Random Variables (rv)

Discrete Random Variables [Ross S4.2]

Definition 9.1: A random variable that can take at most a countable number of possible outcomes is called a discrete random variable.

Definition 9.2: For a discrete random variable X, we define its **Probability Mass Function** (PMF) $p_X(a)$ by

$$p_X(a) = P[X = a].$$

Let $\mathcal{X} = \{x_1, x_2, ...\}$ be the possible outcomes that X takes.

$$\text{Then} \quad \begin{aligned} p_X(x) &\geq 0 && \text{for } x \in \mathcal{X} \\ p_X(x) &= 0 && \text{for all other } x \end{aligned}$$

and, since X must take one of its possible values:

$$\sum_{x \in \mathcal{X}} p_X(x) = 1$$

Example 9.1: Say the PMF of the random variable X is

$$p_X(k) = C \frac{\lambda^k}{k!}, \qquad \text{for } k = 0, 1, 2, \ldots$$

and $\lambda > 0$ is given.

a) Find C in terms of λ

b) Find $P[X = 0]$

c) Find $P[X > 1]$.

Solution:

a) Since $\displaystyle\sum_{k=0}^{\infty} p_X(k) = 1$

$$\Rightarrow \quad C \times \underbrace{\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}}_{\text{power series for } e^{\lambda}} = 1$$

$$\begin{aligned} \Rightarrow \quad & Ce^{\lambda} = 1 \\ \Rightarrow \quad & C = e^{-\lambda} \end{aligned}$$

b) $\quad P[X = 0] = p_X(0) = C \dfrac{\lambda^0}{0!} = e^{-\lambda}.$

c) $$\begin{aligned} P[X > 1] &= 1 - P[X = 0] - P[X = 1] \\ &= 1 - e^{-\lambda} - \lambda e^{-\lambda}. \end{aligned}$$

Say $\mathcal{X} \subset \mathbb{R}$. Instead of specifying $p_X(x)$ for every $x \in \mathcal{X}$, we can specify:

$$F_X(x) = P[X \leq x] \qquad x \in \mathbb{R}$$

instead.

$F_X(x)$ is called the **Cumulative Distribution Function** (CDF) of X.

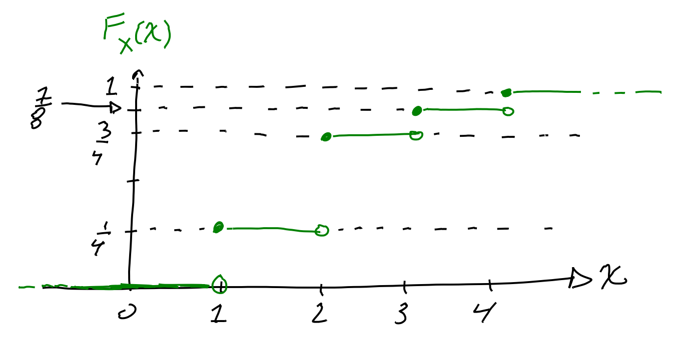
Example 9.2: Let X be such that

$$p_X(1) = \frac{1}{4} \qquad p_X(2) = \frac{1}{2} \qquad p_X(3) = \frac{1}{8} \qquad p_X(4) = \frac{1}{8}$$

Plot the CDF $F_X(x)$.

Solution:

$$\begin{aligned} F_X(-10) &= P[X \leq -10] = 0 \\ F_X(0.999) &= P[X \leq 0.999] = 0 \\ F_X(1) &= P[X \leq 1] = 1/4 \\ F_X(1.999) &= P[X \leq 1.999] = 1/4 \\ F_X(2) &= P[X \leq 2] = 1/4 + 1/2 \\ F_X(2.999) &= P[X \leq 2.999] = 3/4 \\ F_X(3) &= P[X \leq 3] = 7/8 \\ F_X(3.999) &= P[X \leq 3.999] = 7/8 \\ F_X(4) &= P[X \leq 4] = 1 \end{aligned}$$



i) size of jump @ $x = a$ is $P[X = a]$.

ii) open on left side of jump, closed on right side of jump

Expected (Mean) Value [Ross S4.3]

Definition 9.3: The **expected** (or **mean**) value of a random variable X is

$$E[X] = \sum_{x \in \mathcal{X}} x p_X(x) \tag{9.1}$$

This is an "average" where each outcome is weighted by the probability that X assumes that outcome.

Example 9.3: Say $p_X(0) = 1/2, \quad p_X(1) = 1/2.$

$$\text{Then} \quad \begin{aligned} E[X] &= 0 \times 1/2 + 1 \times 1/2 \\ &= 1/2 \end{aligned}$$

Example 9.4: Say $p_X(0) = 1/3, \quad p_X(1) = 2/3.$

$$\text{Then} \quad \begin{aligned} E[X] &= 0 \times 1/3 + 1 \times 2/3 \\ &= 2/3 \end{aligned}$$

Example 9.5: : Let $A \subset S$ be an event. Let the random variable I be such that

$$I = \begin{cases} 1 & A \text{ occurs} \\ 0 & A \text{ does not occur.} \end{cases}$$

Then

$$\begin{aligned} E[I] &= 0 \times P[I = 0] + 1 \times P[I = 1] \\ &= 0 \times P[A^c] + 1 \times P[A] \\ &= P[A] \end{aligned}$$

I is called an **indicator for event** A. We often write I_A or 1_A for this kind of random variable.

Example 9.6: 120 students are driven in 3 buses with 36, 40 and 44 students each. One of the 120 students is chosen randomly.

Let X = # students on bus of randomly chosen student.

What is $E[X]$?

Solution: $\mathcal{X} = \{36, 40, 44\}$.

$P[X = 36] = 36/120$

$P[X = 40] = 40/120$

$P[X = 44] = 44/120$

$$\text{So} \quad \begin{aligned} E[X] &= 36 \times \frac{36}{120} + 40 \times \frac{40}{120} + 44 \times \frac{44}{120} \qquad (\neq 40) \\ &\approx 40.267 \end{aligned}$$