2 Simplexes[1]

Affine Spaces

Exercise 2.1. Note that there is a maximal affine independent subset S of A. This is directly implied by the fact that any set of greater than $n+1$ elements is not affine independent. Hence we can take an affine independent subset of A with maximum size (because the empty set is affine independent).

Write $S = \{p_0, \ldots, p_m\}$. Then let $p_{m+1} \in A \setminus S$. By maximality of S, we know that $S \cup \{p\}$ is not affine independent. Hence there exist s_i not all 0 such that

$$\sum_{i=0}^{m+1} s_i p_i = 0, \quad \sum_{i=0}^{m+1} s_i = 0.$$

Note that the second equation implies $\sum_{i=0}^{m} s_i \neq 0$ for some $i < m+1$. It follows then that

$$\sum_{i=0}^{m} \left(\frac{s_i}{\sum_{i=0}^{m} s_i} p_i \right) = p_{m+1}.$$

But we know that

$$\sum_{i=0}^{m} \frac{s_i}{\sum_{i=0}^{m} s_i} = 1,$$

and so it follows that p_{m+1} is in fact in the affine span of S.

Exercise 2.2. Let φ be the isomorphism from $\mathbb{R}^n$ to a subset of $\mathbb{R}^k$. Suppose $A \subseteq \mathbb{R}^n$ is an affine set containing X. Then $\varphi(X) \subseteq \varphi(A) \subseteq \mathbb{R}^k$.

Moreover, we claim that $\varphi(A)$ is affine. After all, for any $\varphi(x), \varphi(x') \in \varphi(A)$ and any $t \in \mathbb{R}$, the point $t\varphi(x) + (1-t)\varphi(x') = \varphi(tx + (1-t)x') \in \varphi(A)$ because A is affine.

This implies that the intersection of all affine sets in $\mathbb{R}^n$ containing X must contain the intersection of all affine sets in $\varphi(\mathbb{R}^n)$ containing $\varphi(X)$. Because φ is an isomorphism, using φ^{-1} gives the reverse inclusion. Thus the affine set spanned by X in $\mathbb{R}^n$ is precisely the same as that spanned by X in $\mathbb{R}^k$.

Exercise 2.3. This is evident in the case $n = 0$.

Suppose it is true for $n-1$ and consider the canonical injection $\iota : S^{n-1} \hookrightarrow S^n$ which takes $(x_0, \ldots, x_{n-1})$ to $(x_1, \ldots, x_{n-1}, 0)$. It is obvious that we can pick $n+1$ affine independent points $p_0, \ldots, p_n$ in this embedding.

Now consider the point $p_{n+1} = (0, \ldots, 0, 1) \in S^n$. Notice that the last coordinate of each p_i for $i \neq n+1$ is zero. Thus suppose we have s_i with $\sum s_i p_i = 0$ and $\sum s_i = 0$. Then $s_{n+1} = 0$, and so this reduces to the $n-1$ case. Affine independence of $\{p_0, \ldots, p_n\}$ proves the result.

Affine Maps

Exercise 2.4. Consider the map $T'(x) = T(x) - T(0)$. We claim that T' is a linear map.

Observe that $S = \{e_i\} \cup \{0\}$ spans $\mathbb{R}^n$. Thus we can write any point as the affine sum of elements of S. Note that the coefficient of the zero vector is flexible, and so we have effectively no restrictions on the sum of the coefficients.

Consider arbitrary elements $\sum r_i e_i + r \cdot 0$ and $\sum s_i e_i + s \cdot 0$ in $\mathbb{R}^n$, where $r = 1 - \sum r_i$ and similarly for s. Let $R, S \in \mathbb{R}$. Then note that

$$\begin{aligned} T'\left(R \sum r_i e_i + S \sum s_i e_i\right) &= T'\left(\sum (Rr_i + Ss_i) e_i\right) \\ &= T\left(\sum (Rr_i + Ss_i) e_i + \left(1 - \sum (Rr_i + Ss_i)\right) \cdot 0\right) - T(0) \\ &= R \sum r_i T(e_i) + S \sum s_i T(e_i) - R \sum r_i T(0) - S \sum s_i T(0). \end{aligned}$$

[1] I usually use *simplices* as the plural of simplex, but Rotman doesn't; no matter.

