

# **Solutions to Rotman's algebraic topology**

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## 0 Introduction

### Notation

No exercises!

### Brouwer Fixed Point Theorem

**Exercise 0.1.** As per the hint, observe that if  $y \in G$ , then we have  $y = r(y) + (y - r(y))$ . Obviously, we have  $r(y) \in H$ . Moreover, we know that

$$r(y - r(y)) = r(y) - r(r(y)) = 0,$$

and so  $y - r(y) \in \ker r$ . Thus  $G \subseteq H \oplus \ker r$ .

The reverse is obviously true, since  $H$  and  $\ker r$  are both subgroups of  $G$ .

**Exercise 0.2.** Suppose instead that  $f : D^1 \rightarrow D^1$  has no fixed point. Then consider the continuous map  $g : D^1 \rightarrow S^0$  given by

$$g(x) = \begin{cases} 1 & \text{if } f(x) < x \\ -1 & \text{if } f(x) > x \end{cases}.$$

Notice that because  $f(x) \neq x$  for all  $x$ , the function  $g$  is well-defined.

Moreover, we know that  $f(-1) \neq -1$ , since  $f$  has no fixed point, and so  $f(-1) > -1$ . Thus  $g(-1) = -1$ . Similarly, we have  $g(1) = 1$ .

Thus we have  $g(D^1) = S^0$ , which is disconnected. This is a contradiction, so  $f$  must have had a fixed point.

**Exercise 0.3.** Suppose that  $r$  is such a retract. Then we have the following commutative diagram:

$$\begin{array}{ccc} & S^n & \\ i \nearrow & & \searrow r \\ S^{n-1} & \xrightarrow{1} & S^{n-1} \end{array}$$

Applying  $H_{n-1}$ , we get another commutative diagram:

$$\begin{array}{ccc} & H_{n-1}(S^n) & \\ H_{n-1}(i) \nearrow & & \searrow H_{n-1}(r) \\ H_{n-1}(S^{n-1}) & \xrightarrow{H_{n-1}(1)} & H_{n-1}(S^{n-1}) \end{array}$$

We know that  $H_{n-1}(S^n) = 0$ , however, implying that  $H_{n-1}(1) = 0$ . This contradicts the fact that  $H_{n-1}(S^{n-1}) = \mathbb{Z} \neq 0$ . Thus the retraction  $r$  could not have existed.

**Exercise 0.4.** Suppose  $g : D^n \rightarrow X$  is a homeomorphism. Then we know that  $g^{-1} \circ f \circ g$  is a continuous map from  $D^n$  to itself, and so it has a fixed point  $x$ . Then we know that  $g^{-1}(f(g(x))) = x$ , and so it follows that  $f(g(x)) = g(x)$ . Thus  $g(x) \in X$  is a fixed point of  $f$ .

**Exercise 0.5.** Consider the function  $h : \mathbb{I} \times \mathbb{I} \rightarrow \mathbb{I} \times \mathbb{I}$  given by

$$h(s, t) = f(s) - g(t) + (s, t).$$

This is the sum of continuous functions, and so it is itself continuous. Moreover, we know that  $\mathbb{I} \times \mathbb{I}$  is homeomorphic to  $D^1$ , and so it follows that there is a fixed point  $(s, t)$  of  $h$ . But this means that  $f(s) - g(t) = 0$ , and so we are done.

**Exercise 0.6.** Observe that  $x \in \Delta^{n-1}$  must contain some positive coordinate, because  $\sum x_i = 1$  and  $x_i \geq 0$  for all  $i$ . Since  $a_{ij} > 0$  for every  $i, j$ , it follows that  $Ax$  contains only nonnegative coordinates and, moreover, contains at least one positive coordinate. Thus  $\sigma(Ax) > 0$ , and so  $g(x)$  is well-defined.

Moreover, it is continuous because the linear map  $A$ , the map  $\sigma$ , and the division function are all continuous.

Because  $\Delta^{n-1} \approx D^{n-1}$ , it follows that there exists some  $x$  with

$$x = \frac{Ax}{\sigma(Ax)}.$$

Then  $\lambda = \sigma(Ax) > 0$  is a positive eigenvalue for  $A$  and  $x \in \Delta^{n-1}$  is a corresponding eigenvector.

We know that  $x$  contains only nonnegative coordinates. Suppose then that some coordinate, say  $x_1$ , is zero. Then obviously the first coordinate of  $\lambda x$  is zero. However, the first coordinate of  $Ax$  is

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = a_{12}x_2 + \cdots + a_{1n}x_n.$$

Since  $\sum x_i = 1$  and  $x_1 = 0$ , there exists some  $k \neq 1$  such that  $x_k > 0$ . Then  $a_{1k}x_k > 0$ , and since each  $i$  already has  $a_{1i}x_i \geq 0$ , it follows that the first coordinate of  $Ax$  is strictly positive, contradicting that  $Ax = \lambda x$ .

Thus the eigenvector  $x$  has all positive coordinates.

## Categories and Functors

**Exercise 0.7.** We know that

$$g \circ (f \circ h) = g \circ 1_b = g$$

and

$$(g \circ f) \circ h = 1_A \circ h = h,$$

and so associativity implies  $g = h$ .

**Exercise 0.8.**

(i) Notice that if  $1_A$  and  $1'_A$  are both identities, then we must have

$$1_A = 1_A \circ 1'_A = 1'_A,$$

which proves the desired result.

(ii) If  $1'_A$  is the new identity in  $\mathcal{C}'$ , then we know that  $1'_A \in \text{Hom}_{\mathcal{C}'}(A, A) \subseteq \text{Hom}_{\mathcal{C}}(A, A)$ , and so  $1_A \circ 1'_A$  is defined. But we know that

$$1'_A \circ 1_A = 1'_A = 1'_A \circ 1'_A,$$

and so Exercise 0.7 implies the result.

**Exercise 0.9.** Clearly, the Hom-sets are pairwise disjoint, since each  $i_y^x$  appears at most once.

It is also obviously associative. In particular, if  $a \leq b \leq c \leq d$ , then we know that

$$i_d^c \circ (i_c^b \circ i_b^a) = i_d^c \circ i_c^a = i_d^a,$$

and similarly for  $(i_d^c \circ i_c^b) \circ i_b^a$ .

Finally, the map  $i_x^x$  is the identity on  $x \in X$ . To see that it is a left-identity, note that if  $y \leq x$ , then

$$i_x^x \circ i_x^y = i_x^y.$$

Similarly, we can show that this map is a right-identity as well, and so we are done.

**Exercise 0.10.** Disjointness is clear, since there is only one object. Because  $G$  is a monoid, it is associative and has an identity, proving that  $\mathcal{C}$  is a category.

**Exercise 0.11.** It is pretty clear that  $\text{obj}(\mathbf{Top}) \subset \text{obj}(\mathbf{Top}^2)$ . Moreover, a continuous map  $f : X \rightarrow Y$  between two topological spaces corresponds to the map  $(f, \emptyset)$  in  $\mathbf{Top}^2$  from  $(X, \emptyset)$  to  $(Y, \emptyset)$ , which then means that  $\mathbf{Top}$  can be thought of as a subcategory of  $\mathbf{Top}^2$ .

**Exercise 0.12.** It is worth noting that Rotman's definition here is incorrect. The morphisms in  $\mathcal{M}$  should be the commutative squares, not merely the ordered pairs  $(h, k)$ .

Indeed, consider the following counterexample to Rotman's definition. Let  $\mathcal{C}$  be the category of sets. Furthermore, let  $A$  be a set with more than one element. Then the following diagrams are both commutative:

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ \downarrow 1_A & & \downarrow 0 \\ A & \xrightarrow{0} & \{0\} \end{array} \quad \begin{array}{ccc} A & \xrightarrow{0} & A \\ \downarrow 1_A & & \downarrow 0 \\ A & \xrightarrow{0} & \{0\}. \end{array}$$

This implies that the ordered pair  $(1_A, 0)$ , where  $0$  is considered to be the map that sends everything in  $A$  to the zero element, is both in  $\text{Hom}(1_A, 0)$  and in  $\text{Hom}(0, 0)$ , contradicting disjointness.

If we instead consider morphisms of  $\mathcal{M}$  to be the commutative squares, where composition is defined by “stacking” the squares on top of one another, disjointness is clear. After all, the squares contain  $f$  and  $g$ , and so Hom-sets of different objects must be disjoint.

Associativity is clear, as the morphisms of  $\mathcal{C}$  are associative.

Finally, there is an identity  $1_f$  for every  $f \in \text{Hom}_{\mathcal{C}}(A, B)$ , namely the one where  $h = 1_A$  and  $k = 1_B$ .

**Exercise 0.13.** With the hint, this is clear. In particular, we consider  $\mathbf{Top}^2$  to be the subcategory of the arrow category of  $\mathbf{Top}$  in which the objects are inclusions, and  $\text{Hom}_{\mathbf{Top}^2}(i, j) = \text{Hom}_{\mathbf{Top}}(i, j)$ .

**Exercise 0.14.** To see that it is a congruence at all, observe that Property (i) is satisfied because there is only one Hom-set. Moreover, if  $x \sim x'$  and  $y \sim y'$ , then we know that  $x(x')^{-1} = h_x$  and  $y(y')^{-1} = h_y$  for some  $h_x, h_y \in H$ . But then we know that

$$(yx)(y'x')^{-1} = yx(x')^{-1}(y')^{-1} = yh_x(y')^{-1}.$$

However, since  $(y')^{-1} = y^{-1}h_y$ , we know that this is simply

$$(yx)(y'x')^{-1} = yh_xy^{-1}h_y.$$

Because  $H$  is normal, we know that  $yh_xy^{-1} \in H$ . Thus the product of this and  $h_y$  is in  $H$  as well, and so  $xy \sim x'y'$ , as desired.

To see that  $[\ast, \ast] = G/H$  simply requires the observation that  $x \sim y$  if and only if  $x$  and  $y$  are in the same coset of  $H$ .

**Exercise 0.15.** This follows from the fact that functors preserve (or, in the case of contravariant functors, reverse) the directions of the arrows. Thus the resulting diagram still commutes.

**Exercise 0.16.** Note that for (i)–(iv), we can simply use inverses. For instance, for **Set**, it suffices to note that if  $f$  is a bijection, then  $f^{-1}$  is a bijection, which is clearly true. Similarly, the inverse of a homeomorphism is a homeomorphism, and the inverse of a group or ring isomorphism is still an isomorphism.

For (v), note that  $i_y^y$  is defined and satisfies the requirements that  $i_y^y \circ i_x^x = i_x^x$  and  $i_y^x \circ i_x^y = i_y^y$ .

For part (vi), notice that  $f^{-1}$  works because  $f$  is a homeomorphism. In particular, it is a bijection, and so  $f^{-1}(A') = A$ . Moreover, it is (bi)continuous since  $f$  is.

Finally, for the monoid  $G$ , if  $g$  has a two-sided inverse  $h$ , then  $hg = gh = 1$ , which is the identity element of  $\text{Hom}(G, G)$ .

**Exercise 0.17.** To prove that  $T'$  is a functor, first observe that criterion (i) of a functor is satisfied because  $T$  does so. Moreover, if  $[f] \in \text{Hom}_{\mathcal{C}'}(A, B)$ , then  $f \in \text{Hom}_{\mathcal{C}}(A, B)$ , and so  $T'([f]) = Tf$  is a morphism in  $\mathcal{A}$ . In particular, if  $[g] \circ [f] = [g \circ f]$  is defined in  $\mathcal{C}'$ , then  $g \circ f$  is defined in  $\mathcal{C}$ . This means, then, that

$$T'([g] \circ [f]) = T(g \circ f) = (Tg) \circ (Tf) = T'([g]) \circ T'([f]).$$

Finally, it remains to note that  $T'([1_A]) = T_{1_A} = 1_{TA} = 1_{T'([A])}$  for every object  $A$ . Thus  $T'$  is a functor.

**Exercise 0.18.**

- (i) It is clear that  $tG \in \text{obj } \mathbf{Ab}$  for every group  $G$ . Now suppose that we have a homomorphism  $f : G \rightarrow H$ . Then we know that  $t(f)$  is a morphism  $f|_{tG}$  from  $tG$  to  $tH$ . To see this, note that it is the restriction of a homomorphism, and thus is itself a homomorphism. Moreover, if  $x \in f(tG)$ , then  $x = f(y)$  for some  $y \in G$  with finite order. But then there exists some  $n$  so that  $y^n = 1$ . Thus  $x^n = f(y^n) = 1$ , and so  $x$  has finite order. But  $x \in f(G) \subseteq H$  implies that  $x \in tH$ .

Now we must check that  $t$  respects composition. Indeed, if  $g \circ f$  is defined, then

$$t(g \circ f) = (g \circ f)_{tG} = g|_{f(tG)} \circ f|_{tG}.$$

But  $f(tG) \subseteq tH$ , and so this is simply

$$t(g \circ f) = g|_{tH} \circ f|_{tG} = t(g) \circ t(f),$$

which proves that composition is respected.

Finally, note simply that  $t(1_G) = 1|_{tG}$ , which is the identity on  $tG$ .

- (ii) Suppose that  $f$  is an injective homomorphism from  $G$  to  $H$ . Then suppose that  $t(f)(x) = t(f)(y)$ . But  $f(x) = f|_{tG}(x) = t(f)(x)$ , and so it follows that  $f(x) = f(y)$ . Injectivity of  $f$  proves the result.
- (iii) Let  $G = \mathbb{Z}$  and  $H = \mathbb{Z}/2\mathbb{Z}$  and let  $f$  take even integers to 0 and odd integers to 1. This is evidently surjective. But  $tG = \{0\}$  while  $tH = \{0, 1\}$ , and so  $t(f) : tG \rightarrow tH$  cannot be surjective.

**Exercise 0.19.**

- (i) If  $f$  is a surjection, then consider an arbitrary coset  $a + pH$  of  $H/pH$ . We know that there exists some  $b \in G$  with  $f(b) = a$ , and so it follows that  $F(f)$  takes  $b + pG$  to  $a + pH$ , proving surjectivity of  $F(f)$ .
- (ii) Consider the function  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  taking  $x$  to  $2x$ . Then, letting  $p = 2$ , we know that  $F(f) : \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$  has  $F(f)([0]) = F(f)([1])$ .

**Exercise 0.20.**

- (i) This is evident because  $\mathbb{R}$  is a ring, and the operations are pointwise.
- (ii) By the previous part, we know that if  $X$  is a topological space, then  $C(X)$  is a ring. Now suppose that  $f : X \rightarrow Y$  is a continuous map. Then define

$$\begin{aligned} C(f) : C(Y) &\rightarrow C(X) \\ g &\mapsto g \circ f \end{aligned}$$

and note that this is well-defined. Moreover, we know that  $C(g \circ f)(h) = h \circ g \circ f$ , while  $C(f) \circ C(g)$  takes  $h$  to  $C(f) \circ (h \circ g) = h \circ g \circ f$ , which proves that  $C$  reverses composition. Finally, we know that  $C(1_x)$  takes  $g$  to  $g \circ 1_X = g$  and is therefore the identity on  $C(Y)$ . Thus  $C$  (or, rather, the map taking  $X$  to  $C(X)$ , to be precise) gives rise to a contravariant functor.

# 1 Some Basic Topological Notions

## Homotopy

No exercises!

## Convexity, Contractibility, and Cones

**Exercise 1.1.** Suppose  $H : f_0 \simeq f_1$  is a homotopy. Then let  $F(t) = H(x, t)$  for some fixed  $x$ . It is clear that  $F(0) = x_0$  and  $F(1) = 1$ . Moreover, since  $H$  is continuous, it follows that so too is  $F$ . For the converse, simply let the homotopy  $H : f_0 \simeq f_1$  take  $(x, t) \in X \times \mathbb{I}$  to  $F(t)$ .

**Exercise 1.2.**

- (i) There exist functions  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$  such that  $g \circ f \simeq 1_X$  and  $f \circ g \simeq 1_Y$ . Moreover, there is a homotopy  $F : 1_X \simeq c$ , where  $c$  denotes the constant map at some  $x_0 \in X$ . Then consider the map  $G : Y \times \mathbb{I} \rightarrow Y$  which takes  $(y, t)$  to  $f(F(g(y), t))$ . In particular, we know that  $G$  is continuous and that it is thus a homotopy from  $f \circ g$  to the constant map  $c'$  at  $y_0 = f(x_0)$ . But then we find that  $1_Y \simeq f \circ g \simeq c'$ , and so  $Y$  is contractible.
- (ii) Consider, for example, the subsets  $X, Y \subset \mathbb{R}^2$  where

$$X = \{(x, 0) : x \in [0, 1]\},$$

$$Y = \left\{ (x, x) : x \in \left[0, \frac{1}{2}\right] \right\} \cup \left\{ (x, 1-x) : x \in \left[\frac{1}{2}, 1\right] \right\}.$$

It is obvious that  $X$  is convex, but  $Y$  is not, even though there is an obvious homotopy equivalence from  $X$  to  $Y$ .

**Exercise 1.3.** We know that  $R(x) = e^{i\alpha}x$ , and so the continuous map  $F : S^1 \times \mathbb{I} \rightarrow S^1$  given by  $F(x, t) = e^{i\alpha t}x$  is a homotopy  $F : 1_S \simeq R$ . Thus, if  $g : S^1 \rightarrow S^1$  is continuous, then let  $\theta$  be such that  $g(1) = g(e^{i \cdot 0}) = e^{i\theta}$ . Then we know that, letting  $R$  now be the rotation of  $-\theta$  degrees, we must have  $R \circ g \simeq 1_S \simeq g = g$  and  $(R \circ g)(1) = 1$ , as desired.

**Exercise 1.4.**

- (i) Pick  $(x_1, y_1), (x_2, y_2) \in X \times Y$ . Then we know that, for any  $t \in \mathbb{I}$ , we have

$$t(x_1, y_1) + (1-t)(x_2, y_2) = (tx_1 + (1-t)x_2, ty_1 + (1-t)y_2).$$

The result follows from convexity of  $X$  and  $Y$ .

- (ii) If  $F_X : 1_X \simeq c_X$  and  $F_Y : 1_Y \simeq c_Y$ , where  $c_X$  and  $c_Y$  are constant maps at  $c_X$  and  $c_Y$ , respectively, then the map

$$F : (X \times Y) \times \mathbb{I} \rightarrow X \times Y$$

$$(x, y, t) \mapsto (F_X(x, t), F_Y(y, t))$$

is clearly a homotopy from  $1_{X \times Y}$  to  $(c_X, c_Y)$ .

**Exercise 1.5.** It is clear that  $X$  is compact. After all, any open cover of  $X$  must contain some set  $U$  containing 0, and thus containing cofinitely many elements of  $X$ .

If we have a map  $h : X \rightarrow Y$ , then because  $Y$  is discrete, we know that  $\{h^{-1}(y) : y \in Y\}$  is an open covering of  $X$  and thus by compactness admits a finite subcovering. Thus there are only finitely many elements of  $y$  in the image of  $h$ .

Now suppose that  $f : X \rightarrow Y$  is a homotopy equivalence. Then there exists some  $g : Y \rightarrow X$  with a homotopy  $H : f \circ g \simeq 1_Y$ . But  $H(\{y\} \times \mathbb{I})$  is the continuous image of a connected map and is therefore itself connected. Because  $Y$  is discrete, this means that  $H(y, 0) = H(y, 1)$  for all  $y$ . But we know that  $f$  has finite image, and  $Y$  is infinite, so there exists some  $y$  such that  $y \notin \text{im } f$ . In particular, we have  $y \neq f(g(y))$ , and so  $H(y, 0) = f(g(y)) \neq y = 1_Y(y)$ , a contradiction. Thus  $X$  and  $Y$  are not of the same homotopy type.

**Exercise 1.6.** Suppose  $X$  is contractible, with  $F : c \simeq 1_X$ , where  $c$  is the constant map at  $p$ . Note that, for every  $x \in X$ , there is a path  $F(x, t) : \{x\} \times \mathbb{I} \rightarrow X$  taking  $x$  to  $p \in X$ . In particular, this means that every  $x$  is in the same component as  $p$ , proving connectedness.

**Exercise 1.7.** The map  $H : X \rightarrow \mathbb{I} \rightarrow X$  taking  $(x, t)$  to  $x$  and  $(y, t)$  to  $x$  if and only if  $t > \frac{1}{2}$  works. Indeed, note that  $H^{-1}(\{x\} \times \mathbb{I})$  is simply  $\{x\} \times \mathbb{I} \cup \{y\} \times (\frac{1}{2}, 1]$ , which is open in  $X \times \mathbb{I}$ .

**Exercise 1.8.**

- (i) Consider the map taking the unit interval to  $S^1$  given by  $t \mapsto e^{2\pi it}$ .
- (ii) If  $r : Y \rightarrow X$  is a retraction, then we know from  $1_Y \simeq c$  that  $r \circ 1_Y \circ i \simeq r \circ c \circ i$ , where  $i$  is the injection  $X \hookrightarrow Y$ . But the left side is simply  $r \circ i = 1_X$ , while the right side is a constant map, proving the result.

**Exercise 1.9.** We know that there exists some constant map  $c$  with  $f \simeq c$ . But then  $g \circ f \simeq g \circ c$ , and the right side is a constant map. Thus  $g \circ f$  is also nullhomotopic.

**Exercise 1.10.** First, suppose that  $g$  is an identification. Note that  $(gf)^{-1}(U)$  open in  $X$  implies that  $g^{-1}(U)$  is open in  $Y$  because  $f$  is an identification. But the hypothesis on  $g$  implies that  $U$  is open in  $Z$ . Since  $gf$  is clearly a continuous surjection, the result follows.

Now, suppose that  $gf$  is an identification. It suffices to prove that  $g^{-1}(U) \subseteq Y$  open implies that  $U \subseteq Z$  is open. But we know by continuity of  $f$  that  $f^{-1}(g^{-1}(U))$  is open, and so  $gf$  being an identification implies the result.

**Exercise 1.11.** First, note that this is a well-defined function in the sense that  $[x] = [y]$  in  $X/\sim$  implies that  $\bar{f}([x]) = \bar{f}([y])$ .

This is evidently continuous. After all, suppose that  $U \subseteq Y/\square$  is open. Then we know that

$$\bar{f}^{-1}(U) = \{[x] \in X/\sim : [f(x)] \in U\} = U'.$$

If we let  $v : X \rightarrow X/\sim$  and  $u : Y \rightarrow Y/\square$  be the natural maps, then we know that  $U'$  is open in  $X/\sim$  because

$$v^{-1}(U') = \{x \in X : f(x) \in u^{-1}(U)\} = f^{-1}(u^{-1}(U))$$

is open.

Finally, we will show that  $\bar{f}$  is an identification. It is obviously surjective. Moreover, if  $U' = \bar{f}^{-1}(U)$  is open in  $X/\sim$ , then we simply note that a similar argument as above gives us that  $v^{-1}(U') = f^{-1}(u^{-1}(U))$  is open. Since  $f$  and  $u$  are identifications, it follows that  $U$  was an open set in the first place, proving the result.

**Exercise 1.12.** Note that if  $K \subseteq Z$  is closed, then it is compact and so  $h(K)$  is compact in  $X$ , hence itself closed. Thus  $h$  is a closed map, and hence an identification.

Now because  $v : X \rightarrow X/\ker h$  is an identification, Corollary 1.9 applies. Indeed, Corollary 1.9 implies that  $hv^{-1} = \varphi$  is a closed map. Thus it is an identification, i.e., a continuous surjection.

But the same corollary also implies that  $\varphi^{-1} = vh^{-1}$  is continuous. This, combined with Example 1.3, in which it was shown that  $\varphi$  is injective, proves the result, as  $\varphi$  is now a bicontinuous bijection, i.e., a homeomorphism.

**Exercise 1.13.** First observe that  $f(x) = f(y)$  implies that  $[x, t] = [y, t]$  and so  $t = 1$ . Thus  $f$  is injective and hence bijective onto its image  $CX_t = \{[x, t] \in CX : x \in X\}$ . Then open sets in  $CX_t$  are precisely of the form  $U \cap CX_t$  for an open set  $U \subseteq CX$ . But clearly we can assume that  $[x, 1] \notin U$  because  $[x, 1] \notin CX_t$ , and thus we wind up with  $X \times [0, 1)$ , where  $CX_t = X \times \{t\}$ . This is obviously homeomorphic to  $X$ .

**Exercise 1.14.** The functor takes a map  $f : X \rightarrow Y$  to  $Cf : CX \rightarrow CY$  given by  $C([x, t]) = [f(x), t]$ . Note that this is well-defined. Moreover, it is obvious that this satisfies the properties of a functor. Indeed, if  $g : Y \rightarrow Z$ , then

$$C(g \circ f)([x, t]) = [g(f(x)), t] = ((Cg) \circ (Cf))([x, t])$$

and clearly  $C(1_X)$  is the identity on  $CX$ .



## Paths and Path Connectedness

**Exercise 1.15.** Using the hint, suppose that  $g : \mathbb{I} \rightarrow X$  is a path with  $g(0) = (0, a) \in A$  and with  $g(t) \in G$  for all  $t > 0$ . Then note that  $\pi_i \circ g$  is continuous for  $i = 1, 2$ , where  $\pi_i$  are the projections to the  $x$ - and  $y$ -axes. This implies the existence of an  $\epsilon > 0$  such that  $t \in (0, \epsilon)$  implies that  $g(t) = (x(t), \sin(1/x(t)))$  has  $x(t), |\sin(1/x(t)) - a| < \delta$ . But this is obviously impossible, as  $\sin(1/x(t))$  will oscillate wildly between  $-1$  and  $1$ .

**Exercise 1.16.** Let  $(a_i)$  and  $(b_i)$  be points in  $S^n$ . We will construct  $n$  paths which, when joined together in the customary fashion (i.e., by traversing each of the  $n - 1$  subpaths in  $1/(n - 1)$  time), will give us a path from  $(a_i)$  to  $(b_i)$ .

The first path  $f_1$  is defined as

$$f_1(t) = ((1 - t)a_1 + tb_1, c_2, a_3, a_4, \dots, a_n),$$

where  $c_2$  is chosen to be of the same sign as  $a_2$  and in such a way that  $f(t) \in S^n$ . Note that such a  $c_2$  always exists.

In general, for  $1 \leq i \leq n - 1$ , the path  $f_i$  will fix every coordinate except for the  $i$ -th, which it will take to  $b_i$ , and the  $(i + 1)$ -th, which we use as a “free” coordinate to allow for such adjusting. Moreover, observe that if the first  $n - 1$  coordinates of two points on  $S^1$  are the same, then the  $n$ -th coordinates either will be the same or will be negatives.

If joining the paths  $f_1, f_2, \dots, f_{n-1}$  together gives a path from  $(a_i)$  to  $(b_i)$ , then we are done. Note that this occurs if  $a_n$  and  $b_n$  have the same sign.

Otherwise, construct a path  $g$  which adjusts the  $n$ -th coordinate and uses the  $(n - 1)$ -th coordinate as a “free” one, preserving the sign. This effectively allows us to switch the sign of the  $n$ -th coordinate so that the  $n$ -th coordinate is just  $b_n$ . Moreover, because we preserved the sign of the  $(n - 1)$ -th coordinate, it is still equal to  $b_{n-1}$ .

**Exercise 1.17.** It suffices to show the forward direction, so suppose that  $U$  is not path connected. Then there are at least two path components.

We will show that each path component is open, which will prove that  $U$  is not connected. But because  $U$  is open, we know that open sets in  $U$  (as a subspace) or also open in  $\mathbb{R}^n$ . Thus, for every  $x \in U$ , there is a ball  $B_x$  centered at  $x$  and contained in  $U$ . This ball is obviously path-connected. As such, if  $x$  is in the path component  $A$ , it must follow that  $B_x \subseteq A$ , proving that  $A$  is open.

**Exercise 1.18.** We know that if  $X$  is contractible then there exists a point  $c \in X$  such that  $1_X$  is homotopic to the constant map at  $c$  from  $X$  to itself. Now consider the map  $c : \mathbb{I} \rightarrow X$  satisfying  $c(t) = c$  for all  $t$ . In the proof of Theorem 1.13, we saw that any path is homotopic to  $c$ . In particular, the constant maps  $x : \mathbb{I} \rightarrow X$  and  $y : \mathbb{I} \rightarrow X$  at  $x$  and  $y$ , respectively, are both homotopic to  $c$ . Note that these give rise to paths from  $x$  to  $c$  and from  $c$  to  $y$ , respectively, which in turn give rise to a path from  $x$  to  $y$ . This proves path connectedness.

**Exercise 1.19.**

(i) If  $X$  is path connected, then let  $c$  and  $c'$  be constant maps. Let  $f$  be a path from (the point)  $c$  to (the point)  $c'$  and define  $H : X \times \mathbb{I} \rightarrow X$  as  $H(x, t) = f(t)$ . Then  $H$  is a homotopy from  $c$  to  $c'$ .

For the reverse direction, let  $H$  be a homotopy from  $c$  to  $c'$  and define the path  $f : \mathbb{I} \rightarrow X$  as  $f(t) = H(c, t)$ .

(ii) Let  $f : X \rightarrow Y$  be a continuous function. Fix some  $y_0 \in Y$  and consider the map

$$\begin{aligned} H : X \times \mathbb{I} &\rightarrow Y \\ (x, t) &\mapsto p_x(t), \end{aligned}$$

where  $p_x$  is a path from  $f(x)$  to  $y_0$ . This is a homotopy from  $f$  to the constant map mapping  $X$  to  $y_0$ .

But if  $g : X \rightarrow Y$  is another continuous function, then the same argument shows that  $g \simeq y_0$ , and so  $f \simeq g$ , as desired.

**Exercise 1.20.** It suffices to show that if  $a \in A$  and  $b \in B$ , then there is a path from  $a$  to  $b$ . But fix some point  $x \in A \cap B$ . Then there is a path from  $a$  to  $x$ , and a path from  $x$  to  $b$ . Joining the two paths gives a path from  $a$  to  $b$ .

**Exercise 1.21.** This is simply done by noting that for any  $(x, y), (x', y') \in X \times Y$ , we can join the paths  $f(t) = ((1-t)x + tx', y)$  and  $g(t) = (x', (1-t)y + ty')$ .

**Exercise 1.22.** Suppose  $f(a), f(b) \in Y$ . Then let  $p$  be a path from  $a$  to  $b$  in  $X$ . Now simply note that  $q(t) = f(p(t))$  is a path from  $f(a)$  to  $f(b)$ , proving the result.

**Exercise 1.23.**

- (i) We already know that there are at least two path components because the entire space is not path connected. Moreover, both  $A$  and  $G$  are path connected, and so it follows that they must themselves be the path components.
- (ii) Simply note that the sequence  $\{(\frac{1}{n\pi}, \sin(n\pi))\} \subset G$  approaches  $(0, 0) \in A$ .
- (iii) As per the hint, consider  $U$  to be the open disk with center  $(0, \frac{1}{2})$  and radius  $\frac{1}{4}$ . Then  $X \cap U$  is open in  $X$ . But note that  $v(X \cap U)$  is not open in  $X/A \approx [0, \frac{1}{2\pi}]$ . After all, note that any ball  $B_\epsilon$  around the point 0 (which is the image of  $A$  under the natural map in this case) must contain some point  $\frac{1}{n\pi} < \epsilon$ . But  $\frac{1}{n\pi}$ , which corresponds to the point  $(\frac{1}{n\pi}, 0) \in X \setminus U$ , is not contained in  $v(X \cap U)$ .

**Exercise 1.24.** By definition, path components are path connected. Moreover, if  $C$  is a path component and there exists some point  $x \in X$  and  $c \in C$  so that there is a path between  $x$  and  $c$ , then the definition of path components implies that  $x \in C$ . Thus path components are maximally path connected.

Finally, suppose that  $A$  is path connected and pick  $a \in A$ . There exists a unique path component  $C$  such that  $a \in C$ . Then for all  $b \in A$ , we know that there is a path between  $a$  and  $b$ , and so  $b \in C$ . Thus  $A \subseteq C$ , as desired.

