Exceptions to the Ratchet Principle: Rectification of Inhomogenous Fluctuations

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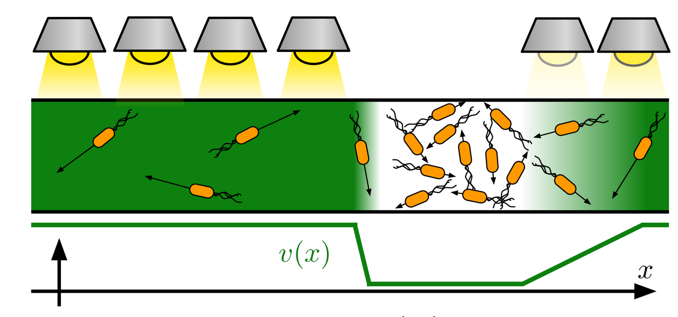
Introduction and phenomenology

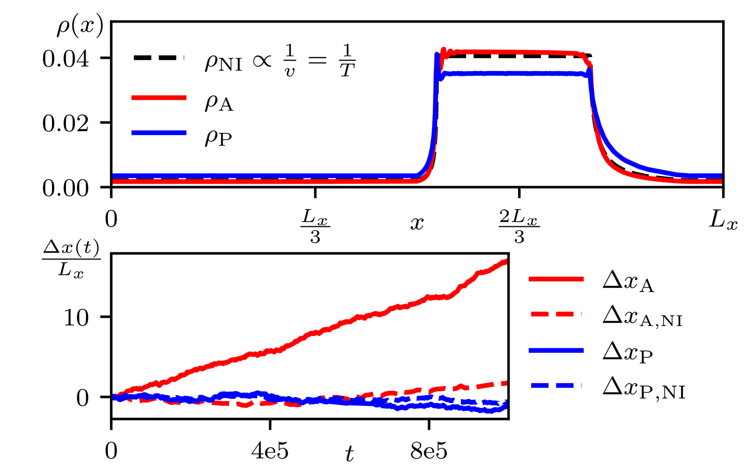
Breaking time-reversal symmetry (TRS) and parity symmetry generically leads to steady-state currents, or – a longstanding heuristic called the "ratchet principle".[1] However, some systems seem to escape this: asymmetric inhomogenous fluctuation landscapes do not produce steady-state currents; e.g. Run-and-Tumble Particles (RTPs) and Active Brownian Particles (ABPs) with asymmetric self-propulsion speed $v(\mathbf{r})$,[2,3] and Passive Brownian Particles (PBPs) in asymmetric temperature field $T(\mathbf{r})$:[4]

RTPs: $\dot{\mathbf{r}}_i = v(\mathbf{r}_i)\mathbf{u}_i(t) - \sum_j \nabla U_{\rm int}(\mathbf{r}_i - \mathbf{r}_j), \quad \mathbf{u}_i \xrightarrow{\alpha} \mathbf{u}_i' \in S^{d-1}. \qquad U_{\rm int} = 0 \implies \mathbf{J} = \langle \dot{\mathbf{r}}_i \rangle = 0$

PBPs: $\dot{\mathbf{r}}_i \overset{\text{Itō}}{=} \sqrt{2\mu T(\mathbf{r}_i)}\boldsymbol{\eta}_i(t) - \sum_j \nabla U_{\rm int}(\mathbf{r}_i - \mathbf{r}_j) \qquad \forall U_{\rm int}, \quad \mathbf{J} = \langle \dot{\mathbf{r}}_i \rangle = 0.$

Adding **symmetric pairwise interactions** between the particles generates a current in ABPs[5] and RTPs, but not in PBPs! Can we amend the ratchet principle to explain these exceptions and their marginal stability?





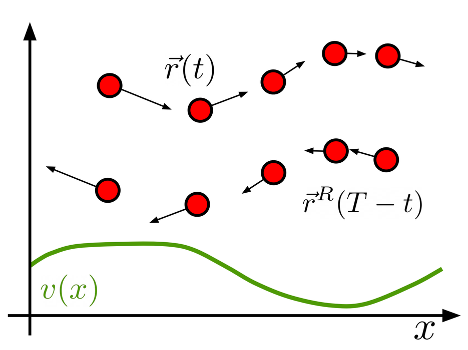
Using the activity $v(x)$ and temperature $T(x)$ landscape above, simulated non-interacting PBPs and ABPs/RTPs converge to the same steady-state density $\rho_{\rm NI}(x) \propto 1/v = 1/T$ (dashed lines). Adding interactions perturbs the density (solid lines); they only induce a ratchet current in the active case.

Time-reversal symmetry of ratchet exceptions

When $U_{\rm int} = 0$, the **entropy-production rate**, marginalized over orientations, for both PBPs in a temperature landscape $T(\mathbf{r})$ and ABPs/RTPs in an activity landscape $v(\mathbf{r})$ is zero:

$$\sigma = \lim_{t_f\to\infty} \frac{1}{t_f} \log \frac{\mathbb{P}[\{\mathbf{r}_i^R(t)\}]}{\mathbb{P}[\{\mathbf{r}_i(t)\}]} = 0 \, .$$

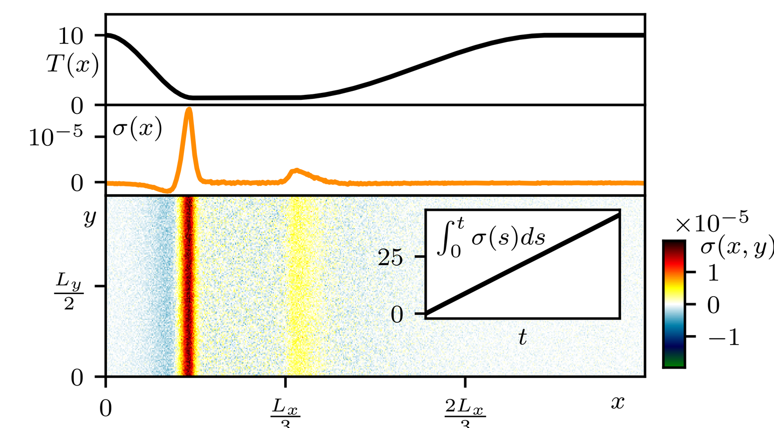
Emergent **time-reversal symmetry (TRS)** $\implies$ no ratchet current.



Interaction-induced TRS violation

For PBPs in $T(\mathbf{r})$, $U_{\rm int} \neq 0$ causes a nonzero entropy production rate, **violating TRS**.

$$\hat{\sigma}(\mathbf{r}) \equiv -\sum_{n,m=1}^{N} \frac{\dot{\mathbf{r}}_n \cdot \nabla U_{\rm int}(\mathbf{r}_n - \mathbf{r}_m)}{T(\mathbf{r}_n)} \delta(\mathbf{r} - \mathbf{r}_n) \, .$$



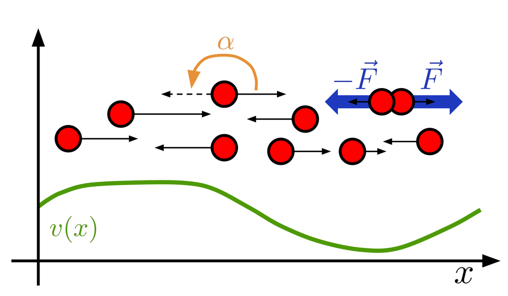
For ABPs/RTPs in $v(\mathbf{r})$, $U_{\rm int} \neq 0 \implies \mathbf{J} \neq 0 \implies$ TRS violation.

Interaction-induced currents in active matter

Consider RTPs in 1d such that $u_i \in \{\pm 1\}$, with $\rho = \langle \sum_i \delta(x - x_i) \rangle$ and $m = \langle \sum_i u_i \delta(x - x_i) \rangle$ evolving as

$$\dot{\rho} = -\partial_x \left[vm - \int dx' \langle \hat{\rho}(x)\hat{\rho}(x') \rangle U_{\rm int}'(x - x') \right]$$

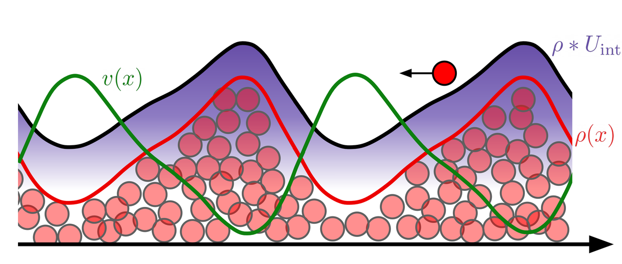
$$\dot{m} = -\partial_x \left[v\rho - \int dx' \langle \hat{m}(x)\hat{\rho}(x') \rangle U_{\rm int}'(x - x') \right] - m/\tau \, .$$



Using a **mean-field** factorization of the correlators $\langle \hat{\rho}(x)\hat{\rho}(x') \rangle \approx \rho(x)\rho(x')$ and $\langle \hat{m}(x)\hat{\rho}(x') \rangle \approx m(x)\rho(x')$,

$$\dot{\rho} = -\partial_x \left[vm - \rho \partial_x V_{\rm eff} \right] \qquad (1)$$

$$\dot{m} = -\partial_x \left[v\rho - m \partial_x V_{\rm eff} \right] - m/\tau \qquad (2)$$



with effective potential $V_{\rm eff}(x) = \int dx' \rho(x') U_{\rm int}(x - x') = (\rho * U_{\rm int})(x)$. This resembles non-interacting RTPs in external potential V, which we solve exactly.

Non-interacting RTPs in potential and activity landscape

Solve Eqs. 1-2 in steady-state ($\dot{\rho} = \dot{m} = 0$) for general V. It can be mapped to passive diffusion with effective mobility, force, and temperature fields:

$$\tilde{\mu}^{-1} = 1 + \frac{V''}{\alpha} - \frac{v'V'}{\alpha v}$$

$$\tilde{F} = -V' + \frac{vv'}{\alpha} - \frac{(V')^2 v'}{\alpha v}$$

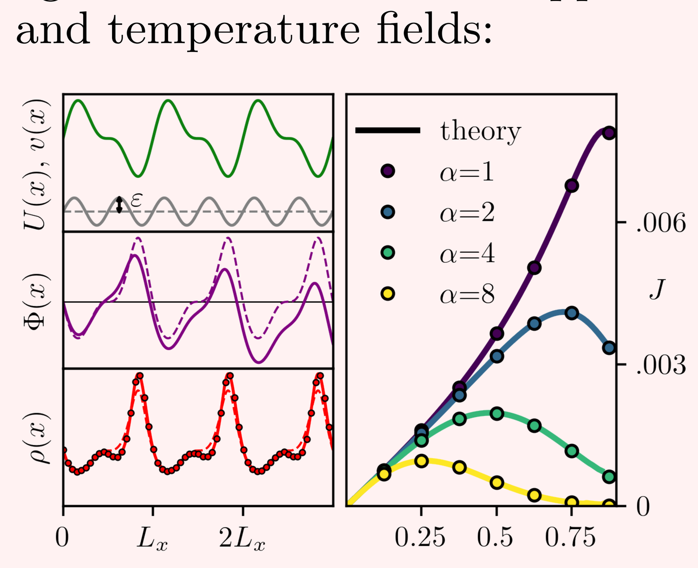
$$\tilde{T} = \frac{v^2}{\alpha} - \frac{(V')^2}{\alpha}.$$

Let $\Phi(x) = \int_0^x du \frac{\tilde{F}(u)}{\tilde{T}(u)}$. Then the exact steady-state density is

$$\rho(x) = \frac{e^{-\Phi(x)}}{\tilde{T}(x)} \left[\tilde{T}(0)\rho(0) + J \int_0^x du \frac{e^{\Phi(u)}}{\tilde{\mu}(u)} \right]$$

and J is known.[4] In fact,

$$\boxed{\text{Nonzero ratchet current} \iff \Phi(L) \neq \Phi(0).}$$

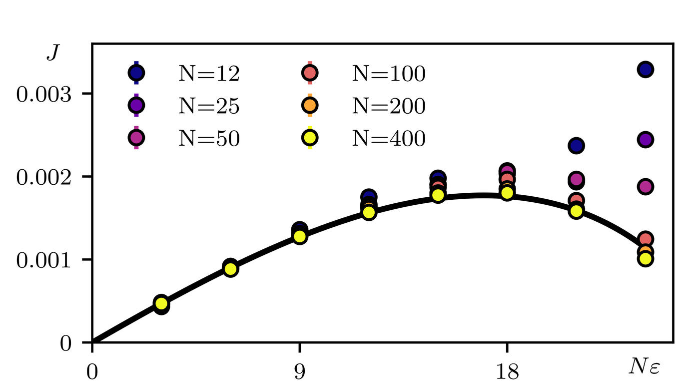


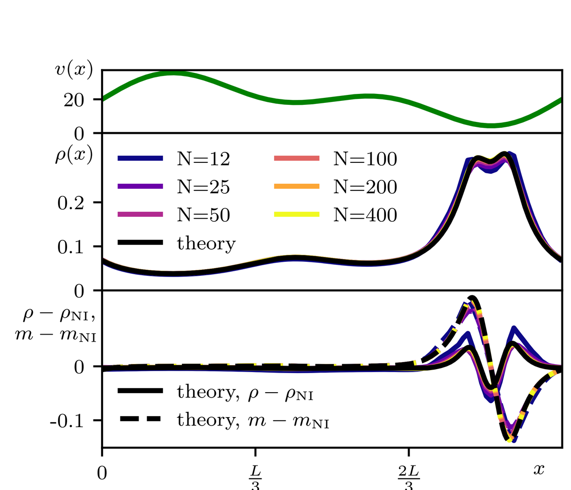
Particle simulations (points) + exact solution (lines). Non-interacting 1d RTPs in asymmetric activity $v(x)$ only demonstrate a ratchet current with an external potential $U(x)$.

Return to mean-field, interacting RTPs. To leading order in $U_{\rm int}$,

$$\Phi(L) = -\int_0^L dx \frac{(U_{\rm int} * \rho_{\rm NI})'(x)}{v(x)^2/\alpha} \propto -\int_0^L dx \int_0^L dx' \frac{U_{\rm int}'(x - x')}{v(x)^2 v(x')} \neq 0 \implies \textbf{interaction-induced current}$$

Solving Eq. 1-2 perturbatively to arbitrary order in $U_{\rm int}$ shows agreement with particle simulations in mean-field limit.





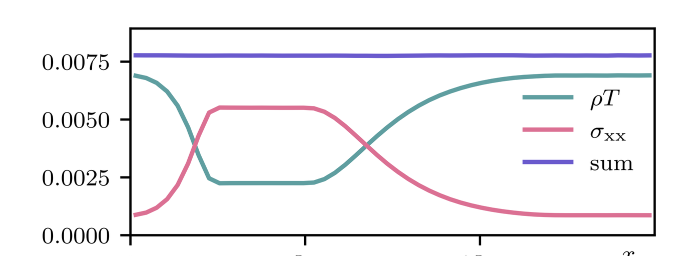
Left: Current induced by soft repulsive $U_{\rm int} \propto \varepsilon$. Simulations (points) converge to perturbative theory (line) as $N \to \infty$, i.e. mean-field. Right: steady-state ρ (middle) and deviation of ρ, m from non-interacting profiles (bottom).

No interaction-induced current in PBPs

PBPs in $T(\mathbf{r})$ obey $\dot{\rho} = -\nabla \cdot \mathbf{J}$, with

$$\mathbf{J} = -\nabla \cdot \boldsymbol{\sigma}_{\rm tot}, \qquad \boldsymbol{\sigma}_{\rm tot}(\mathbf{r}) = \boldsymbol{\sigma}_{\rm IK}(\mathbf{r}) + \rho_p(\mathbf{r})T(\mathbf{r})\mathbb{I}_d \, .$$

Thus $\langle \mathbf{J} \rangle = \int d^d\mathbf{r} \mathbf{J}(\mathbf{r}) = -\int d^d\mathbf{r} \nabla \cdot \sigma_{\rm tot} = 0$.



Equation of State (EOS) for the pressure explains the lack of interaction-induced current in PBPs.

