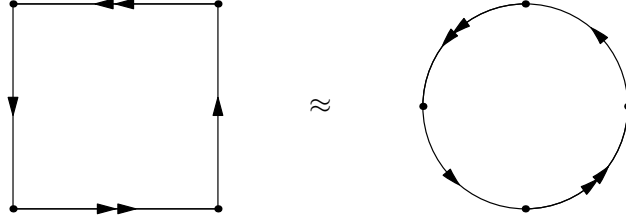


## 7 Simplicial Complexes

### Definitions

**Exercise 7.1.**



**Exercise 7.2.** Consider some (nondegenerate) triangle with vertices  $P, x_0, y_0$  in  $\mathbb{R}^2$ . Then define  $x_i$  to be the midpoint of  $P$  and  $x_{i-1}$ , and similarly define  $y_i$ . Then the union  $X$  of the triangle with all the line segments  $x_i y_i$  is compact and connected.

We claim that it is not a polyhedron. Otherwise, there exists some simplicial complex  $K$  admitting a homeomorphism  $h : |K| \rightarrow X$ . But observe that  $K$  must have an infinite vertex set.

To see this, for each  $i$ , define  $s_i$  to be

$$s_i = \bigcap_{h^{-1}(x_i) \in s} s,$$

where  $s$  ranges over all simplices of  $K$ . Note that this intersection is over a nonempty set because  $\bigcup s = |K|$ , so there must exist some  $s$  containing  $h^{-1}(x_i)$ . Moreover, there are only finitely many simplices, so the intersection exists. Condition (ii) implies that  $s_i$  is a common face of  $s$ , and thus is a simplex. It must be 0-dimensional since the segment  $Px_i$ ,  $x_i y_i$ , and  $x_0 x_i$  cannot all be part of the same 1-simplex. In other words,  $x_i$  must be a common face of two 1-simplices, and so it must be a point.

Hence there are infinitely many vertices of  $K$ , a contradiction.

**Exercise 7.3.** Note that the upper right and lower right triangles are the same.

**Exercise 7.4.**

- (i) The forwards direction is just the definition of the subspace topology. To see the backwards direction, suppose  $F \cap s$  is closed in  $s$  for every  $s \in K$ . Each  $s$  is closed in  $|K|$ , so  $F \cap s$  is closed in  $|K|$ . Since there are finitely many  $s$  and  $\bigcup s = |K|$ , it follows that we can take the union of all  $F \cap s$ . In particular, we have

$$F = \bigcup_{s \in K} (F \cap s)$$

is the finite union of closed sets, hence is itself closed in  $|K|$ .

- (ii) This is obviously true if  $K$  has dimension 0.

If  $K$  (and hence  $s$ ) has dimension  $> 1$ , then consider the complement of  $s^\circ$ :

$$(s^\circ)^c = (|K| - s) \cup \dot{s}.$$

Then notice that

$$[(|K| - s) \cup \dot{s}] \cap s = \dot{s},$$

which is closed in  $s$ . Suppose  $t \in K$  is not equal to  $s$ . Then consider

$$A_t = [(|K| - s) \cup \dot{s}] \cap t.$$

If  $s \cap t = \emptyset$ , then  $A_t = \emptyset$  is closed in  $t$ . Otherwise, we know that  $s \cap t$  is a face of  $t$ . Since  $s$  is of highest dimension, we know that either  $s = t$ , which we already took care of above, or  $s \cap t$  is part of  $\dot{s}$ , in which case we know that

$$\dot{s} \cap t = s \cap t, \quad (|K| - s) \cap t = t - s \cap t.$$

Hence  $A_t = t$ , which is still closed in  $t$ .

The previous part proves the result.

**Exercise 7.5.** We begin by showing  $s^\circ \cap t^\circ = \emptyset$  when  $s \neq t$ . Note that

$$s^\circ \cap t^\circ = (s - \dot{s}) \cap (t - \dot{t}) = s \cap t - \dot{s} \cap t - s \cap \dot{t}.$$

But  $s \cap t$  is a face of both  $s$  and  $t$ . It can't be equal to both  $s$  and  $t$  since  $s \neq t$ . Thus  $s \cap t$  is a *proper* face of at least one of  $s$  and  $t$ , say  $s$ . This means that  $s \cap t$  is part of  $\dot{s}$ , and thus is in  $\dot{s} \cap t$ . This proves disjointness.

To see that  $\bigcup s^\circ = |K|$ , simply do this in the case of  $K$  as a simplex, and take unions. (To do this when  $K$  is a single simplex, use induction.)

**Exercise 7.6.** The backwards direction is obvious by the definition of  $\text{st}$ . For the forwards direction, suppose

$$x \in \text{st}(p) = \bigcup_{p \in \text{Vert}(t)} t^\circ.$$

Then we know that  $x \in t^\circ$  for some  $t$  having  $p$  as a vertex. Uniqueness implies that  $s = t$ , so  $p \in \text{Vert}(s)$ .

**Exercise 7.7.**

- (i) Obviously the union is  $|K|$  because every  $s \in K$  has at least one vertex, hence is contained in at least one star. To see that  $\text{st}(p) \subseteq |K|$  is open, notice that

$$(\text{st}(p))^c = \bigcup_{p \notin \text{Vert}} s^\circ.$$

Intersect this with  $t \in K$ . If  $p \notin \text{Vert}(t)$ , then this intersection is equal to  $t$  since no simplex of  $\dot{t}$  can have  $p$  as a vertex. If  $p \in \text{Vert}(t)$ , then write  $t = [p, p_1, \dots, p_k]$ . The intersection can be seen to simply be  $\{p_1, \dots, p_k\}$ , which is obviously closed. Thus Exercise 7.4 implies the result.

- (ii) If  $x \in \text{st}(p)$ , then  $x \in s^\circ$  for some  $s$  with  $p \in \text{Vert}(s)$ . Since  $x, p \in s$  and  $s$  is convex, it follows that the line segment is also contained in  $\text{st}(p)$ .

**Exercise 7.8.** The forwards direction is because  $[p_0, \dots, p_n]$  is in the intersection. The backwards direction is because there must exist some simplex  $[p_0, \dots, p_n, q_0, \dots, q_m] \in K$ . Since any face of a simplex in  $K$  is also in  $K$ , it follows that  $[p_0, \dots, p_n]$  is a simplex in  $K$ .

## Simplicial Approximation

**Exercise 7.9.** In the forwards direction, suppose  $\varphi$  is a simplicial map. If  $\bigcap \text{st}(p_i) \neq \emptyset$ , then there exists a simplex in  $K$  with vertices  $[p_i]$ . The definition implies that there must exist a simplex with vertices  $[\varphi(p_i)]$ , proving this direction. The backwards direction follows directly from Exercise 7.8.

**Exercise 7.10.** Suppose  $\varphi$  is a simplicial approximation to  $f$ , and suppose  $x \in |K|$  with  $f(x) \in s^\circ$ . Write  $x \in t^\circ$  for  $t \in K$ , and write  $t = [p_1, \dots, p_n]$ . Then we know that  $x \in \text{st}(p_i)$  implies that  $f(x) \in \text{st}(\varphi(p_i))$ , so that  $s^\circ \subseteq \text{st}(\varphi(p_i))$ . Thus  $s$  has  $\varphi(p_i)$  as a vertex for each  $i = 1, \dots, n$ .

Hence  $|\varphi|(x)$ , which is determined by  $\varphi(p_i)$ , is in  $s$  by affineness.

Now suppose that  $f(x) \in s^\circ$  implies  $|\varphi|(x) \in s$ . Let  $p$  be some vertex of  $K$  so that  $x \in \text{st}(p)$ . Then  $f(x) \in s^\circ$ , so  $|\varphi|(x) \in s$ . Hence  $\varphi(p)$  is a vertex of  $s$  by affineness and the definition of  $|\varphi|$ , from which it follows that

$$f(x) \in s^\circ \subseteq \text{st}(\varphi(p)).$$

We can take the union over all  $x \in \text{st}(p)$ :

$$\bigcup_{x \in \text{st}(p)} f(x) \subseteq \text{st}(\varphi(p)).$$

Of course, this left side is exactly  $f(\text{st}(p))$ , and so we're done.

**Exercise 7.11.** Suppose  $\varphi : K \rightarrow L$  is a simplicial approximation. Consider the obvious homotopy:

$$H(t, x) = (1 - t)|\varphi|(x) + tf(x).$$

We can do this because  $|\varphi|(x)$  and  $f(x)$  are, by Exercise 7.10, in the same simplex.

**Exercise 7.12.**

- (i) This is true because it's true for simplices.  
(ii) Order the vertices of  $K$ , and define  $\varphi(b^s)$  to be the smallest vertex of  $s$  under this order. We claim that this gives a simplicial approximation to the identity. Consider a vertex  $b^s$  of  $\text{Sd}(K)$ . Then we know that

$$f(\text{st}(b^s)) = s^\circ \subseteq \text{st}(\varphi(b^s))$$

by the definition of  $\varphi(b^s)$ , where  $f$  is the identity.

- (iii) There exists a homeomorphism  $g : |L| \rightarrow X$ . If  $g(v)$ , then we are done. Otherwise, we know that  $x \in g(s^\circ)$  for some unique  $s \in L$ . Consider the subdivision  $K$  of  $L$  obtained by drawing lines from  $s$  to every vertex of  $s$ . This gives a function  $h : |K| \rightarrow X$  which is equal to  $g$ , and thus is a homeomorphism, as desired.

**Exercise 7.13.** Suppose that  $\sum \lambda_i b^{s_i} = 0$ . Since  $s_0 < \dots < s_q$ , we know that there exists some vertex  $p_q$  which only appears in  $b^{s_q}$ , so  $\lambda_q = 0$ . But then there is a vertex  $p_{q-1}$  which only appears in  $b^{s_{q-1}}$ , so  $\lambda_{q-1} = 0$ , and so on. Thus  $\lambda_i = 0$  for all  $i$ , proving independence.

**Exercise 7.14.** Every point of  $\text{Sd } K$  is contained in a unique open simplex of  $K$ , so it follows that an open simplex of  $\text{Sd } K$  can be contained in at most one open simplex of  $K$ . To see that there is at least one such simplex, note that  $[b^{s_0}, \dots, b^{s_q}]^\circ$  is contained in  $s_q^\circ$ .

**Exercise 7.15.** This follows from the triangle inequality:

$$|x - y| \leq |x - p| + |p - y| \leq 2\mu,$$

because  $x$  and  $p$  are in one simplex, and  $y$  and  $p$  are in another.

**Exercise 7.16.** Write  $s = [b^{s_0}, \dots, b^{s_q}]$ , where  $s_0 < \dots < s_q$ . Then  $\text{diam } s = \sup \|b^{s_i} - b^{s_j}\|$ . If  $i < j$ , then we know that

$$\|b^{s_i} - b^{s_j}\| \leq \frac{n_j}{n_j + 1} \text{diam } s_j,$$

where  $n_j = \dim s_j$ . But  $\text{diam } s_j \leq \text{mesh } K$  since  $s_j \in K$ , and  $\frac{n_j}{n_j + 1} \leq \frac{n}{n + 1}$ , since  $n_j \leq n$ . Hence it follows that

$$\text{diam } s \leq \frac{n}{n + 1} \text{mesh } K,$$

and so  $\text{mesh } \text{Sd } K \leq (n/n + 1) \text{mesh } K$ . Induction implies the general result.

**Exercise 7.17.** If  $s \in K^{(q)}$ , then  $s = [p_0, \dots, p_r]$  for some  $r \leq q$ . Thus  $\varphi(s) = [\varphi(p_0), \dots, \varphi(p_r)] \in L^{(q)}$ , as desired.

**Exercise 7.18.** Let  $b$  be the barycenter of the  $(n + 1)$ -simplex, and consider

$$f(x) = \frac{x - b}{\|x - b\|} + b.$$

This is the desired homeomorphism.

**Abstract Simplicial Complexes**

No exercises!

## Simplicial Homology

**Exercise 7.19.** In general, there are  $\binom{n+2}{q+1}$  total  $q$ -simplices in an  $(n+1)$ -simplex. Since  $S^n$  is the  $n$ -skeleton of such a simplex, it follows that we must simply evaluate

$$\chi(S^n) = \sum_{q=0}^n \binom{n+2}{q+1} (-1)^q = \sum_{q=0}^{n+2} \binom{n+2}{q} (-1)^{q+1} + \binom{n+2}{0} + \binom{n+2}{n+2} = 2$$

when  $q$  is even. When  $q$  is odd, the last term is negative, and we find that  $\chi(S^n) = 0$ .

