

Properties of Expectations

Covariance, Variance of Sums [Ross S7.4]

Proposition 31.1 *If X and Y are independent, then for any functions $g(x)$ and $h(y)$:*

- i) $E[g(X)h(Y)] = E[g(X)]E[h(Y)]$
- ii) $g(X)$ and $h(Y)$ are independent.

Why?

$$\begin{aligned} i) \quad E[g(X)h(Y)] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x)h(y)f_{XY}(x,y)dxdy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x)h(y)f_X(x)f_Y(y)dxdy \\ &= \int_{-\infty}^{\infty} g(x)f_X(x)dx \int_{-\infty}^{\infty} h(y)f_Y(y)dy \\ &= E[h(Y)]E[g(X)] \end{aligned}$$

ii) Let $A = \{x \mid g(x) \leq a\}$ and $B = \{y \mid h(y) \leq b\}$. Then:

$$\begin{aligned} P[g(X) \leq a, h(Y) \leq b] &= P[X \in A, Y \in B] \\ &= P[X \in A] P[Y \in B] && \text{since } X \text{ and } Y \text{ are independent} \\ &= P[g(X) \leq a] P[h(Y) \leq b] \end{aligned}$$

For a single random variable X , its mean and variance give us some information about X .

For two random variables X and Y , **covariance** (and **correlation**) will give us information about the relationship between the pair X and Y .

Definition 31.1: The **covariance** between X and Y , denoted $Cov[X, Y]$, is defined as

$$Cov[X, Y] = E [(X - E[X])(Y - E[Y])]$$

Just as $Var[X] = E[X^2] - (E[X])^2$, we also have:

$$\begin{aligned} Cov[X, Y] &= E [(X - E[X])(Y - E[Y])] \\ &= E [XY - E[X]Y - E[Y]X + E[X]E[Y]] \\ &= E[XY] + E[-E[X]Y] + E[-E[Y]X] + E[E[X]E[Y]] \\ &= E[XY] - E[X]E[Y] - E[Y]E[X] + E[X]E[Y] \\ &= E[XY] - E[X]E[Y] \end{aligned}$$

Note: If X and Y are independent, then $E[XY] = E[X]E[Y]$
so $Cov[X, Y] = 0$.

Example 31.1: Does $Cov[X, Y] = 0$ imply X and Y are independent?

Solution:

Proposition 31.2

$$i) Cov[X, Y] = Cov[Y, X]$$

$$ii) Cov[X, X] = Var[X]$$

$$iii) Cov[aX, Y] = aCov[X, Y] = Cov[X, aY]$$

$$iv) Cov \left[\sum_{i=1}^n X_i, \sum_{j=1}^m Y_j \right] = \sum_{i=1}^n \sum_{j=1}^m Cov[X_i, Y_j]$$

Why? i), ii) and iii) follow from

$$Cov[X, Y] = E[XY] - E[X]E[Y]$$

For iv), let

$$U = \sum_{i=1}^n X_i \qquad V = \sum_{j=1}^m Y_j$$

$$E[X_i] = \mu_i \qquad E[Y_j] = \nu_j$$

Then:

$$E[U] = \sum_{i=1}^n \mu_i \qquad E[V] = \sum_{j=1}^m \nu_j$$

$$\text{So, } Cov[U, V] = E \left[\left(\sum_{i=1}^n X_i - \sum_{i=1}^n \mu_i \right) \left(\sum_{j=1}^m Y_j - \sum_{j=1}^m \nu_j \right) \right]$$

$$\begin{aligned}
&= E \left[\sum_{i=1}^n (X_i - \mu_i) \sum_{j=1}^m (Y_j - \nu_j) \right] \\
&= E \left[\sum_{i=1}^n \sum_{j=1}^m (X_i - \mu_i)(Y_j - \nu_j) \right] \\
&= \sum_{i=1}^n \sum_{j=1}^m E [(X_i - \mu_i)(Y_j - \nu_j)] \\
&= \sum_{i=1}^n \sum_{j=1}^m Cov[X_i, Y_j]
\end{aligned}$$

Now, say we pick $n = m$ and $Y_j = X_j$ in part iv) of Proposition 31.2:

$$\begin{aligned}
Var \left[\sum_{i=1}^n X_i \right] &= Cov \left[\sum_{i=1}^n X_i, \sum_{j=1}^n X_j \right] \\
&= \sum_{i=1}^n \sum_{j=1}^n Cov[X_i, X_j] \\
&= \sum_{i=1}^n \left(Cov[X_i, X_i] + \sum_{\substack{j=1 \\ j \neq i}}^n Cov[X_i, X_j] \right) \\
&= \sum_{i=1}^n Var[X_i] + \sum_{\substack{i,j \\ j \neq i}} Cov[X_i, X_j] \\
&= \sum_{i=1}^n Var[X_i] + 2 \sum_{\substack{i,j \\ i < j}} Cov[X_i, X_j]
\end{aligned}$$

In the special case that each pair X_i, X_j are independent when $i \neq j$, then:

$$\text{Var} \left[\sum_{i=1}^n X_i \right] = \sum_{i=1}^n \text{Var}[X_i]$$

Example 31.2: Recall (from Example 30.3) that $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$ is called the **sample mean**. Let

$$S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$$

be the **sample variance**.

Let X_1, \dots, X_n be iid with (common) mean μ and variance σ^2 .

Find a) $\text{Var}[\bar{X}]$ and b) $E[S^2]$. [b) is hard]

Solution:

Example 31.3: Compute the variance of $X \sim \text{Binomial}(n, p)$.

Solution: