

11 Homotopy Groups

Function Spaces

No exercises!

Group Objects and Cogroup Objects

Exercise 11.1.

- (i) By definition of a product, there is a unique morphism $\theta : (X, q_1, q_2) \rightarrow (C_1 \times C_2, p_1, p_2)$ in \mathcal{C} making the diagram commute, namely $\theta = (q_1, q_2)$.
- (ii) The objects are ordered triples (X, k_1, k_2) where X is a set and $k_i : C_i \rightarrow X$ are functions. Morphisms $\theta : (X, k_1, k_2) \rightarrow (Y, \ell_1, \ell_2)$ are functions $\theta : X \rightarrow Y$ making the following commute:

$$\begin{array}{ccc}
 & X & \\
 k_1 \nearrow & & \nwarrow k_2 \\
 C_1 & & C_1 \\
 \ell_1 \searrow & \theta \downarrow & \swarrow \ell_2 \\
 & Y &
 \end{array}$$

Exercise 11.2.

We first tackle **Ab**.

The map $\theta : X \rightarrow G_1 \oplus G_2$ in the product diagram is given by $\theta(g) = (q_1g, q_2g)$. Commutativity follows from the fact that $p_i(\theta(g)) = q_i(g)$. Uniqueness of θ follows from the fact that any other θ' must satisfy $\theta'(g) = (g_1, g_2)$ where $g_i = p_i(q_1g, q_2g) = q_i(g)$. Hence $\theta' = \theta$.

The map $\eta : G_1 \oplus G_2 \rightarrow X$ in the coproduct diagram is given by $(g, h) \mapsto k_1(g) + k_2(h)$, where $+$ denotes the operation in the abelian group X . We can easily check commutativity and uniqueness using the fact that η must be a group homomorphism.

Now, for **Grp**, note that the free product property on p. 173 is exactly the coproduct property. The same argument as in the abelian case shows that direct product is the product in **Grp**.

Exercise 11.3.

- (i) We will show this for **Top**_{*}. Suppose we have $((X, x), k_1, k_2)$. It is obvious that the map $\theta : (A_1 \vee A_2, *) \rightarrow (X, x)$, if it exists, must take $*$ to x , and $* \neq a_i \in j_i(A_i)$ to $k_i(a_i)$. We need only show that this map θ is continuous. (In contrast, the proof has already been completed for **Set**_{*}; commutativity of the relevant diagram is obvious from the definition of θ .)

Suppose $U \subseteq X$ is closed. Note that $\theta^{-1}(U) \cap A_i = k_i^{-1}(U)$. (This statement is clear if $*$ $\notin U$. If $*$ $\in U$, then

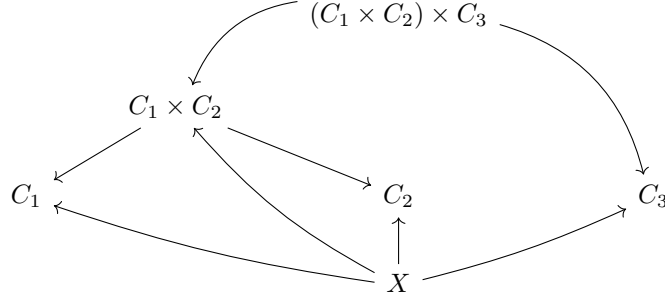
$$\theta^{-1}(U) \cap A_i = (\theta^{-1}(U \setminus \{x\}) \cap A_i) \cup \{*\} = k_i^{-1}(U \setminus \{x\}) \cup \{a_i\} = k_i^{-1}(U),$$

which proves the statement anyway.) The definition of the topology of the wedge (see Example 8.9) implies that $\theta^{-1}(U)$ is closed. Hence θ is continuous, completing the proof.

- (ii) Call this subset S . The map $f : A_1 \vee A_2 \rightarrow S$ which takes $a \in A_i$ (or, more accurately, $a \in j_i(A_i)$) to (a, a_2) if $i = 1$ and to (a_1, a) if $i = 2$ is continuous by the previous argument. It is clearly bijective and closed, since a closed set F in $A_1 \vee A_2$ is still closed in $A_1 \times A_2$. Thus it is a homeomorphism.

