Solutions to Rotman's algebraic topology

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0 Introduction

Notation

No exercises!

Brouwer Fixed Point Theorem

Exercise 0.1. As per the hint, observe that if $y \in G$, then we have $y = r(y) + (y - r(y))$. Obviously, we have $r(y) \in H$. Moreover, we know that

$$r(y - r(y)) = r(y) - r(r(y)) = 0,$$

and so $y - r(y) \in \ker r$. Thus $G \subseteq H \oplus \ker r$.

The reverse is obviously true, since H and $\ker r$ are both subgroups of G.

Exercise 0.2. Suppose instead that $f : D^1 \to D^1$ has no fixed point. Then consider the continuous map $g : D^1 \to S^0$ given by

$$g(x) = \begin{cases} 1 & \text{if } f(x) < x \\ -1 & \text{if } f(x) > x \end{cases}.$$

Notice that because $f(x) \neq x$ for all x, the function g is well-defined.

Moreover, we know that $f(-1) \neq -1$, since f has no fixed point, and so $f(-1) > -1$. Thus $g(-1) = -1$. Similarly, we have $g(1) = 1$.

Thus we have $g(D^1) = S^0$, which is disconnected. This is a contradiction, so f must have had a fixed point.

Exercise 0.3. Suppose that r is such a retract. Then we have the following commutative diagram:

$$\begin{tikzcd} & S^n \arrow[dr, "r"] & \\ S^{n-1} \arrow[ur, "i"] \arrow[rr, "1"] & & S^{n-1}. \end{tikzcd}$$

Applying H_{n-1}, we get another commutative diagram:

$$\begin{tikzcd} & H_{n-1}(S^n) \arrow[dr, "H_{n-1}(r)"] & \\ H_{n-1}(S^{n-1}) \arrow[ur, "H_{n-1}(i)"] \arrow[rr, "H_{n-1}(1)"] & & H_{n-1}(S^{n-1}). \end{tikzcd}$$

We know that $H_{n-1}(S^n) = 0$, however, implying that $H_{n-1}(1) = 0$. This contradicts the fact that $H_{n-1}(S^{n-1}) = \mathbb{Z} \neq 0$. Thus the retraction r could not have existed.

Exercise 0.4. Suppose $g : D^n \to X$ is a homeomorphism. Then we know that $g^{-1} \circ f \circ g$ is a continuous map from D^n to itself, and so it has a fixed point x. Then we know that $g^{-1}(f(g(x))) = x$, and so it follows that $f(g(x)) = g(x)$. Thus $g(x) \in X$ is a fixed point of f.

Exercise 0.5. Consider the function $h : \mathbb{I} \times \mathbb{I} \to \mathbb{I} \times \mathbb{I}$ given by

$$h(s, t) = f(s) - g(t) + (s, t).$$

This is the sum of continuous functions, and so it is itself continuous. Moreover, we know that $\mathbb{I} \times \mathbb{I}$ is homeomorphic to D^1, and so it follows that there is a fixed point (s, t) of h. But this means that $f(s) - g(t) = 0$, and so we are done.

Exercise 0.6. Observe that $x \in \Delta^{n-1}$ must contain some positive coordinate, because $\sum x_i = 1$ and $x_i \geq 0$ for all i. Since $a_{ij} > 0$ for every i, j, it follows that Ax contains only nonnegative coordinates and, moreover, contains at least one positive coordinate. Thus $\sigma(Ax) > 0$, and so $g(x)$ is well-defined.

Moreover, it is continuous because the linear map A, the map σ, and the division function are all continuous.

Because $\Delta^{n-1} \approx D^{n-1}$, it follows that there exists some x with

$$x = \frac{Ax}{\sigma(Ax)}.$$

Then $\lambda = \sigma(Ax) > 0$ is a positive eigenvalue for A and $x \in \Delta^{n-1}$ is a corresponding eigenvector.

We know that x contains only nonnegative coordinates. Suppose then that some coordinate, say x_1, is zero. Then obviously the first coordinate of λx is zero. However, the first coordinate of Ax is

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = a_{12}x_2 + \cdots + a_{1n}x_n.$$

Since $\sum x_i = 1$ and $x_1 = 0$, there exists some $k \neq 1$ such that $x_k > 0$. Then $a_{1k}x_k > 0$, and since each i already has $a_{1i}x_i \geq 0$, it follows that the first coordinate of Ax is strictly positive, contradicting that $Ax = \lambda x$.

Thus the eigenvector x has all positive coordinates.

Categories and Functors

Exercise 0.7. We know that

$$g \circ (f \circ h) = g \circ 1_b = g$$

and

$$(g \circ f) \circ h = 1_A \circ h = h,$$

and so associativity implies $g = h$.

Exercise 0.8.

(i) Notice that if 1_A and $1'_A$ are both identities, then we must have

$$1_A = 1_A \circ 1'_A = 1'_A,$$

which proves the desired result.

(ii) If $1'_A$ is the new identity in $\mathcal{C}'$, then we know that $1'_A \in \mathrm{Hom}_{\mathcal{C}'}(A, A) \subseteq \mathrm{Hom}_{\mathcal{C}}(A, A)$, and so $1_A \circ 1'_A$ is defined. But we know that

$$1'_A \circ 1_A = 1'_A = 1'_A \circ 1'_A,$$

and so Exercise 0.7 implies the result.

Exercise 0.9. Clearly, the Hom-sets are pairwise disjoint, since each i_y^x appears at most once.

It is also obviously associative. In particular, if $a \leq b \leq c \leq d$, then we know that

$$i_d^c \circ \left(i_c^b \circ i_b^a\right) = i_d^c \circ i_c^a = i_d^a,$$

and similarly for $\left(i_d^c \circ i_c^b\right) \circ i_b^a$.

Finally, the map i_x^x is the identity on $x \in X$. To see that it is a left-identity, note that if $y \leq x$, then

$$i_x^x \circ i_x^y = i_x^y.$$

Similarly, we can show that this map is a right-identity as well, and so we are done.

Exercise 0.10. Disjointness is clear, since there is only one object. Because G is a monoid, it is associative and has an identity, proving that $\mathcal{C}$ is a category.

Exercise 0.11. It is pretty clear that obj(Top) $\subset$ obj(Top^2). Moreover, a continuous map $f : X \to Y$ between two topological spaces corresponds to the map $(f, \emptyset)$ in Top^2 from $(X, \emptyset)$ to $(Y, \emptyset)$, which then means that Top can be thought of as a subcategory of Top^2.

Exercise 0.12. It is worth noting that Rotman's definition here is incorrect. The morphisms in $\mathcal{M}$ should be the commutative squares, not merely the ordered pairs (h, k).

Indeed, consider the following counterexample to Rotman's definition. Let $\mathcal{C}$ be the category of sets. Furthermore, let A be a set with more than one element. Then the following diagrams are both commutative:

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "1_A"] & A \arrow[d, "0"] \\ A \arrow[r, "0"] & \{0\} \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "0"] \arrow[d, "1_A"] & A \arrow[d, "0"] \\ A \arrow[r, "0"] & \{0\}. \end{tikzcd}$$

This implies that the ordered pair $(1_A, 0)$, where 0 is considered to be the map that sends everything in A to the zero element, is both in $\mathrm{Hom}(1_A, 0)$ and in $\mathrm{Hom}(0, 0)$, contradicting disjointness.

If we instead consider morphisms of $\mathcal{M}$ to be the commutative squares, where composition is defined by "stacking" the squares on top of one another, disjointness is clear. After all, the squares contain f and g, and so Hom-sets of different objects must be disjoint.

Associativity is clear, as the morphisms of $\mathcal{C}$ are associative.

Finally, there is an identity 1_f for every $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, namely the one where $h = 1_A$ and $k = 1_B$.

Exercise 0.13. With the hint, this is clear. In particular, we consider Top^2 to be the subcategory of the arrow category of Top in which the objects are inclusions, and $\mathrm{Hom}_{\mathsf{Top}^2}(i, j) = \mathrm{Hom}_{\mathsf{Top}}(i, j)$.

Exercise 0.14. To see that it is a congruence at all, observe that Property (i) is satisfied because there is only one Hom-set. Moreover, if $x \sim x'$ and $y \sim y'$, then we know that $x(x')^{-1} = h_x$ and $y(y')^{-1} = h_y$ for some $h_x, h_y \in H$. But then we know that

$$(yx)(y'x')^{-1} = yx(x')^{-1}(y')^{-1} = yh_x(y')^{-1}.$$

However, since $(y')^{-1} = y^{-1}h_y$, we know that this is simply

$$(yx)(y'x')^{-1} = yh_xy^{-1}h_y.$$

Because H is normal, we know that $yh_xy^{-1} \in H$. Thus the product of this and h_y is in H as well, and so $xy \sim x'y'$, as desired.

To see that $[*, *] = G/H$ simply requires the observation that $x \sim y$ if and only if x and y are in the same coset of H.

Exercise 0.15. This follows from the fact that functors preserve (or, in the case of contravariant functors, reverse) the directions of the arrows. Thus the resulting diagram still commutes.

Exercise 0.16. Note that for (i)–(iv), we can simply use inverses. For instance, for Set, it suffices to note that if f is a bijection, then f^{-1} is a bijection, which is clearly true. Similarly, the inverse of a homeomorphism is a homeomorphism, and the inverse of a group or ring isomorphism is still an isomorphism.

For (v), note that i_x^y is defined and satisfies the requirements that $i_x^y \circ i_y^x = i_x^x$ and $i_y^x \circ i_x^y = i_y^y$.

For part (vi), notice that f^{-1} works because f is a homeomorphism. In particular, it is a bijection, and so $f^{-1}(A') = A$. Moreover, it is (bi)continuous since f is.

Finally, for the monoid G, if g has a two-sided inverse h, then $hg = gh = 1$, which is the identity element of $\mathrm{Hom}(G, G)$.

Exercise 0.17. To prove that T' is a functor, first observe that criterion (i) of a functor is satisfied because T does so. Moreover, if $[f] \in \mathrm{Hom}_{\mathscr{C}'}(A, B)$, then $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$, and so $T'([f]) = Tf$ is a morphism in $\mathscr{A}$. In particular, if $[g] \circ [f] = [g \circ f]$ is defined in $\mathscr{C}'$, then $g \circ f$ is defined in $\mathscr{C}$. This means, then, that

$$T'([g] \circ [f]) = T(g \circ f) = (Tg) \circ (Tf) = T'([g]) \circ T'([f]).$$

Finally, it remains to note that $T'([1_A]) = T_{1_A} = 1_{TA} = 1_{T'([A])}$ for every object A. Thus T' is a functor.

Exercise 0.18.

(i) It is clear that $tG \in \operatorname{obj} \mathsf{Ab}$ for every group G. Now suppose that we have a homomorphism $f : G \to H$. Then we know that $t(f)$ is a morphism $f|_{tG}$ from tG to tH. To see this, note that it is the restriction of a homomorphism, and thus is itself a homormophism. Moreover, if $x \in f(tG)$, then $x = f(y)$ for some $y \in G$ with finite order. But then there exists some n so that $y^n = 1$. Thus $x^n = f(y^n) = 1$, and so x has finite order. But $x \in f(G) \subseteq H$ implies that $x \in tH$.

Now we must check that t respects composition. Indeed, if $g \circ f$ is defined, then

$$t(g \circ f) = (g \circ f)_{tG} = g|_{f(tG)} \circ f|_{tG}.$$

But $f(tG) \subseteq tH$, and so this is simply

$$t(g \circ f) = g|_{tH} \circ f|_{tG} = t(g) \circ t(f),$$

which proves that composition is respected.

Finally, note simply that $t(1_G) = 1|_{tG}$, which is the identity on tG.

(ii) Suppose that f is an injective homomorphism from G to H. Then suppose that $t(f)(x) = t(f)(y)$. But $f(x) = f|_{tG}(x) = t(f)(x)$, and so it follows that $f(x) = f(y)$. Injectivity of f proves the result.

(iii) Let $G = \mathbb{Z}$ and $H = \mathbb{Z}/2\mathbb{Z}$ and let f take even integers to 0 and odd integers to 1. This is evidently surjective. But $tG = \{0\}$ while $tH = \{0, 1\}$, and so $t(f) : tG \to tH$ cannot be surjective.

Exercise 0.19.

(i) If f is a surjection, then consider an arbitrary coset $a + pH$ of H/pH. We know that there exists some $b \in G$ with $f(b) = a$, and so it follows that $F(f)$ takes $b + pG$ to $a + pH$, proving surjectivity of $F(f)$.

(ii) Consider the function $f : \mathbb{Z} \to \mathbb{Z}$ taking x to $2x$. Then, letting $p = 2$, we know that $F(f) : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ has $F(f)([0]) = F(f)([1])$.

Exercise 0.20.

(i) This is evident because $\mathbb{R}$ is a ring, and the operations are pointwise.

(ii) By the previous part, we know that if X is a topological space, then $C(X)$ is a ring. Now suppose that $f : X \to Y$ is a continuous map. Then define

$$\begin{aligned} C(f) : C(Y) &\to C(X) \\ g &\mapsto g \circ f \end{aligned}$$

and note that this is well-defined. Moreover, we know that $C(g \circ f)(h) = h \circ g \circ f$, while $C(f) \circ C(g)$ takes h to $C(f) \circ (h \circ g) = h \circ g \circ f$, which proves that C reverses composition. Finally, we know that $C(1_x)$ takes g to $g \circ 1_X = g$ and is therefore the identity on $C(Y)$. Thus C (or, rather, the map taking X to $C(X)$, to be precise) gives rise to a contravariant functor.

1 Some Basic Topological Notions

Homotopy

No exercises!

Convexity, Contractibility, and Cones

Exercise 1.1. Suppose $H : f_0 \simeq f_1$ is a homotopy. Then let $F(t) = H(x,t)$ for some fixed x. It is clear that $F(0) = x_0$ and $F(1) = 1$. Moreover, since H is continuous, it follows that so too is F. For the converse, simply let the homotopy $H : f_0 \simeq f_1$ take $(x,t) \in X \times \mathbb{I}$ to $F(t)$.

Exercise 1.2.

(i) There exist functions $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_Y$. Moreover, there is a homotopy $F : 1_X \simeq c$, where c denotes the constant map at some $x_0 \in X$. Then consider the map $G : Y \times \mathbb{I} \to Y$ which takes (y,t) to $f(F(g(y),t))$. In particular, we know that G is continuous and that it is thus a homotopy from $f \circ g$ to the constant map c' at $y_0 = f(x_0)$. But then we find that $1_Y \simeq f \circ g \simeq c'$, and so Y is contractible.

(ii) Consider, for example, the subsets $X, Y \subset \mathbb{R}^2$ where

$$\begin{aligned} X &= \{(x,0) : x \in [0,1]\}, \\ Y &= \left\{(x,x) : x \in \left[0, \frac{1}{2}\right]\right\} \cup \left\{(x, 1-x) : x \in \left[\frac{1}{2}, 1\right]\right\}. \end{aligned}$$

It is obvious that X is convex, but Y is not, even though there is an obvious homotopy equivalence from X to Y.

Exercise 1.3. We know that $R(x) = e^{i\alpha}x$, and so the continuous map $F : S^1 \times \mathbb{I} \to S^1$ given by $F(x,t) = e^{i\alpha t}x$ is a homotopy $F : 1_S \simeq R$. Thus, if $g : S^1 \to S^1$ is continuous, then let θ be such that $g(1) = g(e^{i \cdot 0}) = e^{i\theta}$. Then we know that, letting R now be the rotation of $-\theta$ degrees, we must have $R \circ g \simeq 1_S \simeq g = g$ and $(R \circ g)(1) = 1$, as desired.

Exercise 1.4.

(i) Pick $(x_1, y_1), (x_2, y_2) \in X \times Y$. Then we know that, for any $t \in \mathbb{I}$, we have

$$t(x_1, y_1) + (1-t)(x_2, y_2) = (tx_1 + (1-t)x_2, ty_1 + (1-t)y_2).$$

The result follows from convexity of X and Y.

(ii) If $F_X : 1_X \simeq c_X$ and $F_Y : 1_Y \simeq c_Y$, where c_X and c_Y are constant maps at c_X and c_Y, respectively, then the map

$$\begin{aligned} F : (X \times Y) \times \mathbb{I} &\to X \times Y \\ (x,y,t) &\mapsto (F_X(x,t), F_Y(y,t)) \end{aligned}$$

is clearly a homotopy from $1_{X \times Y}$ to (c_X, c_Y).

Exercise 1.5. It is clear that X is compact. After all, any open cover of X must contain some set U containing 0, and thus containing cofinitely many elements of X.

If we have a map $h : X \to Y$, then because Y is discrete, we know that $\{h^{-1}(y) : y \in Y\}$ is an open covering of X and thus by compactness admits a finite subcovering. Thus there are only finitely many elements of y in the image of h.

Now suppose that $f : X \to Y$ is a homotopy equivalence. Then there exists some $g : Y \to X$ with a homotopy $H : f \circ g \simeq 1_Y$. But $H(\{y\} \times I)$ is the continuous image of a connected map and is therefore itself connected. Because Y is discrete, this means that $H(y,0) = H(y,1)$ for all y. But we know that f has finite image, and Y is infinite, so there exists some y such that $y \notin \operatorname{im} f$. In particular, we have $y \neq f(g(y))$, and so $H(y,0) = f(g(y)) \neq y = 1_Y(y)$, a contradiction. Thus X and Y are not of the same homotopy type.

Exercise 1.6. Suppose X is contractible, with $F : c \simeq 1_X$, where c is the constant map at p. Note that, for every $x \in X$, there is a path $F(x,t) : \{x\} \times \mathbb{I} \to X$ taking x to $p \in X$. In particular, this means that every x is in the same component as p, proving connectedness.

Exercise 1.7. The map $H : X \to \mathbb{I} \to X$ taking (x,t) to x and (y,t) to x if and only if $t > \frac{1}{2}$ works. Indeed, note that $H^{-1}(\{x\} \times \mathbb{I})$ is simply $\{x\} \times \mathbb{I} \cup \{y\} \times (\frac{1}{2}, 1]$, which is open in $X \times \mathbb{I}$.

Exercise 1.8.

(i) Consider the map taking the unit interval to S^1 given by $t \mapsto e^{2\pi i t}$.

(ii) If $r : Y \to X$ is a retraction, then we know from $1_Y \simeq c$ that $r \circ 1_Y \circ i \simeq r \circ c \circ i$, where i is the injection $X \hookrightarrow Y$. But the left side is simply $r \circ i = 1_X$, while the left side is a constant map, proving the result.

Exercise 1.9. We know that there exists some constant map c with $f \simeq c$. But then $g \circ f \simeq g \circ c$, and the right side is a constant map. Thus $g \circ f$ is also nullhomotopic.

Exercise 1.10. First, suppose that g is an identification. Note that $(gf)^{-1}(U)$ open in X implies that $g^{-1}(U)$ is open in Y because f is an identification. But the hypothesis on g implies that U is open in Z. Since gf is clearly a continuous surjection, the result follows.

Now, suppose that gf is an identification. It suffices to prove that $g^{-1}(U) \subseteq Y$ open implies that $U \subseteq Z$ is open. But we know by continuity of f that $f^{-1}(g^{-1}(U))$ is open, and so gf being an identification implies the result.

Exercise 1.11. First, note that this is a well-defined function in the sense that $[x] = [y]$ in $X/\sim$ implies that $\overline{f}([x]) = \overline{f}([y])$.

This is evidently continuous. After all, suppose that $U \subseteq Y/\square$ is open. Then we know that

$$\overline{f}^{-1}(U) = \{[x] \in X/\sim : [f(x)] \in U\} = U'.$$

If we let $v : X \to X/\sim$ and $u : Y \to Y/\square$ be the natural maps, then we know that U' is open in $X/\sim$ because

$$v^{-1}(U') = \{x \in X : f(x) \in u^{-1}(U)\} = f^{-1}(u^{-1}(U))$$

is open.

Finally, we will show that $\overline{f}$ is an identification. It is obviously surjective. Moreover, if $U' = \overline{f}^{-1}(U)$ is open in $X/\sim$, then we simply note that a similar argument as above gives us that $v^{-1}(U') = f^{-1}(u^{-1}(U))$ is open. Since f and u are identifications, it follows that U was an open set in the first place, proving the result.

Exercise 1.12. Note that if $K \subseteq Z$ is closed, then it is compact and so $h(K)$ is compact in Z, hence itself closed. Thus h is a closed map, and hence an identification.

Now because $v : X \to X/\ker h$ is an identification, Corollary 1.9 applies. Indeed, Corollary 1.9 implies that $hv^{-1} = \varphi$ is a closed map. Thus it is an identification, i.e., a continuous surjection.

But the same corollary also implies that $\varphi^{-1} = vh^{-1}$ is continuous. This, combined with Example 1.3, in which it was shown that φ is injective, proves the result, as φ is now a bicontinuous bijection, i.e., a homeomorphism.

Exercise 1.13. First observe that $f(x) = f(y)$ implies that $[x,t] = [y,t]$ and so $t = 1$. Thus f is injective and hence bijective onto its image $CX_t = \{[x,t] \in CX : x \in X\}$. Then open sets in CX_t are precisely of the form $U \cap CX_t$ for an open set $U \subseteq CX$. But clearly we can assume that $[x,1] \notin U$ because $[x,1] \notin CX_t$, and thus we wind up with $X \times [0,1)$, where $CX_t = X \times \{t\}$. This is obviously homeomorphic to X.

Exercise 1.14. The functor takes a map $f : X \to Y$ to $Cf : CX \to CY$ given by $C([x,t]) = [f(x),t]$. Note that this is well-defined. Moreover, it is obvious that this is satisfies the properties of a functor. Indeed, if $g : Y \to Z$, then

$$C(g \circ f)([x,t]) = [g(f(x)),t] = ((Cg) \circ (Cf))([x,t])$$

and clearly $C(1_X)$ is the identity on CX.

Paths and Path Connectedness

Exercise 1.15. Using the hint, suppose that $g : \mathbb{I} \to X$ is a path with $g(0) = (0, a) \in A$ and with $g(t) \in G$ for all $t > 0$. Then note that $\pi_i \circ g$ is continuous for $i = 1, 2$, where π_i are the projections to the x- and y-axes. This implies the existence of an $\varepsilon > 0$ such that $t \in (0, \varepsilon)$ implies that $g(t) = (x(t), \sin(1/x(t)))$ has $x(t), |\sin(1/x(t)) - a| < \delta$. But this is obviously impossible, as $\sin(1/x(t))$ will oscillate wildly between -1 and 1.

Exercise 1.16. Let (a_i) and (b_i) be points in S^n. We will construct n paths which, when joined together in the customary fashion (i.e., by traversing each of the $n - 1$ subpaths in $1/(n - 1)$ time), will give us a path from (a_i) to (b_i).

The first path f_1 is defined as

$$f_1(t) = ((1 - t)a_1 + tb_1, c_2, a_3, a_4, \dots, a_n),$$

where c_2 is chosen to be of the same sign as a_2 and in such a way that $f(t) \in S^n$. Note that such a c_2 always exists.

In general, for $1 \le i \le n - 1$, the path f_i will fix every coordinate except for the i-th, which it will take to b_i, and the $(i + 1)$-th, which we use as a "free" coordinate to allow for such adjusting. Moreover, observe that if the first $n - 1$ coordinates of two points on S^1 are the same, then the n-th coordinates either will be the same or will be negatives.

If joining the paths $f_1, f_2, \dots, f_{n-1}$ together gives a path from (a_i) to (b_i), then we are done. Note that this occurs if a_n and b_n have the same sign.

Otherwise, construct a path g which adjusts the n-th coordinate and uses the $(n - 1)$-th coordinate as a "free" one, preserving the sign. This effectively allows us to switch the sign of the n-th coordinate so that the n-th coordinate is just b_n. Moreover, because we preserved the sign of the $(n - 1)$-th coordinate, it is still equal to b_{n-1}.

Exercise 1.17. It suffices to show the forward direction, so suppose that U is not path connected. Then there are at least two path components.

We will show that each path component is open, which will prove that U is not connected. But because U is open, we know that open sets in U (as a subspace) or also open in $\mathbb{R}^n$. Thus, for every $x \in U$, there is a ball B_x centered at x and contained in U. This ball is obviously path-connected. As such, if x is in the path component A, it must follow that $B_x \subseteq A$, proving that A is open.

Exercise 1.18. We know that if X is contractible then there exists a point $c \in X$ such that 1_X is homotopic to the constant map at c from X to itself. Now consider the map $c : \mathbb{I} \to X$ satisfying $c(t) = c$ for all t. In the proof of Theorem 1.13, we saw that any path is homotopic to c. In particular, the constant maps $x : \mathbb{I} \to X$ and $y : \mathbb{I} \to X$ at x and y, respectively, are both homotopic to c. Note that these give rise to paths from x to c and from c to y, respectively, which in turn give rise to a path from x to y. This proves path connectedness.

Exercise 1.19.

(i) If X is path connected, then let c and c' be constant maps. Let f be a path from (the point) c to (the point) c' and define $H : X \times \mathbb{I} \to X$ as $H(x, t) = f(t)$. Then H is a homotopy from c to c'.

For the reverse direction, let H be a homotopy from c to c' and define the path $f : \mathbb{I} \to X$ as $f(t) = H(c, t)$.

(ii) Let $f : X \to Y$ be a continuous function. Fix some $y_0 \in Y$ and consider the map

$$\begin{aligned} H : X \times \mathbb{I} &\to Y \\ (x, t) &\mapsto p_x(t), \end{aligned}$$

where p_x is a path from $f(x)$ to y_0. This is a homotopy from f to the constant map mapping X to y_0.

But if $g : X \to Y$ is another continuous function, then the same argument shows that $g \simeq y_0$, and so $f \simeq g$, as desired.

Exercise 1.20. It suffices to show that if $a \in A$ and $b \in B$, then there is a path from a to b. But fix some point $x \in A \cap B$. Then there is a path from a to x, and a path from x to b. Joining the two paths gives a path from a to b.

Exercise 1.21. This is simply done by noting that for any $(x, y), (x', y') \in X \times Y$, we can join the paths $f(t) = ((1-t)x + tx', y)$ and $g(t) = (x', (1-t)y + ty')$.

Exercise 1.22. Suppose $f(a), f(b) \in Y$. Then let p be a path from a to b in X. Now simply note that $q(t) = f(p(t))$ is a path from $f(a)$ to $f(b)$, proving the result.

Exercise 1.23.

(i) We already know that there are at least two path components because the entire space is not path connected. Moreover, both A and G are path connected, and so it follows that they must themselves be the path components.

(ii) Simply note that the sequence $\left\{\left(\frac{1}{n\pi}, \sin(n\pi)\right)\right\} \subset G$ approaches $(0,0) \in A$.

(iii) As per the hint, consider U to be the open disk with center $(0, \frac{1}{2})$ and radius $\frac{1}{4}$. Then $X \cap U$ is open in X. But note that $v(X \cap U)$ is not open in $X/A \approx [0, \frac{1}{2\pi}]$. After all, note that any ball B_ε around the point 0 (which is the image of A under the natural map in this case) must contain some point $\frac{1}{n\pi} < \varepsilon$. But $\frac{1}{n\pi}$, which corresponds to the point $\left(\frac{1}{n\pi}, 0\right) \in X \setminus U$, is not contained in $v(X \cap U)$.

Exercise 1.24. By definition, path components are path connected. Moreover, if C is a path component and there exists some point $x \in X$ and $c \in C$ so that there is a path between x and c, then the definition of path components implies that $x \in C$. Thus path components are maximally path connected.

Finally, suppose that A is path connected and pick $a \in A$. There exists a unique path component C such that $a \in C$. Then for all $b \in A$, we know that there is a path between a and b, and so $b \in C$. Thus $A \subseteq C$, as desired.

Exercise 1.25. Simply use Exercise 1.22 and observe that I is path connected.

Exercise 1.26. Note that, if X is locally path connected, then for all $x \in X$, there exists some open path connected, hence connected, neighborhood V of x. Alternatively, note that if $U \subseteq X$ is open, then its components are unions of its path components and thus open.

Exercise 1.27. Given any open subset U of $X \times Y$ containing a given point $(x, y) \in X \times Y$, there must exist a basic open neighborhood $U_x \times U_y \subseteq U$ of (x, y). Then we know that there exists some path connected V_x with $x \in V_x \subseteq U_x$, and similarly for y. Then $V_x \times V_y$ is path connected by Exercise 1.21. The result follows.

Exercise 1.28. Note that open subsets of open subsets are open in the main space. In particular, let $A \subseteq X$ be open. Given any $x \in A$, let U be an open neighborhood of x in A. Note that this is also an open neighborhood in X, and so there exists an open path connected V in X (and hence open in A as well) such that $x \in V \subseteq U$.

Exercise 1.29. Consider the map $F : (\mathbb{R}^{n+1} \setminus \{0\}) \times \mathbb{I} \to \mathbb{R}^{n+1} \setminus \{0\}$ given by

$$F((x_i), t) = \left[(1-t) + \frac{t}{\sqrt{\sum x_i^2}}\right](x_i).$$

This is evidently a homotopy which makes S^n a deformation retract.

Exercise 1.30. The exact same map as in Exercise 1.29 works for this case.

Exercise 1.31. It is easy to see that the deformation retract of a deformation retract is a deformation retract, either by a direct argument or by applying Theorem 1.22. Thus the previous exercise implies that it suffices to show that $D^n \setminus \{0\}$ is a deformation retract of $S^n \setminus \{a, b\}$. But the map $(x_i) \mapsto (x_1, \ldots, x_{n-1}, 0)$ is exactly the map needed, and so we are done.

Exercise 1.32. If $H : f_0 \simeq f_1$, then the map $H' : (y, t) \mapsto H(r(y), t)$ is a homotopy from $\tilde{f}_0$ to $\tilde{f}_1$.

Exercise 1.33. Let $Y = \{y\}$ and observe that $(x, 1) \sim y$ for all $x \in X$. Thus $(x, 1) \sim (x', 1)$ for alL $x, x' \in X$. Moreover, this is the only equivalence. Thus M_f is precisely the quotient space $(X \times \mathbb{I})/(X \times \{1\}) = CX$.

Exercise 1.34.

(i) We first tackle i. It is obvious that i is injective, and thus a bijection onto its image $i(X) = \{[x, 0] : x \in X\}$. Moreover, the open sets in $i(X)$ are precisely of the $U \cap i(X)$ for open sets U in M_f.

Note that we can suppose without loss of generality that U is contained in $v(X \times [0, 1))$, where v is the natural map. Thus U simply looks like the Cartesian product of an open interval with an open set of X. This proves that i is a homeomorphism, for the open sets of $i(X)$ map exactly to the open sets of X.

We can show that j is a homeomorphism onto $j(Y)$ in a similar manner. The main idea is simply that $y \not\sim y'$ for any $y, y' \in j(Y)$.

(ii) It is obvious that $(rj)(y) = r[y] = y = 1_Y(y)$ for any $y \in Y$. It is also clearly continuous by the gluing lemma. Thus r is indeed a retraction.

(iii) Define $F : M_f \times \mathbb{I} \to M_f$ as suggested in the hint. It is evident that F is continuous. Moreover, for any $[x, t] \in M_f$, we know that

$$F([x, t], 0) = [x, t]$$
$$F([x, t], 1) = [x, 1] = [f(x)] \in Y.$$

Similarly, if $[y] \in Y$, then the definition implies that the remaining criteria for this homotopy to induce a deformation retraction $r(x) = F(x, 1)$ are satisfied.

(iv) Note that Rotman writes that f is homotopic to $r \circ i$; in fact, we can and do prove the stronger statement that f coincides with $r \circ i$.

Let $f : X \to Y$ be continuous. Then it is clear that the map $f = r \circ i$, where $i : X \to M_f$ is an injection and $r : M_f \to Y$ is the retraction taking $[x, t]$ to $[f(x)]$ and taking $[y]$ to itself, proving the result.

2 Simplexes[1]

Affine Spaces

Exercise 2.1. Note that there is a maximal affine independent subset S of A. This is directly implied by the fact that any set of greater than $n+1$ elements is not affine independent. Hence we can take an affine independent subset of A with maximum size (because the empty set is affine independent).

Write $S = \{p_0, \ldots, p_m\}$. Then let $p_{m+1} \in A \setminus S$. By maximality of S, we know that $S \cup \{p\}$ is not affine independent. Hence there exist s_i not all 0 such that

$$\sum_{i=0}^{m+1} s_i p_i = 0, \quad \sum_{i=0}^{m+1} s_i = 0.$$

Note that the second equation implies $\sum_{i=0}^{m} s_i \neq 0$ for some $i < m+1$. It follows then that

$$\sum_{i=0}^{m} \left(\frac{s_i}{\sum_{i=0}^{m} s_i} p_i \right) = p_{m+1}.$$

But we know that

$$\sum_{i=0}^{m} \frac{s_i}{\sum_{i=0}^{m} s_i} = 1,$$

and so it follows that p_{m+1} is in fact in the affine span of S.

Exercise 2.2. Let φ be the isomorphism from $\mathbb{R}^n$ to a subset of $\mathbb{R}^k$. Suppose $A \subseteq \mathbb{R}^n$ is an affine set containing X. Then $\varphi(X) \subseteq \varphi(A) \subseteq \mathbb{R}^k$.

Moreover, we claim that $\varphi(A)$ is affine. After all, for any $\varphi(x), \varphi(x') \in \varphi(A)$ and any $t \in \mathbb{R}$, the point $t\varphi(x) + (1-t)\varphi(x') = \varphi(tx + (1-t)x') \in \varphi(A)$ because A is affine.

This implies that the intersection of all affine sets in $\mathbb{R}^n$ containing X must contain the intersection of all affine sets in $\varphi(\mathbb{R}^n)$ containing $\varphi(X)$. Because φ is an isomorphism, using φ^{-1} gives the reverse inclusion. Thus the affine set spanned by X in $\mathbb{R}^n$ is precisely the same as that spanned by X in $\mathbb{R}^k$.

Exercise 2.3. This is evident in the case $n = 0$.

Suppose it is true for $n-1$ and consider the canonical injection $\iota : S^{n-1} \hookrightarrow S^n$ which takes $(x_0, \ldots, x_{n-1})$ to $(x_1, \ldots, x_{n-1}, 0)$. It is obvious that we can pick $n+1$ affine independent points $p_0, \ldots, p_n$ in this embedding.

Now consider the point $p_{n+1} = (0, \ldots, 0, 1) \in S^n$. Notice that the last coordinate of each p_i for $i \neq n+1$ is zero. Thus suppose we have s_i with $\sum s_i p_i = 0$ and $\sum s_i = 0$. Then $s_{n+1} = 0$, and so this reduces to the $n-1$ case. Affine independence of $\{p_0, \ldots, p_n\}$ proves the result.

Affine Maps

Exercise 2.4. Consider the map $T'(x) = T(x) - T(0)$. We claim that T' is a linear map.

Observe that $S = \{e_i\} \cup \{0\}$ spans $\mathbb{R}^n$. Thus we can write any point as the affine sum of elements of S. Note that the coefficient of the zero vector is flexible, and so we have effectively no restrictions on the sum of the coefficients.

Consider arbitrary elements $\sum r_i e_i + r \cdot 0$ and $\sum s_i e_i + s \cdot 0$ in $\mathbb{R}^n$, where $r = 1 - \sum r_i$ and similarly for s. Let $R, S \in \mathbb{R}$. Then note that

$$\begin{aligned} T'\left(R \sum r_i e_i + S \sum s_i e_i\right) &= T'\left(\sum (Rr_i + Ss_i) e_i\right) \\ &= T\left(\sum (Rr_i + Ss_i) e_i + \left(1 - \sum (Rr_i + Ss_i)\right) \cdot 0\right) - T(0) \\ &= R \sum r_i T(e_i) + S \sum s_i T(e_i) - R \sum r_i T(0) - S \sum s_i T(0). \end{aligned}$$

[1] I usually use *simplices* as the plural of simplex, but Rotman doesn't; no matter.

Considering the R-terms first, simply observe that we can add and subtract $RT(0)$ to give us that

$$R\sum r_iT(e_i) - R\sum r_iT(0) = R\left(T\left(\sum r_iT(e_i) + r\cdot 0\right) - T(0)\right).$$

This is simply $RT'\left(\sum r_ie_i\right)$. A similar result holds for the S-terms, from which we conclude that

$$T'\left(R\sum r_ie_i + S\sum s_ie_i\right) = RT'\left(\sum r_ie_i\right) + ST'\left(\sum s_ie_i\right),$$

proving linearity.

Exercise 2.5. This is obvious from the previous exercise and continuity of linear maps.

Exercise 2.6. Given two m-simplexes $[p_0,\dots,p_m]$ and $[q_0,\dots,q_m]$, the map f taking p_i to q_i for every i is a homeomorphism. Bijectivity is obvious by the definition. Continuity is clear by how we extend f from $\{p_i\}$ to $[p_i]$. Finally, the inverse is of the same form as f, only with the q_i's taking the place of the p_i's and vice versa; thus f^{-1} is also continuous.

Exercise 2.7. The following map works:

$$f : x \mapsto \frac{t_2 - t_1}{s_2 - s_1}(x - s_1) + t_1.$$

Exercise 2.8. Pick arbitrary $T(x), T(x') \in T(X)$ and observe that

$$tT(x) + (1-t)T(x') = T(tx + (1-t)x') \in T(X).$$

Thus $T(X)$ is affine if X is affine, and convex if X is convex. The second statement of the exercise follows by noting that ℓ is convex.

Exercise 2.9. Without loss of generality, we delete p_0. Now suppose that

$$\sum_{i=1}^m s_ip_i + sb = 0, \quad \sum_{i=1}^m s_i + s = 0.$$

Then we know by definition of the barycenter b that

$$\sum_{i=1}^m s_ip_i + \frac{s}{m+1}\sum_{i=0}^m p_i = 0.$$

Moreover, letting s_i' be the coefficient of p_i in the above equation, it is obvious that $\sum_{i=0}^m s_i' = s + \sum_{i=1}^m s_i = 0$. Thus $s_i' = 0$ for all i because $\{p_0,\dots,p_m\}$ was affine independent. But then we conclude that $0 = s_0' = \frac{s}{m+1}$, and so $s = 0$. For every $i \in \{1,\dots,m\}$, we have $0 = s_i' = \frac{s}{m+1} + s_i$. Thus $s = 0$ implies $s_i = 0$ for every i,a nd so it follows that $\{b, p_1,\dots,p_m\}$ is affine independent, as desired.

Exercise 2.10. Once again, suppose without loss of generality that $i = 0$. Then the map taking $\sum t_ip_i \in [p_0, p_1, \dots, p_m]$ to $(\sum_{i=1}^m t_ip_i, t_0)$ works. Note that this actually requires the affine independence of the p_i's, as well as the fact that the coefficients t_i are all between 0 and 1.

Exercise 2.11. Notice that $[0, e_1, \dots, e_n]$, where e_i are the standard basis vectors in $\mathbb{R}^n$, is an n-simplex. Thus there is a homeomorphism $[p_0,\dots,p_n] \to [0, e_1,\dots,e_n]$. If we translate the image by $\mathbf{v} = (-\frac{1}{4},\dots,-\frac{1}{4})$, then we can map the result to D^n by taking a radial mapping. In particular, this map will take

$$p_0 \mapsto \frac{\mathbf{v}}{\|\mathbf{v}\|}$$
$$p_i \mapsto \frac{e_i + \mathbf{v}}{\|e_i + \mathbf{v}\|} \text{ for } i \neq 0.$$

Note that this extends to a homeomorphism.

3 The Fundamental Group

The Fundamental Groupoid

Exercise 3.1. The homotopy $H : X \times \mathbb{I} \to Z$ given by

$$H : (x,t) \mapsto \begin{cases} g_0(F(x,2t)) & \text{if } t \le \frac{1}{2}, \\ G(f_1(x), 2t-1) & \text{if } t \ge \frac{1}{2} \end{cases}$$

works. Continuity follows because $g_0(F(x,1)) = G(f_1(x), 0)$.

Moreover, this homotopy is indeed $\operatorname{rel} A$. For a detailed argument why this is so, simply suppose that $a \in A$ and $t \in I$. If $t \le \frac{1}{2}$, then $F(a, 2t) = f_0(a)$ by definition of F. Hence $H(a,t) = g_0(f_0(a))$.

Similarly, we can show that if $t \ge \frac{1}{2}$, then $H(a,t) = g_1(f_1(a))$. This follows because $f_1(a) \in B$ and G is a homotopy $\operatorname{rel} B$.

It thus suffices to show that $g_0(f_0(a)) = g_1(f_1(a))$. But this is obvious because f_0 and f_1 agree on A, and g_0 and g_1 agree on $B \supseteq f_0(A)$.

Exercise 3.2.

(i) First, note that f' is well-defined because $f(0) = f(1)$. It is obvious by continuity of f and ln that f' is continuous.

Moreover, consider the map

$$H' : \left(e^{2\pi i\theta}, t\right) \mapsto H(\theta, t).$$

This is clearly continuous, for the same reasons that f' was continuous. If $t = 0$, clearly $H'(e^{2\pi i\theta}, t) = H(\theta, 0) = f(\theta) = f'(e^{2\pi i\theta})$, and similarly for $t = 1$. Thus H is indeed a homotopy from f' to g'.

To see that it is a homotopy $\operatorname{rel}\{1\}$, simply note that $e^{2\pi i\theta} = 1$ corresponds to $\theta = 0, 1$. Thus it follows that

$$H'(1,t) = H(1,t) = f(1)$$

for all t, proving the result.

(ii) Theorem 3.1 implies that $f * g \simeq f_1 * g_1 \operatorname{rel} \dot{\mathbb{I}}$. Using the previous part, we find that $(f*g)' \simeq (f_1*g_1)' \operatorname{rel}\{1\}$. Now, using the observation that $(f*g)' = f' * g'$, we find that $f' * g' \simeq f_1' * g_1' \operatorname{rel}\{1\}$, as desired.

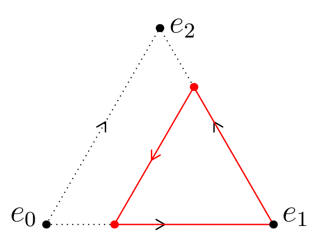
Exercise 3.3. The forward direction is trivial.

For the converse, note that g' is a constant map, and so f' is nullhomotopic. Then Theorem 1.6 implies that $f' \simeq g' \operatorname{rel}\{1\}$. In particular, note that $g' : S^1 \to X$ takes every element of S^1 to $g'(1) = g(0) = x_0$. Observe that $f'(1) = x_0$ as well, and so it follows that $f' \simeq g \operatorname{rel}\{1\}$, as desired.

Exercise 3.4.

(i) Instead of applying Theorem 1.6, I constructed an explicit homotopy. (If you are interested in a proof using Theorem 1.6, my guess would be that it relies on the fact that $\Delta^2 \approx D^2$. However, I have not gone through the details.)

The effective idea of the homotopy I constructed is to, at time $t \in [0,1]$, return the function which traverses the first t units of the face opposite e_0, then goes along a segment to the point t units away from e_1 on the fact opposite e_2, before returning back to e_1, as shown in the red path below.



The specific homotopy $H : \mathbb{I} \times \mathbb{I} \to X$ from $(\sigma_0 * \sigma_1^{-1}) * \sigma_2$ to the constant map at e_1 is as follows:

$$H(x,t) = \begin{cases} \sigma_0(4(1-t)x) & \text{if } x \leq \frac{1}{4}, \\ \sigma((1-x)\varepsilon_0(1-t) + x\varepsilon_2(t)) & \text{if } \frac{1}{4} \leq x \leq \frac{1}{2}, \\ \sigma(2tx - (2t-1)) & \text{if } x \geq \frac{1}{2}. \end{cases}$$

We leave it to the reader to check that this works.

(ii) One can generate a similar homotopy, which we do not do here.

(iii) This time, we use the homotopy which goes up along γ for t units, before going parallel to β and coming back down along δ^{-1}. The particular formula is as follows:

$$H(x,t) = \begin{cases} F(0, 4tx) & \text{if } x \leq \frac{1}{4}, \\ F(4x-1, t) & \text{if } \frac{1}{4} \leq x \leq \frac{1}{2}, \\ F(1, 2t(1-x)) & \text{if } \frac{1}{2} \leq x. \end{cases}$$

Once again, we leave the details to the reader to check.

Exercise 3.5. Simply use the homotopy $H : \mathbb{I} \times \mathbb{I} \to X \times Y$ which takes (s,t) to $(F(s,t), G(s,t))$. This is clearly a homotopy from (f_0, g_0) to (f_1, g_1). To see that it is still $\operatorname{rel} \dot{\mathbb{I}}$, simply observe that $H(0,t) = (F(0,t), G(0,t))$. Because F and G are both $\operatorname{rel} \dot{\mathbb{I}}$, it follows that $H(0,t)$ never changes. A similar argument shows that $H(1,t)$ is always the same, and so H is indeed a homotopy $\operatorname{rel} \dot{\mathbb{I}}$.

Exercise 3.6.

(i) It is obvious that the homotopy $H' : (x,t) \mapsto H(x, 1-t)$ works.

(ii) This is just some slightly annoying manipulation. In particular, note that

$$(f * g)(x) = \begin{cases} f(2x) & \text{if } x \leq \frac{1}{2}, \\ g(2x-1) & \text{if } x \geq \frac{1}{2}. \end{cases}$$

By replacing x with $1-x$ to get the inverse, we find that

$$(f * g)^{-1}(x) = \begin{cases} f(2-2x) & \text{if } x \geq \frac{1}{2}, \\ g(1-2x) & \text{if } x \leq \frac{1}{2}. \end{cases}$$

However, note that

$$\begin{aligned} (g^{-1} * f^{-1})(x) &= \begin{cases} g^{-1}(2x) & \text{if } x \geq \frac{1}{2}, \\ f^{-1}(2x-1) & \text{if } x \geq \frac{1}{2} \end{cases} \\ &= \begin{cases} g(1-2x) & \text{if } x \leq \frac{1}{2}, \\ f(2-2x) & \text{if } x \geq \frac{1}{2}. \end{cases} \end{aligned}$$

Thus the two are indeed the same.

(iii) Take the closed path $f(t) = e^{2\pi i t}$ on S^1. Then note that $(f * f^{-1})(\frac{1}{8}) = f(\frac{1}{4}) = i$, while $(f^{-1} * f)(\frac{1}{8}) = f^{-1}(\frac{1}{4}) = -i$.

(iv) Suppose $i_p * f = f$ and f is not constant. Note that continuity implies that there must exist some $0 < t < 1$ so that $f(t) \neq p$. Thus there exists some $k \in \mathbb{N}$ so that $t < 1 - 2^{-k}$.

We claim, however, that f must be constant on $[0, 1-2^{-n}]$ for every $n \in \mathbb{N}$. We prove this inductively. Clearly, it is true on $n = 0$. If it is true on $n-1$, then we know that $i_p * f$ must be equal to p on $[0, \frac{1}{2}]$, as well as on $[\frac{1}{2}, 1 - 2^{-n}]$ (note that $1 - 2^{-n}$ comes from $2(1 - 2^{-n}) - 1$, which itself comes from the equation for the star operator). Thus f is constant on $[0, 1 - 2^{-n}]$, as desired.

Thus it follows that $f(t) = p$, a contradiction. Thus f must have been constant in the first place.

The Functor π_1

Exercise 3.7. Recall that we defined the $\sin(1/x)$ space as the union of $A = \{(0, y) : -1 \leq y \leq 1\}$ and $G = \{(x, \sin(1/x)) : 0 < x \leq 1/2\pi\}$. We also know that A and G are the path components of the $\sin(1/x)$ space. Moreover, both A and G are contractible, and so every path in either A or G is nullhomotopic. In particular, we conclude that the fundamental group at any basepoint is trivial.

Exercise 3.8. Let X be the $\sin(1/x)$ space. We know that CX is contractible. But consider an open ball around the point $x = ((0, 0), 0)$, that is, the point $(0, 0)$ on the "zeroth" level of the cone. Consider a small neighborhood (not including the points $(t, 1)$, in particular) around this point and pick some element $y = ((\varepsilon, \sin(1/\varepsilon)), 0)$ in the neighborhood. Now observe that any path between x and y can be projected down to a path between $(0, 0)$ and $(\varepsilon, \sin(1/\varepsilon))$ in X, which we know does not exist. Hence CX is contractible but not locally path connected.

Exercise 3.9. Note that composition is associative because $\circ$ is. Moreover, the path class of the trivial loop based at p is the identity on p. Thus this is a category.

To see that each morphism in $\mathscr{C}$, simply note that the inverse path, i.e., the path f^{-1} taking t to $f(1-t)$, gives a path class $[f^{-1}]$ which works as an inverse to $[f] \in \mathrm{Hom}(p, q)$.

Exercise 3.10. We simply let π_0 take $(X, x_0) \in \mathsf{Sets}_*$ to the set of all path components of X, with basepoint equal to the path component containing x_0. It takes a morphism $f \in \mathrm{Hom}((X, x_0), (Y, y_0))$ to the map $\pi_0(f)$ which takes each path component A of X to the path component B of Y which contains $f(A)$.

Note that this is possible because continuous images of path connected spaces are path connected and hence contained within a single path component of Y. Moreover, this is indeed a pointed map because the path component containing x_0 must be contained in the path component containing $f(x_0) = y_0$, which is the basepoint of $\pi_0((Y, y_0))$.

It is easy to check functoriality, completing the proof.

Exercise 3.11. Evidently the only possible path is the constant path at x_0. Hence $\pi_1(X, x_0)$ is the trivial group, i.e., $\{1\}$.

$\pi_1(S^1)$

Exercise 3.12. Note that 1_S is a loop based at 1, i.e., an element of $\pi_1(S^1, 1)$. Thus if $\pi_1(S^1, 1)$ were trivial, then 1_S would be nullhomotopic. The hint gives the rest of the solution.

Exercise 3.13. We know that $\deg u = 1$. Since 1 is a generator for $\mathbb{Z}$, it follows that $[u]$ generates $\pi_1(S^1, 1)$.

Exercise 3.14. Let $\tilde{\gamma}(t) = m\tilde{f}(t)$, where $\tilde{f}$ is the lifting of f satisfying $\tilde{f}(0) = 0$. Now simply observe that

$$\exp \tilde{\gamma}(t) = \left(\exp \tilde{f}(t)\right)^m = f(t)^m$$

and $\tilde{\gamma}(0) = 0$. Thus $\tilde{\gamma}$ is indeed the lifting of f^m taking 0 to 0, and so we conclude that

$$\deg(f^m) = \tilde{\gamma}(1) = m\tilde{f}(1) = m \deg f.$$

Exercise 3.15. Note that Exercise 1.3 implies that there is a homotopy $F : R_f \circ f \simeq f$, where R_f is the rotation associated with f. Moreover, from the proof of that same exercise, it follows that F gives a closed path at every time t. Similarly, we have $G : g \simeq R_g \circ g$. Thus if $H : f \simeq g$ where H gives a closed path at every time t, then the homotopy which follows F, then H, and finally G is a homotopy between $R_f \circ f$ and $R_g \circ g$. Thus Corollary 3.18 implies that f and g have the same degree.

For the converse, simply use Corollary 3.18 to show that $\deg f = \deg g$ implies that there is a homotopy rel $\dot{\mathbb{I}}$ taking $R_f \circ f$ to $R_g \circ g$. Then using F and G defined above, it is clear that $g \simeq R_g \circ g \simeq R_f \circ f \simeq f$.

Exercise 3.16. Theorem 3.7 implies that $\pi_1(T, t_0) = \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$.

Exercise 3.17. Because D^2 is contractible, its fundamental group is trivial. Thus if there were to exist a retraction $r : D^2 \to S^1$, then $r_* : \pi_1(D) \to \pi_1(S^1)$ would be a constant. But then, letting $i : S^1 \to D^2$ be the canonical injection, we would have that $(r \circ i)_* = r_* \circ i_*$ is a constant. However, we also know that $r \circ i$ is the identity on S^1, and so $(r \circ i)_*$ is the identity on $\pi_1(S^1)$, which is *not* a constant. This is a contradiction, from which we conclude that S^1 is not a retract of D^2, as desired.

Exercise 3.18. This was proved in Theorem 0.3, which required only the fact proved in the above problem, namely that S^1 is not a retract of D^2.

Exercise 3.19.

(i) Let $\tilde{f}$ be the unique lifting of f with $\tilde{f}(0) = 0$. Then if $\tilde{f}(1) \geq 1$, the intermediate value theorem implies that every point in the interval $[0, 1] \subset \mathbb{R}$ is in the image of $\tilde{f}$. But this implies that $f = \exp \circ \tilde{f}$ must be surjective, a contradiction.

(ii) Consider the map which traverses the circle once counterclockwise, reaching the point 1 at time $t = \frac{1}{2}$, before looping back and making a clockwise rotation. Clearly it is surjective. However, it is composed of two loops, one of which has degree 1 and one of which has degree -1. Because $\deg(f * g) = \deg f + \deg g$, it follows that this map has degree 0.

Exercise 3.20. As per the hint, consider an arbitrary closed path f in X and let λ be a Lebesgue number of the open cover $\{f^{-1}(U_j) : j \in J\}$ of $\mathbb{I}$. Note that λ exists by the Lebesgue number lemma and compactness of the unit interval. Picking $N \in \mathbb{N}$ with $N > 1/\lambda$, it follows that if we subdivide I into N equal intervals $I_k = [\frac{k}{N}, \frac{k+1}{N}]$, then $f(I_k) \subseteq U_{j_k}$ for some $j_k \in J$.

Define f_k as the path in U_{j_k} obtained by restricting f to I_k and then stretching suitably so that the domain is all of $\mathbb{I}$. With notation, define $f_k(t) = f\left(\frac{k+t}{N}\right) \in U_{j_k}$. Because f_k is a path in U_{j_k}, it follows that $[f_k'] = [i_{j_k} \circ f_k] \in \operatorname{im}(i_j)_*$. But now simply observe that $[f_0' * \cdots * f_{N-1}'] = [f]$, implying that $[f]$ is contained in the group generated by the subsets $\operatorname{im}(i_j)_*$. This proves the result.

Exercise 3.21. Let U_1 and U_2 be defined as in the hint, and let i_k be the injection from U_k to S^n for $k = 1, 2$. Observe that, by the previous exercise, it suffices to show that $\operatorname{im}(i_k)_*$ is trivial for $k = 1, 2$.

Without loss of generality, let $k = 1$. But we know that $(i_1)_*$ takes a closed path $f : \mathbb{I} \to U_1$ to the path class $[i_1 \circ f]$. (Note that the basepoint doesn't really matter for us as long as it is neither the north nor the south pole.Thus we omit it.) Because $U_1 \approx D^n$ and is therefore contractible, it follows that f is nullhomotopic. In particular, we know that $i_1 \circ f$ is nullhomotopic, and so $[i_1 \circ f] = [1]$ for every f. Thus $\operatorname{im}(i_1)_*$ is trivial, and similarly for $k = 2$, proving the result.

Exercise 3.22. Corollary 3.11 implies that path connected spaces of the same homotopy type must have isomorphic fundamental groups. But obviously $\mathbb{Z} \ncong \{1\}$, and so S^1 and S^n do not have the same homotopy type for $n > 1$.

Exercise 3.23. The multiplication map μ on G/H is continuous. After all, if we let v be the natural map, then for any open set $U \subseteq G/H$, we have

$$\mu^{-1}(U) = \{([x], [y]) : xy \in v^{-1}(U)\}.$$

But this set is open in $G/H \times G/H$ because the set consisting of elements $(v^{-1}([x]), v^{-1}([y]))$ for each $([x], [y]) \in \mu^{-1}(U)$ is just $\mu^{-1}(v^{-1}(U))$, which is clearly open.

For the inversion map $i : G/H \to G/H$, a very similar argument holds. In particular, for any open set $U \subseteq G/H$, we have

$$v^{-1}(i^{-1}(U)) = \{x^{-1} : x \in v^{-1}(U)\}.$$

Thus $v^{-1}(i^{-1}(U))$ is open, and so $i^{-1}(U)$ is open, proving continuity.

Exercise 3.24. First, we will show that we can lift a loop $f : (\mathbb{I}, 0) \to (G/H, 1)$ to a unique continuous map $\tilde{f} : (\mathbb{I}, 0) \to (G, h)$ for any $h_0 \in H$, as shown below.

$$\begin{tikzcd} & (G, h_0) \arrow[d, "v"] \\ (\mathbb{I}, 0) \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & (G/H, 1) \end{tikzcd}$$

In the above diagram, the map v is the natural map taking g to the coset $gH \in G/H$.

First, we will find a suitable neighborhood U of 1 such that the family $\{hU : h \in H\}$ is pairwise disjoint. Discreteness of H implies that there exists an open neighborhood V of 1 with $V \cap H = \{1\}$. It is clear that the map $\varphi : (x, y) \mapsto xy^{-1}$ is the composition $\mu \circ (\mathrm{id} \times i)$ and is therefore continuous. Thus $\varphi^{-1}(V) \subseteq G \times G$ is an open neighborhood of $(1, 1)$. This implies that we can find an open neighborhood U of 1 such that $U \times U \subseteq f^{-1}(V)$.

Now suppose that there are $h_1, h_2 \in H$ and $x, y \in U$ with $h_1 x = h_2 y$. But this would require that $xy^{-1} = h_1^{-1} h_2$. It is clear that $xy^{-1} \in \varphi(U) \subseteq V$. Moreover, because H is a subgroup, we know that $h_1^{-1} h_2 \in H$, and so $xy^{-1} \in V \cap H$. Thus $x = y$ and $h_1 = h_2$, proving that the sets hU are disjoint, as desired. Note that any translate $U_g = gU$ of U is a neighborhood of $g \in G$ and has $\{hU_g : h \in H\}$ disjoint.

Note that v is an open map, and so the set $W = v(U) \subseteq G/H$ is open. Moreover, because $v|_U$ is the restriction of a continuous open map to an open set, it follows that $v|_U$ is itself continuous and open. It is also a bijection onto W, and so $v|_U : U \to V$ is a homeomorphism.

Note that the collection of sets $V[g]$ for $[g] \in G/H$ forms an open cover of G/H. Thus, if we are given some $f : (\mathbb{I}, 0) \to (G/H, 1)$, then we can consider the open cover

$$\{f^{-1}(V[g]) : [g] \in G/H\}$$

of $\mathbb{I}$. Note that we can find a finite subcover of this open cover. This means that we can take subsets of the sets in this open cover, given us a finite collection open overlapping subintervals which are, in order of their smaller coordinate, labeled $I_1, \ldots, I_k$. Let the group elements $g_1, \ldots, g_k$ be such that $I_j \subseteq f^{-1}(V[g_j])$. This is simply because $\mathbb{I} = [0, 1]$ is connected compact.

Now we can lift f to each interval $f^{-1}(V[g])$ in this finite subcover. Note that $0 = t_1 \in I_1 \subseteq f^{-1}(V[g_1])$. Moreover, we know that $v^{-1}(V[g_1])$ consists of disjoint unions of U, and so we can pick the one containing h_0. Now, for each $t \in I_1$, we let $\tilde{f}(t)$ to be the unique element in this copy of U such that $v(\tilde{f}(t)) = f(t)$. Because the intervals overlap, we know that there is some $t_2 \in I_2 \cap I_1$, and so we can do the same thing, all the way to t_k. This lets us define $\tilde{f}(t)$ for all $t \in \mathbb{I}$, and it is easy to show that our construction is indeed a lifting satisfying the commutative diagram above.

Now consider the map $d : \pi_1(G/H, 1) \to H$ taking a loop $[f]$ to $d([f]) = \tilde{f}(1)$, where $\tilde{f}$ is the unique lifting of f with $\tilde{f}(0) = 1$. It is obvious that $\operatorname{im} d \subseteq H$ because $v(\tilde{f}(1)) = f(1) = [1]$ implies that $d([f]) = \tilde{f}(1) \in H$. Moreover, the reverse inclusion holds, showing surjectivity. In particular, if $h \in H$, then path connectedness of G implies that there is a path $\tilde{f}$ from 1 to h. Taking its projection $f = v \circ \tilde{f}$, note that f is a loop because $v(\tilde{f}(1)) = v(h) = [1]$. Thus $d([f])$ is defined and equal to h. To show injectivity, simply note that $\tilde{f}(1) = 1$ implies that $\tilde{f}$ is a loop in G. Because G is simply connected, however, it follows that $\tilde{f}$, and hence f, is nullhomotopic. Thus $f \in \ker d$ implies that $[f] = [1]$. Finally, we must show that d is indeed a homomorphism, i.e., that $d(f * g) = d(f)d(g)$. But this is clear if we lift f to $\tilde{f}$ with $\tilde{f}(0) = 1$, and if we lift g to $\tilde{g}$ with $\tilde{g}(0) = d(f)$. This follows the same proof layout as Theorem 3.16, and proves the result.

Exercise 3.25. If $S \subseteq GL(n, \mathbb{R})$ is a subgroup of $GL(n, \mathbb{R})$, then note that $\mu : S \times S \to S$ is continuous. After all, the product, entrywise, is simply a polynomial, and so μ is a polynomial in each of its n^2 entries. Since polynomials are continuous in $\mathbb{R}^2$, it follows that each of the n^2 components of μ is continuous. Hence μ is continuous.

To see that the inversion i is continuous, observe that the determinant $\det A$ is a continuous function, since it too is a polynomial (and is never zero, by definition of GL). It thus suffices to show that the function $A \mapsto \operatorname{adj} A$ is continuous. But it is easy to see that the adjugate matrix, which is the transpose of the cofactor matrix, is also a polynomial in the entries of A, and so $\operatorname{adj} A$ is continuous too. Thus i is continuous, and so S is a topological group, as desired.

Exercise 3.26. As hinted in the exercise, fix $h_0 \in H$ and let $\varphi : G \to H$ be the map taking x to $xh_0x^{-1}h_0^{-1}$. Note that $xh_0x^{-1} \in H$ because H is normal, and so $\varphi(x)$ is indeed an element of H. Moreover, we know that φ is continuous and $\{h\} \subseteq H$ is open for each $h \in H$. Thus $\{\varphi^{-1}(h) : h \in H\}$ is an open cover of G consisting of disjoint open sets.

In particular, if there are two elements $h_1, h_2 \in H$ such that $\varphi^{-1}(h_i) \neq \emptyset$ for $i = 1, 2$, then setting $A = \varphi^{-1}(h_1)$ and $B = \bigcup_{h \neq h_1} \varphi^{-1}(h)$ will give us two disjoint open sets A and B that cover G. This implies that G is disconnected, a contradiction. Thus for all but one element of H, we must have $\varphi^{-1}(h) = \emptyset$, proving that φ is constant. But obviously, setting $x = h_0$, we find that $\varphi(x) = 1$. Thus $xh_0x^{-1}h_0^{-1} = 1$ for all $x \in G$, and so $xh_0 = h_0x$ for each $h_0 \in H$. This proves the result.

4 Singular Homology

Holes and Green's Theorem

No exercises!

Free Abelian Groups

Exercise 4.1. If $\gamma \in F$, then we can write $\gamma = \sum_{b \in B} m_b b$, where $m_b \in \mathbb{Z}$ is zero for almost all b. Now, writing $B = \cup B_\lambda$ for disjoint B_λ, we can define for each λ the value $\gamma_\lambda = \sum_{b \in B_\lambda} m_b b \in F_\lambda$. Then obviously $\gamma = \sum \gamma_\lambda$.

To see that this expression is unique, simply observe that if $\gamma = \sum \gamma'_\lambda$, then because the sums are formal sums only, it follows that $\gamma_\lambda = \gamma'_\lambda$ for every λ. But then it follows that the coefficient for each $b \in B_\lambda$ must be the same in γ_λ and in γ'_λ, and so the two expressions are the same. Moreover, it is clear that almost every γ_λ is zero. After all, only finitely many m_b's are nonzero, and so only finitely many γ_λ contain a nonzero coefficient.

Finally, the converse is clear. In particular, if $\gamma = \sum \gamma_\lambda$ and $\gamma_\lambda = \sum_{b \in B_\lambda} m_b b$, then $\gamma = \sum_{b \in B} m_b b$.

Exercise 4.2. To see the forward direction (isomorphic implies same rank), simply restrict to the basis. In particular, if $\varphi : F \to F'$ is an isomorphism between two free abelian groups, and if B is a basis for F, then $\varphi(B)$ is a basis for F'. But clearly B and $\varphi(B)$ have the same cardinality because φ is injective. Thus F and F' have the same rank.

To see the converse, consider bases B and B' for F and F', respectively. Because B and B' have the same cardinality, there is a bijection $\varphi|_B$ between them. Pick such a bijection and extend it to all of F linearly. Theorem 4.1 tells us that this is a homomorphism; indeed, it is an isomorphism because $\varphi|_B$ was a bijection.

Exercise 4.3.

(i) An arbitrary element of $S_1(X)$ looks like $\sum m_\sigma \sigma$, where σ ranges over paths in X. Then we know that ∂_1 takes $\sum m_\sigma \sigma + \sum n_\sigma \sigma$ to

$$\sum_\sigma m_\sigma \sigma(1) + \sum_\sigma n_\sigma \sigma(1) - \sum_\sigma m_\sigma \sigma(0) - \sum_\sigma n_\sigma \sigma(0) = \partial_1(m) + \partial_1(n),$$

where $m = \sum m_\sigma \sigma$ and similarly for n. Thus this is a homomorphism.

(ii) If x_0 and x_1 lie in the same path component of X, then there is a path σ between them. This path is an element of X (indeed, it is a *basis* element of X), and satisfies $\partial_1(\sigma) = x_1 - x_0$.

The converse is slightly trickier, however. Suppose that x_0 and x_1 belong to different path components, say X_0 and X_1, respectively. Then consider the map $\varphi : S_0(X) \to \mathbb{Z}$ which takes $x \in X$ to 1 if $x \in X_0$ and to 0 otherwise. This defines φ on the basis of $S_0(X)$, so we can linearly extend it to a group homomorphism (Theorem 4.1).

Any element in the image of ∂_1 can be written as $(\sum m_\sigma \sigma)(1) - (\sum m_\sigma \sigma)(0)$. Then we know that

$$\varphi\Big((\sum m_\sigma \sigma)(1) - (\sum m_\sigma \sigma)(0)\Big) = \sum m_\sigma \varphi(\sigma(1) - \sigma(0)).$$

But because σ is a path, obviously $\sigma(1)$ and $\sigma(0)$ are in the same path component. In particular, we have $\varphi(\sigma(1) - \sigma(0)) = 0$, and so $\operatorname{im} \partial_1 \subset \ker \varphi$. Now observe that $\varphi(x_1 - x_0) = -1$. Thus $x_1 - x_0 \notin \operatorname{im} \partial_1$, proving the converse.

(iii) By definition, we have that $\sigma \in \ker \partial_1$ if and only if $\sigma(1) - \sigma(0) = 0$. Because σ is a path, however, this condition is equivalent to saying that σ is a closed path.

To see that the path condition on σ is necessary, note that the sum of two closed paths is in $\ker \partial_1$ but is not itself a closed path.

The Singular Complex and Homology Functors

No exercises!

Dimension Axiom and Compact Supports

Exercise 4.4. Note that $S_n(X) = \emptyset$ for all n, because there is no function $\Delta^n \to X = \emptyset$. Thus $\ker \partial = \operatorname{im} \partial = \emptyset$, and so $H_n(X)$ is trivial.

Exercise 4.5. We know that ∂_0 is the zero map, and so $\ker \partial_0 = S_0(X)$. Moreover, the proof of the dimension axiom shows that ∂_1 is the zero map as well. In particular, we find that $Z_0(X)/B_0(X) \cong S_0(X)$. But we know, once again from the proof of the dimension axiom, that $S_0(X)$ is infinite cyclic and hence $H_0(X) \cong \mathbb{Z}$.

Exercise 4.6. We already know how S_n acts on objects of Top. Defining $S_n(f) = f_\#$ on morphisms, it is easy to see that S_n satisfies the functorial properties $S_n(1_X) = 1_{S_n(X)}$ and $S_n(g \circ f) = S_n(g) \circ S_n(f)$.

Exercise 4.7. We know that S^0 is the disjoint union of two points, and so $H_n(S^0) = H_n(\{0\}) \oplus H_n(\{1\})$. But the dimension axiom and Exercise 4.5 imply that

$$H_n(S^0) = \begin{cases} \mathbb{Z}^2 & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}.$$

Exercise 4.8. Because the Cantor set is the disjoint union of countably many points, it follows that $H_0(X) = \mathbb{Z}^\omega$ and $H_n(X) = 0$ for all $n > 0$.

The Homotopy Axiom

Exercise 4.9.

(i) For $n = 0$, note that $\beta_1 = [a_0, b_0]$, and so $\partial_1\beta_1$ is the constant map taking $e_0 \in \Delta^0$ to $b_0 - a_0 = (e_0, 1) - (e_1, 0)$. On the other hand, we know that P_{-1}^Δ is the zero map, and $\lambda_i^\Delta{}_\#(\delta) = \lambda_i^\Delta$. Thus the right-hand side of the equation is simply

$$\lambda_1^\Delta - \lambda_0^\Delta,$$

which is the map taking $e_0 \in \Delta^0$ to $(e_0, 1) - (e_1, 0)$. The two sides are therefore the same.

For $n = 1$, we first consider the left-hand side. Note that

$$\begin{aligned} \partial_2\beta_2 &= [b_0, b_1] - [a_0, b_1] + [a_0, b_0] - [a_1, b_1] + [a_0, b_1] - [a_0, a_1] \\ &= [b_0, b_1] + [a_0, b_0] - [a_1, b_1] - [a_0, a_1], \end{aligned}$$

and so it is simply the constant map $\Delta^1 \to \Delta^1 \times \mathbb{I}$ taking everything to $b_0 - a_1 = (e_0, 1) - (e_1, 0)$. For the right-hand side, on the other hand, we already know that

$$\lambda_1^\Delta{}_\#(\delta) - \lambda_0^\Delta{}_\#(\delta) = \lambda_1^\Delta - \lambda_0^\Delta : t \mapsto (t, 1) - (t, 0).$$

Moreover, because $\partial_1\Delta^1 = e_1 - e_0$, we know that

$$P_0^\Delta\partial\delta : t \mapsto ((e_1 - e_0)(e_0), t) = (e_1, t) - (e_0, t).$$

Thus the right-hand side takes e_0 to

$$(e_0, 1) - (e_0, 0) - (e_1, 0) + (e_0, 0) = (e_0, 1) - (e_1, 0)$$

and takes e_1 to

$$(e_1, 1) - (e_1, 0) - (e_1, 1) + (e_0, 1) = (e_0, 1) - (e_1, 0).$$

hus the two sides agree on e_0 and e_1, from which we conclude the result.

(ii) We know that

$$\begin{aligned} P_1^X(\sigma) &= (\sigma \times 1)_\#(\beta_2) \\ &= (\sigma \times 1) \circ [a_0, b_0, b_1] - (\sigma \times 1) \circ [a_0, a_1, b_1]. \end{aligned}$$

The first term takes an arbitrary element $(t_0, t_1, t_2) \in \Delta^2$, where we use barycentric coordinates, to the point $(\sigma((t_0+t_1)e_0 + t_2e_1), t_1+t_2)$. By corresponding a point $(1-t)e_0 + te_1 \in \Delta^1$ to t, we find that the first term takes (t_i) to $(\sigma(t_2), t_1+t_2)$. Similarly, the second term takes (t_i) to $(\sigma(t_1+t_2), t_2)$. Thus we find the following explicit formula:

$$P_1^X(\sigma) : (t_0, t_1, t_2) \mapsto (\sigma(t_2), t_1+t_2) + (\sigma(t_1+t_2), t_2).$$

Exercise 4.10. Let $\sigma : \Delta^n \to X$ be a simplex. Then note that $P_n^X(\sigma) = (\sigma \times 1)_\#(\beta_{n+1})$. Thus

$$(f \times 1)_\# P_n^X(\sigma) = (f\sigma \times 1)_\#(\beta_{n+1}).$$

On the other hand, we know that

$$P_n^Y f_\#(\sigma) = (f_\#\sigma \times 1)\#(\beta_{n+1}),$$

which is the same as the previous expression because σ is a simplex and so $f_\#\sigma = f\sigma$.

Exercise 4.11. The inclusion i is a homotopy equivalence, and so Corollary 4.24 implies that i_* is an isomorphism.

Exercise 4.12. Note that the $\sin(1/x)$ space has two path components, both of which are contractible. Thus $H_0(X) = \mathbb{Z}^2$ and $H_n(X) = 0$ for $n > 0$.

The Hurewicz Theorem

Exercise 4.13. We know that $\varphi \circ h_\#$ takes the path class $[f]$ to $\varphi[h \circ f] = \operatorname{cls} hf\eta$. On the flip side, we know that $h_* \circ \varphi$ takes φ to $h_* \operatorname{cls} f\eta$. But because $f\eta$ is a simplex, this is simply $\operatorname{cls} hf\eta$ as well.

Exercise 4.14. We know that

$$f * f^{-1} * (f * f^{-1})^{-1} \simeq c$$

for some constant map c. But note that $(f * f^{-1})^{-1} = f * f^{-1}$. Thus we can apply the Hurewicz map to find that

$$2\operatorname{cls}((f + f^{-1})\eta) = [0].$$

It follows that $f + f^{-1} \in B_1(X)$, where f and f^{-1} are considered as 1-chains. Thus f and $-f^{-1}$ are homologous, as desired.

Exercise 4.15. Note that the boundary of the second triangle is $\alpha * \beta + \gamma - (\alpha * \beta) * \gamma$. Thus $\operatorname{cls}(\alpha * \beta * \gamma) = \operatorname{cls}(\alpha * \beta + \gamma)$. Repeating this procedure on the first triangle, we find that $\operatorname{cls}(\alpha * \beta * \gamma) = \operatorname{cls}(\alpha + \beta + \gamma)$. Note that, in the text, there is a second equality, namely that these expressions equal $\operatorname{cls}\alpha + \operatorname{cls}\beta + \operatorname{cls}\gamma$. However, homology classes are not actually defined for paths which are not closed, so this seems to be an error.

Exercise 4.16. This is proved in Theorem 6.20.

5 Long Exact Sequences

The Category Comp

Exercise 5.1. These results all follow directly from the definition of exactness.

(i) Note that $\ker f = \operatorname{im} 0 = 0$, and so f is injective.

(ii) In this case, we have $\operatorname{im} g = \ker 0 = C$.

(iii) By the previous two parts, we know that f is bijective. Because f is a homomorphism as well, it follows that f is an isomorphism.

(iv) Either observe that $0 \to A \to 0 \to 0$ is exact and apply the previous part, or note that $A \to 0$ is injective while $0 \to A$ is surjective, implying that $A \cong 0$, i.e., that $A = 0$.

Exercise 5.2. Note that f is surjective if and only if $\ker g = \operatorname{im} f = B$. But $\ker g = B$ if and only if g is the zero map, which is itself true exactly when $\ker h = \operatorname{im} g = 0$. Since $\ker h = 0$ if and only if h is injective, we are done.

Exercise 5.3. We know that $0 \to A \xrightarrow{i} B$ implies that i is an injection. But because i is a surjection onto its image, this implies that $iA \cong A$. Moreover, because $\ker p = \operatorname{im} i = iA$, we know that $B/iA = B/\ker p \cong \operatorname{im} p$. Because p is a surjection (see Exercise 5.1), the result follows.

Exercise 5.4. This amounts, effectively, to following the arrows and the equations given by exactness. In more detail, let $f_n : B_n \to C_n$ and $g_n : C_n \to A_{n-1}$. Now observe that $B_n = \operatorname{im} h_n = \ker f_n$. Thus f_n is the zero map. Moreover, because $\ker g_n = \operatorname{im} f_n$, we know that g_n is injective. Finally, we have $\operatorname{im} g_n = \ker h_{n-1}$. But h_{n-1} is an isomorphism, and so its kernel is trivial. Thus $\operatorname{im} g_n = 0$. Because g_n was injective, it follows that $C_n = 0$.

Exercise 5.5.

(i) Let f be the map from A to B and g be the map from B to C. Let $\{a_\alpha\}$ and $\{c_\gamma\}$ be maximal independent sets of A and C, respectively. For every α, let $b_\alpha = f(a_\alpha)$. For every γ, pick some $b_\gamma \in g^{-1}(c_\gamma)$, which is possible by surjectivity of g. If $\sum n_\alpha b_\alpha + \sum n'_\gamma b'_\gamma = 0$, then we know that

$$g\left(\sum n_\alpha b_\alpha + \sum n'_\gamma b'_\gamma\right) = 0.$$

But we also know that $\operatorname{im} f = \ker g$, and so $g(b_\alpha) = 0$. Thus this simply implies that $\sum n'_\gamma c_\gamma = 0$, implying that $n'_\gamma = 0$. But now we know that $\sum n_\alpha b_\alpha = 0$, and so injectivity of f implies that $\sum n_\alpha a_\alpha = 0$ as well. Thus $n_\alpha = 0$ for all α as well, and so $\{b_\alpha\} \cup \{b_\gamma\}$ is independent. Thus $\operatorname{rank} B \geq \operatorname{rank} A + \operatorname{rank} C$.

To show the opposite inequality, it suffices to show that $\{b_\alpha\} \cup \{b_\gamma\}$ is *maximally* independent. Note that $b \notin f(A)$. Otherwise, we could take f^{-1} on $\{b_\alpha\} \cup \{b\}$, which is not independent. Now consider $\{b_\gamma\} \cup \{b\}$. If $g(b) = g(b_\gamma)$ for any γ, then we know by Exercise 5.3 that $b - b_\gamma \in f(A)$. Obviously, we cannot have $b - b_\gamma = b_\alpha$ for any α, otherwise that would give us our linear dependence. Thus $\{b - b_\gamma\} \cup \{b_\alpha\}$ is a subset of $f(A)$ with $\operatorname{rank} A + 1$ elements. This is not independent, a contradiction.

(ii) We prove this by induction. The previous part takes care of the base case. Consider the following commutative diagrams.

$$0 \longrightarrow A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{v} A_{n-1}/\operatorname{im} f_n \longrightarrow 0$$

$$0 \longrightarrow A_{n-1}/\ker f_{n-1} \xrightarrow{\bar{f}_{n-1}} A_{n-2} \xrightarrow{f_{n-2}} \dots \xrightarrow{f_2} A_1 \xrightarrow{f_1} A_0 \longrightarrow 0.$$

Here v is the natural map and $\bar{f}_{n-1}$ is the well-defined map taking $x + \ker f_{n-1}$ to $f_{n-1}(x)$.

Let r_i be the rank of A_i. Then the first diagram implies that $r_n - r_{n-1} + r = 0$, where r is the rank of $A_{n-1}/\operatorname{im} f_n$. Because $\operatorname{im} f_n = \ker f_{n-1}$, the second diagram implies by induction that $r - r_{n-2} + r_{n-3} + \cdots = 0$. Thus we subtract the first from the second to find that $r_n - r_{n-1} + r_{n-2} - \cdots = 0$, as desired.

Exercise 5.6. If $\partial_n = 0$ for all n, then we know that $H_n(S_*) = \ker \partial_n / \operatorname{im} \partial_{n+1} = S_n/\{0\} = S_n$.

Exercise 5.7. If $f : S_* \to S'_*$ is an equivalence, then it has an inverse $g : S'_* \to S_*$. Thus at every n, there is a $g_n : S'_n \to S_n$ so that $g_n \circ f_n = \mathrm{id}_{S_n}$ and $f_n \circ g_n = \mathrm{id}_{S'_n}$. It follows that f_n is an isomorphism for every n.

Conversely, suppose f_n is an isomorphism for every n. Then let $g = \{g_n\}$, where $g_n = f_n^{-1}$. It is clear that f and g are inverses, and so f is indeed an equivalence in Comp.

Exercise 5.8. If the former sequence is exact in Comp, then we know that $\operatorname{im} f = \ker g$. Then the terms of degree n of both $\operatorname{im} f$ and $\ker g$ must be the same. In other words, we must have $\operatorname{im} f_n = \ker g_n$, and so the latter sequence is exact in $\mathscr{A}$.

On the other hand, suppose that the latter sequence is exact for every integer n. Then we know that the degree n terms of $\ker f$ and $\operatorname{im} g$ are the same. Moreover, we know that the differentiation operators are the same because they are defined, in both cases, simply as restrictions of the differentiation operator in S_*. Thus the two complexes are the same, as desired.

Exercise 5.9.

(i) We have the following diagram, where $\bar{\partial}_n$ represents the map taking $s_n + S'_n \mapsto \partial_n(s_n) + S'_{n-1}$.

$$\begin{tikzcd}
\cdots \arrow[r] & S_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "v_{n+1}"] & S_n \arrow[r, "\partial_n"] \arrow[d, "v_n"] & S_{n-1} \arrow[r] \arrow[d, "v_{n-1}"] & \cdots \\
\cdots \arrow[r] & S_{n+1}/S'_{n+1} \arrow[r, "\bar{\partial}_{n+1}"] & S_n/S'_n \arrow[r, "\bar{\partial}_n"] & S_{n-1}/S'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

To show that v is a chain map, we must show that $v_{n-1}\partial_n = \bar{\partial}_n v_n$ for every n. Pick a simplex $\sigma : \Delta^n \to X$. We know that $v_{n-1}(\partial n\sigma) = \partial_n\sigma + S'_{n-1}$. However, we also have $\bar{\partial}_n v_n \sigma = \bar{\partial}_n(\sigma + S'_n) = \partial_n\sigma + S'_{n-1}$. Thus this is indeed a chain map.

Moreover, it is obvious that $\ker v_n = S'_n$ for every n. The definition of a subcomplex implies that the $\partial_n | \ker v_n$ is the operation in S'_*. Thus $\ker v = S'_*$, as desired.

(ii) At each n, we know from the previous part that we have the following commutative diagram in $\mathscr{A}$.

$$\begin{tikzcd}
S_n \arrow[r, "\partial_n"] \arrow[d, "v_n"] & S_{n-1} \arrow[d, "v_{n-1}"] \\
S_n/\ker f_n \arrow[r, "\bar{\partial}_n"] & S_{n-1}/\ker f_{n-1}
\end{tikzcd}$$

By the first isomorphism theorem for groups, however, we know that there is an isomorphism θ_n from $S_n/\ker f_n \to \operatorname{im} f_n$ such that $\theta_n v_n = f_n$.

We claim that $\theta = \{\theta_n\}$ is the desired chain map. To see this, observe that

$$\theta_{n-1}\bar{\partial}_n(\sigma + \ker f_n) = \theta_{n-1}(\partial_n\sigma + \ker f_{n-1}) = f_{n-1}\partial_n\sigma.$$

On the other hand, because $\sigma + \ker f_n = v_n(\sigma)$, we know that

$$\partial'_n\theta_n(\sigma + \ker f_n) = \partial'_n(f_n(\sigma)) = \partial_n f_n \sigma.$$

The two final expressions in the above equations are equal, moreover, because $\{f_n\}$ is itself a chain map.

Exercise 5.10. First, note that both $S'_*/(S'_* \cap S''_*)$ and $(S'_* + S''_*)/S''_*$ are well-defined because everything is abelian. Now consider the map

$$\begin{aligned}\varphi : S'_* &\to \frac{S'_* + S''_*}{S''_*} \\ S'_n &\mapsto S'_n + S''_*.\end{aligned}$$

Note that we have boundary maps

$$\partial'_n : S'_n \to S'_{n-1}$$

and

$$\overline{\partial}_n : \frac{S'_n + S''_n}{S''_n} \to \frac{S'_{n-1} + S''_{n-1}}{S''_{n-1}},$$

where $\overline{\partial}_n$ takes $(s'_n + s''_n) + S''_n$ to $(\partial'_n s'_n + \partial''_n s''_n) + S''_{n-1}$.

We claim that φ is a chain map. To see this, it suffices to show that $\varphi_{n-1}\partial'_n = \overline{\partial}_n\varphi_n$. But for any $\sigma' \in S'_n$, we know that

$$\varphi_{n-1}\partial'_n(\sigma) = \partial'_n\sigma + S''_{n-1} = \overline{\partial}_n(\sigma + S''_n) = \overline{\partial}_n\varphi_n(\sigma),$$

as desired. Moreover, the second isomorphism theorem for groups implies that φ_n is a homomorphism with kernel $S'_n \cap S''_n$, from which it follows that φ is a chain map with $\ker\varphi = S'_* \cap S''_*$. The first isomorphism theorem (Exercise 5.9) implies the result.

Exercise 5.11. Consider the sequence in the problem, namely

$$0 \longrightarrow T_*/U_* \xrightarrow{\ i\ } S_*/U_* \xrightarrow{\ p\ } S_*/T_* \longrightarrow 0\ .$$

Clearly, we have

$$\operatorname{im} i_n = \{t_n + U_n : t_n \in T_n\}.$$

Moreover, we know that $p_n(s_n + U_n) = s_n + T_n$, so

$$\ker p_n = \{s_n + U_n : s_n \in T_n\}.$$

Clearly these are equal.

It now suffices to prove that $\ker i_n = 0$ and $\operatorname{im} p_n = S_n/T_n$. But note that $i_n(t_n + U_n) = 0$ implies that $t_n \in U_n$. Hence $t_n + U_n = 0$ as an element of T_n/U_n as well. Moreover, consider an arbitrary element $s_n + T_n \in S_n/T_n$. It is equal to $p_n(s_n + U_n)$, which proves that p is surjective. Hence the sequence of complexes is exact.

Exercise 5.12. We claim that

$$\ker\left(\sum \partial_n^\lambda\right) = \sum \ker\partial_n^\lambda.$$

If $\sum s_n^\lambda \in \ker\left(\sum \partial_n^\lambda\right)$, then by definition we must have $\partial_n^\lambda(s_n^\lambda) = 0$ for each λ. The converse is also clearly true. Similarly, we find that

$$\operatorname{im}\left(\sum \partial_{n+1}^\lambda\right) = \left\{\sum \partial_{n+1}^\lambda(s_{n+1}^\lambda)\right\} = \sum \operatorname{im}\partial_{n+1}^\lambda.$$

Thus we conclude that

$$H_n\left(\sum S_*^\lambda\right) = \frac{\ker\partial_n}{\operatorname{im}\partial_{n+1}} = \frac{\sum\ker\partial_n^\lambda}{\sum\operatorname{im}\partial_{n+1}^\lambda} = \sum\frac{\ker\partial_n^\lambda}{\operatorname{im}\partial_{n+1}^\lambda} = \sum H_n(S_*^\lambda).$$

Exact Homology Sequences

Exercise 5.13. Suppose $\sum m_\sigma\sigma + S_n(A) = \sum m'_\sigma\sigma + S_n(A)$. This implies that $m_\sigma - m'_\sigma = 0$ for every σ with $\operatorname{im}\sigma \not\subseteq A$. Hence we can pick the unique representative

$$\sum_{\operatorname{im}\sigma\not\subseteq A} m_\sigma\sigma = \sum_{\operatorname{im}\sigma\not\subseteq A} m'_\sigma\sigma,$$

thus showing that this is indeed the free abelian group generated by σ with $\operatorname{im}\sigma \not\subseteq A$.

Exercise 5.14.

(i) It suffices to prove that p_n is surjective and that $\operatorname{im} i_n = \ker p_n$. To see that p_n is surjective, consider the following segment of the long exact sequence:

$$B_n \xrightarrow{p_n} C_n \longrightarrow A_{n-1} \xrightarrow{i_{n-1}} B_{n-1}$$

Note that the map $f_n : C_n \to A_{n-1}$ has $\operatorname{im} f_n = \ker i_{n-1} = 0$, and so $\ker f_n = C_n$. Thus $\operatorname{im} f_n = C_n$, proving surjectivity.

To see that $\operatorname{im} i_n = \ker p_n$, simply consider the following segment:

$$A_n \xrightarrow{i_n} B_n \xrightarrow{p_n} C_n$$

The result immediately follows.

(ii) There exists a map r with $r \circ i = \mathrm{id}_A$. Thus $r_* \circ i_* = \mathrm{id}_{H_n(A)}$, from which it follows that i_* is injective. Theorem 5.8 gives an exact sequence

$$\ldots \longrightarrow H_n(A) \xrightarrow{i_n} H_n(X) \xrightarrow{p_n} H_n(X, A) \xrightarrow{d} \ldots$$

where p_* is induced by the quotient map $S_*(X) \to S_*(X)/S_*(A)$. Since i_* injective implies that i_n is injective, we can apply the previous part to find an exact sequence

$$0 \longrightarrow H_n(A) \xrightarrow{i_n} H_n(X) \xrightarrow{p_n} H_n(X, A) \longrightarrow 0$$

Then Exercise 5.3 implies that $H_n \oplus H_n(X, A) = H_n(X)$, as desired.

(iii) We now have $i \circ r \simeq \mathrm{id}_X$ as well. In particular, since A and X have the same homotopy type, we must have $H_n(A) \cong H_n(X)$ by Corollary 4.24. Thus i_n in the exact sequence given in the previous part must be the identity, and so $\ker p_n = \operatorname{im} i_n = H_n(X)$. But since p_n is surjective, it follows that $H_n(X, A) = 0$, as desired.

Exercise 5.15. We prove this in cases. We will use the follow commutative diagram, where the columns are exact:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\ldots \arrow[r] & S'_{n+1} \arrow[r, "\partial'_{n+1}"] \arrow[d, "i_{n+1}"] & S'_n \arrow[r, "\partial'_n"] \arrow[d, "i_n"] & S'_{n-1} \arrow[r] \arrow[d, "i_{n-1}"] & \ldots \\
\ldots \arrow[r] & S_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "p_{n+1}"] & S_n \arrow[r, "\partial_n"] \arrow[d, "p_n"] & S_{n-1} \arrow[r] \arrow[d, "p_n"] & \ldots \\
\ldots \arrow[r] & S''_{n+1} \arrow[r, "\partial''_{n+1}"] \arrow[d] & S''_n \arrow[r, "\partial''_n"] \arrow[d] & S''_{n-1} \arrow[r] \arrow[d] & \ldots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Case 1. S_* and S'_* are acyclic.

We would like to show than $Z''_n = \ker \partial''_n$ is equal to $B''_n = \operatorname{im} \partial''_{n+1}$. We already know that $B''_n \subseteq Z''_n$. Surjectivity of p implies that we can rewrite Z''_n as

$$Z''_n = p_n(\ker(\partial''_n p_n)) = p_n(\ker(p_{n-1}\partial_n)).$$

This, in turn, can be written as

$$Z''_n = \{p_n s_n : p_{n-1}\partial_n s_n = 0\}.$$

On the other hand, we can rewrite B''_n as

$$B''_n = \operatorname{im}(\partial''_{n+1} p_{n+1} = \operatorname{im}(p_n \partial_{n+1})) = p_n \operatorname{im} \partial_{n+1}.$$

Since S_* is acyclic, we know that $\operatorname{im} \partial_{n+1} = \ker \partial_n$, and so we find that

$$B''_n = \{p_n z_n : \partial_n z_n = 0\}.$$

Now consider an arbitrary element $p_n s_n \in Z''_n$. Since $p_{n-1}\partial_n s_n = 0$, we know that $\partial_n s_n \in \ker p_{n-1} = \operatorname{im} i_{n-1}$, where again we use the fact that S_* is acyclic. Injectivity of i_{n-1} implies the existence of a unique $s'_{n-1} \in S'_{n-1}$ with $i_{n-1}s'_{n-1} = \partial_n s_n$. We know, however, that $\partial_{n-1}\partial_n = 0$, and so

$$0 = \partial_{n-1}\partial_n s = \partial_{n-1} i_{n-1} s'_{n-1} = i_{n-2}\partial'_{n-1} s'_{n-1}.$$

Since i_{n-2} is injective, it follows that $\partial'_{n-1} s_{n-1} = 0$, and so acyclicity of S'_* implies that $s_{n-1} \in \ker \partial'_{n-1} = \operatorname{im} \partial'_n$. In particular, we can write $s'_{n-1} = \partial'_n s'_n$.

Now notice that

$$\partial_n i_n s'_n = i_{n-1}\partial'_n s'_n = i_{n-1} s'_{n-1} = \partial_n s_n,$$

where the last equality follows from the definition of s'_{n-1}. We know that $z_n = s_n - i_n s'_n \in Z_n$ since ∂'_n is a homomorphism. But we also know that

$$p_n z_n = p_n(s_n - i_n s'_n) = p_n s_n - p_n i_n s'_n = p_n s_n,$$

where we use exactness of the columns. In other words, we have a z_n with $\partial_n z_n = 0$, such that $p_n z_n = p_n s_n$. Thus $p_n s_n \in B''_n$, proving that $Z''_n = B''_n$. To be even more explicit, this implies that $H''_n = Z''_n / B''_n = 0$ for all n, proving that S''_* is an acyclic complex as well.

Case 2. S'_* and S''_* are acyclic.

Suppose $s_n \in Z_n$, i.e., that $\partial_n s_n = 0$. Then $\partial''_n p_n = p_{n-1}\partial_n$ implies that $p_n s_n \in \ker \partial''_n = \operatorname{im} \partial''_{n+1}$. Hence write $p_n s_n = \partial''_{n+1} s''_{n+1}$. Since p_{n+1} is surjective, we can find s_{n+1} with $p_{n+1} s_{n+1} = s''_{n+1}$, and so

$$p_n \partial_{n+1} s_{n+1} = \partial''_{n+1} p_{n+1} s_{n+1} = p_n s_n.$$

But then we know that $\partial_{n+1} s_{n+1} - s_n \in \ker p_n = \operatorname{im} i_n$. Thus there exists a unique s'_n with $i_n s'_n = \partial_{n+1} s_{n+1} - s_n$. We can take ∂_n of both sides to find that

$$0 = \partial_n \partial_{n+1} s_{n+1} - \partial_n s_n = \partial_n i_n s'_n = i_{n-1}\partial'_n s'_n,$$

and so it follows that $\partial'_n s'_n = 0$. In particular, we know that $s'_n \in \operatorname{im} \partial'_{n+1}$, so we can find s'_{n+1} whose boundary is s'_n. Recall that we had

$$s_n = \partial_{n+1} s_{n+1} - i_n s'_n.$$

But the last term is equal to $i_n \partial'_{n+1} s'_{n+1} = \partial_{n+1} i_{n+1} s'_{n+1}$, and so this is in turn equal to

$$s_n = \partial_{n+1}(s_{n+1} - i_{n+1} s'_{n+1}).$$

This proves that $s_n \in B_n$, and so $Z_n = B_n$.

Case 3. S_* and S''_* are acyclic.

This final case is handled similarly to the first two, but we lay out the details below. Let $s'_n \in \ker \partial'_n = Z'_n$ be arbitrary. Then $i_n s'_n \in \ker \partial_n = \operatorname{im} \partial_{n+1}$, and so

$$i_n s'_n = \partial_{n+1} s_{n+1}$$

for some s_{n+1}. We know that $p_{n+1} s_{n+1} \in \ker \partial''_{n+1} = \operatorname{im} \partial''_{n+2}$ because $p_n i_n s'_n = 0$. Hence there exists s''_{n+2} with $\partial''_{n+2} s''_{n+2} = p_{n+1} s_{n+1}$. But then it follows that

$$p_{n+1}\partial_{n+2} s_{n+2} = \partial''_{n+2} p_{n+2} s_{n+2} = p_{n+1} s_{n+1},$$

from which it follows that $s_{n+1} - \partial_{n+2} s_{n+2} \in \ker p_{n+1} = \operatorname{im} i_{n+1}$. Thus there exists s'_{n+1} with $i_{n+1} s'_{n+1} = s_{n+1} - \partial_{n+2} s_{n+2}$. We then find that

$$i_n \partial'_{n+1} s'_{n+1} = \partial_{n+1} i_{n+1} s'_{n+1} = \partial_{n+1} s_{n+1} = i_n s'_n.$$

Injectivity implies $s'_n = \partial'_{n+1} s'_{n+1} \in B_n$, thus proving the final case.

Exercise 5.16. To show that $f_\#(Z_n(X,A)) \subseteq Z_n(X',A')$, consider $\gamma \in Z_n(X,A)$. Note that $\gamma \in S_n(X)$, and so Lemma 4.8 implies that

$$\partial'_n f_\# \gamma = f_\# \partial_n \gamma.$$

We know, moreover, that $\partial_n \gamma = \sum m_\sigma \sigma$, where the sum ranges over all σ with $\operatorname{im} \sigma \subseteq A$. Thus it follows that

$$\partial'_n f_\# \gamma = f_\# \left(\sum m_\sigma \sigma \right) = \sum_{\operatorname{im}(f\sigma) \subseteq f(A)} m_\sigma f \sigma.$$

Since $f\sigma$ is a simplex into X' with image contained in A', it follows that this is an element of $S_{n-1}(A')$. Hence we conclude that $f_\# \gamma \in Z_n(X',A')$, as desired.

The proof for boundaries is similar.

Exercise 5.17. As defined, we have that $f_\# : H_n(X,A) \to H_n(X',A')$ is given by

$$f_\# : \overline{\gamma} + \operatorname{im} \overline{\partial}_{n+1} \mapsto f_\# \left(\overline{\gamma} \right) + \operatorname{im} \overline{\partial'}_{n+1},$$

where $\overline{\partial}$ and $\overline{\partial'}$ denote the boundary maps of the quotient complexes $S_*(X)/S_*(A)$ and $S_*(X')/S_*(A')$, respectively, and where

$$\overline{\gamma} \in \ker \overline{\partial}_n = Z_n(X,A)/S_n(A).$$

The third isomorphism theorem gives an isomorphism

$$H_n(X,A) = \frac{Z_n(X,A)/S_n(A)}{B_n(X,A)/S_n(A)} \to \frac{Z_n(X,A)}{B_n(X,A)}$$

which takes

$$\overline{\gamma} + B_n(X,A)/S_n(A) \mapsto \gamma + B_n(X,A).$$

Since $f_\#(\overline{\gamma}) = \overline{f_\#(\gamma)}$, we find that, thinking of $f_\#$ as a map from $Z_n(X,A)/B_n(X,A)$ to a map from $Z_n(X',A')/B_n(X',A')$, it takes

$$\gamma + B_n(X,A) \mapsto f_\#(\gamma) + B_n(X',A'),$$

as desired.

Exercise 5.18. Recall the definition of $\varepsilon_i : \Delta^{n-1} \to \Delta^n$ as taking $\{e_0, \ldots, e_{n-1}\}$ to $\{e_0, \ldots, \hat{e}_i, \ldots, e_{n-1}\}$. Thus we have

$$\partial_n \sigma = \sum_{i=0}^{n} (-1)^i \sigma \varepsilon_i.$$

Since $\sigma\varepsilon_i : \Delta^{n-1} \to X$ has image in A by hypothesis, this is in $S_{n-1}(A)$, as desired.

Reduced Homology

Exercise 5.19. By Theorem 5.6, it is sufficient to show that we have a short exact sequence

$$0 \longrightarrow \widetilde{S}_*(A) \longrightarrow \widetilde{S}_*(X) \longrightarrow S_*(X,A) \longrightarrow 0.$$

When $n \geq 1$, this is clear by Theorem 5.8. When $n = 0$, we have the sequence

$$0 \to \mathbb{Z} \to \mathbb{Z} \to 0 \to 0,$$

which is easily verified to be exact.

Exercise 5.20. Consider the exact sequence

$$H_1(CX) \longrightarrow H_1(CX,X) \longrightarrow \widetilde{H}_0(X) \longrightarrow \widetilde{H}_0(CX).$$

Note that $H_1(CX) = 0$ because CX is contractible. On the other hand, Corollary 5.18 implies that $\widetilde{H}_0(X) \cong \mathbb{Z}^4$, while $\widetilde{H}_0(CX) \cong 0$. Thus the map $H_1(CX,X) \to \widetilde{H}_0(X)$ is surjective. Moreover, its kernel is equal to the image of the map $H_1(CX) \to H_1(CX,X)$, which is simply 0 since $H_1(CX) = 0$. Thus the map is also injective, from which it immediately follows that $H_1(CX,X) \cong \mathbb{Z}^4$.

Exercise 5.21. Consider the exact sequence

$$\widetilde{H}_1(S^0) \longrightarrow \widetilde{H}_1(S^1) \longrightarrow H_1(S^1, S^0) \longrightarrow \widetilde{H}_0(S^0) \longrightarrow \widetilde{H}_0(S^1),$$

which is simply equal to

$$0 \longrightarrow \mathbb{Z} \longrightarrow \boxed{} \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Now note that the first map has im $= 0$, so the second map has ker $= 0$. Thus the second map has im $\cong \mathbb{Z}$, and so the third map has ker $\cong \mathbb{Z}$. Yet we also know that the last map is the zero map, and so the third map has im $= \mathbb{Z}$, from which it follows that the $H_1(S^1, S^0) = \mathbb{Z} \times \mathbb{Z}$.

Exercise 5.22. When $n = 0$, this follows from the exact sequence

$$\widetilde{H}_0(X) \longrightarrow \widetilde{H}_0(X) \longrightarrow H_0(X, X) \longrightarrow 0\ .$$

After all, the first map is the identity, and so the second map is the zero map. But the second map is surjective, and so $H_0(X, X) = 0$.

For $n > 0$, we have the exact sequence

$$\widetilde{H}_n(X) \longrightarrow \widetilde{H}_n(X) \longrightarrow H_n(X, X) \longrightarrow \widetilde{H}_{n-1}(X) \longrightarrow \widetilde{H}_{n-1}(X).$$

The first map is the identity, and so the second map is everywhere zero. Thus the image of the second map, which is the kernel of the third map too, is equal to 0. Since the kernel of the last map, which is 0 (the map is the identity), is equal to the image of the third map, it follows that the third map is everywhere zero. Hence the third map is injective, but also everywhere zero, and so $H_n(X, X)$ must have been 0 in the first place.

6 Excision and Applications

Excision and Mayer–Vietoris

Exercise 6.1. Since A and B are both open, we know that $A^\circ = A$ and $B^\circ = B$. Thus $A^\circ \cup B^\circ = X$, and so we can use the Mayer–Vietoris sequence, along with the fact that $A \cap B = \emptyset$, to find an exact sequence

$$0 = H_n(\emptyset) \xrightarrow{(i_{1*}, i_{2*})} H_n(A) \oplus H_n(B) \xrightarrow{g_* - j_*} H_n(X) \xrightarrow{D} H_{n-1}(\emptyset) = 0 \ .$$

Thus the middle map $g_* - j_*$ is an isomorphism, which proves the result.

Exercise 6.2. We use excision directly. In particular, Excision II gives us an isomorphism $i_* : H_n(B, \emptyset) \to H_n(X, A)$. But $H_n(B, \emptyset) = H_n(B)$ for all $n \geq 0$, and so the conclusion follows.

Exercise 6.3. This is simply a diagram chase. Suppose $D_n([x]) = [x']$. Then by definition of D, we know that there exists some $[z] \in H_n(X_1, X_1 \cap X_2)$ such that $d_n([z]) = [x']$ and $h_n([z]) = q_n([x])$. It thus follows that

$$g_{n-1}(D_n([x])) = [f_{n-1}(x')] = f_{n-1}(d_n([z])).$$

Now set $y = f(x)$, so that $[y] = f_n([x])$. We would like to show that $D'_n([y]) = f_{n-1}(d_n([z]))$. To do this, set $z' = f_n(z)$. Then since d commutes with f by definition (cf. Theorem 5.7), we know that

$$d'_n([z']) = d'_n(f_n(z)) = f_{n-1}(d_n([z])) = g_{n-1}(D_n([x])).$$

Moreover, because h and q are just inclusions, we know that they commute with f. In particular, from the fact that $h_n([z]) = q_n([x])$, and so $f(h_n([z])) = f(q_n([x]))$, we find that

$$h'_n(f([z])) = q'_n(f_n([x])),$$

from which it follows by definition that $h'_n([z']) = q'_n([y])$. Thus it follows that

$$D'(f([x])) = D'([y]) = g_{n-1}(D_n([x])),$$

which proves that the desired diagram commutes.

Exercise 6.4. First note that the condition implies that, for all $n \geq 1$, we have

$$H_n(X_i) = H_n(X_i \cap X_j) = H_n(X_1 \cap X_2 \cap X_3) = 0.$$

Since each X_i is open, we can apply the Mayer–Vietoris sequence. Applying it on X_1 and X_2 gives an exact sequence

$$0 \longrightarrow H_n(X_1 \cup X_2) \longrightarrow 0 \ ,$$

and so we conclude that $H_n(X_1 \cup X_2) = 0$.

Now we can apply Mayer–Vietoris to $X_1 \cup X_2$ and X_3 to find an exact sequence

$$0 \longrightarrow H_n(X) \longrightarrow H_{n-1}((X_1 \cup X_2) \cap X_3). \ .$$

But the last term is exactly $H_{n-1}((X_1 \cap X_3) \cup (X_2 \cap X_3))$.

To see that this is 0, apply Mayer–Vietoris to $X_1 \cap X_3$ and $X_2 \cap X_3$. In particular, setting $H = H_{n-1}((X_1 \cap X_3) \cup (X_2 \cap X_3))$ as the desired homology group, we know that

$$H_{n-1}(X_1 \cap X_3) \oplus H_{n-1}(X_2 \cap X_3) \longrightarrow H \longrightarrow H_{n-2}((X_1 \cap X_3) \cap (X_2 \cap X_3))$$

is exact. If $n > 2$ or if $X_1 \cap X_2 \cap X_3 = \emptyset$, then the first and last terms are clearly 0, which proves that the middle homology group is indeed 0. If, on the other hand, we have $n = 2$ and $X_1 \cap X_2 \cap X_3$ is contractible, then the last term is $\mathbb{Z}$. However, the next map in the Mayer–Vietoris sequence is an injective map, since it is induced by inclusions. Thus we have an exact sequence

$$0 \longrightarrow H \longrightarrow \mathbb{Z} \longrightarrow A \ ,$$

where A is some homology group (in fact, it is $\mathbb{Z}^2$) and the map $\mathbb{Z} \to A$ is injective. Note that the image of the first map is 0, and so the map $H \to \mathbb{Z}$ is injective. But we also know that $\text{im}(H \to \mathbb{Z}) = \ker(\mathbb{Z} \to A) = 0$. Thus $H = 0$, as desired.

Homology of Spheres and Some Applications

No exercises!

Barycentric Subdivision and Proof of Excision

Exercise 6.5. We use induction. In particularly, for Σ^0 we know we have $(0+1)! = 1$ 0-simplexes. Now suppose the statement is true for $n-1$. Using the notation in the definition, note that each n-simplex in Σ^n is spanned by b and an $(n-1)$-simplex in $\operatorname{Sd}\varphi_i$. Since there are $n+1$ total possible φ_i's, and since there are $n!$ total $(n-1)$-simplexes in $\operatorname{Sd}\varphi_i$, it follows that $\operatorname{Sd}\Sigma^n$ has $(n+1)!$ total n-simplexes, as desired,

Exercise 6.6.

(i) By construction, every point is the barycenter of at least one face. Moreover, writing $\Sigma^n = [p_0, \dots, p_{n+1}]$, suppose that b is the barycenter of $[p_{i_1}, \dots, p_{i_k}]$, as well as of $[p_{j_1}, \dots, p_{j_\ell}]$. Then

$$\frac{1}{k+1}(p_{i_1} + \cdots + p_{i_k}) = \frac{1}{\ell+1}(p_{j_1} + \cdots + p_{j_\ell}).$$

This implies linear dependence, unless the two subsimplexes of Σ^n are actually the same simplex.

(ii) This is clear for $n = 0$. Now suppose that the statement is true for $n-1$ Note that, by definition, every n-simplex of Σ^n is spanned by the barycenter b of Σ^n and an $(n-1)$-simplex of $\operatorname{Sd}\varphi_i$. But every face of Σ^n is a subset of Σ^n. Thus we can write an n-simplex of Σ^n as $[b^{\sigma_0}, \dots, b^{\sigma_n}]$ with $\sigma_0 < \cdots < \sigma_{n-1} < \sigma_n = \Sigma^n$.

Exercise 6.7.