**Exercise 7.20.** Here, we have  $\alpha_2 = 18$ ,  $\alpha_1 = 27$ , and  $\alpha_0 = 9$ . Thus  $\chi(T) = 18 - 27 + 9 = 0$ .

**Exercise 7.21.** Note that  $i$  is obviously an injection. Moreover, since the element  $\sum b \in B_1 m_b b + \sum_{c \in B_2} m_c c \in F(b)$  is equal to

$$\sum b \in B_1 m_b b + \sum_{c \in B_2} m_c c \in F(b) = p \left( \sum m_b b, \sum -m_c c \right),$$

we see that  $p$  is surjective. Finally, note that

$$\begin{aligned} \ker p &= \left\{ \left( \sum m_b b, \sum m_c c \right) : \sum m_b b = \sum m_c c \right\} \\ &= \{(x, x) : x \in F(B_1) \cap F(B_2)\} \\ &= \{(x, x) : x \in F(B_1 \cap B_2)\} = \operatorname{im} i, \end{aligned}$$

which completes the proof of exactness.

## Comparison with Singular Homology

**Exercise 7.22.** For  $q \geq 1$ , the complexes are the same. If  $q = 0$ , we use the same argument as in Theorem 5.17, in particular, by restricting our attention to the ending:

$$0 \longrightarrow \ker \tilde{\partial}_0 \hookrightarrow C_0(K) \xrightarrow{\tilde{\partial}_0} C_{-1}(K) \longrightarrow 0.$$

**Exercise 7.23.** This is simply because  $\ker \tilde{\partial}_{-1} = C_{-1}(K)$ .

**Exercise 7.24.**

- (i) We can simply use the straight line homotopy between  $\varphi(p)$  and  $\psi(p)$  for all vertices  $p$  of  $K$ ; the rest of the point follow by affineness. The reason this works is simply because  $\varphi(p)$  and  $\psi(p)$  belong to the same simplex, which is convex.
- (ii) Since  $|\varphi| \simeq |\psi|$ , we know that  $|\varphi|_* = |\psi|_*$ , which in turn implies that  $\varphi_* = \psi_*$  by Theorem 7.22.

**Exercise 7.25.** Let  $L$  be a line segment, along with its endpoints and its midpoints. Thus it is composed of two 1-simplices, and three 0-simplices. Then let  $\varphi_1$  map a 1-simplex to the left side of  $L$ , and  $\varphi_2$  map it to the right side of  $L$ . Finally, if  $\varphi_3$  maps the 1-simplex to the midpoint, it follows that  $\varphi_1 \sim \varphi_3 \sim \varphi_2$ , but obviously  $\varphi_1 \not\sim \varphi_2$ .

**Exercise 7.26.**

- (i) This is clear by mapping the base points together, and mapping a given equivalence class to the corresponding equivalence class. For example, we have some point  $x \in X$ , then the homeomorphism would take  $[[x]] \in (X \vee Y) \vee Z$  to  $[x] \in X \vee (Y \vee Z)$ . Similarly, it would take  $[[y]] \mapsto [[y]]$  and  $[z] \mapsto [[z]]$ .
- (ii) For  $i = 1, 2$ , there exists a simplicial complex  $L_i$  and a homeomorphism  $h_i : |L_i| \rightarrow K_i$ . Fix some vertex  $x_i \in \operatorname{Vert}(L_i)$ . Then let  $L = L_1 \vee L_2$ . Then, identifying each  $L_i$  with the natural corresponding set in  $L$ , we can apply Theorem 7.17 to find the exact sequence

$$\dots \rightarrow H_n(L_1 \cap L_2) \rightarrow H_n(L_1) \oplus H_n(L_2) \rightarrow H_n(L) \rightarrow H_{n-1}(L_1 \cap L_2) \rightarrow \dots$$

Of course, we have  $L_1 \cap L_2$  is a singleton, so the homology groups are 0. Thus, if  $n \geq 2$ , then we know that  $H_n(L_1 \cap L_2) = H_{n-1}(L_1 \cap L_2) = 0$ , and so  $H_n(L) \cong H_n(L_1) \oplus H_n(L_2)$ , as desired. Otherwise, we can simply use the tail:

$$\dots \rightarrow H_1(L) \rightarrow H_0(L_1 \cap L_2) \rightarrow H_0(L_1) \oplus H_0(L_2) \rightarrow H_0(L) \rightarrow 0.$$

