Signals and Systems

Lecture Notes

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Chapter 1

Introduction to Signals and Systems

1.2 Motivations and Applications

The concepts of "Signals and Systems" arise in many areas of science and technology such as communication, control systems, digital signal processing, and many more.

- **Communication Systems**

  A simplified block diagram of a communication system (Radio, TV, mobiles phone, etc):

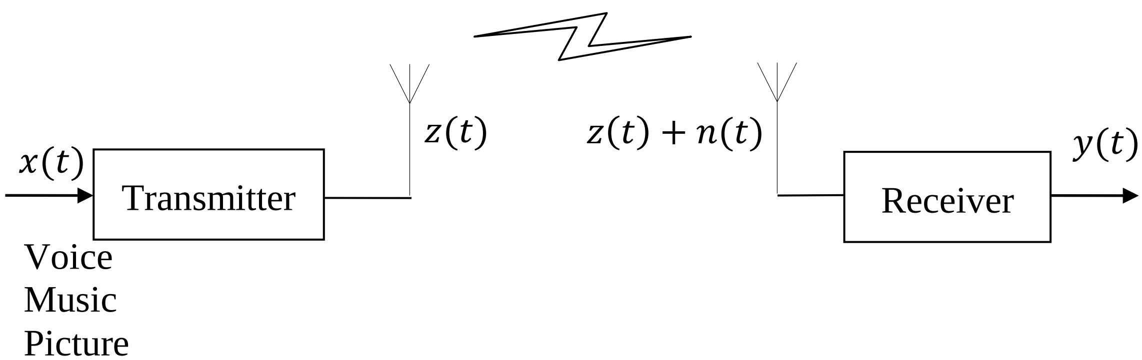

Figure 1

Questions:

– How does the "transmitter system" modify the signal $x(t)$ to get a transmittable signal $z(t)$?

– How does the "receiver system" filter out the noise signal $n(t)$?

– How does the "receiver system" reconstruct the signal $y(t)$ that is close to the original signal $x(t)$?

- **Control Systems**

  The inverted pendulum:

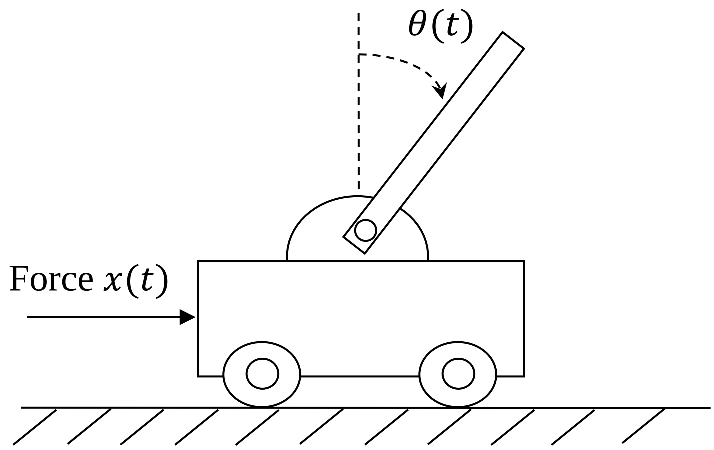

Figure 2

Question:

How do we design a "controller" system that adjusts the force signal $x(t)$ based on measurements of the angle $\theta(t)$ to keep the inverted pendulum close to a vertical position?

- **Digital Signal Processing**
  - How do we sample and digitize an analog signal (voice, music, picture, etc.) to be able to process it using a computer or a digital system?
  - In a CD or DVD player, how do we reconstruct a music or picture signal from its digital samples?
  - How do we design a "digital filter" to process a digital signal based on some given specifications?

1.3 Signals

A "*signal*" is a set of data or information that represents *physical phenomena*.

Examples:

- A voice or TV signal.
- Monthly sales of a company.
- Velocity or position of a car.

Mathematically: "A *signal* is a real or complex valued function of one or more variables."

In this course, we restrict our attention to *one-dimensional signals as functions of time.*

Definition: A continuous- time (CT) signal $x(t)$ is a function defined for all values of time t $(-\infty, \infty)$

Example:

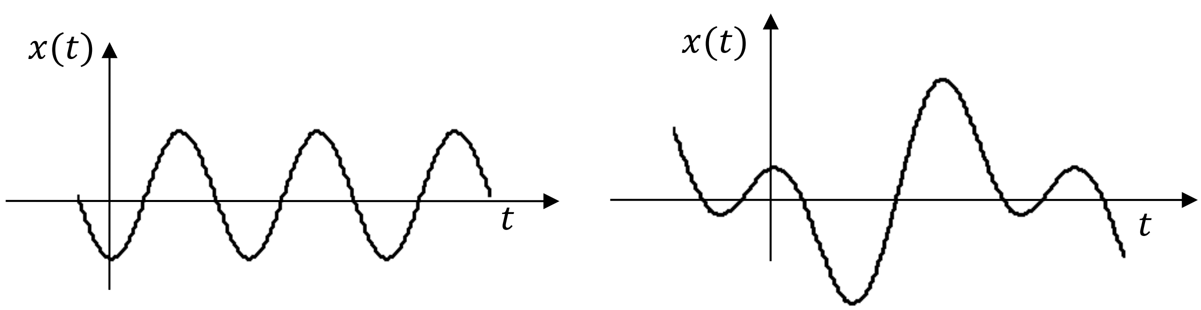

Figure 3

Definition: A discrete-time (DT) signal $x(t_n)$ is a function at discrete instances of time $\{t_n\}$.

Generally, DT signals are derived from CT signals by *sampling* at uniform intervals, i.e.

$$t_n = nT \ , \quad T = \text{ sampling period/interval}$$

then

$$x(t_n) = x(nT) = x[n] \ , \quad n = 0, \pm 1, \pm 2, \dots \dots$$

Example:

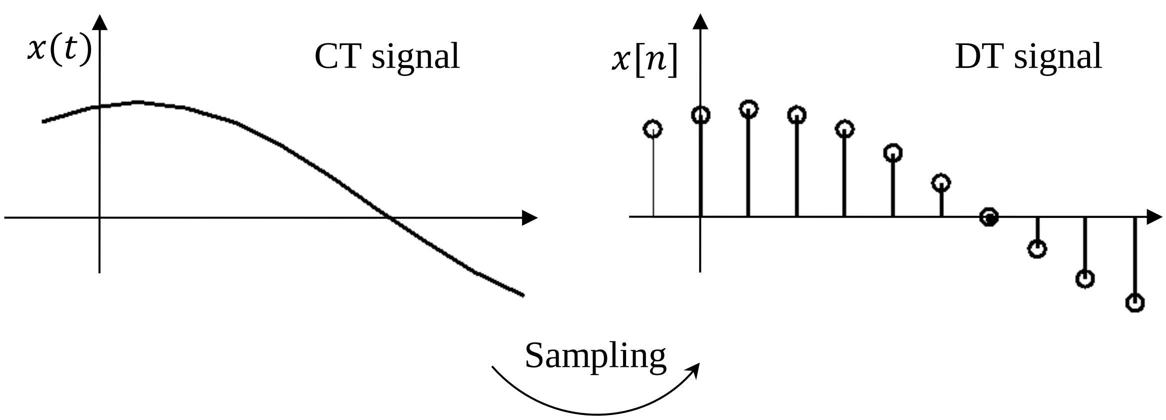

Figure 4

Often and in this course, the notation $x[n]$ is used instead of $x(t_n)$.

1.2 Signal Energy and Power

These are useful measures of the size of a signal.

- The *energy* E of a signal $x(t)$ or $x[n]$ is defined by:

  CT: $E = \int_{-\infty}^{\infty} |x(t)|^2 dt$

  DT: $E = \sum_{n=-\infty}^{\infty} |x[n]|^2$

  Definition: A signal is called an *energy signal* if $0 < E < \infty$.

- The *power* P of a signal $x(t)$ or $x[n]$ is defined by:

  CT: $P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$

  DT: $P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$

  Definition: A signal is called a *power signal* if $0 < P < \infty$.

Remarks:

- A signal can be an *energy signal* (when E is finite), or a *power signal* (when P is finite), or neither (both E and P are not finite). A signal can not be an energy signal and a power signal at the same time.
- Above definitions of P and E are not the actual energy or power of a signal in the conventional sense. They are just measures of the energy or power capability or size of a signal. They are useful in comparing the performance of systems.
- These definitions of E and P have *no measurement units* since $x(t)$ can be any physical quantity, such as a voltage, a current, an angle, or a force.

Example:

$$x(t) = \begin{cases} 0, & t < 0 \\ e^{-t}, & t \geq 0 \end{cases}$$

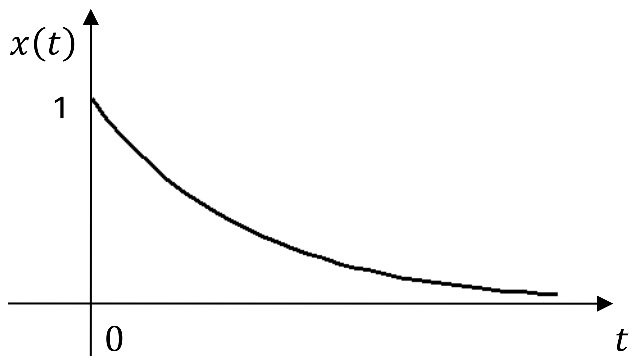

Figure 5

Energy:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{0}^{\infty} e^{-2t} dt = \left.\frac{e^{-2t}}{-2}\right|_0^{\infty} = \frac{1}{-2}(0-1) = \frac{1}{2}$$

Since $0 < E < \infty \quad \Rightarrow \quad x(t)$ is an *energy signal*.

Power:

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt = \lim_{T\to\infty} \frac{1}{2T} \int_{0}^{T} e^{-2t} dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \cdot \left.\frac{e^{-2t}}{-2}\right|_0^{T} = \lim_{T\to\infty} \frac{1}{2T} \cdot \frac{1}{-2}(e^{-2T} - 1)$$

$$= \lim_{T\to\infty} \frac{(1 - e^{-2T})}{4T} = \frac{1-0}{\infty} \ \to \ 0$$

Since $P = 0 \quad \Rightarrow \quad x(t)$ is not a power signal, which is an expected conclusion since the energy E is finite.

Example:

$$x(t) = e^{-t}, \qquad -\infty < t < \infty$$

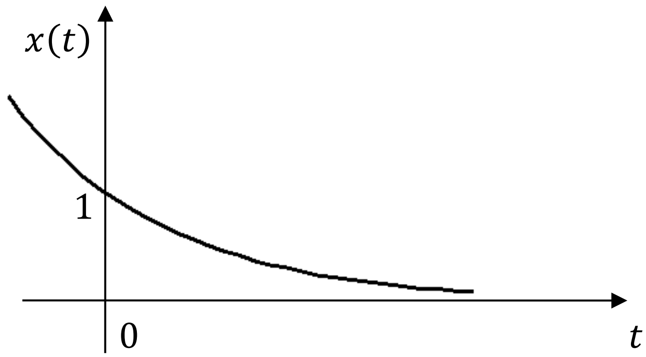

Figure 6

Energy:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} e^{-2t} dt = \left.\frac{e^{-2t}}{-2}\right|_{-\infty}^{\infty} = \frac{1}{-2}(0-\infty) = \infty$$

Power:

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-2t} dt = \lim_{T\to\infty} \frac{1}{2T} \cdot \left.\frac{e^{-2t}}{-2}\right|_{-T}^{T}$$

$$= \lim_{T\to\infty} \frac{e^{2T} - e^{-2T}}{4T} = \lim_{T\to\infty} \frac{2e^{2T} + 2e^{-2T}}{4} = \infty$$

$\therefore$ $x(t)$ is neither an energy nor a power signal.

