Complex Dynamics

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1 Introduction

Consider the quadratic function

$$p_c(z) = z^2 + c, \tag{1}$$

where $c \in \mathbb{C}$ is a constant. Let $p_c^n(z)$ denote the n-fold iteration of p_c, that is

$$p_c^1 = p_c,\ p_c^2 = p_c \circ p_c,\ \ldots\ ,\ p_c^n = p_c \circ \ldots \circ p_c. \tag{2}$$

What happens if you apply p_c to $\mathbb{C}$ over and over again – what is the limiting behavior of an arbitrary point in $\mathbb{C}$, and the limiting global "distortion" of $\mathbb{C}$ as the number of iterations approaches infinity?

Example 1. If we consider the simplest case of the function

$$p_0(z) = z^2, \tag{3}$$

then there are three main regions of distinct behavior under iteration of p:

1. $|z| < 1$: Inside the open unit disk, $|p_0^n(z)| = |z|^{2n}$ approaches 0 as $n \to \infty$; the open unit disk is contracted under p_0^n. There is some rotation, but all points eventually end up arbitrarily close to 0. Moreover, this convergence is uniform in any compact subset of the open disk.

2. $|z| > 1$: Outside the closed unit disk, $|p_0^n(z)| = |z|^{2n}$ approaches ∞ as $n \to \infty$; this region is rotated and expanded under p_0^n.

3. $|z| = 1$: Writing $z = e^{i\theta}$, we have $p_0^n(z) = e^{in\theta}$. For all roots of unity, these "orbits" are periodic, while for irrational arguments they never visit the same point twice. Any neighborhood of any point on this circle has infinite points exhibiting either of these behaviors.

In this project we will restrict our attention to polynomials of the form $p_c(z) = z^2+c$, and will see many repeated instances of these three regions of attraction, escape, and instability; and many equivalent characterizations of this behavior relating to ideas like chaos, periodicity, and normal families of functions.

2 Escape under iteration

Let all polynomials be of the form $p_c(z) = z^2 + c$. One of the first observations we can make about the general dynamics is whether or not points will remain within some bounded region or escape to infinity under iterations of p_c. By the fundamental theorem of algebra, we have that at least one point is fixed and therefore non-escaping, given by the solution of $z^2 + c = z$. We can also observe that points of large $|z|$ will escape to infinity:

Theorem 2.1. *For any p_c, there is some R such that for all $|z| > R$, $\lim_{n\to\infty} |p^n(z)| = \infty$.*

Proof. (Devaney et al. p. 10) Let $R = \max\{3, |c|\}$. Then if $|z| > R$, we have

$$\left|\frac{p_c(z)}{z}\right| \geq |z| - \left|\frac{c}{z}\right| > |z| - 1 > 2. \tag{4}$$

Thus $|p_c^{n+1}(z)| > 2|p_c^n(z)|$, so $|p_c^n(z)|$ approaches infinity as n does.

So both sets are nonempty. We denote $I(p_c)$ as the set of escaping points and $K(p_c)$ as its complement, the points that don't escape. Thus we have as a corollary of Thm. 2.1 that $K(p_c)$ must be bounded. We can also make global statements about the topologies of these sets.

Theorem 2.2. *$I(p_c)$ is open and connected.*

Proof. (Morosawa et al.) Let R be as in Thm. 2.1, and let $V = \{z \in \mathbb{C} \mid |z| > R\}$. Since every point in $I(p_c)$ eventually has modulus greater than R, we can express $I(p_c)$ as the union of iterated preimages of this set, that is

$$I(p_c) = \bigcup_{n=1}^{\infty} (p_c^n)^{-1}(V). \tag{5}$$

This is the countable union of open sets which is open. And each $(p_c^n)^{-1}(V)$ must contain only one connected component stretching to infinity; any other components would be bounded and would cause p_c^n to contradict the maximum modulus principle. Thus $I(p_c)$ is connected.

You can think of p_c^{-1} "pulling back" the escaped points to surround $K(p_c)$. Thus $K(p_c)$ is compact, and each component is simply connected.

It can also be shown, although we will not prove it here, that $I(p_c)$, $K(p_c)$, and their boundaries for a given p_c, are invariant under p_c and p_c^{-1}. To me, it intuitively makes sense, since these sets are already determined by countable iterations of p_c.

Example 2. The polynomial $p_{-1}(z) = z^2 - 1$ has two fixed points at $\frac{1 \pm \sqrt{5}}{2}$, which must be in $K(p_{-1})$. What do $I(p_{-1})$ and $K(p_{-1})$ look like for this polynomial? We consider, as in the proof of Thm. 2.2, the successive preimages of the set $V = \{z : |z| > 3\}$. Actually we really only care about what happens to the boundary of V, since the limit of its preimages will approach $\partial K(p_{-1})$.

So, we want to know what points p_{-1} maps to the circle of radius 3 centered at 0. Taking its inverse, this is precisely $\{\pm\sqrt{z+1} : |z| = 3\} = \{\pm\sqrt{3z+1} : |z| = 1\}$. One can imagine taking the circle of radius 3 and shifting it right by 1, splitting and compressing it along the sector meeting the origin to take the square root, and reflecting it across the real line to form the negative branch. You should end up with a sideways oval with distance from the center to the right edge of 2 (since 2 is the largest possible magnitude of $\sqrt{3z+1}$ if $|z| = 1$). This is depicted in Figure 1, where the gray circle represents the preimage of the circle of radius 3 under p_{-1}, the color represents the argument, and the darkness represents the magnitude.

If you iterate again, you get $\{\pm\sqrt{\pm\sqrt{3z+1}+1} : |z| = 1\} = \{\alpha\sqrt{\sqrt{3z+1}+1} : |z| = 1, \alpha \in \{\pm 1, \pm i\}\}$, which has distance from the center to the right edge $\sqrt{\sqrt{2}+1}$. The operations look similar – you would once again shift, split, compress, and double the circle depicted in Fig. 1. However, because of the doubling, now there are four total branches, so rather than there being two "ridges" (as in the flat sides of the oval), there are four ridges, which show up in Fig. 2.

An intricate, self-similar pattern emerges as the number of iterations increases. At the nth iteration, you can count about 2^n "ridges" corresponding to the 2^n branches of the

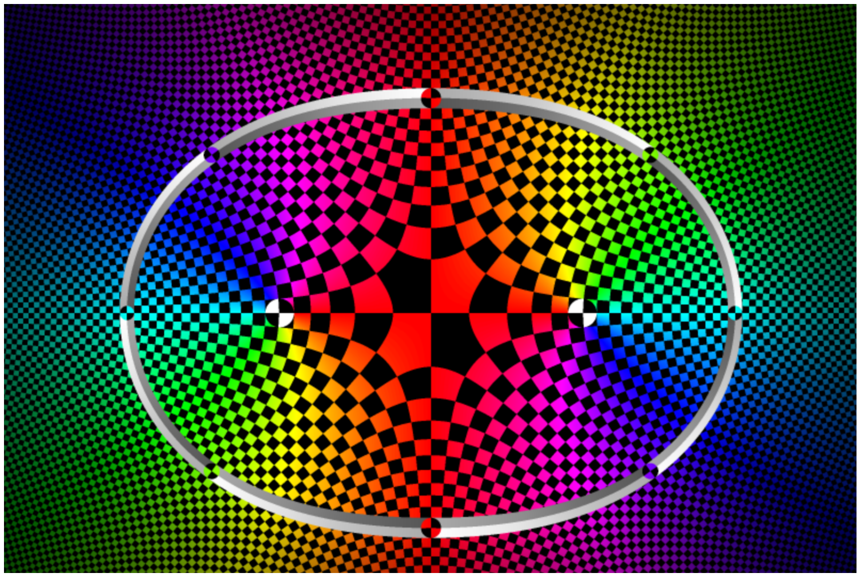

Figure 1: The gray circle is the preimage of the circle of radius 3 under one iteration of p_{-1}. See http://davidbau.com/conformal/ for the plotting program.

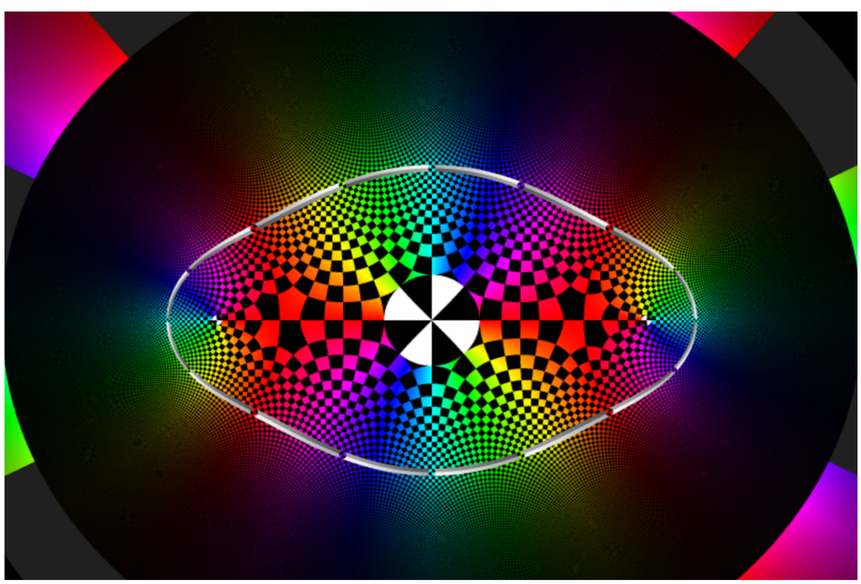

Figure 2: The second preimage of the circle of radius 3 under p_{-1}.

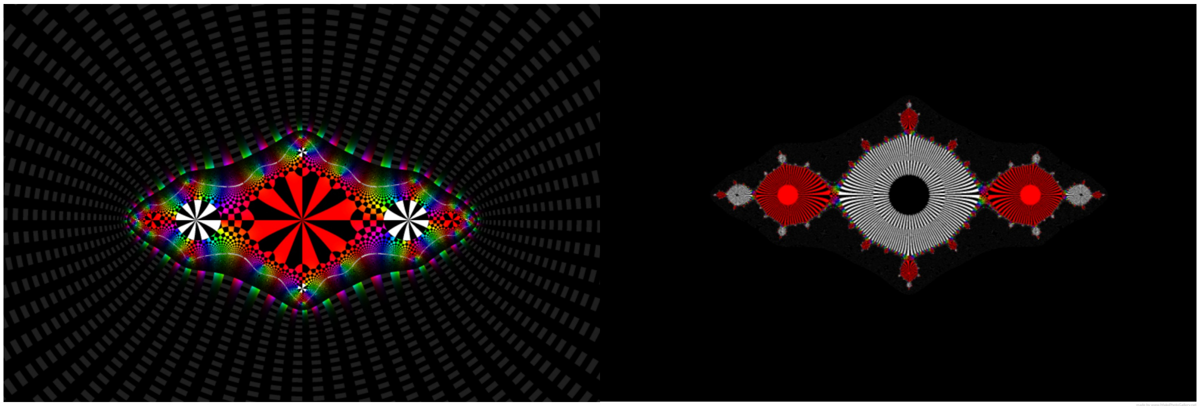

Figure 3: Left: The fifth preimage of the circle of radius 3 under p_{-1}. Right: the twelfth, which begins to resemble The Basilica.

n nested square roots, and the distance from the center to the right edge converges to $\sqrt{\sqrt{\ldots\sqrt{2}+1\ldots}+1}$, which one can show converges to $\frac{1+\sqrt{5}}{2}$, one of our fixed points (which must then lay on the far right edge of $K(p_{-1})$). As depicted in Fig. 3, these "ridges" begin to take the form of bulbs. Evidently, $\partial K(p_{-1})$ must be some sort of fractal.

The final set $K(p_{-1})$, or its boundary, is known as The Basilica. From this example we can make a few observations. First, $K(p_{-1})$ appears to be connected. Every time we iterate p_{-1}^{-1}, we shift it right 1 unit and split it at the origin; this corresponds to splitting it at the center of the left bulb which never "breaks" the circle. And in confirmation of Thm. 2.2, $K(p_{-1})$ appears to be simply connected as well.

Second, if $|c|$ were larger – say, 3 – the shift would be so large that there would be some iteration which "breaks" the circle (corresponding to 0 leaving $K(p_c)$), and every subsequent iteration would double the number of circles. $K(p_c)$ would have about $2^n \to \infty$ connected components. We show in Fig. 4 iterations 1, 2, and 5 of the preimage of $p_{3i}(z) = z^2 + 3i$, applied to the circle of radius 3. After the second iteration, the circle has been split into 2 pieces. With a magnifying glass, you can count 2^4 pieces at the 5th iteration. Being connected, or having infinite distinct connected components, is part of an important and even deeper dichotomy illustrated by Thm. 2.3. It turns out, it all hinges on whether or not 0 escapes to infinity.

Theorem 2.3. *If $I(p_c)$ doesn't contain 0, then $K(p_c)$ is connected. If $I(p_c)$ does contain 0, then $K(p_c)$ is totally disconnected.*

Proof. (Morosawa et al. p. 10, Devaney p. 266-267) First, suppose $I(p_c)$ doesn't contain 0. Let R be the radius given by Thm. 2.1 and $V = \{x \mid |x| > R\}$. Then, since 0 is p_c's only critical point, which never enters V, p_c^n has 2^n continuous, injective inverses defined on V. Since $V \cup \{\infty\}$ is simply connected, each inverse of p_c^n maps it to a simply connected $V_{n,k} \cup \{\infty\}$. And since the $V_{n,k}$ increase with n to cover $I(p_c)$, $I(p_c) \cup \{\infty\}$ must be simply connected; or, $K(p_c)$ is connected.

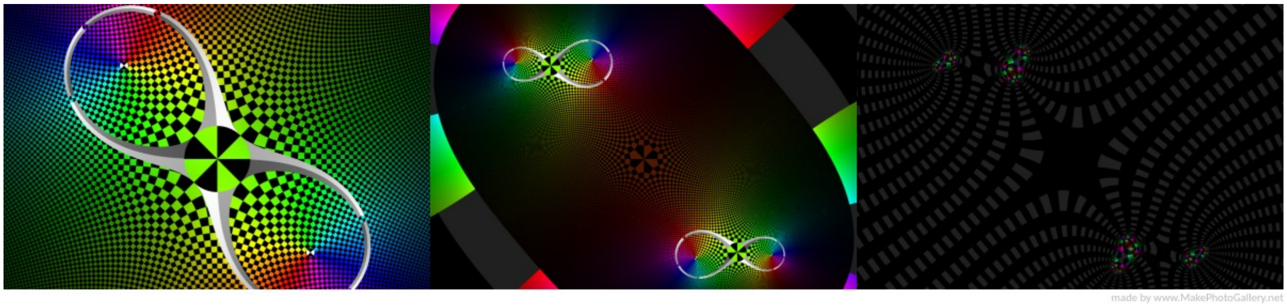

Figure 4: Iterations 1, 2, and 5 of p_{3i}^{-1}, where the gray lines are where it has pulled back the circle of radius 3.

Now suppose $I(p_c)$ contains 0. That means there is some iteration which splits $K(p_c)$ into multiple connected components, as explained in the above observations. As depicted in Fig. 4, each subsequent iteration doubles the number of connected components, putting the subsequent ones inside the "lobes" of the previous ones. It can also be shown that their diameters decrease to 0. Thus $K(p_c)$ is the intersection of proliferating, decreasing subsets, and each "branch" intersects to a single point. So $K(p_c)$ has no non-singleton connected components, and is totally disconnected.

So the points that don't escape either consist of a single connected component like the Basilica in Fig. 3, or infinite singleton sets like in Fig. 4.

Returning to the Basilica, you may notice that most of the points inside $K(p_{-1})$ are either red or white, which means they tend to just two different values (0 and -1, it turns out). And outside $K(p_{-1})$, by definition, everything tends to infinity. The question of what happens along the bifurcation at $\partial K(p_{-1})$ is more difficult, and it helps to look at normal families of functions.

3 Normal families and some history

First, we will give some history, since the history of complex dynamics really begins with the theory of normal families. In 1912, the French mathematician Paul Montel coined the idea of a normal family of functions, given by this definition:
Definition. A family of functions $\mathcal{F}$ defined on an open set U is a *normal family* if any sequence of functions in $\mathcal{F}$ has a subsequence that converges uniformly or to infinity on all compact subsets of U.

This roughly means the family is "clustered together" in the function space. By the Arzela-Ascoli Theorem, it is equivalent to $\mathcal{F}$ being equicontinuous. Montel proved a surprising theorem about families of holormorphic functions:

Theorem 3.1. ***Montel's Theorem****: Let $\mathcal{F}$ be a family of holomorphic functions. If there are two distinct values of $\mathbb{C}$ omitted from the range of every $f \in \mathcal{F}$, then $\mathcal{F}$ is a normal family.*

We skip the proof of this result, but will use it later on. Possibly looking to further develop the theory of normal functions, the Académie des Sciences in Paris announced in 1915 that the topic of the 1918 Grand Prix would be iteration. Pierre Fatou was another French mathematician who at this time, perhaps because of the announcement, began to study this topic (although he ultimately decided not to enter). Looking to determine where the iterations of a complex function were "stable" or "unstable," he made a distinction between the regions where its iterates were normal and where they weren't. We now call the set of points that have a neighborhood where its iterates are normal its *Fatou set* $F(p_c)$.

Gaston Julia, who began studying complex dynamics at the same time as Fatou (and did end up entering – and winning – the Grand Prix), took a different (but ultimately equivalent) approach. He classified the "unstable" parts of an iterated function's domain as being near repelling periodic points. We'll return to this approach later. Occupying a place in this history complementary to Fatou, the complement of the Fatou set – the *Julia set* $J(p_c)$ – is named after him.

It turns out, our approach (which we have borrowed from Morosawa et al. and Lukas Geyer's notes) of dividing up the points based on whether or not they escape is also equivalent to Fatou's, for polynomials.

Theorem 3.2. *For a polynomial $p_c(z) = z^2 + c$, the boundary of the points that don't escape under its iterates, $\partial K(p_c)$, is precisely the set of points for which $\{p_c^n\}$ isn't a normal family in any neighborhood, the Julia set $J(p_c)$. Equivalently, the Fatou set $F(p_c)$ is the union of the escaping points $I(p_c)$ and the interior of the non-escaping points $K(p_c)^\circ$.*

Proof. (Lukas Geyer notes p. 19) First we will show $I(p_c) \cup K(p_c)^\circ \subset F(p_c)$. Suppose z escapes to infinity under iterations of p_c. Then since $I(p_c)$ is open, z has some neighborhood within $I(p_c)$, where by definition all points approach infinity under iterations of p_c. Thus, in this neighborhood, the iterates form a normal family and z is in $F(p_c)$.

Now suppose z is in the interior of $K(p_c)$; then it has a neighborhood inside $K(p_c)^\circ$. Points inside this neighborhood, since they don't escape, can never pass the "point of no return" R (given in Thm. 2.1); thus there are more than two points in $\mathbb{C}$ that are omitted when iterating p_c on this domain. By Montel's Theorem, p_c's iterates is then a normal family in this neighborhood, and z is in $F(p_c)$. So, we have $K(p_c)^\circ \cup I(p_c) \subset F(p_c)$, which implies that $J(p_c) \subset \partial K(p_c)$.

Now suppose $z \in \partial K(p_c)$. Any neighborhood of z will intersect both $I(p_c)$ and $K(p_c)$, thus some points will escape to infinity and some will always be less than R on all sequences of iterates. So, $\{p_c^n\}$ can't be a normal family on any such neighborhood, and $\partial K(p_c) \subset J(p_c)$.

The set $K(p_c)$ is sometimes called the *filled Julia set* of p_c. The boundary between our escaping and non-escaping sets is truly unstable – by Montel's Theorem, every neighborhood of every point in $J(p_c)$ visits all but at most one point of $\mathbb{C}$.

After the developments of Fatou and Julia, the field of complex dynamics lay largely dormant until people like Mandelbrot used computer graphics to visualize the intricate sets in

