Properties of Expectations

Conditional Variance [Ross S7.5.4]

So far, we have defined expectation, conditional expectation and variance. We also define the **conditional variance**

$$Var[X|Y] = E[X^2|Y] - (\ E[X|Y]\ )^2$$

So,

$$\begin{aligned} E[\ Var[X|Y]\ ] &= E\left[\ E[X^2|Y]\ \right] - E\left[\ (\ E[X|Y]\ )^2\ \right] \\ &= E[X^2] - E\left[\ (\ E[X|Y]\ )^2\ \right] \end{aligned} \tag{35.1}$$

Also, $E[X|Y] = g(Y)$ for some function g, so

$$\begin{aligned} Var[\ g(Y)\ ] &= E\left[\ (\ g(Y)\ )^2\right] - (E[\ g(Y)\ ])^2 \\ Var[\ E[X|Y]\ ] &= E\left[\ (\ E[X|Y]\ )^2\right] - (E[\ E[X|Y]\ ])^2 \\ &= E\left[\ (\ E[X|Y]\ )^2\right] - (E[X])^2 \end{aligned} \tag{35.2}$$

Adding (35.1) to (35.2), we get

Proposition 35.1 *Conditional Variance Formula:*

$$Var[X] = E[\ Var[X|Y]\ ] + Var[\ E[X|Y]\ ]$$

Example 35.1: Let $X_1, X_2, \cdots$ be iid and independent of the non-negative integer random variable N. Let's compute

$$Var\left[\sum_{i=1}^{N} X_i\right]$$

by conditioning on N.

Using $Var[X] = E[\ Var[X|Y]\ ] + Var[\ E[X|Y]\ ]$ with

$$\begin{aligned} X &= \sum_{i=1}^{N} X_i \\ Y &= N \end{aligned}$$

then
$$\begin{aligned} E\left[\sum_{i=1}^{N} X_i \middle| N = n\right] &= E\left[\sum_{i=1}^{n} X_i\right] && \text{[Since } N \text{ is ind. of the } X_i\text{]} \\ &= nE[X_1] && \text{[Since } X_i \text{ are iid]} \\ \Rightarrow E\left[\sum_{i=1}^{N} X_i \middle| N\right] &= NE[X_1] \end{aligned}$$

$$\begin{aligned} Var\left[\sum_{i=1}^{N} X_i \middle| N = n\right] &= Var\left[\sum_{i=1}^{n} X_i\right] && \text{[Since } N \text{ is ind. of the } X_i\text{]} \\ &= nVar[X_1] && \text{[Since } X_i \text{ are iid]} \\ \Rightarrow Var\left[\sum_{i=1}^{N} X_i \middle| N\right] &= NVar[X_1] \end{aligned}$$

By the conditional variance formula:

$$\begin{aligned} Var\left[\sum_{i=1}^{N} X_i\right] &= E\left[\ NVar[X_1]\ \right] + Var\left[\ NE[X_1]\ \right] \\ &= E[N]Var[X_1] + (E[X_1])^2 Var[N] \end{aligned}$$