Remark: A signal $\boldsymbol{x(t)}$ can be an energy signal if $|\boldsymbol{x(t)}| \to \mathbf{0}$ as $|\boldsymbol{t}| \to \infty$ (this is a necessary but not sufficient condition). For example, the *unit step* function $\boldsymbol{u(t)}$, defined later, is not an energy signal since $\boldsymbol{u(t)} \to \mathbf{1}$ as $\boldsymbol{t} \to +\infty$.

Example: (Discrete unit step function)

$$x[n] = u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$

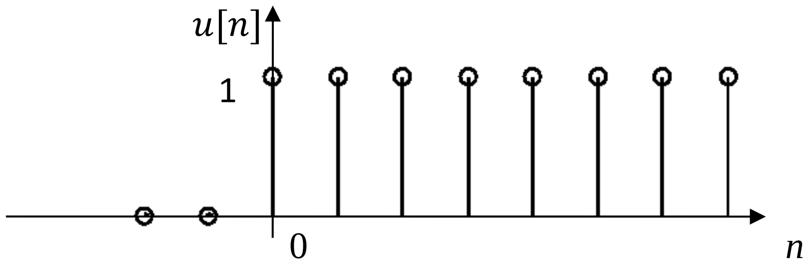

Figure 7

This signal can not be an energy signal since $u[n] \to 1$ when $n \to +\infty$.

Power:

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} |1|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1}(N+1) = \lim_{N\to\infty} \frac{1+\frac{1}{N}}{2+\frac{1}{N}} = \frac{1}{2}$$

$\Rightarrow$ $x[n]$ is a power signal.

Example: (Discrete)

$$x[n] = \begin{cases} \left(\frac{1}{2}\right)^n, & n = 0, 1, 2, 3, \dots, 9 \\ 0, & \text{otherwise} \end{cases}$$

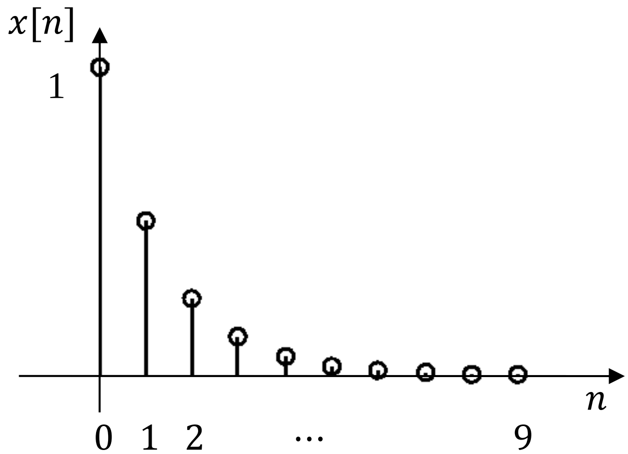

Figure 8

Energy:

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=0}^{9} \left(\frac{1}{2}\right)^{2n} = \sum_{n=0}^{9} \left(\frac{1}{4}\right)^n$$

Using the identity (in text):

$$\sum_{n=0}^{m} r^n = \frac{r^{m+1} - 1}{r - 1}, \quad r \neq 1$$

$$\Rightarrow \quad E = \frac{\left(\frac{1}{4}\right)^{9+1} - 1}{\left(\frac{1}{4}\right) - 1} \approx 1.33$$

$\Rightarrow$ $x[n]$ is an energy signal.

1.4 Some Useful Signal Transformations

Time shifting, scaling, inversion and a combination.

(a) Time Shifting

Delay or advance $x(\mathrm{t})$ by T seconds:

$$y(t) = \quad x(t-T)$$

Example:

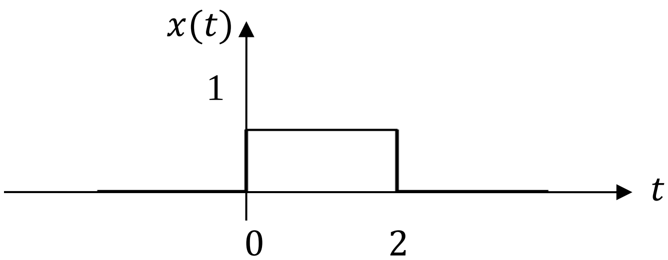

Figure 9

When $T > 0 \;\Rightarrow\;$ delay (shift right)

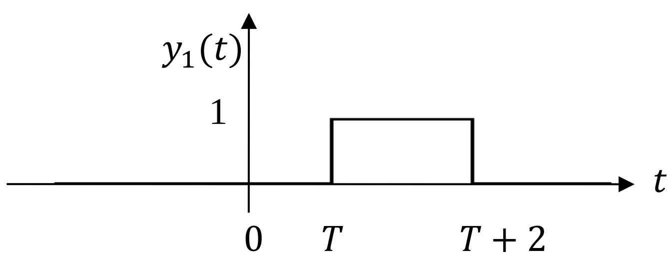

Figure 10

When $T < 0 \Rightarrow$ advance (shift left)

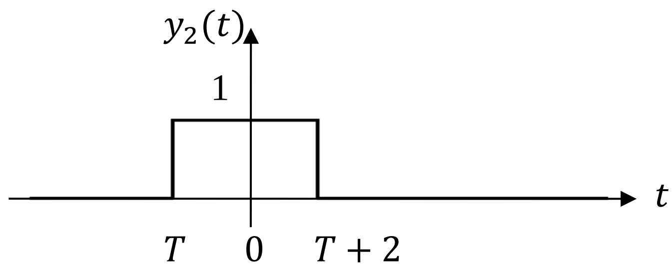

Figure 11

Note: $x(t+T)$ for $T > 0$ represents advance.

$x(t+T)$ for $T < 0$ represents delay.

(b) Time Scaling

Compress or stretch/expand $x(t)$ *in time* by a factor "a" to get $y(t)$:

$$y(t) = x(at) \ , \quad a > 0$$

Example:

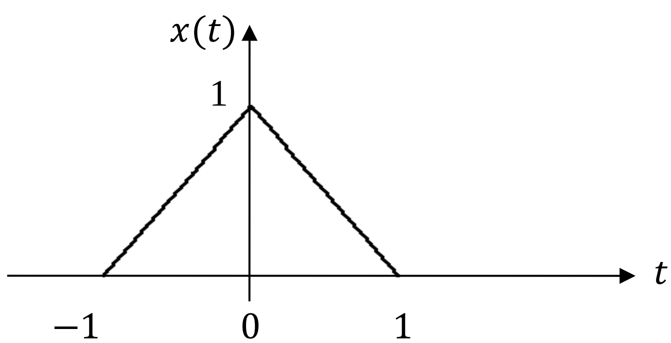

Figure 12

When $a > 1$ (Compression)

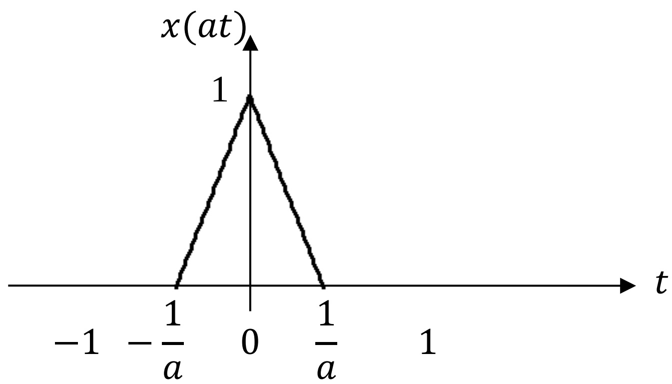

Figure 13

When $0 < a < 1$ (expansion/stretch)

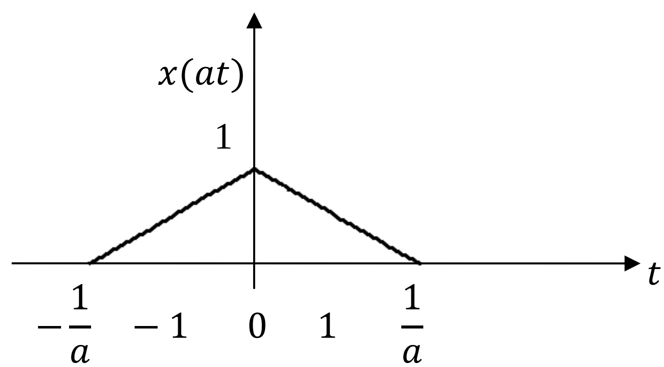

Figure 14

(c) Time Reversal (Reflection)

$$y(t) = x(-t)$$

Example:

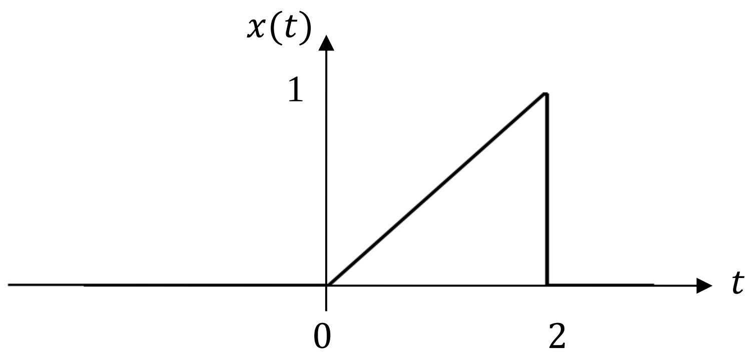

Figure 15

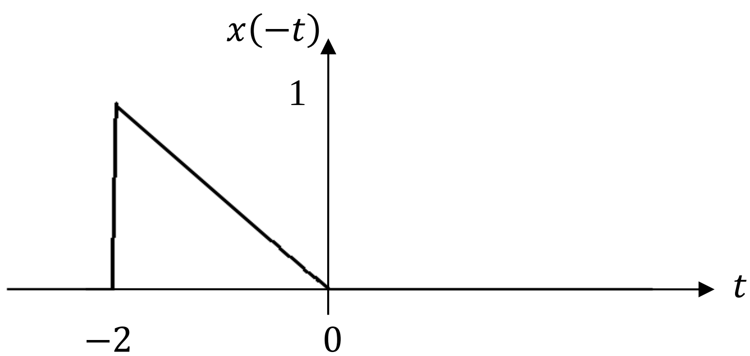

Figure 16

Remark: Time reflection can be considered as a *time-scaling* with $a = -1$.

(d) Combined Operations

Time shifting and scaling with possible reflection (when $a < 0$):

$$y(t) = x(at - T)$$

This can be done in two ways:

1) <u>Time shift</u> $x(t)$ to get $x(t-T)$, then <u>time scale</u> $x(t-T)$ to get $x(at-T)$.

2) <u>Time scale</u> $x(t)$ to get $x(at)$, then <u>time shift</u> $x(at)$ by $\frac{T}{a}$ to get $x\left[a(t-\frac{T}{a})\right] = x(at-T)$.

<u>Example:</u>

Given $x(t)$ in Figure 17, find $y(t) = x(2t+3)$.

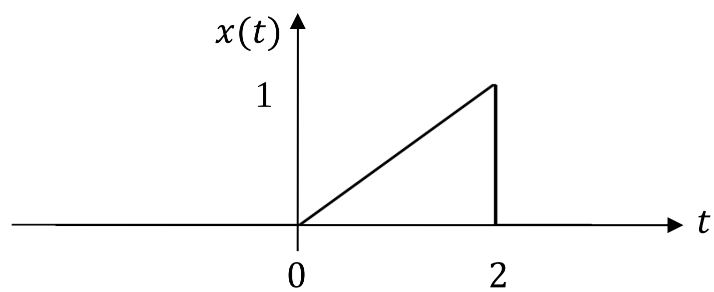

Figure 17

<u>Method 1</u>: First, *time shift* by -3:

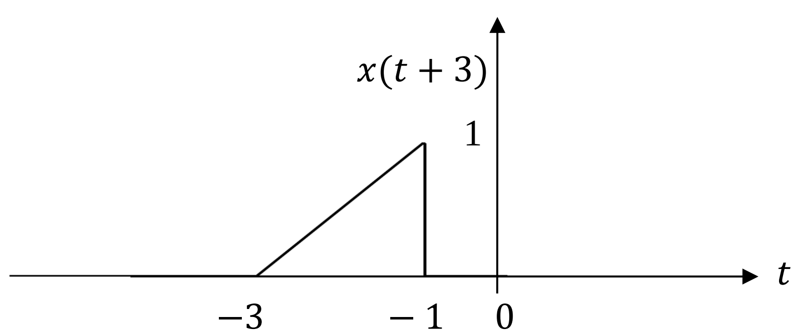

Figure 18

Second, *time scale* by 2:

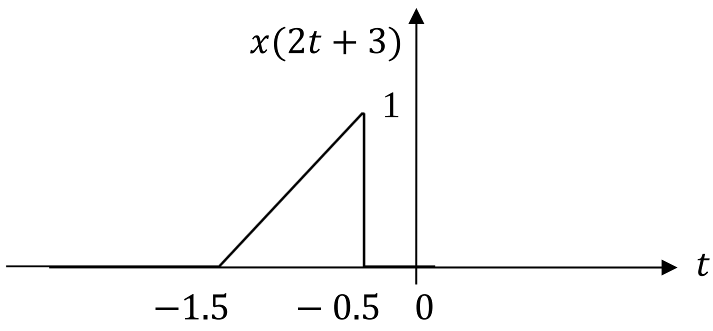

Figure 19

<u>*Method* 2</u>: First, *time scale* by 2:

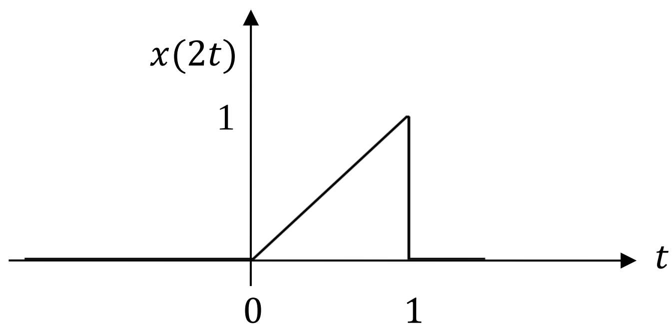

Figure 20

Second, *time shift* by $\frac{-3}{2} = -1.5$:

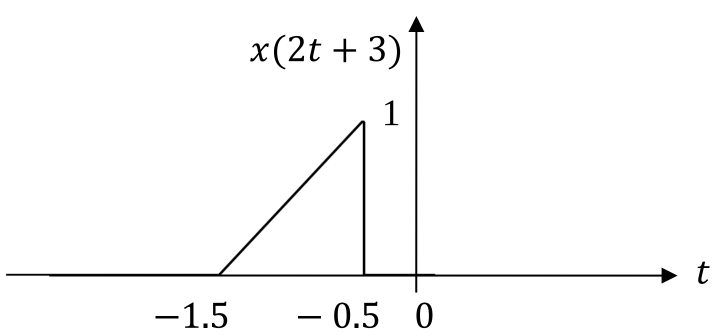

Figure 21

1.5 Some Elementary Signal Models

These models are useful for representing other signals and for the analysis of signals and systems.

(a) Unit Step Function

CT: $$\boldsymbol{u(t)} = \begin{cases} \mathbf{1}, & \boldsymbol{t} \geq \mathbf{0} \\ \mathbf{0}, & \boldsymbol{t} < \mathbf{0} \end{cases}$$

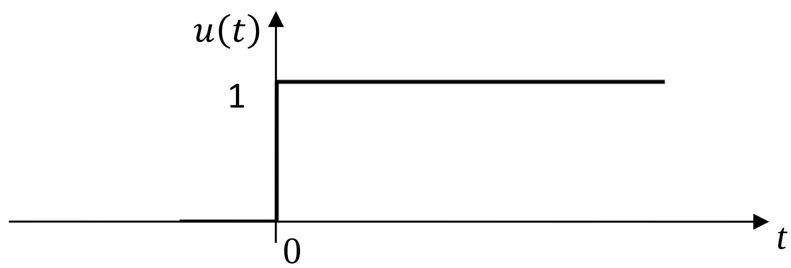

Figure 22

DT: $$\boldsymbol{u[n]} = \begin{cases} \mathbf{1}, & \boldsymbol{n} \geq \mathbf{0} \\ \mathbf{0}, & \boldsymbol{n} < \mathbf{0} \end{cases}$$

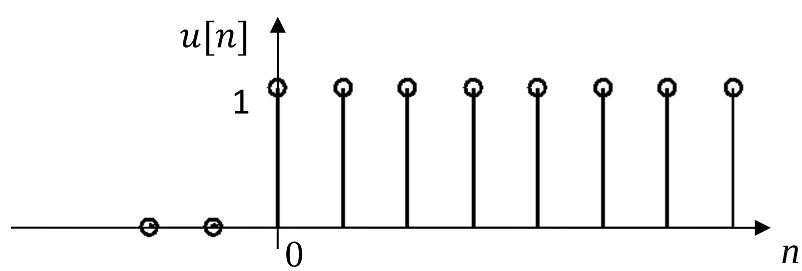

Figure 23

- **Representing Signals Using Unit Step:**

Example: $$x(t) = \begin{cases} 1, & 0 \le t \le 2 \\ 0, & \text{otherwise} \end{cases}$$

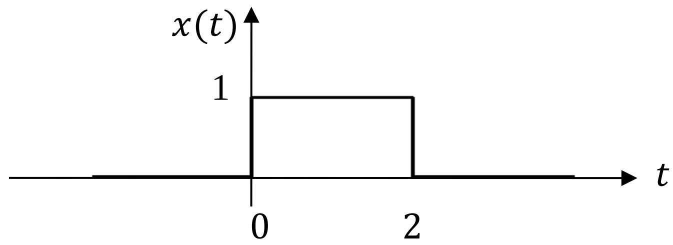

Figure 24

Closed mathematical form: $x(t) = u(t) - u(t-2)$

Example: $$x(t) = \begin{cases} 2t, & 0 \le t < 1 \\ -t+3, & 1 \le t < 3 \\ 0, & \text{otherwise} \end{cases}$$

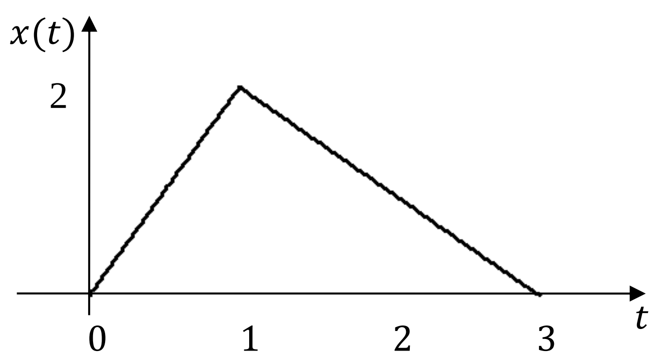

Figure 25

Closed mathematical form:

$$x(t) = 2t[u(t) - u(t-1)] + (-t+3)[u(t-1) - u(t-3)]$$

(b) Unit Impulse Function (Dirac Delta Function)

In continuous-time, this function is defined by two properties:

1) $\boldsymbol{\delta(t) = 0}, \qquad \boldsymbol{t \neq 0}$

2) $\int_{-\infty}^{\infty} \boldsymbol{\delta(t) dt = 1}$ (unit area)

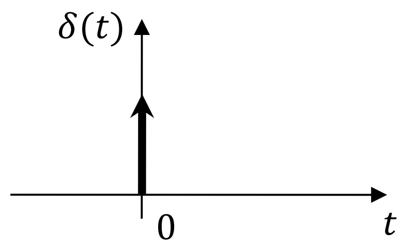

Figure 26

Note: $\boldsymbol{\delta(t)}$ is not defined at $t = 0$, $\boldsymbol{\delta(t)} \rightarrow \infty$.

- **Approximation of $\boldsymbol{\delta(t)}$:**

Let $\delta_\epsilon(t)$ be defined as shown in the figure below:

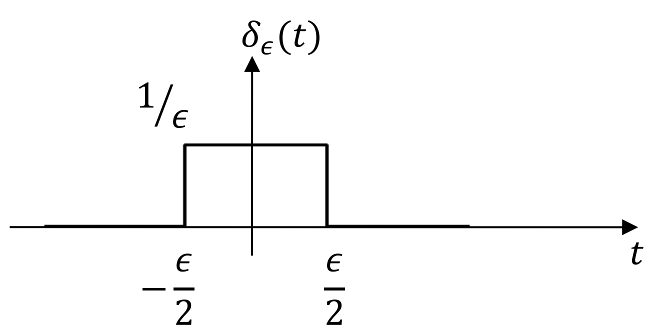

Figure 27

Notice that $\int_{-\infty}^{\infty} \delta_\epsilon(t) dt = 1$ for any ϵ, then $\delta(t)$ can be written:

$$\delta(t) = \lim_{\epsilon \to 0} \delta_\epsilon(\mathrm{t})$$

Remark: There are other approximations (see textbook).

- **Properties of $\boldsymbol{\delta(t)}$**:

1) Multiplication by a function:

$$x(t)\delta(t) = x(0)\delta(t)$$

provided $x(t)$ is continuous at $t = 0$.

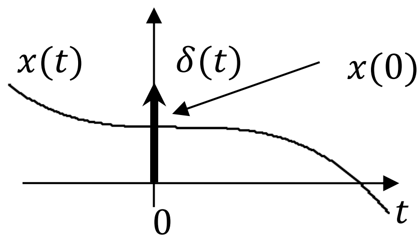

Figure 28

In general, $$x(t)\delta(t-T) = x(T)\delta(t-T)$$

provided $x(t)$ is continuous at $t = T$.

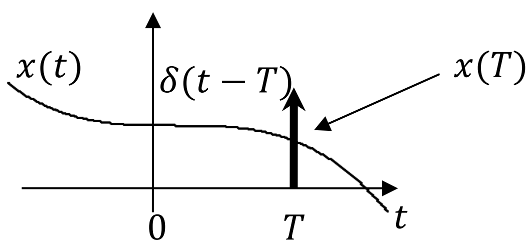

Figure 29

2) Sampling (or Sifting) Property:

$$\int_{-\infty}^{\infty} x(t)\delta(t)dt = x(0)$$

provided $x(t)$ is continuous at $t = 0$.

Proof:

$$\int_{-\infty}^{\infty} x(t)\delta(t)dt = \int_{-\infty}^{\infty} x(0)\delta(t)dt = x(0)\underbrace{\int_{-\infty}^{\infty} \delta(t)dt}_{1} = x(0)$$

In general, $\int_{-\infty}^{\infty} x(t)\delta(t-T)dt = x(T)$

provided $x(t)$ is continuous at $t = T$ (see Figure 29).

