Continuous Random Variables

Appendix [Ross S5.4]

The result below shows that a Gaussian pdf has unit area under its curve.

Proposition 21.1
$$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = 1$$

Why?

Let $u = (x-\mu)/\sigma$ and $du = dx/\sigma$. So

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{u^2}{2}}\,du$$

So we just need to show that

$$I = \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}}\,du = \sqrt{2\pi}$$

We will show that $I^2 = 2\pi$:

$$\begin{aligned}
I^2 &= \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}}\,dxdy \\
&= \int_{0}^{2\pi}\int_{0}^{\infty} e^{-\frac{r^2}{2}}\,rdrd\theta \\
&= \int_{0}^{2\pi}\left[-e^{-\frac{r^2}{2}}\right]_0^{\infty} d\theta \\
&= \int_{0}^{2\pi} 1 d\theta \\
&= 2\pi
\end{aligned}$$

where we switched from Cartesian (x, y) coordinates to polar (r, θ) coordinates.