it is neither the north nor the south pole. Thus we omit it.) Because $U_1 \approx D^n$ and is therefore contractible, it follows that f is nullhomotopic. In particular, we know that $i_1 \circ f$ is nullhomotopic, and so $[i_1 \circ f] = [1]$ for every f . Thus $\text{im}(i_1)_*$ is trivial, and similarly for $k = 2$, proving the result.

Exercise 3.22. Corollary 3.11 implies that path connected spaces of the same homotopy type must have isomorphic fundamental groups. But obviously $\mathbb{Z} \not\cong \{1\}$, and so S^1 and S^n do not have the same homotopy type for $n > 1$.

Exercise 3.23. The multiplication map μ on G/H is continuous. After all, if we let v be the natural map, then for any open set $U \subseteq G/H$, we have

$$\mu^{-1}(U) = \{([x], [y]) : xy \in v^{-1}(U)\}.$$

But this set is open in $G/H \times G/H$ because the set consisting of elements $(v^{-1}([x]), v^{-1}([y]))$ for each $([x], [y]) \in \mu^{-1}(U)$ is just $\mu^{-1}(v^{-1}(U))$, which is clearly open.

For the inversion map $i : G/H \rightarrow G/H$, a very similar argument holds. In particular, for any open set $U \subseteq G/H$, we have

$$v^{-1}(i^{-1}(U)) = \{x^{-1} : x \in v^{-1}(U)\}.$$

Thus $v^{-1}(i^{-1}(U))$ is open, and so $i^{-1}(U)$ is open, proving continuity.

Exercise 3.24. First, we will show that we can lift a loop $f : (\mathbb{I}, 0) \rightarrow (G/H, 1)$ to a unique continuous map $\tilde{f} : (\mathbb{I}, 0) \rightarrow (G, h_0)$ for any $h_0 \in H$, as shown below.

$$\begin{array}{ccc} & (G, h_0) & \\ \tilde{f} \nearrow & \downarrow v & \\ (\mathbb{I}, 0) & \xrightarrow{f} & (G/H, 1) \end{array}$$

In the above diagram, the map v is the natural map taking g to the coset $gH \in G/H$.

First, we will find a suitable neighborhood U of 1 such that the family $\{hU : h \in H\}$ is pairwise disjoint. Discreteness of H implies that there exists an open neighborhood V of 1 with $V \cap H = \{1\}$. It is clear that the map $\varphi : (x, y) \mapsto xy^{-1}$ is the composition $\mu \circ (\text{id} \times i)$ and is therefore continuous. Thus $\varphi^{-1}(V) \subseteq G \times G$ is an open neighborhood of $(1, 1)$. This implies that we can find an open neighborhood U of 1 such that $U \times U \subseteq \varphi^{-1}(V)$.

Now suppose that there are $h_1, h_2 \in H$ and $x, y \in U$ with $h_1x = h_2y$. But this would require that $xy^{-1} = h_1^{-1}h_2$. It is clear that $xy^{-1} \in \varphi(U) \subseteq V$. Moreover, because H is a subgroup, we know that $h_1^{-1}h_2 \in H$, and so $xy^{-1} \in V \cap H$. Thus $x = y$ and $h_1 = h_2$, proving that the sets hU are disjoint, as desired. Note that any translate $U_g = gU$ of U is a neighborhood of $g \in G$ and has $\{hU_g : h \in H\}$ disjoint.

Note that v is an open map, and so the set $W = v(U) \subseteq G/H$ is open. Moreover, because $v|_U$ is the restriction of a continuous open map to an open set, it follows that $v|_U$ is itself continuous and open. It is also a bijection onto W , and so $v|_U : U \rightarrow W$ is a homeomorphism.

Note that the collection of sets $V[g]$ for $[g] \in G/H$ forms an open cover of G/H . Thus, if we are given some $f : (\mathbb{I}, 0) \rightarrow (G/H, 1)$, then we can consider the open cover

$$\{f^{-1}(V[g]) : [g] \in G/H\}$$

of \mathbb{I} . Note that we can find a finite subcover of this open cover. This means that we can take subsets of the sets in this open cover, given us a finite collection open overlapping subintervals which are, in order of their smaller coordinate, labeled I_1, \dots, I_k . Let the group elements g_1, \dots, g_k be such that $I_j \subseteq f^{-1}(V[g_j])$. This is simply because $\mathbb{I} = [0, 1]$ is connected compact.

Now we can lift f to each interval $f^{-1}(V[g])$ in this finite subcover. Note that $0 = t_1 \in I_1 \subseteq f^{-1}(V[g_1])$. Moreover, we know that $v^{-1}(V[g_1])$ consists of disjoint unions of U , and so we can pick the one containing h_0 . Now, for each $t \in I_1$, we let $\tilde{f}(t)$ to be the unique element in this copy of U such that $v(\tilde{f}(t)) = f(t)$. Because the intervals overlap, we know that there is some $t_2 \in I_2 \cap I_1$, and so we can do the same thing, all the way to t_k . This lets us define $\tilde{f}(t)$ for all $t \in \mathbb{I}$, and it is easy to show that our construction is indeed a lifting satisfying the commutative diagram above.

Now consider the map $d : \pi_1(G/H, 1) \rightarrow H$ taking a loop $[f]$ to $d([f]) = \tilde{f}(1)$, where \tilde{f} is the unique lifting of f with $\tilde{f}(0) = 1$. It is obvious that $\text{im } d \subseteq H$ because $v(\tilde{f}(1)) = f(1) = [1]$ implies that $d([f]) = \tilde{f}(1) \in H$. Moreover, the reverse inclusion holds, showing surjectivity. In particular, if $h \in H$, then path connectedness of G implies that there is a path \tilde{f} from 1 to h . Taking its projection $f = v \circ \tilde{f}$, note that f is a loop because $v(\tilde{f}(1)) = v(h) = [1]$. Thus $d([f])$ is defined and equal to h . To show injectivity, simply note that $\tilde{f}(1) = 1$ implies that \tilde{f} is a loop in G . Because G is simply connected, however, it follows that \tilde{f} , and hence f , is nullhomotopic. Thus $f \in \ker d$ implies that $[f] = [1]$. Finally, we must show that d is indeed a homomorphism, i.e., that $d(f * g) = d(f)d(g)$. But this is clear if we lift f to \tilde{f} with $\tilde{f}(0) = 1$, and if we lift g to \tilde{g} with $\tilde{g}(0) = d(f)$. This follows the same proof layout as Theorem 3.16, and proves the result.

Exercise 3.25. If $S \subseteq GL(n, \mathbb{R})$ is a subgroup of $GL(n, \mathbb{R})$, then note that $\mu : S \times S \rightarrow S$ is continuous. After all, the product, entrywise, is simply a polynomial, and so μ is a polynomial in each of its n^2 entries. Since polynomials are continuous in \mathbb{R}^2 , it follows that each of the n^2 components of μ is continuous. Hence μ is continuous.

To see that the inversion i is continuous, observe that the determinant $\det A$ is a continuous function, since it too is a polynomial (and is never zero, by definition of GL). It thus suffices to show that the function $A \mapsto \text{adj } A$ is continuous. But it is easy to see that the adjugate matrix, which is the transpose of the cofactor matrix, is also a polynomial in the entries of A , and so $\text{adj } A$ is continuous too. Thus i is continuous, and so S is a topological group, as desired.

Exercise 3.26. As hinted in the exercise, fix $h_0 \in H$ and let $\varphi : G \rightarrow H$ be the map taking x to $xh_0x^{-1}h_0^{-1}$. Note that $xh_0x^{-1} \in H$ because H is normal, and so $\varphi(x)$ is indeed an element of H . Moreover, we know that φ is continuous and $\{h\} \subseteq H$ is open for each $h \in H$. Thus $\{\varphi^{-1}(h) : h \in H\}$ is an open cover of G consisting of disjoint open sets.

In particular, if there are two elements $h_1, h_2 \in H$ such that $\varphi^{-1}(h_i) \neq \emptyset$ for $i = 1, 2$, then setting $A = \varphi^{-1}(h_1)$ and $B = \bigcup_{h \neq h_1} \varphi^{-1}(h)$ will give us two disjoint open sets A and B that cover G . This implies that G is disconnected, a contradiction. Thus for all but one element of H , we must have $\varphi^{-1}(h) = \emptyset$,

proving that φ is constant. But obviously, setting $x = h_0$, we find that $\varphi(x) = 1$. Thus $xh_0x^{-1}h_0^{-1} = 1$ for all $x \in G$, and so $xh_0 = h_0x$ for each $h_0 \in H$. This proves the result.