Hamilton-Jacobi Formulation of Classical Mechanics

AMATH 456 Report
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1 Introduction

1.1 Background

In addition to the Lagrangian and Hamiltonian formalisms of classical mechanics, Hamilton-Jacobi theory constitutes another alternative approach that is equivalent to Newtonian mechanics. It plays a crucial role in both calculus of variations as well as several fields of physics. This report will introduce the basics of Hamilton-Jacobi theory and use it to tackle some example problems with simple potentials.

Concepts from calculus of variations may be extended with the addition of some new machinery to construct this new method. In order to develop the theory, key concepts will be introduced as necessary. Basic knowledge of variational calculus, as well as some experience with mechanics will be assumed, for example, at the level of the AMATH 456 course notes [1]. Concepts including the Euler-Lagrange equation, functionals, variations, generalized coordinates and momenta, Hamilton's principle of stationary action, as well as both Lagrangian and Hamiltonian mechanics will be relevant.

Transformations of canonical variables, referred to as 'canonical transformations', as well as a special class of such transformations, called 'generating functions', will be defined and used to derive the Hamilton-Jacobi equation. The concept of mechanical action will play a crucial role as it relates to the relevant generating function, referred to as 'Hamilton's principal function'. This will lead to a discussion on solution methods.

1.2 Motivation

The Hamilton-Jacobi equation may be used as calculation tool to solve familiar mechanical problems, as we will soon see. The computation does not necessarily become easier, although there are many advantages to this approach. It is particularly useful in identifying conserved quantities, as we will see in a later example. Since the equation is equivalent to the integral minimization of Hamilton's Principle, it can be useful for solving problems both in calculus of variations, as well as physics at large. It is particularly well suited for transition to quantum or statistical mechanics. This is related to the fact that the Hamilton-Jacobi formulations allow the motion of particles to be represented as waves similar to Schrödinger's equation.

Two different problems using the Hamilton-Jacobi method are solved, comparing the results with the Lagrangian solution. For simplicity, all systems considered will have conserved Hamiltonians. Otherwise the equation becomes difficult to solve using the basic methods. First, to illustrate the basic machinery of the method, the case of simple harmonic motion will be examined. The Hamilton-Jacobi equation will be solved using two different methods. Straight-forward via Hamilton's principle function and with Hamilton's characteristic function. Then central force motion will be used to demonstrate separation of variables. This example will more clearly illustrate some of the key differences between the formalisms. The equations of motion will be compared and shown to be equivalent in each case.

2 Hamilton-Jacobi Theory

2.1 Canonical Transformations

Before deriving the Hamilton-Jacobi equation, we will start by briefly covering canonical transformations as an extension of Hamiltonian dynamics. Every possible trajectory in Hamiltonian dynamics may be described by $\{q_i, p_i\}$, a two dimensional **phase space** made up of canonical (generalized) coordinates and momenta. These describe the system at any point in time. Recall from the course notes, the canonical equations of motion are

$$\dot{q}_i = \frac{\partial H}{\partial q_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

where q is the canonical coordinate and p is the canonical momenta. Similarly to coordinate transformations, we can also perform canonical transformations, where set of canonical coordinates and momenta $\{q_i, p_i\}$ are transformed to a new set $\{Q_i(q,p,t), P_i(q,p,t)\}$. This is also known as a **point transformation of phase space**. This transformation yields a new Hamiltonian, K, and the following equations (Equation 9.5 of Goldstein [4]).

$$\dot{Q}_i = \frac{\partial K}{\partial P_i}, \qquad \dot{P}_i = -\frac{\partial K}{\partial Q_i} \tag{2.1.1}$$

If the new canonical variables Q and P do not explicitly depend on time, we call them **restricted canonical transformations**. Canonical transformations may be used to simplify certain problems. Of interest to us is a special class of canonical transforms called generating functions, which depend on both new and old coordinates. We will assume invertibility throughout. The four basic generating functions are tabulated below.

Properties of the Four Basic Generating Functions

$F = F_1(q, Q, t)$	$\Longrightarrow$	$p = \dfrac{\partial F_1}{\partial q}$,	$P = -\dfrac{\partial F_1}{\partial Q}$
$F = F_2(q, P, t) - QP$	$\Longrightarrow$	$p = \dfrac{\partial F_2}{\partial q}$,	$Q = \dfrac{\partial F_2}{\partial P}$
$F = F_3(q, P, t) + qp$	$\Longrightarrow$	$q = -\dfrac{\partial F_3}{\partial p}$,	$P = -\dfrac{\partial F_3}{\partial Q}$
$F = F_4(p, P, t) + qp - QP$	$\Longrightarrow$	$q = -\dfrac{\partial F_4}{\partial p}$,	$Q = \dfrac{\partial F_4}{\partial P}$

(Table 9.1, pg. 373 [1]). F will always be dependent on either q or p or Q and P.

