Jointly Distributed Random Variables

Sums of Independent Random Variables [Ross S6.3]

Say X and Y are independent continuous random variables. What is the pdf of $Z = X + Y$?

$$
\begin{aligned}
F_Z(z) &= P[X + Y \le z] \\
&= \iint_{x+y\le z} f_{XY}(x,y)\, dxdy \\
&= \iint_{x\le z-y} f_X(x) f_Y(y)\, dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{z-y} f_X(x) f_Y(y)\, dxdy \\
&= \int_{-\infty}^{\infty} f_Y(y) \int_{-\infty}^{z-y} f_X(x)\, dxdy \\
&= \int_{-\infty}^{\infty} f_Y(y) F_X(z-y)\, dy
\end{aligned}
$$

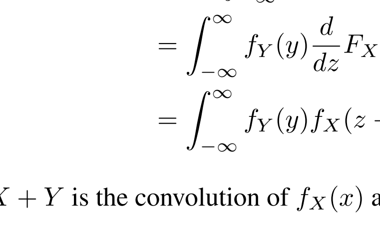
Hence:

$$
\begin{aligned}
f_Z(z) = \frac{d}{dz} F_Z(z) = \frac{d}{dz}\int_{-\infty}^{\infty} f_Y(y) F_X(z-y) dy
&= \int_{-\infty}^{\infty} f_Y(y) \frac{d}{dz} F_X(z-y) dy \\
&= \int_{-\infty}^{\infty} f_Y(y) f_X(z-y) dy
\end{aligned}
$$

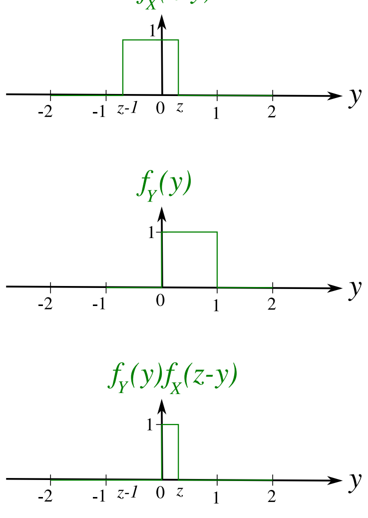
The pdf of $Z = X + Y$ is the convolution of $f_X(x)$ and $f_Y(y)$!

Example 26.1: $X \sim U(0,1)$ and $Y \sim U(0,1)$ are independent. What is the pdf of $Z = X + Y$?

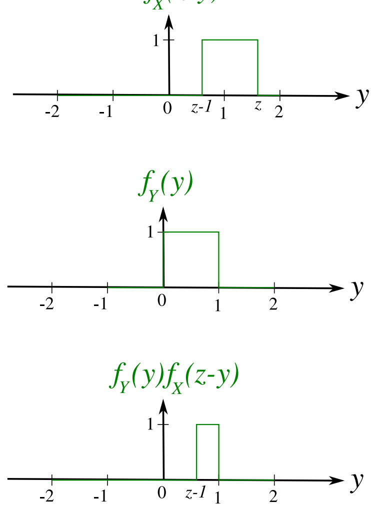
Solution:



case 0 < z < 1:



case 1 < z < 2:



Calculating the area of these rectangles:

$$
\begin{aligned}
f_Z(z) &= \begin{cases} (z-0)\times 1 & 0 \le z \le 1 \\ (1-(z-1))\times 1 & 1 \le z \le 2 \\ 0 & \text{else} \end{cases} \\
&= \begin{cases} z & 0 \le z \le 1 \\ 2-z & 1 \le z \le 2 \\ 0 & \text{else} \end{cases}
\end{aligned}
$$

Sum of Normal (Gaussian) Random Variables

Proposition 26.1 *Let $X_1, X_2, \ldots, X_n$ be independent random variables with $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$.*

Let $Z = X_1 + X_2 + \cdots + X_n$.

Then $Z \sim \mathcal{N}(\mu_Z, \sigma_Z^2)$ where

$$
\begin{aligned}
\mu_Z &= \mu_1 + \mu_2 + \cdots + \mu_N \\
\sigma_Z^2 &= \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_N^2
\end{aligned}
$$

Why?

We prove the result for the sum $Z = X_1 + X_2$. The general case follows by repeatedly applying the 2 variables case.

First determine the pdf of $U = X + Y$ where $X \sim \mathcal{N}(0, \sigma^2)$ and $Y \sim \mathcal{N}(0,1)$.

$$
\begin{aligned}
f_X(u-y) f_Y(y) &= \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(u-y)^2}{2\sigma^2}\right\}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{y^2}{2}\right\} \\
&= \frac{1}{2\pi\sigma}\exp\left\{-\frac{u^2}{2(1+\sigma^2)} - c\left(y - \frac{u}{1+\sigma^2}\right)^2\right\} \\
&\qquad\qquad \left[\text{where } c = \frac{1+\sigma^2}{2\sigma^2}\right] \\
&= \exp\left\{\frac{-u^2}{2(1+\sigma^2)}\right\}\frac{1}{2\pi\sigma}\exp\left\{-c\left(y - \frac{u}{1+\sigma^2}\right)^2\right\}
\end{aligned}
$$

$$
\begin{aligned}
f_U(u) &= \int_{-\infty}^{\infty} f_X(u-y) f_Y(y) dy \\
&= \exp\left\{\frac{-u^2}{2(1+\sigma^2)}\right\}\underbrace{\frac{1}{2\pi\sigma}\int_{-\infty}^{\infty}\exp\left\{-c\left(y-\frac{u}{1+\sigma^2}\right)^2\right\}dy}_{\text{constant } K} \\
&= K\exp\left\{\frac{-u^2}{2(1+\sigma^2)}\right\}
\end{aligned}
$$

But then $U \sim \mathcal{N}(0, 1+\sigma^2)$.

Now, let $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$.

$$
Z = X_1 + X_2 = \sigma_2\left(\underbrace{\frac{X_1-\mu_1}{\sigma_2}}_{X} + \underbrace{\frac{X_2-\mu_2}{\sigma_2}}_{Y}\right) + \mu_1 + \mu_2
$$

$$
\text{where} \quad \begin{aligned} X &\sim \mathcal{N}(0, \sigma_1^2/\sigma_2^2) \\ Y &\sim \mathcal{N}(0,1) \end{aligned}
$$

$$
\text{So} \quad U = X + Y \sim \mathcal{N}(0, 1 + \frac{\sigma_1^2}{\sigma_2^2})
$$

$$
\text{and} \quad \begin{aligned} Z &= \sigma_2 U + (\mu_1 + \mu_2) \\ &\sim \mathcal{N}(\mu_1+\mu_2, \sigma_1^2 + \sigma_2^2) \end{aligned}
$$

Definition 26.1: A random variable Y is called **lognormal** with parameters μ and σ if $\log Y$ is normal with parameter μ and σ^2, i.e., if

$$
Y = e^X,
$$

where $X \sim \mathcal{N}(\mu, \sigma^2)$.

Definition 26.2: If the random variables $X_1, X_2, \ldots, X_n$ are **independent and identically distributed**, we say that they are **i.i.d.**, or **iid**.

Example 26.2: Let $S(n)$ be the value of an investment at the end of week n.

A model for the evolution of $S(n)$ is that

$$
\frac{S(n)}{S(n-1)}
$$

are iid lognormal random variables with parameters μ and σ.

What is the probability that

a) the value increases in each of the next two weeks?

b) the value at the end of two weeks is higher than it is today?

Solution: Let $U_1 \sim \mathcal{N}(\mu, \sigma^2)$, $U_2 \sim \mathcal{N}(\mu, \sigma^2)$, $Z_1 \sim \mathcal{N}(0,1)$ and $Z_2 \sim \mathcal{N}(0,1)$ be independent.

$$
\begin{aligned}
P[S(1) > S(0), S(2) > S(1)] &= P\left[\frac{S(1)}{S(0)} > 1, \frac{S(2)}{S(1)} > 1\right] \\
&= P\left[\ln\frac{S(1)}{S(0)} > 0, \ln\frac{S(2)}{S(1)} > 0\right] \\
&= P[U_1 > 0]\,P[U_2 > 0] \\
&= P\left[\frac{U_1-\mu}{\sigma} > \frac{-\mu}{\sigma}\right] P\left[\frac{U_2-\mu}{\sigma} > \frac{-\mu}{\sigma}\right] \\
&= (1 - \Phi(-\mu/\sigma))^2
\end{aligned}
$$

$$
\begin{aligned}
\text{b)} \quad P[S(2) > S(0)] &= P\left[\frac{S(2)}{S(0)} > 1\right] \\
&= P\left[\frac{S(2)}{S(1)}\frac{S(1)}{S(0)} > 1\right] \\
&= P\left[\ln\frac{S(2)}{S(1)} + \ln\frac{S(1)}{S(0)} > 0\right] \\
&= P\Big[\underbrace{U_2 + U_1}_{\sim\mathcal{N}(\mu+\mu;\sigma^2+\sigma^2)} > 0\Big] \\
&= P\left[\frac{U_2 + U_1 - 2\mu}{\sqrt{2\sigma^2}} > \frac{0 - 2\mu}{\sqrt{2\sigma^2}}\right] \\
&= P\left[Z > -\frac{2\mu}{\sqrt{2\sigma^2}}\right] \\
&= 1 - \Phi\left(-\frac{2\mu}{\sqrt{2\sigma^2}}\right)
\end{aligned}
$$

Example 26.3: Let $X \sim \mathsf{Poisson}(\lambda_1)$ and $Y \sim \mathsf{Poisson}(\lambda_2)$ be independent. What is the pmf of $Z = X + Y$?

Solution:

$$
\begin{aligned}
P[X+Y=n] &= P[\cup_{k=-\infty}^{\infty}\{X=k, Y=n-k\}] \\
&= \sum_{k=-\infty}^{\infty} P[X=k, Y=n-k] \\
&= \sum_{k=-\infty}^{\infty} P[X=k]P[Y=n-k] \\
&= \sum_{k=0}^{n} P[X=k]P[Y=n-k] && [\text{since } X \text{ and } Y \text{ are} \ge 0] \\
&= \sum_{k=0}^{n}\frac{\lambda_1^k}{k!}e^{-\lambda_1}\frac{\lambda_2^{n-k}}{(n-k)!}e^{-\lambda_2} \\
&= e^{-(\lambda_1+\lambda_2)}\sum_{k=0}^{n}\frac{\lambda_1^k}{k!}\frac{\lambda_2^{n-k}}{(n-k)!} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{n!}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\lambda_1^k\lambda_2^{n-k} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{n!}\sum_{k=0}^{n}\binom{n}{k}\lambda_1^k\lambda_2^{n-k} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{n!}(\lambda_1+\lambda_2)^n && [\text{by Binomal Thm}]
\end{aligned}
$$

So $Z \sim \mathsf{Poisson}(\lambda_1 + \lambda_2)$.