3) Relation to the unit step $u(t)$:

$$\int_{-\infty}^{t} \delta(\tau)d\tau = u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

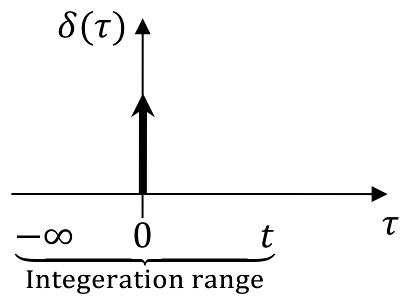

Figure 30

On the other hand, taking the derivative of the above integral gives:

$$\frac{du(t)}{dt} = \delta(t)$$

4) Even Function:

$$\delta(t) = \delta(-t)$$

5) <u>Time Scaling:</u>

$$\delta(at) = \frac{1}{a}\delta(t) \qquad \text{where } \; a > 0$$

Justification:

$$\delta(at) = \lim_{\epsilon \to 0} \delta_\epsilon(at)$$

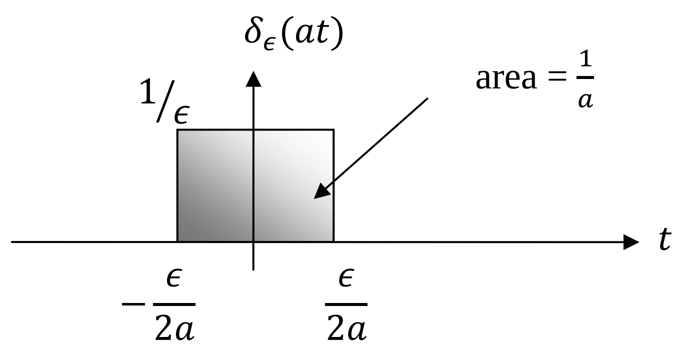

Figure 31

In general, $\qquad \delta(at) = \frac{1}{|a|}\delta(t)$

where a can be positive or negative.

6) <u>Derivative of $\delta(t)$</u>

A non-ordinary derivative. Justification:

$$\dot{\delta}(t) = \frac{d\delta(t)}{dt} = \lim_{\epsilon \to 0} \frac{d\delta_\epsilon(t)}{dt}$$

$$= \lim_{\epsilon \to 0} \frac{d}{dt}\left[\frac{1}{\epsilon}u\left(t + \frac{\epsilon}{2}\right) - \frac{1}{\epsilon}u\left(t - \frac{\epsilon}{2}\right)\right]$$

$$= \lim_{\epsilon \to 0} \left[\frac{1}{\epsilon}\delta\left(t + \frac{\epsilon}{2}\right) - \frac{1}{\epsilon}\delta\left(t - \frac{\epsilon}{2}\right)\right]$$

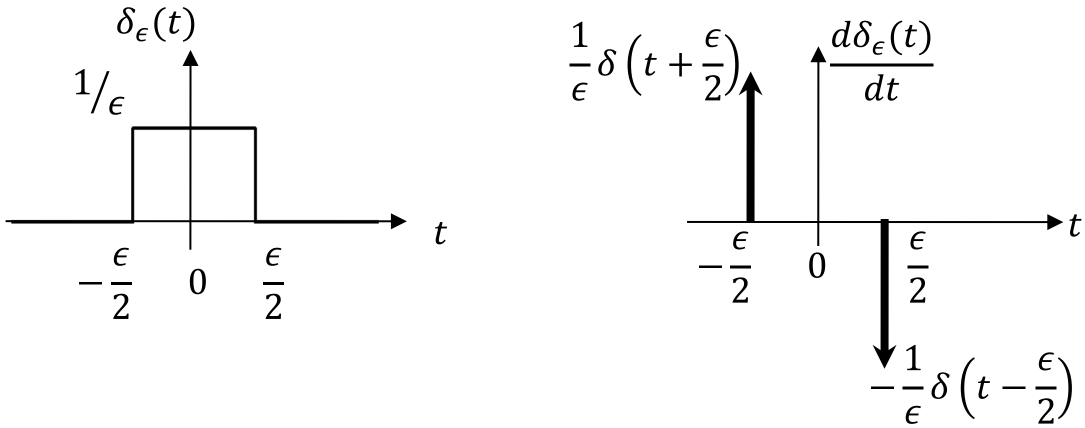

Figure 32

$\dot{\delta}(t)$ is called a *doublet* since it has two impulses at $t = 0$.

<u>Properties of $\dot{\delta}(t)$:</u>

i. $\dot{\delta}(t)$ is an *odd function*, i.e. $\dot{\delta}(t) = -\dot{\delta}(-t)$

Consequently, $\int_{-\infty}^{\infty} \dot{\delta}(t)dt = 0$

ii. Sifting Property: $\int_{-\infty}^{\infty} x(t)\dot{\delta}(t-T)dt = -\dot{x}(T) = -\left.\frac{dx}{dt}\right|_{t=T}$

Provided $x(t)$ and $\dot{x}(t)$ are contineous at $t = T$.

- <u>Discrete-Time Unit Impulse Function</u>

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

(c) CT Exponential Function

$$x(t) = e^{st}, \qquad -\infty < t < \infty$$

In general, it is a *complex function*, where $s = \sigma + j\omega$ is a complex number.

This function can be rewritten:

$$x(t) = e^{st} = e^{(\sigma + j\omega)t} = e^{\sigma t}.e^{j\omega t}$$

$$= \underbrace{e^{\sigma t}\cos(\omega t)}_{x_r(t)} + \underbrace{je^{\sigma t}\sin(\omega t)}_{x_i(t)}$$

real part imaginary part

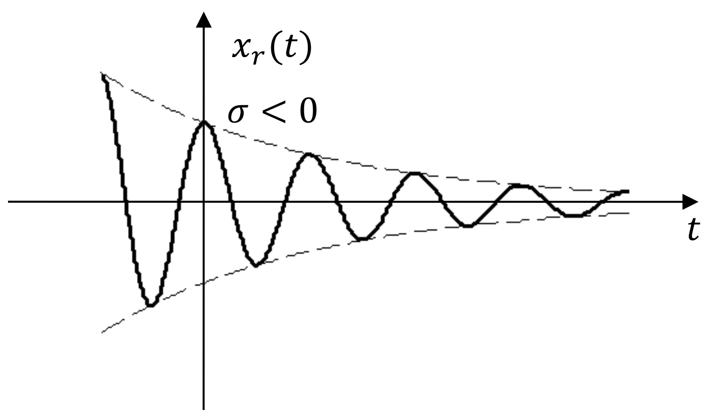

Figure 33

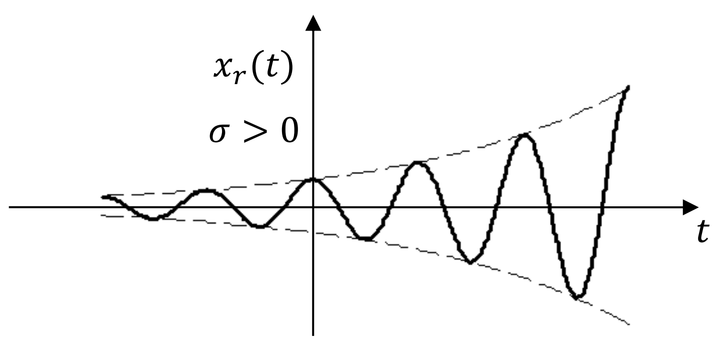

Figure 34

(d) DT Exponential Function

$$\boldsymbol{x[n] = s^n} \;, \qquad \boldsymbol{n = 0, \pm 1, \pm 2, \ldots}$$

In general, this is a *complex function* when $\boldsymbol{s = \sigma + j\omega = re^{j\theta}}$ is a complex number.

<u>Note:</u> Here, we allow $\boldsymbol{r}$ to be a real positive or negative number.

For $\boldsymbol{\theta = 0}$: $\boldsymbol{x[n] = r^n}$, *a real exponential function*.

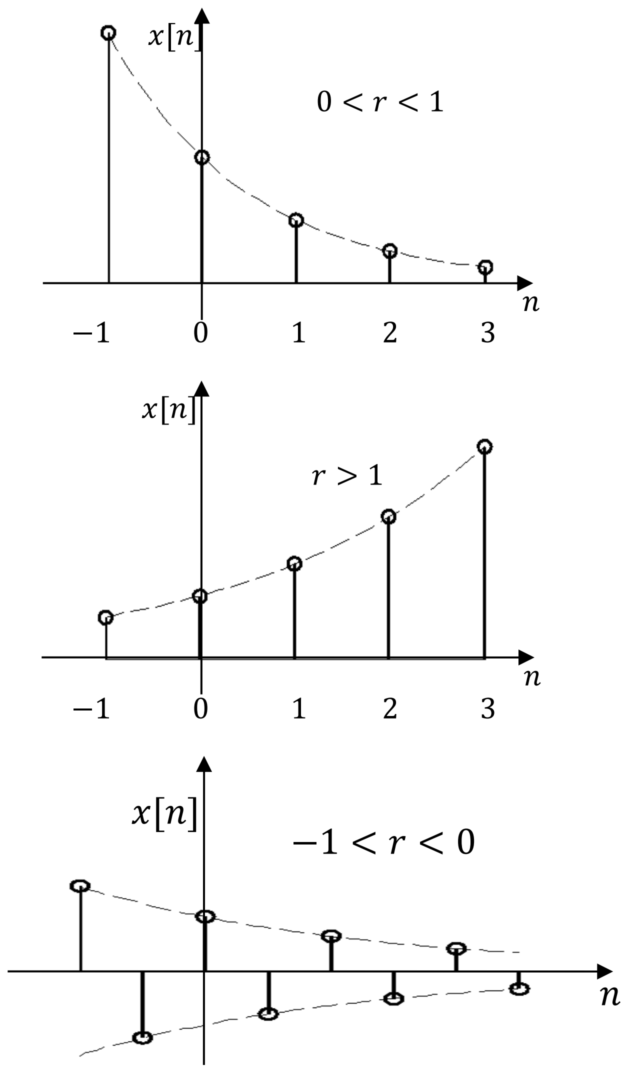

Figure 35

(e) Sinusoidal Function

CT: $$\boldsymbol{x(t) = A\cos(\omega_0 t + \theta)} \qquad -\infty < \boldsymbol{t} < \infty$$

where

$\boldsymbol{A}$ Amplitude

$\boldsymbol{\omega_0}$ Angular frequency (radian/sec.)

$\boldsymbol{f_0 = \frac{\omega_0}{2\pi}}$ Frequency (Hz)

$\boldsymbol{T_0 = \frac{1}{f_0} = \frac{2\pi}{\omega_0}}$ Period (sec.)

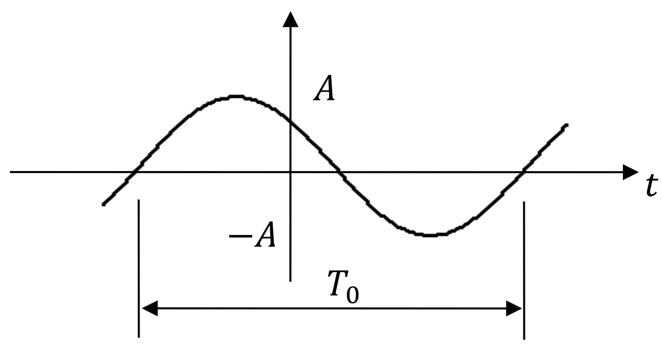

Figure 36

DT: $$\boldsymbol{x[n] = A cos(\Omega_0 n + \theta)}\ , \qquad \boldsymbol{n = 0,\ \pm 1,\ \pm 2,\ \cdots}$$

Remark: CT sinusoids are always periodic, while DT sinusoids are not always periodic, as we will see later.

1.6 Periodic Signals

Definition: A CT signal $x(t)$ is *periodic* if there exists a constant $T > 0$ such that (s.t.):

$$x(t) = x(t + T) \qquad \text{for all } -\infty < t < \infty.$$

The smallest value of T is called the *period* T_0. Otherwise, the signal is *aperiodic* or *non-periodic*.

Example: (Periodic signal with period $T_0 = 2$ sec.)

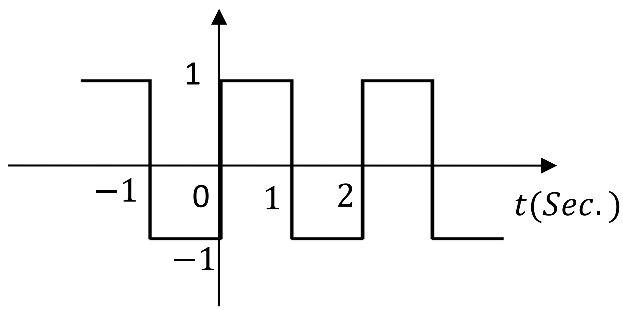

Figure 37

Definition: A DT signal $x[n]$ is *periodic* if there exists an integer N s.t.

$$x[n] = x[n + N] \qquad \text{for all } -\infty < n < \infty.$$

The smallest value of N is called the *period* N_0. Otherwise, the signal is *aperiodic* or *non-periodic*.

Example: (Periodic signal with period $N_0 = 3$)

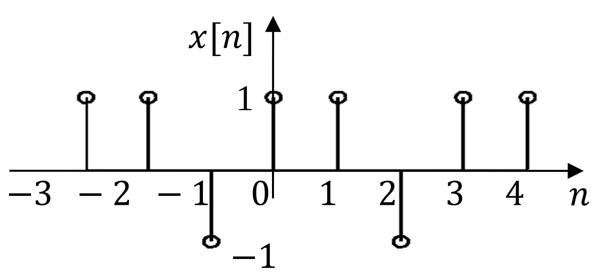

Figure 38

- Periodic signals are power signals whose power can be computed by:

  CT: $P = \frac{1}{T_0}\int_0^{T_0} |x(t)|^2 dt$

  DT: $P = \frac{1}{N_0}\sum_{n=0}^{N_0-1} |x[n]|^2$

  <u>Proof (CT case):</u> From definition of *P*:

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

  Let $T = mT_0$, where T_0 = period, $m =$ integer

$$P = \lim_{m\to\infty} \frac{1}{2mT_0} \int_{-mT_0}^{mT_0} |x(t)|^2 dt = \lim_{m\to\infty} \frac{2m}{2mT_0} \int_{0}^{T_0} |x(t)|^2 dt$$

$$\Rightarrow \quad P = \frac{1}{T_0} \int_{0}^{T_0} |x(t)|^2 dt$$

- Question: Are all sinusoidal signals periodic?

  CT: $x(t) = A\cos(\omega_0 t + \theta), \qquad -\infty < t < \infty$

  DT: $x[n] = A\cos(\Omega_0 n + \theta), \qquad n = 0, \pm 1, \pm 2, \ldots$

  Answer: CT: Yes, always, $T_0 = \frac{2\pi}{\omega_0}$

  DT: Not always.

  Fact: The DT signal $x[n] = A\cos(\Omega_0 n + \theta)$ is periodic if and only if (iff) $\frac{\Omega_0}{2\pi} =$ rational and the period is $N_0 = \frac{2\pi}{\Omega_0} m_0$ where m_0 is the smallest integer s.t. N_0 is an integer.

  Proof: $x[n]$ is periodic iff:

  $$x[n] = x[n+N], \qquad N = \text{integer}$$

  $$\Rightarrow \qquad A\cos(\Omega_0 \text{n} + \theta) = A\cos(\Omega_0 n + \Omega_0 N + \theta)$$

  This is satisfied iff:

  $$\Omega_0 N = 2\pi m, \qquad m = \text{integer}$$

  $$\Rightarrow \qquad \frac{\Omega_0}{2\pi} = \frac{m}{N} = \text{rational}$$

  The period is $N_0 = \frac{2\pi}{\Omega_0} m_0$, where $m_0 =$ smallest integer s.t. $N_0 =$ integer.

  Example: $x[n] = \cos\left(\frac{8\pi}{31} n\right)$

  $$\Omega_0 = \frac{8\pi}{31} \quad \Rightarrow \quad \frac{\Omega_0}{2\pi} = \frac{4}{31} = \text{rational} \quad \Rightarrow \quad x[n] \text{ periodic}$$

  Period: $N_0 = \frac{2\pi}{\Omega_0} m_0 = \frac{31}{4} m_0 = 31$ for $m_0 = 4$

Example: $x[n] = \cos(n), \qquad \Omega_0 = 1$

$$\frac{\Omega_0}{2\pi} = \frac{1}{2\pi} = \text{non-rational} \quad \Rightarrow \quad x[n] \text{ non-periodic}$$

- Question: Is the sum of periodic signals periodic?

  i.e. Is (periodic + periodic = periodic) ?

Answer: CT: Not always

DT: Yes, always

Fact (CT): Let $x_1(t)$ be periodic with period T_1 and $x_2(t)$ be periodic with period T_2. The sum $x(t) = x_1(t) + x_2(t)$ is periodic iff

$$\frac{T_1}{T_2} = \text{rational}$$

The period of the sum $x(t)$ is $T_0 = m_1 T_1 = m_2 T_2$, where m_1 and m_2 are integers with no common factors.

Proof: Since $x_1(t)$ and $x_2(t)$ are periodic, then:

$$x_1(t) = x_1(t + T_1) = x_1(t + m_1 T_1)$$

$$x_2(t) = x_2(t + T_2) = x_2(t + m_2 T_2)$$

where m_1 and m_2 are integers.

Now, the sum $x(t) = x_1(t) + x_2(t)$ is periodic iff $x(t) = x(t + T)$ for some value T and all t. Substituting gives:

$$x_1(t + m_1 T_1) + x_2(t + m_2 T_2) = x_1(t + T) + x_2(t + T)$$

This equality holds iff $T = m_1 T_1 = m_2 T_2$. Or,

$$\frac{T_1}{T_2} = \frac{m_2}{m_1} = rational$$

And, the period of the sum $x(t)$ is $T_0 = m_1 T_1 = m_2 T_2$, where m_1 and m_2 are integers with no common factor.

<u>Example:</u>

$$x(t) = \cos\left(\frac{2\pi}{3}t\right) + \sin\left(\frac{2\pi}{5}t\right)$$

Is $x(t)$ periodic?

$\cos\left(\frac{2\pi}{3}t\right)$ is periodic with period $T_1 = \frac{2\pi}{\omega_1} = \frac{2\pi}{2\pi/3} = 3$

$\sin\left(\frac{2\pi}{5}t\right)$ is periodic with period $T_2 = \frac{2\pi}{\omega_2} = \frac{2\pi}{2\pi/5} = 5$

Since $\frac{T_1}{T_2} = \frac{3}{5} =$ rational $\Rightarrow$ $x(t)$ periodic

Period: Since $m_1 = 5$ and $m_2 = 3$ have no common factor:

$$T_0 = m_1 T_1 = m_2 T_2 = 5(3) = 3(5) = 15$$

<u>Example:</u> $x(t) = \sin(2t) + \sin(3\pi t)$

$$T_1 = \frac{2\pi}{2} = \pi, \qquad T_2 = \frac{2\pi}{3\pi} = \frac{2}{3}$$

$$\Rightarrow \quad \frac{T_1}{T_2} = \frac{\pi}{2/3} = \frac{3\pi}{2} \text{ non-rational}$$

$\Rightarrow$ $x(t)$ non-periodic.

Fact (DT): The sum of DT periodic signals is always periodic. This is true since if $x_1[n]$ is periodic with period N_1, and $x_2[n]$ is periodic with period N_2 then $x[n] = x_1[n] + x_2[n]$ is periodic since $\frac{N_1}{N_2}$ is always rational. The period of the sum is $N_0 = m_1 N_1 + m_2 N_2$, where m_1 and m_2 are integers with no common factors.

- Definition: A signal $x(t)$ is called a *causal signal* if $x(t) = 0$ for $t < 0$.

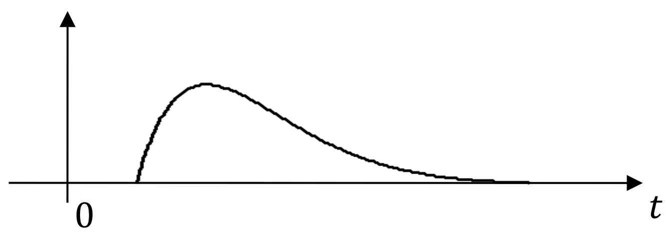

Figure 39

Otherwise, it is *non-causal signal*.

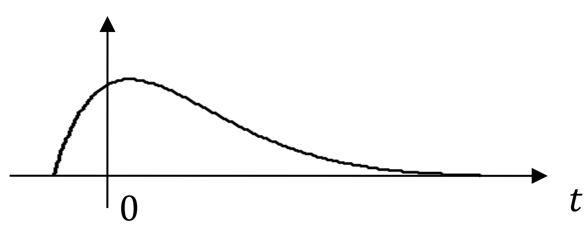

Figure 40

- Definition: A signal $x(t)$ is called *an everlasting signal* if it exists over the entire interval $-\infty < t < \infty$.

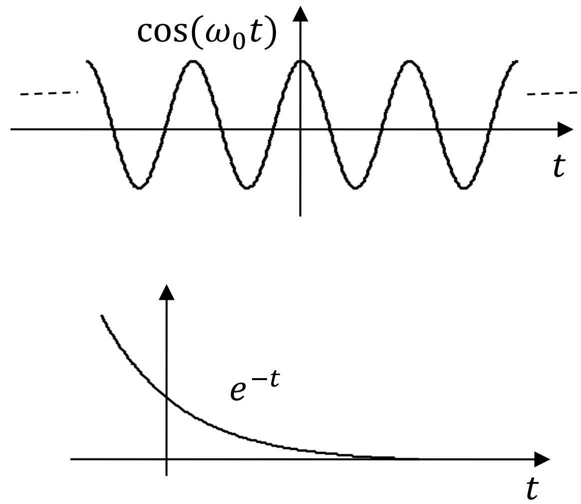

Figure 41

1.7 Even and Odd Signals

- Definition: A real signal is called an *even signal* if it is symmetric w.r.t. the *y*-axis, i.e:

  CT: $x(t) = x(-t)$ for all t

  DT: $x[n] = x[-n]$ for all n

  Examples: $x_1(t) = \cos(\omega_0 t)$

  $x_2(t) = |t|$

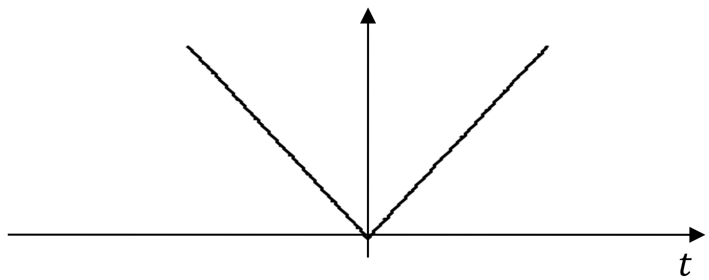

Figure 42

- Definition: A real signal is called an *odd signal* if it is anti-symmetric w.r.t. the *y*-axis, i.e.:

  CT: $x(t) = -x(-t)$ for all t

  DT: $x[n] = -x[-n]$ for all n

  Examples: $x_1(t) = \sin(\omega_0 t)$

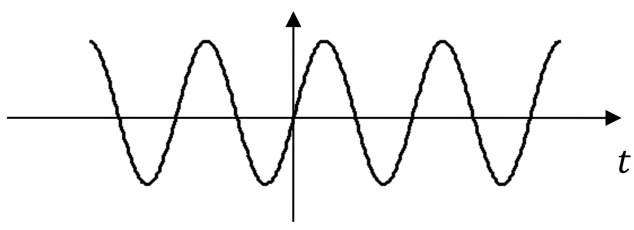

Figure 43

$$x_2(t) = t$$

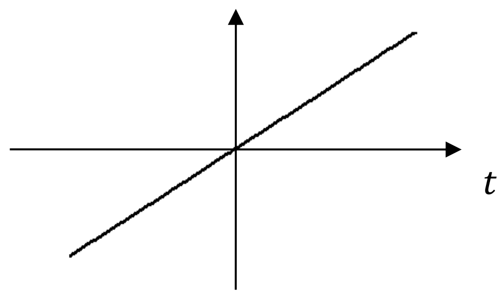

Figure 44

- Properties:

  i. even × even = even

  ii. odd × odd = even

  iii. even × odd = odd

Remark: Even and odd properties are useful in many applications, such as integrations (CT) or summations (DT):

Even:

$$\int_{-a}^{a} x(t)dt = 2\int_{0}^{a} x(t)dt$$

$$\sum_{n=-N}^{N} x[n] = x[0] + 2\sum_{n=1}^{N} x[n]$$

Odd:

$$\int_{-a}^{a} x(t)dt = 0$$

$$\sum_{n=-N}^{N} x[n] = 0$$

Example:

$$\int_{-100}^{100} \underbrace{\underbrace{t}_{\text{odd}} \underbrace{\cos(t)}_{\text{even}}}_{\text{odd}} \, dt = 0$$

Remark: Odd signals have always:

$$x(0) = 0\,, \qquad\qquad x[0] = 0$$

- Fact: Every signal can be written uniquely as a sum of even and odd signals, i.e.:

$$x(t) = \underbrace{x_e(t)}_{\text{even}} + \underbrace{x_o(t)}_{\text{odd}}$$

where:

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)]$$

$$x_o(t) = \frac{1}{2}[x(t) - x(-t)]$$

Same relations hold for DT signals.

