Properties of Expectations

Expectation of Sums of Random Variables [Ross S7.2]

Recall that the mean value of X is

$$E[X] = \begin{cases} \sum_x x p_X(x) & X \text{ is discrete} \\ \\ \int_{-\infty}^{\infty} x f_X(x) dx & X \text{ is continuous} \end{cases}$$

Proposition 30.1 *Let X and Y be two random variables. Let $g(x, y)$ be a function. Then*

$$E[g(X,Y)] = \begin{cases} \sum_y \sum_x g(x,y) p_{XY}(x,y) & X, Y \textit{ are discrete} \\ \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{XY}(x,y) dx dy & X, Y \textit{ are continuous} \end{cases}$$

Why?

[Only show for continuous case and $g(x, y)$ is non-negative]

Recall from Proposition 16.2:

$$E[Z] = \int_0^{\infty} P[Z > t] dt$$

So

$$
\begin{aligned}
E[g(X,Y)] &= \int_0^\infty P[g(X,Y) > t]dt \\
&= \int_0^\infty \iint\limits_{(x,y):g(x,y)>t} f_{XY}(x,y)dxdy\ dt \\
&= \iint\limits_{\mathbb{R}^2} \int_0^{g(x,y)} f_{XY}(x,y)dt\ dxdy \\
&= \iint\limits_{\mathbb{R}^2} g(x,y) f_{XY}(x,y)dxdy
\end{aligned}
$$

Example 30.1: The positions $X \sim U(0,L)$ and $Y \sim U(0,L)$ of two persons on a road are independent. What is the mean distance between them?

Solution: Here,

$$
f_{XY}(x,y) = \begin{cases} \frac{1}{L^2} & 0 < x < L, 0 < y < L \\ 0 & \text{else} \end{cases}
$$

We want

$$
\begin{aligned}
E[|X-Y|] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x-y| f_{XY}(x,y)dxdy \\
&= \frac{1}{L^2}\int_0^L\int_0^L |x-y|dxdy
\end{aligned}
$$

Now $\quad \displaystyle\int_0^L |x-y|dx = \int_0^y (y-x)dx + \int_y^L (x-y)dx$

$$= \frac{L^2}{2} + y^2 - yL$$

So $\quad E[|X-Y|] = \frac{1}{L^2}\int_0^L \frac{L^2}{2} + y^2 - yL\, dy$
$$= \frac{L}{3}$$

Example 30.2: Show $E[X+Y] = E[X] + E[Y]$.

Solution: [Continuous case only, discrete is similar]

With $g(x,y) = x+y$:

$$\begin{aligned} E[X+Y] &= E[g(X,Y)] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{XY}(x,y) dx dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y) f_{XY}(x,y) dx dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{XY}(x,y) dx dy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{XY}(x,y) dx dy \\ &= E[X] + E[Y] \end{aligned}$$

Note: by induction, $E[X_1 + \cdots + X_n] = E[X_1] + \cdots + E[X_n]$.

Example 30.3: Let $X_1, X_2, \ldots, X_n$ be iid with (common) mean μ. The quantity

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

is called the **sample mean**. What is $E[\bar{X}]$?

Solution:

$$\begin{aligned}
E[\bar{X}] &= E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] \\
&= \frac{1}{n}E\left[\sum_{i=1}^{n} X_i\right] \\
&= \frac{1}{n}\sum_{i=1}^{n} E\left[X_i\right] \\
&= \frac{1}{n}\sum_{i=1}^{n} \mu \\
&= \mu
\end{aligned}$$

Example 30.4: 10 friends play a game. Each has a ball and picks one of 10 targets randomly, independently of the others. Each has probability p of hitting their chosen target.

What is the expected number of targets not hit?

Solution:

Let $X_i = 1$ if target i is not hit, and 0 otherwise; $X =$ # targets not hit.

Each person independently hits target i with probability $p/10$.

So
$$\begin{aligned} P[X_i = 1] &= \left(1 - \frac{p}{10}\right)^{10} \\ E[X_i] &= 1 \times P[X_i = 1] + 0 \times P[X_i = 0] \\ &= \left(1 - \frac{p}{10}\right)^{10} \end{aligned}$$

$$\begin{aligned} E[X] &= E[X_1 + \cdots + X_{10}] \\ &= E[X_1] + \cdots + E[X_{10}] \\ &= 10\left(1 - \frac{p}{10}\right)^{10} \end{aligned}$$