

Properties of Expectations

Conditional Variance [Ross S7.5.4]

So far, we have defined expectation, conditional expectation and variance. We also define the **conditional variance**

$$Var[X|Y] = E[X^2|Y] - (E[X|Y])^2$$

So,

$$\begin{aligned} E[Var[X|Y]] &= E[E[X^2|Y]] - E[(E[X|Y])^2] \\ &= E[X^2] - E[(E[X|Y])^2] \end{aligned} \tag{35.1}$$

Also, $E[X|Y] = g(Y)$ for some function g , so

$$\begin{aligned} Var[g(Y)] &= E[(g(Y))^2] - (E[g(Y)])^2 \\ Var[E[X|Y]] &= E[(E[X|Y])^2] - (E[E[X|Y]])^2 \\ &= E[(E[X|Y])^2] - (E[X])^2 \end{aligned} \tag{35.2}$$

Adding (35.1) to (35.2), we get

Proposition 35.1 *Conditional Variance Formula:*

$$Var[X] = E[Var[X|Y]] + Var[E[X|Y]]$$

Example 35.1: Let X_1, X_2, \dots be iid and independent of the non-negative integer random variable N . Let's compute

$$Var\left[\sum_{i=1}^N X_i\right]$$

by conditioning on N .

Using $Var[X] = E[Var[X|Y]] + Var[E[X|Y]]$ with

$$\begin{aligned} X &= \sum_{i=1}^N X_i \\ Y &= N \end{aligned}$$

$$\begin{aligned} \text{then } E\left[\sum_{i=1}^N X_i \middle| N = n\right] &= E\left[\sum_{i=1}^n X_i\right] && \text{[Since } N \text{ is ind. of the } X_i\text{]} \\ &= nE[X_1] && \text{[Since } X_i \text{ are iid]} \\ \Rightarrow E\left[\sum_{i=1}^N X_i \middle| N\right] &= NE[X_1] \end{aligned}$$

$$\begin{aligned} Var\left[\sum_{i=1}^N X_i \middle| N = n\right] &= Var\left[\sum_{i=1}^n X_i\right] && \text{[Since } N \text{ is ind. of the } X_i\text{]} \\ &= nVar[X_1] && \text{[Since } X_i \text{ are iid]} \\ \Rightarrow Var\left[\sum_{i=1}^N X_i \middle| N\right] &= NVar[X_1] \end{aligned}$$

By the conditional variance formula:

$$\begin{aligned} Var\left[\sum_{i=1}^N X_i\right] &= E[NVar[X_1]] + Var[NE[X_1]] \\ &= E[N]Var[X_1] + (E[X_1])^2Var[N] \end{aligned}$$