Axioms (or Laws) of Probability

Sample Space and Events [Ross S2.2]

Random experiments do not have predictable outcomes.

The set of all possible outcomes is called the **sample space**, and denoted S (or sometimes Ω).

Example 2.1: If we toss two 2 coins, then $S = \{hh, ht, th, tt\}$.

Example 2.2: If we toss two 6-sided dice, then

$$S = \{(i, j) \in \mathbb{Z}^2 \mid i = 1, 2, \ldots, 6, j = 1, 2, \ldots, 6\} \tag{2.1}$$

Example 2.3: In roulette, $S = \{00, 0, 1, \ldots, 36\}$.

Example 2.4: In an experiment measuring the lifetime of a solid-state drive,

$$S = \{x \in \mathbb{R} \mid x \geq 0\}.$$

Example 2.5: Two persons will meet. Each will arrive with a delay that is between 0 and 1 hour:

$$S = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}.$$

Definition 2.1: A subset $E \subset S$ is called an **event**.

Example 2.6: In Example 2.1,

$$E = \{hh, tt\}$$

is the event that both coins come up identical.

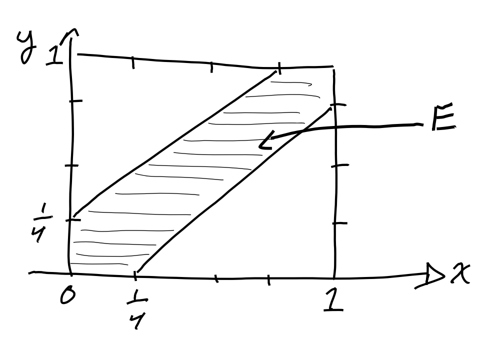
Example 2.7: In Example 2.2, the event that the dice add up to 9 is

$$F = \{(3, 6), (4, 5), (5, 4), (6, 3)\}$$

Example 2.8: In roulette, even $= \{2, 4, 6, \ldots, 36\}$ is called an even outcome and odd $= \{1, 3, 5, \ldots, 35\}$ is called an odd outcome.

Example 2.9: In Example 2.5, the event that both arrive within 1/4 hour of each other is:

$$E = \{(x, y) \in S \mid |x - y| \leq 1/4\}$$



Definition 2.2: For 2 events E and F:

- $E \cup F$ is the event that either E <u>or</u> F occurs
  $E \cup F = \{x \in S \mid x \in E \text{ or } x \in F\}$

- $E \cap F$ is the event that both E <u>and</u> F occur
  $E \cap F = \{x \in S \mid x \in E \text{ and } x \in F\}$
  We also write EF.

- If $EF = \underbrace{\emptyset}_{\text{empty set}}$ then E and F are said to be **mutually exclusive** or **disjoint**.

- E^c is the event that E does not occur
  $E^c = \{x \in S \mid x \notin E\}$
  We also write $\overline{E}$

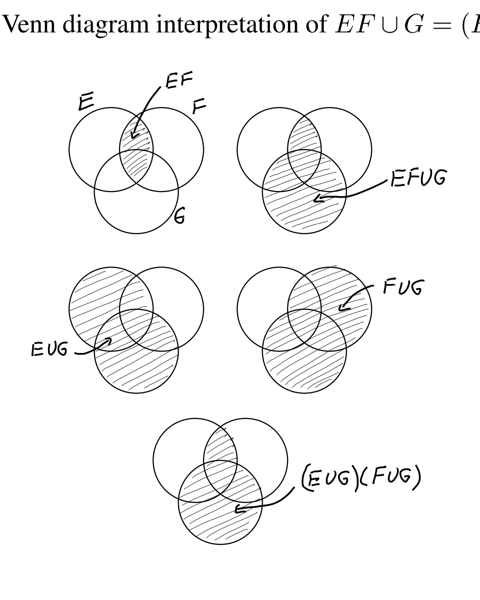
- Given F and $E_1, E_2, \ldots, E_n$, if
  - $E_1, E_2, \ldots, E_n$ are disjoint (i.e., $E_i E_j = \emptyset$ for $i \neq j$)
  - $F = \cup_{i=1}^n E_i$

then $E_1, E_2, \ldots, E_n$ are said to **partition** F.

Properties:

Commutative Laws:	$E \cup F = F \cup E$	$EF = FE$
Associative Laws:	$(E \cup F) \cup G = E \cup (F \cup G)$	$(EF)G = E(FG)$
Distributive Laws:	$(E \cup F)G = EG \cup FG$	$EF \cup G = (E \cup G)(F \cup G)$

Example 2.10: Venn diagram interpretation of $EF \cup G = (E \cup G)(F \cup G)$:



DeMorgan's Laws:

$$\left(\bigcup_{i=1}^n E_i\right)^c = \bigcap_{i=1}^n E_i^c$$
$$\left(\bigcap_{i=1}^n E_i\right)^c = \bigcup_{i=1}^n E_i^c$$

Example 2.11: Prove the 1st law: $\left(\bigcup_{i=1}^n E_i\right)^c = \bigcap_{i=1}^n E_i^c$

Solution:

<u>Step 1:</u> We will show $(\cup_i E_i)^c \subset \cap_i E_i^c$

Let $x \in (\cup_i E_i)^c$

Then $x \notin \cup_i E_i$

Then, for each i, $x \notin E_i$

Then, for each i, $x \in E_i^c$

Then, $x \in \cap_i E_i^c$

<u>Step 2:</u> We will show $\cap_i E_i^c \subset (\cup_i E_i)^c$

Let $x \in \cap_i E_i^c$

Then, for each i, $x \in E_i^c$

Then, for each i, $x \notin E_i$

Then, $x \notin E_1 \cup E_2 \cup \cdots \cup E_n$

Then, $x \in \underbrace{(E_1 \cup E_2 \cup \cdots \cup E_n)^c}_{(\cup_i E_i)^c}$

Home Exercises: Verify other properties with Venn diagrams; prove 2nd DeMorgan Law.

Given two sets A and B, the Cartesian product $A \times B$ is:

$$A \times B = \{(x, y) \mid x \in A, y \in B\}$$

We used the shorthand $A^2 = A \times A$.

Example 2.12:

$$\{0, 1\} \times \{0, 1, 2\} = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2)\}$$
$$\neq \{0, 1, 2\} \times \{0, 1\}$$

$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) \mid x \in \mathbb{R}, y \in \mathbb{R}\}$

$\{0, 1\}^{10} = \{\text{all binary strings of length 10}\}$