11 Homotopy Groups

Function Spaces

No exercises!

Group Objects and Cogroup Objects

Exercise 11.1.

(i) By definition of a product, there is a unique morphism $\theta : (X, q_1, q_2) \to (C_1 \times C_2, p_1, p_2)$ in $\mathscr{C}$ making the diagram commute, namely $\theta = (q_1, q_2)$.

(ii) The objects are ordered triples (X, k_1, k_2) where X is a set and $k_i : C_i \to X$ are functions. Morphisms $\theta : (X, k_1, k_2) \to (Y, \ell_1, \ell_2)$ are functions $\theta : X \to Y$ making the following commute:

$$\begin{tikzcd} & X \arrow[dd, "\theta"] & \\ C_1 \arrow[ur, "k_1"] \arrow[dr, "\ell_1"'] & & C_1 \arrow[ul, "k_2"'] \arrow[dl, "\ell_2"] \\ & Y & \end{tikzcd}$$

Exercise 11.2. We first tackle **Ab**.

The map $\theta : X \to G_1 \oplus G_2$ in the product diagram is given by $\theta(g) = (q_1 g, q_2 g)$. Commutativity follows from the fact that $p_i(\theta(g)) = q_i(g)$. Uniqueness of θ follows from the fact that any other θ' must satisfy $\theta'(g) = (g_1, g_2)$ where $g_i = p_i(g_1, g_2) = q_i(g)$. Hence $\theta' = \theta$.

The map $\eta : G_1 \oplus G_2 \to X$ in the coproduct diagram is given by $(g, h) \mapsto k_1(g) + k_2(h)$, where $+$ denotes the operation in the abelian group X. We can easily check commutativity and uniqueness using the fact that η must be a group homomorphism.

Now, for **Grp**, note that the free product property on p. 173 is exactly the coproduct property. The same argument as in the abelian case shows that direct product is the product in **Grp**.

Exercise 11.3.

(i) We will show this for $\mathbf{Top}_*$. Suppose we have $((X, x), k_1, k_2)$. It is obvious that the map $\theta : (A_1 \vee A_2, *) \to (X, x)$, if it exists, must take $*$ to x, and $* \neq a_i \in j_i(A_i)$ to $k_i(a_i)$. We need only show that this map θ is continuous. (In contrast, the proof has already been completed for $\mathbf{Set}_*$; commutativity of the relevant diagram is obvious from the definition of θ.)

Suppose $U \subseteq X$ is closed. Note that $\theta^{-1}(U) \cap A_i = k_i^{-1}(U)$. (This statement is clear if $* \notin U$. If $* \in U$, then

$$\theta^{-1}(U) \cap A_i = (\theta^{-1}(U \setminus \{x\}) \cap A_i) \cup \{*\} = k_i^{-1}(U \setminus \{x\}) \cup \{a_i\} = k_i^{-1}(U),$$

which proves the statement anyway.) The definition of the topology of the wedge (see Example 8.9) implies that $\theta^{-1}(U)$ is closed. Hence θ is continuous, completing the proof.

(ii) Call this subset S. The map $f : A_1 \vee A_2 \to S$ which takes $a \in A_i$ (or, more accurately, $a \in j_i(A_i)$) to (a, a_2) if $i = 1$ and to (a_1, a) if $i = 2$ is continuous by the previous argument. It is clearly bijective and closed, since a closed set F in $A_1 \vee A_2$ is still closed in $A_1 \times A_2$. Thus it is a homeomorphism.

Exercise 11.4. Commutativity follows from the interchanging of C_1 and C_2 in the definition. To see associativity, consider the following diagram:

$$\begin{tikzcd}
& & (C_1 \times C_2) \times C_3 \arrow[dl, bend right] \arrow[ddrr, bend left] & & \\
& C_1 \times C_2 \arrow[dl] \arrow[dr] & & & \\
C_1 & & C_2 & & C_3 \\
& & X \arrow[ull] \arrow[uul, bend left] \arrow[u] \arrow[urr] & &
\end{tikzcd}$$

There is a unique map $X \to (C_1 \times C_2) \times C_3$ making this diagram commute.

Now define $p_1 : (C_1 \times C_2) \times C_3 \to C_1$ to be the composition of the red arrows below. Furthermore, the product property of $C_2 \times C_3$ implies the existence of the following blue and green arrows:

$$\begin{tikzcd}
& & (C_1 \times C_2) \times C_3 \arrow[dl, bend right, red] \arrow[dr, blue] \arrow[ddrr, bend left] & & \\
& C_1 \times C_2 \arrow[dl, red] \arrow[dr] & & C_2 \times C_3 \arrow[dl, green] \arrow[dr, green] & \\
C_1 & & C_2 & & C_3 \\
& & X \arrow[ull] \arrow[uul, bend left] \arrow[u] \arrow[urr] & &
\end{tikzcd}$$

Let p_2 be the blue arrow. The fact that there is still the same unique map $X \to (C_1 \times C_2) \times C_2$ making this commute, then, implies that $(C_1 \times C_2) \times C_3$ is the product of C_1 and $C_2 \times C_3$, thus proving associativity.

Exercise 11.5.

(i) We would like to find $f_1 \times f_2$ making the following commute:

$$\begin{tikzcd}
C_2 \arrow[r, "f_2"] & D_2 \\
C_1 \times C_2 \arrow[u] \arrow[d] \arrow[r, "f_1 \times f_2"] & D_1 \times D_2 \arrow[u] \arrow[d] \\
C_1 \arrow[r, "f_1"] & D_1
\end{tikzcd}$$

But the existence of maps $C_1 \times C_2 \to C_i \to D_i$ implies, by the product property of $D_1 \times D_2$, a unique map $f_1 \times f_2$ into $D_1 \times D_2$ making the diagram commute.

(ii) Same idea.

Exercise 11.6.

(i) Note that Δ_X is the unique map making the red part of the diagram commute, while $q_1 \times q_2$ is the unique map making the blue part commute:

$$\begin{tikzcd}
& D_1 \times D_2 \arrow[dl] \arrow[dr] & \\
D_1 & & D_2 \\
& X \times X \arrow[uu, dashed, "q_1 \times q_2"] \arrow[dl] \arrow[dr] & \\
X \arrow[uu] & & X \arrow[uu] \\
& X \arrow[uu, dashed, "\Delta_X"] \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

But of course, since the maps $q_i \circ 1_X = X \to X \to D_i$ are equal to simply the maps $q_i : X \to D_i$, we know that the unique map $X \mapsto D_1 \times D_2$ making this entire diagram commute is (q_1, q_2). Uniqueness implies that (q_1, q_2) must be equal to $(q_1 \times q_2)\Delta_X$.

(ii) This is the same idea.

(iii) We already showed the first statement. For the second, notice that $\nabla_B(f \times g) = (f, g)$. But $(f, g)\Delta_A(a) = (f(a), g(a)) = (f + g)(a)$ because $A \oplus B = A \amalg B$.

Exercise 11.7.

(i) Everything follows from the hint, except that we must verify that $1_{X\times Z}$ and $\theta\lambda$ complete the given diagram. Commutativity of the left triangle is obvious in both cases. To see that $q1_{X\times Z} = t$, note that Z being terminal implies that $q = t$. To show that $q\theta\lambda = t$, note that $q\theta\lambda : X \times Z \to X \to X \times Z \to Z$. Thus Z being terminal again implies the result.

Now θ and λ are inverses, and so $X \times Z$ and X are equivalent.

(ii) This is the dualized version of the previous part.

Exercise 11.8. We will use the definition of a group object. If G is a group object, then the terminal object Z is the one-element group $\{z\}$. With standard notation, let $e \in G$ be $\varepsilon(z)$. Note that $\mu(g, e) = \mu(e, g) = g$. Now the fact that μ is a homomorphism implies that

$$\mu(g_1, g_2) = \mu(g_1, e)\mu(e, g_2) = g_1 g_2,$$

so that μ must be the multiplication operation of G. Using this, we can show that η is indeed the inverse operation: $\eta(g) = g^{-1}$. In particular, we know that

$$g \cdot \eta(g) = (\mu \circ (1, \eta))(g) = e$$

for any g.

Now we know that η must be a homomorphism. Thus

$$g^{-1}h^{-1} = \eta(g)\eta(h) = \eta(gh) = (gh)^{-1} = h^{-1}g^{-1}.$$

Obviously this proves that G is abelian.

Exercise 11.9. The initial object A in both cases is the empty set. The existence of a morphism $e : C \to A$ implies that $C = \emptyset$. It is easy to verify that $\emptyset$ is a cogroup object, which completes the proof.

Exercise 11.10. This time we use the co-identity property. Let $x \in C$. Then $m(x)$ is either in the first coordinate or the second (or it is the basepoint $*$). Thus either $1 \amalg e$ or $e \amalg 1$ will take $m(x)$ to $e(C) = * \in A \subseteq C \amalg A$.

Now we compare this with the maps in the co-identity triangles. In particular, if $x \neq *$ is an element of C, then the maps $C \to C \amalg A$ and $C \to A \amalg C$ take x to itself, not $x \mapsto *$. This contradicts commutativity, so $C = \{*\}$.

Exercise 11.11.

(i) We prove this only for group objects; the result for cogroup objects simply involves oppositely oriented arrows.

Identities follow from the commutativity of the following:

$$\begin{array}{ccc} G\times G & \xrightarrow{1_G\times 1_G} & G\times G \\ \mu\big\downarrow & & \big\downarrow\mu \\ G & \xrightarrow[1_G]{} & G. \end{array}$$

Associativity follows from associativity of $\mathscr{C}$. Composition follows from commutativity of the following:

$$\begin{array}{ccccc} G\times G & \xrightarrow{f\times f} & H\times H & \xrightarrow{g\times g} & J\times J \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ G & \xrightarrow[f]{} & H & \xrightarrow[g]{} & J, \end{array}$$

as well as the fact that

$$(g\times g)\circ(f\times f) = gf\circ gf.$$

(ii) The first statement follows from Theorem 11.4.

For the second statement, we must show that

$$f_*(M_X^G(p,q)) = M_X^H(f_*(p), f_*(q)),$$

where $p, q \in \mathrm{Hom}(X, G)$. But we know that

$$M_X^G(p,q) = \mu^G(p,q) \in \mathrm{Hom}(X,G),$$

so that

$$f\circ M_X^G(p,q) : x \mapsto f(\mu^G(p(x), q(x))).$$

On the other hand we know that

$$M_X^H(f_*(p), f_*(q)) = \mu^H(fp, fq)$$

is the map taking

$$x \mapsto \mu^H(fp(x), fq(x)).$$

It thus suffices to show that

$$f(\mu^G(p(x), q(x)) = \mu^H(fp(x), fq(x)).)$$

But following $(p(x), q(x)) \in G\times G$ in the special diagram implies the result.

Exercise 11.12. That every abelian group is a group object is clear by Exercise 11.8. To see that it is a cogroup object, define $e : g \mapsto a$ where $A = \{a\}$, $m : g \mapsto (g,g)$, and $h : g \mapsto -g$. The axioms are easy to check.

Exercise 11.13. We will show that $\mathrm{Hom}(F, -)$ takes values in groups, where F is a finitely generated free group. Let $\{x_1, \ldots, x_n\}$ be a basis for F. Now consider the following function $P_G : \mathrm{Hom}(F,G)\times\mathrm{Hom}(F,G) \to \mathrm{Hom}(F,G)$:

$$P_G : (f,g) \mapsto (x_i \mapsto f(x_i)g(x_i)).$$

We will show that this gives $\mathrm{Hom}(F,G)$ a group structure.

Note that an element of $\mathrm{Hom}(F,G)$ is completely determined by where it sends each x_i. Thus P_G is well-defined. Now suppose $\varphi : G \to H$, so that $\varphi_* : \mathrm{Hom}(F,G) \to \mathrm{Hom}(F,H)$. Then we need to show that

$$\varphi_*(P_G(f,g)) = P_H(\varphi_*(f), \varphi_*(g)).$$

The left side takes

$$x_i \mapsto f(x_i)g(x_i) \mapsto \varphi(f(x_i)g(x_i)).$$

On the other hand, the right side takes

$$x_i \mapsto (\varphi f(x_i), \varphi g(x_i)) \mapsto \varphi f(x_i) \cdot \varphi g(x_i).$$

But these are equal because φ is a homeomorphism.

Exercise 11.14. This is easy; we can even use the same functions/morphisms.

Loop Space and Suspension

Exercise 11.15. We would like to show that the following commutes for all $f : A' \to A$:

$$\begin{array}{ccc} \mathrm{Hom}(A \otimes Y, C) & \xrightarrow{(f\otimes 1)^*} & \mathrm{Hom}(A' \otimes Y, C) \\ \tau_{AC}\big\downarrow & & \big\downarrow\tau_{A'C} \\ \mathrm{Hom}(A, \mathrm{Hom}(Y, C)) & \xrightarrow[f^*]{} & \mathrm{Hom}(A', \mathrm{Hom}(Y, C)), \end{array}$$

where $\tau_{AC}(\varphi) = \varphi^{\#}$.

First, we look at the lower path $f^* \circ \tau_{AC}$. If $\varphi : A \otimes Y \to C$ takes (a, y) to $\varphi(a, y)$, then $\tau_{AC}(\varphi) = \varphi^{\#}$ takes $a \in A$ to the map $\varphi_a \in \mathrm{Hom}(Y, C)$ defined by $\varphi_a(y) = \varphi(a, y)$. Thus $\varphi^{\#} f : A' \to \mathrm{Hom}(Y, C)$ is defined by

$$\varphi^{\#} f : f' \mapsto a \mapsto \varphi_a,$$

where $a = f(a')$.

On the other hand, the upper path takes φ to the map

$$[\varphi(f \otimes 1)]^{\#} : A' \to \mathrm{Hom}(Y, C)$$

defined by taking

$$a' \mapsto [\psi_{a'} : y \mapsto \varphi(f(a'), 1(y))].$$

Of course, these are the same since $f(a') = a$, proving commutativity.

The second square is similar.

Exercise 11.16. We'll show the first square, namely commutativity of

$$\begin{array}{ccc} \mathrm{Hom}(GA, C) & \xrightarrow{(Gf)^*} & \mathrm{Hom}(GA', C) \\ \tau_{AC}\big\downarrow & & \big\downarrow\tau_{A'C} \\ \mathrm{Hom}(A, C) & \xrightarrow[f^*]{} & \mathrm{Hom}(A', C), \end{array}$$

where τ_{AC} takes $\varphi : GA \to C$ to $\varphi|_A$. But commutativity is obvious, since both paths end up taking φ to $\varphi f : A' \to C$, where the maps are as sets.

Exercise 11.17. We will show this for G; the statement for F amounts to dualizing the following argument.

Adjointness implies that there is a bijection τ_{AC} between $\mathrm{Hom}(FA, C)$ and $\mathrm{Hom}(A, GC)$. Hence consider the two diagrams below; the left one is in $\mathscr{C}$ and the right one is in $\mathscr{A}$:

$$\begin{tikzcd} & FX \arrow[dl, "\tilde{q}_1"'] \arrow[dr, "\tilde{q}_2"] \arrow[dd, dashed, "\tilde{\theta}"] & \\ C_1 & & C_2 \\ & C \arrow[ul, "p_1"] \arrow[ur, "p_2"'] & \end{tikzcd} \qquad \begin{tikzcd} & X \arrow[dl, "q_1"'] \arrow[dr, "q_2"] \arrow[dd, dashed, "\theta"] & \\ GC_1 & & GC_2 \\ & GC \arrow[ul, "Gp_1"] \arrow[ur, "Gp_2"'] & \end{tikzcd}$$

Here, we let $\theta : X \to GC$ be the morphism corresponding to $\tilde{\theta}$ under τ_{XC}, and we let $\tilde{q}_i$ be the morphism corresponding to q_i under the bijection τ_{XC_i}. We claim that θ completes the diagram on the right. To see this, use the fact that $(Gg)_*\tau = \tau g_*$. Now if $g = p_1$, then

$$\tau g_*(\tilde{\theta}) = \tau(p_1\tilde{\theta}) = \tau(\tilde{q}_1) = q_1.$$

Now we must show that θ is the unique map making the product diagram on the right commute. Suppose η were another possible map. Define $\tilde{\eta} = \tau^{-1}(\eta)$. We will show that $\tilde{\eta} = \tilde{\theta}$, so the product diagram in $\mathscr{C}$ and the fact that τ is a bijection will imply that $\eta = \theta$.

But notice that

$$((Gp_1)_* \circ \tau)(\tilde{\eta}) = (Gp_1) * (\eta) = (Gp_1) * (\theta) = ((Gp_1)_* \circ \tau)(\tilde{\theta}) = (\tau \circ (p_1)_*)(\tilde{\theta}).$$

But naturality implies that

$$((Gp_1)_* \circ \tau)(\tilde{\eta}) = (\tau \circ (p_1)_*)(\tilde{\eta}).$$

It thus follows that

$$\tau(p_1\tilde{\theta}) = \tau(p_1\tilde{\eta}).$$

Since τ is a bijection, it follows that

$$p_1\tilde{\theta} = p_1\tilde{\eta} = \tilde{q}_1.$$

Thus $\tilde{\eta}$ completes the product diagram in $\mathscr{C}$, so that $\tilde{\eta} = \tilde{\theta}$, proving the result.

