Continuous Random Variables

Common continuous random variables

A) Uniform random variables [Ross 5.3]

We say X is uniform on the interval (a, b), denoted $X \sim U(a, b)$, if

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{else} \end{cases}$$

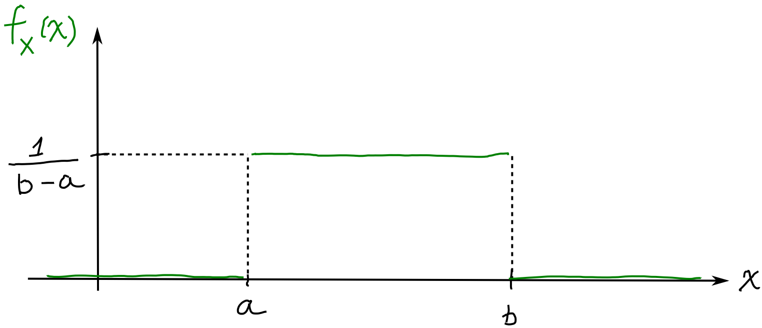
So, $$F_X(x) = \begin{cases} 0 & x \leq a \\ \frac{x}{b-a} - \frac{a}{b-a} & a \leq x \leq b \\ 1 & b \leq x \end{cases}$$

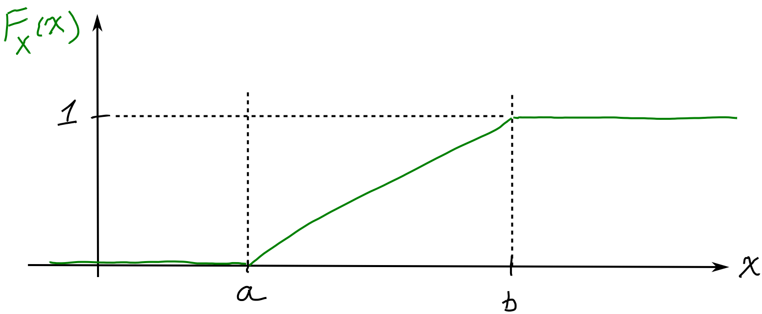
Note: If X has units of kg, then a and b have units of kg, and $1/(b - a)$ has units kg^{-1}.

Example 17.1: Buses arrive at a stop at 7:00, 7:15 and 7:30. If a person arrives between 7:00 and 7:30 uniformly, what is probability that they wait less than 5 minutes?

Solution: Let X = # of minutes past 7:00 that person arrives. Then $X \sim U(0, 30)$.

$$\begin{aligned} P[\text{wait less than 5 min}] &= P[\{10 < X < 15\} \cup \{25 < X < 30\}] \\ &= \int_{10}^{15} \frac{1}{30} dx + \int_{25}^{30} \frac{1}{30} dx \\ &= 1/3 \end{aligned}$$





Example 17.2: Let $X \sim U(a,b)$. Find $E[X]$ and $Var[X]$.

Solution:

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{else} \end{cases}$$

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

$$
\begin{aligned}
&= \int_a^b \frac{x}{b-a} dx \\
&= \frac{1}{2} \frac{b^2 - a^2}{b-a} \\
&= \frac{a+b}{2}
\end{aligned}
$$

$$
\begin{aligned}
Var[X] &= E[X^2] - (E[X])^2 \\
&= \int_{-\infty}^{\infty} x^2 f_X(x) dx - (E[X])^2 \\
&= \int_a^b \frac{x^2}{b-a} dx - \left(\frac{a+b}{2}\right)^2 \\
&= \frac{1}{3} \frac{b^3 - a^3}{b-a} - \left(\frac{a+b}{2}\right)^2 \\
&= \frac{1}{3}(b^2 + ab + a^2) - \left(\frac{a+b}{2}\right)^2 \\
&= \frac{1}{12}(b-a)^2
\end{aligned}
$$

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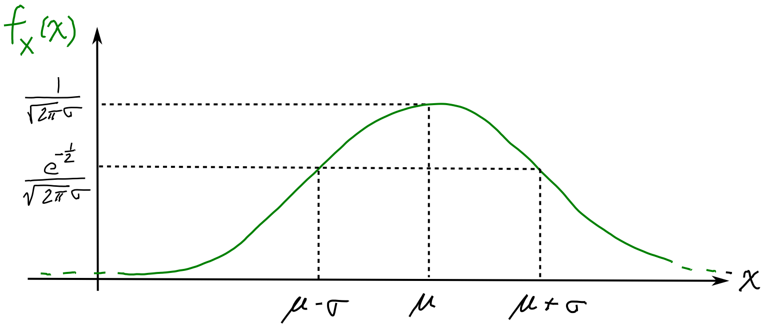
2) Normal (Gaussian) random variables [Ross 5.4]

Definition 17.1: X is normal (or Gaussian) with parameters μ and σ^2 if

$$
f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{17.1}
$$

This is denoted $X \sim \mathcal{N}(\mu, \sigma^2)$.



To verify that $f_X(x)$ has unit area, see Notes #21.

Note: If X has units of kg, then μ has units of kg and σ^2 has units of kg^2.

Proposition 17.1 *If* $X \sim \mathcal{N}(\mu, \sigma^2)$*, then* $Y = aX + b$ *is* $\mathcal{N}(a\mu + b, a^2\sigma^2)$

Why? [Assume $a > 0$; $a < 0$ is similar]

$$\begin{aligned} F_Y(u) &= P[Y \leq u] \\ &= P[aX + b \leq u] \\ &= P[X \leq (u-b)/a] \\ &= F_X\left(\frac{u-b}{a}\right) \end{aligned}$$

Then

$$
\begin{aligned}
f_Y(u) &= \frac{d}{du} F_Y(u) \\
&= \frac{d}{du} F_X\left(\frac{u-b}{a}\right) \\
&= f_X\left(\frac{u-b}{a}\right) \times \frac{1}{a} \\
&= \frac{1}{\sqrt{2\pi} a\sigma} \exp\left(-\frac{(\frac{u-b}{a} - \mu)^2}{2\sigma^2}\right) \\
&= \frac{1}{\sqrt{2\pi} a\sigma} \exp\left(-\frac{(u-b-a\mu)^2}{2(a\sigma)^2}\right)
\end{aligned}
$$

So $Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$.