If  $L_i$  has  $c_i$  components, then notice that  $L$  has  $c_1 + c_2 - 1$  components. Since the map  $H_0(L_1) \oplus H_0(L_2) \rightarrow H_0(L)$  is surjective, it follows that its kernel is  $\mathbb{Z}$  (or, more accurately, a free abelian group of rank 1). Hence the image of  $H_0(L_1 \cap L_2) \rightarrow H_0(L_1) \oplus H_0(L_2)$  is  $\mathbb{Z}$ . The fact that  $H_0(L_1 \cap L_2) = \mathbb{Z}$  implies that this map is an isomorphism, thus with empty kernel. Finally, we conclude that the image of  $H_1(L) \rightarrow H_0(L_1 \cap L_2)$  is trivial, and so we again have the exact sequence

$$0H_1(L_1) \oplus H_1(L_2) \rightarrow H_1(L) \rightarrow 0.$$

The result follows.

- (iii) Use Corollary 7.19. In particular, let  $K_q$  consist of all proper faces of an oriented  $(q+1)$ -simplex. Then the corollary implies that  $H_q(K_q) = \tilde{H}_q(K_q) = \mathbb{Z}$  and  $H_r(K_q) = 0$  for any  $r \neq q$ . (Note that reduced homology matches the regular homology since  $q \geq 1$ .) Thus the previous part shows that the space

$$\bigvee_{q=1}^n \bigvee_{i=1}^{m_q} K_q,$$

where the wedge occurs at some identified vertex, satisfies the desired properties.

**Exercise 7.27.**

- (i) This follows directly from the five lemma and Theorem 7.22, namely by looking at the following commutative diagram with exact rows:

$$\begin{array}{ccccccccccc} \dots & \rightarrow & H_n(L) & \rightarrow & H_n(K) & \rightarrow & H_n(K, L) & \rightarrow & H_{n-1}(L) & \rightarrow & H_{n-1}(K) & \rightarrow \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \dots & \rightarrow & H_n(|L|) & \rightarrow & H_n(|K|) & \rightarrow & H_n(|K|, |L|) & \rightarrow & H_{n-1}(|L|) & \rightarrow & H_{n-1}(|K|) & \rightarrow \dots \end{array}$$

- (ii) This follows from the previous part, Corollary 7.17, and Theorem 7.22.

**Exercise 7.28.** We can simply use the straight line homotopy to  $p$ . Exercise 7.7 implies that this is well-defined.

**Exercise 7.29.** In particular, we must show that

$$\left( \bigcap L_{\alpha_i} \right) \cap \left( \bigcap L_{\beta_i} \right) \neq \emptyset.$$

But notice that  $\sigma_0 < \sigma_1 < \dots < \sigma_q$  implies that  $\sigma_0 \in L_{\beta_i}$  for each  $\beta_i$ . We also know that  $\sigma_0 \in L_{\alpha_0} \cap \dots \cap L_{\alpha_q}$ , and so it follows that  $\sigma_0$  is in the displayed intersection above. Hence  $g$  and  $f$  are contiguous.

**Exercise 7.30.** We have the following exact sequence:

$$H_q(M \cap L_1) \rightarrow H_q(M) \oplus H_q(L_1) \rightarrow H_q(M \cup L_1) \rightarrow H_{q-1}(M \cap L_1).$$

The conditions imply that  $H_q(M) \oplus H_q(L_1) = H_q(M)$  and the two outermost terms are both trivial. Thus  $H_q(M) \cong H_q(M \cup L_1)$ .

Now consider the following exact sequence:

$$H_q((M \cup L_1) \cap L_2) \rightarrow H_q(M \cup L_1) \oplus H_q(L_2) \rightarrow H_q(M \cup L_1 \cup L_2) \rightarrow H_{q-1}((M \cup L_1) \cap L_2).$$

But notice that

$$(M \cup L_1) \cap L_2 = (M \cap L_2) \cup (L_1 \cap L_2) = M \cap L_2$$

since  $L_1 \cap L_2 \subseteq M$ . Hence the flanking terms of the exact sequence displayed above are again 0. Since  $L_2$  is acyclic, it follows that  $H_q(M \cup L_1) \cong H_q(M \cup L_1 \cup L_2)$ . Repeating this proves the result.

## Calculations

**Exercise 7.31.** Consider the Klein bottle, as in Figure 1. Let  $P$  be the entire square. Then we can define

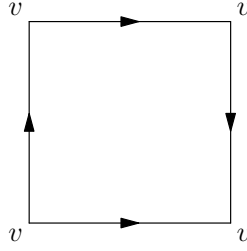


Figure 1: The Klein bottle

the adequate subcomplex with chains

$$E_2 = \langle P \rangle, \quad E_1 = \langle a \rangle \oplus \langle b \rangle, \quad E_0 = \langle v \rangle.$$

We have

$$\begin{aligned} \partial P &= a + b + a - b \\ \partial a &= \partial b = 0 \\ \partial v &= 0. \end{aligned}$$

Hence it follows that we have

$$\begin{aligned} Z_2 &= 0, & Z_1 &= \langle a \rangle \oplus \langle b \rangle, & Z_0 &= \langle v \rangle, \\ B_2 &= 0, & B_1 &= \langle 2a \rangle, & B_0 &= 0. \end{aligned}$$

The results are obvious.

**Exercise 7.32.** This time, if we let  $a$  denote each edge and  $v$  denote each vertex, we have

$$\partial P = ka, \quad \partial a = 0, \quad \partial v = 0.$$

Thus we now have

$$\begin{aligned} Z_2 &= 0, & Z_1 &= \langle a \rangle \oplus \langle b \rangle, & Z_0 &= \langle v \rangle, \\ B_2 &= 0, & B_1 &= \langle ka \rangle, & B_0 &= 0. \end{aligned}$$

This gives the desired homology groups.

## Fundamental Groups of Polyhedra

**Exercise 7.33.** This is true because equality is an equivalence relation.

**Exercise 7.34.**

- (i) It suffices to show that  $o(\alpha)$  cannot be changed in a single move. But this is clear. In particular, using the definition, note that  $o(\alpha)$  is  $o(\beta)$  if  $\beta \neq \emptyset$ , and is  $p$  if  $\beta$  is empty. The same holds for  $o(\alpha')$ , so  $o(\alpha)$  is preserved. Similarly,  $e(\alpha) = e(\alpha')$ .
- (ii) Again, it suffices to show this for a single elementary move. We can further assume that  $\beta = \beta'$ . Write  $\alpha = \gamma(p, q)(q, r)\delta$  and  $\alpha' = \gamma(p, r)\delta$ . Then

$$\alpha\beta = \gamma(p, q)(q, r)\delta\beta = \gamma(p, r)\delta\beta = \alpha'\beta'.$$

(Recall  $\beta = \beta'$ .)

**Exercise 7.35.** An edge path, by definition, only goes along the 1-skeleton. Thus  $K$  being connected automatically implies that  $K^{(1)}$  is.

If  $K^{(1)}$  is connected, then let  $x, y \in |K|$ . There are unique open simplices  $s^\circ, t^\circ$  with  $x \in s^\circ$  and  $y \in t^\circ$ . Pick vertices  $v$  and  $w$  of  $s$  and  $t$ , respectively. Then consider the path taken by going straight line from  $x$  to  $v$ , then along the edges to  $w$ , then along a straight line to  $y$ . Hence  $|K|$  is connected (indeed, path-connected).

If  $|K|$  is connected, then  $|K|$  is clearly path-connected.

Finally, if  $|K|$  is path-connected, then we can find edge paths between any two vertices of  $K$  in the following manner: Each time the path crosses the 1-skeleton, say along the edge between  $v$  and  $w$ , pick either  $v$  and  $w$  and append that vertex (or, rather, the edge between that vertex and the previous one) to the edge path. That this works is clear.

**Exercise 7.36.** This is exactly the proof of Theorem 3.6, with  $\gamma$  as the edge path from  $p_0$  to  $p_1$ .

**Exercise 7.37.** Since an elementary move only moves across a 2-simplex, it follows that the edge path group is only dependent on the 2-skeleton.