Considering the R-terms first, simply observe that we can add and subtract $RT(0)$ to give us that

$$R\sum r_iT(e_i) - R\sum r_iT(0) = R\left(T\left(\sum r_iT(e_i) + r\cdot 0\right) - T(0)\right).$$

This is simply $RT'\left(\sum r_ie_i\right)$. A similar result holds for the S-terms, from which we conclude that

$$T'\left(R\sum r_ie_i + S\sum s_ie_i\right) = RT'\left(\sum r_ie_i\right) + ST'\left(\sum s_ie_i\right),$$

proving linearity.

Exercise 2.5. This is obvious from the previous exercise and continuity of linear maps.

Exercise 2.6. Given two m-simplexes $[p_0,\dots,p_m]$ and $[q_0,\dots,q_m]$, the map f taking p_i to q_i for every i is a homeomorphism. Bijectivity is obvious by the definition. Continuity is clear by how we extend f from $\{p_i\}$ to $[p_i]$. Finally, the inverse is of the same form as f, only with the q_i's taking the place of the p_i's and vice versa; thus f^{-1} is also continuous.

Exercise 2.7. The following map works:

$$f : x \mapsto \frac{t_2 - t_1}{s_2 - s_1}(x - s_1) + t_1.$$

Exercise 2.8. Pick arbitrary $T(x), T(x') \in T(X)$ and observe that

$$tT(x) + (1-t)T(x') = T(tx + (1-t)x') \in T(X).$$

Thus $T(X)$ is affine if X is affine, and convex if X is convex. The second statement of the exercise follows by noting that ℓ is convex.

Exercise 2.9. Without loss of generality, we delete p_0. Now suppose that

$$\sum_{i=1}^m s_ip_i + sb = 0, \quad \sum_{i=1}^m s_i + s = 0.$$

Then we know by definition of the barycenter b that

$$\sum_{i=1}^m s_ip_i + \frac{s}{m+1}\sum_{i=0}^m p_i = 0.$$

Moreover, letting s_i' be the coefficient of p_i in the above equation, it is obvious that $\sum_{i=0}^m s_i' = s + \sum_{i=1}^m s_i = 0$. Thus $s_i' = 0$ for all i because $\{p_0,\dots,p_m\}$ was affine independent. But then we conclude that $0 = s_0' = \frac{s}{m+1}$, and so $s = 0$. For every $i \in \{1,\dots,m\}$, we have $0 = s_i' = \frac{s}{m+1} + s_i$. Thus $s = 0$ implies $s_i = 0$ for every i,a nd so it follows that $\{b, p_1,\dots,p_m\}$ is affine independent, as desired.

Exercise 2.10. Once again, suppose without loss of generality that $i = 0$. Then the map taking $\sum t_ip_i \in [p_0, p_1,\dots,p_m]$ to $(\sum_{i=1}^m t_ip_i, t_0)$ works. Note that this actually requires the affine independence of the p_i's, as well as the fact that the coefficients t_i are all between 0 and 1.

Exercise 2.11. Notice that $[0, e_1,\dots,e_n]$, where e_i are the standard basis vectors in $\mathbb{R}^n$, is an n-simplex. Thus there is a homeomorphism $[p_0,\dots,p_n] \to [0, e_1,\dots,e_n]$. If we translate the image by $\mathbf{v} = (-\frac{1}{4},\dots,-\frac{1}{4})$, then we can map the result to D^n by taking a radial mapping. In particular, this map will take

$$\begin{aligned} p_0 &\mapsto \frac{\mathbf{v}}{\|\mathbf{v}\|} \\ p_i &\mapsto \frac{e_i + \mathbf{v}}{\|e_i + \mathbf{v}\|} \text{ for } i \neq 0. \end{aligned}$$

Note that this extends to a homeomorphism.