Complex Dynamics and Renormalization

Jessica Metzger PHYS 252 project

June 3 2021

Contents

In this project we'll look at holomorphic maps of the complex place $\mathbb{C}$, in search of developing a "taxonomy" of the Julia and Mandelbrot set structures formed by their dynamics, and explaining the prevalent self-similarity.

A holomorphic map $f : \mathbb{C} \to \mathbb{C}$ is differentiable with respect to its complex variable in some neighborhood of every point in its domain. A background on holomorphic functions, and some useful results, is included in the appendix.

1 Quadratic maps of $\mathbb{C}$

Start in the restricted case: the one-parameter family of holomorphic maps $f_c : \mathbb{C} \to \mathbb{C}$ where

$$f_c(z) = z^2 + c \tag{1}$$

for some $c \in \mathbb{C}$.

1.1 Complex Dynamics Preliminaries

We can observe that points of large $|z|$ will escape to infinity under iteration of f_c:

Proposition 1.1. *For any quadratic map f_c, there is some R such that for all $|z| > R$, $\lim_{n\to\infty} |f_c^n(z)| = \infty$.*

Proof. ([9] p. 10) Let $R = \max\{3, |c|\}$. Then if $|z| > R$, we have

$$\left|\frac{f_c(z)}{z}\right| \geq |z| - \left|\frac{c}{z}\right| > |z| - 1 > 2. \tag{2}$$

Thus $|f_c^{n+1}(z)| > 2|f_c^n(z)|$, so $|f_c^n(z)|$ approaches infinity as n does. $\square$

So the set of escaping points is nonempty.

We also have that at least one point (given by the solution of $z^2 + c = z$) is fixed and therefore non-escaping. We can find other examples of non-escaping points by finding periodic points of all periods, given by solutions of $f_c^n(z) = 0$. Moreover, if any periodic cycle is attracting (has multiplier $|(f_c^n)'| < 1$), which, intuitively and observably, tends to be possible for small values of c, it will have some open basin of attraction. So in these cases, the non-escaping *sets* are nonempty too, and possess their own connected components. We call this set of non-escaping points $K(f_c)$. We have as a corollary of Prop. 1.1 that $K(f_c)$ must be bounded. We can also show that it is closed (by expressing its complement as the countable union of f_c-preimages of $\{z : |z| > R\}$).

The boundary of the set of nonescaping points $K(f_c)$ will be called $J(f_c) = \partial K(f_c)$, the **Julia set**. The filled-in set $K(f_c)$ is called the **filled Julia set**. It turns out that the points on the Julia set are exactly those around which f exhibits chaotic behavior.

Finally, we will call the complement of this chaotic boundary the **Fatou set**, $F(f_c) = \mathbb{C} \setminus J(f_c)$. These will be the points around which f_c isn't chaotic. It

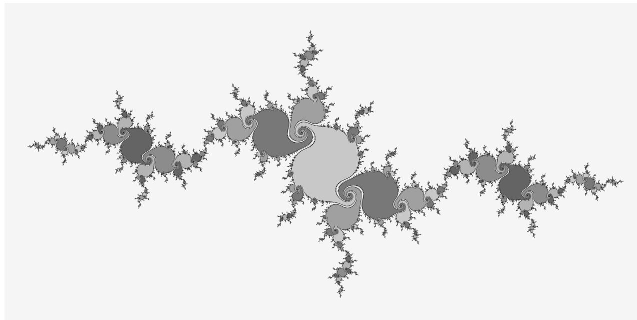

Figure 1: The filled Julia set formed by the escape dynamics of the map $f_c(z) = z^2 + c$ where $c = -1.12 + 0.222i$. $J(f_c)$ is the boundary, and $K(f_c)$ is the boundary plus interior.

includes the set of escaping points, and the interior of the set of non-escaping points $K(f_c)$. An example of a filled Julia sets is given in Fig. 1. It seems to have connected components.

1.2 A dichotomy in the parameter space

We noted that there are two sets in the domain of f_c with drastically different behavior (escape, $\mathbb{C} \setminus K(f_c)$ or non-escape, $K(f_c)$). There are also two regimes of drastically different behavior in the space of parameters c. Either the filled Julia set $K(f_c)$ will be connected, or totally disconnected with empty interior. Serendipitously, the set of parameters in the phase plane $\mathbb{C}$ that display the first behavior has structures *suspiciously similar to some Julia sets.* We'll say more on this later.

It turns out, the "phase" of a map f_c is governed by the behavior of its critical point 0:

Theorem 1.1. *Let $f_c(z) = z^2 + c$.*

1. *0 doesn't escape to infinity $\iff$ $K(f_c)$ is connected.*
2. *0 escapes to infinity $\iff$ $K(f_c)$ is totally disconnected, with empty interior.*

Proof. Suppose 0 doesn't escape. If we take R as in Prop. 1.1, we can express $K(f_c)^c$ as the iterated preimages of $V = \{z : |z| > R\}$, then each f_c^n has 2^n continuous, injective inverses defined on V. Choosing one for each n, because $V \cup \{\infty\}$ is simply-connected (in the topology of the Riemann sphere $\mathbb{C} \cup \{\infty\}$, $f_c^{-n}(V \cup \{\infty\})$ will be simply-connected. Then $K(f_c)^c = \bigcup_n f_c^{-n}(V \cup \infty)$ is the increasing union of simply-connected sets, which is simply-connected. So $K(f_c)$ is connected. Likewise, if $K(f_c)$ is totally disconnected, 0 must escape, otherwise this argument would be contradicted.

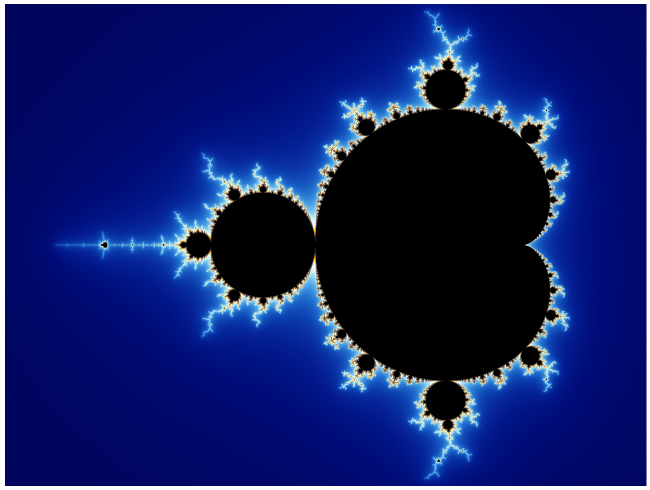

Figure 2: The Mandelbrot set $\mathcal{M}$, values of c for which 0 doesn't escape.

Now suppose 0 escapes. Then there is some iteration which splits $K(p_c)$ into multiple connected components, as explained in the above observations. Each subsequent iteration doubles the number of connected components, putting the subsequent ones inside the "lobes" of the previous ones. It can also be shown that their diameters decrease to 0. Thus $K(p_c)$ is the intersection of proliferating, decreasing subsets, and each "branch" intersects to a single point. So $K(p_c)$ has no non-singleton connected components, and is totally disconnected. Likewise, if $K(f_c)$ is connected, 0 can't escape, proving the converse of (1). $\square$

We call the set of c such that 0 doesn't escape (so that $K(f_c)$ is connected) the **Mandelbrot set** $\mathcal{M}$, named after the mathematicion Benoit Mandelbrot.

We also observed that if there is an attracting periodic orbit present, it must have some open basin of attraction and the filled Julia set must be "significant" in some sense. It turns out that this is a condition for c to belong to $\mathcal{M}$.

Proposition 1.2. *If f_c has an attracting periodic orbit, 0 must be attracted to it.*

We skip the proof of this, but it also uses the existence of injective preimages in the absence of critical points in the basin of attraction. Then by Thm. 1.1, if f_c has an attractive periodic orbit, $K(p_c)$ must be connected. This is similar to our observation above, that if there is an attractive orbit $K(p_c)$ must be "significant" in some sense. We speculated that this should be true for small enough c. It is true that $\mathcal{M}$ is bounded, but its shape is quite complicated. See Figure 2.

Looking at a few Julia sets with their associated c labeled in $\mathcal{M}$ (Fig. 3), we note that Julia sets farther in the interior of $\mathcal{M}$ seem "thicker", with sets getting

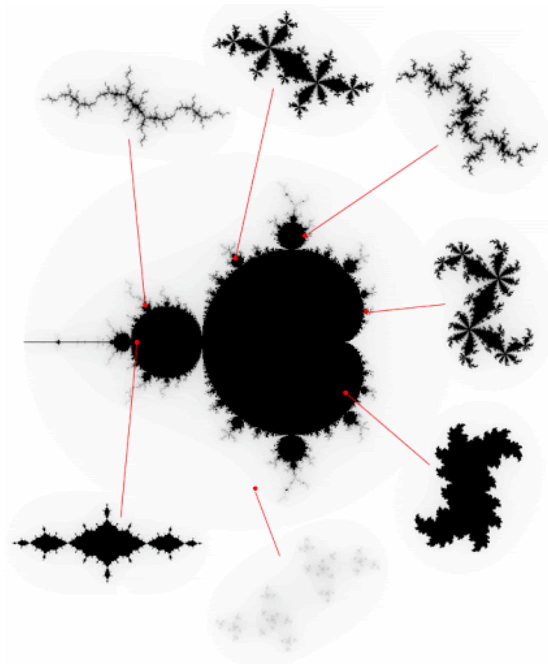

Figure 3: Julia sets for f_c with various values of c labeled in $\mathcal{M}$. From http://paulbourke.net/fractals/juliaset/.

thinner as they approach the boundary, before becoming a "dendrite" at the boundary and then becoming totally disconnected outside $\mathcal{M}$.