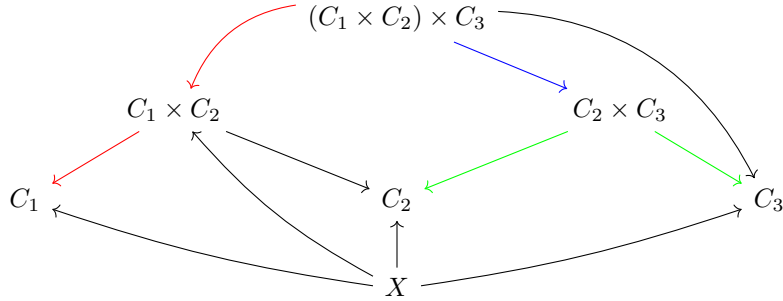
Exercise 11.4. Commutativity follows from the interchanging of C_1 and C_2 in the definition. To see

associativity, consider the following diagram:



There is a unique map $X \rightarrow (C_1 \times C_2) \times C_3$ making this diagram commute.

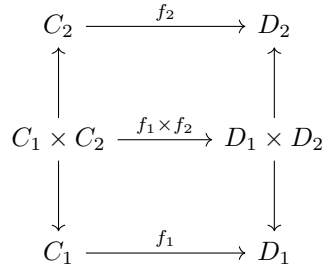
Now define $p_1 : (C_1 \times C_2) \times C_3 \rightarrow C_1$ to be the composition of the red arrows below. Furthermore, the product property of $C_2 \times C_3$ implies the existence of the following blue and green arrows:



Let p_2 be the blue arrow. The fact that there is still the same unique map $X \rightarrow (C_1 \times C_2) \times C_3$ making this commute, then, implies that $(C_1 \times C_2) \times C_3$ is the product of C_1 and $C_2 \times C_3$, thus proving associativity.

Exercise 11.5.

(i) We would like to find $f_1 \times f_2$ making the following commute:



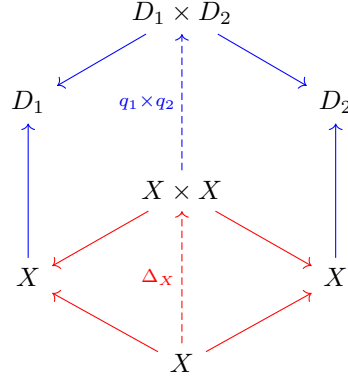
But the existence of maps $C_1 \times C_2 \rightarrow C_i \rightarrow D_i$ implies, by the product property of $D_1 \times D_2$, a unique map $f_1 \times f_2$ into $D_1 \times D_2$ making the diagram commute.

(ii) Same idea.

Exercise 11.6.

(i) Note that Δ_X is the unique map making the red part of the diagram commute, while $q_1 \times q_2$ is the unique

map making the blue part commute:



But of course, since the maps $q_i \circ 1_X = X \rightarrow X \rightarrow D_i$ are equal to simply the maps $q_i : X \rightarrow D_i$, we know that the unique map $X \rightarrow D_1 \times D_2$ making this entire diagram commute is (q_1, q_2) . Uniqueness implies that (q_1, q_2) must be equal to $(q_1 \times q_2)\Delta_X$.

- (ii) This is the same idea.
- (iii) We already showed the first statement. For the second, notice that $\nabla_B(f \times g) = (f, g)$. But $(f, g)\Delta_A(a) = (f(a), g(a)) = (f + g)(a)$ because $A \oplus B = A \amalg B$.

Exercise 11.7.

- (i) Everything follows from the hint, except that we must verify that $1_{X \times Z}$ and $\theta\lambda$ complete the given diagram. Commutativity of the left triangle is obvious in both cases. To see that $q1_{X \times Z} = t$, note that Z being terminal implies that $q = t$. To show that $q\theta\lambda = t$, note that $q\theta\lambda : X \times Z \rightarrow X \rightarrow X \times Z \rightarrow Z$. Thus Z being terminal again implies the result.

Now θ and λ are inverses, and so $X \times Z$ and X are equivalent.

- (ii) This is the dualized version of the previous part.