Proof:

$$x(t) = \underbrace{\frac{1}{2}x(t) + \frac{1}{2}x(t)}_{x(t)} + \underbrace{\frac{1}{2}x(-t) - \frac{1}{2}x(-t)}_{0}$$

$$= \underbrace{\frac{1}{2}[x(t) + x(-t)]}_{x_e(t)} + \underbrace{\frac{1}{2}[x(t) - x(-t)]}_{x_o(t)}$$

Example: $x(t) = e^{-\alpha t}\cos(\omega_0 t), \quad -\infty < t < \infty$

Where α and ω_0 are real constants.

Notice that $x(t)$ is neither even nor odd. It can be written as $x(t) = x_e(t) + x_o(t)$ where:

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)]$$

$$= \frac{1}{2}[e^{-\alpha t}\cos(\omega_0 t) + e^{\alpha t}\cos(\omega_0 t)]$$

$$= \frac{1}{2}[e^{-\alpha t} + e^{\alpha t}]\cos(\omega_0 t)$$

$$= \underbrace{\cosh(\alpha t)}_{\text{even}}\underbrace{\cos(\omega_0 t)}_{\text{even}}$$

$$x_o(t) = \frac{1}{2}[x(t) - x(-t)]$$

$$= \frac{1}{2}[e^{-\alpha t}\cos(\omega_0 t) - e^{\alpha t}\cos(\omega_0 t)]$$

$$= \frac{1}{2}[e^{-\alpha t} - e^{\alpha t}]\cos(\omega_0 t)$$

$$= \underbrace{-\sinh(\alpha t)}_{\text{odd}}\underbrace{\cos(\omega_0 t)}_{\text{even}}$$

1.8 Systems

- Definition: A *system* is an entity that transforms or processes one or more input signals to produce one or more output signals. A system may consists of a *physical* entity (hardware components) or an *algorithm* (software components).

  Systems are used to extract information from signals (Signal Analysis) or to modify signals into more useful signals (Signal Processing).

- A system is characterized by its:
  - Inputs
  - Outputs
  - Rules of operation or mathematical model

  In this course, we mainly consider *single-input single output* (SISO) systems.

- Notation: The relationship between the input and the output of a system can be written as:

$$y(t) = \mathrm{S}[x(t)]$$

or

$$\mathrm{S}: \quad x(t) \rightarrow y(t)$$

  Mathematically, the system can be thought of as an *operator* S which operates on the input $x(t)$ to produce the output $y(t)$. Graphically, the system is represented as a black box.

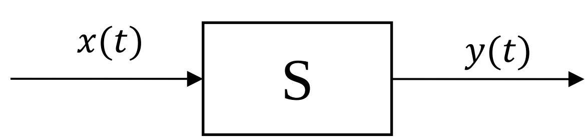

Figure 45

- Classification of Systems: There are several classifications of systems, as we will see later. This helps in the simplification of analysis and design of systems. For example, systems can be classified as *linear* or *non-linear*, *time-invariant* or *time-varying*, *continuous-time*, *discrete-time*, or *hybrid*:

  – CT systems: $x(t) \to y(t)$

  – DT systems: $x[n] \to y[n]$

  – Hybrid systems: $x(t) \to y[n]$ (A/D Converter)

  or $x[n] \to y(t)$ (D/A Converter)

  Several other classifications or properties of systems will be discussed later in more details.

  Example: (CT system)

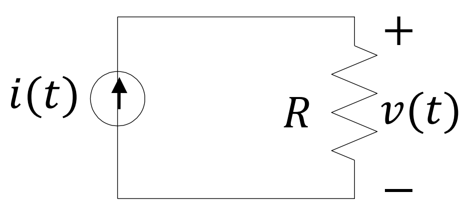

Figure 46

Input: $x(t) = i(t)$

Output: $y(t) = v(t)$

Mathematical model: $v(t) = Ri(t)$

or: $y(t) = Rx(t)$

Example: (CT system)

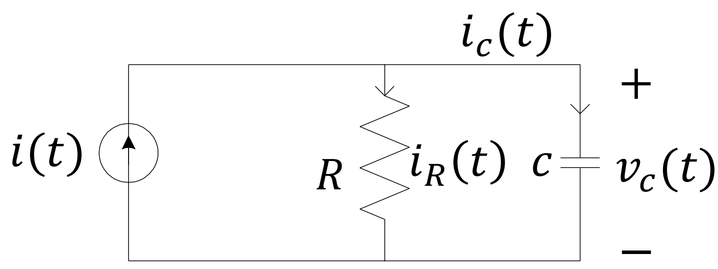

Figure 47

Input: $x(t) = i(t)$

Output: $y(t) = v_c(t)$

Model: KCL: $i_c(t) + i_R(t) = i(t)$ (*)

where: $i_c(t) = C\frac{dv_c(t)}{dt} = C\frac{dy(t)}{dt}$

$$i_R(t) = \frac{1}{R}v_c(t) = \frac{1}{R}y(t)$$

Substituting in (*):

$$C\frac{dy(t)}{dt} + \frac{1}{R}y(t) = x(t) \quad \text{(System model)}$$

- The differential equation represents an *implicit relationship* between the input $x(t)$ and the output $y(t)$.
- The solution of the differential equation gives an *explicit relationship* of $y(t)$ in terms of $x(t)$. We study the solution later.

Example: (DT system)

$$y[n] = \frac{1}{3}[x[n] + x[n-1] + x[n-2]]$$

This is called *moving average filter* which is a simple way to filter out noise in the input signal.

- **Linear and Nonlinear Systems**

  Definition: A system is *linear* if it satisfies two properties: *Additivity* and *Homogeneity*.

  Consider the system: S: $x(t) \rightarrow y(t)$

  And, let $x_1(t) \rightarrow y_1(t)$

  $x_2(t) \rightarrow y_2(t)$

  – The system S is *additive* if, for all x_1 and x_2:

  $$x(t) = x_1(t) + x_2(t) \quad \rightarrow \quad y(t) = y_1(t) + y_2(t)$$

  – The system S is *homogeneous* if, for all real or complex constants c:

  $$x(t) = cx_1(t) \quad \rightarrow \quad y(t) = cy_1(t)$$

  – Additivity and homogeneity can be combined into one property, the *superposition property/principle*. i.e. a system is linear if, for all real or complex constants c_1 and c_2:

  $$x(t) = c_1x_1(t) + c_2x_2(t) \quad \rightarrow \quad y(t) = c_1y_1(t) + c_2y_2(t)$$

  – Linearity is the same for DT systems.

  – A system is *nonlinear* if it does not satisfy linearity.

  Example: A system is described by: $y(t) = 5x(t)$

  Let $x_1(t) \rightarrow y_1(t) = 5x_1(t)$

  $x_2(t) \rightarrow y_2(t) = 5x_2(t)$

  Then, for $x(t) = c_1x_1(t) + c_2x_2(t) \rightarrow y(t) = 5x(t) = 5[c_1x_1(t) + c_2x_2(t)] = c_1[5x_1(t)] + c_2[5x_2(t)] = c_1y_1(t) + c_2y_2(t)$

  ∴ System is linear.

Example: A system is described by: $y(t) = x^2(t)$

Let $x_1(t) \to y_1(t) = x_1^2(t)$

$x_2(t) \to y_2(t) = x_2^2(t)$

For $x(t) = c_1x_1(t) + c_2x_2(t) \;\to\; y(t) = x^2(t) = [c_1x_1(t) + c_2x_2(t)]^2$

$= c_1^2x_1^2(t) + c_2^2x_2^2(t) + 2c_1c_2x_1(t)x_2(t)$

$\neq c_1y_1(t) + c_2y_2(t) = c_1x_1^2(t) + c_2x_2^2(t)$

$\therefore$ System is nonlinear.

Example: A system is described by: $y(t) = 2x(t) + 3$

Let $x_1(t) \;\to\; y_1(t) = 2x_1(t) + 3$

$x_2(t) \;\to\; y_2(t) = 2x_2(t) + 3$

For $x(t) = c_1x_1(t) + c_2x_2(t) \;\to\; y(t) = 2x(t) + 3 = 2c_1x_1(t) +$ $2c_2x_2(t) + 3 \neq c_1y_1(t) + c_2y_2(t) = 2c_1x_1(t) + 3c_1 + 2c_2x_2(t) + 3c_2$

$\therefore$ System is nonlinear.

More Examples:

- $y(t) = x(t - T)$ Linear
- $y(t) = x(at)$ Linear
- $y(t) = t^2x(t)$ Linear
- $\frac{dy(t)}{dt} + 5y(t) = 2x(t)$ Linear
- $\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + x(t)$ Linear
- $\frac{dy(t)}{dt} + 2y(t) = x(t)\frac{dx(t)}{dt}$ Nonlinear

– $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$ Linear

Remark: Real physical systems are approximately linear within a range of operation. For example, linear resistors become nonlinear for large currents.

Remark (Response of a Linear System to a Zero Input Signal):

Assume a linear system S: $x(t) \rightarrow y(t)$. Since the system is linear, then it must satisfy the homogeneity property $cx(t) \rightarrow cy(t)$ for any constant c. Taking $c = 0$ gives:

$$x(t) = 0 \rightarrow y(t) = 0 \quad \text{for all } t$$

This means, for a linear system, if there is no input, then there must be no output.

- **Time-Invariant and Time-Varying Systems**

Definition: A system is *time-invariant* if its response does not change with a shift in time. i.e.

If $x(t) \rightarrow y(t)$

then $x_1(t) = x(t-T) \rightarrow y_1(t) = y(t-T)$

for all time shift T and all $x(t)$.

Graphically, if a system has a response as shown in Figure 48.

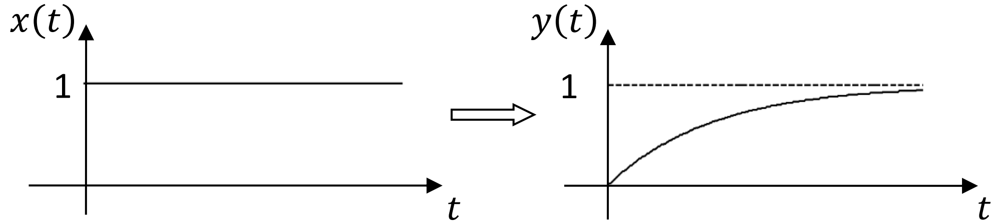

Figure 48

Then, for time-invariant system, if the input is delayed by T, then the output must be delayed by the same interval T, as shown in Figure 49.

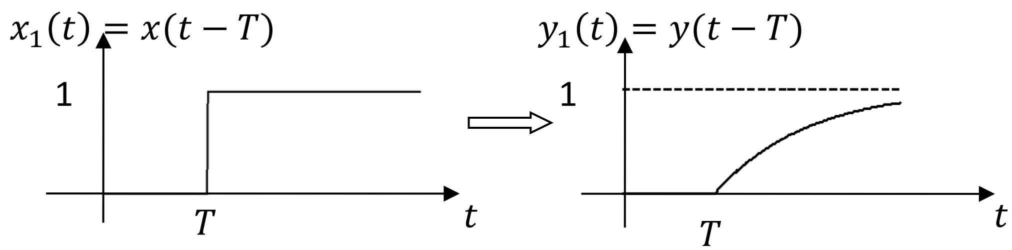

Figure 49

<u>Example</u>: $y(t) = x^2(t)$

Let's shift the output by replacing t by $t-T$: $y(t-T) = x^2(t-T)$

Now, let's shift the input: $x_1(t) = x(t-T)$

then $y_1(t) = x_1^2(t) = x^2(t-T) = y(t-T)$

∴ The system is time-invariant.

<u>Example:</u> $y(t) = tx(t)$

Replace t by $t-T$: $y(t-T) = (t-T)x(t-T)$

Let $x_1(t) = x(t-T)$

then $y_1(t) = tx_1(t) = tx(t-T) \neq y(t-T)$

∴ The system is not time-invariant. It is time-varying.

<u>Example:</u> $y(t) = x(2t)$ (time-scaling system)

Shift the output: $y(t-T) = x\big(2(t-T)\big) = x(2t-2T)$

Shift the input: $x_1(t) = x(t-T)$

then $y_1(t) = x_1(2t) = x(2t-T) \neq y(t-T)$

∴ The system is time-varying.

More system examples: Let $\dot{x}(t) = \frac{dx(t)}{dt}$ and $\dot{y}(t) = \frac{dy(t)}{dt}$, then:

$\dot{y}(t) + \underbrace{y^2(t)}_{nonlinear} = \underbrace{t}_{time-varying} x(t)$ (Nonlinear and Time-Varying)

$\dot{y}(t) + \underbrace{t^2}_{time-varying} y(t) = \underbrace{\sin[x(t)]}_{nonlinear}$ (Nonlinear and Time-Varying)

$\dot{y}(t) + 2y(t) = x(t) + \underbrace{1}_{nonlinear}$ (Nonlinear and Time-Invariant)

$\dot{y}(t) + \underbrace{x(t)y(t)}_{nonlinear} = \dot{x}(t)$ (Nonlinear and Time-Invariant)

$\ddot{y}(t) + 2\dot{y}(t) + y(t) = \dot{x}(t) + x(t)$ (Linear and Time-Invariant)

$y(t) = \int_{-\infty}^{t} x(\tau) d\tau$ (Linear and Time-Invariant)

- **Response of Linear and Time-Invariant (LTI) Systems**

The class of LTI systems has an interesting property: Knowing the response of an LTI system to a single input enables us to find the response to any other input. We will discuss this later in more details. For now, we show this by a simple example.

Example:

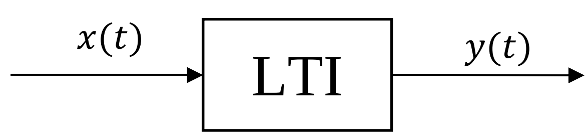

Figure 50

Suppose, we know the response of $x_1(t) = u(t)$ to be $y_1(t)$, as shown in Figure 51.

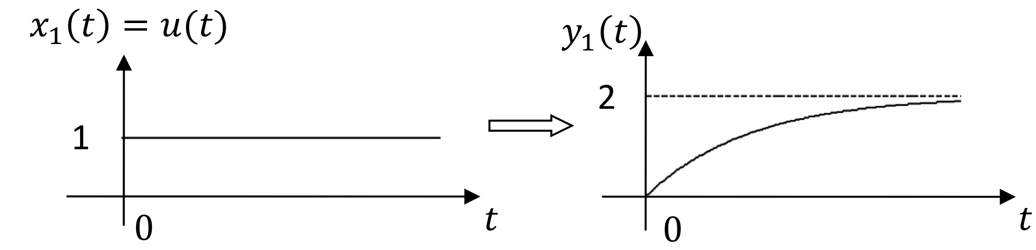

Figure 51

Then, the response $y(t)$ due to the new input $x(t) = u(t) - u(t - T)$ shown in Figure 52 can be found by applying:

Time-Invariance: $u(t - T) \rightarrow y_1(t - T)$

Linearity: $x(t) = u(t) - u(t - T) \rightarrow y(t) = y_1(t) - y_1(t - T)$

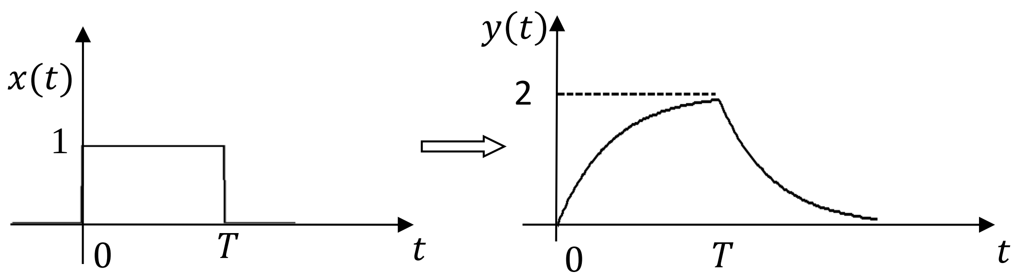

Figure 52

<u>Remarks</u>:

- RLC circuits are LTI systems.
- Electronic circuits are LTI systems under small-signal operation.
- Systems described by linear differential equations with constant coefficients are LTI.

- **<u>Causal and Noncausal Systems</u>**

<u>Definition</u>: A system is *causal* if the output $y(t)$ at any instant t_0 depends on the input $x(t)$ for $t \leq t_0$ only.

This means the present value of the output depends on past and/or present values of the input $x(t)$ only, not on future values. This must be valid for all $-\infty < t_0 < \infty$. This also implies that the output can not start before the input is applied.

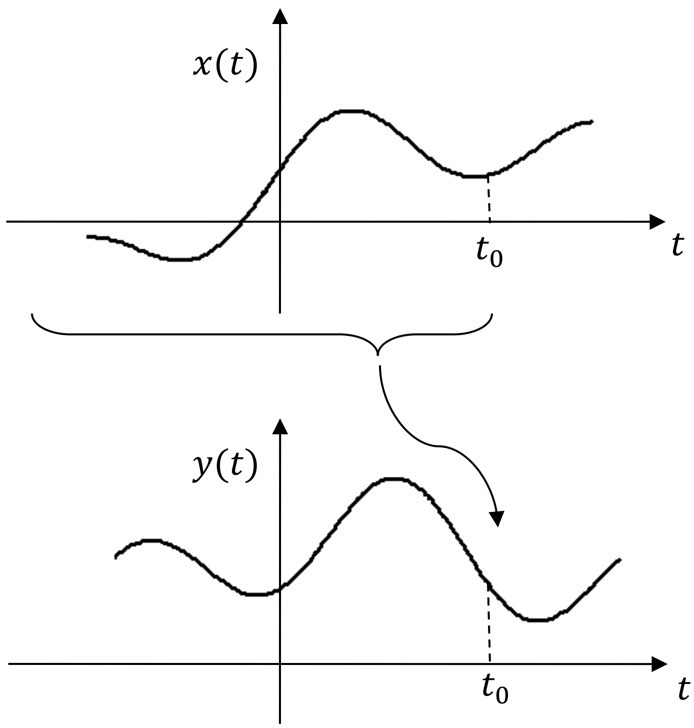

Figure 53

<u>Example</u>: A system is described by:

$$y(t) = x(t+2)$$

For $t = 0 \quad \Rightarrow \quad y(0) = x(2)$

Since $y(0)$ depends on the future value $x(2)$, then the system is noncausal.

More Examples:

$y(t) = x(t-2)$	Causal
$y(t) = x(-t)$	Noncausal
$y(t) = Rx(t) + \frac{1}{C}\int_{-\infty}^{t} x(\tau)d\tau$	Causal
$y[n] = \frac{1}{3}[x[n] + x[n-1] + x[n-2]]$	Causal
$y[n] = \frac{1}{3}[x[n+1] + x[n] + x[n-1]]$	Noncausal

Remarks:

1. RLC circuits and electronic circuits are causal systems.
2. Systems described by linear differential equations with constant coefficients are causal systems.
3. Temporal systems, whose independent variable is time *t*, can be realized in real-time if they are causal. This means practical systems that operate in real-time must necessarily be causal.
4. Noncausal temporal systems are not realizable in real-time, but can be realized in non real-time. For example, record a speech and process it later.
5. Non-temporal systems can be realized in real time if they are noncausal, e.g. optical or image systems.
6. Noncausal systems, such as *ideal filters*, are useful in studying the performance of causal systems. We will discuss ideal filters at the end of the course.

- **Other System Properties**

  - Instantaneous (Memoryless) and Dynamic Systems: A system is *memoryless* if its output $y(t)$ at any instant t depends on the input $x(t)$ at the same instant t, not on the past or future values. For example, resistive circuits are memoryless systems, while RLC circuits are not memoryless (dynamic) systems.

  - Invertible Systems: A system is *invertible* if its input $x(t)$ can be obtained *uniquely* from its given output $y(t)$. For example, the system described by $y(t) = 2x(t) + 1$ is an invertible system since $x(t) = \frac{1}{2}y(t) - \frac{1}{2}$ is unique. On the other hand, the *rectifier* system $y(t) = |x(t)|$ is *noninvertible* since there are two input values that give the same output.

  - Stable and Unstable Systems: There are two types of stability: *external* or *internal*. A system is *externally stable* if every *bounded-input* $x(t)$ produces a *bounded-output* $y(t)$. This external stability is called *bounded-input bounded-output* (BIBO) stability. For example, the system $y(t) = 2x(t) + 1$ is BIBO stable. While, the system $y(t) = tx(t)$ is not BIBO stable (*unstable*) since $y(t) \to \infty$ when $t \to \infty$. The internal stability is called *asymptotic* stability, which will be discussed later in chapter 2.

1.9 System Models

There are two models: 1 – Input-Output Model (External Description)

2 – State-Space Model (Internal Description)

Here, we give examples for how to derive the system models for simple *electrical* and *mechanical* LTI systems.

Example: (Input-Output Model)

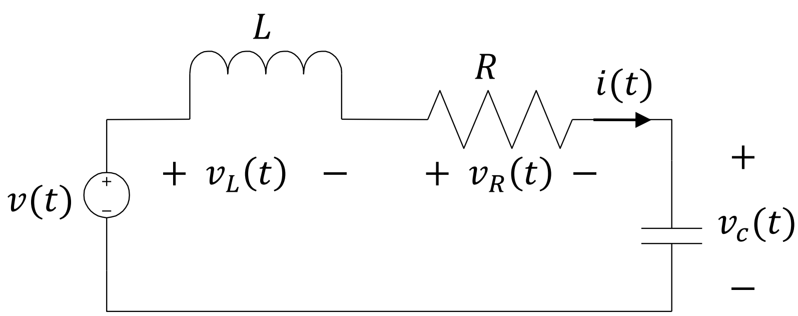

Figure 54

Let input: $x(t) = v(t)$

Output: $y(t) = i(t)$

KVL: $v_L + v_R + v_C = v$

$$L\frac{dy}{dt} + Ry + \frac{1}{C}\int_{-\infty}^{t} y(\tau)d\tau = x$$

Differentiating:

$$L\frac{d^2y(t)}{dt^2} + R\frac{dy(t)}{dt} + \frac{1}{C}y(t) = \frac{dx(t)}{dt}$$

This *second-order differential equation* is the (Input-Output Description) of the system. It is called an *implicit model*. The solution for $y(t)$ is called an *explicit model*. We will discuss the solution later in chapter 2.

<u>Example:</u> (State-Space Model)

For the same previous example, shown in Figure 54.

Input $x(t) = v(t)$

Output $y(t) = i(t)$

Define the *state-variables*:

$$q_1(t) = i(t) \qquad \text{(current through inductor)}$$

$$q_2(t) = v_c(t) \qquad \text{(voltage across capacitor)}$$

From $v_L + v_R + v_C = v$

$$L\frac{di}{dt} + Ri + v_c = v$$

$$L\dot{q}_1 + Rq_1 + q_2 = x \qquad \text{where} \quad \dot{q}_1 = \frac{dq_1}{dt}$$

$$\dot{q}_1 = -\frac{R}{L}q_1 - \frac{1}{L}q_2 + \frac{1}{L}x \tag{1}$$

For *C*: $i(t) = C\frac{dv_c(t)}{dt}$

$$q_1 = c\dot{q}_2 \qquad \Rightarrow \qquad \dot{q}_2 = \frac{1}{C}q_1 \tag{2}$$

Equations (1) and (2) in matrix form:

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} x \tag{3}$$

The output is:

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + [0]x \tag{4}$$

Equations (3) and (4) are the *state-space representation/model* of the system.

Remarks:

1) State-space equation (3) and (4) are solved using techniques that are beyond the scope of this course. For those who are interested, see chapter 10 in the textbook.

2) The general form of the state-space representation is given by:

$$\dot{q} = \mathrm{A}q + \mathrm{B}x$$

$$y = \mathrm{C}q + \mathrm{D}x$$

where x is a vector of one or more inputs, y is a vector of one or more outputs, q is a vector of states, and $\mathrm{A}, \mathrm{B}, \mathrm{C}$ and D are matrices.

3) State-space representation is more powerful tool than input-output representation. It can be applied to both SISO systems and MIMO systems, and it is also very useful in the study of stability. However, in this course, we mainly consider the solutions for the input-output models since they are much easier than the solutions for the state-space models.

- **<u>Review of Basic Mechanical Components</u>**

  – <u>Mass</u>:

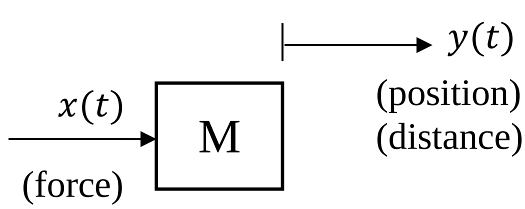

Figure 55

Relation: $x(t) = M\frac{d^2y(t)}{dt^2}$ (Newton′s Law)

For more than one force: $\sum x(t) = M \frac{d^2y(t)}{dt^2}$

Where the force x is in Newton (N), the distance y is in meter (m), and the mass M is in kilogram (kg).

– <u>Linear Spring</u>:

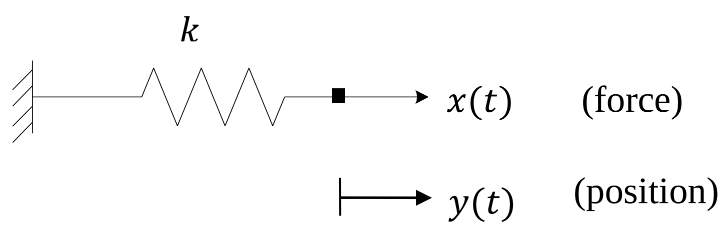

Figure 56

Relation: $x(t) = ky(t)$ where $k =$ *Stiffness* of the spring

– <u>Linear Dashpot (Shock Absorber)</u>:

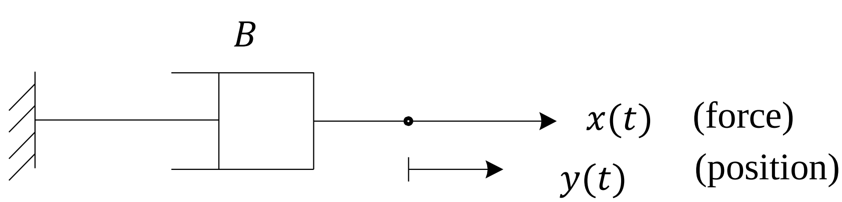

Figure 57

Relation: $x(t) = B \frac{dy(t)}{dt}$

Where $B =$ *damping coefficient* or *viscous friction coefficient*.

Example:

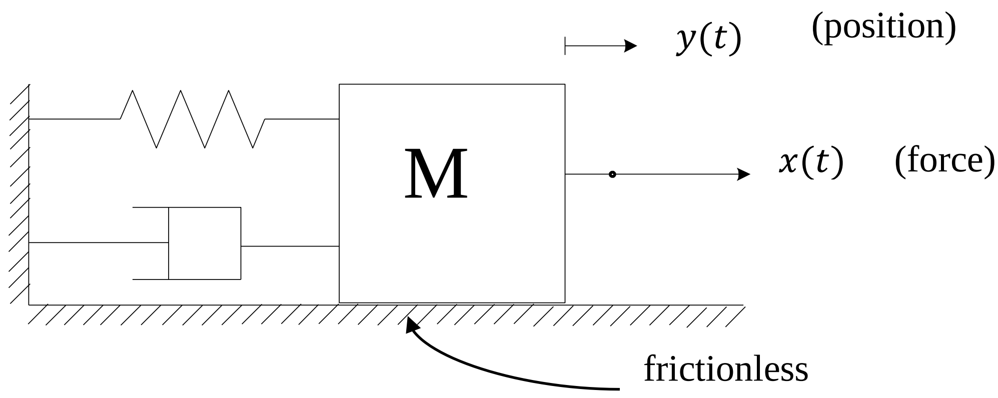

Figure 58

Free-body diagram

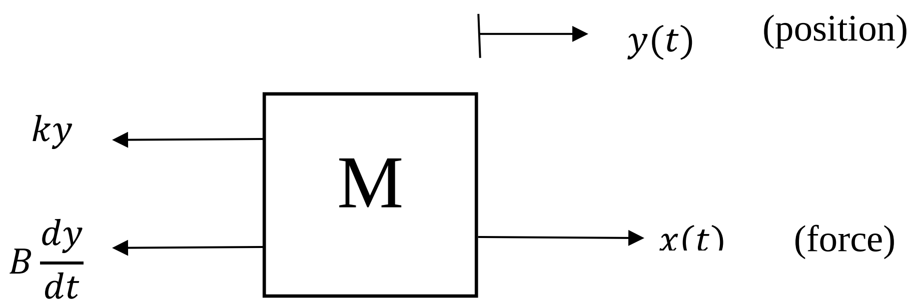

Figure 59

Newton's law:

$$M\frac{d^2y}{dt^2} = x - ky - B\frac{dy}{dt}$$

or

$$M\frac{d^2y}{dt^2} + B\frac{dy}{dt} + ky = x$$

Again, we get a differential equation, similar to the differential equation we found before for the RLC circuit.

- **D-operator Notation**

$$\frac{dy(t)}{dt} = \dot{y} = \mathrm{D}y$$

$$\frac{d^2y(t)}{dt^2} = \ddot{y} = \mathrm{D}^2y$$

$$\vdots$$

$$\frac{d^ny(t)}{dt^n} = y^{(n)} = \mathrm{D}^ny$$

The D-operator is very useful in the manipulation and solution of differential equations. For example, the D-operator is very convenient in finding the input-output model for systems described by a set of differential equations, as shown in the next example.

Example: (Simultaneous Differential Equations)

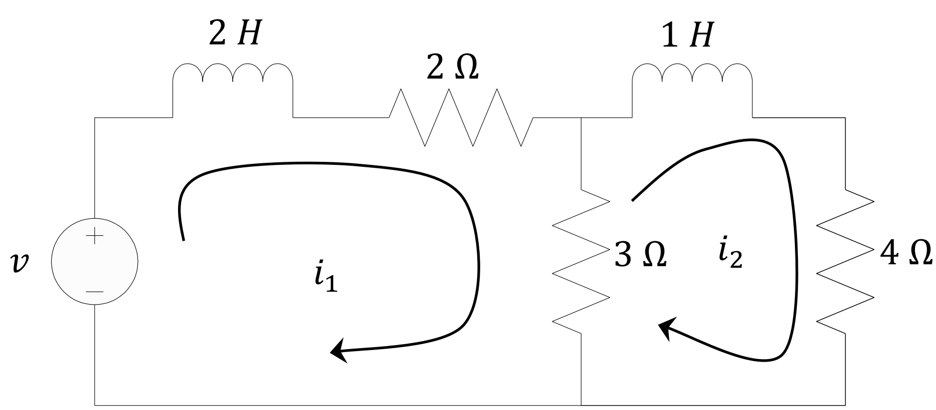

Figure 60

Find the differential equation that relates the output $y = i_1$ and the input $x = v$?

Using loop analysis,

KVL1:

$$-v + 2\frac{di_1}{dt} + 2i_1 + 3(i_1 - i_2) = 0$$

$$\Rightarrow \quad (2D + 5)i_1 - 3i_2 = v \qquad (1)$$

KVL2:

$$3(i_2 - i_1) + \frac{di_2}{dt} + 4i_2 = 0$$

$$\Rightarrow \quad -3i_1 + (\mathrm{D} + 7)i_2 = 0 \qquad (2)$$

Differential equation (1) and (2) can be written in matrix form as:

$$\begin{bmatrix} (2\mathrm{D} + 5) & -3 \\ -3 & (\mathrm{D} + 7) \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} v \\ 0 \end{bmatrix} \qquad (3)$$

We are not going to solve these equations since D is an operator, not a constant number. Instead, we will use a modified version of the Cramer's rule to find a single differential equation relating the output $y = i_1$ and the input $x = v$.

<u>Recall</u>: (Cramer's Rule)

$$\underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}}_{\mathrm{A}} \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{\mathrm{x}} = \underbrace{\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}}_{\mathrm{B}}$$

Define A_i = matrix A with i^{th} column replaced with B, $i = 1,2, \dots, n$

Cramer's rule: $\quad x_i = \frac{\det(A_i)}{\det(A)} \quad i = 1,2, \dots, n$

This can be written in a modified form as:

$$\det(A)\, x_i = \det(A_i)$$

Applying the modified Cramer's rule for i_1 in equation (3):

$$\det\begin{bmatrix}(2\mathrm{D}+5) & -3\\ -3 & (\mathrm{D}+7)\end{bmatrix} i_1 = \det\begin{bmatrix} v & -3\\ 0 & (\mathrm{D}+7)\end{bmatrix}$$

$$\Rightarrow \quad [(2\mathrm{D}+5)(\mathrm{D}+7)-9]\underbrace{i_1}_{y} = (\mathrm{D}+7)\underbrace{v}_{x}$$

$$\Rightarrow \quad (2\mathrm{D}^2+19\mathrm{D}+26)y = (\mathrm{D}+7)x$$

or

$$2\frac{d^2y}{dt^2}+19\frac{dy}{dt}+26y = \frac{dx}{dt}+7x$$

This linear differential equation is the system model for the circuit.

<u>Outline of the rest of topics in this course</u>:

We will study the class of "LTI" and "Causal" systems and described by:

- Differential equations (CT)
- Difference equations (DT)

Chapter 2: Time-Domain Analysis of CT systems

Chapter 3: Time-Domain Analysis of DT systems

Chapter 4: Frequency-Domain Analysis of CT systems (Laplace)

Chapter 5: Frequency-Domain Analysis of DT systems (Z-Transform)

Chapter 6: Fourier Series

Chapter 7: Fourier Transform