1.3 Periodic Orbits

For which parameters c does f_c have an orbit of period n? When is a periodic orbit attracting, and when is it repulsive?

1.3.1 Repulsive

There can be multiple repulsive periodic orbit. In fact, the Julia set $J(f_c) = \partial K(f_c)$ is exactly the closure of the repelling periodic orbits [8]. This makes intuitive sense: it repels points inwards as the inner Fatou components are sucked in, outwards as the outer Fatou component is flung out, and laterally for a connected $J(f_c)$ as f_c stretches and doubles it like a rubber band. (Or, generalizing to an order-n map, stretches and loops it around n-fold.)

1.3.2 Attracting

There can only be one attracting periodic orbit, since by Prop. 1.2, 0 must be attracted to it and it can't be attracted to multiple orbits. On the other

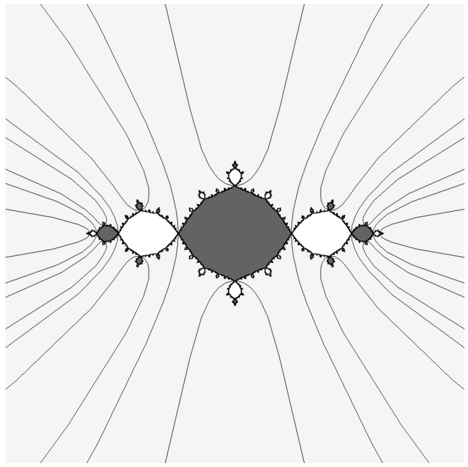

Figure 4: The Julia set of f_{-1}, called the "basilica", with the alternating components labeled.

hand, we can find examples of f_c with no attracting orbits (and therefore empty interior) such that $c \in \mathcal{M}$. (An example is the Feigenbaum polynomial, which is the self-similar period-free limit of the period-doubling cascade that occurs along $\mathbb{R}$ as c is decreased.) But the only c where $K(f_c)$ has empty interior are in the boundary $\partial\mathcal{M}$: suppose f_c has $K(f_c)$ with empty interior. Intuitively, the Julia sets evolve "continuously" [9], so some perturbation to c would decimate it.

Overall, we can observe a pattern: at the center of each "bulb" of the Mandelbrot set, there is a value of c for which 0 is periodic and attracting under f_c. It must have multiplier $|(f_c^n)'(0)| = 0$, so it is super-attracting. An example is the period-2 bulb which contains at its center $c = -1$. For this $f_c(z) = z^2 - 1$, the filled Julia set $K(f_c)$ is the famous Basilica. This is shown in Fig. 4 with the alternating components labeled: points in the basin of attraction of $\{-1, 0\}$ hop from gray to white components and so on.

1.3.3 A pattern

As you move outwards in the bulb from the c where 0 is periodic, there remains a periodic orbit with the same period (and 0 remains in its basin of attraction). The multiplier $|(f_c^n)'(z)|$ increases from 0 towards 1. However, it still hasn't been rigorously proven that this occurs in every bulb of $\mathcal{M}$. The following is known regarding this conjecture:

- If some c in some connected component of $\mathcal{M}^\circ$ has an attracting periodic orbit, then all other parameter values in that bulb have one too (it's a *structurally stable* property).

- All bulbs intersecting the real axis have an attracting periodic orbit (this was proven by McMullen in [1] in 1994).
- The conjecture is equivalent to the f_c with attracting periodic cycles being dense in the space of all f_c.
- If $\mathcal{M}$ is *locally connected*, then the conjecture is true.

Therefore a lot of attention has been placed on the local connectivity of $\mathcal{M}$.

2 Equivalences and Stability

To appreciate the universality of this serendipity, let's generalize to more types of holomorphic maps. We will start this section by noting the famous Riemann mapping theorem.

Theorem 2.1. *(Riemann Mapping). Given any nonempty, simply-connected, open set $U \subsetneq \mathbb{C}$, there is a biholomorphic map between it and the open unit disk.*

This theorem is used widely in complex analysis and we'll assume it throughout.

2.1 Definitions

We know about topological conjugacy and the various "topological invariants" (limit sets, etc.) that are preserved by it. We can also define a stronger equivalence for holomorphic maps. Two holomorphic maps f, g are **holomorphically conjugate** if there is a biholomorphic map ϕ such that $g \circ \phi = \phi \circ f$. In this case, the dynamics of f^n and g^n are, for all purposes, equivalent. For example, we know that

Proposition 2.1. *If f and g are holomorphically conjugate by ϕ, then ϕ maps period-p points to period-p points with the same multipliers $(f^p)'(z_0), (g^p)'(w_0)$.*

Proof. Suppose f, g, ϕ are holomorphic with $g \circ \phi = \phi \circ f$. Then if $f^p(z) = z$, we have $g^p(\phi(z) = \phi(f^p(z)) = \phi(z)$.

Likewise, if $w = \phi(z)$ for a fixed p-period point z of f, we have by the chain rule

$$g'(w) = [\phi(f^p(\phi^{-1}(w)))]' = \phi'(f^p(z))(f^p)'(z)\phi^{-1}(w) = (f^p)'(z). \tag{3}$$

$\square$

So we don't have to worry about duplicates from the same conjugacy class. This justifies our restriction to quadratic polynomials of the form $f_c(z) = z^2 + c$, because:

Proposition 2.2. *Any complex quadratic polynomial of the form $F(z) = \alpha z^2 + \beta z + \gamma$ with $\alpha \neq 0$ is holomorphically conjugate to the map*

$$\tilde{F}(z) = z^2 + \left(\alpha\gamma + \frac{\beta}{2} - \frac{\beta^2}{4}\right) \equiv z^2 + c \qquad (4)$$

by the affine conjugacy $\phi(z) = \alpha z + \beta/2$.

So dynamically speaking, the space of quadratic maps is really a 1-complex-dimensional family of maps.

We can introduce another equivalence between maps that is weaker than holomorphic but stronger than topological. Two polynomial-like maps f, g are **hybrid equivalent** if there is a quasiconformal map q between neighborhoods U_f, U_g of their filled Julia sets such that $q \circ f = g \circ q$ and $\partial_{\overline{z}} q = 0$ almost everywhere on f's filled Julia set. (A quasiconformal map, rather than being independent of the complex conjugate $\overline{z}$ as a conformal map is, is weakly dependent.)

It turns out that, for polynomials of the same degree and with connected Julia sets, hybrid equivalence implies holomorphic equivalence [5].

2.2 Polynomial-like maps

What if our map isn't polynomial? A map f, with open sets $U, V \subset \mathbb{C}$ such that $\overline{U} \subset V$, is **polynomial-like** on (U, V) with degree d if $f : U \to V$ is a proper map where each point in V has d preimages in U, when counted with multiplicity. (For *proper* maps, preimages of compact sets are compact.) When a map is polynomial-like with degree 2, we say it is **quadratic-like**. We can also define the filled Julia set of a polynomial-like map as $K(f) = \cap_n f^{-n}(U) = \cap_n f^{-n}(V)$.

Theorem 2.2. ***The Straightening Theorem*** *Every polynomial-like mapping (f, U, V) of degree d is hybrid-equivalent to a polynomial P of degree d. If K_f is connected, then P is unique up to affine conjugation.*

This proof, which we skip, uses the guaranteed Riemann maps between U, $K(f)$, and the open unit disk. This result implies that the Julia sets of f and P are homeomorphic. This suggests an explanation for why we often see structures that look like quadratic map Julia sets, for non-quadratic maps, e.g. Fig 5. This also highlights the importance of our conjectures about the existence of attracting orbits in $\mathcal{M}^\circ$: if each point in $\mathcal{M}^\circ$ has an attracting periodic orbit, then the same holds for a much larger class of maps than the quadratic ones. It holds for all classes of maps with a hybrid equivalence between their family and the quadratic ones.

Consider the following examples of polynomial-like maps:

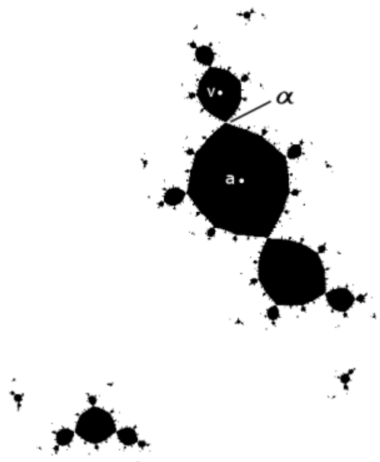

Figure 5: The filled Julia set of a cubic map which is hybrid equivalent to $f_{-1}(z) = z^2 - 1$ [10]. That's why the components of its components look like the Basilica, $K(f_{-1})$.

- Any small enough perturbation of a polynomial with degree d is a polynomial-like map of degree d. [5]
- The one-parameter family of maps

$$\lambda \cos(z) \tag{5}$$

  is full of quadratic-like maps, e.g. $\pi \cos(z)$ on the region $U = \{z : |\Re(z) + \pi| < 2, |\Im(z)| < 1.7\}$. In fact, if you look at the "Mandelbrot set" of this family, it looks exactly like that of the quadratic maps, except that it is centered at π and reflected about the imaginary axis. This is displayed in 6. Under this correspondence, the map $\pi \cos(z)$ is equivalent to z^2, in addition to being hybrid equivalent to it. Under the hybrid equivalence, $-\pi$'s basin of attraction is homeomorphic to the open unit disk.

- Many polynomials with degree higher than 2 are quadratic-like. For example, for the map $F(z) \approx z^3 - 1.688z + 0.984$, we have F^2 is quadratic-like with the critical point obeying a period-2 orbit. In fact, this map is hybrid equivalent to f_{-1}, for which 0 is also 2-periodic. And as suggested by the straightening theorem, the connected component of its filled Julia set that contains this critical point is homeomorphic to $J(f_{-1})$, the famous "basilica" shown in Fig. 4.

  Indeed, the maps "near" F in some subspace of the degree-3 polynomials all have 2-periodic critical points. Another example of a filled Julia

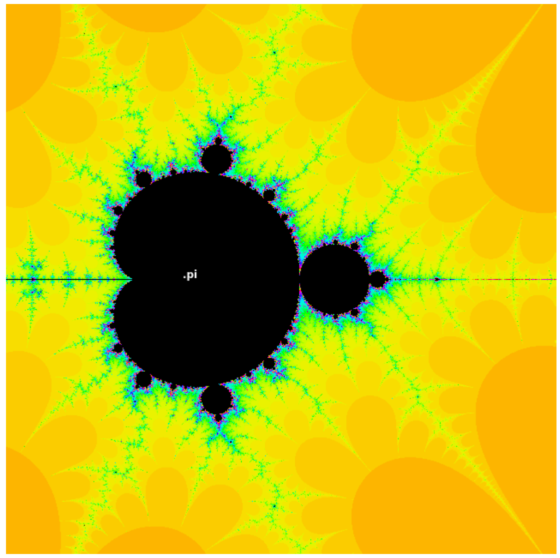

Figure 6: The Mandelbrot set for the family of maps $\lambda \cos(z)$. From https://www.ibiblio.org/e-notes/MSet/Fagella.htm, part of the wonderful series of notes by Prof. Nuria Fagella.

set for one such cubic polynomial is shown in Fig. 5. This subspace of polynomials is in fact homeomorphic to the original period-2 bulb of $\mathcal{M}$ by the map induced by this hybrid equivalence [10]. You can wonder what 1-complex-dimensional subspace of the space of degree-3 polynomials will have Mandelbrot set homeomorphic to the quadratic $\mathcal{M}$: i.e. what subspace of cubic polynomials is dynamically equivalent to the quadratic ones.

- Likewise, iterates of quadratic maps are sometimes themselves quadratic-like. For instance, consider the map $f_c(z)$ with $c \approx -1.75778+0.0137961i$. Let U be the shrunken unit disk. We have that $f_c^3(0) = 0$, and that this occurs with multiplicity 2: $f_c(1) = f_c(-1)$ so U overlaps itself on the first iterate, but in the regimes of $f_c(U)$ and $f_c^2(U)$, multiplicity isn't increased any more (the 1st and 2nd images each belong to the same multiplicity "sheet"). The images of ∂U are displayed over f_c's Julia set in Fig. 7. [2]

This is also why many different quadratic maps have "copies" of each others' Julia sets– for example, f_c^3 from the last example is hybrid-equivalent to some $f_{c'}$ with c' near $-0.123 + 0.745i$, whose Julia set is the famous Douady rabbit. The Douady rabbit is displayed, along with the Julia set of another *rational* map with an iterate also hybrid equivalent to this quadratic, in Figure 8.

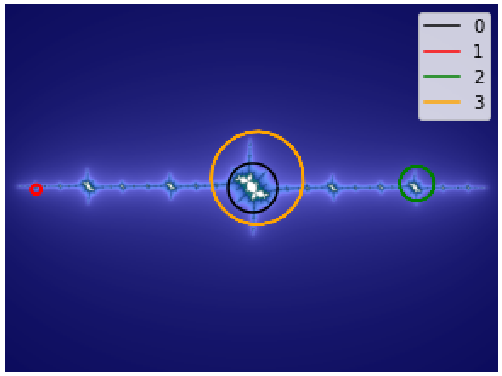

Figure 7: Iterates of the shrunken unit circle plotted over the Julia set of $f_c(z)$ with $c = -1.75778 + 0.0137961i$. Julia set image was generated by http://usefuljs.net/fractals/index.html.

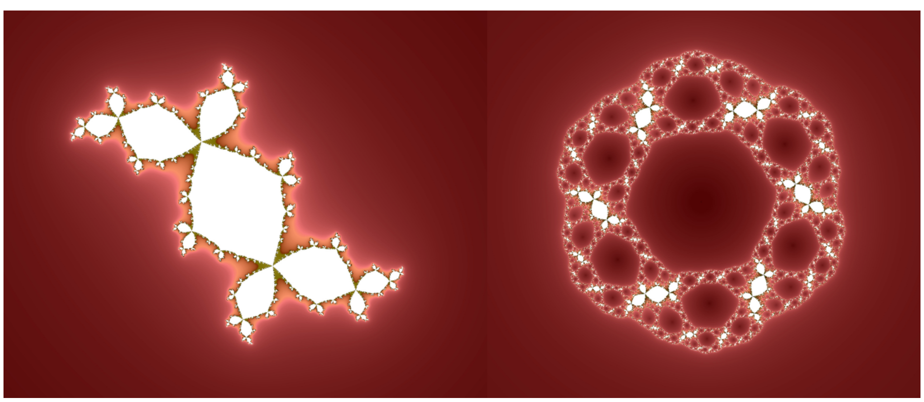

Figure 8: The Douady rabbit $J(f_c)$ and the Julia set of a rational map with an iterate hybrid equivalent to f_c. Notice the copies of the Douady rabbit inside it.

3 Renormalization

The idea of hybrid equivalence of iterates to the original maps reminds us of the renormalization of quadratic maps of $\mathbb{R}$. We can define an analogous concept for holomorphic maps.

A quadratic holomorphic map f_c is **renormalizable** if there exist open neighborhoods (U, V) conformally equivalent to the open disk such that $0 \in U$, and some $n \in \mathbb{N}$, such that $f_c^n : U \to V$ is a quadratic-like map with connected Julia set.

Equivalently, we could simply require there to be such U, V, n for which $f_c^{nk}(0) \in U$ for all $k \geq 0$, by the connectedness/disconnectedness dichotomy contingent on 0's escape. This is useful, because for any f_c with an attracting p-periodic orbit with the point z_0 and a suitable neighborhood $U \ni z_0$, there is some n for which $f_c^{np}(0) \in U$. Then for all $k \geq n$, $f_c^{kp}(0) \in U$. So f^{np} is p-renormalizable.

By the straightening theorem (thm. 2.2), a quadratic-like f^n is hybrid equivalent to some quadratic map, which by Prop. 2.2 is holomorphically equivalent to some $f_c(z) = z^2 + c$. So renormalization induces a "renormalization map" on hybrid equivalence classes of $\mathcal{M}$, which is interesting.

3.1 Renormalization Examples

Let's start looking at examples of renormalizable f_c. Consider f_{-1}. This map sends the neighborhood of 0 to -1, then back over itself, with multiplicity 2. Therefore f_{-1}^2 is renormalizable. Because 0 is a fixed point of f_{-1}^2, it is hybrid equivalent to z^2, via a homeomorphism that sends the main bulb of the Basilica to the open unit disk.

There is also a map that is sent to f_{-1} by renormalization. Let $c = -1.772892...$ [3]. It belongs to the same Mandelbrot bulb (the period-6 one) as the map whose Julia set is displayed in Fig. 7, but the Julia set is "twisted" back to preserve $i \mapsto -i$ symmetry. See Fig. 9.

First of all, we know that this f_c is 6-renormalizable, since $f^6(0) = 0$. This once again indicates the correspondence between $J(f_c)$'s central bulb with the open unit disk, via f_c^6's hybrid equivalence with z^2. But f_c is also 3-renormalizable, by an observation that looks almost identical to Fig. 7. f_c maps the shrunken unit disk to the left, to the right, and then back over itself. This establishes a hybrid equivalence with f_{-1} (for which 0 has period $2 = 6/3$, which checks out; see Fig. 4 for its Julia set the Basilica). A shrunken copy of the Basilica lies in the center of $J(f_c)$ (Fig. 9).

We could really look at any point for which 0 is attracted to a periodic cycle; that is, any interior of a Mandelbrot bulb. If 0 is attracted to a p-periodic orbit, then f is p-renormalizable. But the more interesting cases happen when f is n-renormalizable for some n less than p. For example, above, we observed a map f_c with a period-6 attractive orbit that was 3-renormalizable. It actually

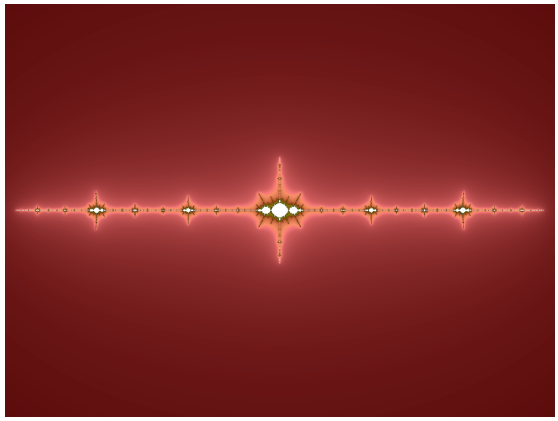

Figure 9: The Julia set for f_c with $c = -1.772892$. This is near the center of the period-6 bulb of $\mathcal{M}$.

belongs to the **period-3 window**. We observed how the entire period-doubling bifurcation is reiterated in this window along the real axis, with a smaller copy of the cascade embedded in this part of the bifurcation diagram. In fact, an entire copy of the Mandelbrot set lies in this window. There are many such copies, at least one for each prime, in fact.

4 Miscellaneous

There is a wealth of fascinating topics related to this. Hopefully I'll get a chance to expand on them some other time.

4.1 Scaling

We can use renormalization to predict scaling between the Mandelbrot set and its copies. For example, if f_c^p is renormalizable via the hybrid equivalence $\Lambda_p[z^2 + \beta(c' - c)]$, then the Mandelbrot set is shrunk down by a factor of $\beta\Lambda_p^2$ in its copy, and the renormalized Julia sets are shrunk down by a factor of Λ_p in their copies from the original set.

4.2 Asymptotic self-similarity

We noticed the presence of small copies of $\mathcal{M}$ embedded in itself. In fact, for some $c \in \partial\mathcal{M}$, there is embedded a copy of the corresponding $J(f_c)$ (more precisely, $\mathcal{M}$ and $J(f_c)$ are *asymptotically self-similar* in that region; this is proven in [6]). This occurs at the **Misiurewicz points**, for which 0 is strictly pre-periodic. These are countable, determined by the c constraint $f_c^{k+n}(z) = f_c^n(z)$ for period n and preperiod k. In fact, these sets are dense on the boundary of $\mathcal{M}$.

Figure 10: The Douady Rabbit bulb tuned with the segment, f_{-2}. From [7].

4.3 Tuning

We briefly mentioned the map induced by renormalization on the hybrid equivalence classes of $\mathcal{M}$. This is related to the "tuning" technique of Douady and Hubbard [9]. Intuitively, given a bulb in the Mandelbrot set and a point in the main cardioid, tuning generates the point in the bulb that corresponds to the point. The tuned function will look similar to the point you started with, but in the "style" of the bulb you're using. See for example Fig. 10, see also Fig. 3.

More precisely, if $f_c(z) = z^2 + c$ is such that 0 is periodic with period p (i.e. c is at the center of its Mandelbrot bulb), we can take the unique Riemann mapping from 0's basin of attraction A under f_c to the open unit disk, sending c to 0. We can associate each point in A with some *internal angle* via this mapping. If we do the same thing with some other $f_{c'}(z) = z^2 + (c')^2$ with connected Julia set, we can establish a correspondence between $K(f_{c'})$ and A. We can similarly glue copies of $K(f_{c'})$ to all other connected components of $K(f_c)$. This generates a map with an attractive p-period cycle whose pth iterate is hybrid equivalent to $f_{c'}$. Additionally, it is p-renormalizable.

5 Appendix

The next few sections will be one big illustration of how huge a restriction holomorphic-ness is.

5.1 Holomorphic functions are conservative

First, we note that a holomorphic map is "conservative" or path-independent. If we write a holomorphic map as $f(z) = f(x+iy) = u(x,y) + iv(x,y)$, then differentiability with respect to $x+iy$ requires, for any (x_0, y_0), and require that the limit definition of the derivative exists and is the same for all directions of approach $(x,y) \to (x_0, y_0)$, we find the restrictions (by looking at directions of approach along the coordinate axes)

$$\begin{aligned} \frac{\partial u}{\partial x} = \frac{i}{i}\frac{\partial v}{\partial y} \quad &\implies \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} \cdot \frac{1}{i} = i\frac{\partial v}{\partial x} \quad &\implies \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \end{aligned} \tag{6}$$

If we think of f as a map from $\mathbb{R}^2$ to itself, then these equations require that f's Jacobian at each point is a rotation. So f locally preserves angles. We can use this to show that lots of functions are holomorphic: all polynomials of a complex variable, sin, cosine, exponentials, etc. are. Sums, products, and compositions of holomorphic functions are also holomorphic; and quotients of holomorphic functions holomorphic are where the denominator is nonzero.

Now we show that a holomorphic function has a zero integral along a closed curve γ, or, is conversative. We can write

$$\begin{aligned} \oint_\gamma f(z)dz = \oint_\gamma (u+iv)(dx+idy) = \oint_\gamma \left[(udx - vdy) + i(vdx + udy)\right] \\ = \int\int_D \left[\left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + i\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\right] dxdy \end{aligned} \tag{7}$$

by Stokes' Theorem, where D is the area enclosed by γ. By the restrictions we derived on the partial derivatives of u and v, this is zero. Note that this only works when f is holomorphic on all of D. So holomorphic functions are like conservative fields, and their integrals are path-independent.