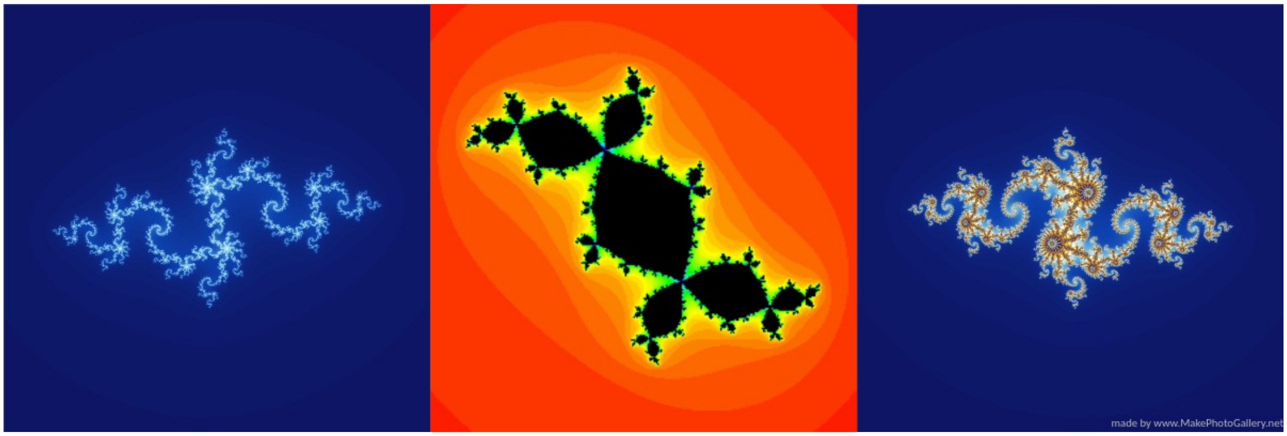

Figure 5: Left: the Julia set for $c = -0.835 - 0.2321i$, taken from Wikipedia. Middle: the "Douady Rabbit," the filled Julia Set for $c = -0.13 - 0.75i$, taken from Robert Devaney's website. Right: the Julia set for $c = -0.8 + 0.156i$, taken from Wikipedia.

consideration, leading to a "2nd wave" that has continued to this day. Now might be a good time to look at some examples of Julia sets. The unit circle of example 1 is the Julia set for $c = 0$. The Basilica in Fig. 3 and the "dust" in Fig. 4 are some approximations of filled Julia sets for $c = -1$ and $c = 3i$. We include some more examples in Fig. 5.

4 Periodic points

Gaston Julia's approach to complex dynamics started not from normal families, but from periodic points. A polynomial p_c is periodic at z_0 with period n if $p_c^n(z_0) = z_0$ (and n is the lowest number such that this is true). A fixed point is a periodic point of period 1. The dynamics around a periodic point depend on p_c^n's derivative at that point, and the points are classified as such:

Definition. (Devaney et al. p. 3) Let p_c be periodic at z_0 with period n. Then z_0 is

1. a *superattracting* periodic point if $|(p_c^n)'(z_0)| = 0$
2. an *attracting* periodic point if $0 < |(p_c^n)'(z_0)| < 1$
3. an *indifferent* periodic point if $|(p_c^n)'(z_0)| = 1$
4. a *repelling* periodic point if $|(p_c^n)'(z_0)| > 1$.

Note that since periodic points occur in cycles, and by the chain rule the derivative of a cycle is the same everywhere on the cycle, we can talk about attracting/repelling cycles, rather than points. These classifications have these names for a reason:

Lemma 4.1. *If z_0 is an (super-)attracting fixed point of p_c, it has a neighborhood, often called its "basin of attraction," where all points converge to it under iterations of p_c.*

Proof. Since $p_c'(z_0) < c$ for some $c < 1$, there is some neighborhood U where for all $z \in U$, $\left|\frac{p_c(z)-p_c(z_0)}{z-z_0}\right| < c$, or, $|p_c(z) - p_c(z_0)| < c|z - z_0|$. Then by the Contraction Mapping Theorem, $p_c^n(z) \to z_0$ as $n \to \infty$.

The same thing clearly applies to attracting periodic points of period n – we will instead have that $p_c^{nj}(z) \to z_0$ as j approaches infinity, and each point on the cycle will have its own basin. And in the case of repelling periodic points, by the inverse function theorem, p_c will have an inverse derivative less than 1 in one of its neighborhoods, so by applying the lemma to the backwards iterates, we see that all points in some neighborhood are repelled from a repelling periodic point.

We can immediately observe that any attractive periodic point, and its basin of attraction, must lie in the Fatou set of p_c, since p_c's iterates automatically form a normal family. And by Theorem 2.3, if 0 escapes under p_c, $K(p_c)$ is totally disconnected and thus admits no basin of attraction. So, if 0 escapes, there are no attracting periodic points. In fact, something stronger can be said:

Lemma 4.2. ***Fatou's Lemma****: If p_c has an attracting periodic orbit, then 0 is in its basin of attraction.*

Proof. (Milnor problem 8-g) Suppose p_c has some attracting fixed point z_0 (the periodic case will follow), and suppose 0 isn't attracted to it. Then we could find some open set A_0 containing z_0 inside the basin of attraction, and such that $p_c(A_0) \subset A_0$. Since A_0 contains no critical value of p_c (its only critical value is 0), $p_c^{-1} : A_0 \to A_1$, with $A_0 \subset A_1$, has some analytic branch such that $p_c^{-1}(A_0) \subset A_1$. Similarly, if A_1 doesn't contain 0, $p_c^{-1} : A_1 \to A_2$ (where $A_1 \subset A_2$) has an analytic branch mapping A_1 into A_2.

Continuing like this, we can construct a sequence of iterated inverses $p_c^{-n} : U_0 \to U_n$ where $U_n \subset U_{n+1}$ and all are contained inside the bounded part of the Fatou set (since they're inside z_0's basin of attraction). Thus $\{p_c^{-n}\}$ is a normal family, and must have some subsequence converging to an analytic function. But $|(p_c^{-n})'(z_0)| = |\lambda|^n$ where $|\lambda| > 1$, so the derivatives approach infinity, a contradiction.

As a result, there can only be one attractive periodic orbit, since 0 can't be attracted to more than one thing. This theorem adds another layer to the already deep bifurcation between the two cases – either 0 escapes and $J(p_c)$ is totally disconnected, or 0 remains bounded and all of $K(p_c)^\circ$ is one connected basin of attraction surrounded by $J(p_c)$.

If, on the other hand, we look at the repelling periodic points, we learn about the Julia set:

Theorem 4.3. *The Julia set of p_c is the closure of its repelling periodic points.*

Proof. (Devaney 269, Lukas Geyer notes p. 29) First suppose z_0 is a repelling periodic point of p_c with period n. Since $p_c^{nj}(z_0) = z_0$, the iterates can't converge to infinity on any subsequence of this sequence. If it had a subsequence $p_c^{nj_k}$ converging to an analytic function f, the derivatives $(p_c^{nj_k})'$ would also converge to its derivative f'. But since z_0 is a repelling periodic point, by the chain rule, $|(p_c^{nj_k})'(z_0)|$ is the j_k-fold product of $|p_c'(z_0)| > 1$, so it

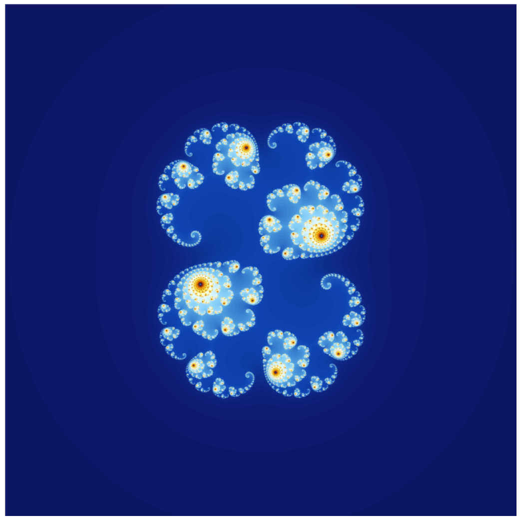

Figure 6: The Julia set for $c = 0.285 + 0.01i$, a Cantor set since 0 escapes under iteration of p_c. (Stolen from Wikipedia)

approaches infinity, a contradiction. So the closure of the repelling fixed points is inside the Julia set.

Now suppose z_0 is in the Julia set of p_c, or, $\{p_c^n\}$ isn't normal in any neighborhood of a point z_0. Let U be any neighborhood of z_0. Since p_c is a degree-2 polynomial, z_0 has two distinct preimages, z_1 and z_2, under p_c. Thus p_c has two locally analytic inverse branches g_1 and g_2 such that $g_1(z_0) = z_1$ and $g_2(z_0) = z_2$, defined on some open $V \ni z_0$ such that $V \subset U$ and $g_1(V) \cap g_2(V) = \emptyset$. By Montel's Theorem, since p_c's iterates aren't normal on V, there must be some N such that Nth iteration of p_c intersects V with $g_1(V)$ or $g_2(V)$. In other words, for some $z \in V$ and $j \in \{1, 2\}$, we have $p_c^N(z) = g_j(z)$. But this means $p_c^{N+1}(z) = p_c(g_j(z)) = z$, so z is a periodic point of p_c inside U.

Since (super-)attracting periodic points are always in the Fatou set, the Julia set must be inside the closure of the repelling and indifferent periodic points. Fatou showed (although we'll skip the proof) that the indifferent periodic points are finite. So, the Julia set is the closure of the repelling periodic points.

This was one of Gaston Julia's main conclusions. One can imagine the repelling points flinging any neighborhood of $J(p_c)$ across the entire complex plane. From this theorem, we also have that for any point in $J(p_c)$, any neighborhood surrounding it has a repelling periodic point. Consequently,

Corollary 4.3.1. *$J(p_c)$ is a compact, perfect set.*

Together with Thm. 2.3, this implies that the Julia set of every p_c where 0 escapes is a Cantor set. See, for example, Fig. 4 or Fig. 6.

Example 3. Looking back at $p_0(z) = z^2$ from Example 1, we see that the Julia set, easily identifiable as the unit circle (as it is the boundary between the escaping and bounded sets), is the closure of the repelling periodic points. Solving the equation $z^{2n} = z$, we find that p_0 has one superattracting fixed point at $z = 0$, since $p_0'(0) = 0$. This is exhibited in the contraction of the unit disk. All other periodic points are roots of unity; thus, they're dense in the unit circle. In addition, they're repelling, since $|p_0'(e^{i\theta})| = 2$.

Example 4. Consider the polynomial $p_{-1}(z) = z^2 - 1$ from Example 2. Its two fixed points, $\frac{1 \pm \sqrt{5}}{2}$, are repelling since the derivative of p_{-1} at each has magnitude greater than 1. In confirmation of Thm. 4.3, you can find them located in $J(p_{-1})$ – we found, for instance, the Basilica's radius to be $\frac{1+\sqrt{5}}{2}$, which means that fixed point was located on the right edge of $J(p_{-1})$.

