Jointly Distributed Random Variables

The bivariate normal distribution [Ross S6.5]

Two random variables X and Y are **jointly Gaussian** (normal) or **bivariate Gaussian** (normal) with parameters:

$$\mu_X, \mu_Y, \sigma_X > 0, \sigma_Y > 0, -1 < \rho < 1$$

when

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \tag{28.1}$$
$$\times \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - 2\rho\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}\right]\right\}$$

It is customary to denote

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \boldsymbol{\mu} = \begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix} \qquad \Sigma = \begin{bmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{bmatrix}$$

then

$$f_{XY}(x,y) = \frac{1}{2\pi|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T\Sigma^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right\}$$

$\boldsymbol{\mu}$ is called the **mean vector** of (X,Y).

Σ is called the **covariance matrix** of (X,Y).

We say that the pair $(X,Y) \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$.

Note: Σ is symmetric and +ve definite.

Marginal Distributions

To find the marginal

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy$$

after completing the square for y in (28.1) + lots of algebra:

$$f_X(x) = Ce^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}}$$

where C doesn't depend on x.

So, X is normal with mean μ_X and variance σ_X^2!

Likewise

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_Y}e^{-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}}. \tag{28.2}$$

Conditional distribution

To get the conditional $f_{X|Y}(x|y)$, combining (28.1) + (28.2) + lots of algebra:

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f_{XY}(x,y)}{f_Y(y)} \\ &= K\exp\left\{\frac{-1}{2\sigma_X^2(1-\rho^2)}\left[x - \left(\mu_X + \rho\frac{\sigma_X}{\sigma_Y}(y-\mu_Y)\right)\right]^2\right\} \end{aligned}$$

and K is a constant that doesn't depend on x or y.

So we recognize $f_{X|Y}(x|y)$ is the pdf of a Gaussian when X has mean

$$\mu_X + \rho\frac{\sigma_X}{\sigma_Y}(y-\mu_Y)$$

and variance $\sigma_X^2(1-\rho^2)$.

Note that we have

$$f_{XY}(x,y) = f_X(x)f_Y(y) \Leftrightarrow f_{X|Y}(x|y) = f_X(x)$$

and the latter happens when $\rho = 0$.

So for bivariate normal X and Y: X and Y are independent $\Leftrightarrow \rho = 0$.

Remark: ρ is called the **correlation coefficient** between X and Y.