Properties of Expectations

Conditional Expectation [Ross S7.5]

Recall that for 2 discrete random variables X and Y with $P[Y=y]>0$:

$$p_{X|Y}(x|y) = P[X=x|Y=y] = \frac{p_{XY}(x,y)}{p_Y(y)}$$

We can define the **conditional expectation**:

$$E[X|Y=y] = \sum_x x p_{X|Y}(x|y)$$

Similarly, if X and Y are continuous, then provided $f_Y(y)>0$:

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)},$$

and

$$E[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)dx$$

Example 33.1: Say X and Y have joint pdf [see Example 27.3]

$$f_{XY}(x,y) = \begin{cases} \frac{e^{-x/y}e^{-y}}{y} & 0<x<\infty,\ 0<y<\infty \\ 0 & \text{else} \end{cases}$$

Find $E[X|Y=y]$.

Solution: From Example 27.3, for $x>0, y>0$

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{1}{y}e^{-x/y}$$

So, $\quad E[X|Y=y] = \int_0^{\infty} \frac{x}{y}e^{-x/y}dx = y$

Note: Conditional expectations satisfy all the properties of ordinary expectation, e.g.,

$$E[g(X) \mid Y=y] = \begin{cases} \sum_x g(x)p_{X|Y}(x|y) & \text{discrete case} \\ \int_{-\infty}^{\infty} g(x)f_{X|Y}(x|y)dx & \text{continuous case} \end{cases}$$

and

$$E\left[\sum_{i=1}^n X_i \,\middle|\, Y=y\right] = \sum_{i=1}^n E[X_i|Y=y]$$

Computing Expectations by Conditioning

$E[X|Y=y]$ is a function of y, say $g(y)$.
Let $E[X|Y]$ be $g(Y)$, i.e., in Example 33.1:

$$E[X|Y=y] = y$$

So, $\quad E[X|Y] = Y$

Proposition 33.1 $E[X] = E[\,E[X|Y]\,]$, *i.e.,*

$$E[X] = \sum_y E[X|Y=y]p_Y(y) \qquad \textit{[discrete case]}$$

$$E[X] = \int_{-\infty}^{\infty} E[X|Y=y]f_Y(y)dy \qquad \textit{[continuous case]}$$

Why? [Continuous Case]

$$\begin{aligned}\int_{-\infty}^{\infty} E[X|Y=y]f_Y(y)dy &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x f_{X|Y}(x|y)dx\right]f_Y(y)dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{X|Y}(x|y)f_Y(y)\,dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{XY}(x,y)\,dxdy \\ &= E[X]\end{aligned}$$

Example 33.2: You are in a room with 3 doors.

The 1st door exits the building after 3 min of travel.

The 2nd door returns to where you are after 5 min.

The 3rd door returns to where you are after 7 min.

Each time you enter the room, you are equally likely to pick each of the 3 doors. What is the expected time until you leave the building?

Solution: Let X = time to leave building, and Y = door choice.

$$\begin{aligned}E[X] &= E[X|Y=1]P[Y=1] \\ &\quad + E[X|Y=2]P[Y=2] \\ &\quad + E[X|Y=3]P[Y=3] \\ &= \frac{1}{3}(E[X|Y=1]+E[X|Y=2]+E[X|Y=3])\end{aligned}$$

Also, $\quad E[X|Y=1] = 3$
$\quad E[X|Y=2] = 5+E[X]$
$\quad E[X|Y=3] = 7+E[X]$

Combining, $\quad E[X] = \frac{1}{3}(3+5+E[X]+7+E[X])$

$$\Rightarrow E[X] = 15$$

Example 33.3: The number of people that enter a store in a day is random with mean 50.

The amount spent by each person is iid with mean \$8, and independent of the number of people that enter.

What is the expected amount spent in the store in one day? [Hard]

Solution:

Let N = # customers that enter store in one day.

Let X_i = amount spent by ith customer.

Total amount spent is $Y = \sum_{i=1}^N X_i$.

$$E\left[\sum_{i=1}^N X_i\right] = E\left[E\left[\sum_{i=1}^N X_i \middle| N\right]\right]$$

and $\quad E\left[\sum_{i=1}^N X_i \middle| N=n\right] = E\left[\sum_{i=1}^n X_i \middle| N=n\right] = E\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n E[X_i] = nE[X_1]$

so $\quad E\left[\sum_{i=1}^N X_i \middle| N\right] = NE[X_1]$

Therefore $\quad E\left[\sum_{i=1}^N X_i\right] = E[\,NE[X_1]\,] = E[N]E[X_1] = 50\times 8$