

## Random Variables (rvs)

### Expectation of sums of random variables [Ross S4.9]

Recall, a random variable  $X$  is a function  $X(s)$  of the outcome  $s$  of a random experiment.

We can have two functions of the same outcome  $s$ , say  $X(s)$  and  $Y(s)$ .

**Example 14.1:** Flip a coin 5 times.

Let  $X = \#$  heads in first 3 flips;  $Y = \#$  heads in last 2 flips.

Since  $X$  and  $Y$  are numbers, we can add them:  $Z(s) = X(s) + Y(s)$ .

In other words,  $Z$  is also a random variable.

Here,  $Z = \#$  of heads in all 5 flips.

Now, for each  $s \in S$ , let  $p(s) = P[\{s\}]$ .

Then  $P[A] = \sum_{s \in A} p(s)$

Let  $X \in \mathcal{X} = \{x_1, \dots, x_n\}$

$$A_k = \{s \in S \mid X(s) = x_k\}$$

$$\begin{aligned}
 \text{Then} \quad E[X] &= \sum_{k=1}^n x_k P[X = x_k] \\
 &= \sum_{k=1}^n x_k P[A_k] \\
 &= \sum_{k=1}^n x_k \sum_{s \in A_k} p(s) \\
 &= \sum_{k=1}^n \sum_{s \in A_k} x_k p(s) \\
 &= \sum_{k=1}^n \sum_{s \in A_k} X(s) p(s) \\
 &= \sum_{s \in S} X(s) p(s)
 \end{aligned}$$

**Example 14.2:** Two independent flips of a fair coin are made.

Let  $X = \#$  heads.

$$\begin{aligned}
 \text{Then} \quad P[X = 0] &= 1/4 \\
 P[X = 1] &= 1/2 \\
 P[X = 2] &= 1/4
 \end{aligned}$$

$$\text{So} \quad E[X] = 0 \times \frac{1}{4} + 1 \times \frac{1}{2} + 2 \times \frac{1}{4} = 1.$$

Also,  $S = \{tt, th, ht, hh\}$ , and each outcome has probability  $1/4$ .

$$\begin{aligned}
\text{So } E[X] &= X(tt) \times \frac{1}{4} + X(th) \times \frac{1}{4} + X(ht) \times \frac{1}{4} + X(hh) \times \frac{1}{4} \\
&= 0 \times \frac{1}{4} + 1 \times \frac{1}{4} + 1 \times \frac{1}{4} + 2 \times \frac{1}{4} \\
&= 1
\end{aligned}$$


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**Proposition 14.1** For random variables  $X_1, X_2, \dots, X_n$ :

$$E[X_1 + \dots + X_n] = E[X_1] + \dots + E[X_n]$$

Why?

Let  $Z = X_1 + \dots + X_n$ . Then

$$\begin{aligned}
E[Z] &= \sum_{s \in S} Z(s)p(s) \\
&= \sum_{s \in S} (X_1(s) + \dots + X_n(s))p(s) \\
&= \sum_{s \in S} X_1(s)p(s) + \dots + \sum_{s \in S} X_n(s)p(s) \\
&= E[X_1] + \dots + E[X_n]
\end{aligned}$$

**Example 14.3:** Let  $X \sim \text{Binomial}(n, p)$ . Then

$$X = X_1 + \dots + X_n$$

where each  $X_k \sim \text{Bernoulli}(p)$  and is an independent trial.

Then:

$$\begin{aligned}
E[X] &= E[X_1 + \cdots + X_n] \\
&= E[X_1] + \cdots + E[X_n] \\
&= p + \cdots + p \\
&= np
\end{aligned}$$

$$\begin{aligned}
E[X^2] &= E \left[ \left( \sum_{k=1}^n X_k \right) \left( \sum_{\ell=1}^n X_{\ell} \right) \right] \\
&= E \left[ \sum_{k=1}^n \left( \sum_{\ell=1}^n X_k X_{\ell} \right) \right] \\
&= E \left[ \sum_{k=1}^n \left( X_k X_k + \sum_{\substack{\ell=1 \\ \ell \neq k}}^n X_k X_{\ell} \right) \right] \\
&= E \left[ \sum_{k=1}^n X_k^2 + \sum_{k=1}^n \sum_{\substack{\ell=1 \\ \ell \neq k}}^n X_k X_{\ell} \right] \\
&= \sum_{k=1}^n E[X_k^2] + \sum_{k=1}^n \sum_{\substack{\ell=1 \\ \ell \neq k}}^n E[X_k X_{\ell}]
\end{aligned}$$

Now

$$\begin{aligned}
P[X_k^2 = 1] &= P[X_k = 1] = p \\
P[X_k X_{\ell} = 1] &= P[X_k = 1, X_{\ell} = 1] \\
&= P[X_k = 1]P[X_{\ell} = 1] \quad [\text{since trials are independent}] \\
&= p^2
\end{aligned}$$

So

$$E[X^2] = np + n(n-1)p^2$$

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### Properties of CDFs [Ross 4.10]

Recall  $F_X(x) = P[X \leq x]$

Therefore:

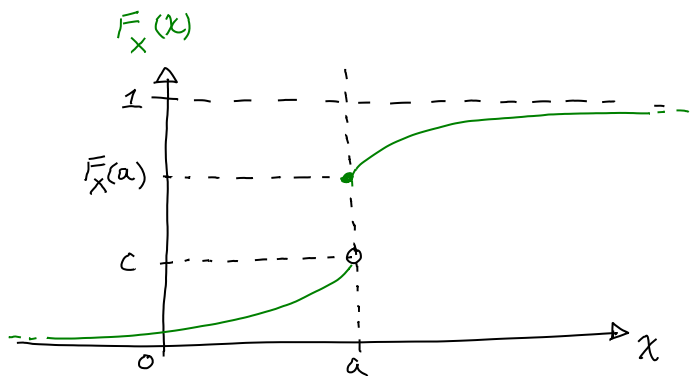
- 1)  $0 \leq F_X(x) \leq 1$
- 2) If  $a < b$  then  $\{X \leq a\} \subset \{X \leq b\}$   
 $\Rightarrow P[X \leq a] \leq P[X \leq b]$   
 $\Rightarrow F_X(a) \leq F_X(b)$   
or,  $F_X(x)$  is non-decreasing in  $x$ .

It can also be show that:

- 3)  $\lim_{x \rightarrow \infty} F_X(x) = 1$
- 4)  $\lim_{x \rightarrow -\infty} F_X(x) = 0$
- 5)  $\lim_{x \downarrow b} F_X(x) = F_X(b)$   
[i.e.,  $F_X(x)$  is continuous from the right]
- 6)  $\lim_{x \uparrow b} F_X(x)$  exists  
[i.e.,  $F_X(x)$  has left limits]

A function with properties 5) and 6) is called **càdlàg** [continue à droite, limite à gauche].

### Example 14.4:



Here:

$$\lim_{x \downarrow a} F_X(x) = F_X(a)$$

$$\lim_{x \uparrow a} F_X(x) = c \neq F_X(a)$$