Properties of Expectations

Conditional Expectation [Ross S7.5]

Example 34.1: Recall that X and Y are jointly (bivariate) Gaussian (normal) with parameters:

$$\mu_X, \mu_Y, \sigma_X > 0, \sigma_Y > 0, -1 < \rho < 1$$

when $f_{XY}(x, y)$ is given by

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \times \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - 2\rho\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}\right]\right\} \tag{28.1}$$

We now show that ρ is the correlation between X and Y.

From Notes #28:

$$\begin{aligned} E[X] &= \mu_X \\ E[Y] &= \mu_Y \\ Var[X] &= \sigma_X^2 \\ Var[Y] &= \sigma_Y^2 \end{aligned}$$

Therefore $\quad \rho(X,Y) = \dfrac{Cov[X,Y]}{\sigma_X\sigma_Y} = \dfrac{E[XY] - \mu_X\mu_Y}{\sigma_X\sigma_Y}$

To determine $E[XY]$, recall from Notes #28 that $f_{X|Y}(x|y)$ is Gaussian pdf where X has mean

$$\mu_X + \rho\frac{\sigma_X}{\sigma_Y}(y - \mu_Y).$$

So

$$E[X|Y=y] = \mu_X + \rho\frac{\sigma_X}{\sigma_Y}(y-\mu_Y)$$

Now, $\qquad E[XY] = E[\, E[XY|Y]\,]$

and $\quad$
$$\begin{aligned} E[XY|Y=y] &= E[Xy|Y=y] \\ &= yE[X|Y=y] \\ &= y\left(\mu_X + \rho\frac{\sigma_X}{\sigma_Y}(y-\mu_Y)\right) \\ &= \mu_X y + \rho\frac{\sigma_X}{\sigma_Y}(y^2 - \mu_Y y) \end{aligned}$$

$$\Rightarrow E[XY|Y] = \mu_X Y + \rho\frac{\sigma_X}{\sigma_Y}(Y^2 - \mu_Y Y)$$

Therefore
$$\begin{aligned} E[XY] &= E[\, E[XY|Y]\,] \\ &= E\left[\mu_X Y + \rho\frac{\sigma_X}{\sigma_Y}(Y^2 - \mu_Y Y)\right] \\ &= \mu_X E[Y] + \rho\frac{\sigma_X}{\sigma_Y}(E[Y^2] - \mu_Y E[Y]) \\ &= \mu_X\mu_Y + \rho\frac{\sigma_X}{\sigma_Y}(E[Y^2] - \mu_Y^2) \\ &= \mu_X\mu_Y + \rho\frac{\sigma_X}{\sigma_Y}Var[Y] \\ &= \mu_X\mu_Y + \rho\frac{\sigma_X}{\sigma_Y}\sigma_Y^2 \\ &= \mu_X\mu_Y + \rho\sigma_X\sigma_Y \end{aligned}$$

$$\begin{aligned} \Rightarrow \rho(X,Y) &= \frac{E[XY] - \mu_X\mu_Y}{\sigma_X\sigma_Y} \\ &= \frac{\rho\sigma_X\sigma_Y}{\sigma_X\sigma_Y} \\ &= \rho \end{aligned}$$

Computing Probabilities by Conditioning

We can use conditioning to compute probabilities:

Let A be an event.

Let random variable $Y \in \{y_1, y_2, \ldots\}$ and $B_i = \{Y = y_i\}$.

Then $B_1, B_2, \ldots$ partition the sample space S. So by law of total probability:

$$\begin{aligned} P[A] &= P[A|B_1]P[B_1] + P[A|B_2]P[B_2] + \cdots \\ &= P[A|Y=y_1]P[Y=y_1] + P[A|Y=y_2]P[Y=y_2] + \cdots \\ &= \sum_n P[A|Y=y_n]P[Y=y_n] \end{aligned}$$

Similarly, if Y is continuous:

$$P[A] = \int_{-\infty}^{\infty} P[A \mid Y=y] f_Y(y)dy$$

Example 34.2: Say X and Y are independent random variables with densities $f_X(x)$ and $f_Y(y)$.

Find $P[X < Y]$.

Solution: Method 1:

$$\begin{aligned} P[X<Y] &= \iint_{x<y} f_X(x)f_Y(y)dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{y} f_X(x)f_Y(y)dxdy \\ &= \int_{-\infty}^{\infty} F_X(y)f_Y(y)dxdy \end{aligned}$$

Method 2:

$$\begin{aligned} P[X<Y] &= \int_{-\infty}^{\infty} P[X<Y \mid Y=y]f_Y(y)dy \\ &= \int_{-\infty}^{\infty} P[X<y \mid Y=y]f_Y(y)dy \\ &= \int_{-\infty}^{\infty} P[X<y]f_Y(y)dy \\ &= \int_{-\infty}^{\infty} F_X(y)f_Y(y)dy \end{aligned}$$

Example 34.3: Say X and Y are independent with densities $f_X(x)$ and $f_Y(y)$. Find the cdf and pdf of $X+Y$ by conditioning on Y.

Solution:

$$\begin{aligned} P[X+Y \le a] &= \int_{-\infty}^{\infty} P[X+Y \le a \mid Y=y]f_Y(y)dy \\ &= \int_{-\infty}^{\infty} P[X+y \le a \mid Y=y]f_Y(y)dy \\ &= \int_{-\infty}^{\infty} P[X+y \le a]f_Y(y)dy \\ &= \int_{-\infty}^{\infty} P[X \le a-y]f_Y(y)dy \\ &= \int_{-\infty}^{\infty} F_X(a-y)f_Y(y)dy \end{aligned}$$

and, taking derivatives:

$$\begin{aligned} f_{X+Y}(a) &= \frac{d}{da}P[X+Y \le a] \\ &= \frac{d}{da}\int_{-\infty}^{\infty} F_X(a-y)f_Y(y)dy \\ &= \int_{-\infty}^{\infty} \frac{d}{da}F_X(a-y)f_Y(y)dy \\ &= \int_{-\infty}^{\infty} f_X(a-y)f_Y(y)dy \end{aligned}$$