We also pointed out that on the right panel of Figure 3, most of the points in its filled Julia set are alternately red or white; that is, near one of two values. You can find that the center of the middle bulb is 0, the next two are ± 1, the next two are $\pm\sqrt{2}$, and so on. Since $p_{-1}(0) = -1$ and $p_{-1}(-1) = 0$, and $|(p_{-1}^2)'(0)| = |(p_{-1}^2)'(-1)| = 0$, the points 0 and -1 are in a superattracting periodic cycle of period 2. Note that $p_{-1}(1) = 0$ and $p_{-1}^2(\pm\sqrt{2}) = 0$. Evidently, points at the centers of the bulbs join this periodic cycle. Because it has a basin of attraction, points near the bulbs' centers must approach 0 and -1, too.

5 Chaos

We now give our final characterization of the Julia and Fatou sets, as the chaotic and non-chaotic sets of p_c. Robert Devaney defines chaos as "sensitivity to initial conditions," formally stated in the following definition from Devaney et al.

Definition. A map $p_c : \mathbb{C} \to \mathbb{C}$ is *chaotic* on a set $A \subset \mathbb{C}$ if it is sensitively dependent on initial conditions; that is, if there is some $\beta > 0$ such that for any $z_0 \in A$ and any neighborhood U of z_0, there exist $n > 0$ and $z \in U$ such that $|p_c^n(z_0) - p_c^n(z)| > \beta$.

Thus, no matter how close two points start out (i.e. no matter how small a neighborhood U), they always end up at least β away from each other, at some point. There are two conditions which imply a map is chaotic, which require the concept of transitivity:

Definition. A continuous map $p_c : \mathbb{C} \to \mathbb{C}$ is *transitive* if, for any open sets U and V, there is some iteration n where $p_c^n(U) \cap V \neq \emptyset$.

So a transitive map thoroughly scrambles up the domain. We then have:

Theorem 5.1. *Let $p_c : \mathbb{C} \to \mathbb{C}$. If p_c satisfies*

1. *Periodic points of p_c are dense in $\mathbb{C}$*
2. *p_c is transitive on $\mathbb{C}$*

Then p_c is chaotic on $\mathbb{C}$.

Robert Devaney originally defined chaotic maps as having all three properties in his 1986 book *An Introduction to Chaotic Dynamical Systems*. However, in 1992, Banks et al. proved that in most cases, only the first two were necessary (hence the theorem). We skip the proof, but it can be found in Banks et al. 1992.

Chaotic behavior is reminiscent of the dynamics around the Julia set, so it's no surprise that

Theorem 5.2. *The Julia set is the set where p_c is chaotic.*

Proof. We have that, by Theorem 4.3, repelling periodic points are dense in the Julia set, so condition 1 is fulfilled. Additionally, since any point in the Julia set has no neighborhood where the iterates form a normal family (Theorem 3.2), condition 2 is fulfilled. So by Theorem 5.1, p_c is chaotic on its Julia set.

First, we pointed out in Example 4 that points in the interior of the filled Julia set $K(p_c)$ seemed to be attracted to an attractive periodic orbit. This turns out to be true of every Julia set with nonempty interior (let's just assume the result), and it follows that p_c isn't chaotic at these points. For the unbounded part of the Fatou set, we cheat (following Robert Devaney, Devaney et al. p. 5) – let the metric be such that points get closer together as they approach infinity. Then p_c isn't chaotic on $I(p_c)$.

Note that it's pretty common, when dealing with maps from $\mathbb{C}$ to $\mathbb{C}$, to "close" the plane by including ∞, and define a metric such that escaping points of iterated maps get "closer" as they approach ∞ and treat ∞ as a superattracting fixed point. It fits with the full picture of the dynamical behavior (convergence to infinity is a type of convergence), so the cheating is warranted.

6 The Parameter Space

Let's go a level up and focus on the structure of the parameter space; i.e., what happens when we vary c. We already saw that different values of c can give wildly different behaviors – Theorem 2.3 showed that $K(p_c)$ is connected if 0's orbit is bounded, and totally disconnected if it isn't. So let's start by considering the set of points c where 0 doesn't escape under p_c.

We have by Lem. 4.2 that if there is any attractive periodic cycle, 0 must be attracted to it and must therefore be bounded. So let's start by identifying values of c that give a periodic cycle. To be an attractive cycle of period 1 (a fixed point), p_c must satisfy the following criteria:

$$p_c(z) = z^2 + c = z \tag{6}$$

$$|p_c'(z)| = |2z| < 1 \tag{7}$$

The solution to this is bounded by the curve (Devaney et al. p. 18)

$$c = \frac{1}{2}e^{i\theta} - \frac{1}{4}e^{2i\theta}, \tag{8}$$

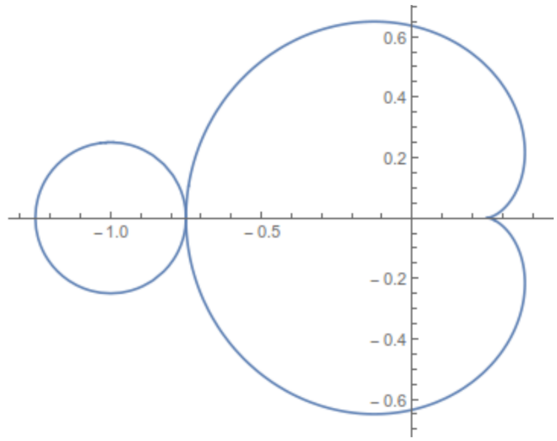

Figure 7: The bounds of the regions for the parameter c with an attractive fixed point (right) and an attractive 2-step orbit (left). Picture from Lukas Geyer's notes.

where $0 \leq \theta < 2\pi$. The curve forms a sideways "cardioid." Actually, you can check that for parameters c lying directly on the curve, p_c has an indifferent fixed point, and its derivative there is $e^{i\theta}$ for the given θ. So both the "rationally" (root of unity) and "irrationally" indifferent derivatives are dense along the boundary of the attractive ones, a difference that ends up being important in the dynamics.

