Continuous Random Variables

Appendix [Ross S5.4]

The result below shows that a Gaussian pdf has unit area under its curve.

Proposition 21.1

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = 1$$

Why?

Let $u = (x-\mu)/\sigma$ and $du = dx/\sigma$. So

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{u^2}{2}}\,du$$

So we just need to show that

$$I = \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}}\,du = \sqrt{2\pi}$$

We will show that $I^2 = 2\pi$:

$$
\begin{aligned}
I^2 &= \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}} dx dy \\
&= \int_{0}^{2\pi} \int_{0}^{\infty} e^{-\frac{r^2}{2}} r dr d\theta \\
&= \int_{0}^{2\pi} \left[-e^{-\frac{r^2}{2}} \right]_0^{\infty} d\theta \\
&= \int_{0}^{2\pi} 1 d\theta \\
&= 2\pi
\end{aligned}
$$

where we switched from Cartesian (x, y) coordinates to polar (r, θ) coordinates.