Signals and Systems

Lecture Notes

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Chapter 2

Time-Domain Analysis of Continuous-Time Systems

2.1 Introduction

A large class of linear and time-invariant (LTI) systems (electrical, mechanical, etc) is described by linear differential equations with constant coefficients of the form:

$$\frac{d^N y(t)}{dt^N} + a_{N-1}\frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t)$$
$$= b_M \frac{d^M x(t)}{dt^M} + b_{M-1}\frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_1 \frac{dx(t)}{dt} + b_0 x(t)$$

where $a_0, \cdots, a_{N-1}, b_0, \cdots, b_M$ are constants and M, N are integers.

Using the D-operator notation $\left(D^n \triangleq \frac{d^n}{dt^n}\right)$:

$$(D^N + a_{N-1}D^{N-1} + \cdots + a_1 D + a_0)y(t)$$
$$= (b_M D^M + b_{M-1}D^{M-1} + \cdots + b_1 D + b_0)x(t)$$

or in short notation:

$$Q(D)y(t) = P(D)x(t)$$

where

$$Q(D) = D^N + a_{N-1}D^{N-1} + \cdots + a_1 D + a_0$$

$$P(D) = b_M D^M + b_{M-1}D^{M-1} + \cdots + b_1 D + b_0$$

are polynomials in *D*.

Remark**:** The case $M > N$ is impractical due to reasons that will be shown later. Practical systems have $M \leq N$. We will use $M = N$ for generality.

- **Total System Response**

The response or solution $y(t)$ for $t \geq 0$ of an LTI system described by the differential equation given above can be expressed as the sum of two components:

$$y(t) = y_{zi}(t) + y_{zs}(t), \qquad t \geq 0$$

where

$y_{zi}(t)$ = *Zero-input response* due to internal *initial conditions* (ICs), usually given at $t = 0$, with $x(t) = 0$ for $t \geq 0$

$y_{zs}(t)$ = *Zero-state response* due to $x(t)$ for $t \geq 0$ when the system is in *zero state*, i.e. ICs(0) = 0

2.2 Zero-Input Response

For $t \geq 0$, $y(t) = y_{zi}(t)$ when $x(t) = 0$, i.e.

$$Q(D)y_{zi}(t) = 0, \qquad t \geq 0$$

or

$$(D^N + a_{N-1}D^{N-1} + \cdots + a_1 D + a_0)y_{zi}(t) = 0, \qquad t \geq 0 \qquad (1)$$

with *N* initial conditions.

To find $y_{zi}(t)$, we assume a solution of the form:

$$y_{zi}(t) = ce^{\lambda t}$$

where c and λ are constants (real or complex) that need to be determined.

then

$$Dy_{zi} = c\lambda e^{\lambda t}$$

$$D^2 y_{zi} = c\lambda^2 e^{\lambda t}$$

$$\vdots$$

$$D^N y_{zi} = c\lambda^N e^{\lambda t}$$

Substituting in (1):

$$c(\lambda^N + a_{N-1}\lambda^{N-1} + \cdots + a_1\lambda + a_0)e^{\lambda t} = 0$$

For non-trivial solution, $\{c \neq 0,\ e^{\lambda t} \neq 0\}$, we must have:

$$Q(\lambda) = \underbrace{\lambda^N + a_{N-1}\lambda^{N-1} + \cdots + a_1\lambda + a_0}_{\text{Characteristic polynomial in } \lambda} = 0$$

This $Q(\lambda) = 0$ is called the *characteristic equation*. It has *N* roots, and can be written in factored form:

$$Q(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_N) = 0$$

where $\{\lambda_1, \lambda_2, \cdots, \lambda_N\}$ are called the *characteristic roots* of the system. They are also called the *characteristic values*, *eigenvalues*, or *natural frequencies*.

- **Case 1 (Distinct Roots)**

  When the roots $\{\lambda_1, \lambda_2, \cdots, \lambda_N\}$ are distinct, the system in (1) has *N* possible solutions:

$$c_1 e^{\lambda_1 t},\ c_2 e^{\lambda_2 t},\ \cdots\ ,\ c_N e^{\lambda_N t}$$

  and the sum is the general solution of $Q(D)y_{zi}(t) = 0$, i.e.

$$y_{zi}(t) = \sum_{i=1}^{N} c_i e^{\lambda_i t} = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \cdots + c_N e^{\lambda_N t}, \quad t \geq 0 \qquad (2)$$

where $c_1, c_2, \cdots, c_N$ are constants that are determined from *N* constraints (initial conditions).

Remark: The initial conditions $\{y(0), Dy(0), \cdots, D^{N-1}y(0)\}$ at $t = 0$ are sometimes derived from $\{y(0^-), Dy(0^-), \cdots, D^{N-1}y(0^-)\}$ at $t = 0^-$ as we will see later.

Remark: The exponentials $\{e^{\lambda_1 t}, e^{\lambda_2 t}, \cdots, e^{\lambda_N t}\}$ are called the *characteristic modes* or *natural modes* of the system.

- **Case 2 (Repeated Roots)**

If there are repeated roots, the form of the solution is modified slightly. For example, if $Q(\lambda) = (\lambda - \lambda_1)^2$, then $c_1 e^{\lambda_1 t}$ and $c_2 t e^{\lambda_1 t}$ are two independent possible solutions, and the zero-input response is given by:

$$y_{zi}(t) = c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_1 t}$$

In general, for *r*-repeated root, i.e. $Q(\lambda) = (\lambda - \lambda_1)^r$, the solution is:

$$y_{zi}(t) = c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_1 t} + \cdots + c_r t^{r-1} e^{\lambda_1 t} = \sum_{i=1}^{r} c_i t^{i-1} e^{\lambda_1 t}$$

Example:

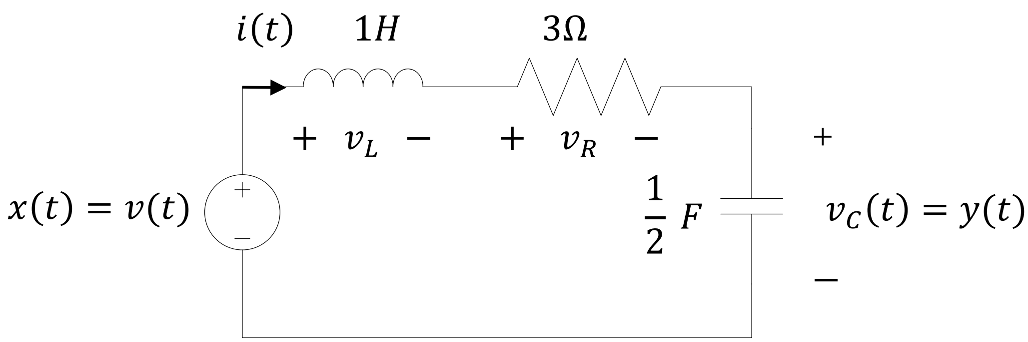

Figure 1

Given $i(0^-) = 2A$ and $v_C(0^-) = 5V$, find the system response $y_{zi}(t)$?

Solution:

Step1: (Derive the differential equation or system model)

KVL: $v_L + v_R + v_C = v$

or $L\frac{di}{dt} + Ri + y = x$ (3)

where, from capacitor:

$$i = C\frac{dv_c}{dt} = \frac{1}{2}\frac{dy}{dt} = \frac{1}{2}Dy$$

$$\frac{di}{dt} = C\frac{d^2v_C}{dt^2} = \frac{1}{2}\frac{d^2y}{dt^2} = \frac{1}{2}D^2y$$

Substituting in (3):

$$\frac{1}{2}D^2y + 3\frac{1}{2}Dy + y = x$$

$$\Rightarrow \quad \underbrace{(D^2 + 3D + 2)}_{Q(D)}y(t) = \underbrace{2}_{P(D)}x(t)$$

Step2: (Find the characteristic modes)

From the characteristic equation $Q(\lambda) = \lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2) = 0$, the system roots are $\{\lambda_1 = -1,\ \lambda_2 = -2\}$ and the system modes are $\{e^{-t},\ e^{-2t}\}$.

And, the solution is:

$$y_{zi}(t) = c_1 e^{-t} + c_2 e^{-2t}, \qquad t \geq 0$$

where c_1 and c_2 are constants that need to be determined.

Step3: (Find initial conditions $\{y_{zi}(0), Dy_{zi}(0)\}$)

They need to be derived from the given $i(0^-)$ and $v_C(0^-)$.

Remark: Inductor current cannot change instantaneously (i_L has no jumps), i.e. $i_L(0^-) = i_L(0)$. Also, capacitor voltage cannot change instantaneously (v_C has no jumps), i.e. $v_C(0^-) = v_C(0)$.

