# 11 Homotopy Groups

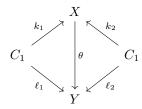
## **Function Spaces**

No exercises!

## **Group Objects and Cogroup Objects**

#### Exercise 11.1.

- (i) By definition of a product, there is a unique morphism  $\theta: (X, q_1, q_2) \to (C_1 \times C_2, p_1, p_2)$  in  $\mathscr{C}$  making the diagram commute, namely  $\theta = (q_1, q_2)$ .
- (ii) The objects are ordered triples  $(X, k_1, k_2)$  where X is a set and  $k_i : C_i \to X$  are functions. Morphisms  $\theta : (X, k_1, k_2) \to (Y, \ell_1, \ell_2)$  are functions  $\theta : X \to Y$  making the following commute:



#### Exercise 11.2. We first tackle Ab.

The map  $\theta: X \to G_1 \oplus G_2$  in the product diagram is given by  $\theta(g) = (q_1g, q_2g)$ . Commutativity follows from the fact that  $p_i(\theta(g)) = q_i(g)$ . Uniqueness of  $\theta$  follows from the fact that any other  $\theta'$  must satisfy  $\theta'(g) = (g_1, g_2)$  where  $g_i = p_i(g_1, g_2) = q_i(g)$ . Hence  $\theta' = \theta$ .

The map  $\eta: G_1 \oplus G_2 \to X$  in the coproduct diagram is given by  $(g,h) \mapsto k_1(g) + k_2(h)$ , where + denotes the operation in the abelian group X. We can easily check commutativity and uniqueness using the fact that  $\eta$  must be a group homomorphism.

Now, for **Grp**, note that the free product property on p. 173 is exactly the coproduct property. The same argument as in the abelian case shows that direct product is the product in **Grp**.

#### Exercise 11.3.

(i) We will show this for  $\mathbf{Top}_*$ . Suppose we have  $((X, x), k_1, k_2)$ . It is obvious that the map  $\theta : (A_1 \vee A_2, *) \to (X, x)$ , if it exists, must take \* to x, and  $* \neq a_i \in j_i(A_i)$  to  $k_i(a_i)$ . We need only show that this map  $\theta$  is continuous. (In contrast, the proof has already been completed for  $\mathbf{Set}_*$ ; commutativity of the relevant diagram is obvious from the definition of  $\theta$ .)

Suppose  $U \subseteq X$  is closed. Note that  $\theta^{-1}(U) \cap A_i = k_i^{-1}(U)$ . (This statement is clear if  $* \notin U$ . If  $* \in U$ , then

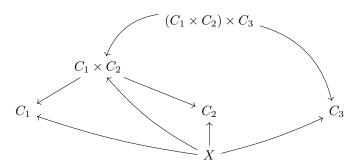
$$\theta^{-1}(U) \cap A_i = (\theta^{-1}(U \setminus \{x\}) \cap A_i) \cup \{*\} = k_i^{-1}(U \setminus \{x\}) \cup \{a_i\} = k_i^{-1}(U)$$

which proves the statement anyway.) The definition of the topology of the wedge (see Example 8.9) implies that  $\theta^{-1}(U)$  is closed. Hence  $\theta$  is continuous, completing the proof.

(ii) Call this subset S. The map  $f: A_1 \vee A_2 \to S$  which takes  $a \in A_i$  (or, more accurately,  $a \in j_i(A_i)$ ) to  $(a, a_2)$  if i = 1 and to  $(a_1, a)$  if i = 2 is continuous by the previous argument. It is clearly bijective and closed, since a closed set F in  $A_1 \vee A_2$  is still closed in  $A_1 \times A_2$ . Thus it is a homeomorphism.

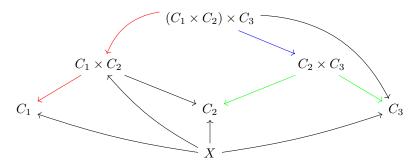
**Exercise 11.4.** Commutativity follows from the interchanging of  $C_1$  and  $C_2$  in the definition. To see

associativity, consider the following diagram:



There is a unique map  $X \to (C_1 \times C_2) \times C_3$  making this diagram commute.

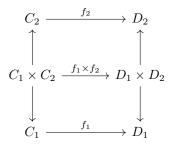
Now define  $p_1:(C_1\times C_2)\times C_3\to C_1$  to be the composition of the red arrows below. Furthermore, the product property of  $C_2\times C_3$  implies the existence of the following blue and green arrows:



Let  $p_2$  be the blue arrow. The fact that there is still the same unique map  $X \to (C_1 \times C_2) \times C_2$  making this commute, then, implies that  $(C_1 \times C_2) \times C_3$  is the product of  $C_1$  and  $C_2 \times C_3$ , thus proving associativity.

### Exercise 11.5.

(i) We would like to find  $f_1 \times f_2$  making the following commute:



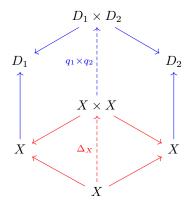
But the existence of maps  $C_1 \times C_2 \to C_i \to D_i$  implies, by the product property of  $D_1 \times D_2$ , a unique map  $f_1 \times f_2$  into  $D_1 \times D_2$  making the diagram commute.

(ii) Same idea.

### Exercise 11.6.

(i) Note that  $\Delta_X$  is the unique map making the red part of the diagram commute, while  $q_1 \times q_2$  is the unique

map making the blue part commute:



But of course, since the maps  $q_i \circ 1_X = X \to X \to D_i$  are equal to simply the maps  $q_i : X \to D_i$ , we know that the unique map  $X \mapsto D_1 \times D_2$  making this entire diagram commute is  $(q_1, q_2)$ . Uniqueness implies that  $(q_1, q_2)$  must be equal to  $(q_1 \times q_2)\Delta_X$ .

- (ii) This is the same idea.
- (iii) We already showed the first statement. For the second, notice that  $\nabla_B(f \times g) = (f, g)$ . But  $(f, g)\Delta_A(a) = (f(a), g(a)) = (f + g)(a)$  because  $A \oplus B = A \coprod B$ .

#### Exercise 11.7.

(i) Everything follows from the hint, except that we must verify that  $1_{X\times Z}$  and  $\theta\lambda$  complete the given diagram. Commutativity of the left triangle is obvious in both cases. To see that  $q1_{X\times Z}=t$ , note that Z being terminal implies that q=t. To show that  $q\theta\lambda=t$ , note that  $q\theta\lambda: X\times Z\to X\to X\times Z\to Z$ . Thus Z being terminal again implies the result.

Now  $\theta$  and  $\lambda$  are inverses, and so  $X \times Z$  and X are equivalent.

(ii) This is the dualized version of the previous part.

**Exercise 11.8.** We will use the definition of a group object. If G is a group object, then the terminal object Z is the one-element group  $\{z\}$ . With standard notation, let  $e \in G$  be  $\varepsilon(z)$ . Note that  $\mu(g,e) = \mu(e,g) = g$ . Now the fact that  $\mu$  is a homomorphism implies that

$$\mu(q_1, q_2) = \mu(q_1, e)\mu(e, q_2) = q_1q_2,$$

so that  $\mu$  must be the multiplication operation of G. Using this, we can show that  $\eta$  is indeed the inverse operation:  $\eta(g) = g^{-1}$ . In particular, we know that

$$q \cdot \eta(q) = (\mu \circ (1, \eta))(q) = e$$

for any g.

Now we know that  $\eta$  must be a homomorphism. Thus

$$g^{-1}h^{-1} = \eta(g)\eta(h) = \eta(gh) = (gh)^{-1} = h^{-1}g^{-1}.$$

Obviously this proves that G is abelian.

**Exercise 11.9.** The initial object A in both cases is the empty set. The existence of a morphism  $e: C \to A$  implies that  $C = \emptyset$ . It is easy to verify that  $\emptyset$  is a cogroup object, which completes the proof.

**Exercise 11.10.** This time we use the co-identity property. Let  $x \in C$ . Then m(x) is either in the first coordinate or the second (or it is the basepoint \*). Thus either 1 II e or  $e \coprod 1$  will take m(x) to  $e(C) = * \in A \subset C \coprod A$ .

Now we compare this with the maps in the co-identity triangles. In particular, if  $x \neq *$  is an element of C, then the maps  $C \to C \coprod A$  and  $C \to A \coprod C$  take x to itself, not  $x \mapsto *$ . This contradicts commutativity, so  $C = \{*\}$ .

#### Exercise 11.11.

 We prove this only for group objects; the result for cogroup objects simply involves oppositely oriented arrows.

Identities follow from the commutativity of the following:

$$\begin{array}{ccc} G \times G \xrightarrow{1_G \times 1_G} G \times G \\ \downarrow^{\mu} & \downarrow^{\mu} \\ G \xrightarrow{1_G} G. \end{array}$$

Associativity follows from associativity of  $\mathscr{C}$ . Composition follows from commutativity of the following:

$$\begin{array}{cccc} G \times G & \xrightarrow{f \times f} & H \times H & \xrightarrow{g \times g} & J \times J \\ \downarrow & & \downarrow & & \downarrow \\ G & \xrightarrow{f} & H & \xrightarrow{g} & J, \end{array}$$

as well as the fact that

$$(g \times g) \circ (f \times f) = gf \circ gf.$$

(ii) The first statement follows from Theorem 11.4.

For the second statement, we must show that

$$f_*(M_X^G(p,q)) = M_X^H(f_*(p), f_*(q)),$$

where  $p, q \in \text{Hom}(X, G)$ . But we know that

$$M_X^G(p,q)=\mu^G(p,q)\in \operatorname{Hom}(X,G),$$

so that

$$f \circ M_X^G(p,q) : x \mapsto f(\mu^G(p(x),q(x))).$$

On the other hand we know that

$$M_X^H(f_*(p), f_*(q)) = \mu^H(fp, fq)$$

is the map taking

$$x \mapsto \mu^H(fp(x), fq(x)).$$

It thus suffices to show that

$$f(\mu^G(p(x),q(x)) = \mu^H(fp(x),fq(x)).)$$

But following  $(p(x), q(x)) \in G \times G$  in the special diagram implies the result.

**Exercise 11.12.** That every abelian group is a group object is clear by Exercise 11.8. To see that it is a cogroup object, define  $e: g \mapsto a$  where  $A = \{a\}, m: g \mapsto (g,g),$  and  $h: g \mapsto -g$ . The axioms are easy to check.

**Exercise 11.13.** We will show that  $\operatorname{Hom}(F, -)$  takes values in groups, where F is a finitely generated free group. Let  $\{x_1, \ldots, x_n\}$  be a basis for F. Now consider the following function  $P_G : \operatorname{Hom}(F, G) \times \operatorname{Hom}(F, G) \to \operatorname{Hom}(F, G)$ :

$$P_G:(f,q)\mapsto (x_i\mapsto f(x_i)q(x_i)).$$

We will show that this gives Hom(F, G) a group structure.

Note that an element of  $\operatorname{Hom}(F,G)$  is completely determined by where it sends each  $x_i$ . Thus  $P_G$  is well-defined. Now suppose  $\varphi:G\to H$ , so that  $\varphi_*:\operatorname{Hom}(F,G)\to\operatorname{Hom}(F,H)$ . Then we need to show that

$$\varphi_*(P_G(f,g)) = P_H(\varphi_*(f), \varphi_*(g)).$$

The left side takes

$$x_i \mapsto f(x_i)g(x_i) \mapsto \varphi(f(x_i)g(x_i)).$$

On the other hand, the right side takes

$$x_i \mapsto (\varphi f(x_i), \varphi g(x_i)) \mapsto \varphi f(x_i) \cdot \varphi g(x_i).$$

But these are equal because  $\varphi$  is a homeomorphism.

Exercise 11.14. This is easy; we can even use the same functions/morphisms.

## **Loop Space and Suspension**

**Exercise 11.15.** We would like to show that the following commutes for all  $f: A' \to A$ :

$$\begin{array}{ccc} \operatorname{Hom}(A \otimes Y, C) & \xrightarrow{\quad (f \otimes 1)^* \quad} \operatorname{Hom}(A' \otimes Y, C) \\ & & \downarrow \tau_{A'C} \\ \operatorname{Hom}(A, \operatorname{Hom}(Y, C)) & \xrightarrow{\quad f^* \quad} \operatorname{Hom}(A', \operatorname{Hom}(Y, C)), \end{array}$$

where  $\tau_{AC}(\varphi) = \varphi^{\#}$ .

First, we look at the lower path  $f^* \circ \tau_{AC}$ . If  $\varphi : A \otimes Y \to C$  takes (a, y) to  $\varphi(a, y)$ , then  $\tau_{AC}(\varphi) = \varphi^{\#}$  takes  $a \in A$  to the map  $\varphi_a \in \text{Hom}(Y, C)$  defined by  $\varphi_a(y) = \varphi(a, y)$ . Thus  $\varphi^{\#}f : A' \to \text{Hom}(Y, C)$  is defined by

$$\varphi^{\#}f: f' \mapsto a \mapsto \varphi_a,$$

where a = f(a').

On the other hand, the upper path takes  $\varphi$  to the map

$$[\varphi(f\otimes 1)]^\#:A'\to \operatorname{Hom}(Y,C)$$

defined by taking

$$a' \mapsto [\psi_{a'} : y \mapsto \varphi(f(a'), 1(y))].$$

Of course, these are the same since f(a') = a, proving commutativity. The second square is similar.

Exercise 11.16. We'll show the first square, namely commutativity of

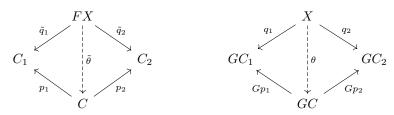
$$\operatorname{Hom}(GA,C) \xrightarrow{(Gf)^*} \operatorname{Hom}(GA',C)$$

$$\downarrow^{\tau_{A'C}} \qquad \qquad \downarrow^{\tau_{A'C}}$$

$$\operatorname{Hom}(A,C) \xrightarrow{f^*} \operatorname{Hom}(A',C),$$

where  $\tau_{AC}$  takes  $\varphi: GA \to C$  to  $\varphi|_A$ . But commutativity is obvious, since both paths end up taking  $\varphi$  to  $\varphi f: A' \to C$ , where the maps are as sets.

**Exercise 11.17.** We will show this for G; the statement for F amounts to dualizing the following argument. Adjointness implies that there is a bijection  $\tau_{AC}$  between  $\operatorname{Hom}(FA,C)$  and  $\operatorname{Hom}(A,GC)$ . Hence consider the two diagrams below; the left one is in  $\mathscr C$  and the right one is in  $\mathscr A$ :



Here, we let  $\theta: X \to GC$  be the morphism corresponding to  $\tilde{\theta}$  under  $\tau_{XC}$ , and we let  $\tilde{q}_i$  be the morphism corresponding to  $q_i$  under the bijection  $\tau_{XC_i}$ . We claim that  $\theta$  completes the diagram on the right. To see this, use the fact that  $(Gg)_*\tau = \tau g_*$ . Now if  $g = p_1$ , then

$$\tau g_*(\tilde{\theta}) = \tau(p_1\tilde{\theta}) = \tau(\tilde{q}_1) = q_1.$$

Now we must show that  $\theta$  is the unique map making the product diagram on the right commute. Suppose  $\eta$  were another possible map. Define  $\tilde{\eta} = \tau^{-1}(\eta)$ . We will show that  $\tilde{\eta} = \tilde{\theta}$ , so the product diagram in  $\mathscr{C}$  and the fact that  $\tau$  is a bijection will imply that  $\eta = \theta$ .

But notice that

$$((Gp_1)_* \circ \tau)(\tilde{\eta}) = (Gp_1) * (\eta) = (Gp_1) * (\theta) = ((Gp_1)_* \circ \tau)(\tilde{\theta}) = (\tau \circ (p_1)_*)(\tilde{\theta}).$$

But naturality implies that

$$((Gp_1)_* \circ \tau)(\tilde{\eta}) = (\tau \circ (p_1)_*)(\tilde{\eta}).$$

It thus follows that

$$\tau(p_1\tilde{\theta}) = \tau(p_1\tilde{\eta}).$$

Since  $\tau$  is a bijection, it follows that

$$p_1\tilde{\theta} = p_1\tilde{\eta} = \tilde{q}_1.$$

Thus  $\tilde{\eta}$  completes the product diagram in  $\mathscr{C}$ , so that  $\tilde{\eta} = \tilde{\theta}$ , proving the result.

Exercise 11.18. This is exactly stereographic projection (or the reverse of it).

**Exercise 11.19.** Note that  $J^n$  is homeomorphic to  $\mathbb{I}^n \setminus \{N\}$ , where N is some fixed point. Hence  $(J^n)^\infty \approx \mathbb{I}^n$ .

**Exercise 11.20.** Consider the map taking A to  $\infty$  and taking  $x \in X \setminus A$  to itself. This is obviously a homeomorphism.

Exercise 11.21. We have the following:

$$S^m \wedge S^n = (\mathbb{R}^m)^{\infty} \wedge (\mathbb{R}^n)^{\infty} = (\mathbb{R}^{m+n})^{\infty} = S^{m+n}.$$

Exercise 11.22. We have

$$\mathbb{I}^n \wedge \mathbb{I} = (J^n)^{\infty} \wedge J^{\infty} = (J^{n+1})^{\infty} = \mathbb{I}^{n+1}.$$

### **Homotopy Groups**

**Exercise 11.23.** Let  $F: \beta \simeq y_0$  be a homotopy. We would like to show that

$$\beta_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$$
  
 $[\alpha] \mapsto [\beta \circ \alpha].$ 

To do so, we must show that  $\beta \circ \alpha$  is nullhomotopic rel  $\dot{\mathbb{I}}^n$ . Consider the map

$$F \circ (\alpha \times \mathrm{id}_{\mathbb{I}}) : \mathbb{I}^n \times \mathbb{I} \to Y$$
  
 $(u, t) \mapsto F(\alpha(u), t).$ 

Obviously, this is a homotopy between  $\beta(\alpha(u))$  and the constant map at  $y_0$ . To see that this is rel $\dot{\mathbb{I}}^n$ , simply note that  $u \in \dot{\mathbb{I}}^n$  implies that  $F(\alpha(u), t) = F(x_0, t) = y_0$ , since  $\alpha$  and F are pointed maps.

Exercise 11.24. We have the following chain of equalities (note that some equalities are up to isomorphism or homotopy, depending on the category):

$$\pi_n(X \times Y) = [S^n, X \times Y]$$

$$= \Omega(X \times Y)$$

$$= \Omega X \times \Omega Y$$

$$= [S^n, X] \times [S^n, Y]$$

$$= [S^n, X] \oplus [S^n, Y] = \pi_n(X) \oplus \pi_n(Y).$$

Since  $\pi_n(S^1) = 0$ , it follows that  $\pi_n(T) = \pi_n(S^1) \times \pi_n(S^1) = 0$ .

**Exercise 11.25.** This follows from Theorem 11.29 and the fact that  $S^n$  covers  $\mathbb{R}P^n$ .

**Exercise 11.26.** Note that Theorem 10.54(i) applies because locally path-connected and contractible implies connected. Thus X is a covering space for X/G, and so Theorem 11.29 implies that  $\pi_n(X) \cong \pi_n(X/G)$ . But  $\pi_n(X) = 0$  for  $n \geq 2$  since X is contractible.

Exercise 11.27. This follows almost immediately from the hint. To see that \* and o coincide, note that

$$f * g = (f \circ e) * (e \circ g) = (f * e) \circ (e * g) = f \circ g.$$

To see commutativity, we need only check that  $f * g = g \circ f$ . But this follows because

$$f * g = (e \circ f) * (g \circ e) = (e * g) \circ (f * e) = g \circ f,$$

as desired.

#### Exercise 11.28.

(i) We follow the path laid out in the hints. First, note that, if  $q \in Q$ , then

$$\mu(f,e)(q) = \mu(f(q),p_0) = (\mu(-,p_0) \circ f)(q).$$

Thus it follows that

$$[f] * [e] = [\mu(f, e)] = [\mu(-, p_0) \circ f] = [1_P \circ f] = [f],$$

where we use the property of an *H*-space. Similarly, we can show that [e] \* [f] = [f].

Now we must show that  $[f] \circ [e] = [f]$ . But  $[f] \circ [e] = [(f, e)m]$ , and (f, e)m takes q to f(q) if m(q) is in the first coordinate of  $Q \vee Q$ , and takes q to  $p_0$  if m(q) is in the second coordinate. Letting  $q_1$  be as in the definition of an H'-group, i.e., letting  $q_1$  be the projection to the first coordinate, we see that  $q_1m$  takes q to q if m(q) is in the first coordinate and takes q to  $q_0$  otherwise. Thus  $fq_1m$  takes q to either f(q) or  $p_0$ , depending on the coordinate of q, and so it follows that  $fq_1m = (f, e)m$ . But of course  $q_1m \simeq 1_Q$ , from which the conclusion follows.