5.2 Cauchy integral formula

We can use this fact to derive the Cauchy integral formula. If we let f be holomorphic and $g(z) = (f(z) - f(z_0))/(z - z_0)$ with $g(z_0) = f'(z_0)$, then g is also holomorphic. Thus, by conservativity,

$$0 = \oint_\gamma g(z)dz = \oint_\gamma \frac{f(z)}{z - z_0}dz - f(z_0)\oint_\gamma \frac{1}{z - z_0}dz = \oint_\gamma \frac{f(z)}{z - z_0}dz - f(z_0)2\pi i. \tag{8}$$

The last equality can be found by evaluating the integral of $1/(z - z_0)$ about a circle with the parameter $z = z_0 + e^{2\pi i t}$, also using conservativity to extrapolate this to any γ. In summary,

$$f(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0}. \tag{9}$$

Thus $f(z_0)$ is determined entirely by the values of f on any curve surrounding it, as long as f is holomorphic on all values bounded by the curve.

5.3 Holomorphic functions are analytic

We can use this to show that any holomorphic function is equal to its own Taylor expansion (it's "analytic"). Let γ be a closed curve within which f is holomorphic, and let z_0 is inside the region bounded by γ. Without loss of generality let 0 be inside γ and look at the behavior around 0. We have that

$$\begin{aligned} 2\pi i f(z_0) &\overset{1}{=} \oint_\gamma \frac{f(z)}{z - z_0} = \oint_\gamma \frac{f(z)}{z} \frac{1}{1 - z_0/z} dz \overset{3}{=} \oint_\gamma \frac{f(z)}{z} \sum_{n=0}^{\infty} \left(\frac{z_0}{z}\right)^n dz \\ &\overset{4}{=} \sum_{n=0}^{\infty} \oint_\gamma \frac{z_0^n}{z^{n+1}} f(z) dz = \sum_{n=0}^{\infty} z_0^n \oint_\gamma \frac{f(z)}{z^{n+1}} dz \equiv \sum_{n=0}^{\infty} z_0^n a_n 2\pi i. \end{aligned} \tag{10}$$

(For the equality $\overset{1}{=}$, we used the Cauchy integral formula. For $\overset{3}{=}$, we used geometric series expansion. For $\overset{4}{=}$, we used the uniform convergence of $f(z) z_0^n / z^{n+1}$ on γ to switch the summation and integration.)

Note that the converse also holds– a Taylor expansion about a point is holomorphic.

5.4 Invertible holomorphic maps on $\mathbb{C}$ are linear

Say we have $f(z_0) = \sum_n z_0^n a_n$ for complex coefficients a_n, and define the map $g(z) = f(1/z) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots$. This g is injective and holomorphic except at $z = 0$ since it is a composition of f and $1/z$. For g to be injective, it must avoid some neighborhood of some point b. So the function

$$h(z) = \frac{1}{g(z) - b} \tag{11}$$

is holomorphic everywhere except at 0, bounded, and has zeroes at the poles of g. (A pole is a singularity at z_0 that can be removed by multiplication by $(z - z_0)^n$ for some finite n). Thus it can be analytically continued to a holomorphic function at 0 too, and we can write $g(z) = b + 1/h(z)$. Now if our analytic continuation $h(0)$ is zero, $g(z)$ has a pole at zero (otherwise $h(z)$ would not be analytic there). And if $h(0)$ is nonzero, $g(z)$ has a singularity removable by cancellation. Both cases imply that f is a polynomial with finite degree.

And if $\deg(f) > 1$, f can't be injective, so f must be linear.

6 References

1. McMullen 1994. *Complex Dynamics and Renormalization.*
2. Fagella, Nuria 1995a. *LIMITING DYNAMICS FOR THE COMPLEX STANDARD FAMILY.* International Journal of Bifurcation and Chaos, Vol. 05, No. 03, pp. 673-699.
3. Fagella, Nurla 1996. *Notes on Renormalization of Complex Polynomials.*
4. Fagella, Nuria. https://www.ibiblio.org/e-notes/MSet
5. A. Douady and John H. Hubbard (1985). *On the dynamics of polynomial-like mappings.* Annales scientifiques de l'É.N.S.
6. Tan, Lei 1990. *Similarity between the Mandelbrot set and Julia sets.* Commun. Math. Phys. 134, 587-617.
7. Bielefeld, Ben. Quasiconformal Surgery and Deformations.
8. Devaney 1986. *An Introduction to Chaotic Dynamical Systems.*
9. Devaney et al. 1991. *Complex Dynamical Systems: The Mathematics Behind the Mandelbrot and Julia Sets.* Proceedings of Symposia in Applied Mathematics vol. 49.
10. Estabrooks 2018. *Identifications in Escape Regions of the Parameter Space of Cubic Polynomials.* Open Access Dissertations. Paper 734.