Therefore,

$$y_{zi}(0) = v_C(0) = v_C(0^-) = 5V$$

To find $Dy_{zi}(0)$:

$$Dy_{zi}(0) = \frac{1}{C} i(0) = \frac{1}{C} i(0^-) = \frac{1}{1/2}.2 = 4 \ \text{V/S}$$

Step4: (Find the constants $c_1 \ and \ c_2$)

From $$y_{zi}(t) = c_1 e^{-t} + c_2 e^{-2t}, \tag{4}$$

$$\Rightarrow \quad Dy_{zi}(t) = -c_1 e^{-t} - 2c_2 e^{-2t} \tag{5}$$

Set $t = 0$ in (4) and (5):

$$y_{zi}(0) = 5 = c_1 + c_2 \tag{6}$$

$$Dy_{zi}(0) = 4 = -c_1 - 2c_2 \tag{7}$$

Solving the linear equations (6) and (7):

$$c_1 = 14, \quad c_2 = -9$$

Substituting in (4):

$$y_{zi}(t) = 14e^{-t} - 9e^{-2t}, \qquad t \geq 0$$

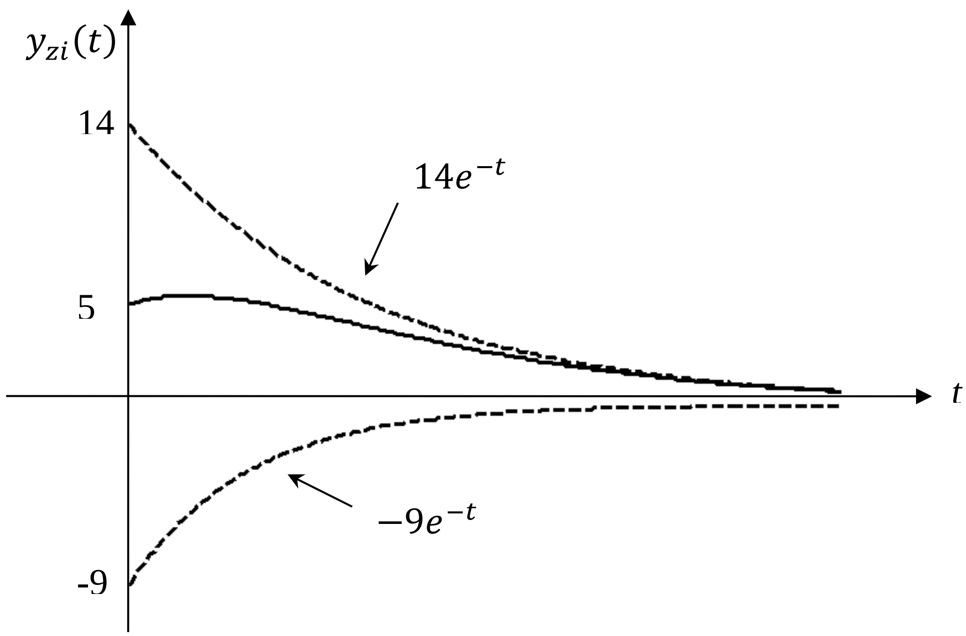

Figure 2

Example: An LTI system is specified by:

$$(D^2 + 2D + 1)y(t) = (D + 3)x(t)$$

Find $y_{zi}(t)$ if $y_{zi}(0) = Dy_{zi}(0) = 1$

Solution:

$$Q(\lambda) = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2$$

$$\Rightarrow \quad \lambda_1 = \lambda_2 = -1 \qquad \text{(repeated root)}$$

$$\Rightarrow \quad y_{zi}(t) = c_1 e^{-t} + c_2\, te^{-t}, \qquad t \geq 0$$

Find c_1 and c_2:

$$y_{zi}(0) = 1 = c_1 + 0, \qquad \Rightarrow \quad c_1 = 1$$

$$Dy_{zi}(t) = -c_1 e^{-t} + c_2(e^{-t} - te^{-t})$$

$$Dy_{zi}(0) = 1 = -c_1 + c_2(1 - 0) \qquad \Rightarrow \quad c_2 = 2$$

$$\Rightarrow \quad y_{zi}(t) = (e^{-t} + 2te^{-t})u(t)$$

- **Case 3 (Complex Roots)**

  For complex roots, the procedure is the same as for real roots. This leads in general to complex modes and complex solution. However, for *real systems*, the final solution must be also real. In this case, the complex form of the solution can be avoided as shown next.

  For real systems, complex roots occur in pairs of conjugates, i.e. if $\lambda_1 = \alpha + j\beta$ is a root of $Q(\lambda)$, then $\lambda_2 = \alpha - j\beta$ is also a root. For these two roots:

$$y_{zi}(t) = c_1 e^{(\alpha+j\beta)t} + c_2 e^{(\alpha-j\beta)t}, \qquad t \geq 0 \qquad \text{(Complex form)}$$

  Also, for real systems, since $y_{zi}(t)$ is real, then the constants c_1 and c_2 must be complex conjugates, i.e. if $c_1 = \frac{c}{2}e^{j\theta}$, then $c_2 = \frac{c}{2}e^{-j\theta}$

  The complex form of $y_{zi}(t)$ can be rewritten in a real form:

$$y_{zi}(t) = \frac{c}{2}e^{j\theta}e^{\alpha t}e^{j\beta t} + \frac{c}{2}e^{-j\theta}e^{\alpha t}e^{-j\beta t}$$

$$= \frac{c}{2}e^{\alpha t}\left[e^{j(\beta t+\theta)} + e^{-j(\beta t+\theta)}\right]$$

$$\Rightarrow \qquad y_{zi}(t) = ce^{\alpha t}\cos(\beta t + \theta), \qquad t \geq 0 \qquad \text{(Real form)}$$

  <u>Example</u>: $(D^2 + 2D + 5)y = 3x$ with $y(0) = 2$ and $Dy(0) = -6$

  <u>Solution:</u>

$$Q(\lambda) = \lambda^2 + 2\lambda + 5$$

$$\text{Roots:} \quad \lambda_1, \lambda_2 = \underbrace{-1}_{\alpha} \pm j\underbrace{2}_{\beta}$$

$$\Rightarrow \quad y_{zi}(t) = ce^{-t}\cos(2t + \theta), \qquad t \geq 0$$

Find the constants c and θ:

$$Dy_{zi}(t) = -ce^{-t}\cos(2t+\theta) - 2ce^{-t}\sin(2t+\theta)$$

$$y_{zi}(0) = 2 = c\cos(\theta) \qquad (8)$$

$$Dy_{zi}(0) = -6 = -\underbrace{c\cos(\theta)}_{2} - 2c\sin(\theta)$$

$$\Rightarrow \quad c\sin(\theta) = 2 \qquad (9)$$

From (8) and (9):

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{2}{2} = 1 \qquad \Rightarrow \quad \theta = \frac{\pi}{4}$$

And from (8):

$$c = \frac{2}{\cos\theta} = \frac{2}{\cos\left(\frac{\pi}{4}\right)} = \frac{2}{1/\sqrt{2}} = 2\sqrt{2}$$

$$\Rightarrow \quad y_{zi}(t) = 2\sqrt{2}e^{-t}\cos\left(2t+\frac{\pi}{4}\right), \qquad t \geq 0$$

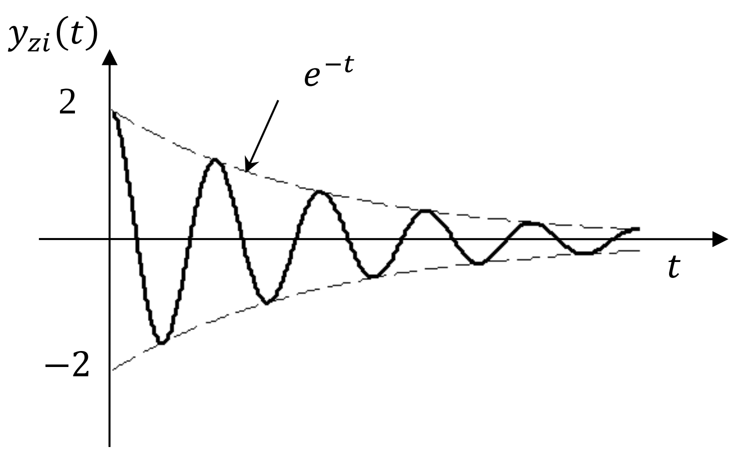

Figure 3

<u>Remark</u>: The angle $\theta = \frac{\pi}{4} - \pi = -\frac{3\pi}{4}$ is also valid; but it gives $c = -2\sqrt{2}$.

2.3 Unit Impulse Response

Given an LTI system described by $Q(D)y(t) = P(D)x(t)$; or (for $M = N$)

$$(D^N + a_{N-1}D^{N-1} + \cdots + a_1 D + a_0)y(t)$$
$$= (b_N D^N + b_{N-1}D^{N-1} + \cdots + b_1 D + b_0)x(t)$$

The *unit impulse response* $h(t)$ is the *zero-state response* $y_{zs}(t) = h(t)$ due to a unit impulse input; i.e.

$$x(t) = \delta(t) \quad \rightarrow \quad y(t) = y_{zs}(t) = h(t)$$

or
$$Q(D)h(t) = P(D)\delta(t)$$

<u>Remark</u>: Zero-state response implies <u>zero initial conditions</u> at $t = 0$, or for convenience at $t = 0^-$ since $\delta(t)$ is applied at $t = 0$.

- **Nature of $\boldsymbol{h(t)}$**

Applying $\delta(t)$ at $t = 0$ creates *non-zero initial conditions* at $t = 0^+$.

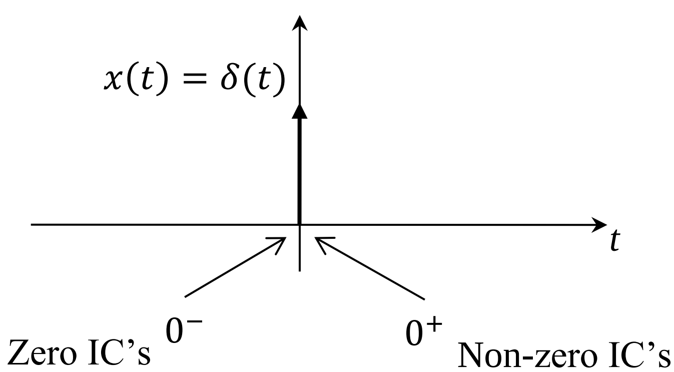

Figure 4

For $t \geq 0^+$, $x(t) = \delta(t) = 0$ and $y(t) = h(t)$ is similar to a zero-input response due to the ICs created by $\delta(t)$ at $t = 0^+$, i.e. $h(t)$ contains characteristics modes for $t \geq 0^+$ since $y_{zi}(t)$ contains characteristic modes.

For $t = 0$, $h(t)$ can have $\delta(t), \dot{\delta}(t), \ldots\ldots$ However, for $M \le N$, $h(t)$ can have at most $\delta(t)$ component (when $M = N$) only. The derivatives do not appear.

Above discussion gives:

$$\boxed{h(t) = A_0\delta(t) + \text{characteristic modes}\,, \;\; t \ge 0}$$

where A_0 is a constant.

<u>Example:</u> (Finding $h(t)$ by matching functions)

Let $(D+1)y(t) = (D+2)x(t)$, find $h(t)$?

<u>Solution:</u>

$Q(\lambda) = \lambda + 1 = 0 \;\; \Rightarrow \;\; \lambda = -1 \;\; \Rightarrow$ Characteristic mode = e^{-t}.

$$\therefore \;\; h(t) = A_0\delta(t) + [Ce^{-t}]u(t)$$

where A_0 and C are constants. Notice that we must multiply by $u(t)$ since characteristic modes are valid for $t \ge 0$. Taking the derivative:

$$\dot{h}(t) = A_0\dot{\delta}(t) + [-Ce^{-t}]u(t) + \underbrace{[Ce^{-t}]\delta(t)}_{=C\delta(t)}$$

Substituting in the system equation $(D+1)h(t) = (D+2)\delta(t)$, gives:

$$\dot{h}(t) + h(t) = \dot{\delta}(t) + 2\delta(t)$$

$$\Rightarrow \;\; A_0\dot{\delta}(t) + [-Ce^{-t}]u(t) + C\delta(t) + A_0\delta(t) + [Ce^{-t}]u(t) = \dot{\delta}(t) + 2\delta(t)$$

$$\Rightarrow \;\; A_0\dot{\delta}(t) + C\delta(t) + A_0\delta(t) = \dot{\delta}(t) + 2\delta(t)$$

Matching terms in LHS and RHS gives:

$$A_0 = 1$$

$$C + A_0 = 2 \quad \Rightarrow \quad C = 1$$

$$\Rightarrow \quad h(t) = \delta(t) + e^{-t}u(t)$$

<u>Remark</u>: The term $A_0\delta(t)$ in $h(t)$ does not exist if $M < N$, i.e. when order of $P(D) <$ order of $Q(D)$.

<u>Remark</u>: It can be shown that $A_0 = b_N$, where b_N is the coefficient of D^N in $P(D)$ (see textbook for justification).

- **Alternate Derivation of $\boldsymbol{h}(\boldsymbol{t})$**

It is shown in section 2.8 of the textbook that the impulse response for $M \leq N$ is given by:

$$h(t) = b_N\delta(t) + [P(D)y_n(t)]u(t)$$

where $b_N =$ coefficients of N^{th} order term in $P(D)$, and $y_n(t) =$ specific zero input response due to the following specific initial conditions:

$$\left\{\begin{matrix} y_n(0) = 0 \\ Dy_n(0) = 0 \\ \vdots \\ D^{N-2}y_n(0) = 0 \\ D^{N-1}y_n(0) = 1 \end{matrix}\right\}$$

<u>Remark</u>: For any M and N, the impulse response is given by:

$$h(t) = P(D)[y_n(t)u(t)]$$

Example: An LTI system is given by $(D^2 + 3D + 2)y(t) = (D - 2)x(t)$. Find the impulse response $h(t)$?

Solution:

$$Q(\lambda) = \lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2)$$

$$\Rightarrow \lambda_1 = -1, \qquad \lambda_2 = -2$$

$$\Rightarrow y_n(t) = C_1 e^{-t} + C_2 e^{-2t} \tag{10}$$

$$Dy_n(t) = -C_1 e^{-t} - 2C_2 e^{-2t} \tag{11}$$

Substituting ICs $\{ y_n(0) = 0$ and $Dy_n(0) = 1\}$ in (10) and (11):

$$0 = C_1 + C_2$$
$$1 = -C_1 - 2C_2$$

$$\Rightarrow \quad C_1 = 1,\ C_2 = -1 \quad \Rightarrow \quad y_n(t) = e^{-t} - e^{-2t}$$

Using $h(t) = b_N \delta(t) + [P(D) y_n(t)]u(t)$ with $b_N = 0$ {since there is no D^2 term in $P(D)$}:

$$h(t) = [P(D)y_n(t)]u(t) = [(D-2)y_n(t)]u(t) = [Dy_n(t) - 2y_n(t)]u(t)$$

$$= [-e^{-t} + 2e^{-2t} - 2e^{-t} + 2e^{-2t}]u(t)$$

$$\Rightarrow \quad h(t) = (-3e^{-t} + 4e^{-2t})u(t)$$

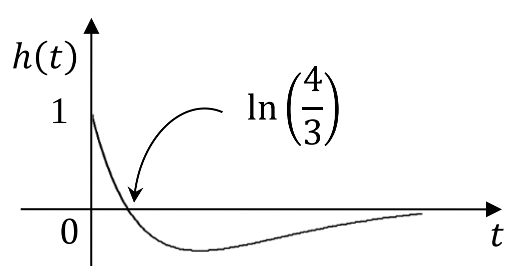

Figure 5

- **Unit Step Response**

  Fact: For an LTI system S where $h(t) = S[\delta(t)]$ is the unit impulse response and $g(t) = S[u(t)]$ is the unit step response:

$$g(t) = \int_{-\infty}^{t} h(\tau)\, d\tau$$

$$h(t) = \frac{dg(t)}{dt}$$

  Proof: Change variable $t \ to \ \tau$: $h(\tau) = S[\delta(\tau)]$

  Integrate both sides: $\int_{-\infty}^{t} h(\tau)\, d\tau = \int_{-\infty}^{t} S[\delta(\tau)]\, d\tau$

  Interchange $\int$ and S due to linearity of both operators:

$$\int_{-\infty}^{t} h(\tau)\, d\tau = S\underbrace{\left[\int_{-\infty}^{t} \delta(\tau)\, d\tau\right]}_{u(t)} = S[u(t)] = g(t)$$

  Taking the derivative of $g(t)$, gives: $h(t) = \frac{dg(t)}{dt}$

- **Causality and $\boldsymbol{h(t)}$**

  If the system is LTI and causal, then $h(t) = 0$ for $t < 0$. This is because the system cannot respond before applying $\delta(t)$ at $t = 0$.

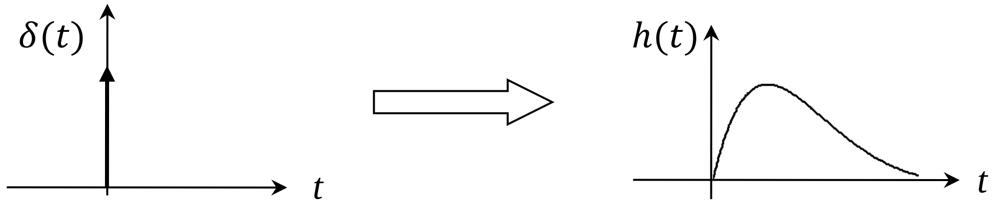

Figure 6

2.4 Zero-State Response

Let S be an LTI system $y(t) = S[x(t)]$. The zero state response $y(t) = y_{zs}(t)$ is the response of the system due to the input $x(t)$ only with ICs = 0.

The impulse response of the system S is: $h(t) = S[\delta(t)]$

By time-invariance: $h(t-\tau) = S[\delta(t-\tau)]$

Recall the sampling/sifting property: $x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$

then

$$y(t) = S[x(t)] = S\left[\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau\right]$$

Since S is a linear operator on functions of t, and the integration is a linear operator on function of τ, then we can interchange S and $\int_{-\infty}^{\infty}$, i.e.

$$y(t) = \int_{-\infty}^{\infty} S[x(\tau)\delta(t-\tau)d\tau]$$

Since $x(\tau)$ and $d\tau$ are not functions of t, and by linearity of S:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\underbrace{S[\delta(t-\tau)]}_{h(t-\tau)}d\tau$$

Remark: In last step, we used the linearity property of S: $S[az(t)] = aS[z(t)]$ for a = constant.

Finally,

$$\boxed{y(t) = y_{zs}(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau}$$

This relation is called the "*Convolution Integral*". It is valid for all values of t, $-\infty < t < \infty$.

- **Total Response**

When an LTI system has both non-zero input $x(t)$ and non-zero ICs, the total response is:

$$y(t) = y_{zi}(t) + y_{zs}(t) = y_{zi}(t) + \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Remark: The zero-input response $y_{zi}(t)$ contains the system characteristic modes only, while the zero-state response $y_{zs}(t)$ contains the system characteristic modes due to $h(t)$ in addition to other modes due to $x(t)$.

Remark: In practice, $y_{zs}(t)$ is more important than $y_{zi}(t)$ since most systems start from rest-state (IC's=0). Also, for large t, $y_{zi}(t) \to 0$ for stable practical systems.

Remark: If the LTI system is causal, i.e. $h(t) = 0$ for $t < 0$, and the input $x(t)$ is a causal signal, i.e. $x(t) = 0$ for $t < 0$, then

$$y(t) = \int_{-\infty}^{\infty} \underbrace{x(\tau)}_{\substack{=0 \\ \text{for } \tau<0}} \underbrace{h(t-\tau)}_{\substack{=0 \\ \text{for } t-\tau<0 \\ \text{or } \tau>t}} d\tau = \int_{0}^{t} x(\tau)h(t-\tau)d\tau\,, \quad t \geq 0$$

where $y(t) = 0$ for $t < 0$.

This means $y(t)$ is a causal signal, too.

2.5 The Convolution Integral

The convolution of $x_1(t)$ and $x_2(t)$ is defined by:

$$y(t) = x_1(t) * x_2(t) = (x_1 * x_2)(t) = \int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau) d\tau$$

<u>Properties:</u>

(1) <u>Commutative Property</u>: $x_1 * x_2 = x_2 * x_1$

This can be proved easily by a change of variables (see textbook).

(2) <u>Distributive Property</u>:

$$x_1 * [x_2 + x_3] = x_1 * x_2 + x_1 * x_3$$

(3) <u>Associative Property</u>:

$$x_1 * [x_2 * x_3] = [x_1 * x_2] * x_3$$

(4) <u>Shift Property</u>:

If $x_1(t) * x_2(t) = y(t)$

then $x_1(t) * x_2(t-T) = x_1(t-T) * x_2(t) = y(t-T)$

$x_1(t-T_1) * x_2(t-T_2) = y(t-T_1-T_2)$

(5) <u>Convolution with an Impulse</u>:

$$x(t) * \delta(t) = x(t)$$

This can be proved easily by using the sampling property of the impulse.

$$x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau) \delta(t-\tau) d\tau = x(t)$$

(6) Width Property:

If $x_1(t)$ has the finite duration (width) T_1 and $x_2(t)$ has the finite width T_2, then $x_1 * x_2$ has the finite width $T_1 + T_2$.

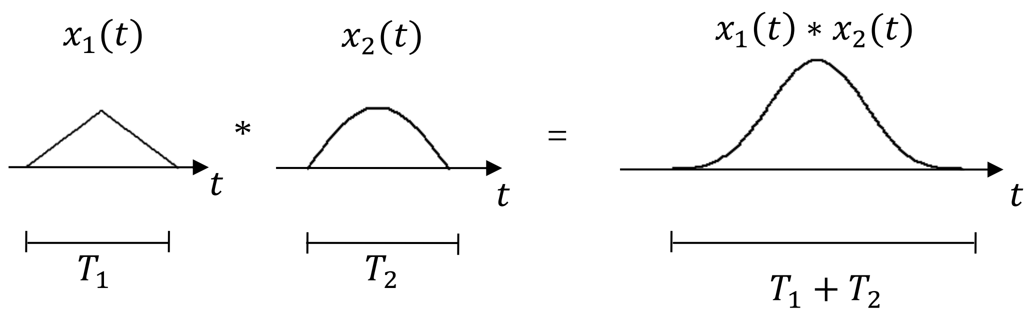

Figure 7

Remark: Convolution causes *spreading* and *smoothness*.

Above properties are useful when using the table of convolution (Table 2.1) given in the textbook.

Example: (Non-graphical Convolution)

Given: $x_1(t) = e^{\lambda_1 t}u(t), \quad x_2(t) = e^{\lambda_2 t}u(t)$

Find $x_1 * x_2$ where $\lambda_1 \neq \lambda_2$.

Solution:

$$x_1 * x_2 = \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau = \int_{-\infty}^{\infty} e^{\lambda_1 \tau}u(\tau).\, e^{\lambda_2 (t-\tau)}u(t-\tau)d\tau$$

Since

$$u(\tau) = \begin{cases} 0, & \tau < 0 \\ 1, & \tau \geq 0 \end{cases}$$

and

$$u(t-\tau) = \begin{cases} 0, & t-\tau < 0 \quad \text{or} \quad \tau > t \\ 1, & t-\tau \geq 0 \quad \text{or} \quad \tau \leq t \end{cases}$$

then

$$x_1 * x_2 = \int_0^t e^{\lambda_1 \tau}e^{\lambda_2 t}e^{-\lambda_2 \tau}d\tau = e^{\lambda_2 t}\int_0^t e^{(\lambda_1-\lambda_2)\tau}d\tau, \qquad t \geq 0$$

$$= e^{\lambda_2 t}\frac{1}{\lambda_1-\lambda_2}\left[e^{(\lambda_1-\lambda_2)t} - 1\right], \qquad t \geq 0$$

$$\Rightarrow \quad x_1 * x_2 = \frac{1}{\lambda_1-\lambda_2}\left[e^{\lambda_1 t} - e^{\lambda_2 t}\right]u(t)$$

Remark: This result is the entry 4 in Table 2.1 in textbook.

Example: (Graphical Convolution)

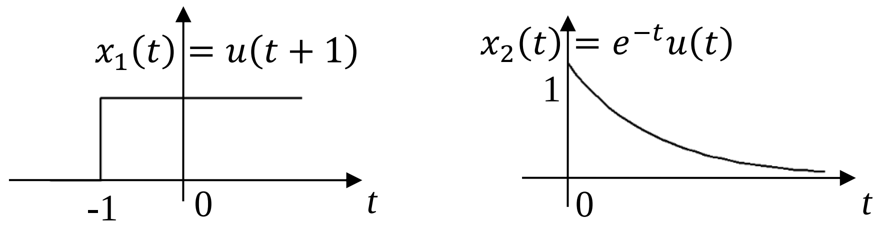

Figure 8

Find $x_1 * x_2$.

Solution:

From $\quad y(t) = x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau) d\tau$

Step 1: Graph $x_1(\tau)$ and $x_2(\tau)$, i.e. replace t by τ.

Step2: Flip $x_2(\tau)$ horizontally to get $x_2(-\tau)$:

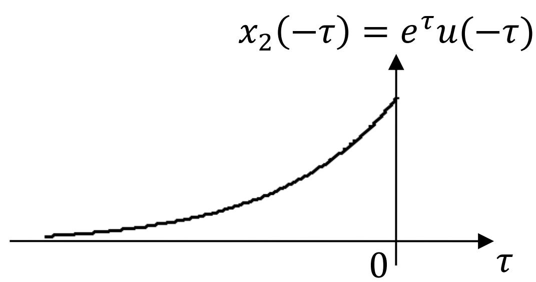

Figure 9

Step3: Shift $x_2(-\tau)$ to get $x_2(t-\tau)$:

Note: $\begin{cases} t > 0 & \text{shift to right} \\ t < 0 & \text{shift to left} \end{cases}$

$$x_2(t-\tau) = e^{-(t-\tau)}u(t-\tau)$$

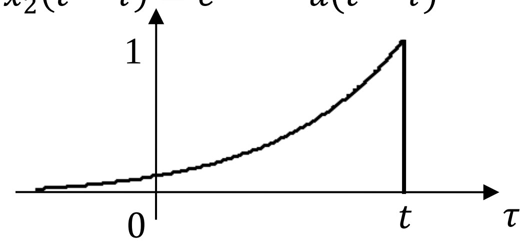

Figure 10

Step4: Multiply $x_1(\tau)$ by $x_2(t-\tau)$

$$x_1(\tau).\,x_2(t-\tau) = e^{\tau-t}u(t-\tau)u(\tau+1)$$

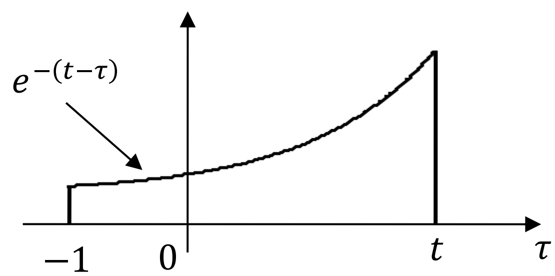

Figure 11

Step5: Integrate and consider various regions for t.

For $t < -1$: $x_1(\tau)x_2(t-\tau) = 0$

$$\Rightarrow \quad y(t) = x_1 * x_2 = 0, \qquad t < -1$$

For $t \geq -1$:

$$y(t) = x_1 * x_2 = \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau = \int_{-1}^{t} e^{\tau-t}d\tau = e^{-t}\int_{-1}^{t} e^{\tau}d\tau$$

$$= e^{-t}[e^{t} - e^{-1}] = 1 - e^{-(t+1)}$$

So,

$$y(t) = \begin{cases} 0, & t < -1 \\ 1 - e^{-(t+1)}, & t \geq -1 \end{cases}$$

Or, in closed form: $y(t) = [1 - e^{(t+1)}]u(t+1)$

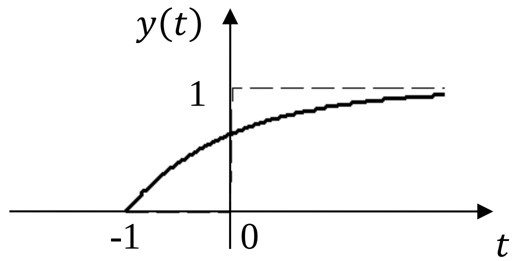

Figure 12

<u>Example</u>: (Using Convolution Table and Properties)

$$x_1(t) = 2[u(t) - u(t-2)]$$

$$x_2(t) = u(t) - 2u(t-1) + u(t-2)$$

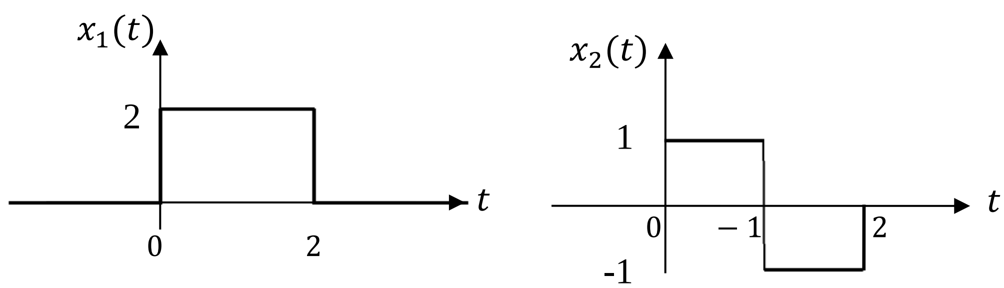

Figure 13

Find $x_1 * x_2$?

Solution:

$$x_1 * x_2 = [2u(t) - 2u(t-2)] * [u(t) - 2u(t-1) + u(t-2)]$$

$$= 2u(t) * u(t) - 4u(t) * u(t-1) + 2u(t) * u(t-2) - 2u(t-2) * u(t)$$

$$+4u(t-2) * u(t-1) - 2u(t-2) * u(t-2)$$

From convolution table: $u(t) * u(t) = tu(t)$

And, using convolution properties: $u(t) * u(t-T) = (t-T)u(t-T)$

$$u(t-T_1) * u(t-T_2) = (t-T_1-T_2)u(t-T_1-T_2)$$

This gives:

$$x_1 * x_2 = 2tu(t) - 4(t-1)u(t-1) + 2(t-2)u(t-2)$$

$$-2(t-2)u(t-2) + 4(t-3)u(t-3) - 2(t-4)u(t-4)$$

$$= 2tu(t) + (-4t+4)u(t-1) + (4t-12)u(t-3) + (-2t+8)u(t-4)$$

This result can be rewritten as:

$$x_1 * x_2 = \begin{cases} 2t & , \quad 0 \le t < 1 \\ -2t+4 & , \quad 1 \le t < 3 \\ 2t-8 & , \quad 3 \le t < 4 \\ 0 & , \quad \text{otherwise} \end{cases}$$

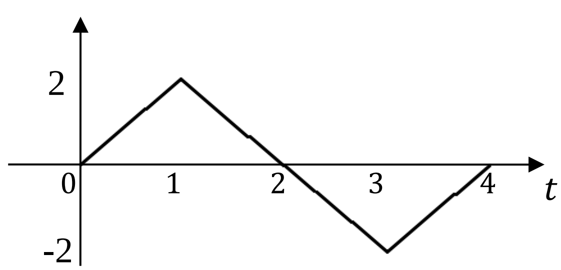

Figure 14

2.6 Interconnection of systems

Large LTI systems can be viewed as an interconnection of smaller subsystems.

Parallel Connection:

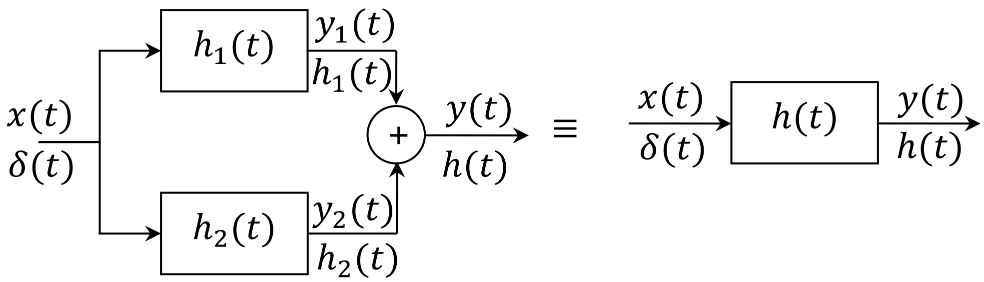

Figure15

Equivalent impulse response: $h(t) = h_1(t) + h_2(t)$

Cascade Connection:

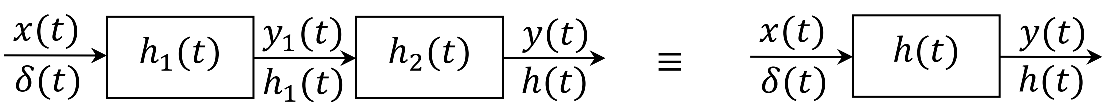

Figure 16

Equivalent impulse response: $h(t) = h_1(t) * h_2(t)$

Remark: Feedback connection will be discussed later in chapter 4.

2.7 LTI System Response to the Everlasting Exponential e^{st}

Given an LTI system with the impulse response $h(t)$.

Let the input be the everlasting complex exponential $x(t) = e^{st},\ -\infty < t < \infty$ where s is a complex constant, then:

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau$$

$$= e^{st} \underbrace{\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau}_{H(s)\ \ \text{function of } s} = H(s)e^{st}$$

The function $H(s)$ is called the *transfer function* of the system. Notice, it is a function of s only.

For a specific value of s, the function $H(s)$ is just a constant. Therefore, the output $y(t) = H(s)e^{st}$ is also an everlasting complex exponential; i.e.:

Everlasting Exponential Input $\Rightarrow$ Everlasting Exponential output

- **LTI Systems Described by Differential Equation**

For LTI systems described by: $\quad Q(D)y(t) = P(D)x(t)$

If the input is the everlasting exponential $x(t) = e^{st}$, then the output is the everlasting exponential $y(t) = H(s)e^{st}$. Substituting gives:

$$Q(D)H(s)e^{st} = P(D)e^{st}$$

or
$$H(s)[Q(D)e^{st}] = P(D)e^{st}$$

Since $\quad D^r e^{st} = s^r e^{st} \quad$ (Notice the replacement of D with s)

then

$$Q(D)e^{st} = Q(s)e^{st}$$

$$P(D)e^{st} = P(s)e^{st}$$

Substituting gives:

$$H(s)Q(s)e^{st} = P(s)e^{st}$$

$$\Rightarrow \quad H(s) = \frac{P(s)}{Q(s)}$$

<u>Remark</u>: This last result enables us to find the transfer function $H(s)$ by inspection from the differential equation.

<u>Example</u>: An LTI system is described by $(D^2 + D + 1)y(t) = (D + 1)x(t)$. The input signal is $x(t) = e^{jt}, \ -\infty < t < \infty$. Find the output $y(t)$?

<u>Solution</u>:

By inspection: $\quad H(s) = \frac{P(s)}{Q(s)} = \frac{s+1}{s^2+s+1}$

For $s = j$: $\quad H(j) = \frac{j+1}{j^2+j+1} = 1 - j$

$$\Rightarrow \quad y(t) = H(j)e^{jt} = (1-j)e^{jt}, \quad -\infty < t < \infty$$

2.8 System Stability

There are two types of system stability:

(1) Internal (Asymptotic) Stability: Behavior of internal signals (system modes) as $t \to \infty$ due to disturbances, i.e. non-zero initial conditions.

(2) External (BIBO) Stability: Behavior of output $y(t)$ due to input $x(t)$.

- **Analogy (Ball system)**

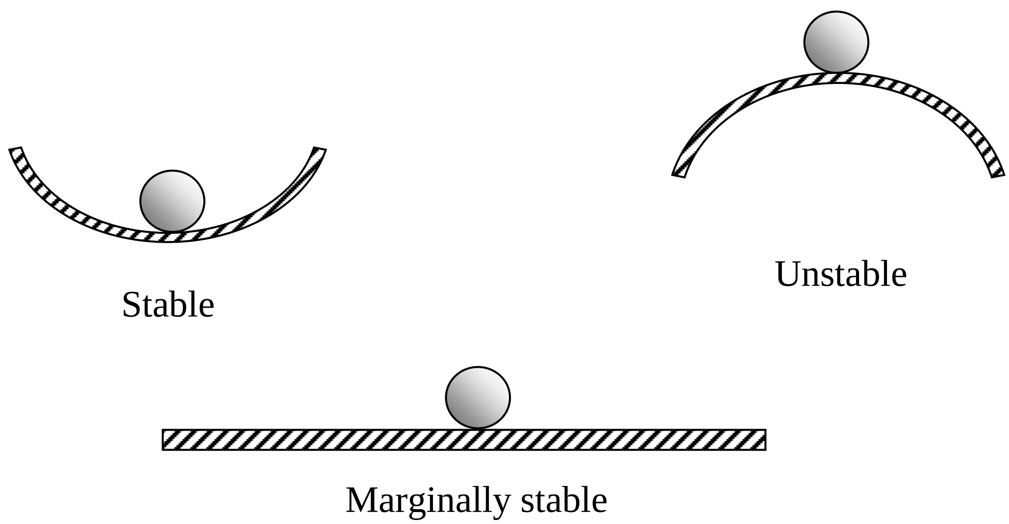

Figure 17

- **Internal (Asymptotic) Stability**

Assume an LTI system is in *zero-state* or *rest state* [ICs = 0, $x(t) = 0$, $y(t) = 0$] which is a stable state. Due to the application of small disturbances (non-zero ICs), we say:

(1) The system is <u>*asymptotically stable*</u> (AS) if all system modes $\to 0$ as $t \to \infty$, i.e. system goes back to its rest state.

(2) The system is <u>*unstable*</u> if at least one system mode $\to \infty$ as $t \to \infty$.

(3) The system is *marginally stable* (MS) if some modes are bounded, while the remaining modes $\rightarrow 0$ as $t \rightarrow \infty$.

For an LTI system described by $Q(D)y(t) = P(D)x(t)$ with N roots:

$$\lambda_1,\ \lambda_2, \dots,\ \lambda_N \quad \text{(complex in general)}$$

The system modes are of the forms:

$$e^{\lambda_i t} \ \text{(Distinct roots)} \qquad \text{or} \qquad t^r e^{\lambda_i t} \ \text{(Repeated roots)}$$

Let $\lambda_i = \alpha_i + j\beta_i$ then:

$$\lim_{t\rightarrow\infty} e^{\lambda_i t} = \lim_{t\rightarrow\infty} e^{\alpha_i t} e^{j\beta_i t} = \begin{cases} 0 & \text{if} \quad \alpha_i < 0 \\ \infty & \text{if} \quad \alpha_i > 0 \end{cases}$$

This is also valid for the modes $t^r e^{\lambda_i t}$. Therefore, we conclude:

System is asymptotically stable (AS) if $\text{Re}[\lambda_i] < 0$ for all roots

System is unstable if $\text{Re}[\lambda_i] > 0$ for at least one root

For the case $\text{Re}[\lambda_i] = 0$,

If $\lambda_i = j\beta_i$ is not repeated, the corresponding mode $e^{j\beta_i t}$ is bounded since:

$$e^{j\beta_i t} = \underbrace{\cos\beta_i t}_{\text{bounded}} + j \underbrace{\sin\beta_i t}_{bounded}$$

If $\lambda_i = j\beta_i$ is repeated, the corresponding modes $t^r e^{\beta_i t}$ are unbounded since:

$$t^r e^{\beta_i t} = \underbrace{t^r \cos\beta_i t}_{\text{unbounded}} + j \underbrace{t^r \sin\beta_i t}_{\text{unbounded}} \rightarrow \infty \quad \text{as} \quad t \rightarrow \infty$$

Therefore, we conclude:

System is marginally stable (MS) if for some roots $\text{Re}[\lambda_i] = 0$ and unrepeated, while the remaining roots are in the left-half plane (LHP), i.e. $\text{Re}[\lambda_k] < 0$.

System is unstable if for at least one root $\text{Re}[\lambda_i] = 0$ and repeated.

Summary: (Stability and location of roots of $\boldsymbol{Q(\lambda)}$ in the complex plane)

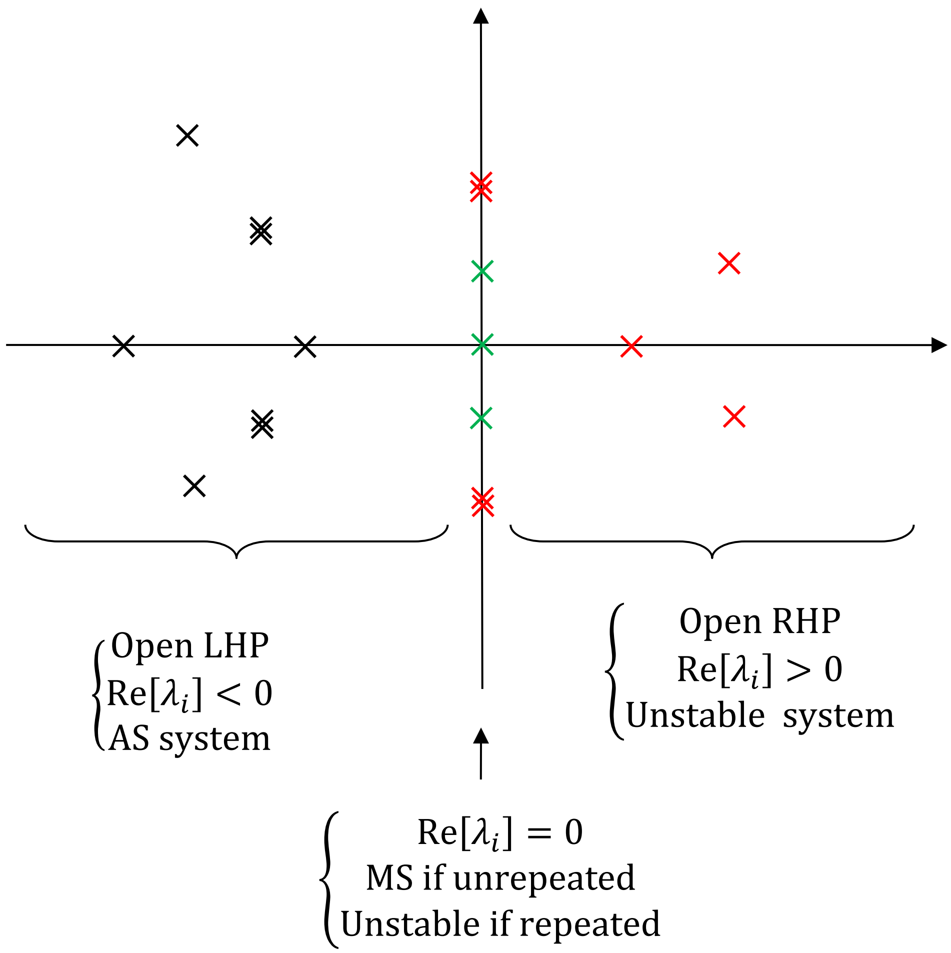

Figure 18

Example: The RC circuit shown in Figure 19 below is described by the differential equation (DE): $\left(D + \frac{1}{RC}\right) y(t) = \frac{1}{RC} x(t)$

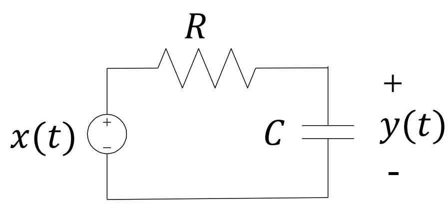

Figure 19

The system has a single root $\lambda = -\frac{1}{RC}$ which is always real and negative.

Therefore, the system is AS.

Example: (Mechanical system)

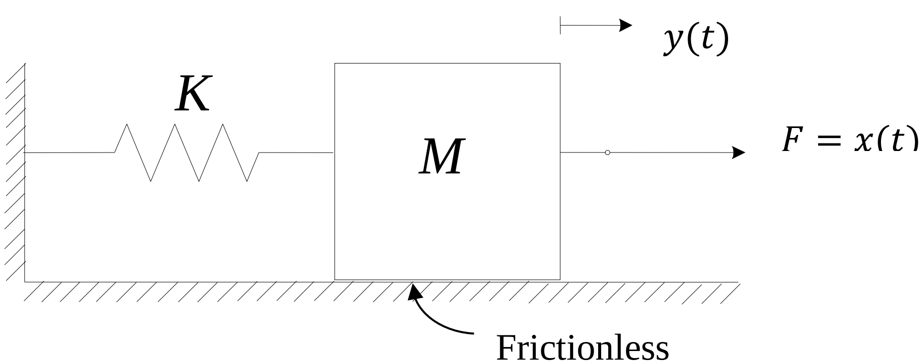

Figure 20

The system is described by the differential equation: $\left(D^2 + \frac{K}{M}\right) y(t) = \frac{1}{M} x(t)$.

The system has two roots $\lambda_{1,2} = \pm j\sqrt{\frac{K}{M}}$ which are pure complex and unrepeated.

Therefore, the system is MS (System keeps oscillating with no damping).

Remark: A pure LC circuit with no resistance behaves similarly.

Example: (Rocket System)

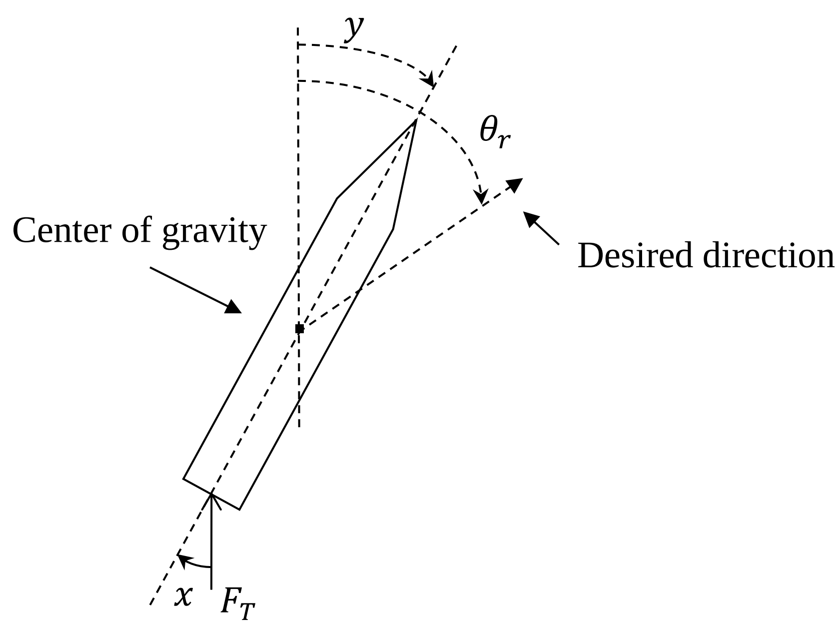

Figure 21

A simplified model of the rocket system is given by: $(D^2 - 1)y = 2x$

Where $F_T =$ Engine Thrust, $x(t) =$ Angle of engine, $y(t) =$ Angle of rocket, $\theta_r =$ Desired angle.

System roots: $\lambda_1 = -1, \quad \lambda_2 = +1$

The system is unstable due to λ_2 (in RHP).

Remark: To stabilize the rocket, a controller subsystem and feedback are used. We will see in chapter 4 an example for how to stabilize an unstable system with feedback.

- **External Bounded-Input Bounded-Output (BIBO) Stability**

  Definition: A system is *BIBO stable* if every bounded input $x(t)$ produces bounded output $y(t)$. If even one bounded input produces an unbounded output, the system is *BIBO unstable*.

  In general, this definition is impractical to apply directly since it is impossible to test all possible inputs to a system. However, for LTI systems, BIBO stability reduces to a single test, as shown below.

  For an LTI system:

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

$$|y(t)| = \left| \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \right| \leq \int_{-\infty}^{\infty} |h(\tau)|.|x(t-\tau)|d\tau$$

  If input is bounded, i.e. $|x(t-\tau)| < K_1 < \infty$, where $K_1 = \text{constant}$, then:

$$|y(t)| \leq K_1 \int_{-\infty}^{\infty} |h(\tau)|d\tau$$

  The output will be bounded if the integral in last relation is absolutely integrable, i.e.

$$\int_{-\infty}^{\infty} |h(\tau)|d\tau \leq K_2 < \infty$$

  where $K_2 = \text{constant}$. We conclude:

An LTI system is BIBO stable iff its impulse response $h(t)$ is absolutely integrable.

Remark: In general, BIBO stability criteria is weaker than asymptotic stability criteria since BIBO stability is an external test while asymptotic stability is an internal test. If the system has no hidden modes (*controllable* and *observable* system), the two criteria are equivalent as we will see later. BIBO criteria is useful when the internal description of a system is unknown.

- **Relationship between Asymptotic Stability and BIBO Stability**

LTI system: $\underbrace{Q(D)}_{\text{order N}}\, y(t) = \underbrace{P(D)}_{\text{order M}}\, x(t)$

Case 1 ($M < N$): The impulse response $h(t)$ is a scaled sum of modes of the form $e^{\lambda_i t}$ (distinct) or $t^r e^{\lambda_i t}$ (repeated). These modes are always absolutely integrable if $\mathrm{Re}[\lambda_i] < 0$. To show this, let $h(t) = e^{\lambda_i t} u(t)$, where $\lambda_i = \alpha_i + j\beta_i$, then

$$\int_{-\infty}^{\infty} |h(t)| dt = \int_{-\infty}^{\infty} \left| e^{\lambda_i t} u(t) \right| dt = \int_{0}^{\infty} |e^{\alpha_i t}| \underbrace{\left| e^{j\beta_i t} \right|}_{1} dt$$

$$= \int_{0}^{\infty} e^{\alpha_i t} dt = \begin{cases} -\dfrac{1}{\alpha_i}, & \text{if } \alpha_i < 0 \\ \infty, & \text{if } \alpha_i \geq 0 \end{cases}$$

This result also applies to the mode $t^r e^{\lambda_i t}$. Therefore, if a system is AS, then $\mathrm{Re}[\lambda_i] < 0,\ i = 1, \ldots, N$ $\Rightarrow$ all modes are absolutely integrable $\Rightarrow$ $h(t)$ is absolutely integrable. We conclude:

$$\boxed{\text{AS system} \quad \Rightarrow \quad \text{BIBO stable}}$$

However, a BIBO stable system is not necessary AS, i.e.

$$\boxed{\text{BIBO stable} \quad \not\Rightarrow \quad \text{AS system.}}$$

Example: (BIBO stable system that is AS unstable)

$$\underbrace{(D+1)(D-2)}_{Q(D)}\,y(t) = \underbrace{(D-2)}_{P(D)}\,x(t)$$

Roots of $Q(D)$: $-1,\ +2$

⇒ System is asymptotically unstable due to the root at +2.

However, the impulse response $h(t) = e^{-t}u(t)$ is absolutely integrable.

⇒ System is BIBO stable.

This is due to the hidden mode e^{2t} which is not observable at the output or not controllable at the input. This can be viewed as:

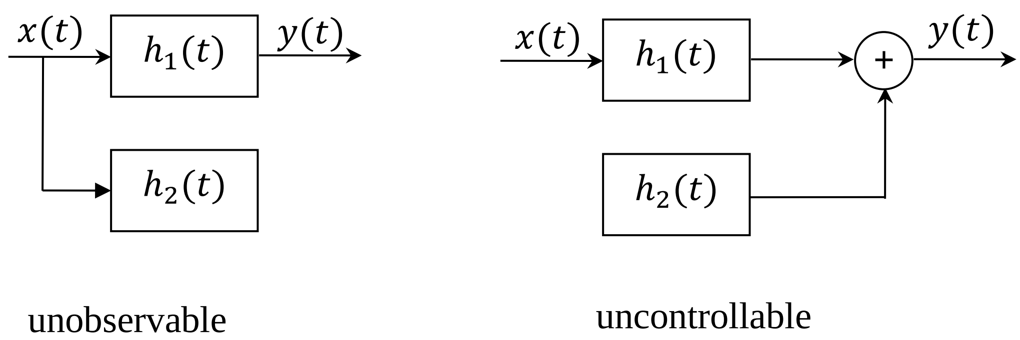

Figure 22

Remarks:

– The cancellation of $(D-2)$ term between $Q(D)$ and $P(D)$ is equivalent to the cancellation of the mode e^{2t}, which leads to the wrong conclusion from stability point of view.

– See text (example 2.13) for another point of view.

– Analogy: Car with leaking gas. Input: gas pedal, Output: tires' rotation. The driver is not aware that the car is about to explode.

Case 2 ($M = N$): The impulse response $h(t)$ contains $\delta(t)$ mode in addition to the modes $e^{\lambda_i t}$ and $t^r e^{\lambda_i t}$. Let us consider the effect of $\delta(t)$ mode only:

$$h(t) = \delta(t)$$

then $$y(t) = h(t) * x(t) = \delta(t) * x(t) = x(t)$$

$$\therefore \quad \text{For every } x(t) \text{ bounded} \quad \Rightarrow \quad y(t) \text{ bounded.}$$

Again, the system is BIBO stable if is it is AS; i.e. $\mathrm{Re}[\lambda_i] < 0$ for all λ_i.

Case 3 ($M > N$): The impulse response $h(t)$ contains the modes $\dot{\delta}(t), \ddot{\delta}(t), \ldots$ in addition to the modes $\{\delta(t),\ e^{\lambda_i t}, t^r e^{\lambda_i t}\}$.

Consider the effect of $\dot{\delta}(t)$ mode by letting $h(t) = \dot{\delta}(t)$, then:

$$y(t) = x(t) * h(t) = x(t) * \dot{\delta}(t) = \int_{-\infty}^{\infty} x(\tau)\dot{\delta}(t-\tau)d\tau = \dot{x}(t)$$

If the input $x(t)$ is bounded, $y(t) = \dot{x}(t)$ is not necessarily bounded.

For example, $x(t) = u(t)$ (bounded) gives $y(t) = \dot{x}(t) = \delta(t)$ (unbounded).

So, for $M > N$, system is BIBO unstable.

Remark: This is the reason we only consider the case $M \leq N$ in $Q(D)y(t) = P(D)x(t)$, and ignore the case $M > N$ since the system becomes unstable. Another reason for ignoring the case $M > N$ is amplification of *noise*, as we will see later.

Question: Is MS system BIBO stable?

Answer: For MS system with no hidden MS modes, $h(t)$ contains modes of the form $e^{j\beta_i t}$ or $\cos(\beta_i t + \theta)$. Let $h(t) = e^{j\beta_i t}u(t)$, then:

$$\int_{-\infty}^{\infty} |h(t)|dt = \int_{0}^{\infty} \left|e^{j\beta_i t}\right|dt = \int_{0}^{\infty} dt \to \infty$$

Similarly, $\int_0^\infty |\cos(\beta_i t + \theta)|dt \to \infty$

∴ MS system (with no hidden MS modes) → BIBO unstable.

Example: (Spring-Mass system. See Figure 20)

D.E.: $\left(D^2 + \frac{k}{M}\right) y(t) = \frac{1}{M} x(t)$ ⇒ Roots: $\lambda_{1,2} = \pm j\sqrt{\frac{k}{M}}$ ⇒ MS system

$$h(t) = \frac{1}{\sqrt{kM}} \sin\left(\sqrt{\frac{k}{M}}\, t\right) u(t)$$

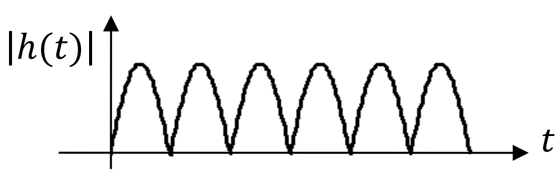

Figure 23

$$\Rightarrow \quad \int_{-\infty}^{\infty} |h(t)|dt \to \infty$$

∴ system is BIBO unstable.