Exercise 11.18. This is exactly stereographic projection (or the reverse of it).

Exercise 11.19. Note that J^n is homeomorphic to $\mathbb{I}^n \setminus \{N\}$, where N is some fixed point. Hence $(J^n)^\infty \approx \mathbb{I}^n$.

Exercise 11.20. Consider the map taking A to ∞ and taking $x \in X \setminus A$ to itself. This is obviously a homeomorphism.

Exercise 11.21. We have the following:

$$S^m \wedge S^n = (\mathbb{R}^m)^\infty \wedge (\mathbb{R}^n)^\infty = (\mathbb{R}^{m+n})^\infty = S^{m+n}.$$

Exercise 11.22. We have

$$\mathbb{I}^n \wedge \mathbb{I} = (J^n)^\infty \wedge J^\infty = (J^{n+1})^\infty = \mathbb{I}^{n+1}.$$

Homotopy Groups

Exercise 11.23. Let $F : \beta \simeq y_0$ be a homotopy. We would like to show that

$$\begin{aligned} \beta_* : \pi_n(X, x_0) &\to \pi_n(Y, y_0) \\ [\alpha] &\mapsto [\beta \circ \alpha]. \end{aligned}$$

To do so, we must show that $\beta \circ \alpha$ is nullhomotopic rel $\dot{\mathbb{I}}^n$. Consider the map

$$\begin{aligned} F \circ (\alpha \times \mathrm{id}_{\mathbb{I}}) : \mathbb{I}^n \times \mathbb{I} &\to Y \\ (u, t) &\mapsto F(\alpha(u), t). \end{aligned}$$

Obviously, this is a homotopy between $\beta(\alpha(u))$ and the constant map at y_0. To see that this is rel $\dot{\mathbb{I}}^n$, simply note that $u \in \dot{\mathbb{I}}^n$ implies that $F(\alpha(u), t) = F(x_0, t) = y_0$, since α and F are pointed maps.

Exercise 11.24. We have the following chain of equalities (note that some equalities are up to isomorphism or homotopy, depending on the category):

$$\begin{aligned} \pi_n(X \times Y) &= [S^n, X \times Y] \\ &= \Omega(X \times Y) \\ &= \Omega X \times \Omega Y \\ &= [S^n, X] \times [S^n, Y] \\ &= [S^n, X] \oplus [S^n, Y] = \pi_n(X) \oplus \pi_n(Y). \end{aligned}$$

Since $\pi_n(S^1) = 0$, it follows that $\pi_n(T) = \pi_n(S^1) \times \pi_n(S^1) = 0$.

Exercise 11.25. This follows from Theorem 11.29 and the fact that S^n covers $\mathbb{R}P^n$.

Exercise 11.26. Note that Theorem 10.54(i) applies because locally path-connected and contractible implies connected. Thus X is a covering space for X/G, and so Theorem 11.29 implies that $\pi_n(X) \cong \pi_n(X/G)$. But $\pi_n(X) = 0$ for $n \geq 2$ since X is contractible.

Exercise 11.27. This follows almost immediately from the hint. To see that $*$ and $\circ$ coincide, note that

$$f * g = (f \circ e) * (e \circ g) = (f * e) \circ (e * g) = f \circ g.$$

To see commutativity, we need only check that $f * g = g \circ f$. But this follows because

$$f * g = (e \circ f) * (g \circ e) = (e * g) \circ (f * e) = g \circ f,$$

as desired.

Exercise 11.28.

(i) We follow the path laid out in the hints. First, note that, if $q \in Q$, then

$$\mu(f, e)(q) = \mu(f(q), p_0) = (\mu(-, p_0) \circ f)(q).$$

Thus it follows that

$$[f] * [e] = [\mu(f, e)] = [\mu(-, p_0) \circ f] = [1_P \circ f] = [f],$$

where we use the property of an H-space. Similarly, we can show that $[e] * [f] = [f]$.