Momentum sources in active ratchets

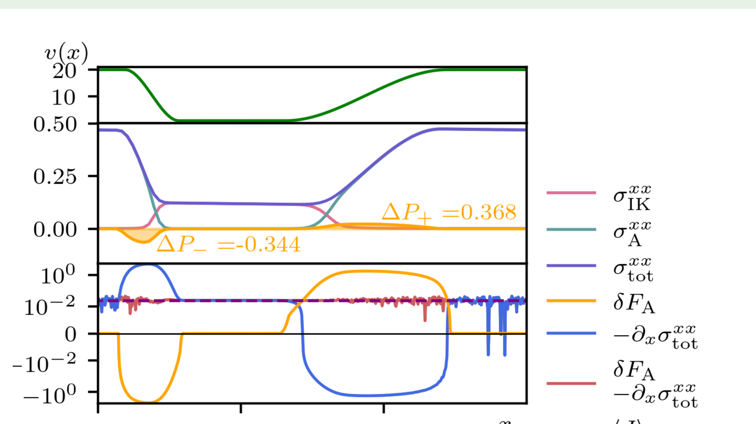
ABPs/RTPs in $v(\mathbf{r})$ obey $\dot{\rho} = -\nabla \cdot \mathbf{J}$, where

$$\mathbf{J} = -\nabla \cdot \boldsymbol{\sigma}_{\rm tot} + \delta\mathbf{F}_{\rm A}$$

$$\boldsymbol{\sigma}_{\rm tot} = \boldsymbol{\sigma}_{\rm IK} + \boldsymbol{\sigma}_{\rm A}, \quad \boldsymbol{\sigma}_{\rm A} = \tau v \langle \dot{\mathbf{r}} \otimes \mathbf{u} \rangle,$$

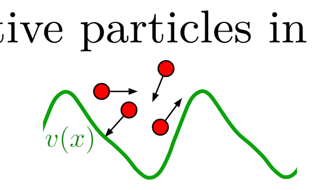
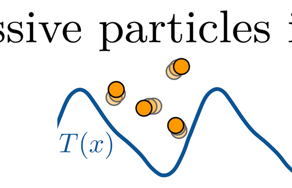
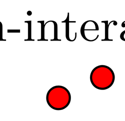
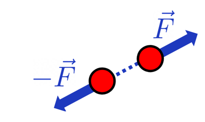
$$\delta\mathbf{F}_{\rm A} = \tau \langle \mathbf{u} \otimes \dot{\mathbf{r}} \rangle \cdot \nabla v.$$

Thus $\langle \mathbf{J} \rangle = \int d^d\mathbf{r} \mathbf{J}(\mathbf{r}) = \int d^d\mathbf{r} \, \delta\mathbf{F}_{\rm A} \neq 0$.



No EOS for the pressure $\implies$ **interaction-induced currents in RTPs/ABPs**.

Conclusion

Ratchet Principle Exceptions	Active particles in $v(\mathbf{r})$ 	Passive particles in $T(\mathbf{r})$ 
Non-interacting 	TRS ✓ ~~EOS~~ $\implies$ $\mathbf{J} = 0$	TRS ✓ EOS ✓ $\implies$ $\mathbf{J} = 0$
Interacting 	~~TRS~~ ~~EOS~~ $\implies$ $\mathbf{J} \neq 0$ **Interaction-induced current**	~~TRS~~ EOS ✓ $\implies$ $\mathbf{J} = 0$ **No interaction-induced current**

Non-equilibrium systems can become "exceptions" to the ratchet principle by an emergent spatial **Time-Reversal Symmetry**. They may remain protected from a ratchet current, despite interaction-induced TRS violation, by an **Equation of State (EOS)**.

References

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[3] M. E. Cates and J. Tailleur. *When are active Brownian particles and run-and-tumble particles equivalent?...* Europhys. Letters (2013)
[4] N. G. van Kampen. *Relative Stability in Nonuniform Temperature.* IBM J. Res. Dev. (1988)
[5] J. Stenhammar et al. *Light-induced self-assembly of active rectification devices.* Sci. Adv. (2016)

Acknowledgements

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