

Properties of Expectations

Conditional Expectation [Ross S7.5]

Example 34.1: Recall that X and Y are jointly (bivariate) Gaussian (normal) with parameters:

$$\mu_X, \mu_Y, \sigma_X > 0, \sigma_Y > 0, -1 < \rho < 1$$

when $f_{XY}(x, y)$ is given by

$$\begin{aligned} f_{XY}(x, y) = & \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \\ & \times \exp \left\{ -\frac{1}{2(1-\rho^2)} \left[\left(\frac{x-\mu_X}{\sigma_X} \right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y} \right)^2 - 2\rho \frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} \right] \right\} \end{aligned} \quad (28.1)$$

We now show that ρ is the correlation between X and Y .

From Notes #28:

$$\begin{aligned} E[X] &= \mu_X \\ E[Y] &= \mu_Y \\ Var[X] &= \sigma_X^2 \\ Var[Y] &= \sigma_Y^2 \end{aligned}$$

$$\begin{aligned} \text{Therefore } \rho(X, Y) &= \frac{Cov[X, Y]}{\sigma_X\sigma_Y} \\ &= \frac{E[XY] - \mu_X\mu_Y}{\sigma_X\sigma_Y} \end{aligned}$$

To determine $E[XY]$, recall from Notes #28 that $f_{X|Y}(x|y)$ is Gaussian pdf where X has mean

$$\mu_X + \rho \frac{\sigma_X}{\sigma_Y}(y - \mu_Y).$$

So

$$E[X|Y = y] = \mu_X + \rho \frac{\sigma_X}{\sigma_Y}(y - \mu_Y)$$

Now,
$$E[XY] = E[E[XY|Y]]$$

and
$$\begin{aligned} E[XY|Y = y] &= E[Xy|Y = y] \\ &= yE[X|Y = y] \\ &= y \left(\mu_X + \rho \frac{\sigma_X}{\sigma_Y}(y - \mu_Y) \right) \\ &= \mu_X y + \rho \frac{\sigma_X}{\sigma_Y}(y^2 - \mu_Y y) \end{aligned}$$

$$\Rightarrow E[XY|Y] = \mu_X Y + \rho \frac{\sigma_X}{\sigma_Y}(Y^2 - \mu_Y Y)$$

Therefore
$$\begin{aligned} E[XY] &= E[E[XY|Y]] \\ &= E \left[\mu_X Y + \rho \frac{\sigma_X}{\sigma_Y}(Y^2 - \mu_Y Y) \right] \\ &= \mu_X E[Y] + \rho \frac{\sigma_X}{\sigma_Y}(E[Y^2] - \mu_Y E[Y]) \\ &= \mu_X \mu_Y + \rho \frac{\sigma_X}{\sigma_Y}(E[Y^2] - \mu_Y^2) \end{aligned}$$

$$\begin{aligned}
&= \mu_X \mu_Y + \rho \frac{\sigma_X}{\sigma_Y} \text{Var}[Y] \\
&= \mu_X \mu_Y + \rho \frac{\sigma_X}{\sigma_Y} \sigma_Y^2 \\
&= \mu_X \mu_Y + \rho \sigma_X \sigma_Y \\
\Rightarrow \rho(X, Y) &= \frac{E[XY] - \mu_X \mu_Y}{\sigma_X \sigma_Y} \\
&= \frac{\rho \sigma_X \sigma_Y}{\sigma_X \sigma_Y} \\
&= \rho
\end{aligned}$$

Computing Probabilities by Conditioning

We can use conditioning to compute probabilities:

Let A be an event.

Let random variable $Y \in \{y_1, y_2, \dots\}$ and $B_i = \{Y = y_i\}$.

Then B_1, B_2, \dots partition the sample space S . So by law of total probability:

$$\begin{aligned}
P[A] &= P[A|B_1]P[B_1] + P[A|B_2]P[B_2] + \dots \\
&= P[A|Y = y_1]P[Y = y_1] + P[A|Y = y_2]P[Y = y_2] + \dots \\
&= \sum_n P[A|Y = y_n]P[Y = y_n]
\end{aligned}$$

Similarly, if Y is continuous:

$$P[A] = \int_{-\infty}^{\infty} P[A | Y = y] f_Y(y) dy$$

Example 34.2: Say X and Y are independent random variables with densities $f_X(x)$ and $f_Y(y)$.

Find $P[X < Y]$.

Solution:

Example 34.3: Say X and Y are independent with densities $f_X(x)$ and $f_Y(y)$. Find the cdf and pdf of $X + Y$ by conditioning on Y .

Solution:

