Jointly Distributed Random Variables

Examples [Ross S6.1]

Example 23.1: The joint pdf of X and Y is given by

$$f_{XY}(x,y) = \begin{cases} 2e^{-x}e^{-2y} & x > 0 \text{ and } y > 0 \\ 0 & \text{else} \end{cases}$$

Compute

a) $P[X > 1, Y < 1]$

b) $P[X < Y]$

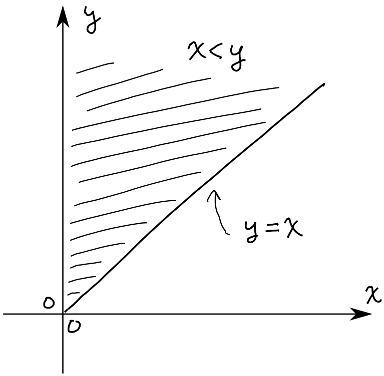
c) $P[X < a]$ (assume $a > 0$)

Solution:

a)
$$\begin{aligned} P[X > 1, Y < 1] &= P[X \in (1,\infty), Y \in (-\infty, 1)] \\ &= \int_{-\infty}^{1} \int_{1}^{\infty} f_{XY}(x,y)dxdy \\ &= \int_{0}^{1} \int_{1}^{\infty} 2e^{-x}e^{-2y}dxdy \\ &= \int_{0}^{1} \left[-2e^{-x}e^{-2y}\right]_{x=1}^{x=\infty} dy \\ &= \int_{0}^{1} 2e^{-1}e^{-2y}dy \\ &= \left[-e^{-1}e^{-2y}\right]_{y=0}^{y=1} \\ &= e^{-1} - e^{-3} \end{aligned}$$

b) $$P[X<Y] = \iint\limits_{x<y} f_{XY}(x,y)\,dx\,dy$$

$$= \iint\limits_{\substack{x<y\\x>0\\y>0}} 2e^{-x}e^{-2y}\,dx\,dy$$



So $$P[X<Y] = \int_0^\infty \int_0^y 2e^{-x}e^{-2y}\,dx\,dy$$

$$= \int_0^\infty \left[-2e^{-x}e^{-2y}\right]_{x=0}^{x=y} dy$$

$$= \int_0^\infty 2e^{-2y} - 2e^{-3y}\,dy$$

$$= \left[-e^{-2y} + \frac{2}{3}e^{-3y}\right]_0^\infty$$

$$= 1 - \frac{2}{3}$$

$$= \frac{1}{3}$$

c) $$\begin{aligned}P[X < a] &= P[X \in (-\infty, a), Y \in (-\infty, \infty)] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{a} f_{XY}(x,y)dxdy \\ &= \int_{0}^{\infty}\int_{0}^{a} 2e^{-x}e^{-2y}dxdy \\ &= \int_{0}^{\infty} \left[-2e^{-x}e^{-2y}\right]_{x=0}^{x=a} dy \\ &= \int_{0}^{\infty} 2(1-e^{-a})e^{-2y}dy \\ &= -(1-e^{-a})e^{-2y}\Big|_{y=0}^{y=\infty} \\ &= (1-e^{-a})\end{aligned}$$

Example 23.2: Given $R > 0$, consider the joint pdf

$$f_{XY}(x,y) = \begin{cases} c & \text{if } x^2 + y^2 \le R^2 \\ 0 & \text{else} \end{cases}$$

for some $c > 0$.

a) Find c.

b) Find the marginal pdf of X.

c) Let $D = \sqrt{X^2 + Y^2}$ be the distance of the pair (X, Y) from the origin. Find $P[D \le a]$.

d) Find $E[D]$.

Note: This is the uniform distribution on a disk of radius R.

Solution:

a) $$1 = \iint\limits_{\mathbb{R}^2} f_{XY}(x,y)dxdy$$
$$= \iint\limits_{x^2+y^2 \leq R^2} c\ dxdy$$
$$= c \iint\limits_{x^2+y^2 \leq R^2} 1\ dxdy$$
$$= c \times \pi R^2$$

So, $c = 1/\pi R^2$.

b) $$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy$$
$$= \int_{y:x^2+y^2 \leq R^2} c\ dy$$
$$= \int_{y:y^2 \leq R^2-x^2} c\ dy \tag{23.1}$$
$$= \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} c\ dy \qquad \text{assuming } x^2 \leq R^2$$
$$= c\sqrt{R^2 - x^2}$$

If $x^2 > R^2$, then the set of y in (23.1) is empty and the integral is 0. So

$$f_X(x) = \begin{cases} \frac{2}{\pi R^2}\sqrt{R^2 - x^2} & x^2 \leq R^2 \\ 0 & \text{else} \end{cases}$$

c) Assuming $0 \leq a \leq R$:

$$P[D \leq a] = P[X^2 + Y^2 \leq a^2]$$

$$
\begin{aligned}
&= \iint\limits_{x^2+y^2\le a^2} f_{XY}(x,y)\,dxdy \\
&= \iint\limits_{x^2+y^2\le a^2} c\,dxdy && \text{since } a^2 \le R^2 \\
&= c \times \pi a^2 \\
&= \frac{a^2}{R^2}
\end{aligned}
$$

If $a > R$, since $X^2 + Y^2$ cannot be larger than R^2, then $P[D \le a] = 1$. Formally:

$$
\begin{aligned}
P[D \le a] &= P[X^2 + Y^2 \le a^2] \\
&= \iint\limits_{x^2+y^2\le a^2} f_{XY}(x,y)\,dxdy \\
&= \iint\limits_{x^2+y^2\le R^2} c\,dxdy + \iint\limits_{R^2<x^2+y^2\le a^2} 0\,dxdy \\
&= c \times \pi R^2 \\
&= 1
\end{aligned}
$$

If $a < 0$, since D can't be negative, $P[D \le a] = 0$.

d) The pdf of D for $0 \le a \le R$ is

$$f_D(a) = \frac{d}{da}\frac{a^2}{R^2} = \frac{2a}{R^2}$$

and 0 otherwise. Therefore

$$E[D] = \int_{-\infty}^{\infty} a f_D(a) da$$

$$
\begin{aligned}
&= \int_0^R \frac{2a^2}{R^2} da \\
&= \frac{2R}{3}
\end{aligned}
$$

Example 23.3: The joint pdf of X and Y is

$$
f_{XY}(x,y) = \begin{cases} e^{-(x+y)} & x > 0 \text{ and } y > 0 \\ 0 & \text{else} \end{cases}
$$

Find the pdf of $Z = X/Y$.

Solution:

X and Y only take +ve values $\Rightarrow$ X/Y only takes +ve values.

Assume $a > 0$:

$$
\begin{aligned}
F_Z(a) &= P\left[\frac{X}{Y} \leq a\right] \\
&= P\left[X \leq aY\right] \\
&= \iint\limits_{x \leq ay} f_{XY}(x,y) dx dy \\
&= \int_0^\infty \int_0^{ay} e^{-x} e^{-y} \, dx dy \\
&= \int_0^\infty (1 - e^{-ay}) e^{-y} \, dy \\
&= \int_0^\infty e^{-y} - e^{-(1+a)y} \, dy \\
&= 1 - \frac{1}{1+a}
\end{aligned}
$$

and $F_Z(a) = 0$ for $a \leq 0$.

Therefore
$$f_Z(a) = \frac{d}{da} F_Z(a)$$
$$= \begin{cases} \frac{1}{(1+a)^2} & a > 0 \\ 0 & \text{else} \end{cases}$$