(i) Note that $\operatorname{Sd}_1(\delta^1)$ is exactly $b_1. \operatorname{Sd}_0(\partial\delta^1)$. Since Sd_0 is the identity, we know that this is $b_1.(\partial\delta^1)$. But $\partial\delta^1 = e_1 - e_0$, while $b_1 = \frac{1}{2}(e_0 + e_1)$, and so it follows that

$$\begin{aligned}
\operatorname{Sd}_1(\delta^1)(t) &= b_1.e_1(t) - b_1.e_0(t) \\
&= \left(\frac{t}{2}(e_0 + e_1) + (1-t)e_1\right) - \left(\frac{t}{2}(e_0 + e_1) + (1-t)e_0\right).
\end{aligned}$$

Note that both terms within the large parentheses are 1-simplexes, and so we cannot "cancel" the $\frac{t}{2}(e_0 + e_1)$ terms.

For $n = 2$, we would like to evaluate $b_2. \operatorname{Sd}_1(\partial\delta^2)$. Note that $\partial\delta^2 = [e_1, e_2] - [e_0, e_2] + [e_0, e_1]$. Thus we know, either by using the same argument as before or by appealing to the case $n = 1$ in part (ii) below, that

$$\begin{aligned}
\operatorname{Sd}_1(\partial\delta^2)(t) = &\left(\frac{t}{2}(e_1 + e_2) + (1-t)e_2\right) - \left(\frac{t}{2}(e_1 + e_2) + (1-t)e_1\right) \\
&- \left(\frac{t}{2}(e_0 + e_2) + (1-t)e_2\right) + \left(\frac{t}{2}(e_0 + e_2) + (1-t)e_0\right) \\
&+ \left(\frac{t}{2}(e_0 + e_1) + (1-t)e_1\right) - \left(\frac{t}{2}(e_0 + e_1) + (1-t)e_0\right).
\end{aligned}$$

Thus we may evaluate $\operatorname{Sd}_2(\delta^2)$ on a term-by-term basis. For example, the first term of $\operatorname{Sd}_2(\delta^2)(t_1, t_2)$ is

$$t_1 b_2 + (1-t_1)\left(\frac{t_2/(1-t_1)}{2}(e_1 + e_2) + (1 - t_2/(1-t_1))e_2\right) = t_1 b_2 + \left(\frac{t_2}{2}(e_1 + e_2) + (1 - t_1 - t_2)e_2\right).$$

We can do this with each term to find $\operatorname{Sd}_2(\delta^2)$.

(ii) We can simply evaluate this using the previous part. In particular, we have

$$\begin{aligned}
\operatorname{Sd}_1(\sigma) &= \sigma_\# \operatorname{Sd}_1(\delta^1) \\
&= \left(\frac{t}{2}(\sigma(e_0) + \sigma(e_1)) + (1-t)\sigma(e_1)\right) - \left(\frac{t}{2}(\sigma(e_0) + \sigma(e_1)) + (1-t)\sigma(e_0)\right).
\end{aligned}$$

Similarly, by replacing each e_i in $\operatorname{Sd}_2(\delta^2)$ with $\sigma(e_i)$, we have $\operatorname{Sd}_2(\sigma)$.

Exercise 6.8. It is sufficient to show commutativity for generators $\sigma : \Delta^n \to X$. But note that $f_\# \operatorname{Sd}_n(\sigma) = f_\#\sigma_\# \operatorname{Sd}_n(\delta^n)$. However, since $f_\#\sigma_\# = (f \circ \sigma)_\#$, it follows that this is in turn equal to $(f \circ \sigma)_\# \operatorname{Sd}_n(\delta^n) = \operatorname{Sd}_n(f \circ \sigma) = \operatorname{Sd}_n f_\#\sigma$. This proves commutativity, as desired.

Exercise 6.9. Recall that the j-th face of σ is $\sigma\varepsilon_j : [e_0, \ldots, e_{n-1}] \to [e_0, \ldots, \hat{e}_i, \ldots, e_n]$. Now observe that ε_j is clearly affine. After all, we know that

$$\varepsilon_j\left(\sum_i t_i e_i\right) = \sum_{i<j} t_i e_i + \sum_{i\geq j} t_i e_{i+1} = \sum_i t_i \varepsilon_j(e_i).$$

Thus we know that

$$\sigma\varepsilon_j\left(\sum t_i e_i\right) = \sigma\left(\sum t_i \varepsilon_j(e_i)\right) = \sum t_i \sigma(\varepsilon_j(e_i)).$$

Thus $\sigma\varepsilon_j$ is affine. Since $\sigma\varepsilon_j(e_i)$ is either e_i (if $i < j$) or e_{i+1} (if $i \geq j$), it follows that the vertex set of $\sigma\varepsilon_j$ is a (proper) subset of the vertex set of σ. Since $\partial\sigma$ is just an alternating sum of faces of σ, it follows that $\partial\sigma$ is affine whenever σ is.

Exercise 6.10. Recall the definition of a cone:

$$b.\sigma(t_0, \ldots, t_{n+1}) = \begin{cases} b & \text{if } t_0 = 1, \\ t_0 b + (1-t_0)\sigma\left(\frac{t_1}{1-t_0}, \ldots, \frac{t_{n+1}}{1-t_0}\right) & \text{if } t_0 < 1. \end{cases}$$

It is clear that b is affine. Since the case $t_0 < 1$ results in the sum of affine maps, it follows that this is also affine.

Now note that $b.\sigma(e_0) = b$ and $b.\sigma(e_i) = \sigma(e_i)$ for $i \neq 0$. Thus the vertex set of $b.\sigma$ is the union of $\{b\}$ and the vertex set of σ. Note that $\operatorname{Sd}_0 \sigma$ is affine whenever $\sigma : \Delta^0 \to E$ is affine. If Sd_{n-1} preserves affineness, then note that Sd_n must as well. After all, we know that $\operatorname{Sd}_n \sigma = b_n . \operatorname{Sd}_{n-1}(\partial\sigma)$ is the cone of some point $b_n \in E$ and the affine function $\operatorname{Sd}_{n-1}(\partial\sigma)$. (Note that this last function is affine because $\partial\sigma$ is, by Exercise 6.9, affine.) The result follows.

More Applications to Euclidean Space

Exercise 6.11. Writing $(1 + a^n_\#)\gamma$ as $\gamma + a^n_\#\gamma$, note that

$$\begin{aligned}\partial(\gamma + a^n_\#\gamma) &= \partial\gamma + a^{n-1}_\#\partial\gamma \\ &= \gamma(e_1) - \gamma(e_0) + (-\gamma(e_1)) - (-\gamma(e_0)).\end{aligned}$$

But recall that $-\gamma(e_1) = \gamma(e_0)$ and $-\gamma(e_0) = \gamma(e_1)$, and so we know that the terms cancel out to 0. Thus $(1 + a^n_\#)\gamma$ is a 1-cycle.

Exercise 6.12. We can simply compute evaluate $(1 + a^n_\#)(1 - a^n_\#)$ on a simplex σ. In particular, we find that

$$\begin{aligned}(1 + a^n_\#)(1 - a^n_\#)\sigma &= (1 + a^n_\#)(\sigma - a^n\sigma) \\ &= \sigma + a^n_\#\sigma - a^n\sigma - a^n_\#(a^n\sigma) \\ &= \sigma + a^n\sigma - a^n\sigma - a^n a^n\sigma.\end{aligned}$$

But of course we have $\sigma = a^n a^n \sigma$, and so the terms all cancel out. We can simply extend to any 1-chain γ.

Exercise 6.13. Again, we evaluate the expression on a simplex σ and find that

$$\begin{aligned}(1 + a^n_\#)(1 + a^n_\#)\sigma &= (1 + a^n_\#)(\sigma + a^n\sigma) \\ &= \sigma + a^n\sigma + a^n\sigma + a^n a^n\sigma \\ &= 2\sigma + 2a^n\sigma = 2(1 + a^n_\#)\sigma.\end{aligned}$$

As before, we can extend.

Exercise 6.14. Letting τ be, as in Theorem 6.22, the southerly path in S^1 from $a^1(y)$ to y, recall that the homology class of the cycle $\sigma + \tau$ generates all of $H_1(S^1)$. But notice that $[(1 + a^1_{\#})\sigma] = [\sigma + a^1\sigma] = [\sigma + \tau]$, which proves the result.

Exercise 6.15. Suppose $f : S^1 \to \mathbb{R}$ is continuous. Note that such a function is effectively a continuous function $f : [0, 1] \to \mathbb{R}$ with $f(0) = f(1)$, and so the intermediate value theorem implies the result.

Exercise 6.16. Suppose $S \subseteq \mathbb{R}^2$ is homeomorphic to S^2. Then there is a function $\varphi : S^2 \to S \hookrightarrow \mathbb{R}^2$, where the $S^2 \to S$ part is a homeomorphism. The Borsuk-Ulam theorem, however, implies that there exists some point $x \in S^2$ with $\varphi(x) = \varphi(-x)$. But φ is an injection, a contradiction.

Exercise 6.17. This is obvious from Borsuk-Ulam. Since there exists an x with $f(x) = f(-x)$, but $f(-x) = -f(x)$ for all x, it follows that there exists an x with $f(x) = -f(x)$, i.e., with $f(x) = 0$.

Exercise 6.18. This follows the same proof as that of Borsuk-Ulam. In particular, we use the function

$$\begin{aligned} g(x) : S^n &\to S^{n-1} \\ x &\mapsto \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}. \end{aligned}$$

This would be an antipodal map, a contradiction.

Exercise 6.19. Suppose $a^n(F_i) \cap F_i = \emptyset$ for $i = 1, \ldots, n$. There exist functions $g_i : S^n \to I$ with $g_i(F_i) = 0$ and $g_i(a^n F_i) = 1$. Then define $f : S^n \to \mathbb{R}^n$ by $f(x) = (g_1(x), \ldots, g_n(x))$. Note that Exercise 6.18 implies that there exists some $x_0 \in S^n$ with $f(x_0) = f(-x_0)$. Thus it follows that

$$g_i(x_0) = g_i(-x_0) = g_i(a^n x_0)$$

for all i. Hence if $x_0 \in F_i$ then the left side of the equation is 0 while the right side is 1, and if $x_0 \in a^n F_i$ then the left side is 1 while the right side is 0. Either way, this is a contradiction, and so it follows that $x_0, a^n x_0 \notin F_i$ for all i. Thus $x_0 a^n x_0 \in F_{n+1}$.

I have not come up with a counterexample in the $n + 2$ case, unfortunately.

Exercise 6.20. Suppose $A \subseteq S^n$ is a subspace, and suppose that $h : S^n \to A$ is a homeomorphism. Invariance of domain implies that A is open in S^n. But we also know by compactness of S^n that A must be compact, and hence closed in its ambient space. Thus A is clopen. Since A is obviously nonempty, it follows by connectedness that $A = S^n$.

Exercise 6.21. This follows because we can just write $S^n = \mathbb{R}^n \cup \{\infty\}$. Hence any open set in $\mathbb{R}^n$, including $\mathbb{R}^n$ itself, is just an open set in S^n.

Walking through this in more detail, suppose $U, V \subseteq \mathbb{R}^n$ with a homeomorphism $h : U \to V$ and with U open. Then U is an open subset of S^n because $\mathbb{R}^n$ is open in S^n. Hence invariance of domain on S^n implies that V is open in S^n, and so $V \cap \mathbb{R}^n = V$ is open in $\mathbb{R}^n$ as well.

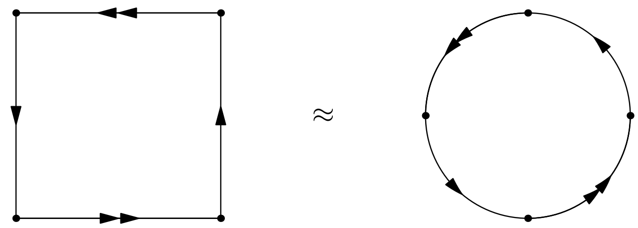
Exercise 6.22. Suppose $\varphi : X \to Y$ is a homeomorphism. Then we can simply pass to a homeomorphism between U and V in X to a homeomorphism between $\varphi(U)$ and $\varphi(V)$ in Y. Since every open set in Y is of the form $\varphi(U)$ for some open U in X, the result follows immediately.

Exercise 6.23. Consider the map $h : D^n \to \overline{D_{\frac{1}{2}}(0)}$ defined by $h(x) = \frac{x}{2}$. It effectively shrinks $D^n n$ down to the closed ball with radius $\frac{1}{2}$. Obviously the two disks are homeomorphic. But D^n is open while $\overline{D_{\frac{1}{2}}(0)}$ is not.

7 Simplicial Complexes

Definitions

Exercise 7.1.



Exercise 7.2. Consider some (nondegenerate) triangle with vertices P, x_0, y_0 in $\mathbb{R}^2$. Then define x_i to be the midpoint of P and x_{i-1}, and similarly define y_i. Then the union X of the triangle with all the line segments $x_i y_i$ is compact and connected.

We claim that it is not a polyhedron. Otherwise, there exists some simplicial complex K admitting a homeomorphism $h : |K| \to X$. But observe that K must have an infinite vertex set.

To see this, for each i, define s_i to be

$$s_i = \bigcap_{h^{-1}(x_i) \in s} s,$$

where s ranges over all simplices of K. Note that this intersection is over a nonempty set because $\bigcup s = |K|$, so there must exist some s containing $h^{-1}(x_i)$. Moreover, there are only finitely many simplices, so the intersection exists. Condition (ii) implies that s_i is a common face of s, and thus is a simplex. It must be 0-dimensional since the segment Px_i, $x_i y_i$, and $x_0 x_i$ cannot all be part of the same 1-simplex. In other words, x_i must be a common face of two 1-simplices, and so it must be a point.

Hence there are infinitely many vertices of K, a contradiction.

Exercise 7.3. Note that the upper right and lower right triangles are the same.

Exercise 7.4.

(i) The forwards direction is just the definition of the subspace topology. To see the backwards direction, suppose $F \cap s$ is closed in s for every $s \in K$. Each s is closed in $|K|$, so $F \cap s$ is closed in $|K|$. Since there are finitely many s and $\bigcup s = |K|$, it follows that we can take the union of all $F \cap s$. In particular, we have

$$F = \bigcup_{s \in K} (F \cap s)$$

is the finite union of closed sets, hence is itself closed in $|K|$.

(ii) This is obviously true if K has dimension 0.

If K (and hence s) has dimension > 1, then consider the complement of s°:

$$(s^\circ)^c = (|K| - s) \cup \dot{s}.$$

Then notice that

$$\big[(|K| - s) \cup \dot{s}\big] \cap s = \dot{s},$$

which is closed in s. Suppose $t \in K$ is not equal to s. Then consider

$$A_t = \big[(|K| - s) \cup \dot{s}\big] \cap t.$$

If $s \cap t = \emptyset$, then $A_t = \emptyset$ is closed in t. Otherwise, we know that $s \cap t$ is a face of t. Since s is of highest dimension, we know that either $s = t$, which we already took care of above, or $s \cap t$ is part of $\dot{s}$, in which case we know that

$$\dot{s} \cap t = s \cap t, \quad (|K| - s) \cap t = t - s \cap t.$$

Hence $A_t = t$, which is still closed in t.

The previous part proves the result.

Exercise 7.5. We begin by showing $s^\circ \cap t^\circ = \emptyset$ when $s \neq t$. Note that

$$s^\circ \cap t^\circ = (s - \dot{s}) \cap (t - \dot{t}) = s \cap t - \dot{s} \cap t - s \cap \dot{t}.$$

But $s \cap t$ is a face of both s and t. It can't be equal to both s and t since $s \neq t$. Thus $s \cap t$ is a *proper* face of at least one of s and t, say s. This means that $s \cap t$ is part of $\dot{s}$, and thus is in $\dot{s} \cap t$. This proves disjointness.

To see that $\bigcup s^\circ = |K|$, simply do this in the case of K as a simplex, and take unions. (To do this when K is a single simplex, use induction.)

Exercise 7.6. The backwards direction is obvious by the definition of st. For the forwards direction, suppose

$$x \in \operatorname{st}(p) = \bigcup_{p \in \operatorname{Vert}(t)} t^\circ.$$

Then we know that $x \in t^\circ$ for some t having p as a vertex. Uniqueness implies that $s = t$, so $p \in \operatorname{Vert}(s)$.

Exercise 7.7.

(i) Obviously the union is $|K|$ because every $s \in K$ has at least one vertex, hence is contained in at least one star. To see that $\operatorname{st}(p) \subseteq |K|$ is open, notice that

$$(\operatorname{st}(p))^c = \bigcup_{p \notin \operatorname{Vert}} s^\circ.$$

Intersect this with $t \in K$. If $p \notin \operatorname{Vert}(t)$, then this intersection is equal to t since no simplex of $\dot{t}$ can have p as a vertex. If $p \in \operatorname{Vert}(t)$, then write $t = [p, p_1, \ldots, p_k]$. The intersection can be seen to simply be $\{p_1, \ldots, p_k\}$, which is obviously closed. Thus Exercise 7.4 implies the result.

(ii) If $x \in \operatorname{st}(p)$, then $x \in s^\circ$ for some s with $p \in \operatorname{Vert}(s)$. Since $x, p \in s$ and s is convex, it follows that the line segment is also contained in $\operatorname{st}(p)$.

Exercise 7.8. The forwards direction is because $[p_0, \ldots, p_n]$ is in the intersection. The backwards direction is because there must exist some simplex $[p_0, \ldots, p_n, q_0, \ldots, q_m] \in K$. Since any face of a simplex in K is also in K, it follows that $[p_0, \ldots, p_n]$ is a simplex in K.

Simplicial Approximation

Exercise 7.9. In the forwards direction, suppose φ is a simplicial map. If $\bigcap \operatorname{st}(p_i) \neq \emptyset$, then there exists a simplex in K with vertices $[p_i]$. The definition implies that there must exist a simplex with vertices $[\varphi(p_i)]$, proving this direction. The backwards direction follows directly from Exercise 7.8.

Exercise 7.10. Suppose φ is a simplicial approximation to f, and suppose $x \in |K|$ with $f(x) \in s^\circ$. Write $x \in t^\circ$ for $t \in K$, and write $t = [p_1, \ldots, p_n]$. Then we know that $x \in \operatorname{st}(p_i)$ implies that $f(x) \in \operatorname{st}(\varphi(p_i))$, so that $s^\circ \subseteq \operatorname{st}(\varphi(p_i))$. Thus s has $\varphi(p_i)$ as a vertex for each $i = 1, \ldots, n$.

Hence $|\varphi|(x)$, which is determined by $\varphi(p_i)$, is in s by affineness.

Now suppose that $f(x) \in s^\circ$ implies $|\varphi|(x) \in s$. Let p be some vertex of K so that $x \in \operatorname{st}(p)$. Then $f(x) \in s^\circ$, so $|\varphi|(x) \in s$. Hence $\varphi(p)$ is a vertex of s by affineness and the definition of $|\varphi|$, from which it follows that

$$f(x) \in s^\circ \subseteq \operatorname{st}(\varphi(p)).$$

We can take the union over all $x \in \operatorname{st}(p)$:

$$\bigcup_{x \in \operatorname{st}(p)} f(x) \subseteq \operatorname{st}(\varphi(p)).$$

Of course, this left side is exactly $f(\operatorname{st}(p))$, and so we're done.

Exercise 7.11. Suppose $\varphi : K \to L$ is a simplicial approximation. Consider the obvious homotopy:

$$H(t, x) = (1 - t)|\varphi|(x) + t f(x).$$

We can do this because $|\varphi|(x)$ and $f(x)$ are, by Exercise 7.10, in the same simplex.

Exercise 7.12.

(i) This is true because it's true for simplices.

(ii) Order the vertices of K, and define $\varphi(b^s)$ to be the smallest vertex of s under this order. We claim that this gives a simplicial approximation to the identity. Consider a vertex b^s of $\mathrm{Sd}(K)$. Then we know that

$$f(\mathrm{st}(b^s)) = s^\circ \subseteq \mathrm{st}(\varphi(b^s))$$

by the definition of $\varphi(b^s)$, where f is the identity.

(iii) There exists a homeomorphism $g : |L| \to X$. If $g(v)$, then we are done. Otherwise, we know that $x \in g(s^\circ)$ for some unique $s \in L$. Consider the subdivision K of L obtained by drawing lines from s to every vertex of s. This gives a function $h : |K| \to X$ which is equal to g, and thus is a homeomorphism, as desired.

Exercise 7.13. Suppose that $\sum \lambda_i b^{s_i} = 0$. Since $s_0 < \cdots < s_q$, we knwo that there exists some vertex p_q which only appears in b^{s_q}, so $\lambda_q = 0$. But then there is a vertex p_{q-1} which only appears in $b^{s_{q-1}}$, so $\lambda_{q-1} = 0$, and so on. Thus $\lambda_i = 0$ for all i, proving independence.

Exercise 7.14. Every point of $\mathrm{Sd}\, K$ is contained in a unique open simplex of K, so it follows that an open simplex of $\mathrm{Sd}\, K$ can be contained in at most one open simplex of K. To see that there is at least one such simplex, note that $[b^{s_0}, \dots, b^{s_q}]^\circ$ is contained in s_q°.

Exercise 7.15. This follows from the triangle inequality:

$$|x - y| \leq |x - p| + |p - y| \leq 2\mu,$$

because x and p are in one simplex, and y and p are in another.

Exercise 7.16. Write $s = [b^{s_0}, \dots, b^{s_q}]$, where $s_0 < \cdots < s_q$. Then $\operatorname{diam} s = \sup ||b^{s_i} - b^{s_j}||$. If $i < j$, then we know that

$$||b^{s_i} - b^{s_j}|| \leq \frac{n_j}{n_j + 1} \operatorname{diam} s_j,$$

where $n_j = \dim s_j$. But $\operatorname{diam} s_j \leq \operatorname{mesh} K$ since $s_j \in K$, and $\frac{n_j}{n_j+1} \leq \frac{n}{n+1}$, since $n_j \leq n$. Hence it follows that

$$\operatorname{diam} s \leq \frac{n}{n+1} \operatorname{mesh} K,$$

and so $\operatorname{mesh} \mathrm{Sd}\, K \leq (n/n+1) \operatorname{mesh} K$. Induction implies the general result.

Exercise 7.17. If $s \in K^{(q)}$, then $s = [p_0, \dots, p_r]$ for some $r \leq q$. Thus $\varphi(s) = [\varphi(p_0), \dots, \varphi(p_r)] \in L^{(q)}$, as desired.

Exercise 7.18. Let b be the barycenter of the $(n+1)$-simplex, and consider

$$f(x) = \frac{x - b}{||x - b||} + b.$$

This is the desired homeomorphism.

Abstract Simplicial Complexes

No exercises!

Simplicial Homology

Exercise 7.19. In general, there are $\binom{n+2}{q+1}$ total q-simplices in an $(n+1)$-simplex. Since S^n is the n-skeleton of such a simplex, it follows that we must simply evaluate

$$\chi(S^n) = \sum_{q=0}^{n} \binom{n+2}{q+1}(-1)^q = \sum_{q=0}^{n+2} \binom{n+2}{q}(-1)^{q+1} + \binom{n+2}{0} + \binom{n+2}{n+2} = 2$$

when q is even. When q is odd, the last term is negative, and we find that $\chi(S^n) = 0$.

Exercise 7.20. Here, we have $\alpha_2 = 18$, $\alpha_1 = 27$, and $\alpha_0 = 9$. Thus $\chi(T) = 18 - 27 + 9 = 0$.

Exercise 7.21. Note that i is obviously an injection. Moreover, since the element $\sum b \in B_1 m_b b + \sum_{c \in B_2} m_c c \in F(b)$ is equal to

$$\sum b \in B_1 m_b b + \sum_{c \in B_2} m_c c \in F(b) = p\left(\sum m_b b, \sum -m_c c\right),$$

we see that p is surjective. Finally, note that

$$\begin{aligned}\ker p &= \left\{\left(\sum m_b b, \sum m_c c\right) : \sum m_b b = \sum m_c c\right\}\\ &= \{(x,x) : x \in F(B_1) \cap F(B_2)\}\\ &= \{(x,x) : x \in F(B_1 \cap B_2)\} = \operatorname{im} i,\end{aligned}$$

which completes the proof of exactness.

Comparison with Singular Homology

Exercise 7.22. For $q \geq 1$, the complexes are the same. If $q = 0$, we use the same argument as in Theorem 5.17, in particular, by restricting our attention to the ending:

$$0 \longrightarrow \ker \tilde{\partial}_0 \hookrightarrow C_0(K) \xrightarrow{\tilde{\partial}_0} C_{-1}(K) \longrightarrow 0 \ .$$

Exercise 7.23. This is simply because $\ker \tilde{\partial}_{-1} = C_{-1}(K)$.

Exercise 7.24.

(i) We can simply use the straight line homotopy between $\varphi(p)$ and $\psi(p)$ for all vertices p of K; the rest of the point follow by affineness. The reason this works is simply because $\varphi(p)$ and $\psi(p)$ belong to the same simplex, which is convex.

(ii) Since $|\varphi| \simeq |\psi$, we know that $|\varphi|_* = |\psi|_*$, which in turn implies that $\varphi_* = \psi_*$ by Theorem 7.22.

Exercise 7.25. Let L be a line segment, along with its endpoints and its midpoints. Thus it is composed of two 1-simplices, and three 0-simplices. Then let φ_1 map a 1-simplex to the left side of L, and φ_2 map it to the right side of L. Finally, if φ_3 maps the 1-simplex to the midpoint, it follows that $\varphi_1 \sim \varphi_3 \sim \varphi_2$, but obviously $\varphi_1 \not\sim \varphi_2$.

Exercise 7.26.

(i) This is clear by mapping the base points together, and mapping a given equivalence class to the corresponding equivalence class. For example, we have some point $x \in X$, then the homeomorphism would take $[[x]] \in (X \vee Y) \vee Z$ to $[x] \in X \vee (Y \vee Z)$. Similarly, it would take $[[y]] \mapsto [[y]]$ and $[z] \mapsto [[z]]$.

(ii) For $i = 1, 2$, there exists a simplicial complex L_i and a homeomorphism $h_i : |L_i| \to K_i$. Fix some vertex $x_i \in \mathrm{Vert}(L_i)$. Then let $L = L_1 \vee L_2$. Then, identifying each L_i with the natural corresponding set in L, we can apply Theorem 7.17 to find the exact senuence

$$\ldots \longrightarrow H_n(L_1 \cap L_2) \longrightarrow H_n(L_1) \oplus H_n(L_2) \longrightarrow H_n(L) \longrightarrow H_{n-1}(L_1 \cap L_2) \longrightarrow \ldots.$$

Of course, we have $L_1 \cap L_2$ is a singleton, so the homology groups are 0. Thus, if $n \geq 2$, then we know that $H_n(L_1 \cap L_2) = H_{n-1}(L_1 \cap L_2) = 0$, and so $H_n(L) \cong H_n(L_1) \oplus H_n(L_2)$, as desired. Otherwise, we can simply use the tail:

$$\ldots \longrightarrow H_1(L) \longrightarrow H_0(L_1 \cap L_2) \longrightarrow H_0(L_1) \oplus H_0(L_2) \longrightarrow H_0(L) \longrightarrow 0.$$

If L_i has c_i components, then notice that L has c_1+c_2-1 components. Since the map $H_0(L_1)\oplus H_0(L_2) \to H_0(L)$ is surjective, it follows that its kernel is $\mathbb{Z}$ (or, more accurately, a free abelian group of rank 1). Hence the image of $H_0(L_1 \cap L_2) \to H_0(L_1) \oplus H_0(L_2)$ is $\mathbb{Z}$. The fact that $H_0(L_1 \cap L_2) = \mathbb{Z}$ implies that this map is an isomorphism, thus with empty kernel. Finally, we conclude that the image of $H_1(L) \to H_0(L_1 \cap L_2)$ is trivial, and so we again have the exact sequence

$$0 H_1(L_1) \oplus H_1(L_2) \longrightarrow H_1(L) \longrightarrow 0\ .$$

The result follows.

(iii) Use Corollary 7.19. In particular, let K_q consist of all proper faces of an oriented $(q+1)$-simplex. Then the corollary implies that $H_q(K_q) = \tilde{H}_q(K_q) = \mathbb{Z}$ and $H_r(K_q) = 0$ for any $r \neq q$. (Note that reduced homology matches the regular homology since $q \geq 1$.) Thus the previous part shows that the space

$$\bigvee_{q=1}^{n} \bigvee_{i=1}^{m_q} K_q,$$

where the wedge occurs at some identified vertex, satisfies the desired properties.

Exercise 7.27.

(i) This follows directly from the five lemma and Theorem 7.22, namely by looking at the following commutative diagram with exact rows:

$$\begin{tikzcd}
\ldots \arrow[r] & H_n(L) \arrow[r] \arrow[d] & H_n(K) \arrow[r] \arrow[d] & H_n(K,L) \arrow[r] \arrow[d] & H_{n-1}(L) \arrow[r] \arrow[d] & H_{n-1}(K) \arrow[r] \arrow[d] & \ldots \\
\ldots \arrow[r] & H_n(|L|) \arrow[r] & H_n(|K|) \arrow[r] & H_n(|K|,|L|) \arrow[r] & H_{n-1}(|L|) \arrow[r] & H_{n-1}(|K|) \arrow[r] & \ldots
\end{tikzcd}.$$

(ii) This follows from the previous part, Corollary 7.17, and Theorem 7.22.

Exercise 7.28. We can simply use the straight line homotopy to p. Exercise 7.7 implies that this is well-defined.

Exercise 7.29. In particular, we must show that

$$\left(\bigcap L_{\alpha_i}\right) \cap \left(\bigcap L_{\beta_i}\right) \neq \emptyset.$$

But notice that $\sigma_0 < \sigma_1 < \cdots < \sigma_q$ implies that $\sigma_0 \in L_{\beta_i}$ for each β_i. We also know that $\sigma_0 \in L_{\alpha_0} \cap \cdots \cap L_{\alpha_q}$, and so it follows that σ_0 is in the displayed intersection above. Hence g and f are contiguous.

Exercise 7.30. We have the following exact sequence:

$$H_q(M \cap L_1) \longrightarrow H_q(M) \oplus H_q(L_1) \longrightarrow H_q(M \cup L_1) \longrightarrow H_{q-1}(M \cap L_1).$$

The conditions imply that $H_q(M) \oplus H_q(L_1) = H_q(M)$ and the two outermost terms are both trivial. Thus $H_q(M) \cong H_q(M \cup L_1)$.

Now consider the following exact sequence:

$$H_q((M \cup L_1) \cap L_2) \longrightarrow H_q(M \cup L_1) \oplus H_q(L_2) \longrightarrow H_q(M \cup L_1 \cup L_2) \longrightarrow H_{q-1}((M \cup L_1) \cap L_2).$$

But notice that

$$(M \cup L_1) \cap L_2 = (M \cap L_2) \cup (L_1 \cap L_2) = M \cap L_2$$

since $L_1 \cap L_2 \subseteq M$. Hence the flanking terms of the exact sequence displayed above are again 0. Since L_2 is acyclic, it follows that $H_q(M \cup L_1) \cong H_q(M \cup L_1 \cup L_2)$. Repeating this proves the result.

Calculations

Exercise 7.31. Consider the Klein bottle, as in Figure 1. Let P be the entire square. Then we can define

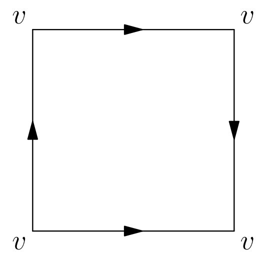


Figure 1: The Klein bottle

the adequate subcomplex with chains

$$E_2 = \langle P\rangle, \quad E_1 = \langle a\rangle \oplus \langle b\rangle, \quad E_2 = \langle v\rangle.$$

We have

$$\begin{aligned} \partial P &= a + b + a - b \\ \partial a = \partial b &= 0 \\ \partial v &= 0. \end{aligned}$$

Hence it follows that we have

$$\begin{array}{lll} Z_2 = 0, & Z_1 = \langle a\rangle \oplus \langle b\rangle, & Z_0 = \langle v\rangle, \\ B_2 = 0, & B_1 = \langle 2a\rangle, & B_0 = 0. \end{array}$$

The results are obvious.

Exercise 7.32. This time, if we let a denote each edge and v denote each vertex, we have

$$\partial P = ka, \quad \partial a = 0, \quad \partial v = 0.$$

Thus we now have

$$\begin{array}{lll} Z_2 = 0, & Z_1 = \langle a\rangle \oplus \langle b\rangle, & Z_0 = \langle v\rangle, \\ B_2 = 0, & B_1 = \langle ka\rangle, & B_0 = 0. \end{array}$$

This gives the desired homology groups.

Fundamental Groups of Polyhedra

Exercise 7.33. This is true because equality is an equivalence relation.

Exercise 7.34.

(i) It suffices to show that $o(()\alpha)$ cannot be changed in a single move. But this is clear. In particular, using the definition, note that $o(()\alpha)$ is $o(()\beta)$ if $\beta \neq \emptyset$, and is p if β is empty. The same holds for $o(()\alpha')$, so $o(()\alpha)$ is preserved. Similarly, $\mathrm{e}(()\alpha) = \mathrm{e}(()\alpha')$.

(ii) Again, it suffices to show this for a single elementary move. We can further assume that $\beta = \beta'$. Write $\alpha = \gamma(p,q)(q,r)\delta$ and $\alpha' = \gamma(p,r)\delta$. Then

$$\alpha\beta = \gamma(p,q)(q,r)\delta\beta = \gamma(p,r)\delta\beta = \alpha'\beta'.$$

(Recall $\beta = \beta'$.)

Exercise 7.35. An edge path, by definition, only goes along the 1-skeleton. Thus K being connected automatically implies that $K^{(1)}$ is.

If $K^{(1)}$ is connected, then let $x, y \in |K|$. There are unique open simplices s°, t° with $x \in s^\circ$ and $y \in t^\circ$. Pick vertices v and w of s and t, respectively. Then consider the path taken by going straight line from x to v, then along the edges to w, then along a straight to y. Hence $|K|$ is connected (indeed, path-connected).

If $|K|$ is connected, then $|K|$ is clearly path-connected.

Finally, if $|K|$ is path-connected, then we can find edge paths between any two vertices of K in the following manner: Each time the path crosses the 1-skeleton, say along the edge between v and w, pick either v and w and append that vertex (or, rather, the edge between that vertex and the previous one) to the edge path. That this works is clear.

Exercise 7.36. This is exactly the proof of Theorem 3.6, with γ as the edge path from p_0 to p_1.

Exercise 7.37. Since an elementary move only moves across a 2-simplex, it follows that the edge path group is only dependent on the 2-skeleton.

Exercise 7.38.

(i) This is clear.

(ii) If v and w are in the same component as some point x, then by taking an edge path from v to x, then from x to w, we have an edge path between v and w. This proves that components are connected.

Obviously the union of the components is K. To see that the unions are disjoint, suppose $v \in [x] \cap [y]$ and $w \in [x]$. Then the path $w \to x \to v \to y$ implies that $w \in [y]$. Since w was arbitrary, and since $w \in [y]$ would similarly imply $w \in [x]$, it follows that $[x] = [y]$. This proves disjointness.

(iii) Suppose $[\alpha] \in \pi(K, x)$. Then we claim that $[\alpha] \in \pi(L, x)$. But this is simply because any vertex along α is necessarily connected to p via an edge path, hence belongs to L.

Exercise 7.39.

(i) Write $\alpha = e_1 \dots e_m$ and $\beta = e_{m+1} \dots e_{m+n}$. Then $(\alpha\beta)^\circ : I_{m+n} \to K$ takes v_i to p_i, where $p_i = \alpha^\circ(v_i)$ for $0 \leq i \leq m$ and $p_i = \beta^\circ(v_{i-m-1})$ otherwise. This is exactly γ.

(ii) It suffices to show this if α and β are separated by one step. But, writing $\alpha = \gamma(p, q)(q, r)\delta = \gamma(p, r)\delta = \beta$, simply note that we can use the straight line homotopy from the center of (p, r) to go to q. Resizing intervals as necessary, as in the previous part, gives the result.

Exercise 7.40. It suffices to show that trees are contractible. This is true for zero or one 1-simplices. For $(n + 1)$ total 1-simplices, simply pick an edge one of whose endpoints is a leaf. Then we can contract that edge to the other vertex, which is connected to the rest of the tree. Induction implies the result.

Exercise 7.41. Suppose $e_1 \dots e_n$ were a circuit in $T_1 \cup T_2$. Suppose without loss of generality that $e_1 \in T_1$. Let i and j be the first and last indices, respectively, such that $e_i, e_j \in T_1 \cap T_2$. There is a path α which starts with e_i and ends with e_j contained in $T_1 \cap T_2$. Now notice that $e_1 \dots e_{i-1}\alpha e_{j+1} \dots e_n$ is a circuit contained entirely within T_1, contradicting that T_1 is a tree.

Exercise 7.42. Let G be any abelian group, and let $\varphi : \{xF' : x \to X\} \to G$. Our goal is to show that there is a unique homomorphism $\psi : F/F' \to G$ with $\psi(xF') = \varphi(xF')$ for all $xF' \in F/F'$. (See Theorem 4.1(i).)

As in the definition of a free group, let $\tilde{\varphi}$ be the unique homomorphism from F to G with $\tilde{\varphi}(x) = \varphi(xF')$ for all $x \in X$. Now define

$$\begin{aligned} \psi : F/F' &\to G \\ fF' &\mapsto \tilde{\varphi}(f). \end{aligned}$$

To see that this is well-defined, notice that $f \in F'$ implies that $f = g^{-1}h^{-1}gh$ for some $g, h \in F$. Thus

$$\tilde{\varphi}(f) = \tilde{\varphi}(g)^{-1}\tilde{\varphi}(h)^{-1}\tilde{\varphi}(g)\tilde{\varphi}(h).$$

But G is abelian, so this is exactly 1, which proves well-definedness.

To see that ψ does indeed satisfy that $\psi(xF') = \varphi(xF')$, simply notice that $\psi(xF') = \tilde{\varphi}(x)$, which is defined to be $\varphi(xF')$.

Finally, to see that ψ is the *unique* homomorphism with this property, note that any other function ψ' would have to have $\psi'(fF') = \tilde{\varphi}(f)$, and thus be exactly equal to ψ.

Hence F/F' is indeed free abelian, with the desired basis.

Exercise 7.43. Exercise 7.42 shows that the rank of the free group F is the rank of the free abelian group F/F'. But this latter rank is invariant with respect to X.

Exercise 7.44.

(i) By picking a maximal tree T, and setting some edge not in the tree to be x, we can see that every other edge becomes either x, x^{-1}, or 1. Hence $G_{\mathbb{R}P^2,T} \cong \mathbb{Z}/2\mathbb{Z}$, and Corollary 7.37 implies the result.

(ii) Hurewicz's theorem applies since $\mathbb{R}P^2$ is obviously path-connected. Moreover, since $\mathbb{Z}/2\mathbb{Z}$ is abelian, its commutator subgroup is trivial. Thus $H_1(\mathbb{R}P^2) \cong \mathbb{Z}/2\mathbb{Z}$, as desired.

Exercise 7.45.

(i) Pick points $x, y \in X$. Then consider vertices p and q of the simplices containing x and y, respectively. Consider the following path: Take the straight line from x to p, then take the path mapped out by $F(p,t)$ as $t \in \mathbb{I}$, then the path mapped out by $F(q, 1-t)$, and finally the straight line from q to y.

(ii) Let $F : X \times \mathbb{I} \to X$ have $F(v, 0)$ for all $v \in X^{(1)}$ and $F(\ ,1)$ a constant function. Then by taking the homotopy along F, we can go from (p,q) to the constant point, then back to some arbitrary edge of T, where T is a maximal tree of X. Hence $(p,q) = 1$, implying a trivial edge path group. Thus the fundamental group is trivial too.

Exercise 7.46. Since there are n vertices, we know that there are $n-1$ edges of a maximal tree. The result follows from Corollary 7.35.

Exercise 7.47. If X has m edges and n vertices, then $\chi(X) = -m + n$. Thus $1 - \chi(X) = m - n + 1$. Now use Hurewicz's theorem, Exercise 7.35, Exercise 7.42, and Corollary 7.35 to find the result for H_1. Note that $H_0(X) = \mathbb{Z}$ because X is connected, and $H_q(X) = 0$ for $q \geq 2$ because X has dimension 1.

Exercise 7.48. We know that S^m is the boundary of an $(m+1)$-simplex. Thus there is an edge between any two vertices, so we can fix one vertex p and let T be the star consisting of all edges (p,q). Now consider any other edge (q,r). Note that $\{p,q,r\}$ forms a simplex, so $(p,q)(q,r) = (p,r)$. But in $G_{K,T}$, we know that $(p,q) = (p,r) = 1$, so $(q,r) = 1$ as well. Thus $\pi(K,p) \cong G_{K,T} = 1$, and so $\pi_1(S^m) = 1$. Hence S^m is simply connected.

Exercise 7.49.

(i) Since every vertex is contained in $K^{(q)}$, we can pick any simplex of maximal dimension. Its vertices are contained in $\mathrm{Vert}(K^{(q)})$, but it does not itself belong in the q-skeleton.

(ii) If a full subcomplex L exists, we know that it would need to include every simplex of K with vertices in A. Moreover, adding any other simplex would introduce new vertices. Thus such a subcomplex would be unique. Note that the set thus described is indeed a subcomplex, since any faces of $s \in L$ would have to have vertices in A as well.

The second part of the statement follows from the description of L.

Exercise 7.50. Consider some element $[\alpha] =\in \pi(K, v_0)$. Then there is some path $\alpha' \simeq \alpha$ with $\alpha' \in \pi(L, v_0)$. Thus $i[\alpha'] = [\alpha]$, proving surjectivity.

If K is the 2-simplex and L is its boundary, then obviously any closed edge path in K is also in L (and, in particular, is homotopic to a closed edge path in L). But the fact that K is simply connected while L is not implies that there cannot be an isomorphism.

The Seifert–van Kampen Theorem

No exercises!

8 CW Complexes

Hausdorff Quotient Spaces

Exercise 8.1. For any $x, y \neq 0$, let $\lambda = xy^{-1}$. Then $x = \lambda y$, so $[x] = [y]$. Hence FP^0 is just a single point.

Exercise 8.2. In each case, first use the fact that each space is a division ring to map $[x_0, x_1] \mapsto [1, x] = [1, x_0^{-1}x_1]$, then divide each term by $\sqrt{1 + ||x||^2}$ so that the result has magnitude 1. This gives the desired homeomorphisms.

Exercise 8.3. Note that $U(\mathbb{R}) = \{\pm 1\} \approx S^0$. To see the homeomorphism for $\mathbb{C}$, use the map $e^{i\theta} \mapsto \cos\theta + i \sin\theta$. Finally, to see the homeomorphism for $\mathbb{H}$, write a given quaternion as an ordered quadruple, and divide by its magnitude.

Exercise 8.4. Note that the real projective plane is just the quotient of S^2, where antipodal points are identified. This is in turn equal to the quotient of $\mathbb{R}^3$ where points on a line through the origin are identified, i.e., $\mathbb{R}P^2$.

Exercise 8.5. Use the map $[x] \mapsto [x/|x|]$ to get a homeomorphism $\mathbb{R}P^n \mapsto S^n/\sim$.

Exercise 8.6. Consider the map f taking $[x_1, \dots, x_{2n+2}] \in S^{2n+1}/\sim$ to $[z_1, \dots, z_{n+1}] \in \mathbb{C}P^n$, where $z_j = x_{2j-1} + ix_{2j}$ for each j. This is easily seen to be well-defined. If $x, y \in S^{2n+1}$ with $x \sim y$, then $x = \lambda y$ for some λ with $|\lambda| = 1$. If $f(x) = [z_i]$ and $f(y) = [w_i]$, then notice that $(z_i) = \lambda(w_i)$ as well, so $[z_i] = [w_i]$.

Exercise 8.7. The same argument as above holds, this time by defining $z_n = x_{4n-3} + ix_{4n-2} + jx_{4n-1} + kx_{4n}$.

Attaching Cells

Exercise 8.8. By Corollary 1.9, it suffices to show that $\alpha \amalg \beta$ is constant on the fibers of v. Thus suppose $v(s) = v(t)$. It is sufficient to suppose that $t = f(s)$ and $s \in A$, since the relation $\sim$ is generated by all $(a, f(a))$. But if $t = f(s)$, then we have

$$(\alpha \amalg \beta)(s) = \alpha(s) = \beta(f(s)) = \beta(t) = (\alpha \amalg \beta)(t).$$

This proves the result.

Exercise 8.9.

(i) It is clear that B and B^{-1} are contained in the equivalence relation generated by B. Note that D is as well due to reflexivity. Finally, K is attained by $(a, f(a))(f(a), a') = (a, a')$, where $f(a') = f(a)$. Now note that repeating this with $(a', f(a'))$ simply takes us back to $(a, f(a))$, so there are no other elements in the equivalence relation.

(ii) Simply note that

$$\begin{aligned} K &= \{(a, a') : f(a) = f(a')\} \\ &= (f \times f)^{-1}\{(x, y) \in \operatorname{im}(f \times f) : x = y\} \\ &= (f \times f)^{-1}(\Delta \cap \operatorname{im}(f \times f)). \end{aligned}$$

This is exactly what we wanted.

Exercise 8.10. It is easy to verify that the diagram commutes. Now suppose we have some Z with functions $\alpha : X \to Z$ and $\beta : Y \to Z$ so that $\beta \circ f \alpha \circ i$. We would like to find a function $\varphi : X \amalg_f Y \to Z$ making the pushout diagram commute.

We will first show that, if such a function exists, then it must be unique. Since the maps $X \to X \amalg_f Y$ and $Y \to X \amalg_f Y$ are induced by v, it follows that $\varphi \circ v|X = \alpha$ and similarly for Y and β. Hence $\varphi \circ v$ would have to be equal to $\alpha \amalg \beta$, i.e., $\varphi = (\alpha \amalg \beta) \circ v^{-1}$. Note that Exercise 8.8 applies, so this is indeed a well-defined map.

Exercise 8.11. The only case too check is if $x \in X$ and $y \in Y$. Note that there exists some $a \in A$, so consider the path from x to $a = f(a)$ to y. Hence $X \amalg_f Y$ is path-connected.

Exercise 8.12.

(i) The equation given in the hint follows from Exercise 8.9. Now, for the forwards direction, observe that $v(C)$ closed implies $v^{-1}v(C)$ closed in $X \amalg Y$. Hence its intersection with Y is closed in Y. But notice that $f(C \cap A) \cap Y = f(C \cap A)$. Furthermore, we know that $f^{-1}(f(C \cap A))$ and $f^{-1}(C \cap Y)$ are completely disjoint from Y. Thus

$$v^{-1}v(C) \cap Y = (C \cap Y) \cup f(C \cap A)$$

is closed in Y, as desired.

Going backwards, observe that $v^{-1}v(C) \cap Y$ is closed in Y, using the hypothesis and the argument above. Moreover, since f is continuous, the hypothesis implies that

$$f^{-1}((C \cap Y) \cup f(C \cap A)) = f^{-1}(C \cap Y) \cup f^{-1}(f(C \cap A))$$

is closed. Since $C \cap X$ is closed in X and $f(C \cap A) \cap X = \emptyset$, this implies that $v^{-1}v(C) \cap X$ is closed in X. Hence $v^{-1}v(C)$ is closed in $X \amalg Y$. Since v is an identification, it follows that $v(C)$ is closed in $X \amalg_f Y$.

(ii) The function is clearly bijective and continuous. To see that it is a homeomorphism, we will show that it is a closed map. Thus suppose $C \subseteq Y$ is closed. Obviously, $i(C) \subseteq X \amalg Y$ is closed and has empty intersection with X. Then to see that $v(i(C))$ is closed in $X \amalg_f Y$, simply use the previous part. In particular, observe that

$$i(C) \cap Y = i(C),$$

which is closed in Y, while

$$f(i(C) \cap A) = \emptyset,$$

which is also closed, so that their union is closed. Thus $v(i(C))$ is closed in $X \amalg_f Y$, proving that the given function is a homeomorphism.

(iii) Again, this is clearly bijective and continuous. Since $X - A$ is open in X, if U is open in $X - A$, then it is also open in X. Note that $i(U)$ is open in $X \amalg Y$. Now note that $i(U)^c$ is closed in $X \amalg Y$, and its intersection with X is U^c, which is closed in X. Moreover, we know that its intersection with Y is Y itself, while $f(i(U)^c \cap A) = f(A)$. Since $Y \cup f(A) = Y$, which is closed in Y, part (i) implies that $v(i(U)^c)$, which, by surjectivity of v, is exactly $v(i(U))^c$, is closed. Thus $v(i(U))$ is open, proving that this is an open, bijective, continuous map, thus a homeomorphism.

(iv) Note that Φ takes $A \subseteq X$ to $A \subseteq X \amalg Y$, which is then exactly equal to the attached region of $X \amalg_f Y$.

Exercise 8.13.

(i) Since f is from a compact set to a Hausdorff set, it is closed. Let C be closed. Then A being compact implies that it is closed, so $C \cap A$ is closed in X. Thus $f(C \cap A)$ is closed in Y. Since $C \cap Y$ is closed in Y, it follows from part (i) that $v(C)$ is closed in $X \amalg_f Y$.

(ii) First, suppose that $z \in \operatorname{im} \Phi|A$. Then there is some $x \in X$ with $v(i(x)) = z$, so $i(x)$ is in the fiber. We know that $\{z\}$ is closed because Y is Hausdorff, so $v^{-1}(z)$ is also closed. Since $v^{-1}(z) \subseteq A$, and closed subsets of compact sets are compact, it follows that $v^{-1}(z)$ is compact.

Otherwise, we know that we can use either the homeomorphism in Exercise 8.12(ii) or the homeomorphism $\Phi|(X - A)$ to show that $v^{-1}(z) = \{z\}$, which is indeed a nonempty compact subset of X.

Exercise 8.14. This is just invariance of boundary. Alternatively, see the proof of Lemma 8.15.

Exercise 8.15. If $n = 0$, this is obviously true. Otherwise, let $e = s - \dot{s} \approx D^{n-1} - S^{n-1}$ and let $Y = |K^{(n-1)}|$ be a closed subset of $|K|$ (since it's the finite union of (closed) simplices). Then $e \cap Y = \emptyset$ and e is an n-cell. Hence Theorem 8.7 says that we need only exhibit a relative homeomorphism $\Phi : (D^n, S^{n-1}) \to (e \cup Y, Y)$. But letting Φ be the obvious homeomorphism from D^n to s works.

Exercise 8.16. Write $Y = \{y\}$. Then define the relative homeomorphism $\Phi : (D^n, S^{n-1}) \to (e^n \cup Y, Y)$ which takes $D^n - S^{n-1}$ to e^n in the obvious way, and takes $x \in S^{n-1}$ to y. Theorrem 8.7 tells us that the attachment of D^n to Y along $f = \Phi|S^{n-1}$ is a homeomorphisms between $D^n/\partial D^n = S^n$ and $e^n \cup Y \approx e^n \cup e^0$.

Homology and Attaching Cells

Exercise 8.17. Note that $\chi(K) = 1 - 2 + 1 = 0$, so $\operatorname{rank} H_2(K) + 1 = \operatorname{rank} H_1(K)$. Furthermore, doing the same thing as with the torus in Example 8.7, we see that the projections are $f\alpha * f\alpha_1^{-1}$, which has degree 0, and $f\beta * f\beta_1$, which has degree 2. Thus, since $H_1(S^1 \vee S^1) \cong H_1(S^1) \oplus H_1(S^1)$, we can consider f_* to be the map $x \mapsto (0, 2x)$. It has trivial kernel and image isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Working through the exact sequence in Theorem 8.11 gives the result.

Exercise 8.18. There are a couple typos here: In the first part, the wedge for M is of $2h$ circles, and in the second part, we should have $\chi(M) = 2 - 2h$, not $\chi(M) = h$.

(i) Note that each $(\alpha_i, \alpha_i^{-1})$ and (β_i, β_i^{-1}) pair gives a circle. Since all the vertices are identified with each other, this gives us the desired wedge product. More formally, we can define a function Φ from a polygon P to W, and let $f = \Phi|\partial P$. Then $f\alpha_i = (f\alpha_i^{-1})^{-1}$, and similarly for β, which gives us our $2h$ circles. A similar argument can be done for M'.

(ii) Note that $H_2(S^1 \vee \cdots \vee S^1) = 0$. Thus we have the following exact sequence:

$$0 \longrightarrow H_2(M) \longrightarrow H_1(S^1) \xrightarrow{f_*} H_1(S^1 \vee \cdots \vee S^1) \xrightarrow{i_*} H_1(M) \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z} \longrightarrow 0\ ,$$

where the last few terms are just $H_0(S^1)$, $\mathbb{Z} \oplus H_0(S^1 \vee \cdots \vee S^1)$, and $H_0(M)$, since all three spaces are path-connected. The fact that this sequence is exact implies that $\mathbb{Z}^2 \to \mathbb{Z}$ is a surjection, so the map $\mathbb{Z} \to \mathbb{Z}^2$ is an injection. Hence $H_1(M) \to \mathbb{Z}$ is the zero map. Thus i_* is surjective and has kernel (isomorphic to) $\mathbb{Z}^{2n}/H_1(M)$.

Looking at the maps from left to right now, observe that $H_2(M) \to H_1(S^1) = \mathbb{Z}$ is injective. Thus $\ker f_* = H_2(M)$, so $\operatorname{im} f_* = H_1(S^1)/H_2(M)$. But $\ker i_* = \operatorname{im} f_*$, and so it follows that

$$\chi(M) = \operatorname{rank} H_2(M) - \operatorname{rank} H_1(M) + \operatorname{rank} H_0(M) = 2 - 2h,$$

where we use the fact that $\operatorname{rank} H_0(M) = 1$.

Now notice that the same argument as in Example 8.7 implies that f_* is the zero map. Thus

$$H_2(M) = \ker f_* = H_1(S^1) = \mathbb{Z}.$$

For $H_1(M)$, since the flanking terms are torsion-free, it follows that $H_1(M)$ is also torsion-free. Since it has rank $2h$, the result follows.

(iii) The same argument as before shows that $\chi(M') = 2 - n$. This time, however, the map f_* is not the zero map. In particular, by composing with projections, we find that $f_* : H_1(S^1) \to H_1(S^1 \vee \cdots \vee S^1)$ takes $x \mapsto (2x, \ldots, 2x)$, where we have identified $H_1(S^1 \vee \cdots \vee S^1)$ with $H_1(S^1) \oplus \cdots \oplus H_1(S^1)$.

In particular, we have $\ker f_* = 0$ and $\operatorname{im} f_* = (\mathbb{Z}/2\mathbb{Z})^n$. The argument before shows that $\ker f_* = H_2(M')$, and so $H_2(M') = 0$. Using the Euler characteristic (i.e., a rank argument), we can conclude that $\operatorname{rank} H_1(M') = n - 1$. (Note that, this time, the first homology group isn't torsion-free, thanks to the $\mathbb{Z}/2\mathbb{Z}$ terms.)

(iv) We first consider M. Note that it only has one vertex, say v. Thus, with chains

$$E_2 = \langle W \rangle, \quad E_1 = \langle \alpha_1 \rangle \oplus \langle \beta_1 \rangle \oplus \cdots \oplus \langle \alpha_n \rangle \oplus \langle \beta_n \rangle, \quad E_0 = \langle v \rangle,$$

we have

$$\partial W = \alpha_1 + \beta_1 - \alpha_1 - \beta_1 + \cdots = 0, \quad \partial \alpha_i = \partial \beta_i = v - v = 0, \quad \partial v = 0.$$

Hence it follows that

$$\begin{aligned} &Z_2 = \langle P \rangle, \quad &&Z_1 = \langle \alpha_1 \rangle \oplus \langle \beta_1 \rangle \oplus \ldots, \quad &&Z_0 = \langle v \rangle, \\ &B_2 = 0, &&B_1 = 0, &&B_0 = 0. \end{aligned}$$

Thus we have

$$H_2(M) = \mathbb{Z}, \quad H_1(M) = \mathbb{Z}^{2h}, \quad H_0(M) = \mathbb{Z}.$$

Now, for M', with the natural chains, we have

$$\partial P = 2\alpha_1 + \cdots + 2\alpha_n, \quad \partial\alpha_i = 0, \quad \partial v = 0.$$

Thus we conclude that

$$\begin{aligned} Z_2 &= 0 \quad & Z_1 &= \langle\alpha_1\rangle \oplus \cdots \oplus \langle\alpha_n\rangle, \quad & Z_0 &= \langle v\rangle, \\ B_2 &= 0, \quad & B_1 &= \langle 2(\alpha_1 + \cdots + \alpha_n)\rangle, \quad & B_0 &= 0. \end{aligned}$$

This gives us that

$$H_2(M') = 0, \quad H_1(M') = \mathbb{Z}^n/2\mathbb{Z} = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^{n-1}, \quad H_0(M') = \mathbb{Z},$$

which coincides with the previous parts.

CW Complexes

Exercise 8.19. Note that $U \subseteq X$ is open if and only if $U^c \subseteq X$ is closed, which is in turn the case if and only if $U^c \cap A_j$ is closed in A_j for all $j \in J$. But $U \cap A_j = A_j - U^c \cap A_j$, so this last conditiono is true if and only if $U \cap A_j$ is open in A_j for $j \in J$.

Exercise 8.20. It is obvious that $\{Y \cap A_j\}$ fits the conditions (i)–(iii). Now note that if $F \subseteq Y$ is closed in the subspace topology, then $F = Y \cap F'$ for some closed $F' \subseteq X$. Hence

$$F \cap (Y \cap A_j) = Y \cap F' \cap Y \cap F_j = (Y \cap A_j) \cap F'.$$

Of course, this is closed in $Y \cap A_j$, so F must be closed in the weak topology.

Now suppose F is closed in the weak topology. Then $F \cap (Y \cap A_j)$ is closed in all $Y \cap A_j$. Since Y is closed, we know that $F \cap Y \cap A_j$ must also be closed in A_j. This is true for all j, so $F \cap Y$ is closed in the weak topology on X, i.e., as a subset of X. Thus $F = F \cap Y$ is closed as a subspace of Y.

Exercise 8.21. By Lemma 8.20, we know that A closed in X implies that $A \cap X'$ is closed in X' for all finite subcomplexes X'. Theorem 8.19 says that this implies that, for all compact K of X, we must have $A \cap K$ closed in K. Thus, by definition, it follows that A is closed in the weak topology generated by compact subsets.

Now suppose $A \cap K$ is closed in K for all compact K. Since finite subcomplexes are compact, it follows that $A \cap X'$ is closed in X' for all such X'. Hence A is closed in X.

Exercise 8.22. To see that $X^{(0)}$ is discrete, simply let $A \subseteq X^{(0)}$. Then $A \cap \bar{e}$ is the finite union of 0-cells, and hence is closed.

To see that the 0-skeleton is closed, note that $X^{(0)} \cap \bar{e}$ is a finite union of 0-cells, and thus is closed in $\bar{e}$. This is true for all e, so $X^{(0)}$ is closed.

Exercise 8.23. If A is closed, then obviously $A \cap X^{(n)}$ is closed in $X^{(n)}$. Now if $A \cap X^{(n)}$ is closed in $X^{(n)}$ for each n, pick X' to be any finite complex. Let n be the highest dimension of any cell in X'. Then $A \cap X' = (A \cap X^{(n)}) \cap X'$ must be closed in X'. Lemma 8.20 implies the result.

Finally, the corresponding statement for open sets follows from Exercise 8.19.

Exercise 8.24. This is simply because each n-cell is just $D^n - S^{n-1}$; the attachment is given by $\Phi_e(S^{n-1})$ according to Exercise 8.12(iv).

Exercise 8.25. This is visually clear. Alternatively, with α and β as the edges, and v as the vertex, we can notice that there is a relative homeomorphism

$$\Phi : (D^2, S^1) \to (T, \alpha \cup \beta \cup \{v\})$$

since $D^2 \cong \mathbb{I} \times \mathbb{I}$. Since α and β are 1-cells, and v is a 0-cell, it follows that this map gives T as the union of two 1-cells, one 0-cell, and one 2-cell (namely $\operatorname{im} \Phi|(D^2 - S^1)$).

The same argument can be done for the Klein bottle.

Exercise 8.26.

(i) They both violate closure finiteness since the closure of the base point intersects infinitely many cells.

(ii) The set of all $\{1/n\}$ is closed in the weak topology, but not as a subspace.

Exercise 8.27. The same proof as Theorem 7.1 holds, but with D^n in place of Δ^n.

Exercise 8.28. First, observe that the 1-skeleton is always nonempty (as long as X is nonempty). In particular, suppose e is an n-cell in X, where n is the smallest dimension of a cell in e. Then the relative homeomorphism $(D^n, S^{n-1}) \to (e \cup X^{(n-1)}, X^{(n-1)})$ implies that there is a map between S^{n-1} and $X^{(n-1)} = \emptyset$, which is impossible. Thus there must be some 0-cell, and so the 1-skeleton is nonempty.

In fact, there must be some part of the 1-skeleton in each path component. Thus if X is disconnected, then its 1-skeleton must be as well.

Now suppose that the 1-skeleton is disconnected. We can easily show that $X^{(n)}$ disconnected implies that $X^{(n+1)}$ is disconnected. Since X is the union of all its skeletons, and since $X^{(n)} \subseteq X^{(n+1)}$, it follows that X being connected would have to imply that there is some n with $X^{(n)}$ connected. Since $X^{(0)}$ is discrete, hence disconnected (unless it has one element only, in which case the 1-skeleton would be connected), it follows that $n \geq 1$, and so this provides the desired contradiction.

Exercise 8.29. The forward direction is obvious. Now suppose that $f\Phi_e$ is continuous for all e. Let $K \subseteq Y$ be closed and let e be a k-cell. We want to show that

$$f^{-1} \cap \Phi_e(D^k)$$

is closed in $\bar{e} = \Phi_e(D^k)$. But Φ_e is a relative homeomorphism and is, in particular, a closed map on $D^k - S^{k-1}$. Now, because

$$\Phi_e^{-1}(f^{-1}(K) \cap \Phi_e(D^k)) = (f\Phi_e)^{-1}(K) \cap D^k$$

is closed in D^k, we're done.

Exercise 8.30.

(i) Consider attaching the (closed) top half of the circle to the topologist's sine curve (which maps 0 to 0 and x to $\sin(1/x)$ for $x \in (0, 2\pi]$). Then attach the (closed) bottom half of the circle to the same curve, but running backwards. Obviously this is a CW complex. But it is connected and not path-connected, violating Exercise 7.35. Hence this is not a polyhedron.

(ii) If $n = 0$, this is obvious. Suppose this is true for n. Say we attach k total $(n+1)$-cells. (Note that this kind of inductive creation of CW complexes is made possible byy Theorem 8.24.) Note that an $(n+1)$-cell is homeomorphic to an open $(n+1)$-simplex. Furthermore, the attachment map can be approximated by a simplicial map. Since simplicial approximations are homotopic to the original maps, the result follows.

Exercise 8.31. Here we can use the same cells and attaching maps, only with the basepoints all identified. For any cell not equal to the basepoint, its closure is contained in whichever X_λ the cell was originally in, and thus intersects only finitely many cells. The closure of the basepoint is itself, and only intersects itself. This proves closure finiteness.

To see that this has the weak topology, simply note that if A is closed, then $A \cap X_\lambda$ is closed, where we identify X_λ with its natural image in $\bigvee X_\lambda$. Thus, since the closure of any cell is contained within X_λ for some λ, it follows that $A \cap \bar{e} = A \cap X_\lambda \cap \bar{e}$ is closed in $\bar{e}$ for every e.

Exercise 8.32. The first and second conditions of a CW complex are clearly satisfied since D^{i+j} is homeomorphic to $[0,1]^{i+j}$.

To see the third condition holds, use the equation in the hint. Notice that all four expressions on the right side intersect finitely many cells in X or X'. In particular, it follows that $(\bar{e} - e) \times \bar{e}'$ intersects finitely many cells of E'', and similarly for the other term. Thus $\overline{e \times e'}$ intersects $e \times e'$, plus these finitely many other cells. This proves closure finiteness.

Finally, for the fourth condition, note that the weak topology is just the product topology when working with finitely many factors.

Exercise 8.33. With notation as suggested in the hint, suppose the intersection of A and every cell in E'' is closed in the cell. Now observe that

$$\overline{e \times a^0} = \bar{e} \times a^0,$$

and similarly for b^0. Moreover, we know that

$$\overline{e \times c^1} = \left[(\bar{e} - e) \times \mathbb{I}\right] \cup \left(\bar{e} \times (a^0 \cup b^0)\right) \cup (e \times c^1).$$

But now observe that $\bar{e} = e \cup (\bar{e} - e)$, so that the middle term can be rewritten as

$$\bar{e} \times (a^0 \cup b^0) = \left(e \times (a^0 \cup b^0)\right) \cup \left((\bar{e} - e) \times (a^0 \cup b^0)\right).$$

Since $a^0 \cup b^0 \cup c^1 = \mathbb{I}$, it then follows that

$$\overline{e \times c^1} = \left[(\bar{e} - e) \times \mathbb{I}\right] \cup (e \times \mathbb{I}) \cup \left((\bar{e} - e) \times (a^0 \cup b^0)\right) = \bar{e} \times \mathbb{I}.$$

Now let π_X and $\pi_{\mathbb{I}}$ be the projections to X and $\mathbb{I}$, respectively. We know that $\pi_X(A) \cap \bar{e}$ is closed in each e, since $\pi_X(A) \cap \bar{e}$ is closed in each $\bar{e}$, and similarly for $\pi_{\mathbb{I}}(A) \cap \bar{a^0}$ and $\pi_{\mathbb{I}}(A) \cap \bar{b^0}$. Moreover, since $\pi_{\mathbb{I}}(A) \cap \bar{c^1} = \pi_{\mathbb{I}}(A) \cap \mathbb{I}$, and since $A \cap (\bar{e} \times \mathbb{I})$ is closed in $\bar{e} \times \mathbb{I}$, it follows that the intersection $\pi_{\mathbb{I}}(A) \cap \bar{c^1}$ is also closed in $\mathbb{I}$.

Thus $\pi_X(A)$ and $\pi_{\mathbb{I}}(A)$ are closed, so A is closed, as desired.

Exercise 8.34. Let $i : Z \to Y$ and $j : Y \to X$ be the injections. We can now easily check the criteria for a strong deformation retraction. In particular, note that

$$r_1 r_2 j i = r_1 1_Y i = r_1 i = 1_Z,$$

while

$$j i r_1 r_2 = j(i r_1) r_2 \simeq j_r 2 \operatorname{rel} Z \simeq 1_X \operatorname{rel} Z,$$

since $Y \subseteq Z$.

Cellular Homology

Exercise 8.35. ☹

Exercise 8.36.

(i) If X is compact, then it is finite. Thus $W_k(X, Y) = H_k(X_Y^k, X_Y^{k-1})$ is free abelian of rank equal to the number of k-cells in $E - E'$. Say this rank is r_k. Then we know that

$$H_k(X, Y) \cong H_k(W_*(X, Y)) = \ker d_k / \operatorname{im} d_{k+1},$$

but both $\ker d_k$ and $\operatorname{im} d_{k+1}$ are subsets of $\mathbb{Z}^{r_k}$. Thus $H_k(X, Y)$ is finitely generated.

(ii) This is the same proof, since r_k is at most the number of cells of dimension k.

Exercise 8.37. Using the cellular decomposition for $\mathbb{R}P^\infty = \bigcup \mathbb{R}P^n$, we find that

$$W_k(\mathbb{R}P^\infty) = H_k(e^0 \cup \cdots \cup e^k, e^0 \cup \cdots \cup e^{k-1}),$$

which is obviously free abelian of rank 1. It follows that the we get a chain $\cdots \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to \ldots$, so the kernels and images of each map must be 0 or $\mathbb{Z}$. Hence $H_k(\mathbb{R}P^\infty) = H_k(W_*(\mathbb{R}P^\infty))$ is either 0 or $\mathbb{Z}$.

Exercise 8.38. Let b_λ be the basepoint of X_λ. Then we know that $\bigvee X_\lambda = \coprod X_\lambda / \{b_\lambda\}$. Let v be the natural map. Then Theorem 8.41 implies that v_* induces an isomorphism from

$$H_k\left(\coprod X_\lambda, \{b_\lambda\}\right) \to \tilde{H}_k\left(\bigvee X_\lambda\right).$$

Of course, Theorems 5.13 and 5.17 also imply that the left side is equal to

$$\sum_\lambda \tilde{H}_k(X_\lambda),$$

which implies the result.

Exercise 8.39. To do this, we simply compute d_2, d_1, d_0. In particular, since $W_2(T)$ is generated by e^2, we know that $d_2 = \partial e^2 = 0$. Similarly, we find that $d_1 = d_0 = 0$. This gives the result.

Exercise 8.40. Use Exercise 7.19. In particular, this implies that

$$\chi(S^m \times S^n) = \begin{cases} 0 & \text{if } m \text{ or } n \text{ odd} \\ 4\text{otherwise} & \end{cases}.$$

Exercise 8.41. Use the cellular decomposition of $\mathbb{C}P^n$. In particular, we know that $\mathbb{C}P^n = e^0 \cup \cdots \cup e^{2n}$, and so the only nonzero α_i are for even i. Thus

$$\chi(\mathbb{C}P^n) = \sum (-1)^i \alpha_i = 1 + 1 + \cdots + 1 = n + 1.$$

The same argument holds for $\mathbb{H}P^n$.

Exercise 8.42. This is again obvious:

$$\chi(\mathbb{R}P^n) = 1 - 1 + 1 - 1 + \dots$$

is equal to 0 if n is odd and 2 if n is even. This is exactly $\frac{1}{2}(1 + (-1)^n)$.

Exercise 8.43. This is simply the principle of inclusion-exclusion.

Exercise 8.44. It is sufficient to show that $\sim$ is closed. Notice that

$$\{(z_0, z_1, z_2, z_3) : h^m(z_0, z_1) = (z_2, z_3)\} = \bigcup_{m=1}^{p} \{(z_0, z_1, z_2, z_3) : h^m(z_0, z_1) = (z_2, z_3)\} = \bigcup_{m=1}^{p} S_m.$$

because $h^p = h$. Thus it suffices to check that each S_m is closed. Suppose that $(z_0, z_1, z_2, z_3) \notin S_m$. Say that $z_2 \neq \zeta^m z_0$; note that a similar argument can be given if $z_3 \neq \zeta^{mq} z_1$. Then there is an open neighborhood with coordinates (x_0, x_1, x_2, x_3) on which $\zeta^m(x_0) \neq x_3$ since $\zeta^m x - y$ is continuous. Thus $(S_m)^c$ is open, which proves that S_m is closed, as desired.

Exercise 8.45.

(i) Note that $\zeta = 1$, so h is just the identity. Thus $S^3/\!\sim\, = S^3$

(ii) Now we have $\zeta = -1$, so h maps antipodal points to each other. Thus $S^3/\!\sim\, = \mathbb{R}P^3$.

(iii) In this case, we know that $\zeta^q = \zeta^{q'}$, so $h_q = h_{q'}$. Thus $L(p, q) = L(p, q')$.

Exercise 8.46.

(i) Since this is a finite decomposition, we only need to verify the first two conditions for a CW complex. The first is clear by definition. For the second condition, the maps are obvious for e_r^0 and e_r^1. For e_r^2, we use the fact that $z_1 = z_1(z_0)$ is determined by z_0. Thus the map

$$z_0 \mapsto (z_0, z_1(z_0))$$

works. Finally, for e_r^3, take (z_0, θ) and map θ linearly onto $(2\pi r/p, 2\pi(r+1)/p)$.

(ii) It is easy to check that $e_r^i \sim e_{r'}^i$ for each i.

Exercise 8.47.

(i) This is the cellular boundary formula, or just a generalization of the argument for Lemma 8.46.

(ii) For $D(\gamma_1)$, simply notice that

$$D(\gamma_1) = v_\# d_1 v_\#^{-1}(\gamma_1) = v_\# d_1 e_r^1 = v_\#(e_r^0 - e_{r+1}^0) = 0.$$

A similar argument holds for the other differentiations.

(iii) This is obvious from the chain complex:

$$W_4 = 0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0 = W_{-1}$$

9 Natural Transformations

Definitions and Examples

Exercise 9.1. This is obvious from Lemma 4.8.

Exercise 9.2. This is exactly the statement of Exercise 4.10.

Exercise 9.3. The commutative diagram in Exercise 4.13 is exactly the statement that the map is natural.

Exercise 9.4. Commutativity of the diagram

$$\begin{tikzcd} H_n(X,A) \arrow[r, "f_*"] \arrow[d, "\partial"'] & H_n(Y,B) \arrow[d, "\partial"] \\ H_{n-1}(A,\emptyset) \arrow[r, "(f|A)_*"'] & H_{n-1}(B,\emptyset) \end{tikzcd}$$

follows from the exact sequence in Theorem 5.8, since $H_{n-1}(A,\emptyset) = H_{n-1}(A)$.

Exercise 9.5. This is again precisely the statement from Exercise 6.8.

Exercise 9.6.

(i) Suppose $\sigma : F \to G$ and $\tau : G \to H$ are natural. Then we can "stack" the commutative diagrams:

$$\begin{tikzcd} F(C) \arrow[r, "Ff"] \arrow[d, "\sigma_C"'] & F(D) \arrow[d, "\sigma_D"] \\ G(C) \arrow[r, "Gf"] \arrow[d, "\tau_C"'] & G(D) \arrow[d, "\tau_D"] \\ H(C) \arrow[r, "Hf"] & H(D) \end{tikzcd}$$

Hence it follows that $\tau\sigma = (\tau_C\sigma_C)$ gives a natural transformation.

(ii) Reflexivity is due to the commutativity of the following diagram:

$$\begin{tikzcd} F(C) \arrow[r, "Ff"] \arrow[d, "1_{F(C)}"'] & G(C) \arrow[d, "1_{G(C)}"] \\ F(C) \arrow[r, "Ff"] & G(C) \end{tikzcd}$$

To see symmetry, simply choose τ_C^{-1} for each object C. This can be done because each τ_C is an equivalence. Finally, transitivity follows from the previous part and the fact that the composition of equivalences is an equivalence.

Exercise 9.7.

(i) If $\varphi \in \mathrm{Nat}(\mathrm{Hom}(\ ,A), F)$, then φ_A is a map from $\mathrm{Hom}(A,A)$ to $F(A)$. Since $1_A \in \mathrm{Hom}(A,A)$, it follows that $\varphi_A(1_A) \in F(A)$. Thus y is a well-defined function.

(ii) We must check that $\tau \in \mathrm{Nat}(\mathrm{Hom}(\ ,A), F)$ whenever $\mu \in F(A)$. First, observe that τ_X is indeed a morphism from $\mathrm{Hom}(X,A)$ to $F(X)$. After all, if $f : X \to A$, then $Ff : FA \to FX$. Hence $\tau_X(f) = (Ff)(\mu)$ is an element of $F(X)$.

To see that τ is natural, we must show that the following diagram commutes for all $f : X \to Y$.

$$\begin{tikzcd} \mathrm{Hom}(X,A) \arrow[d, "\tau_X"'] & \mathrm{Hom}(Y,A) \arrow[l, "\mathrm{Hom}(f,A)"'] \arrow[d, "\tau_Y"] \\ F(X) & F(Y) \arrow[l, "Ff"] \end{tikzcd}$$

But for each $g \in \mathrm{Hom}(X, A)$, we know that

$$(Ff) \circ \tau_Y(g) = (Ff)(Fg(\mu)) = F(g \circ f)\mu,$$

while we have $\mathrm{Hom}(f, A) = f^*$, so that

$$\tau_X \circ \mathrm{Hom}(f, A)(g) = \tau_X \circ f^*(g) = \tau_X(g \circ f) = F(g \circ f)\mu.$$

These are equal, so τ is a natural transformation.

(iii) First, we will show that $y \circ y' : F(A) \to F(A)$ is the identity. Let $\mu \in F(A)$. Then we know that

$$y'(\mu) = \{\tau_X : f \mapsto (Ff)(\mu)\},$$

and so we have that

$$y(y'(\mu)) = (y'(\mu))_A(1_A) = F(1_A)(\mu).$$

But F is a functor, so $F(1_A) = 1_{F(A)}$, and so this is exactly equal to $1_{F(A)}(\mu) = \mu$, which proves that $y \circ y'$ is the identity on $F(A)$.

Now to check $y'y$, suppose $\varphi \in \mathrm{Nat}(\mathrm{Hom}(\ , A), F)$. Then we know that

$$y'(y(\varphi)) = \{\tau_X : f \mapsto (Ff)(\varphi_A(1_A)).$$

We would like to show that

$$(Ff)(\varphi_A(1_A)) = \varphi_X f,$$

since this will imply that $y'(y(\varphi)) = \varphi$. But we know that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}(X, A) \arrow[d, "\varphi_X"'] & \mathrm{Hom}(A, A) \arrow[l, "f^*"'] \arrow[d, "\varphi_A"] \\
F(X) & F(A) \arrow[l, "Ff"]
\end{tikzcd}$$

Thus we know, in particular, that

$$Ff \circ \varphi_A(1_A) = \varphi_X f^*(1_A) = \varphi_X(1_A \circ f) = \varphi_x \circ f.$$

This is what we wanted.

(iv) If $\varphi : \mathrm{Hom}(\ , A) \to \mathrm{Hom}(\ , B)$ is natural, then we have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}(X, A) \arrow[d, "\varphi_X"'] & \mathrm{Hom}(Y, A) \arrow[l, "\mathrm{Hom}(f,A)"'] \arrow[d, "\varphi_Y"] \\
\mathrm{Hom}(X, B) & \mathrm{Hom}(Y, B) \arrow[l, "\mathrm{Hom}(f,B)"]
\end{tikzcd}$$

Let F be the functor $\mathrm{Hom}(\ , B)$. Then we know that

$$\begin{aligned}
\varphi_X(f) &= y'(y(\varphi))_X(f) \\
&= (Ff)(\varphi_A(1_A)) \\
&= \mathrm{Hom}(f, B)(\varphi_A(1_A)) \\
&= \varphi_A(1_A) \circ f,
\end{aligned}$$

where $f : X \to A$. Thus $\varphi_X(f) = \mu f$, as desired.

(v) Same proof.

Exercise 9.8. We must verify the properties of a category. To see that the family of $\mathrm{Hom}(F, G)$'s, where F and G are functors $\mathscr{C} \to \mathscr{A}$, is disjoint, notice that this means that there exists some $\tau = (\tau_C : F(C) \to G(C))$ and $\sigma = (\sigma_C : F'(C) \to G'(C))$ which are equal. Hence $F(C) = F'(C)$ and $G(C) = G'(C)$ for all C, since $\tau_C = \sigma_C$ is in both $\mathrm{Hom}(F(C), G(C))$ and $\mathrm{Hom}(F'(C), G'(C))$. Since this is true for all $C \in \mathscr{C}$, it follows that $F = F'$ and $G = G'$.

Composition of natural transformations reduces to composition of morphisms, which is associative.

Finally, note that $1_A \in \mathrm{Hom}(F, F)$ given by

$$1_A = \{(1_A)_C = 1_{F(C)}\}$$

works as an identity morphism.

Exercise 9.9.

(i) We shall verify the properties of a contravariant functor. The functor gives us a complex

$$\dots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \dots \longrightarrow C_0 \longrightarrow C_{-1} \longrightarrow \dots$$

Since C_n is abelian, we know that $n \in \mathbb{Z}$ implies that $C(n) = C_n \in \mathscr{A}$.

The only morphisms in $\mathbb{Z}$ are ι_y^x when $x \leq y$. Note that $C(\iota_y^x)$ is the composition $\partial_{x+1} \circ \dots \circ \partial y : C_y \to C_x$. We must verify that composition is reversed and identities are respected. But this is clear from the definition:

$$]iota_z^y \circ \iota_y^x = \iota_z^x = \partial_{x+1} \circ \dots \circ \partial_z$$

is exactly $C(\iota_y^x) \circ C(\iota_z^y)$, and $C(\iota_x^x)$ is the composition of an empty set of differentiation operators, and thus is the identity.

(ii) The chain map condition is exactly the condition of commutativity.

Eilenberg–Steenrod Axioms

No exercises!

Chain Equivalences

Exercise 9.10. To prove (i) implies (ii), note that $ps = 1_C$ implies s is injective. Then the same argument as in Corollary 9.2 implies that $B = \ker p \oplus \operatorname{im} s$. Of course, we have $\ker p = \operatorname{im} i$ and $C' = \operatorname{im} s = s(C) \cong C$. Since $p(C') = C$, this proves the first implication.

The second implication is clear. In particular, consider $q : B \to A$ defined by $(i(x), c) \mapsto x$. Then $qi(a) = q(i(a)) = a$.

Finally, to show (iii) implies (i), define $s(c)$ as

$$s(c) = p^{-1}(c) - iqp^{-1}(c).$$

To see that this is well-defined, pick $b \in \ker p = \operatorname{im} i$, so $b = i(a)$. Thus $b - iq(b) = i(a) - iqi(a) = 0$. Hence $p(b) = p(b')$ means that $b - iqb = b' - iqb'$, proving well-definedness. To see that this choice of s gives a split exact sequence, simply verify that

$$ps(c) = p(p^{-1}(c) - iqp^{-1}(c)) = c - piqp^{-1}(c).$$

Since $pi = 0$, this is equal to c.

Acyclic Models

Exercise 9.11. First we show that the diagram given by Rotman commutes, i.e., that

$$\partial_n(t_n - t;_n - s_{n-1}d_n) = 0.$$

We know that

$$\partial_n t_n - \partial_n t_n' - \partial_n s_{n-1} d_n = t_{n-1} d_n - t_{n-1}' d_n - \partial_n s_{n-1} d_n.$$

The inductive hypothesis implies that

$$\partial_n s_{n-1} = t_{n-1} - t_{n-1}' - s_{n-2} d_{n-1}.$$

Plugging this value in and canceling gives us

$$\partial_n t_n - \partial_n t_n' - \partial_n s_{n-1} d_n = s_{n-2} d_{n-1} d_n = 0,$$

because $dd = 0$.

Thus the diagram commutes. In particular, we know that

$$\operatorname{im}(t_n - t_n' - s_{n-1} d_n) \subseteq \ker \partial_n = \operatorname{im} \partial_{n+1},$$

where the final equality comes from the fact that E_* is an acyclic complex. This means that we can rewrite the diagram as follows:

$$\begin{tikzcd} & F_n \arrow[d, "t_n - t_n' - s_{n-1} d_n"'] & \\ E_{n+1} \arrow[r, "\partial_{n+1}"'] & \begin{array}{c}\operatorname{im}(t_n - t_n' - s_{n-1} d_n) \\ = \operatorname{im} \partial_{n+1} = \ker \partial_n\end{array} \arrow[r, "\partial_n = 0"'] & 0 \end{tikzcd}$$

Thus Theorem 9.1 implies that we can find s_n with the desired properties.

Exercise 9.12. We have $F(g) = F(0 + g) = F(0) + F(g)$, so $F(0)$ acts as the 0 element. If A is the zero group, then its identity is the zero homomorphism. Hence $1_{F(A)} = F(1_A)$ is the zero homomorphism, so $F(A) = 0$.

Exercise 9.13.

(i) We'll prove the covariant case. By Exercise 9.10, we have a morphism $q : A \to B$ with $qi = 1_A$. Note that $(Fp) \circ (Fs) = F(p \circ s) = F(1_C) = 1_{F(C)}$, and similarly for q and i, so that we still have a split sequence, as long as it is exact. Moreover, these imply that Fp is surjective and Fi is injective.

It now suffices to check that $\operatorname{im} Fi = \ker Fp$. But notice that $B \cong iq(B) \oplus sp(B)$ implies that $F(B)$ is equal to the functored version of the right side, thus making the center of the short functored sequence exact.

(ii) This simply uses induction on $|I| = n + 1$ and the following short exact sequence:

$$0 \longrightarrow \textstyle\sum_{i=1}^{n} A_i \xrightarrow{\ i\ } \textstyle\sum_{i=1}^{n+1} A_i \xrightarrow{\ p\ } A_{n+1} \longrightarrow 0.$$

Note that this is split exact with $s : a_{n+1} \mapsto (0, \ldots, 0, a_{n+1})$. Thus the previous part applies, and Exercise 9.10 implies that

$$F\left(\sum_{i=1}^{n+1} A_i\right) \cong F\left(\sum_{i=1}^{n} A_i\right) \oplus F(A_{n+1}) = \sum_{i=1}^{n+1} F(A_i),$$

where the last equality follows from the inductive hypothesis.

Exercise 9.14.

(i) If $\partial_n \partial_{n+1} = 0$, then $F(\partial_n \partial_{n+1}) = 0$ thanks to additivity. This proves that the functored complex is a chain complex too.

(ii) Note that $f_{n-1}\partial_n = \partial_n' f_n$ implies that $F(f_{n-1}\partial_n) = F(\partial_n' f_n)$. Since functors respect composition, this proves the result.

(iii) Note that additive functors respect homotopy because they rerspect both composition and addition. Hence if $g : B_* \to A_*$ makes f an equivalence, i.e., if $g \circ f \simeq 1_{A_*}$ and $f \circ g \simeq 1_{B_*}$, then it follows that

$$Fg \circ Ff \simeq F1_{A_*} = 1_{FA_*},$$

and similarly for B. Hence Fg is an inverse for Ff, so Ff is a chain equivalence.

Exercise 9.15.

(i) This simply involves applying Corollary 9.13(ii). In particulars, we know that F_p and S_p arer both free with basis in $\mathcal{M} = \{\Delta^p\}$. We want to show that Δ^p is totally S- and F-acyclic. But notice that $\tilde{H}_n(S_*(\Delta^k)) = 0$ because Δ^k is contractible, and similarly for F_p, since it coincides with C_p ono Δ. This proves acyclicity, and so the two are naturally chain equivalent.

(ii) Theorem 9.8 implies that singular and large simplicial homology are the same, while Theorem 7.22 implies that normal simplicial and singular homology are the same.

Lefschetz Fixed Point Theorem

Exercise 9.16. Notice that 1_G induces $1_{G/tG} : x + tG \mapsto x + tG$. Hence, with any basis $\{x_1, \dots, x_n\}$ of G/tG, we have $1_{G/tG}$ equal to the identity matrix whose dimension is $\operatorname{rank} G/tG$.

Exercise 9.17. A basis $\{x_1, \dots, x_k\}$ of G'/tG' can be extended to $\{x_1, \dots, x_n\}$ of G/tG. Since G'' is just G/G' and $f''(g + G') = pf(g) = f(g) + G'$. Thus $f''(x_i + G') = f(x_i) + G'$ for $i = k + 1, \dots, n$. Thus the matrix of $\bar{f}$ is diagonal, of the form shown on p. 259 of the textbook, which implies the result.

Exercise 9.18. If $f : S^n \to S^n$, then f_{0*} and f_{n*} are maps $\mathbb{Z} \to \mathbb{Z}$. Note that f_{0*} is the identity, and thus has trace 1. If $\operatorname{tr} f_{n*} = 1$ as well, then the whole map is homotopic to either the identity or the antipodal map, implying that f is a homotopy equivalence. Thus $\operatorname{tr} f_{n*} = 0$, and so $\lambda(f) = 1 \neq 0$. The Lefschetz fixed point theorem implies the result.

Tensor Products

Exercise 9.19. Note that

$$\begin{aligned} a \otimes 0 + a' \otimes b' &= a \otimes 0 + (a \otimes b' + (a' - a) \otimes b') \\ &= a \otimes b' + (a' - a) \otimes b' \\ &= a' \otimes b', \end{aligned}$$

and similarly for $0 \otimes b$.

Exercise 9.20. We would like to show that $m(a, b) \sim (ma, b)$ for $m \in \mathbb{Z}$. It is true for $m > 0$ by induction, true for $m = 0$ by Exercise 9.19, and true for $m < 0$ by inverses.

Exercise 9.21. The hint gives the full solution. If $a \in A$ then there exists some $m > 0$ so that $ma = 0$. Hence $a \otimes q = ma \otimes (q/m) = 0$. Since this is true fora all generators of $A \otimes \mathbb{Q}$, the result follows.

Exercise 9.22. Let m be the order of $a \in A$ and n the order of $b \in B$. Then we know that $\gcd(m, n) = 1$, so that there exist integers x, y with $mx + ny = 1$. Hence we have that

$$\begin{aligned} a \otimes b &= a \otimes (mx + ny)b \\ &= (mx + ny)(a \otimes b) \\ &= mx(a \otimes b) + ny(a \otimes b) \\ &= (mxa \otimes b) + (a \otimes nyb) = 0. \end{aligned}$$

Exercise 9.23. This is the exact same argument as Corollary 9.27.

Exercise 9.24.

(i) Use Theorem 9.25(ii) with the fact that $A \times B \cong B \times A$.

(ii) To see this, simply consider the following commutative diagram:

$$\begin{array}{ccc} A \otimes B & \xrightarrow{1_A \otimes f} & A \otimes C \\ \downarrow & & \downarrow \\ B \otimes A & \xrightarrow[f \otimes 1_A]{} & C \otimes A \end{array}$$

Exercise 9.25. Note that $T_A(f+g) = 1_A \otimes f + 1_A \otimes g$, since both maps complete the diagram

$$\begin{tikzcd} A \times B \arrow[rr] \arrow[dr, "\varphi"'] & & A \otimes B \arrow[dl, dashed] \\ & A \otimes B & \end{tikzcd}$$

where $\varphi(a, b) = (a, (f+g)(b))$. But note that $1_A \otimes f + 1_A \otimes g$ is just $T_A(f) + T_A(g)$, proving additivity.

Exercise 9.26. This is clear, since we have

$$\begin{aligned} 1_A \otimes f : A \otimes B &\to A \times B \\ a \otimes b &\mapsto a \otimes fb = a \otimes mb = m(a \otimes b). \end{aligned}$$

Exercise 9.27.

(i) This is easy to show directly. In particular, we show that $a \mapsto 1 \otimes a$ is an isomorphism. We would like to show that $1 \otimes a = n \otimes b$ if $nb = a$. But $n \otimes b = n(1 \otimes b) = 1 \otimes (nb) = 1 \otimes a$, as desired. Hence this map is surjective. It is injective because, otherwise, every $1 \otimes a$ would be 0, which would violate the universal property of tensor products given by Theorem 9.25. Hence this is an isomorphism.

(ii) We must show that the following commutes:

$$\begin{array}{ccc} \mathbb{Z} \otimes A & \xrightarrow{1_A \otimes f} & \mathbb{Z} \otimes B \\ {\scriptstyle \tau_A}\downarrow & & \downarrow{\scriptstyle \tau_B} \\ A & \xrightarrow[f]{} & B \end{array}$$

This commutes because

$$\tau_B \circ (1_A \otimes f) : (n, a) \mapsto (n, f(a)) \mapsto nf(a),$$

while

$$f \circ \tau_A : (n, a) \mapsto na \mapsto f(na),$$

and $nf(a) = f(na)$ since f is a homomorphism.

Universal Coefficients

Exercise 9.28.

(i) We can write $F = \sum A_j$ where $A_j = \mathbb{Z}x_j$, and $F' = \sum A'_k$ where $A'_k = \mathbb{Z}x'_k$. Then $F \otimes F'$ is just

$$\begin{aligned} F \otimes F' &= F \otimes \sum A'_k = \sum (F \otimes A'_k) \\ &= \sum \left(\sum A_j \otimes A'_k \right) \\ &= \sum_{j,k} A_j \otimes A'_k. \end{aligned}$$

But it is easy to verify that $A_j \otimes A'_k = \mathbb{Z}(x_j \otimes x'_k) \cong \mathbb{Z}$, which proves the result.

(ii) This is obvious from the previous part since

$$\operatorname{rank} F \otimes F' = |J \times K| = |J||K| = \operatorname{rank} F \operatorname{rank} F'$$

Exercise 9.29. This is simply an application of Theorem 9.28 and Corollary 9.30, along with Exercise 9.27. We end up with

$$A \otimes B = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}.$$

Exercise 9.30.

(i) Using coordinate-wise addition and scalar multiplication of the form

$$p \sum (q_i, g_i) = \sum (pq_i, g_i)$$

shows that $\mathbb{Q} \otimes G$ is a $\mathbb{Q}$-vector space. Hence $\dim \mathbb{Q} \otimes G$ is defined.

(ii) This follows immediately from the Tor exact sequence, along with the fact that $\operatorname{Tor}(\mathbb{Q}, B) = 0$ for all B.

Exercise 9.31. This is simply a calculation using the properties of Tor. We end up with

$$\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}.$$

Exercise 9.32. Using Exercise 9.30 with the short exact sequence

$$0 \to F \to G \to G/F \to 0$$

gives us

$$\dim \mathbb{Q} \otimes G = \dim \mathbb{Q} \otimes F + \dim \mathbb{Q} \otimes G/F.$$

But $\dim \mathbb{Q} \otimes G/F = 0$ by Exercise 9.21 and the fact that G/F is torsion. Moreover, we know that $\mathbb{Q} \otimes F$ has basis $(1, x_i)$, where x_i is a generator of F, so $\dim \mathbb{Q} \otimes F = \operatorname{rank} F = \operatorname{rank} G$, which proves the result.

Exercise 9.33. Note that [Tor 1] and [Tor 5] imply that there is an exact sequence

$$0 \to \operatorname{Tor}(B', A) \to \operatorname{Tor}(B, A) \to \operatorname{Tor}(B'', A) \to B' \otimes A \to B \otimes A \to B'' \otimes A \to 0,$$

since $B \otimes A \cong A \otimes B$ by Exercise 9.24. But if A is torsion-free, then $\operatorname{Tor}(B'', A) = 0$ by [Tor 2], which gives us the desired exact sequence.

Exercise 9.34. This is false! Consider, for example, when $F = \mathbb{Z}$ and $H = \mathbb{Z}/2\mathbb{Z}$, and $a = 2$, $h = 1$. In general, we need the condition that if $a = \sum m_j x_j$, where $\{x_j\}$ is a basis for F, then $m_j h \neq 0$ for at least some j. After all, we need that

$$a \otimes h = (m_j x_j \otimes h)_j = (m_j h) \neq 0.$$

Exercise 9.35. Let α be the map $(\operatorname{cls} z) \otimes g \mapsto \operatorname{cls}(z \otimes g)$. Then the Universal Coefficients Theorem implies that

$$0 \longrightarrow H_n(X) \otimes G \xrightarrow{\alpha} H_n(X; G) \longrightarrow \operatorname{Tor}(H_{n-1}(X), G) \longrightarrow 0$$

is exact. Of course, since G is torsion-free, we know that $\operatorname{Tor}(H_{n-1}(X), G) = 0$. Hence α is an isomorphism.

Exercise 9.36. Use the second part of the Universal Coefficients Theorem. In particular, it gives us that

$$H_n(X; \mathbb{Z}/m\mathbb{Z}) \cong (H_n(X) \otimes \mathbb{Z}/m\mathbb{Z}) \oplus H_{n-1}(X)[m],$$

since

$$\operatorname{Tor}(H_{n-1}(X), \mathbb{Z}/m\mathbb{Z}) = H_{n-1}(X)[m]$$

by [Tor 4]. If $H_{n-1}(X)$ is torsion-free, the second term is zero, which gives the conclusion.

Eilenberg–Zilber Theorem and the Künneth Formula

Exercise 9.37. This is a straightforward calculation. In particular, we find that

$$\begin{aligned}(\lambda\otimes\mu)_{n-1}D_n(c_i\otimes e_j) &= (\lambda\otimes\mu)_{n-1}(dc_i\otimes e_j+(-1)^i c_i\otimes\partial e_j)\\ &= (\lambda_{i-1}\otimes\mu_j)(dc_i\otimes e_j)+(\lambda_i\otimes\mu_{j-1})((-1)^i c_i\otimes\partial e_j)\\ &= \lambda_{i-1}dc_i\otimes\mu_j e_j+(-1)^i\lambda_i c_i\otimes\mu_{j-1}\partial e_j.\end{aligned}$$

A similar calculation gives

$$\begin{aligned}D_n'(\lambda\otimes\mu)_n(c_i\otimes e_j) &= D_n'(\lambda_i\otimes\mu_j)(c_i\otimes e_j)\\ &= D_n'(\lambda_i c_i\otimes\mu_j e_j)\\ &= d\lambda_i c_i\otimes\mu_j e_j+(-1)^i\lambda_i c_i\otimes\partial(\mu_j e_j).\end{aligned}$$

Of course, we know that $d\lambda=\lambda d$ and $\partial\mu=\mu\partial$, which implies the result.

Exercise 9.38. Note that it suffices to prove the hint, since transitivity will finish the proof. The proof of the hint is a routine, if long, computation.

Exercise 9.39. Suppose $\lambda : C_*\to C_*'$ and $\lambda' : C_*'\to C_*$ with $\lambda\circ\lambda'\simeq 1_{C_*'}$ and $\lambda'\circ\lambda\simeq 1_{C_*}$. Similarly define μ and μ'. Then Exercise 9.38 implies that

$$\lambda\otimes\mu : C_*\otimes E_*\to C_*'\otimes E_*',$$

and similarly for $\lambda'\otimes\mu'$). But

$$(\lambda\otimes\mu)\circ(\lambda'\otimes\mu')=(\lambda\lambda')\otimes(\mu\mu')\simeq 1_{C_*'}\otimes 1_{E_*'}=1_{C_*'\otimes E_*'}.$$

The same calculation holds for the other composition, which proves chain equivalence.

Exercise 9.40. Each n (i.e., each $0\to S_n'\to S_n\to S_n''\to 0$) works because E_* is a chain complex, hence E_n is free.

Exercise 9.41. For $n\geq 1$, we know that $H_n(X)=0=H_n(Y)$. Hence the Künneth formula implies that

$$H_n(X\times Y)\cong\sum_{i+j=n}H_i(X)\otimes H_j(Y)\oplus\sum_{p+q=n-1}\mathrm{Tor}(H_p(X),H_q(Y)).$$

But the first term is 0 since one of i,j is at least 1, and thus one of $H_i(X),H_j(Y)$ is 0. The second term is zero since the only way for $H_p(X)$ and $H_q(Y)$ to both be nonzero is if $p=q=0$, in which case both homology groups are free. Hence the torsion $\mathrm{Tor}(H_0(X),H_0(Y))$ is zero in that case too.

Exercise 9.42. For path-connected X and Y, we have

$$H_1(X\times Y)=H_0(X)\otimes H_1(Y)\oplus H_1(X)\otimes H_0(Y)\oplus\mathrm{Tor}(H_0(X),H_0(Y)).$$

But $H_0(X)=H_0(Y)=\mathbb{Z}$, and so using Exercise 9.27 gives us that the first two terms are $H_1(Y)$ and $H_1(X)$, respectively, while [Tor 2] implies that the last term is 0. This gives the first equation.

For H_2, notice that the Tor terms have either $H_0(X)$ or $H_0(Y)$, so [Tor 2] implies that they are 0. Hence

$$H_2(X\times Y)=[H_0(X)\otimes H_2(Y)]\oplus[H_1(X)\otimes H_1(Y)]\oplus[H_2(X)\otimes H_0(Y)].$$

Using $H_0(X)=H_0(Y)=\mathbb{Z}$ again gives the result.

Exercise 9.43. This splits into multiple cases and is slightly annoying. We end up with the following:

$$H_p(K\times\mathbb{R}P^n)=\begin{cases}0 & p\geq n+2\\ \mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z} & p=n+1, n\text{ odd}\\ \mathbb{Z}/2\mathbb{Z} & p=n+1, n\text{ even}\\ \mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z} & p=n, n\text{ odd}\\ \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z} & p=n, n\text{ even}\\ \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z} & 1<p<n\\ \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z} & p=1, p\neq n\\ \mathbb{Z} & p=0\end{cases}$$

Exercise 9.44. We once again have many, many cases.

$$H_p(\mathbb{R}P^n \times S^m) = \begin{cases} \mathbb{Z} & p = 0, m \neq 0 \\ \mathbb{Z} \oplus \mathbb{Z} & p = m = 0 \\ \mathbb{Z}/2\mathbb{Z} & p \text{ odd}, p < \min(m, n) \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & m \text{ odd}, p = m \\ \mathbb{Z}/2\mathbb{Z} & m \text{ odd}, p \text{ odd between } m \text{ and } n \\ \mathbb{Z} & m \text{ odd}, p \text{ even}, p \leq m + n \\ \mathbb{Z} & m \text{ even}, p = m \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & m \text{ even}, p \text{ odd between } m \text{ and } n \\ \mathbb{Z} & m \text{ even}, p \text{ odd}, p \leq m + n \\ 0 & \text{otherwise} \end{cases}$$

(Something like that, I can't quite read my work anymore.)

Exercise 9.45. This is the exact same idea, but I'll admit I didn't work it all out.

Exercise 9.46. It turns out that the machinery we have (i.e., fundamental groups) isn't sufficient to distinguish $S^1 \vee S^2 \vee S^3$ from $S^1 \times S^2$, as they both have fundamental group $\mathbb{Z}$. In fact, this seems to require cohomology (see **??**).

Exercise 9.47. We use the Künneth formula here. Note that the homology groups of S^1 are all cyclic or zero, so the Tor terms are zero. Hence

$$H_n(S^1 \times S^1) = \sum_{i+j=n} H_i(S^1) \otimes H_j(S^1).$$

If $n > 2$, then one of $H_i(S^1)$ and $H_j(S^1)$ is zero, so

$$H_n(S^1 \times S^1) = 0 \quad n > 2.$$

When $n = 0$, then we have $i = 0, j = 0$, so

$$H_0(S^1 \times S^1) = \mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z}$$

If $n = 1$, then we have $(i, j) = (0, 1), (1, 0)$, and so

$$H_1(S^1 \times S^1) = \mathbb{Z} \otimes \mathbb{Z} \oplus \mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z} \oplus \mathbb{Z}.$$

Finally, if $n = 2$, then we only have to consider $(i, j) = (1, 1)$, so that

$$H_2(S^1 \times S^1) = \mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z}.$$

Now recall that

$$H_n(K_1 \vee K_2) \cong H_n(K_1) \oplus H_n(K_2),$$

so we know that

$$H_n(S^2 \vee S^1 \vee S^1) \cong H_n(S^2) \oplus H_n(S^1) \oplus H_n(S^1).$$

We know the homology groups of S^1 and S^2, and working them out gives the same homology groups as those of $S^1 \times S^1$.

(Interestingly, the fundamental groups of these two spaces are different from one another, which one can show using Seifert–Van Kampen. I wonder if Rotman mixed up this problem with Exercise 9.46.)

Exercise 9.48.

(i) This is straightforward using the fact that the homology of wedges is the direct sum of homology groups. Hence both homology groups are $\mathbb{Z}$ when $n = 0, 3$, $\mathbb{Z}/2\mathbb{Z}$ when $n = 1$, and 0 otherwise.

(ii) According to a cursory search online, this requires universal coverings.

(iii) This seems to be another mistake on Rotman's part, as he seems to have thought that $\mathbb{R}P^3$ and $\mathbb{R}P^2 \vee S^3$ had different fundamental groups. In fact, they both have $\mathbb{Z}/2\mathbb{Z}$ as their fundamental group, and so it is obvious that $\mathbb{R}P^3 \times \mathbb{R}P^2$ and $(\mathbb{R}P^2 \vee S^3) \times \mathbb{R}P^2$ have the same homology groups and fundamental group.

Exercise 9.49. Since the homology groups of S^1 are all cyclic or zero, the Tor terms in the Künneth formula don't count. Suppose that

$$H_n(T^{r-1}) = \mathbb{Z}^{\binom{r-1}{n}}.$$

Note that this is true for $r = 1$. Then we have that

$$H_n(S^1 \times T^{r-1}) \cong \sum_{i+j=n} H_i(S^1) \otimes H_j(T^{r-1}) = H_n(T^{r-1}) \oplus H_{n-1}(T^{r-1}).$$

But of course we know that

$$\binom{r-1}{n} + \binom{r-1}{n-1} = \binom{r}{n},$$

and so it follows that

$$H_n(T^r) = H_n(S^1 \times T^{r-1}) = \mathbb{Z}^{\binom{r}{n}}.$$

10 Covering Spaces

Basic Properties

Exercise 10.1.

Exercise 10.2.

Exercise 10.3.

Exercise 10.4.

Exercise 10.5.

Exercise 10.6.

Exercise 10.7.

Exercise 10.8.

Exercise 10.9.

Exercise 10.10.

Exercise 10.11.

Exercise 10.12.

Exercise 10.13.

Exercise 10.14.

Exercise 10.15.

Exercise 10.16.

Exercise 10.17.

Covering Transformations

Exercise 10.18.

Exercise 10.19.

Exercise 10.20.

Exercise 10.21.

Exercise 10.22.

Exercise 10.23.

Existence

Exercise 10.24.

Exercise 10.25.

Exercise 10.26.

Exercise 10.27.

Exercise 10.28.

Orbit Spaces

Exercise 10.29. We know by Theorem 10.54 that $\mathrm{Cov}(\tilde{X}/(\tilde{X}/H)) = H$, and so we can think of G as a subgroup of $\mathrm{Cov}(\tilde{X}/(\tilde{X}/H))$. Now use Theorem 10.52, with $X = \tilde{X}/H$. We know, in particular, that G is a subgroup of $\mathrm{Cov}(\tilde{X}/X)$, and thus is exactly a covering space $(\tilde{X}/G, v)$ of $X = \tilde{X}/H$, as desired.

Exercise 10.30.

(i) Suppose $gx = x$ and consider a proper neighborhood V of x. Then we know that $gV \cap V = \emptyset$, but $x = gx \in gV \cap V$, contradiction.

(ii) If $G = \{e, g_1, \ldots, g_n\}$ and $x \in X$, then, since X is Hausdorff and since $g_i x \neq x$, there exists a neighborhood V of X which does not contain any $g_i x$. Obviously, this V is a proper neighborhood.

Exercise 10.31. This is exactly the argument in the proof of Theorem 10.2, namely in the first full paragraph on p. 276.

Exercise 10.32.

(i) The group $\mathbb{Z}/p\mathbb{Z}$ acts on S^3 via $m \bullet (z_0, z_1) = (\zeta^m z_0, \zeta^{mq} z_1)$. This action is proper because part (ii) of Exercise 10.30 obviously applies.

(ii) Note that $S^3/(\mathbb{Z}/p\mathbb{Z})$ is exactly $L(p,q)$. Thanks to the previous part, Theorem 10.54(ii) applies, which implies that
$$\pi_1(L(p,q)) = \pi_1(S^3/(\mathbb{Z}/p\mathbb{Z})) = \mathbb{Z}/p\mathbb{Z}.$$

(iii) We know that $L(p,q)$ inherits the local properties of S^3, since there is a local homeomorphism between them. Thus $L(p,q)$ is a 3-manifold.

If $\mathcal{U}$ is an open cover of $L(p,q)$, then $p^{-1}(\mathcal{U})$ is an open cover of S^3. Hence finitely many elements of $p^{-1}(\mathcal{U})$, say $p^{-1}(U_i)$ for $i = 1, \ldots, n$, cover S^3. But then $\{U_1, \ldots, U_n\}$ is a finite subcover of $\mathcal{U}$ which covers $L(p,q)$, proving compactness.

Finally, note that $A \subseteq L(p,q)$ clopen implies that $p^{-1}(A)$ is clopen in S^3. Hence $p^{-1}(A) = \emptyset, S^3$, and so $A = \emptyset, L(p,q)$. Thus $L(p,q)$ is connected too.

Exercise 10.33. Notice that $T \to T/G$ is a universal covering space since T is simply connected. Moreover, since T/G is a connected 1-complex, we know by Corollary 7.35 that $\pi_1(T/G)$ is free. But Theorem 10.54(iii) implies that $\pi_1(T/G) \cong G$, and so G is free.