Rotman algebraic topology solutions

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0 Introduction

Brouwer Fixed Point Theorem

Exercise 0.1. As per the hint, observe that if $y \in G$, then we have $y = r(y) + (y - r(y))$. Obviously, we have $r(y) \in H$. Moreover, we know that

$$r(y - r(y)) = r(y) - r(r(y)) = 0,$$

and so $y - r(y) \in \ker r$. Thus $G \subseteq H \oplus \ker r$.

The reverse is obviously true, since H and $\ker r$ are both subgroups of G.

Exercise 0.2. Suppose instead that $f : D^1 \to D^1$ has no fixed point. Then consider the continuous map $g : D^1 \to S^0$ given by

$$g(x) = \begin{cases} 1 & \text{if } f(x) < x \\ -1 & \text{if } f(x) > x \end{cases}.$$

Notice that because $f(x) \neq x$ for all x, the function g is well-defined.

Moreover, we know that $f(-1) \neq -1$, since f has no fixed point, and so $f(-1) > -1$. Thus $g(-1) = -1$. Similarly, we have $g(1) = 1$.

Thus we have $g(D^1) = S^0$, which is disconnected. This is a contradiction, so f must have had a fixed point.

Exercise 0.3. Suppose that r is such a retract. Then we have the following commutative diagram:

$$\begin{tikzcd} & S^n \arrow[dr, "r"] & \\ S^{n-1} \arrow[ur, "i"] \arrow[rr, "1"] & & S^{n-1}. \end{tikzcd}$$

Applying H_{n-1}, we get another commutative diagram:

$$\begin{tikzcd} & H_{n-1}(S^n) \arrow[dr, "H_{n-1}(r)"] & \\ H_{n-1}(S^{n-1}) \arrow[ur, "H_{n-1}(i)"] \arrow[rr, "H_{n-1}(1)"] & & H_{n-1}(S^{n-1}). \end{tikzcd}$$

We know that $H_{n-1}(S^n) = 0$, however, implying that $H_{n-1}(1) = 0$. This contradicts the fact that $H_{n-1}(S^{n-1}) = \mathbb{Z} \neq 0$. Thus the retraction r could not have existed.

Exercise 0.4. Suppose $g : D^n \to X$ is a homeomorphism. Then we know that $g^{-1} \circ f \circ g$ is a continuous map from D^n to itself, and so it has a fixed point x. Then we know that $g^{-1}(f(g(x))) = x$, and so it follows that $f(g(x)) = g(x)$. Thus $g(x) \in X$ is a fixed point of f.

Exercise 0.5. Consider the function $h : \mathbb{I} \times \mathbb{I} \to \mathbb{I} \times \mathbb{I}$ given by

$$h(s,t) = f(s) - g(t) + (s,t).$$

This is the sum of continuous functions, and so it is itself continuous. Moreover, we know that $\mathbb{I} \times \mathbb{I}$ is homeomorphic to D^1, and so it follows that there is a fixed point (s,t) of h. But this means that $f(s) - g(t) = 0$, and so we are done.

Exercise 0.6. Observe that $x \in \Delta^{n-1}$ must contain some positive coordinate, because $\sum x_i = 1$ and $x_i \geq 0$ for all i. Since $a_{ij} > 0$ for every i, j, it follows that Ax contains only nonnegative coordinates and, moreover, contains at least one positive coordinate. Thus $\sigma(Ax) > 0$, and so $g(x)$ is well-defined.

Moreover, it is continuous because the linear map A, the map σ, and the division function are all continuous.

Because $\Delta^{n-1} \approx D^{n-1}$, it follows that there exists some x with

$$x = \frac{Ax}{\sigma(Ax)}.$$

Then $\lambda = \sigma(Ax) > 0$ is a positive eigenvalue for A and $x \in \Delta^{n-1}$ is a corresponding eigenvector.

We know that x contains only nonnegative coordinates. Suppose then that some coordinate, say x_1, is zero. Then obviously the first coordinate of λx is zero. However, the first coordinate of Ax is

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = a_{12}x_2 + \cdots + a_{1n}x_n.$$

Since $\sum x_i = 1$ and $x_1 = 0$, there exists some $k \neq 1$ such that $x_k > 0$. Then $a_{1k}x_k > 0$, and since each i already has $a_{1i}x_i \geq 0$, it follows that the first coordinate of Ax is strictly positive, contradicting that $Ax = \lambda x$.

Thus the eigenvector x has all positive coordinates.

Categories and Functors

Exercise 0.7. We know that

$$g \circ (f \circ h) = g \circ 1_b = g$$

and

$$(g \circ f) \circ h = 1_A \circ h = h,$$

and so associativity implies $g = h$.

Exercise 0.8.

(i) Notice that if 1_A and $1'_A$ are both identities, then we must have

$$1_A = 1_A \circ 1'_A = 1'_A,$$

which proves the desired result.

(ii) If $1'_A$ is the new identity in $\mathcal{C}'$, then we know that $1'_A \in \mathrm{Hom}_{\mathcal{C}'}(A, A) \subseteq \mathrm{Hom}_{\mathcal{C}}(A, A)$, and so $1_A \circ 1'_A$ is defined. But we know that

$$1'_A \circ 1_A = 1'_A = 1'_A \circ 1'_A,$$

and so **??** implies the result.

Exercise 0.9. Clearly, the Hom-sets are pairwise disjoint, since each i_y^x appears at most once.

It is also obviously associative. In particular, if $a \leq b \leq c \leq d$, then we know that

$$i_d^c \circ \left(i_c^b \circ i_b^a\right) = i_d^c \circ i_c^a = i_d^a,$$

and similarly for $\left(i_d^c \circ i_c^b\right) \circ i_b^a$.

Finally, the map i_x^x is the identity on $x \in X$. To see that it is a left-identity, note that if $y \leq x$, then

$$i_x^x \circ i_x^y = i_x^y.$$

Similarly, we can show that this map is a right-identity as well, and so we are done.

Exercise 0.10. Disjointness is clear, since there is only one object. Because G is a monoid, it is associative and has an identity, proving that $\mathcal{C}$ is a category.

Exercise 0.11. It is pretty clear that $\text{obj}(\mathbf{Top}) \subset \text{obj}(\mathbf{Top}^2)$. Moreover, a continuous map $f : X \to Y$ between two topological spaces corresponds to the map $(f, \emptyset)$ in $\mathbf{Top}^2$ from $(X, \emptyset)$ to $(Y, \emptyset)$, which then means that $\mathbf{Top}$ can be thought of as a subcategory of $\mathbf{Top}^2$.

Exercise 0.12. It is worth noting that Rotman's definition here is incorrect. The morphisms in $\mathcal{M}$ should be the commutative squares, not merely the ordered pairs (h, k).

Indeed, consider the following counterexample to Rotman's definition. Let $\mathcal{C}$ be the category of sets. Furthermore, let A be a set with more than one element. Then the following diagrams are both commutative:

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "1_A"] & A \arrow[d, "0"] \\ A \arrow[r, "0"] & \{0\} \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "0"] \arrow[d, "1_A"] & A \arrow[d, "0"] \\ A \arrow[r, "0"] & \{0\}. \end{tikzcd}$$

This implies that the ordered pair $(1_A, 0)$, where 0 is considered to be the map that sends everything in A to the zero element, is both in $\text{Hom}(1_A, 0)$ and in $\text{Hom}(0, 0)$, contradicting disjointness.

If we instead consider morphisms of $\mathcal{M}$ to be the commutative squares, where composition is defined by "stacking" the squares on top of one another, disjointness is clear. After all, the squares contain f and g, and so Hom-sets of different objects must be disjoint.

Associativity is clear, as the morphisms of $\mathcal{C}$ are associative.

Finally, there is an identity 1_f for every $f \in \text{Hom}_{\mathcal{C}}(A, B)$, namely the one where $h = 1_A$ and $k = 1_B$.

Exercise 0.13. With the hint, this is clear. In particular, we consider $\mathbf{Top}^2$ to be the subcategory of the arrow category of $\mathbf{Top}$ in which the objects are inclusions, and $\text{Hom}_{\mathbf{Top}^2}(i, j) = \text{Hom}_{\mathbf{Top}}(i, j)$.