**Exercise 1.25.** Simply use Exercise 1.22 and observe that  $I$  is path connected.

**Exercise 1.26.** Note that, if  $X$  is locally path connected, then for all  $x \in X$ , there exists some open path connected, hence connected, neighborhood  $V$  of  $x$ . Alternatively, note that if  $U \subseteq X$  is open, then its components are unions of its path components and thus open.

**Exercise 1.27.** Given any open subset  $U$  of  $X \times Y$  containing a given point  $(x, y) \in X \times Y$ , there must exist a basic open neighborhood  $U_x \times U_y \subseteq U$  of  $(x, y)$ . Then we know that there exists some path connected  $V_x$  with  $x \in V_x \subseteq U_x$ , and similarly for  $y$ . Then  $V_x \times V_y$  is path connected by Exercise 1.21. The result follows.

**Exercise 1.28.** Note that open subsets of open subsets are open in the main space. In particular, let  $A \subseteq X$  be open. Given any  $x \in A$ , let  $U$  be an open neighborhood of  $x$  in  $A$ . Note that this is also an open neighborhood in  $X$ , and so there exists an open path connected  $V$  in  $X$  (and hence open in  $A$  as well) such that  $x \in V \subseteq U$ .

**Exercise 1.29.** Consider the map  $F : (\mathbb{R}^{n+1} \setminus \{0\}) \times \mathbb{I} \rightarrow \mathbb{R}^{n+1} \setminus \{0\}$  given by

$$F((x_i), t) = \left[ (1-t) + \frac{t}{\sqrt{\sum x_i^2}} \right] (x_i).$$

This is evidently a homotopy which makes  $S^n$  a deformation retract.

**Exercise 1.30.** The exact same map as in Exercise 1.29 works for this case.

**Exercise 1.31.** It is easy to see that the deformation retract of a deformation retract is a deformation retract, either by a direct argument or by applying Theorem 1.22. Thus the previous exercise implies that it suffices to show that  $D^n \setminus \{0\}$  is a deformation retract of  $S^n \setminus \{a, b\}$ . But the map  $(x_i) \mapsto (x_1, \dots, x_{n-1}, 0)$  is exactly the map needed, and so we are done.

**Exercise 1.32.** If  $H : f_0 \simeq f_1$ , then the map  $H' : (y, t) \mapsto H(r(y), t)$  is a homotopy from  $\tilde{f}_0$  to  $\tilde{f}_1$ .

**Exercise 1.33.** Let  $Y = \{y\}$  and observe that  $(x, 1) \sim y$  for all  $x \in X$ . Thus  $(x, 1) \sim (x', 1)$  for all  $x, x' \in X$ . Moreover, this is the only equivalence. Thus  $M_f$  is precisely the quotient space  $(X \times \mathbb{I}) / (X \times \{1\}) = CX$ .

**Exercise 1.34.**

- (i) We first tackle  $i$ . It is obvious that  $i$  is injective, and thus a bijection onto its image  $i(X) = \{[x, 0] : x \in X\}$ . Moreover, the open sets in  $i(X)$  are precisely of the form  $U \cap i(X)$  for open sets  $U$  in  $M_f$ .

Note that we can suppose without loss of generality that  $U$  is contained in  $v(X \times [0, 1))$ , where  $v$  is the natural map. Thus  $U$  simply looks like the Cartesian product of an open interval with an open set of  $X$ . This proves that  $i$  is a homeomorphism, for the open sets of  $i(X)$  map exactly to the open sets of  $X$ .

We can show that  $j$  is a homeomorphism onto  $j(Y)$  in a similar manner. The main idea is simply that  $y \not\sim y'$  for any  $y, y' \in j(Y)$ .

- (ii) It is obvious that  $(rj)(y) = r[y] = y = 1_Y(y)$  for any  $y \in Y$ . It is also clearly continuous by the gluing lemma. Thus  $r$  is indeed a retraction.
- (iii) Define  $F : M_f \times \mathbb{I} \rightarrow M_f$  as suggested in the hint. It is evident that  $F$  is continuous. Moreover, for any  $[x, t] \in M_f$ , we know that

$$\begin{aligned} F([x, t], 0) &= [x, t] \\ F([x, t], 1) &= [x, 1] = [f(x)] \in Y. \end{aligned}$$

Similarly, if  $[y] \in Y$ , then the definition implies that the remaining criteria for this homotopy to induce a deformation retraction  $r(x) = F(x, 1)$  are satisfied.

- (iv) Note that Rotman writes that  $f$  is homotopic to  $r \circ i$ ; in fact, we can and do prove the stronger statement that  $f$  coincides with  $r \circ i$ .

Let  $f : X \rightarrow Y$  be continuous. Then it is clear that the map  $f = r \circ i$ , where  $i : X \rightarrow M_f$  is an injection and  $r : M_f \rightarrow Y$  is the retraction taking  $[x, t]$  to  $[f(x)]$  and taking  $[y]$  to itself, proving the result.

## 2 Simplexes<sup>1</sup>

### Affine Spaces

**Exercise 2.1.** Note that there is a maximal affine independent subset  $S$  of  $A$ . This is directly implied by the fact that any set of greater than  $n + 1$  elements is not affine independent. Hence we can take an affine independent subset of  $A$  with maximum size (because the empty set is affine independent).

