Random Variables (rv)

Functions of a Random Variable [Ross S4.4]

Say we have a random variable X. Let $Y = g(X)$ for some function $g(.)$. Then:

- X is a function of the outcome $s \in S$
- Y is a function of X

$\Rightarrow$ Y is a function of the outcome $s \in S$

$\Rightarrow$ Y is a random variable.

Y has a PMF $p_Y(y)$. We can find it from $p_X(x)$.

Example 10.1: Let X be a random variable such that

$$P[X = -1] = 0.1, \qquad P[X = 0] = 0.3, \qquad P[X = 1] = 0.6.$$

Let $Y = X^2$. What are $E[X]$ and $E[Y]$?

Solution:

So $E[g(X)] \neq g(E[X])$ in general.

Proposition 10.1 *If X is a rv with possible values* $\mathcal{X} = \{x_1, x_2, \ldots\}$ *then*

$$E[g(X)] = \sum_{i \geq 1} g(x_i) p_X(x_i)$$

Why is this true? Let $Y = g(X)$.

Let $\mathcal{Y} = \{y_1, y_2, \ldots\}$ be all possible values of Y.

$$
\begin{aligned}
\sum_{i \geq 1} g(x_i) p_X(x_i) &= \sum_{j \geq 1} \sum_{i : g(x_i) = y_j} g(x_i) p_X(x_i) \\
&= \sum_{j \geq 1} \sum_{i : g(x_i) = y_j} y_j p_X(x_i) \\
&= \sum_{j \geq 1} y_j \sum_{i : g(x_i) = y_j} p_X(x_i) \\
&= \sum_{j \geq 1} y_j P[g(X) = y_j] \\
&= \sum_{j \geq 1} y_j P[Y = y_j] \\
&= E[Y] \\
&= E[g(X)]
\end{aligned}
$$

Example 10.2: In Example 10.1,

$$
\begin{aligned}
E[X^2] &= \sum_i x_i^2 p_X(x_i) \\
&= (-1)^2 \times p_X(-1) + 0^2 \times p_X(0) + 1^2 \times p_X(1) \\
&= 1 \times 0.1 + 0 \times 0.3 + 1 \times 0.6 \\
&= 0.7
\end{aligned}
$$

Corollary 10.1 *If a and b are constants, then $E[aX + b] = aE[X] + b$.*

Why?

$$
\begin{aligned}
E[aX + b] &= \sum_{x \in \mathcal{X}} (ax + b) p_X(x) \\
&= a \sum_{x \in \mathcal{X}} x p_X(x) + b \sum_{x \in \mathcal{X}} p_X(x)
\end{aligned}
$$

$$= aE[X] + b$$

Example 10.3: Say $E[X] = 3$. Then $E[10X + 4] = 10 \times 3 + 4 = 34$.

Note: $E[X]$ is called **mean** of X. $E[X^n]$ is called the n-th **moment** of X.

Often write $\mu_X = E[X]$.

Variance [Ross S4.5]

Given X, it is useful to summarize some essential properties of X.

$E[X]$ tells us about the "center" of how X is distributed.

Example 10.4: Let

$$P[W = 0] = 1$$
$$P[Y = 1] = P[Y = -1] = \frac{1}{2}$$
$$P[Z = 100] = P[Z = -100] = \frac{1}{2}$$

Then $E[W] = 0 = E[Y] = E[Z]$, but these are not equally spread...

Definition 10.1: The **variance** of X is

$$\begin{aligned} Var[X] &= E[\ (X - E[X])^2\ ] \\ &= E[\ (X - \mu_X)^2\ ] \end{aligned}$$

We often write $\sigma_X^2 = Var[X]$.

Note: Since $(X - \mu_X)^2 \geq 0$, then $Var[X] \geq 0$. $\qquad (*)$

Also
$$\begin{aligned}
Var[X] &= E[\,(X - \mu_X)^2\,] \\
&= \sum_{x \in \mathcal{X}} (x - \mu_X)^2 \, p_X(x) \\
&= \sum_{x \in \mathcal{X}} \left(x^2 - 2\mu_X x + \mu_X^2\right) p_X(x) \\
&= \sum_{x \in \mathcal{X}} x^2 p_X(x) - 2\mu_X \underbrace{\sum_{x \in \mathcal{X}} x p_X(x)}_{\mu_X} + \mu_X^2 \sum_{x \in \mathcal{X}} p_X(x) \\
&= E[X^2] - 2\mu_X^2 + \mu_X^2 \qquad\qquad (\ \mu_X = E[X]\ ) \\
&= E[X^2] - (E[X])^2 \qquad\qquad (10.1)
\end{aligned}$$

Also, combinining $(*)$ with (10.1), we get

$$E[X^2] \geq (E[X])^2 \qquad (10.2)$$

and, if $E[X] > 0$, then

$$\frac{E[X^2]}{E[X]} \geq E[X] \qquad (10.3)$$

Example 10.5: Let X be the the outcome of a dice roll. What is $Var[X]$?

Solution:

Example 10.6: The distance from Vancouver to Boston is 4200km. If the wind is good (with probability 0.7), the speed of a plane is $V = 700$ km/h. If the wind is not good (probability 0.3), the speed is $V = 600$ km/h.

What is the average flight time?

Solution: