Continuous Random Variables

C) Exponential Random Variable [Ross S5.5]

A random variable X with pdf

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{else} \end{cases}$$

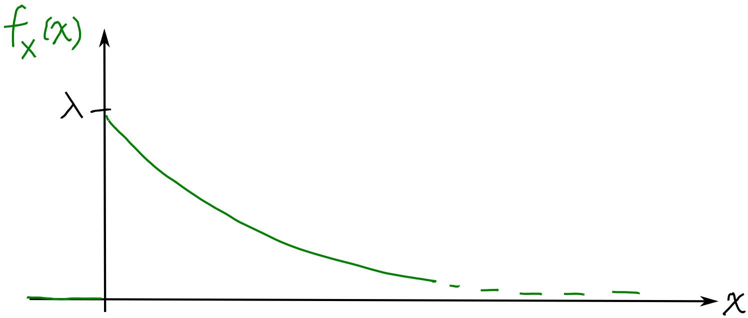
is called **exponential** with **rate parameter** $\lambda > 0$ and denoted $X \sim \mathsf{Exp}(\lambda)$.

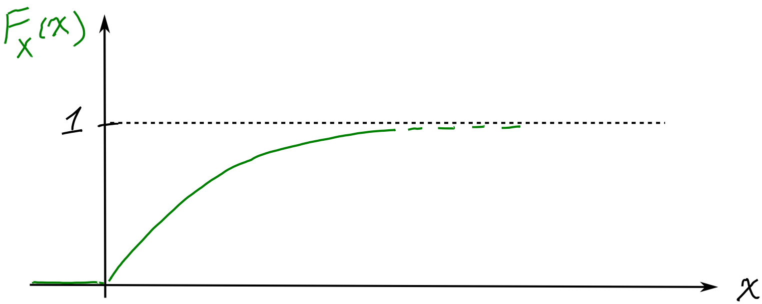
Note: If X has units of min then λ has units min^{-1}.

$$\begin{aligned} F_X(a) &= \int_{-\infty}^{a} f_X(u)du \\ &= \begin{cases} 1 - e^{-\lambda a} & a \geq 0 \\ 0 & a < 0 \end{cases} \end{aligned}$$

Example 19.1: For $X \sim \mathsf{Exp}(\lambda)$, what are $E[X]$ and $Var[X]$?

Solution: We compute $E[X^n]$ first.





$$
\begin{aligned}
E[X^n] &= \int_{-\infty}^{\infty} x^n f_X(x)dx \\
&= \int_0^{\infty} \underbrace{x^n}_{u} \underbrace{\lambda e^{-\lambda x} dx}_{dv} & u = x^n \qquad dv = \lambda e^{-\lambda x} dx \\
&= \left[uv\Big|_0^{\infty} - \int_0^{\infty} v du \right] & du = nx^{n-1}dx \quad v = -e^{-\lambda x} \\
&= \left[-x^n e^{-\lambda x}\Big|_0^{\infty} - \int_0^{\infty} -e^{-\lambda x} n x^{n-1} dx \right] \\
&= \frac{n}{\lambda} \int_0^{\infty} x^{n-1} \lambda e^{-\lambda x} dx
\end{aligned}
$$

$$= \frac{n}{\lambda} E[X^{n-1}]$$

Since $E[X^0] = E[1] = 1$, then

$$E[X] = \frac{1}{\lambda} E[X^0] = \frac{1}{\lambda}$$
$$E[X^2] = \frac{2}{\lambda} E[X^1] = \frac{2}{\lambda^2}$$

Hence

$$\begin{aligned} Var[X] &= E[X^2] - (E[X])^2 \\ &= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 \\ &= \frac{1}{\lambda^2} \end{aligned}$$

Example 19.2: The time someone uses an ATM machine is an exponential random variable with $\lambda = 1/3 \text{ min}^{-1}$. Someone arrives at the ATM just before you. What is the probability that you wait

a) more than 3 min,

b) between 3 and 6 min?

Solution: $X \sim \mathsf{Exp}(1/3)$.

a) $P[X > 3] = 1 - F_X(3) = \exp(-3\lambda) = \exp(-1) \approx 0.36788$

b) $P[3 < X < 6] = F_X(6) - F_X(3) = [1 - \exp(-6\lambda)] - [1 - \exp(-3\lambda)]$
$= \exp(-1) - \exp(-2) \approx 0.23254$

Definition 19.1: A non-negative random variable X is called **memoryless** if for all $s > 0$ and all $t > 0$

$$P[X > s + t \mid X > t] = P[X > s]$$

In words: The probability of waiting s seconds more given you have already waited t seconds is the same as waiting s seconds from the start. In other words, no matter how long you have waited, time to wait still has the same distribution.

Example 19.3: Does $\mathsf{Exp}(\lambda)$ have the memoryless property?

Solution: Let $X \sim \mathsf{Exp}(\lambda)$. Then

$$\begin{aligned}
P[X > s + t \mid X > t] &= \frac{P[X > s + t, X > t]}{P[X > t]} \\
&= \frac{P[X > s + t]}{P[X > t]} \\
&= \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}} \\
&= e^{-\lambda s} \\
&= P[X > s]
\end{aligned}$$

Yes, $\mathsf{Exp}(\lambda)$ has the memoryless property.

Example 19.4: Persons A and B are each being served by a teller. Person C arrives, and waits for one of the two tellers. All service times are exponential with the same parameter λ. What is the probability that C is the last to leave?

Solution: C starts being served as soon as one of A or B is finished.

Once this happens, the time to go for remaining person and C has the same distribution $\mathsf{Exp}(\lambda)$ due to memoryless property.

By symmetry, each has a probability 1/2 of finishing last.

Example 19.5: A car battery has a lifetime that is exponentially distributed with mean $10,000$ km.

a) What is the probability of completing a 5000 km trip without replacing the battery?

b) What can we say if lifetime is not exponential?

Solution: Let $X \sim \mathsf{Exp}(\lambda)$ with $\lambda = 1/10000$.

Let $d =$ # km that battery has been for operating so far.

a) Since battery has operated for d km so far,

$$\begin{aligned} P[X > 5000 + d \mid X > d] &= P[X > 5000] \qquad \text{[by memoryless property]} \\ &= 1 - F_X(5000) \\ &= \exp(-5000/10000) \\ &\approx 0.607 \end{aligned}$$

b)

$$\begin{aligned} P[X > d + 5000 \mid X > d] &= \frac{P[X > d + 5000,\ X > d]}{P[X > d]} \\ &= \frac{P[X > d + 5000]}{P[X > d]} \\ &= \frac{1 - F_X(d + 5000)}{1 - F_X(d)}. \end{aligned}$$

The exponential distribution can be used:

- to model service times in queuing systems
- time between radioactive decays
- credit risk modeling in finance
- is maximum entropy distribution on $[0, \infty]$ subject to a specified mean.