To show the second condition of Exercise 11.27, note first that

$$([f] \circ [h]) * ([g] \circ [j]) = [\mu((f,h)m,(g,j)m)].$$

The map on the right side takes q to either  $\mu(fq, gq)$  or  $\mu(hq, jq)$ , depending on where m(q) is in the first or second coordinate of  $Q \vee Q$ . On the other hand, we have

$$([f] * [g]) \circ ([h] * [j]) = [(\mu(f, g), \mu(h, j))m],$$

which takes

$$q\mapsto m(q)\mapsto \begin{cases} \mu(fq,gq)\\ \mu(hq,jq) \end{cases},$$

which is the exact same. Thus condition (ii) is satisfied, and the previous exercise proves the result.

(ii) Note that  $[\Sigma^2 X, Y] = [\Sigma X, \Omega Y]$  since  $\Sigma$  and  $\Omega$  are adjoint functors. Now since  $\Sigma X$  is an H-group and  $\Omega Y$  is an H-group in the category, hence an H-space, it follows from the previous part that  $[\Sigma X, \Omega Y]$  is abelian.

**Exercise 11.29.** First, we must show that this is well-defined. Suppose  $F: f \simeq g \operatorname{rel}\{s_n\}$ . We claim that  $\Sigma f \simeq \Sigma g$  with the map  $G: (a, b, t) \mapsto (F(a, t), b)$ . But this is easy to verify because  $G(a, b, 0) = (F(a, 0), b) = (f(a), b) = (\Sigma f)(a, b)$ , and similarly for G(a, b, 1).

Now, we will show that it is a homomorphism. Let  $m_n: S^n \to S^n \vee S^n$  be comultiplication. Then [f][g] = [(f,g)m]. We would like to show that

$$[\Sigma((f,q)m_n)] = [\Sigma f][\Sigma q] = [(\Sigma f, \Sigma q)m_{n+1}].$$

To see this, note that the left side takes (a,b) to (f(a),b) or (g(a),b), depending on which  $S^n \wedge S^1$ -component  $m_{n+1}(a,b)$  belongs to in  $(S^n \wedge S^1) \vee (S^n \wedge S^1)$ . On the other hand, we know that  $\Sigma((f,g)m_n)$  takes (a,b) to  $(((f,g)m_n)(a),b)$ . This first coordinate  $((f,g)m_n)(a)$  is f(a) or g(a), depending on which "half"  $m_n$  takes a to. Using a rotation to make sure the two halves which  $m_n$  and  $m_{n+1}$  determine line up (after projecting  $S^{n+1}$  down to the equator, which is  $S^n$ ), it is easy to show that these maps are homotopic.

**Exercise 11.30.** Any map  $Y \to X$  is homotopic to some simplicial approximation  $Sd^qL \to K$ . Obviously, there are only countably many simplicial approximations, since there are only finitely many vertices of each  $Sd^qL$  and of K. Hence [Y,X] is countable. Thus  $\pi_n(X) = [S^n,X]$  must be countable.

### **Exact Sequences**

Exercise 11.31. This is the exact same argument as part (ii) of ??.

**Exercise 11.32.** The same diagram chase remark applies, just changing H to  $\pi$ .

Exercise 11.33. We have the following long exact sequence:

$$\ldots \to \pi_{n+1}(X,X) \xrightarrow{d} \pi_n(X) \xrightarrow{\mathrm{id}} \pi_n(X) \xrightarrow{j_*} \pi_n(X,X) \xrightarrow{d} \pi_{n-1}(X) \xrightarrow{\mathrm{id}} \ldots$$

Now we know that  $\ker j_* = \operatorname{imid} = \pi_n(X)$ , so that  $\operatorname{im} j_* = 0$ . Hence  $\ker d = 0$ . But  $\operatorname{im} d = \ker \operatorname{id} = 0$ , and so  $\ker d = \pi_n(X,X)$ . The result now follows.

### **Fibrations**

Exercise 11.34.