Axioms (or Laws) of Probability [Ross S2.3, S2.4]

We wish to assign to each event E a probability, denoted $P[E]$ (or $P(E)$).

How do we determine it?

Frequentist approach: Let $n(E)$ be number of occurences of E in n repeated experiments. Then define

$$P[E] = \lim_{n \to \infty} \frac{n(E)}{n}. \tag{3.1}$$

Does this limit exist? In what sense?

Modern Approach: Instead, assume that certain rules (axioms) must hold.

[A1] $0 \le P[E] \le 1$

[A2] $P[S] = 1$

[A3] If $E_1, E_2, \dots$ are disjoint (i.e., mutually exclusive), then

$$P[E_1 \cup E_2 \cup \dots] = \sum_{i=1}^{\infty} P[E_i]$$

Consequences of axioms:

Corollary 3.1 $P[\emptyset] = 0$.

Why? Let $E_1 = S, E_2 = \emptyset, E_3 = \emptyset, \dots$.

Then $E_1, E_2, E_3, \dots$ are disjoint.

Hence,

$$\begin{aligned} P[E_1 \cup E_2 \cup E_3 \cup \cdots] &= P[E_1] + P[E_2] + P[E_3] + \cdots \\ &= P[S] + P[\emptyset] + P[\emptyset] + \cdots \\ &= 1 + P[\emptyset] + P[\emptyset] + \cdots \end{aligned}$$

But this sum must be ≤ 1, so $P[\emptyset] = 0$.

Corollary 3.2 *Say $E_1, E_2, \dots, E_n$ are disjoint. Then*

$$P[\cup_{i=1}^n E_i] = \sum_{i=1}^n P[E_i]$$

Why? Take $\emptyset = E_{n+1} = E_{n+2} = \cdots$. Then

$$\begin{aligned} P[\cup_{i=1}^n E_i] &= P[\cup_{i=1}^\infty E_i] \\ &= \sum_{i=1}^\infty P[E_i] \\ &= \sum_{i=1}^n P[E_i] + \sum_{i=n+1}^\infty P[E_i] \\ &= \sum_{i=1}^n P[E_i] \end{aligned}$$

Example 3.1: If each of roulette's 38 possible outcomes are equally likely, then

1) $P[00] = P[0] = P[1] \cdots = P[36]$
2) $1 = P[\{00, 0, 1, \cdots, 36\}] = P[00] + P[0] + \cdots + P[36]$

Hence,

$$P[00] = P[0] = \cdots = P[36] = 1/38$$

So,

$$\begin{aligned} P[\text{even}] &= P[\{2, 4, \dots, 36\}] \\ &= P[2] + P[4] + \cdots + P[36] \\ &= 18/38 = 9/19 \end{aligned}$$

Corollary 3.3 $P[E^c] = 1 - P[E]$

Why? E and E^c are disjoint, and $E \cup E^c = S$.

$$\Rightarrow \quad 1 = P[S] = P[E \cup E^c] = P[E] + P[E^c]$$

So $P[E^c] = 1 - P[E]$

Corollary 3.4 *If $E \subset F$ then $P[E] \le P[F]$.*

Why? Since $E \subset F$, then

- $F = SF = (E \cup E^c)F = EF \cup E^cF = E \cup E^cF$
- E and E^cF are disjoint

Then

$$\begin{aligned} P[F] &= P[E] + \underbrace{P[E^cF]}_{\ge 0} \\ \Rightarrow \quad P[F] &\ge P[E] \end{aligned}$$

Example 3.2: In roulette, odd $\subset$ even^c, so

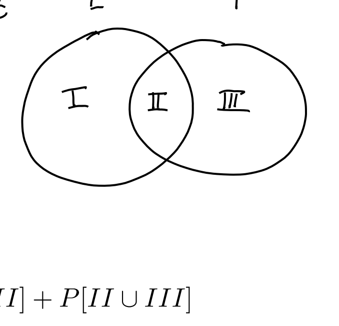
$$\underbrace{P[\text{odd}]}_{9/19} \le \underbrace{P[\text{even}^c]}_{10/19}$$

Corollary 3.5 $P[E \cup F] = P[E] + P[F] - P[E \cap F]$

Why?

$$\left.\begin{aligned} I &= EF^c \\ II &= EF \\ III &= E^cF \end{aligned}\right\} \text{disjoint}$$

$$\begin{aligned} E &= I \cup II \\ F &= II \cup III \\ E \cup F &= I \cup II \cup III \end{aligned}$$



$$\begin{aligned} P[E] + P[F] &= P[I \cup II] + P[II \cup III] \\ &= P[I] + P[II] + P[II] + P[III] \\ &= P[I \cup II \cup III] + P[II] \\ &= P[E \cup F] + P[EF] \end{aligned}$$

Example 3.3: After 5 years, a car may need

i) new brakes with prob. 0.5
ii) new tires with prob. 0.4
iii) both with prob. 0.3

What is probability it needs neither?

Solution: $B = \{\text{needs brakes}\}, T = \{\text{needs tires}\}$

$P[B] = 0.5 \qquad P[T] = 0.4 \qquad P[BT] = 0.3$

$$\begin{aligned} P[\{\text{needs neither}\}] &= P[B^cT^c] \\ &= P[(B \cup T)^c] \\ &= 1 - P[B \cup T] \\ &= 1 - (P[B] + P[T] - P[BT]) \\ &= 1 - (0.5 + 0.4 - 0.3) \\ &= 0.4 \end{aligned}$$

Can we generalize the $P[E \cup F]$ idea of Corollary 3.5? Yes!

$$\begin{aligned} P[E \cup F \cup G] &= P[(E \cup F) \cup G] \\ &= P[(E \cup F)] + P[G] - P[(E \cup F)G] \\ &= P[E] + P[F] - P[EF] + P[G] - P[EG \cup FG] \\ &= P[E] + P[F] + P[G] - P[EF] \\ &\quad - (P[EG] + P[FG] - P[EGFG]) \\ &= P[E] + P[F] + P[G] \\ &\quad - P[EF] - P[EG] - P[FG] \\ &\quad + P[EFG] \end{aligned}$$

Proposition 3.1 *Inclusion/Exclusion Principle*

$$\begin{aligned} &P[E_1 \cup E_2 \cup \ldots \cup E_n] \\ &\quad = P[E_1] + P[E_2] + \cdots P[E_n] && \textit{include all events} \\ &\qquad - \sum_{1 \le i_1 < i_2 \le n} P[E_{i_1} E_{i_2}] && \textit{exclude intersections of pairs} \\ &\qquad + \sum_{1 \le i_1 < i_2 < i_3 \le n} P[E_{i_1} E_{i_2} E_{i_3}] && \textit{include triple intersections} \\ &\qquad \vdots && \vdots \\ &\qquad + (-1)^{r+1} \sum_{1 \le i_1 < \cdots < i_r \le n} P[E_{i_1} E_{i_2} \cdots E_{i_r}] && \textit{(in/ex)clude } r\textit{-way intersections} \\ &\qquad \vdots && \vdots \\ &\qquad + (-1)^{n+1} P[E_1 E_2 \cdots E_n] && \textit{(in/ex)clude } n\textit{-way intersection} \end{aligned}$$

Proof: See textbook.