Jointly Distributed Random Variables

Conditional Distributions:

<u>Discrete Case</u> [Ross S6.4]

Recall that for $P[F] > 0$:

$$P[E|F] = \frac{P[EF]}{P[F]}$$

Say $p_Y(y) > 0$. The **conditional pmf** for X given Y is

$$\begin{aligned} p_{X|Y}(x|y) &= P[X = x \,|\, Y = y] \\ &= \frac{P[X = x, Y = y]}{P[Y = y]} \\ &= \frac{p_{XY}(x, y)}{p_Y(y)} \end{aligned}$$

The **conditional cdf** for X given Y is

$$\begin{aligned} F_{X|Y}(x|y) &= P[X \leq x \,|\, Y = y] \\ &= \frac{P[X \leq x, Y = y]}{P[Y = y]} \\ &= \sum_{a \leq x} \frac{P[X = a, Y = y]}{P[Y = y]} \\ &= \sum_{a \leq x} p_{X|Y}(a|y) \end{aligned}$$

If X and Y are independent:

$$\begin{aligned}
p_{X|Y}(x|y) &= \frac{p_{XY}(x,y)}{p_Y(y)} \\
&= \frac{p_X(x)p_Y(y)}{p_Y(y)} \\
&= p_X(x)
\end{aligned}$$

Example 27.1: Let $X \sim \mathsf{Poisson}(\lambda_1)$ and $Y \sim \mathsf{Poisson}(\lambda_2)$ be independent. Find the conditional pmf for X given $X+Y=n$.

Solution:

$$\begin{aligned}
P[X=k \,|\, X+Y=n] &= \frac{P[X=k, X+Y=n]}{P[X+Y=n]} \\
&= \frac{P[X=k, Y=n-k]}{P[X+Y=n]} \\
&= \frac{P[X=k]P[Y=n-k]}{P[X+Y=n]} \qquad (27.1)
\end{aligned}$$

If $k>n$: $P[Y=n-k]=0 \Rightarrow P[X=k \,|\, X+Y=n]=0$.

If $k<0$: $P[X=k]=0 \Rightarrow P[X=k \,|\, X+Y=n]=0$.

From Ex. 26.3, $X+Y$ is $\sim \mathsf{Poisson}(\lambda_1+\lambda_2)$. So for $0 \le k \le n$:

$$\begin{aligned}
P[X=k|X+Y=n] &= \frac{\lambda_1^k e^{-\lambda_1}}{k!}\frac{\lambda_2^{n-k}e^{-\lambda_2}}{(n-k)!}\left[\frac{(\lambda_1+\lambda_2)^n e^{-(\lambda_1+\lambda_2)}}{n!}\right]^{-1} \\
&= \frac{n!}{k!(n-k)!}\frac{\lambda_1^k\lambda_2^{n-k}}{(\lambda_1+\lambda_2)^n} \\
&= \binom{n}{k}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}
\end{aligned}$$

This is binomial with parameters n and $\lambda_1/(\lambda_1+\lambda_2)$.

Example 27.2: Let $X_1, X_2, \ldots, X_n$ be iid and $\sim \mathsf{Bernoulli}(p)$.

Say these result in k ones. Show that each of the $\binom{n}{k}$ possible orderings of k ones are then equally likely.

Solution: Let $Z = X_1 + \cdots + X_n$. We are conditioning on $Z = k$.

Let $x_1, x_2, \ldots, x_n$ be binary, and such that $x_1 + x_2 + \cdots + x_n = k$.

$$
\begin{aligned}
P[X_1 = x_1, \ldots, X_n = x_n | Z = k] &= \frac{P[X_1 = x_1, \ldots, X_n = x_n, Z = k]}{P[Z = k]} \\
&= \frac{P[X_1 = x_1, \ldots, X_n = x_n]}{P[Z = k]} \qquad (*) \\
&= \frac{p^k (1-p)^{n-k}}{\binom{n}{k} p^k (1-p)^{n-k}} \\
&= \frac{1}{\binom{n}{k}}
\end{aligned}
$$

(*) since $\{X_1 = x_1, \ldots, X_n = x_n\} \subset \{Z = k\}$ when $x_1 + \cdots + x_n = k$.

Continuous Case [Ross S6.5]

If X and Y are continuous, for $f_Y(y) > 0$, the **conditional pdf** of X given $Y = y$ is

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

We also define:

$$P[X \in A | Y = y] = \int_A f_{X|Y}(x|y) dx$$

and then

$$\begin{aligned}
\int_{-\infty}^{\infty} P[X \in A|Y=y] f_Y(y)dy &= \int_{-\infty}^{\infty} \left[\int_A f_{X|Y}(x|y)dx\right] f_Y(y)dy \\
&= \int_{-\infty}^{\infty} \int_A f_{X|Y}(x|y) f_Y(y)dydx \\
&= \int_A \int_{-\infty}^{\infty} f_{XY}(x,y)dydx \\
&= P[X \in A] \qquad (27.2)
\end{aligned}$$

With $A = (-\infty, a]$, we get the **conditional cdf**

$$F_{X|Y}(a|y) = P[X \le a|Y=y] = \int_{-\infty}^{a} f_{X|Y}(x|y)dx$$

If X and Y are independent and $f_Y(y) > 0$:

$$\begin{aligned}
f_{X|Y}(x|y) &= \frac{f_{XY}(x,y)}{f_Y(y)} \\
&= \frac{f_X(x) f_Y(y)}{f_Y(y)} \\
&= f_X(x)
\end{aligned}$$

Example 27.3: The joint pdf of X and Y is

$$f_{XY}(x,y) = \begin{cases} \frac{e^{-x/y}e^{-y}}{y} & 0 < x < \infty,\ 0 < y < \infty \\ 0 & \text{else} \end{cases}$$

Find $P[X > 1|Y=1]$.

Solution: For $y > 0$:

$$
\begin{aligned}
f_{X|Y}(x|y) &= \frac{f_{XY}(x,y)}{f_Y(y)} \\
&= \frac{f_{XY}(x,y)}{\int_{-\infty}^{\infty} f_{XY}(x,y)dx} \\
&= \frac{\frac{1}{y}e^{-x/y}e^{-y}}{e^{-y}\int_0^{\infty} \frac{1}{y}e^{-x/y}dx} \\
&= \frac{\frac{1}{y}e^{-x/y}e^{-y}}{e^{-y} \times 1} \\
&= \frac{1}{y}e^{-x/y}
\end{aligned}
$$

Hence:

$$
\begin{aligned}
P[X > 1|Y = y] &= \int_1^{\infty} f_{X|Y}(x|y)dx \\
&= \int_1^{\infty} \frac{1}{y}e^{-x/y}dx \\
&= - \left. e^{-x/y} \right|_1^{\infty} \\
&= e^{-1/y}
\end{aligned}
$$