2.2 The Hamilton-Jacobi Equation

The Hamilton-Jacobi Equation results by taking the derivative of the generating function with respect to time (and using the chain rule), F, in equation (2.3.2) and applying the transformations (partial derivatives of the generating function) for $p \to P$ and $q \to Q$.

$$K = H + \frac{\partial F_n}{\partial t}, \qquad \text{where } n = 1, 2, 3, 4 \tag{2.2.1}$$

In order to guarantee that the new variables are constant it time, we set the new Hamiltonian $K = 0$. The new equations of motion for the canonical variables are (equation 10.1 [1]).

$$\frac{\partial K}{\partial P} = \dot{Q} = 0, \qquad -\frac{\partial K}{\partial Q} = \dot{P} = 0 \qquad \Longrightarrow \qquad Q = \beta, \qquad P = \alpha \tag{2.2.2}$$

The Hamilton-Jacobi Equation

$$-\frac{\partial S}{\partial t} = H(q_1, ..., q_n; \frac{\partial S}{\partial q_1}, ..., \frac{\partial S}{\partial q_n}; t) \tag{2.2.3}$$

We now refer to the generating function F as Hamilton's principal function, S. The purpose of the Hamilton-Jacobi equation is to transform the original Hamiltonian to a new form such that the problem becomes easier to solve. The equation will vary depending on the type of generating function used. For different problems, we require different generating functions to obtain the Hamilton-Jacobi equation. For example, the canonical transformations for a harmonic oscillator require the generating function of a second kind. To determine the generating function, we can solve for F using the Hamilton-Jacobi equation and substituting in the new Hamiltonian, K, and original Hamiltonian, H. However, if the Hamiltonian is time-independent, we can use the solution methods outlined in the next section.

Mathematically speaking, the Hamilton-Jacobi approach is a departure from the other formulations. The Hamilton-Jacobi equation is a single first-order nonlinear partial differential equation in Hamilton's principal function S, involving N generalized coordinates and time, where N is the dimension of the system ($N + 1$ coordinates total). As opposed to Hamiltonian mechanics, canonical momenta and conjugate momenta do not appear directly in the equations of motion, only as derivatives of S. The familiar equations of motion result from solving the equation and utilizing the relations between S and the canonical variables.

Newtonian: system of ordinary differential equations arising from analysis of all the forces acting in the system (Based on Newton's Laws).

Lagrangian: system of N, generally second-order ordinary differential equations for the time evolution of the generalized coordinates (stationary action).

Hamiltonian: system of $2N$ coupled first-order ordinary differential equations for the time evolution of the generalized coordinates and their conjugate momenta.

Compared to Newtonian mechanics, Lagrangian and Hamiltonian mechanics are generally more useful at eliminating forces of constraint and in systems without friction. They take same form in nearly any coordinate system. They differ in description, operating under a variational principle as opposed to the cause and effect relationships of Newton's laws. Hamilton-Jacobi theory extends Hamiltonian mechanics and presents new advantages, some of which will be touched on in the upcoming examples. Before we are able to discuss solution methods, we will briefly cover Hamilton's principal function and mechanical action.

2.3 Hamilton's Principal Function

Express S using the chain rule. Using the definition of the Hamiltonian and $\frac{\partial S}{\partial q_i} = p_i$,

$$\frac{dS}{dt} = \sum_i \frac{\partial S}{\partial q_i} q_i + \frac{\partial S}{\partial t} dt = \sum_i p_i q_i - H = L$$

$$S = \int L dt$$

The mechanical action of a system with Lagrangian L is the functional of t, $y(t)$, and $y'(t)$

$$A(y) = \int_a^b L(t, y, \dot{y}) dt \tag{2.3.1}$$

as seen in the course notes (Equation 5.2 [1]). The physical meaning of S is now clear. Hamilton's principal function is closely related to the mechanical action. They are in fact the same, separated by an integration constant.

Hamilton's Principle of Stationary Action

$$\delta A(y) = 0$$

Between fixed times a and b, a system moves along the trajectory that makes stationary the action integral over all admissible trajectories. [1]

$$S(q) = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt = \int_{t_1}^{t_2} (p dq - H(q, p, t)) dt = \int_{t_1}^{t_2} (p\dot{q} - H(q, p, t)) dt$$

$$\delta \int_{t_1}^{t_2} (p\dot{q} - H(q, p, t)) dt = 0 \tag{2.3.2}$$

(Equation 9.7 [4]). In order for Q and P to be valid canonical coordinates and momenta, they must satisfy Hamilton's principal function. Extending this to our new Hamiltonian, K, with our canonical transformations, we have (Equation 9.6 [4]).

2.4 Solution Methods

There are three solution methods that be will be examined: Hamilton's Principal Function, Hamilton's Characteristic Function, and Separation of Variables. We will then see two examples using these solution methods. For Hamilton's Principal Function and Hamilton's Characteristic Function, the main procedure will be outlined (pgs. 442-444 [4]) for the case of time-independent Hamiltonians. In the example of the simple harmonic oscillator, we will see that either method will work since the Hamiltonian does not explicitly depend on time. The two methods are related by $S(q,\alpha,t) = W(q,\alpha) - \alpha t$ where S is Hamilton's Principal Function, W is Hamilton's Characteristic Function, and α is a constant of integration.

2.4.1 Hamilton's Principal Function Recipe

Requirements: The original Hamiltonian, H, of a system is a function of q, p, t
Goal: $q \to Q$ and $p \to P$ such that Q, P are constants and $H \to K$ such that $K = 0$
Generating Function: Hamilton's Principal Function denoted $F = S(q, P, t)$

1. Write down the Hamilton-Jacobi equation with $p = \dfrac{\partial S}{\partial q}$
2. We can assume some solution S. If the Hamiltonian does not explicitly depend on time, we can take S to be $S(q,\alpha,t) = W(q,\alpha) - \alpha t$
3. Set the new canonical momenta equal to the constants of integration, i.e. $P = \alpha$.
4. Solve for the equations of motion, q and p using the following relations

$$q, \qquad p = \frac{\partial S}{\partial q}, \qquad P = \alpha, \qquad Q = \frac{\partial S}{\partial \alpha} = \beta$$

2.4.2 Hamilton's Characteristic Function Recipe

Requirements: The original Hamiltonian, H, is not explicitly dependant on time
Goal: $q \to Q$ and $p \to P$ such that P is constant and $H \to K$ such that $K = H(P) = \alpha$
Generating Function: Hamilton's Characteristic Function denoted $F = W(q, P)$.
The new equations of motion for the canonical variables are (equations 10.45, 10.47 [4])

$$\frac{\partial K}{\partial P} = \dot{Q} = v, \qquad -\frac{\partial K}{\partial Q} = \dot{P} = 0 \qquad \Longrightarrow \qquad Q = vt + \beta, \qquad P = \alpha$$

1. Write down the Hamilton-Jacobi equation with $p = \dfrac{\partial W}{\partial q}$ and $K = \alpha$
2. Set the new canonical momenta equal to the constants of integration, i.e. $P = \alpha$.
3. Find Q by using $\dot{Q} = \dfrac{\partial K}{\partial P}$
4. Solve for the equations of motion, q and p using the following relations

$$q, \qquad p = \frac{\partial W}{\partial q}, \qquad P = \alpha, \qquad Q = \frac{\partial W}{\partial \alpha} = vt + \beta$$

2.4.3 Separation of Variables

The Hamilton-Jacobi equation is most useful when it can be solved with **separation of variables**. This technique separates the equation as a sum of all variables in the system. In doing so we are able to solve each equation individually.

Consider a Hamilton-Jacobi equation of the form

$$H\Big(q_1, ..., q_n; \frac{\partial S}{\partial q_1}, ..., \frac{\partial S}{\partial q_n}; \alpha_1, ..., \alpha_n; t\Big) + \frac{\partial S}{\partial t} = 0 \tag{2.4.1}$$

Now consider a separable coordinate q_1 with a Hamilton's principal function S, the separation of q_i can be written in the form (10.48 [4])

$$S(q_1, ..., q_n; \alpha_1, ..., \alpha_n; t) = S_1(q_1; \alpha_1, ..., \alpha_n; t) + S'(q_2, ..., q_n; \alpha_1, ..., \alpha_n; t) \tag{2.4.2}$$

By definition, if an Hamilton-Jacobi equation is said to be separable, all coordinates are separable, or

$$S = \sum_i S_i(q_i; \alpha_1, \ldots, \alpha_n; t)$$

And the H-J equation simplifies to n equations of

$$H_i\Big(q_i; \frac{\partial S_i}{\partial q_i}; \alpha_1, ..., \alpha_n; t\Big) + \frac{\partial S_i}{\partial t} = 0 \tag{2.4.3}$$

If the Hamiltonian is not explicitly dependent on t, then we can write each S_i as

$$S_i(q_i; \alpha_1, \ldots, \alpha_n; t) = W_i(q_j; \alpha_1, ..., \alpha_n; t) - \alpha_i t \tag{2.4.4}$$

And hence each equation can be further simplified as

$$H_i\Big(q_i; \frac{\partial W_i}{\partial q_i}; \alpha_1, ..., \alpha_n; t\Big) + \frac{\partial S_i}{\partial t} = \alpha_i \tag{2.4.5}$$

This is a much more workable form compared to the original equation. The physical properties of the variables may vary, but ultimately it gives an elegant way to work with a multi-coordinate system. We will see how this solution method is used in Kepler's problem later.

Note that not every H-J equation is said to be separable, nor does it require separation to solve. However, in many physical systems the Hamiltonian is not explicitly dependent on time, and hence t will be separable. When allowed, this is a useful tool to simplify the Hamilton-Jacobi equation and take advantage of Hamilton's characteristic function W.

3 Example Problems

3.1 Simple Harmonic Motion

3.1.1 Simple Harmonic Motion Lagrangian Method

As a basis for comparison, let us briefly talk about one dimensional simple harmonic oscillator using the Lagrangian Method. Starting from the Lagrangian for simple harmonic motion in free space

$$L = T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \tag{3.1.1}$$

The Euler-Lagrange equation then gives the equation of motion,

$$\ddot{x} + \frac{k}{m}x = 0 \qquad \Longrightarrow \qquad \ddot{x} + \omega^2 x = 0 \tag{3.1.2}$$

$$x = c_1\cos(\omega t) + c_2\sin(\omega t) \qquad \Longrightarrow \qquad x = A\sin(\omega t - \varphi) \tag{3.1.3}$$

is the solution, where $\omega = \sqrt{\frac{k}{m}}$, A is a the amplitude, φ is the phase of the motion and c_1, c_2 are constants to be determined by the initial conditions.

3.1.2 Simple Harmonic Motion Hamilton-Jacobi Method

Next we will examine the simple harmonic oscillator in one dimension using the Hamilton-Jacobi Method. The equation of motion of the oscillator will be solved for using Hamilton's Principal Function and Hamilton's Characteristic Function, the two solution methods earlier described. In the end, equivalence of both solutions will be demonstrated. The solutions should also be equivalent to what we solved for using the Lagrangian method.

In order to apply either of the two solution methods, first obtain an equation for the Hamiltonian, H, of the one dimensional simple harmonic oscillator. Recall that the Lagrangian from the previous section. We can replace x with q in order to match the notation used in Goldstein [4], where q is the only generalized/canonical coordinate of the problem. Recall the general form of the Hamiltonian (Theorem 5.6 of the Course Notes).

$$H = -L + \sum_{i=0}^{n} \dot{y}_i L_{\dot{y}_i} \Longrightarrow H = -L + \sum_{i=0}^{n} \dot{q}_i L_{\dot{q}_i}$$

The canonical momenta are given by $p_i = L_{\dot{q}_i} \longrightarrow p = m\dot{q}$. We have an expression for the Hamiltonian in the generalized coordinate, q, which can be simplified by substituting in our expression for p.

$$H = \frac{1}{2}kq^2 + \frac{1}{2}m\dot{q}^2 \qquad \Longrightarrow \qquad H = \frac{1}{2m}(m^2\omega^2q^2 + p^2) \tag{3.1.4}$$

Now that we have established a Hamiltonian for the one dimensional harmonic oscillator, we can apply the Hamilton-Jacobi Theory. We will first solve for the equation of motion of the oscillator using the principal function, then the characteristic function.

Using Hamilton's Principal Function

Using Hamilton's Principal Function, we can write our Hamilton-Jacobi equation as (Equation 10.20 [1]) based on the form of the Hamilton-Jacobi equation outlined in section 2.4.1

$$H\left(q, \frac{\partial S}{\partial q}, t\right) + \frac{\partial S}{\partial t} = 0 \qquad \Longrightarrow \qquad \frac{1}{2m}\left[m^2\omega^2q^2 + \left(\frac{\partial S}{\partial q}\right)^2\right] + \frac{\partial S}{\partial t} = 0 \tag{3.1.5}$$

where $p = \frac{\partial S}{\partial q}$. Since the Hamiltonian is independent of time, we can use the fact that Hamilton's principle (of least action) function can be written as (Equation 10.14 [4]) $S(q, \alpha, t) = W(q, \alpha) - \alpha t$. Taking the partial derivative of S with respect to q and t and substituting it into our Hamilton-Jacobi Equation leaves us with (Equation 10.21 [4])

$$\frac{\partial S}{\partial q} = \frac{\partial W}{\partial Q}, \qquad \frac{\partial S}{\partial t} = -\alpha \qquad \Longrightarrow \qquad \frac{1}{2m}\left[m^2\omega^2q^2 + \left(\frac{\partial W}{\partial q}\right)^2\right] = \alpha \tag{3.1.6}$$

Using the relation $\frac{\partial S}{\partial t} + H = 0$ we can see that $-\alpha + H = 0 \Longrightarrow H = \alpha$.

We know that the Hamiltonian contains the energy of the system, E, thus α must be equal to E. Since the principal function method requires that we take our new canonical momenta, P, to be equal to our constant of integration, α, we have that $P = E$. Next, we can rearrange equation (3.1.6) for $\frac{\partial W}{\partial q}$.

$$m^2\omega^2q^2 + \left(\frac{\partial W}{\partial q}\right)^2 = 2m\alpha \Longrightarrow \frac{\partial W}{\partial q} = \sqrt{2m\alpha}\sqrt{1 - \frac{m\omega^2q^2}{2\alpha}}$$

Integrating both sides with respect to q gives us an expression for W (equation 10.22 [4]). This can then be substituted into S, leaving us with (equation 10.23 [4])

$$W = \sqrt{2m\alpha}\int\sqrt{1 - \frac{m\omega^2q^2}{2\alpha}}dq \qquad \Longrightarrow \qquad S = \sqrt{2m\alpha}\int\sqrt{1 - \frac{m\omega^2q^2}{2\alpha}}dq - \alpha t \tag{3.1.7}$$

Another requirement of the principal function method is that the new canonical coordinate, Q, is found by $Q = \frac{\partial S}{\partial \alpha} = \beta$. So we have (Equation 10.24 [4])

$$Q = \beta = \frac{\partial S}{\partial \alpha} = \sqrt{\frac{m}{2\alpha}}\int\frac{dq}{\sqrt{1 - \frac{m\omega^2q^2}{2\alpha}}} - t \Longrightarrow \beta + t = \frac{1}{\omega}\arcsin q\sqrt{\frac{m\omega^2}{2\alpha}} \tag{3.1.8}$$

$\beta\omega$ can be absorbed into β. Rearranging for q as a function of t leaves us with (Equation 10.25 [4]). For the original canonical momentum, we require $p = \frac{\partial S}{\partial q}$. So we get (Equation 10.26 [4]). Our equations of motion are then

$$q = \sqrt{\frac{2\alpha}{m\omega^2}} \sin(\omega t + \beta), \qquad p = \frac{\partial S}{\partial q} = \sqrt{2m\alpha - m^2\omega^2 q^2} \tag{3.1.9}$$

We can simplify our result for p by substituting in our solution for q (Equation 10.27 [4])

$$\boxed{p = \sqrt{2m\alpha - m^2\omega^2\left(\sqrt{\frac{2\alpha}{m\omega^2}}\sin(\omega t + \beta)\right)^2} = \sqrt{2m\alpha}\cos(\omega t + \beta) \tag{3.1.10}}$$

So our equations of motion are equations (3.1.9) and (3.1.10). We notice that q has the same form as the equation of motion we solved for using the Lagrangian method. Let $A = \sqrt{\frac{2\alpha}{m\omega^2}}$ and $\phi = -\beta$.

Using Hamilton's Characteristic Function

Our Hamilton-Jacobi equation should be of the form

$$H\left(q, \frac{\partial W}{\partial q}\right) - \alpha = 0 \longrightarrow \frac{1}{2m}\left[m^2\omega^2 q^2 + \left(\frac{\partial W}{\partial q}\right)^2\right] = \alpha \tag{3.1.11}$$

where W is the characteristic function and $p = \frac{\partial W}{\partial q}$. The new Hamiltonian K is constant in all coordinates as required. Since the characteristic function method requires that we take our new canonical momenta, P, to be equal to our constant of integration, α, we have that $P = \alpha$. Again, α is the energy of the system, E. Our new canonical coordinate, Q, given by

$$\dot{Q} = \frac{\partial K}{\partial P} = \frac{\partial \alpha}{\partial P} = \frac{\partial P}{\partial P} = 1$$

Integrating gives $Q = t + \beta$. Instead of $Q = \beta$ (from the principal function method), we have $Q = t + \beta$ (from the characteristic function method). And we know that equation $Q = t + \beta$ is equivalent to equation (3.1.8). Rearranging for q as a function of t, we arrive at the same q and p as earlier. Since the Hamiltonian we had for the one dimensional simple harmonic oscillator did not explicitly depend on time, we were able to use two different methods to set up the problem and obtain their equations of motion. The transformed variables are slightly different (i.e. Q is different, P is the same), but they still generate the same equations of motion. Note that we could have also found the equations of motion using the separation of variables method.

3.2 Central Potential & Kepler's Orbits

3.2.1 Kepler's Orbits Lagrangian Method

The 2-Dimensional Kepler's orbit is defined by a central force, $\vec{F}$, and potential energy, V

$$\vec{F} = -\frac{GMm}{r^2}\hat{r}, \qquad V = -\frac{GMm}{r} \tag{3.2.1}$$

where G (gravitational constant), M (mass of central object) , m (mass of interest) are constants. Let's express the kinetic energy using polar coordinates $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$, the Lagrangian is therefore

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{GMm}{r} \tag{3.2.2}$$

The Euler-Lagrange equation gives

$$\begin{cases} mr\dot{\theta}^2 - \frac{GMm}{r^2} - m\ddot{r} = 0 \\ mr^2\dot{\theta} = l \end{cases}$$

where l denotes the constant as the angular momentum. The first Euler-Lagrange equation can be rewritten as

$$m\ddot{r} - \frac{(mr^2\dot{\theta}^2)^2}{mr^3} + \frac{GMm}{r^2} = 0 \qquad \Longrightarrow \qquad m\ddot{r}\dot{r} - \frac{l^2}{mr^3}\dot{r} + \frac{GMm}{r^2}\dot{r} = 0$$

Since, $m\ddot{r}\dot{r} = \frac{d}{dt}(\frac{1}{2}m\dot{r}^2)$, $\frac{l^2}{mr^3}\dot{r} = \frac{d}{dt}(-\frac{l^2}{2mr^2})$, $\frac{GMm}{r^2}\dot{r} = \frac{d}{dt}(-\frac{GMm}{r})$, we have

$$\frac{d}{dt}\left[\frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} - \frac{GMm}{r}\right] = 0 \qquad \Longrightarrow \qquad \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} - \frac{GMm}{r} = c$$

where c is some constant. Plugging in l, we get

$$\frac{1}{2}m\dot{r}^2 + \frac{(mr^2\dot{\theta})^2}{2mr^2} - \frac{GMm}{r} = c \qquad \Longrightarrow \qquad \left[\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)\right] + \left[-\frac{GMm}{r}\right] = c$$

which corresponds to the total energy of the system $T + U = E =$ constant. We now know the total energy of the system is conserved. Replacing c with E and rearranging for $\dot{r}$

$$\dot{r} = \sqrt{\frac{2}{m}(E + \frac{GMm}{r} - \frac{mr^2\dot{\theta}^2}{2})} \tag{3.2.3}$$

which is the equation of motion for r. Rewriting $\dot{r}$ as $\frac{dr}{dt}$, (3.2.3) can then be written as

$$\boxed{dt = \frac{dr}{\sqrt{\frac{2}{m}(E + \frac{GMm}{r} - \frac{mr^2\dot{\theta}^2}{2})}} \qquad \Longrightarrow \qquad t = \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{m}(E + \frac{GMm}{r} - \frac{mr^2\dot{\theta}^2}{2})}}}$$

where $r_0 = r(t = 0)$. We can solve this equation for t, then invert to solve for r.

Now we wish to solve for θ, we notice that the equation for l can be written as:

$$d\theta = \frac{l}{mr^2}dt \qquad \Longrightarrow \qquad \theta = l\int \frac{1}{mr^2}dt \tag{3.2.4}$$

3.2.2 Kepler's Orbits Hamilton-Jacobi Method

Define $r^2 = x^2 + y^2 + z^2$. By equation (3.3.2), the Lagrangian is:

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin(\theta)^2(\dot{\phi})^2) + V(r)$$

The Hamilton-Jacobi equation can be used to find the canonical coordinates which can help to solve the Kepler's problem. The Hamilton is given by in a Cartesian coordinate system

$$H = \frac{p^2}{2m} + V(r) \tag{3.2.5}$$

Next, the canonical coordinates are converted to spherical form. This is suitable as the potential energy depends only on r. Thus, the form of Hamiltonian in spherical coordinates

$$H_s = \frac{1}{2m}[p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2\sin(\theta)^2}] + V(r) \tag{3.2.6}$$

From here we will be following reference [7]. The Hamilton–Jacobi equation is

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\frac{\partial S}{\partial r})^2 + \frac{1}{2mr^2}(\frac{\partial S}{\partial \theta})^2 + \frac{1}{2mr^2\sin(\theta)^2}(\frac{\partial S}{\partial \phi})^2 + V(r) = 0 \tag{3.2.7}$$

This Hamilton–Jacobi equation can be simplified by using separation of variables.

$$S(t, r, \theta, \phi) = S_1(t) + S_2(r) + S_3(\theta) + S_4(\phi) \tag{3.2.8}$$

We can rewrite the Hamilton-Jacobi equation:

$$\frac{\partial S_1}{\partial t} + \frac{1}{2m}(\frac{\partial S_2}{\partial r})^2 + \frac{1}{2mr^2}(\frac{\partial S_3}{\partial \theta})^2 + \frac{1}{2mr^2\sin(\theta)^2}(\frac{\partial S_4}{\partial \phi})^2 + V(r) = 0 \tag{3.2.9}$$

Since the equation does not depend on t and ϕ, so $\frac{\partial S_1}{\partial t}$ and $\frac{\partial S_4}{\partial \phi}$ will be constant. Then we can assume:

$$\frac{\partial S_1}{\partial t} = -E\frac{\partial S_4}{\partial \phi} = L_z$$

By a Legendre transformation of the original action S, it will then be

$$S_2(r) + S_3(\theta) = S(t, r, \theta, \phi) + Et - L_z\phi \tag{3.2.10}$$

The Hamilton–Jacobi equation becomes

$$\frac{1}{2m}(\frac{\partial S_2}{\partial r})^2 + \frac{1}{2mr^2}(\frac{\partial S_3}{\partial \theta})^2 + \frac{L_z^2}{2mr^2\sin(\theta)^2} + V(r) = E \tag{3.2.11}$$

Since the dependence of θ is only in the second and third terms in the left-hand side of the equation, so we can have

$$(\frac{\partial S_3}{\partial \theta})^2 + \frac{L_z^2}{\sin(\theta)^2} = L^2 \tag{3.2.12}$$

Since L is the orbital angular momentum, $S_3(\theta)$ will be

$$\frac{\partial S_3}{\partial \theta} = \sqrt{L^2 - \frac{L_z^2}{\sin(\theta)^2}} \tag{3.2.13}$$

If the solution exists, we need $L^2 \geq L_z^2$ because the change from S to $S_2 + S_3$(equation 3.2.10) is a Legendre transform. Also, ϕ can be defined by using the inverse transform, by a derivative with respect to L_z. Thus, L_z will only depend on S_3.