Exercise 0.14. To see that it is a congruence at all, observe that Property (i) is satisfied because there is only one Hom-set. Moreover, if $x \sim x'$ and $y \sim y'$, then we know that $x(x')^{-1} = h_x$ and $y(y')^{-1} = h_y$ for some $h_x, h_y \in H$. But then we know that

$$(yx)(y'x')^{-1} = yx(x')^{-1}(y')^{-1} = yh_x(y')^{-1}.$$

However, since $(y')^{-1} = y^{-1}h_y$, we know that this is simply

$$(yx)(y'x')^{-1} = yh_xy^{-1}h_y.$$

Because H is normal, we know that $yh_xy^{-1} \in H$. Thus the product of this and h_y is in H as well, and so $xy \sim x'y'$, as desired.

To see that $[*, *] = G/H$ simply requires the observation that $x \sim y$ if and only if x and y are in the same coset of H.

Exercise 0.15. This follows from the fact that functors preserve (or, in the case of contravariant functors, reverse) the directions of the arrows. Thus the resulting diagram still commutes.

Exercise 0.16. Note that for (i)–(iv), we can simply use inverses. For instance, for **Set**, it suffices to note that if f is a bijection, then f^{-1} is a bijection, which is clearly true. Similarly, the inverse of a homeomorphism is a homeomorphism, and the inverse of a group or ring isomorphism is still an isomorphism.

For (v), note that i_x^y is defined and satisfies the requirements that $i_x^y \circ i_y^x = i_x^x$ and $i_y^x \circ i_x^y = i_y^y$.

For part (vi), notice that f^{-1} works because f is a homeomorphism. In particular, it is a bijection, and so $f^{-1}(A') = A$. Moreover, it is (bi)continuous since f is.

Finally, for the monoid G, if g has a two-sided inverse h, then $hg = gh = 1$, which is the identity element of $\text{Hom}(G, G)$.

Exercise 0.17. To prove that T' is a functor, first observe that criterion (i) of a functor is satisfied because T does so. Moreover, if $[f] \in \text{Hom}_{\mathcal{C}'}(A, B)$, then $f \in \text{Hom}_{\mathcal{C}}(A, B)$, and so $T'([f]) = Tf$ is a morphism in $\mathcal{A}$. In particular, if $[g] \circ [f] = [g \circ f]$ is defined in $\mathcal{C}'$, then $g \circ f$ is defined in $\mathcal{C}$. This means, then, that

$$T'([g] \circ [f]) = T(g \circ f) = (Tg) \circ (Tf) = T'([g]) \circ T'([f]).$$

Finally, it remains to note that $T'([1_A]) = T_{1_A} = 1_{TA} = 1_{T'([A])}$ for every object A. Thus T' is a functor, as desired.

Exercise 0.18.

(i) It is clear that $tG \in \operatorname{obj} \mathbf{Ab}$ for every group G. Now suppose that we have a homomorphism $f : G \to H$. Then we know that $t(f)$ is a morphism $f|_{tG}$ from tG to tH. To see this, note that it is the restriction of a homomorphism, and thus is itself a homormophism. Moreover, if $x \in f(tG)$, then $x = f(y)$ for some $y \in G$ with finite order. But then there exists some n so that $y^n = 1$. Thus $x^n = f(y^n) = 1$, and so x has finite order. But $x \in f(G) \subseteq H$ implies that $x \in tH$.

Now we must check that t respects composition. Indeed, if $g \circ f$ is defined, then

$$t(g \circ f) = (g \circ f)_{tG} = g|_{f(tG)} \circ f|_{tG}.$$

But $f(tG) \subseteq tH$, and so this is simply

$$t(g \circ f) = g|_{tH} \circ f|_{tG} = t(g) \circ t(f),$$

which proves that composition is respected.

Finally, note simply that $t(1_G) = 1|_{tG}$, which is the identity on tG.

(ii) Suppose that f is an injective homomorphism from G to H. Then suppose that $t(f)(x) = t(f)(y)$. But $f(x) = f|_{tG}(x) = t(f)(x)$, and so it follows that $f(x) = f(y)$. Injectivity of f proves the result.

(iii) Let $G = \mathbb{Z}$ and $H = \mathbb{Z}/2\mathbb{Z}$ and let f take even integers to 0 and odd integers to 1. This is evidently surjective. But $tG = \{0\}$ while $tH = \{0, 1\}$, and so $t(f) : tG \to tH$ cannot be surjective.

Exercise 0.19.

(i) If f is a surjection, then consider an arbitrary coset $a + pH$ of H/pH. We know that there exists some $b \in G$ with $f(b) = a$, and so it follows that $F(f)$ takes $b + pG$ to $a + pH$, proving surjectivity of $F(f)$.

(ii) Consider the function $f : \mathbb{Z} \to \mathbb{Z}$ taking x to $2x$. Then, letting $p = 2$, we know that $F(f) : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ has $F(f)([0]) = F(f)([1])$.

Exercise 0.20.

(i) This is evident because $\mathbb{R}$ is a ring, and the operations are pointwise.

(ii) By the previous part, we know that if X is a topological space, then $C(X)$ is a ring. Now suppose that $f : X \to Y$ is a continuous map. Then define

$$\begin{aligned} C(f) : C(Y) &\to C(X) \\ g &\mapsto g \circ f \end{aligned}$$

and note that this is well-defined. Moreover, we know that $C(g \circ f)(h) = h \circ g \circ f$, while $C(f) \circ C(g)$ takes h to $C(f) \circ (h \circ g) = h \circ g \circ f$, which proves that C reverses composition. Finally, we know that $C(1_x)$ takes g to $g \circ 1_X = g$ and is therefore the identity on $C(Y)$. Thus C (or, rather, the map taking X to $C(X)$, to be precise) gives rise to a contravariant functor.

1 Homotopy

Exercise 1.1. Suppose $H : f_0 \simeq f_1$ is a homotopy. Then let $F(t) = H(x, t)$ for some fixed x. It is clear that $F(0) = x_0$ and $F(1) = 1$. Moreover, since H is continuous, it follows that so too is F. For the converse, simply let the homotopy $H : f_0 \simeq f_1$ take $(x, t) \in X \times \mathbb{I}$ to $F(t)$.

Exercise 1.2.

(i) There exist functions $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_Y$. Moreover, there is a homotopy $F : 1_X \simeq c$, where c denotes the constant map at some $x_0 \in X$. Then consider the map $G : Y \times \mathbb{I} \to Y$ which takes (y, t) to $f(F(g(y), t))$. In particular, we know that G is continuous and that it is thus a homotopy from $f \circ g$ to the constant map c' at $y_0 = f(x_0)$. But then we find that $1_Y \simeq f \circ g \simeq c'$, and so Y is contractible.

(ii) Consider, for example, the subsets $X, Y \subset \mathbb{R}^2$ where

$$X = \{(x, 0) : x \in [0, 1]\},$$
$$Y = \left\{(x, x) : x \in \left[0, \frac{1}{2}\right]\right\} \cup \left\{(x, 1 - x) : x \in \left[\frac{1}{2}, 1\right]\right\}.$$

It is obvious that X is convex, but Y is not, even though there is an obvious homotopy equivalence from X to Y.

Exercise 1.3. We know that $R(x) = e^{i\alpha}x$, and so the continuous map $F : S^1 \times \mathbb{I} \to S^1$ given by $F(x, t) = e^{i\alpha t}x$ is a homotopy $F : 1_S \simeq R$. Thus, if $g : S^1 \to S^1$ is continuous, then let θ be such that $g(1) = g(e^{i\cdot 0}) = e^{i\theta}$. Then we know that, letting R now be the rotation of $-\theta$ degrees, we must have $R \circ g \simeq 1_S \simeq g = g$ and $(R \circ g)(1) = 1$, as desired.

Exercise 1.4.

(i) Pick $(x_1, y_1), (x_2, y_2) \in X \times Y$. Then we know that, for any $t \in \mathbb{I}$, we have

$$t(x_1, y_1) + (1 - t)(x_2, y_2) = (tx_1 + (1 - t)x_2, ty_1 + (1 - t)y_2).$$

The result follows from convexity of X and Y.

(ii) If $F_X : 1_X \simeq c_X$ and $F_Y : 1_Y \simeq c_Y$, where c_X and c_Y are constant maps at c_X and c_Y, respectively, then the map

$$\begin{aligned} F : (X \times Y) \times \mathbb{I} &\to X \times Y \\ (x, y, t) &\mapsto (F_X(x, t), F_Y(y, t)) \end{aligned}$$

is clearly a homotopy from $1_{X\times Y}$ to (c_X, c_Y).

Exercise 1.5. It is clear that X is compact. After all, any open cover of X must contain some set U containing 0, and thus containing cofinitely many elements of X.

If we have a map $h : X \to Y$, then because Y is discrete, we know that $\{h^{-1}(y) : y \in Y\}$ is an open covering of X and thus by compactness admits a finite subcovering. Thus there are only finitely many elements of y in the image of h.

Now suppose that $f : X \to Y$ is a homotopy equivalence. Then there exists some $g : Y \to X$ with a homotopy $H : f \circ g \simeq 1_Y$. But $H(\{y\} \times I)$ is the continuous image of a connected map and is therefore itself connected. Because Y is discrete, this means that $H(y, 0) = H(y, 1)$ for all y. But we know that f has finite image, and Y is infinite, so there exists some y such that $y \notin \operatorname{im} f$. In particular, we have $y \neq f(g(y))$, and so $H(y, 0) = f(g(y)) \neq y = 1_Y(y)$, a contradiction. Thus X and Y are not of the same homotopy type.

Exercise 1.6. Suppose X is contractible, with $F : c \simeq 1_X$, where c is the constant map at p. Note that, for every $x \in X$, there is a path $F(x, t) : \{x\} \times \mathbb{I} \to X$ taking x to $p \in X$. In particular, this means that every x is in the same component as p, proving connectedness.

Exercise 1.7. The map $H : X \to \mathbb{I} \to X$ taking (x, t) to x and (y, t) to x if and only if $t > \frac{1}{2}$ works. Indeed, note that $H^{-1}(\{x\} \times \mathbb{I})$ is simply $\{x\} \times \mathbb{I} \cup \{y\} \times (\frac{1}{2}, 1]$, which is open in $X \times \mathbb{I}$.

Exercise 1.8.

(i) Consider the map taking the unit interval to S^1 given by $t \mapsto e^{2\pi i t}$.

(ii) If $r : Y \to X$ is a retraction, then we know from $1_Y \simeq c$ that $r \circ 1_Y \circ i \simeq r \circ c \circ i$, where i is the injection $X \hookrightarrow Y$. But the left side is simply $r \circ i = 1_X$, while the left side is a constant map, proving the result.

Exercise 1.9. We know that there exists some constant map c with $f \simeq c$. But then $g \circ f \simeq g \circ c$, and the right side is a constant map. Thus $g \circ f$ is also nullhomotopic.

Exercise 1.10. First, suppose that g is an identification. Note that $(gf)^{-1}(U)$ open in X implies that $g^{-1}(U)$ is open in Y because f is an identification. But the hypothesis on g implies that U is open in Z. Since gf is clearly a continuous surjection, the result follows.

Now, suppose that gf is an identification. It suffices to prove that $g^{-1}(U) \subseteq Y$ open implies that $U \subseteq Z$ is open. But we know by continuity of f that $f^{-1}(g^{-1}(U))$ is open, and so gf being an identification implies the result

Exercise 1.11. First, note that this is a well-defined function in the sense that $[x] = [y]$ in $X/\sim$ implies that $\overline{f}([x]) = \overline{f}([y])$.

This is evidently continuous. After all, suppose that $U \subseteq Y/\square$ is open. Then we know that

$$\overline{f}^{-1}(U) = \{[x] \in X/\sim : [f(x)] \in U\} = U'.$$

If we let $v : X \to X/\sim$ and $u : Y \to Y/\square$ be the natural maps, then we know that U' is open in $X/\sim$ because

$$v^{-1}(U') = \{x \in X : f(x) \in u^{-1}(U)\} = f^{-1}(u^{-1}(U))$$

is open.