To look for attractive points of period 2, we similarly solve

$$p_c^2(z) = z \tag{9}$$

$$|(p_c^2)'(z)| < 1 \tag{10}$$

and find it is bounded by the circle of radius $1/4$ around -1. The plot of these two retions is shown in Fig. 7.

This is the beginning approximation of the Mandelbrot set, which is the set we have defined:

Definition. The *Mandelbrot Set* is the set of all parameters c such that 0 is bounded under iterations of p_c.

You may notice that our method of finding points in the Mandelbrot set might miss some. Because we are only using Lem. 4.2 as a heuristic to find parameters that give an attractive periodic orbit, it might not capture all the parameters c inside the Mandelbrot set–there may be some that don't give an attractive orbit. Actually, it is still unknown whether or not this is true–one of the largest conjectures in the field is whether or not every point in the interior of every filled Julia set converges to some periodic orbit, even though years of extensive computer simulations haven't found any counterexamples.

At this point, we'll give up plotting the Mandelbrot set by hand, since it starts to get tedious. We include a computer-generated version in Fig. 8, along with a zoomed-in picture of the boundary, where you find another baby Mandelbrot set. Evidently the Mandelbrot set, too, is self-similar in some way.

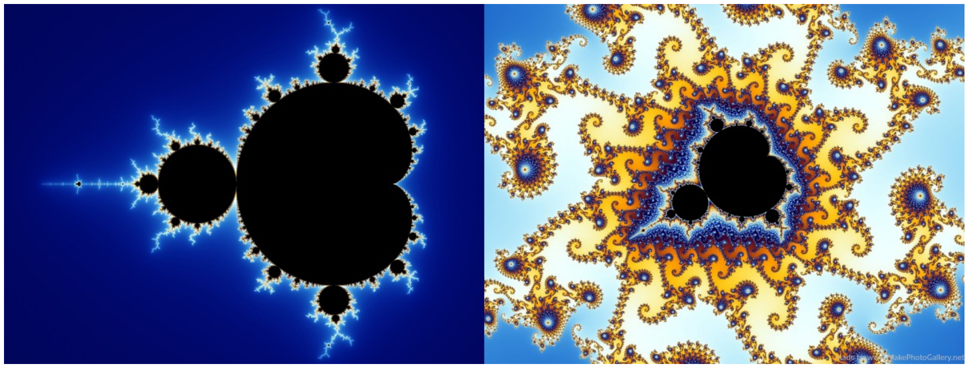

Figure 8: Left: the Mandelbrot set. Right: a zoomed-in picture of near the boundary of the Mandelbrot set. (Stolen from Wikipedia)

7 Conclusion

So why is pretty much everything here a fractal? Since iterated maps basically repeatedly distort $\mathbb{C}$, it's no surprise that there is some chaos in the final product of those infinite distortions. And it's no surprise that there's so much chaos in the bifurcation between two wildly different cases – in the dynamical plane, the boundary of the Julia set divides points that become indefinitely periodic from those that go out to infinity. And in parameter space, the boundary of the Mandelbrot set divides cases where the Julia set is connected from those where it is totally disconnected.

You might be surprised to see that the similarities between the two domains (dynamical plane and parameter space) are more than just philosophical: if you zoom in on a part of the Mandelbrot set and produce a Julia set with a parameter near there, you may find that the two have local similarities. See the example in Fig. 9. No one is quite certain the reason for this spooky similarity, or if there is any deep reason at all.

We've reached the point where you have to use fancy methods like renormalization and Blottcher functions to go any further. But there are many people in complex dynamics working on the problems like this local similarity, the attractive periodic orbit conjecture, and much more. There will likely be many more neat results in years to come.

8 References

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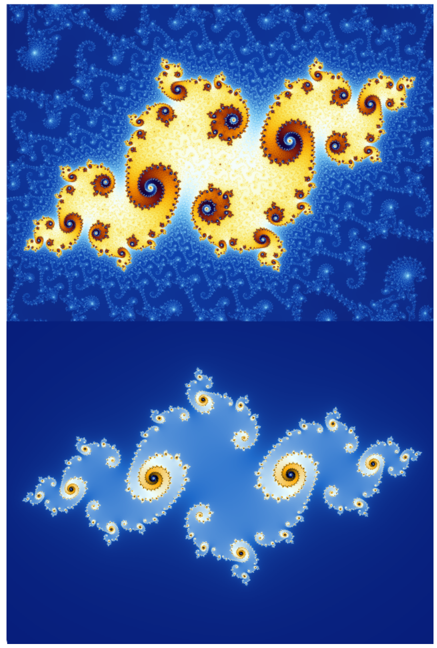

Figure 9: Top: A zoomed-in piece of the Mandelbrot set. Bottom: A similar Julia set. (Stolen from Wikipedia).

6. Lukas Geyer's Complex Dynamics lecture notes: http://www.math.montana.edu/geyer/2016-fall-m597/materials/s2016-complex-dynamics-notes.pdf

7. David Bau's complex number plotting website: http://davidbau.com/conformal/#z