To confirm this, consider the specific input:

$$x(t) = \sin\left(\sqrt{\frac{k}{M}}\ t\right)u(t) \quad \text{(Bounded)}$$

then

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

$$= \int_0^t \frac{1}{\sqrt{kM}} \sin\left(\sqrt{\frac{k}{M}}\tau\right) \sin\left(\sqrt{\frac{k}{M}}(t-\tau)\right) d\tau$$

Using $\sin\alpha \,.\, \sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$ gives:

$$y(t) = \frac{1}{2\sqrt{kM}} \int_0^t \cos\left(\sqrt{\frac{k}{M}}(t-2\tau)\right) d\tau - \frac{1}{2\sqrt{kM}} \underbrace{\int_0^t \cos\left(\sqrt{\frac{k}{M}}t\right) d\tau}_{\substack{z(t)=\cos\left(\sqrt{\frac{k}{M}}t\right)\int_0^t d\tau \\ =t\cos\left(\sqrt{\frac{k}{M}}t\right)}}$$

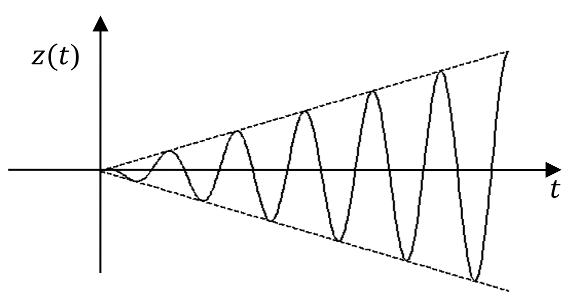

Figure 24

So, $z(t)$ unbounded $\Rightarrow$ $y(t)$ unbounded $\Rightarrow$ system is BIBO unstable.

Remark: Above example represents the *external resonance* between $x(t)$ and $h(t)$, which is different from the *internal resonance* due to initial conditions, External resonance can have severe consequences. For example, the Tacoma Narrows suspension bridge built in the US in 1940 was an MS system and collapsed within 4 months due to a mild but resonating wind.

- **Summary:**

AS system $\Rightarrow$ BIBO stable
MS system (with no hidden MS modes) $\Rightarrow$ BIBO unstable
BIBO stable $\nRightarrow$? (cannot say, system can be AS, MS or even unstable).

2.9 System Behavior

(This section corresponds to section 2.6 in the textbook, which is a reading material).

The basic attributes of an LTI system are: its *characteristic roots*, *characteristic modes*, and the *impulse response* $h(t)$.

These basic attributes determine all system behavior, including: zero-input response $y_{zi}(t)$, zero-state response $y_{zs}(t)$, stability, and other behaviors.

Question: How strong is a system response to an input?

If the input $x(t)$ has a mode similar or close to a mode in $h(t)$, then output is strong or high. For example, let $h(t) = e^{\lambda_1 t}u(t)$ and $x(t) = e^{\lambda_2 t}u(t)$, then:

$$y(t) = h(t) * x(t) = \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}u(t)$$

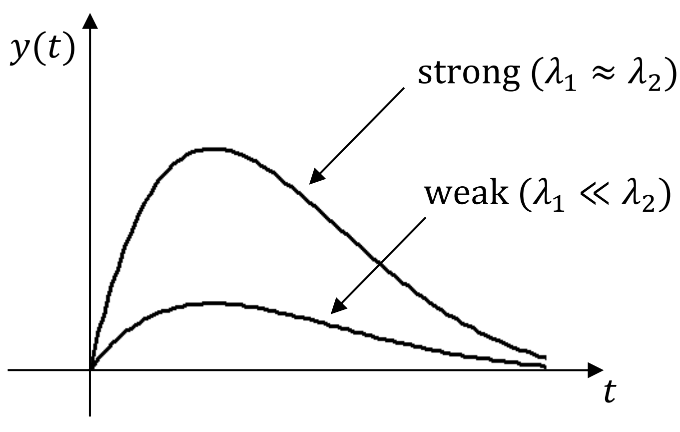

Figure 25

Question: How fast does a system respond to an input?

The *time-constant* T_h of a system is equal to the time-width of its equivalent rectangular impulse response $h(t)$. The time-constant is an indication of the speed of system response.

The *rise-time* T_r of a system is defined practically to be the time required for the unit step response to rise from 10% to 90% of its steady-state value. The rise-time is also an indication of the speed of system response. The *time-constant* T_h is similar to the *rise-time* T_r ($T_h \approx T_r$).

Consider two systems with impulse responses $h_1(t)$ and $h_2(t)$ with time-constants T_{h_1} and T_{h_2} where $T_{h_2} < T_{h_1}$ (or $T_{r_2} < T_{r_1}$), as shown in the figure below.

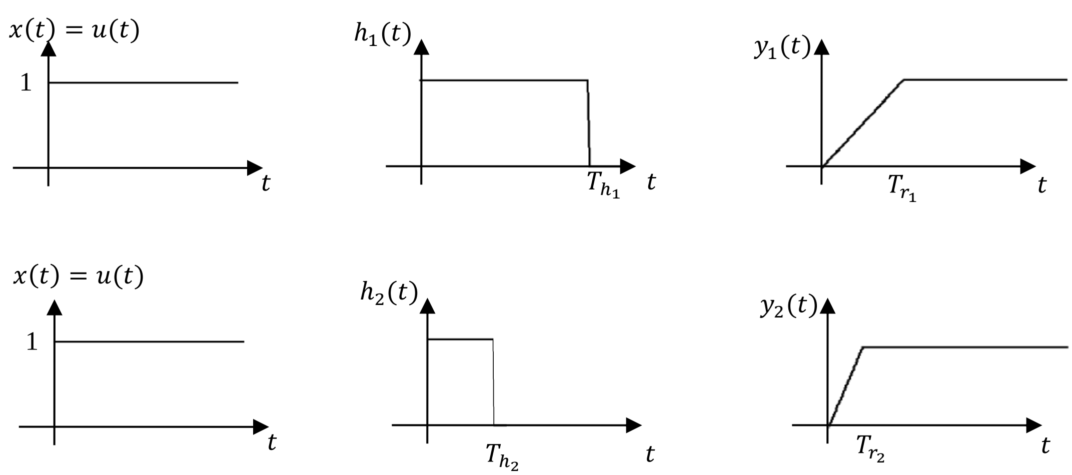

Figure 26

The system $h_2(t)$ is faster since $T_{h_2} < T_{h_1}$ (or $T_{r_2} < T_{r_1}$).

Question: In Digital Communication systems, what is the highest rate of pulse transmission through a channel?

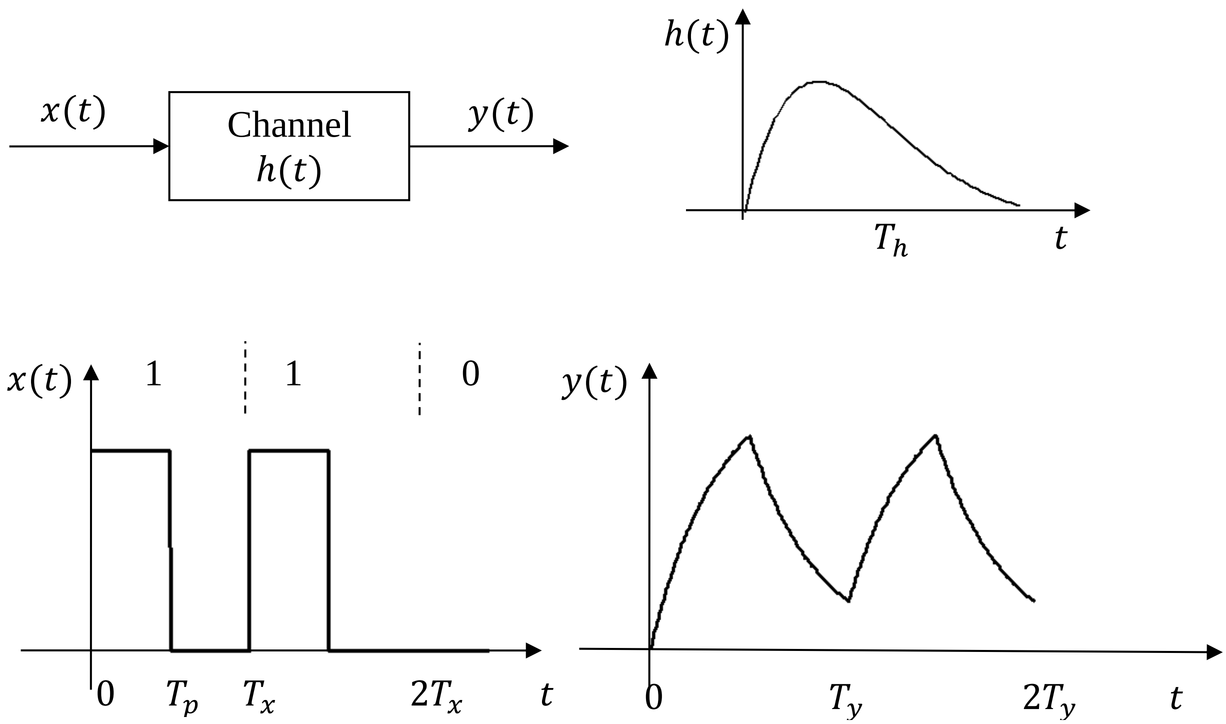

Figure 27

Convolution causes pulse dispersion (spreading).

T_p = Pulse width of $x(t)$.

$f_x = \frac{1}{T_x}$ = Rate of pulse transmission (Pulses/Second).

T_h = System time-constant or width of $h(t)$.

Due to pulse spreading, $y(t)$ pulses have width: $T_y = T_x = T_p + T_h$

To avoid interference between pulses, we must have: $T_x \geq T_h$

$$\Rightarrow \quad f_x \leq \frac{1}{T_h} = f_c \ (Channel\ Bandwidth)$$