8 CW Complexes

Hausdorff Quotient Spaces

Exercise 8.1. For any $x, y \neq 0$, let $\lambda = xy^{-1}$. Then $x = \lambda y$, so $[x] = [y]$. Hence FP^0 is just a single point.

Exercise 8.2. In each case, first use the fact that each space is a division ring to map $[x_0, x_1] \mapsto [1, x] = [1, x_0^{-1}x_1]$, then divide each term by $\sqrt{1 + ||x||^2}$ so that the result has magnitude 1. This gives the desired homeomorphisms.

Exercise 8.3. Note that $U(\mathbb{R}) = \{\pm 1\} \approx S^0$. To see the homeomorphism for $\mathbb{C}$, use the map $e^{i\theta} \mapsto \cos\theta + i\sin\theta$. Finally, to see the homeomorphism for $\mathbb{H}$, write a given quaternion as an ordered quadruple, and divide by its magnitude.

Exercise 8.4. Note that the real projective plane is just the quotient of S^2, where antipodal points are identified. This is in turn equal to the quotient of $\mathbb{R}^3$ where points on a line through the origin are identified, i.e., $\mathbb{R}P^2$.

Exercise 8.5. Use the map $[x] \mapsto [x/|x|]$ to get a homeomorphism $\mathbb{R}P^n \mapsto S^n/\!\sim$.

Exercise 8.6. Consider the map f taking $[x_1, \ldots, x_{2n+2}] \in S^{2n+1}/\!\sim$ to $[z_1, \ldots, z_{n+1}] \in \mathbb{C}P^n$, where $z_j = x_{2j-1} + ix_{2j}$ for each j. This is easily seen to be well-defined. If $x, y \in S^{2n+1}$ with $x \sim y$, then $x = \lambda y$ for some λ with $|\lambda| = 1$. If $f(x) = [z_i]$ and $f(y) = [w_i]$, then notice that $(z_i) = \lambda(w_i)$ as well, so $[z_i] = [w_i]$.

Exercise 8.7. The same argument as above holds, this time by defining $z_n = x_{4n-3} + ix_{4n-2} + jx_{4n-1} + kx_{4n}$.

Attaching Cells

Exercise 8.8. By Corollary 1.9, it suffices to show that $\alpha \amalg \beta$ is constant on the fibers of v. Thus suppose $v(s) = v(t)$. It is sufficient to suppose that $t = f(s)$ and $s \in A$, since the relation $\sim$ is generated by all $(a, f(a))$. But if $t = f(s)$, then we have

$$(\alpha \amalg \beta)(s) = \alpha(s) = \beta(f(s)) = \beta(t) = (\alpha \amalg \beta)(t).$$

This proves the result.

Exercise 8.9.

(i) It is clear that B and B^{-1} are contained in the equivalence relation generated by B. Note that D is as well due to reflexivity. Finally, K is attained by $(a, f(a))(f(a), a') = (a, a')$, where $f(a') = f(a)$. Now note that repeating this with $(a', f(a'))$ simply takes us back to $(a, f(a))$, so there are no other elements in the equivalence relation.

(ii) Simply note that

$$\begin{aligned} K &= \{(a, a') : f(a) = f(a')\} \\ &= (f \times f)^{-1}\{(x, y) \in \operatorname{im}(f \times f) : x = y\} \\ &= (f \times f)^{-1}(\Delta \cap \operatorname{im}(f \times f)). \end{aligned}$$

This is exactly what we wanted.

Exercise 8.10. It is easy to verify that the diagram commutes. Now suppose we have some Z with functions $\alpha : X \to Z$ and $\beta : Y \to Z$ so that $\beta \circ f\alpha \circ i$. We would like to find a function $\varphi : X \amalg_f Y \to Z$ making the pushout diagram commute.

We will first show that, if such a function exists, then it must be unique. Since the maps $X \to X \amalg_f Y$ and $Y \to X \amalg_f Y$ are induced by v, it follows that $\varphi \circ v|X = \alpha$ and similarly for Y and β. Hence $\varphi \circ v$ would have to be equal to $\alpha \amalg \beta$, i.e., $\varphi = (\alpha \amalg \beta) \circ v^{-1}$. Note that Exercise 8.8 applies, so this is indeed a well-defined map.

Exercise 8.11. The only case too check is if $x \in X$ and $y \in Y$. Note that there exists some $a \in A$, so consider the path from x to $a = f(a)$ to y. Hence $X \amalg_f Y$ is path-connected.

Exercise 8.12.

(i) The equation given in the hint follows from Exercise 8.9. Now, for the forwards direction, observe that $v(C)$ closed implies $v^{-1}v(C)$ closed in $X \amalg Y$. Hence its intersection with Y is closed in Y. But notice that $f(C \cap A) \cap Y = f(C \cap A)$. Furthermore, we know that $f^{-1}(f(C \cap A))$ and $f^{-1}(C \cap Y)$ are completely disjoint from Y. Thus

$$v^{-1}v(C) \cap Y = (C \cap Y) \cup f(C \cap A)$$

is closed in Y, as desired.

Going backwards, observe that $v^{-1}v(C) \cap Y$ is closed in Y, using the hypothesis and the argument above. Moreover, since f is continuous, the hypothesis implies that

$$f^{-1}((C \cap Y) \cup f(C \cap A)) = f^{-1}(C \cap Y) \cup f^{-1}(f(C \cap A))$$

is closed. Since $C \cap X$ is closed in X and $f(C \cap A) \cap X = \emptyset$, this implies that $v^{-1}v(C) \cap X$ is closed in X. Hence $v^{-1}v(C)$ is closed in $X \amalg Y$. Since v is an identification, it follows that $v(C)$ is closed in $X \amalg_f Y$.

(ii) The function is clearly bijective and continuous. To see that it is a homeomorphism, we will show that it is a closed map. Thus suppose $C \subseteq Y$ is closed. Obviously, $i(C) \subseteq X \amalg Y$ is closed and has empty intersection with X. Then to see that $v(i(C))$ is closed in $X \amalg_f Y$, simply use the previous part. In particular, observe that

$$i(C) \cap Y = i(C),$$

which is closed in Y, while

$$f(i(C) \cap A) = \emptyset,$$

which is also closed, so that their union is closed. Thus $v(i(C))$ is closed in $X \amalg_f Y$, proving that the given function is a homeomorphism.

(iii) Again, this is clearly bijective and continuous. Since $X - A$ is open in X, if U is open in $X - A$, then it is also open in X. Note that $i(U)$ is open in $X \amalg Y$. Now note that $i(U)^c$ is closed in $X \amalg Y$, and its intersection with X is U^c, which is closed in X. Moreover, we know that its intersection with Y is Y itself, while $f(i(U)^c \cap A) = f(A)$. Since $Y \cup f(A) = Y$, which is closed in Y, part (i) implies that $v(i(U)^c)$, which, by surjectivity of v, is exactly $v(i(U))^c$, is closed. Thus $v(i(U))$ is open, proving that this is an open, bijective, continuous map, thus a homeomorphism.

(iv) Note that Φ takes $A \subseteq X$ to $A \subseteq X \amalg Y$, which is then exactly equal to the attached region of $X \amalg_f Y$.

Exercise 8.13.

(i) Since f is from a compact set to a Hausdorff set, it is closed. Let C be closed. Then A being compact implies that it is closed, so $C \cap A$ is closed in X. Thus $f(C \cap A)$ is closed in Y. Since $C \cap Y$ is closed in Y, it follows from part (i) that $v(C)$ is closed in $X \amalg_f Y$.

(ii) First, suppose that $z \in \operatorname{im} \Phi | A$. Then there is some $x \in X$ with $v(i(x)) = z$, so $i(x)$ is in the fiber. We know that $\{z\}$ is closed because Y is Hausdorff, so $v^{-1}(z)$ is also closed. Since $v^{-1}(z) \subseteq A$, and closed subsets of compact sets are compact, it follows that $v^{-1}(z)$ is compact.

Otherwise, we know that we can use either the homeomorphism in Exercise 8.12(ii) or the homeomorphism $\Phi|(X - A)$ to show that $v^{-1}(z) = \{z\}$, which is indeed a nonempty compact subset of X.

Exercise 8.14. This is just invariance of boundary. Alternatively, see the proof of Lemma 8.15.

Exercise 8.15. If $n = 0$, this is obviously true. Otherwise, let $e = s - \dot{s} \approx D^{n-1} - S^{n-1}$ and let $Y = |K^{(n-1)}|$ be a closed subset of $|K|$ (since it's the finite union of (closed) simplices). Then $e \cap Y = \emptyset$ and e is an n-cell. Hence Theorem 8.7 says that we need only exhibit a relative homeomorphism $\Phi : (D^n, S^{n-1}) \to (e \cup Y, Y)$. But letting Φ be the obvious homeomorphism from D^n to s works.

Exercise 8.16. Write $Y = \{y\}$. Then define the relative homeomorphism $\Phi : (D^n, S^{n-1}) \to (e^n \cup Y, Y)$ which takes $D^n - S^{n-1}$ to e^n in the obvious way, and takes $x \in S^{n-1}$ to y. Theorrem 8.7 tells us that the attachment of D^n to Y along $f = \Phi|S^{n-1}$ is a homeomorphisms between $D^n/\partial D^n = S^n$ and $e^n \cup Y \approx e^n \cup e^0$.

Homology and Attaching Cells

Exercise 8.17. Note that $\chi(K) = 1 - 2 + 1 = 0$, so $\operatorname{rank} H_2(K) + 1 = \operatorname{rank} H_1(K)$. Furthermore, doing the same thing as with the torus in Example 8.7, we see that the projections are $f\alpha * f\alpha_1^{-1}$, which has degree 0, and $f\beta * f\beta_1$, which has degree 2. Thus, since $H_1(S^1 \vee S^1) \cong H_1(S^1) \oplus H_1(S^1)$, we can consider f_* to be the map $x \mapsto (0, 2x)$. It has trivial kernel and image isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Working through the exact sequence in Theorem 8.11 gives the result.

Exercise 8.18. There are a couple typos here: In the first part, the wedge for M is of $2h$ circles, and in the second part, we should have $\chi(M) = 2 - 2h$, not $\chi(M) = h$.

(i) Note that each $(\alpha_i, \alpha_i^{-1})$ and (β_i, β_i^{-1}) pair gives a circle. Since all the vertices are identified with each other, this gives us the desired wedge product. More formally, we can define a function Φ from a polygon P to W, and let $f = \Phi|\partial P$. Then $f\alpha_i = (f\alpha_i^{-1})^{-1}$, and similarly for β, which gives us our $2h$ circles. A similar argument can be done for M'.

(ii) Note that $H_2(S^1 \vee \cdots \vee S^1) = 0$. Thus we have the following exact sequence:

$$0 \longrightarrow H_2(M) \longrightarrow H_1(S^1) \xrightarrow{f_*} H_1(S^1 \vee \cdots \vee S^1) \xrightarrow{i_*} H_1(M) \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z} \longrightarrow 0 \ ,$$

where the last few terms are just $H_0(S^1)$, $\mathbb{Z} \oplus H_0(S^1 \vee \cdots \vee S^1)$, and $H_0(M)$, since all three spaces are path-connected. The fact that this sequence is exact implies that $\mathbb{Z}^2 \to \mathbb{Z}$ is a surjection, so the map $\mathbb{Z} \to \mathbb{Z}^2$ is an injection. Hence $H_1(M) \to \mathbb{Z}$ is the zero map. Thus i_* is surjective and has kernel (isomorphic to) $\mathbb{Z}^{2n}/H_1(M)$.

Looking at the maps from left to right now, observe that $H_2(M) \to H_1(S^1) = \mathbb{Z}$ is injective. Thus $\ker f_* = H_2(M)$, so $\operatorname{im} f_* = H_1(S^1)/H_2(M)$. But $\ker i_* = \operatorname{im} f_*$, and so it follows that

$$\chi(M) = \operatorname{rank} H_2(M) - \operatorname{rank} H_1(M) + \operatorname{rank} H_0(M) = 2 - 2h,$$

where we use the fact that $\operatorname{rank} H_0(M) = 1$.

Now notice that the same argument as in Example 8.7 implies that f_* is the zero map. Thus

$$H_2(M) = \ker f_* = H_1(S^1) = \mathbb{Z}.$$

For $H_1(M)$, since the flanking terms are torsion-free, it follows that $H_1(M)$ is also torsion-free. Since it has rank $2h$, the result follows.

(iii) The same argument as before shows that $\chi(M') = 2 - n$. This time, however, the map f_* is not the zero map. In particular, by composing with projections, we find that $f_* : H_1(S^1) \to H_1(S^1 \vee \cdots \vee S^1)$ takes $x \mapsto (2x, \dots, 2x)$, where we have identified $H_1(S^1 \vee \cdots \vee S^1)$ with $H_1(S^1) \oplus \cdots \oplus H_1(S^1)$.

In particular, we have $\ker f_* = 0$ and $\operatorname{im} f_* = (\mathbb{Z}/2\mathbb{Z})^n$. The argument before shows that $\ker f_* = H_2(M')$, and so $H_2(M') = 0$. Using the Euler characteristic (i.e., a rank argument), we can conclude that $\operatorname{rank} H_1(M') = n - 1$. (Note that, this time, the first homology group isn't torsion-free, thanks to the $\mathbb{Z}/2\mathbb{Z}$ terms.)

(iv) We first consider M. Note that it only has one vertex, say v. Thus, with chains

$$E_2 = \langle W \rangle, \quad E_1 = \langle \alpha_1 \rangle \oplus \langle \beta_1 \rangle \oplus \cdots \oplus \langle \alpha_n \rangle \oplus \langle \beta_n \rangle, \quad E_0 = \langle v \rangle,$$

we have

$$\partial W = \alpha_1 + \beta_1 - \alpha_1 - \beta_1 + \cdots = 0, \quad \partial \alpha_i = \partial \beta_i = v - v = 0, \quad \partial v = 0.$$

Hence it follows that

$$\begin{aligned} &Z_2 = \langle P \rangle, && Z_1 = \langle \alpha_1 \rangle \oplus \langle \beta_1 \rangle \oplus \dots, && Z_0 = \langle v \rangle, \\ &B_2 = 0, && B_1 = 0, && B_0 = 0. \end{aligned}$$

Thus we have

$$H_2(M) = \mathbb{Z}, \quad H_1(M) = \mathbb{Z}^{2h}, \quad H_0(M) = \mathbb{Z}.$$

Now, for M', with the natural chains, we have

$$\partial P = 2\alpha_1 + \cdots + 2\alpha_n, \quad \partial\alpha_i = 0, \quad \partial v = 0.$$

Thus we conclude that

$$\begin{aligned} &Z_2 = 0 \quad && Z_1 = \langle\alpha_1\rangle \oplus \cdots \oplus \langle\alpha_n\rangle, \quad && Z_0 = \langle v\rangle, \\ &B_2 = 0, \quad && B_1 = \langle 2(\alpha_1 + \cdots + \alpha_n)\rangle, \quad && B_0 = 0. \end{aligned}$$

This gives us that

$$H_2(M') = 0, \quad H_1(M') = \mathbb{Z}^n/2\mathbb{Z} = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^{n-1}, \quad H_0(M') = \mathbb{Z},$$

which coincides with the previous parts.

CW Complexes

Exercise 8.19. Note that $U \subseteq X$ is open if and only if $U^c \subseteq X$ is closed, which is in turn the case if and only if $U^c \cap A_j$ is closed in A_j for all $j \in J$. But $U \cap A_j = A_j - U^c \cap A_j$, so this last conditiono is true if and only if $U \cap A_j$ is open in A_j for $j \in J$.

Exercise 8.20. It is obvious that $\{Y \cap A_j\}$ fits the conditions (i)–(iii). Now note that if $F \subseteq Y$ is closed in the subspace topology, then $F = Y \cap F'$ for some closed $F' \subseteq X$. Hence

$$F \cap (Y \cap A_j) = Y \cap F' \cap Y \cap F_j = (Y \cap A_j) \cap F'.$$

Of course, this is closed in $Y \cap A_j$, so F must be closed in the weak topology.

Now suppose F is closed in the weak topology. Then $F \cap (Y \cap A_j)$ is closed in all $Y \cap A_j$. Since Y is closed, we know that $F \cap Y \cap A_j$ must also be closed in A_j. This is true for all j, so $F \cap Y$ is closed in the weak topology on X, i.e., as a subset of X. Thus $F = F \cap Y$ is closed as a subspace of Y.

Exercise 8.21. By Lemma 8.20, we know that A closed in X implies that $A \cap X'$ is closed in X' for all finite subcomplexes X'. Theorem 8.19 says that this implies that, for all compact K of X, we must have $A \cap K$ closed in K. Thus, by definition, it follows that A is closed in the weak topology generated by compact subsets.

Now suppose $A \cap K$ is closed in K for all compact K. Since finite subcomplexes are compact, it follows that $A \cap X'$ is closed in X' for all such X'. Hence A is closed in X.

Exercise 8.22. To see that $X^{(0)}$ is discrete, simply let $A \subseteq X^{(0)}$. Then $A \cap \bar{e}$ is the finite union of 0-cells, and hence is closed.

To see that the 0-skeleton is closed, note that $X^{(0)} \cap \bar{e}$ is a finite union of 0-cells, and thus is closed in $\bar{e}$. This is true for all e, so $X^{(0)}$ is closed.

Exercise 8.23. If A is closed, then obviously $A \cap X^{(n)}$ is closed in $X^{(n)}$. Now if $A \cap X^{(n)}$ is closed in $X^{(n)}$ for each n, pick X' to be any finite complex. Let n be the highest dimension of any cell in X'. Then $A \cap X' = (A \cap X^{(n)}) \cap X'$ must be closed in X'. Lemma 8.20 implies the result.

Finally, the corresponding statement for open sets follows from Exercise 8.19.

Exercise 8.24. This is simply because each n-cell is just $D^n - S^{n-1}$; the attachment is given by $\Phi_e(S^{n-1})$ according to Exercise 8.12(iv).

Exercise 8.25. This is visually clear. Alternatively, with α and β as the edges, and v as the vertex, we can notice that there is a relative homeomorphism

$$\Phi : (D^2, S^1) \to (T, \alpha \cup \beta \cup \{v\})$$

since $D^2 \cong \mathbb{I} \times \mathbb{I}$. Since α and β are 1-cells, and v is a 0-cell, it follows that this map gives T as the union of two 1-cells, one 0-cell, and one 2-cell (namely $\operatorname{im} \Phi|(D^2 - S^1)$).

The same argument can be done for the Klein bottle.

Exercise 8.26.

(i) They both violate closure finiteness since the closure of the base point intersects infinitely many cells.

(ii) The set of all $\{1/n\}$ is closed in the weak topology, but not as a subspace.

Exercise 8.27. The same proof as Theorem 7.1 holds, but with D^n in place of Δ^n.

Exercise 8.28. First, observe that the 1-skeleton is always nonempty (as long as X is nonempty). In particular, suppose e is an n-cell in X, where n is the smallest dimension of a cell in e. Then the relative homeomorphism $(D^n, S^{n-1}) \to (e \cup X^{(n-1)}, X^{(n-1)})$ implies that there is a map between S^{n-1} and $X^{(n-1)} = \emptyset$, which is impossible. Thus there must be some 0-cell, and so the 1-skeleton is nonempty.

In fact, there must be some part of the 1-skeleton in each path component. Thus if X is disconnected, then its 1-skeleton must be as well.

Now suppose that the 1-skeleton is disconnected. We can easily show that $X^{(n)}$ disconnected implies that $X^{(n+1)}$ is disconnected. Since X is the union of all its skeletons, and since $X^{(n)} \subseteq X^{(n+1)}$, it follows that X being connected would have to imply that there is some n with $X^{(n)}$ connected. Since $X^{(0)}$ is discrete, hence disconnected (unless it has one element only, in which case the 1-skeleton would be connected), it follows that $n \geq 1$, and so this provides the desired contradiction.

Exercise 8.29. The forward direction is obvious. Now suppose that $f\Phi_e$ is continuous for all e. Let $K \subseteq Y$ be closed and let e be a k-cell. We want to show that

$$f^{-1} \cap \Phi_e(D^k)$$

is closed in $\bar{e} = \Phi_e(D^k)$. But Φ_e is a relative homeomorphism and is, in particular, a closed map on $D^k - S^{k-1}$. Now, because

$$\Phi_e^{-1}(f^{-1}(K) \cap \Phi_e(D^k)) = (f\Phi_e)^{-1}(K) \cap D^k$$

is closed in D^k, we're done.

Exercise 8.30.

(i) Consider attaching the (closed) top half of the circle to the topologist's sine curve (which maps 0 to 0 and x to $\sin(1/x)$ for $x \in (0, 2\pi]$). Then attach the (closed) bottom half of the circle to the same curve, but running backwards. Obviously this is a CW complex. But it is connected and not path-connected, violating **??**. Hence this is not a polyhedron.

(ii) If $n = 0$, this is obvious. Suppose this is true for n. Say we attach k total $(n+1)$-cells. (Note that this kind of inductive creation of CW complexes is made possible byy Theorem 8.24.) Note that an $(n+1)$-cell is homeomorphic to an open $(n+1)$-simplex. Furthermore, the attachment map can be approximated by a simplicial map. Since simplicial approximations are homotopic to the original maps, the result follows.

Exercise 8.31. Here we can use the same cells and attaching maps, only with the basepoints all identified. For any cell not equal to the basepoint, its closure is contained in whichever X_λ the cell was originally in, and thus intersects only finitely many cells. The closure of the basepoint is itself, and only intersects itself. This proves closure finiteness.

To see that this has the weak topology, simply note that if A is closed, then $A \cap X_\lambda$ is closed, where we identify X_λ with its natural image in $\bigvee X_\lambda$. Thus, since the closure of any cell is contained within X_λ for some λ, it follows that $A \cap \bar{e} = A \cap X_\lambda \cap \bar{e}$ is closed in $\bar{e}$ for every e.

Exercise 8.32. The first and second conditions of a CW complex are clearly satisfied since D^{i+j} is homeomorphic to $[0,1]^{i+j}$.

To see the third condition holds, use the equation in the hint. Notice that all four expressions on the right side intersect finitely many cells in X or X'. In particular, it follows that $(\bar{e} - e) \times \bar{e}'$ intersects finitely many cells of E'', and similarly for the other term. Thus $\overline{e \times e'}$ intersects $e \times e'$, plus these finitely many other cells. This proves closure finiteness.

Finally, for the fourth condition, note that the weak topology is just the product topology when working with finitely many factors.

Exercise 8.33. With notation as suggested in the hint, suppose the intersection of A and every cell in E'' is closed in the cell. Now observe that

$$\overline{e \times a^0} = \bar{e} \times a^0,$$

and similarly for b^0. Moreover, we know that

$$\overline{e \times c^1} = \left[(\bar{e} - e) \times \mathbb{I}\right] \cup \left(\bar{e} \times (a^0 \cup b^0)\right) \cup (e \times c^1).$$

But now observe that $\bar{e} = e \cup (\bar{e} - e)$, so that the middle term can be rewritten as

$$\bar{e} \times (a^0 \cup b^0) = \left(e \times (a^0 \cup b^0)\right) \cup \left((\bar{e} - e) \times (a^0 \cup b^0)\right).$$

Since $a^0 \cup b^0 \cup c^1 = \mathbb{I}$, it then follows that

$$\overline{e \times c^1} = \left[(\bar{e} - e) \times \mathbb{I}\right] \cup (e \times \mathbb{I}) \cup \left((\bar{e} - e) \times (a^0 \cup b^0)\right) = \bar{e} \times \mathbb{I}.$$

Now let π_X and $\pi_{\mathbb{I}}$ be the projections to X and $\mathbb{I}$, respectively. We know that $\pi_X(A) \cap \bar{e}$ is closed in each e, since $\pi_X(A) \cap \bar{e}$ is closed in each $\bar{e}$, and similarly for $\pi_{\mathbb{I}}(A) \cap \bar{a^0}$ and $\pi_{\mathbb{I}}(A) \cap \bar{b^0}$. Moreover, since $\pi_{\mathbb{I}}(A) \cap \bar{c^1} = \pi_{\mathbb{I}}(A) \cap \mathbb{I}$, and since $A \cap (\bar{e} \times \mathbb{I})$ is closed in $\bar{e} \times \mathbb{I}$, it follows that the intersection $\pi_{\mathbb{I}}(A) \cap \bar{c^1}$ is also closed in $\mathbb{I}$.

Thus $\pi_X(A)$ and $\pi_{\mathbb{I}}(A)$ are closed, so A is closed, as desired.

Exercise 8.34. Let $i : Z \to Y$ and $j : Y \to X$ be the injections. We can now easily check the criteria for a strong deformation retraction. In particular, note that

$$r_1 r_2 j i = r_1 1_Y i = r_1 i = 1_Z,$$

while

$$j i r_1 r_2 = j(i r_1) r_2 \simeq j_r 2 \operatorname{rel} Z \simeq 1_X \operatorname{rel} Z,$$

since $Y \subseteq Z$.

Cellular Homology

Exercise 8.35. ☹

Exercise 8.36.

(i) If X is compact, then it is finite. Thus $W_k(X, Y) = H_k(X_Y^k, X_Y^{k-1})$ is free abelian of rank equal to the number of k-cells in $E - E'$. Say this rank is r_k. Then we know that

$$H_k(X, Y) \cong H_k(W_*(X, Y)) = \ker d_k / \operatorname{im} d_{k+1},$$

but both $\ker d_k$ and $\operatorname{im} d_{k+1}$ are subsets of $\mathbb{Z}^{r_k}$. Thus $H_k(X, Y)$ is finitely generated.

(ii) This is the same proof, since r_k is at most the number of cells of dimension k.

Exercise 8.37. Using the cellular decomposition for $\mathbb{R}P^\infty = \bigcup \mathbb{R}P^n$, we find that

$$W_k(\mathbb{R}P^\infty) = H_k(e^0 \cup \cdots \cup e^k, e^0 \cup \cdots \cup e^{k-1}),$$

which is obviously free abelian of rank 1. It follows that the we get a chain $\cdots \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to \ldots$, so the kernels and images of each map must be 0 or $\mathbb{Z}$. Hence $H_k(\mathbb{R}P^\infty) = H_k(W_*(\mathbb{R}P^\infty))$ is either 0 or $\mathbb{Z}$.

Exercise 8.38. Let b_λ be the basepoint of X_λ. Then we know that $\bigvee X_\lambda = \coprod X_\lambda / \{b_\lambda\}$. Let v be the natural map. Then Theorem 8.41 implies that v_* induces an isomorphism from

$$H_k\left(\coprod X_\lambda, \{b_\lambda\}\right) \to \tilde{H}_k\left(\bigvee X_\lambda\right).$$

Of course, Theorems 5.13 and 5.17 also imply that the left side is equal to

$$\sum_\lambda \tilde{H}_k(X_\lambda),$$

which implies the result.

Exercise 8.39. To do this, we simply compute d_2, d_1, d_0. In particular, since $W_2(T)$ is generated by e^2, we know that $d_2 = \partial e^2 = 0$. Similarly, we find that $d_1 = d_0 = 0$. This gives the result.

Exercise 8.40. Use **??**. In particular, this implies that

$$\chi(S^m \times S^n) = \begin{cases} 0 & \text{if } m \text{ or } n \text{ odd} \\ 4 \text{otherwise} & \end{cases}.$$

Exercise 8.41. Use the cellular decomposition of $\mathbb{C}P^n$. In particular, we know that $\mathbb{C}P^n = e^0 \cup \cdots \cup e^{2n}$, and so the only nonzero α_i are for even i. Thus

$$\chi(\mathbb{C}P^n) = \sum (-1)^i \alpha_i = 1 + 1 + \cdots + 1 = n + 1.$$

The same argument holds for $\mathbb{H}P^n$.

Exercise 8.42. This is again obvious:

$$\chi(\mathbb{R}P^n) = 1 - 1 + 1 - 1 + \ldots$$

is equal to 0 if n is odd and 2 if n is even. This is exactly $\frac{1}{2}(1 + (-1)^n)$.

Exercise 8.43. This is simply the principle of inclusion-exclusion.

Exercise 8.44. It is sufficient to show that $\sim$ is closed. Notice that

$$\{(z_0, z_1, z_2, z_3) : h^m(z_0, z_1) = (z_2, z_3)\} = \bigcup_{m=1}^{p} \{(z_0, z_1, z_2, z_3) : h^m(z_0, z_1) = (z_2, z_3)\} = \bigcup_{m=1}^{p} S_m.$$

because $h^p = h$. Thus it suffices to check that each S_m is closed. Suppose that $(z_0, z_1, z_2, z_3) \notin S_m$. Say that $z_2 \neq \zeta^m z_0$; note that a similar argument can be given if $z_3 \neq \zeta^{mq} z_1$. Then there is an open neighborhood with coordinates (x_0, x_1, x_2, x_3) on which $\zeta^m(x_0) \neq x_3$ since $\zeta^m x - y$ is continuous. Thus $(S_m)^c$ is open, which proves that S_m is closed, as desired.

Exercise 8.45.

(i) Note that $\zeta = 1$, so h is just the identity. Thus $S^3/{\sim} = S^3$

(ii) Now we have $\zeta = -1$, so h maps antipodal points to each other. Thus $S^3/{\sim} = \mathbb{R}P^3$.

(iii) In this case, we know that $\zeta^q = \zeta^{q'}$, so $h_q = h_{q'}$. Thus $L(p, q) = L(p, q')$.

Exercise 8.46.

(i) Since this is a finite decomposition, we only need to verify the first two conditions for a CW complex. The first is clear by definition. For the second condition, the maps are obvious for e_r^0 and e_r^1. For e_r^2, we use the fact that $z_1 = z_1(z_0)$ is determined by z_0. Thus the map

$$z_0 \mapsto (z_0, z_1(z_0))$$

works. Finally, for e_r^3, take (z_0, θ) and map θ linearly onto $(2\pi r/p, 2\pi(r+1)/p)$.

(ii) It is easy to check that $e_r^i \sim e_{r'}^i$ for each i.

Exercise 8.47.

(i) This is the cellular boundary formula, or just a generalization of the argument for Lemma 8.46.

(ii) For $D(\gamma_1)$, simply notice that

$$D(\gamma_1) = v_\# d_1 v_\#^{-1}(\gamma_1) = v_\# d_1 e_r^1 = v_\#(e_r^0 - e_{r+1}^0) = 0.$$

A similar argument holds for the other differentiations.

(iii) This is obvious from the chain complex:

$$W_4 = 0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0 = W_{-1}$$