$$\phi = -\frac{\partial S_3}{\partial L_z} = -\frac{\partial}{\partial L_z} \int \sqrt{L^2 - \frac{L_z^2}{\sin(\theta)^2}} = \int \frac{2L_z d\theta}{\sqrt{L^2 - \frac{L_z^2}{\sin(\theta)^2}} \sin(\theta)^2} = \arctan \frac{L_z \cos(\theta)}{\sqrt{L^2 \sin(\theta)^2 - L_z^2}} + \phi_0$$

By simplifying this equation, we can get

$$\cos(\theta)^2 = \frac{(L^2 - L_z^2)\tan(\phi - \phi_0)^2}{(L^2 + L_z^2)\tan(\phi - \phi_0)^2} \tag{3.2.14}$$

As $\phi \in [0, 2\pi], \cos(\theta) \in [-\frac{L^2+L_z^2}{L}, \frac{L^2+L_z^2}{L}]$,
when $L_z = L$, then $\cos(\theta) = 0$, so the motion would be in the xy plane. When $L_z < L$, the orbit would not be in xy plane, but the polar angle still changes uniquely as the azimuth is changed which would be a closed orbit. As $L_z \Longrightarrow 0$, the azimuth $\phi \Longrightarrow \phi_0$, while $\cos(\theta) \in [-1, 1]$. This behavior happens to be our expectation of a closed orbit with the fixed angular momenta. Then the Hamilton–Jacobi equation will become

$$\frac{1}{2m}(\frac{\partial S_2}{\partial r})^2 + \frac{L^2}{2mr^2} + V(r) = E \longrightarrow \frac{\partial S_2}{\partial r} = \sqrt{2mE - 2mV(r) - \frac{L^2}{r^2}} \tag{3.2.15}$$

Therefore, the 3-D motion problem of Hamilton– Jacobi equation has been simplified to a single integral. Note that this is the same result as before. The time-dependence of the motion can be expressed by the inverse Legendre transformation:

$$t = \frac{\partial S_2}{\partial E} = \int \frac{mdr}{\sqrt{2mE - 2mV(r) - \frac{L^2}{r^2}}} \tag{3.2.16}$$

The Hamilton–Jacobi equation we solved can be applied to the Kepler motion problem. Take $V(r) = \frac{GMm}{r}$ and substitute $V(r)$ to equation(3.2.15). The Hamilton–Jacobi equation is

$$\frac{1}{2m}(\frac{\partial S_2}{\partial r})^2 + \frac{L^2}{2mr^2} + \frac{GMm}{r} = E \qquad \Longrightarrow \qquad \frac{\partial S_2}{\partial r} = \sqrt{2mE + \frac{2GMm^2}{r} - \frac{L^2}{r^2}} \tag{3.2.17}$$

In order to solve the integration easier, we choose the z-axis. Thus the Kepler motion will be in the $x - y$ plane, which means this will change to a 2-D problem. Then $sin(\theta) = 1$ and $L = L_z$, so ϕ can be solved by

$$\phi = -\frac{\partial S_2}{\partial L} = L \int \frac{dr}{r\sqrt{2mEr^2 + 2GMm^2r - L^2}} = \arccos(\frac{L^2 - GMm^2r}{r\sqrt{2mEL^2 + G^2M^2m^4}}) + \phi_0$$

$$L^2 = r[GMm^2 + \sqrt{2mEL^2 + G^2M^2m^4}\cos(\phi - \phi_0)] \tag{3.2.18}$$

Let $\phi_0 = \pi$. Since polar coordinate formula of the conic sections is $r = \frac{ed}{1-e\cos(\theta)}$, we obtain

$$e = \frac{\sqrt{2mEL^2 + G^2M^2m^4}}{GMm^2}, \qquad d = \frac{GMm^2L^2}{\sqrt{2mEL^2 + G^2M^2m^4}} \tag{3.2.19,20}$$

When there is an ellipse ($e < 1$), $E < 0$ (bound state).
When there is a parabola ($e = 1$), $E = 0$. (unbound state)
When there is a hyperbola ($e > 1$), $E > 0$. (unbound state)

Some insight about conserved quantities and functional dependence is gleaned via these solution methods. Even these relatively simple examples begin to shed light on the strengths of the Hamilton-Jacobi formalism.

"In this simple example, some of the power and elegance of the Hamilton-Jacobi method begins to be apparent. A few short steps suffice to obtain the dependence of r and t and the orbit equation. Eqs, (10.69a and b), results derived earlier with only considerable labor. The conserved quantities of the central force system also appear automatically. Separation of variables for the purely central force problem can also be performed in other coordinate systems, for example, parabolic coordinates, and the conserved quantities appear there in forms appropriate to the particular coordinates." (Goldstein pg. 451 [4]) (Where Eqs 10.69a and b refer to the integrals achieved earlier by the Lagrangian method.)

When considering the Lagrangian solution, the variable dependence of the solution was not immediately clear. The integral reduction of the problem was arrived at with some difficulty. One of the greatest strengths of the Hamilton-Jacobi equation is in its ability to identify conserved quantities in systems. Since the new variables are constant in time by construction, it can be easily see that the total energy E, as well as the angular momentum about the orbital axis L are conserved. A deeper dive into conservation and symmetries is another topic beyond the scope of this report.

4 Conclusion

4.1 Extensions

Action-Angle Variables

Sometimes we do not want to solve the equations of motion of a system. We can use the Hamilton-Jacobi equation and its theory to solve for the frequency of motion or oscillations of a system. The solution method that is used to solve for these frequencies is called 'action-angle variables'. Using this method, it is possible to find the frequency of oscillations of a system without having to solve the equations of motion. There are times that the equations of motion of a system are too difficult to solve even with the canonical transformations. For example, it is very difficult to solve the exact equations of motion for Kepler's problem without solving it numerically. It becomes even more difficult to solve when we introduce perturbed dynamics into the system. However, the action-angle variables can be applied to our earlier examples of simple harmonic motion and Kepler's problem. These two examples are considered 'periodic', so the solution method can be applied. There are two types periodic motion: *libration* and *rotation*. The one dimensional simple harmonic oscillator we examined would fall under librational motion.

Instead of setting the canonical momentum, P, from the Hamilton-Jacobi equation to be equal to the constants of integration, α, we define new variables to be equal to α. These new variables are called **action variables** and act as the "transformed (constant) momentum [4]" ($J = \oint pdq$). An example of the simple harmonic oscillator and action-angle variables was presented in Avery Broderick's PHYS 363 Course Notes [5]. In this example, there's a string pendulum of length l hanging from the ceiling carrying a mass m at the end. From behind the ceiling, there is a gentle tug at the string, so the length/height of the swinging pendulum decreases as a function of time. This simple pendulum would exhibit both properties of libration and rotation, giving it the name *bifurication*.

4.2 Final Words

The report began with elementary calculus of variations and Hamilton's principle of least action to introduce the basics of Hamilton-Jacobi theory. It has been shown how Hamiltonian mechanics may be extended to a fourth equivalent formulation. It can be used to find the differential equations governing a mechanical system. Even when an exact solutions can not be reached, the Hamilton-Jacobi equation can provide insight into the solutions. Functional dependence in curvilinear coordinates and conserved quantities may be identified.

The theory addressed includes canonical transforms, generating functions, the derivation of the Hamilton-Jacobi equation, Hamilton's principal function, classical action, Hamilton's characteristic function, and separation of variables. This is of course only a surface level analysis developed with simple problems in mind. There are many caveats to be issued is such a short treatment of the topic, but the content here may serve at as a start.

The examples of simple harmonic motion and central potential systems were solved in new ways. For each of these problems, solving the Hamilton-Jacobi equation was shown to yield solutions equivalent to the Lagrangian method (and thus Newtonian and Hamiltonian also). Three methods of solution were addressed, with examples. The forms of the Hamiltonians in the sample problems were simple enough to allow for diction solution methods. This is not the case in general. Most often the equation is not separable, and the conditions for an equation to be guaranteed separable are quite complex. If the Hamiltonian takes a more complicated form, or is not conserved, separation of variables will prove ineffective. To solve more general problems, computational methods for partial differential equations are needed. In the case of the Kepler orbits, the problem was reduced to an integral. The details of how to solve this integral are omitted.

Since Hamiltonian mechanics places position and momentum on equal footing, transformations can be used to greater simply problems by changing the form of the Hamiltonian. It further abstracts the Hamiltonian method, making its usage unclear on the basis of Hamiltonians that take a simple form. Problems with Hamiltonians that may be simplified by taking a different form, for example by separating the radial and angular spherical components may be solved more easily. Canonical transforms is a large topic which was only scratched upon this report. To do it justice, much more preliminary work is required. For more on canonical transforms, see Goldstein chapter 9. For a more detailed take on the theory, as well as the problems covered, including the use of action-angle variables on the Kepler problem, see Goldstein chapter 10. The strengths of the theory are many and greater than could be shown here. For further details and a more thorough analysis of Hamilton-Jacobi theory, including more examples, see the references cited.

The example of a free particle was originally intended to be in this report, but was omitted due to page and word number constraints. The idea was to simply take the Lagrangian of a free particle and find it's equations of motion. Then use the 'Hamilton's principal function recipe' outlined in the solutions section to solve the Hamilton-Jacobi equation. In both cases the motion would be show to be linear. The example would demonstrate the process in action before moving onto simple harmonic motion. The reference for this is included below [7]. (There are others that were discarded). A more detailed comparison was originally planned. It was going to have its own section about advantages, disadvantages and equivalence. This was eventually delegated to some basic comments. Finally, simulations and figures were in the outline, but did not make the final cut. A reasonable choice would have been to numerically solve the Kepler integral and plot the orbits. There is some detail on how this is done in [4].

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