**Exercise 7.38.**

- (i) This is clear.
- (ii) If  $v$  and  $w$  are in the same component as some point  $x$ , then by taking an edge path from  $v$  to  $x$ , then from  $x$  to  $w$ , we have an edge path between  $v$  and  $w$ . This proves that components are connected.  
Obviously the union of the components is  $K$ . To see that the unions are disjoint, suppose  $v \in [x] \cap [y]$  and  $w \in [x]$ . Then the path  $w \rightarrow x \rightarrow v \rightarrow y$  implies that  $w \in [y]$ . Since  $w$  was arbitrary, and since  $w \in [y]$  would similarly imply  $w \in [x]$ , it follows that  $[x] = [y]$ . This proves disjointness.
- (iii) Suppose  $[\alpha] \in \pi(K, x)$ . Then we claim that  $[\alpha] \in \pi(L, x)$ . But this is simply because any vertex along  $\alpha$  is necessarily connected to  $p$  via an edge path, hence belongs to  $L$ .

**Exercise 7.39.**

- (i) Write  $\alpha = e_1 \dots e_m$  and  $\beta = e_{m+1} \dots e_{m+n}$ . Then  $(\alpha\beta)^\circ : I_{m+n} \rightarrow K$  takes  $v_i$  to  $p_i$ , where  $p_i = \alpha^\circ(v_i)$  for  $0 \leq i \leq m$  and  $p_i = \beta^\circ(v_{i-m-1})$  otherwise. This is exactly  $\gamma$ .
- (ii) It suffices to show this if  $\alpha$  and  $\beta$  are separated by one step. But, writing  $\alpha = \gamma(p, q)(q, r)\delta = \gamma(p, r)\delta = \beta$ , simply note that we can use the straight line homotopy from the center of  $(p, r)$  to go to  $q$ . Resizing intervals as necessary, as in the previous part, gives the result.

**Exercise 7.40.** It suffices to show that trees are contractible. This is true for zero or one 1-simplices. For  $(n+1)$  total 1-simplices, simply pick an edge one of whose endpoints is a leaf. Then we can contract that edge to the other vertex, which is connected to the rest of the tree. Induction implies the result.

**Exercise 7.41.** Suppose  $e_1 \dots e_n$  were a circuit in  $T_1 \cup T_2$ . Suppose without loss of generality that  $e_1 \in T_1$ . Let  $i$  and  $j$  be the first and last indices, respectively, such that  $e_i, e_j \in T_1 \cap T_2$ . There is a path  $\alpha$  which starts with  $e_i$  and ends with  $e_j$  contained in  $T_1 \cap T_2$ . Now notice that  $e_1 \dots e_{i-1} \alpha e_{j+1} \dots e_n$  is a circuit contained entirely within  $T_1$ , contradicting that  $T_1$  is a tree.

**Exercise 7.42.** Let  $G$  be any abelian group, and let  $\varphi : \{xF' : x \rightarrow X\} \rightarrow G$ . Our goal is to show that there is a unique homomorphism  $\psi : F/F' \rightarrow G$  with  $\psi(xF') = \varphi(xF')$  for all  $xF' \in F/F'$ . (See Theorem 4.1(i).)

As in the definition of a free group, let  $\tilde{\varphi}$  be the unique homomorphism from  $F$  to  $G$  with  $\tilde{\varphi}(x) = \varphi(xF')$  for all  $x \in X$ . Now define

$$\begin{aligned} \psi : F/F' &\rightarrow G \\ fF' &\mapsto \tilde{\varphi}(f). \end{aligned}$$

To see that this is well-defined, notice that  $f \in F'$  implies that  $f = g^{-1}h^{-1}gh$  for some  $g, h \in F$ . Thus

$$\tilde{\varphi}(f) = \tilde{\varphi}(g)^{-1}\tilde{\varphi}(h)^{-1}\tilde{\varphi}(g)\tilde{\varphi}(h).$$

But  $G$  is abelian, so this is exactly 1, which proves well-definedness.

To see that  $\psi$  does indeed satisfy that  $\psi(xF') = \varphi(xF')$ , simply notice that  $\psi(xF') = \tilde{\varphi}(x)$ , which is defined to be  $\varphi(xF')$ .

Finally, to see that  $\psi$  is the *unique* homomorphism with this property, note that any other function  $\psi'$  would have to have  $\psi'(fF') = \tilde{\varphi}(f)$ , and thus be exactly equal to  $\psi$ .

Hence  $F/F'$  is indeed free abelian, with the desired basis.

**Exercise 7.43.** Exercise 7.42 shows that the rank of the free group  $F$  is the rank of the free abelian group  $F/F'$ . But this latter rank is invariant with respect to  $X$ .