Write  $S = \{p_0, \dots, p_m\}$ . Then let  $p_{m+1} \in A \setminus S$ . By maximality of  $S$ , we know that  $S \cup \{p\}$  is not affine independent. Hence there exist  $s_i$  not all 0 such that

$$\sum_{i=0}^{m+1} s_i p_i = 0, \quad \sum_{i=0}^{m+1} s_i = 0.$$

Note that the second equation implies  $\sum_{i=0}^m s_i \neq 0$  for some  $i < m + 1$ . It follows then that

$$\sum_{i=0}^m \left( \frac{s_i}{\sum_{i=0}^m s_i} p_i \right) = p_{m+1}.$$

But we know that

$$\sum_{i=0}^m \frac{s_i}{\sum_{i=0}^m s_i} = 1,$$

and so it follows that  $p_{m+1}$  is in fact in the affine span of  $S$ .

**Exercise 2.2.** Let  $\varphi$  be the isomorphism from  $\mathbb{R}^n$  to a subset of  $\mathbb{R}^k$ . Suppose  $A \subseteq \mathbb{R}^n$  is an affine set containing  $X$ . Then  $\varphi(X) \subseteq \varphi(A) \subseteq \mathbb{R}^k$ .

Moreover, we claim that  $\varphi(A)$  is affine. After all, for any  $\varphi(x), \varphi(x') \in \varphi(A)$  and any  $t \in \mathbb{R}$ , the point  $t\varphi(x) + (1-t)\varphi(x') = \varphi(tx + (1-t)x') \in \varphi(A)$  because  $A$  is affine.

This implies that the intersection of all affine sets in  $\mathbb{R}^n$  containing  $X$  must contain the intersection of all affine sets in  $\varphi(\mathbb{R}^n)$  containing  $\varphi(X)$ . Because  $\varphi$  is an isomorphism, using  $\varphi^{-1}$  gives the reverse inclusion. Thus the affine set spanned by  $X$  in  $\mathbb{R}^n$  is precisely the same as that spanned by  $X$  in  $\mathbb{R}^k$ .

**Exercise 2.3.** This is evident in the case  $n = 0$ .

Suppose it is true for  $n - 1$  and consider the canonical injection  $\iota : S^{n-1} \hookrightarrow S^n$  which takes  $(x_0, \dots, x_{n-1})$  to  $(x_1, \dots, x_{n-1}, 0)$ . It is obvious that we can pick  $n + 1$  affine independent points  $p_0, \dots, p_n$  in this embedding.

Now consider the point  $p_{n+1} = (0, \dots, 0, 1) \in S^n$ . Notice that the last coordinate of each  $p_i$  for  $i \neq n + 1$  is zero. Thus suppose we have  $s_i$  with  $\sum s_i p_i = 0$  and  $\sum s_i = 0$ . Then  $s_{n+1} = 0$ , and so this reduces to the  $n - 1$  case. Affine independence of  $\{p_0, \dots, p_n\}$  proves the result.

**Exercise 2.4.** Consider the map  $T'(x) = T(x) - T(0)$ . We claim that  $T'$  is a linear map.

Observe that  $S = \{e_i\} \cup \{0\}$  spans  $\mathbb{R}^n$ . Thus we can write any point as the affine sum of elements of  $S$ . Note that the coefficient of the zero vector is flexible, and so we have effectively no restrictions on the sum of the coefficients.

Consider arbitrary elements  $\sum r_i e_i + r \cdot 0$  and  $\sum s_i e_i + s \cdot 0$  in  $\mathbb{R}^n$ , where  $r = 1 - \sum r_i$  and similarly for  $s$ . Let  $R, S \in \mathbb{R}$ . Then note that

$$\begin{aligned} T' \left( R \sum r_i e_i + S \sum s_i e_i \right) &= T' \left( \sum (Rr_i + Ss_i) e_i \right) \\ &= T \left( \sum (Rr_i + Ss_i) e_i + \left( 1 - \sum (Rr_i + Ss_i) \right) \cdot 0 \right) - T(0) \\ &= R \sum r_i T(e_i) + S \sum s_i T(e_i) - R \sum r_i T(0) - S \sum s_i T(0). \end{aligned}$$

Considering the  $R$ -terms first, simply observe that we can add and subtract  $RT(0)$  to give us that

$$R \sum r_i T(e_i) - R \sum r_i T(0) = R \left( T \left( \sum r_i T(e_i) + r \cdot 0 \right) - T(0) \right).$$

<sup>1</sup>I usually use *simplices* as the plural of simplex, but Rotman doesn't; no matter.

This is simply  $RT'(\sum r_i e_i)$ . A similar result holds for the  $S$ -terms, from which we conclude that

$$T' \left( R \sum r_i e_i + S \sum s_i e_i \right) = RT' \left( \sum r_i e_i \right) + ST' \left( \sum s_i e_i \right),$$

proving linearity.

**Exercise 2.5.** This is obvious from the previous exercise and continuity of linear maps.

**Exercise 2.6.** Given two  $m$ -simplexes  $[p_0, \dots, p_m]$  and  $[q_0, \dots, q_m]$ , the map  $f$  taking  $p_i$  to  $q_i$  for every  $i$  is a homeomorphism. Bijectivity is obvious by the definition. Continuity is clear by how we extend  $f$  from  $\{p_i\}$  to  $[p_i]$ . Finally, the inverse is of the same form as  $f$ , only with the  $q_i$ 's taking the place of the  $p_i$ 's and vice versa; thus  $f^{-1}$  is also continuous.

**Exercise 2.7.** The following map works:

$$f : x \mapsto \frac{t_2 - t_1}{s_2 - s_1}(x - s_1) + t_1.$$

**Exercise 2.8.**