Now we must show that $[f] \circ [e] = [f]$. But $[f] \circ [e] = [(f, e)m]$, and $(f, e)m$ takes q to $f(q)$ if $m(q)$ is in the first coordinate of $Q \vee Q$, and takes q to p_0 if $m(q)$ is in the second coordinate. Letting q_1 be as in the definition of an H'-group, i.e., letting q_1 be the projection to the first coordinate, we see that $q_1 m$ takes q to q if $m(q)$ is in the first coordinate and takes q to q_0 otherwise. Thus fq_1m takes q to either $f(q)$ or p_0, depending on the coordinate of q, and so it follows that $fq_1m = (f, e)m$. But of course $q_1 m \simeq 1_Q$, from which the conclusion follows.

To show the second condition of Exercise 11.27, note first that

$$([f] \circ [h]) * ([g] \circ [j]) = [\mu((f, h)m, (g, j)m)].$$

The map on the right side takes q to either $\mu(fq, gq)$ or $\mu(hq, jq)$, depending on where $m(q)$ is in the first or second coordinate of $Q \vee Q$. On the other hand, we have

$$([f] * [g]) \circ ([h] * [j]) = [(\mu(f, g), \mu(h, j))m],$$

which takes

$$q \mapsto m(q) \mapsto \begin{cases} \mu(fq, gq) \\ \mu(hq, jq) \end{cases},$$

which is the exact same. Thus condition (ii) is satisfied, and the previous exercise proves the result.

(ii) Note that $[\Sigma^2 X, Y] = [\Sigma X, \Omega Y]$ since Σ and Ω are adjoint functors. Now since ΣX is an H'-group and ΩY is an H-group in the category, hence an H-space, it follows from the previous part that $[\Sigma X, \Omega Y]$ is abelian.

Exercise 11.29. First, we must show that this is well-defined. Suppose $F : f \simeq g \operatorname{rel}\{s_n\}$. We claim that $\Sigma f \simeq \Sigma g$ with the map $G : (a, b, t) \mapsto (F(a, t), b)$. But this is easy to verify because $G(a, b, 0) = (F(a, 0), b) = (f(a), b) = (\Sigma f)(a, b)$, and similarly for $G(a, b, 1)$.

Now, we will show that it is a homomorphism. Let $m_n : S^n \to S^n \vee S^n$ be comultiplication. Then $[f][g] = [(f, g)m]$. We would like to show that

$$[\Sigma((f, g)m_n)] = [\Sigma f][\Sigma g] = [(\Sigma f, \Sigma g)m_{n+1}].$$

To see this, note that the left side takes (a, b) to $(f(a), b)$ or $(g(a), b)$, depending on which $S^n \wedge S^1$-component $m_{n+1}(a, b)$ belongs to in $(S^n \wedge S^1) \vee (S^n \wedge S^1)$. On the other hand, we know that $\Sigma((f, g)m_n)$ takes (a, b) to $(((f, g)m_n)(a), b)$. This first coordinate $((f, g)m_n)(a)$ is $f(a)$ or $g(a)$, depending on which "half" m_n takes a to. Using a rotation to make sure the two halves which m_n and m_{n+1} determine line up (after projecting S^{n+1} down to the equator, which is S^n), it is easy to show that these maps are homotopic.

Exercise 11.30. Any map $Y \to X$ is homotopic to some simplicial approximation $Sd^q L \to K$. Obviously, there are only countably many simplicial approximations, since there are only finitely many vertices of each $\mathrm{Sd}^q\, L$ and of K. Hence $[Y, X]$ is countable. Thus $\pi_n(X) = [S^n, X]$ must be countable.

Exact Sequences

Exercise 11.31. This is the exact same argument as part (ii) of **??**.

Exercise 11.32. The same diagram chase remark applies, just changing H to π.

Exercise 11.33. We have the following long exact sequence:

$$\ldots \longrightarrow \pi_{n+1}(X, X) \xrightarrow{d} \pi_n(X) \xrightarrow{\mathrm{id}} \pi_n(X) \xrightarrow{j_*} \pi_n(X, X) \xrightarrow{d} \pi_{n-1}(X) \xrightarrow{\mathrm{id}} \ldots$$

Now we know that $\ker j_* = \operatorname{im} \mathrm{id} = \pi_n(X)$, so that $\operatorname{im} j_* = 0$. Hence $\ker d = 0$. But $\operatorname{im} d = \ker \mathrm{id} = 0$, and so $\ker d = \pi_n(X, X)$. The result now follows.

Fibrations

Exercise 11.34.