**Exercise 7.44.**

- (i) By picking a maximal tree  $T$ , and setting some edge not in the tree to be  $x$ , we can see that every other edge becomes either  $x$ ,  $x^{-1}$ , or 1. Hence  $G_{\mathbb{R}P^2, T} \cong \mathbb{Z}/2\mathbb{Z}$ , and Corollary 7.37 implies the result.
- (ii) Hurewicz's theorem applies since  $\mathbb{R}P^2$  is obviously path-connected. Moreover, since  $\mathbb{Z}/2\mathbb{Z}$  is abelian, its commutator subgroup is trivial. Thus  $H_1(\mathbb{R}P^2) \cong \mathbb{Z}/2\mathbb{Z}$ , as desired.

**Exercise 7.45.**

- (i) Pick points  $x, y \in X$ . Then consider vertices  $p$  and  $q$  of the simplices containing  $x$  and  $y$ , respectively. Consider the following path: Take the straight line from  $x$  to  $p$ , then take the path mapped out by  $F(p, t)$  as  $t \in \mathbb{I}$ , then the path mapped out by  $F(q, 1 - t)$ , and finally the straight line from  $q$  to  $y$ .
- (ii) Let  $F : X \times \mathbb{I} \rightarrow X$  have  $F(v, 0)$  for all  $v \in X^{(1)}$  and  $F(\cdot, 1)$  a constant function. Then by taking the homotopy along  $F$ , we can go from  $(p, q)$  to the constant point, then back to some arbitrary edge of  $T$ , where  $T$  is a maximal tree of  $X$ . Hence  $(p, q) = 1$ , implying a trivial edge path group. Thus the fundamental group is trivial too.

**Exercise 7.46.** Since there are  $n$  vertices, we know that there are  $n - 1$  edges of a maximal tree. The result follows from Corollary 7.35.

**Exercise 7.47.** If  $X$  has  $m$  edges and  $n$  vertices, then  $\chi(X) = -m + n$ . Thus  $1 - \chi(X) = m - n + 1$ . Now use Hurewicz's theorem, Exercise 7.35, Exercise 7.42, and Corollary 7.35 to find the result for  $H_1$ . Note that  $H_0(X) = \mathbb{Z}$  because  $X$  is connected, and  $H_q(X) = 0$  for  $q \geq 2$  because  $X$  has dimension 1.

**Exercise 7.48.** We know that  $S^m$  is the boundary of an  $(m + 1)$ -simplex. Thus there is an edge between any two vertices, so we can fix one vertex  $p$  and let  $T$  be the star consisting of all edges  $(p, q)$ . Now consider any other edge  $(q, r)$ . Note that  $\{p, q, r\}$  forms a simplex, so  $(p, q)(q, r) = (p, r)$ . But in  $G_{K, T}$ , we know that  $(p, q) = (p, r) = 1$ , so  $(q, r) = 1$  as well. Thus  $\pi(K, p) \cong G_{K, T} = 1$ , and so  $\pi_1(S^m) = 1$ . Hence  $S^m$  is simply connected.

**Exercise 7.49.**

- (i) Since every vertex is contained in  $K^{(q)}$ , we can pick any simplex of maximal dimension. Its vertices are contained in  $\text{Vert}(K^{(q)})$ , but it does not itself belong in the  $q$ -skeleton.
- (ii) If a full subcomplex  $L$  exists, we know that it would need to include every simplex of  $K$  with vertices in  $A$ . Moreover, adding any other simplex would introduce new vertices. Thus such a subcomplex would be unique. Note that the set thus described is indeed a subcomplex, since any faces of  $s \in L$  would have to have vertices in  $A$  as well.

The second part of the statement follows from the description of  $L$ .

**Exercise 7.50.** Consider some element  $[\alpha] \in \pi(K, v_0)$ . Then there is some path  $\alpha' \simeq \alpha$  with  $\alpha' \in \pi(L, v_0)$ . Thus  $i[\alpha'] = [\alpha]$ , proving surjectivity.

If  $K$  is the 2-simplex and  $L$  is its boundary, then obviously any closed edge path in  $K$  is also in  $L$  (and, in particular, is homotopic to a closed edge path in  $L$ ). But the fact that  $K$  is simply connected while  $L$  is not implies that there cannot be an isomorphism.

## The Seifert–van Kampen Theorem

No exercises!