Random Variables (rv)

Bernoulli and Binomial [Ross S4.6]

A) Let

$$p_X(k) = \begin{cases} 1-p & \text{if } k = 0 \\ p & \text{if } k = 1 \end{cases}$$

with $0 \le p \le 1$.

Then X is called **Bernoulli** with parameter p, denoted $X \sim \mathsf{Bernoulli}(p)$.

This random variable models binary conditions:

- coin flip outcome
- state of a connection
- preference for/against politician

B) Consider n independent trials of $\mathsf{Bernoulli}(p)$.

Let $X =$ # of ones in the n trials.

Then X is called **binomial** with parameters n and p, denoted $X \sim \mathsf{Binomial}(n, p)$.

Note: $\mathsf{Bernoulli}(p) = \mathsf{Binomial}(1, p)$.

For $0 \le k \le n$, there are $\binom{n}{k}$ ways to get k ones from n Bernoulli trials.

Each has probability $p^k(1-p)^{n-k}$. So

$$p_X(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & 0 \le k \le n \\ 0 & \text{else} \end{cases}$$

Note: Since X must be between 0 and n:

$$1 = \sum_{k=0}^{n} p_X(k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}$$

Example 12.1: A company sells screw in packs of 10. Each screw has a prob. 0.01 of being defective. There is a money-back guarantee if *more* than 1 screw is defective. What is the prob. that a pack will be replaced?

Solution: $X \sim \mathsf{Binomial}(10, 0.01)$.

$$\begin{aligned} P[\text{not replaced}] &= P[X=0] + P[X=1] \\ &= \binom{10}{0}(0.01)^0(0.99)^{10} + \binom{10}{1}(0.01)^1(0.99)^9 \\ &\approx 0.996 \end{aligned}$$

So

$$\begin{aligned} P[\text{replaced}] &= 1 - P[\text{not replaced}] \\ &\approx 0.004 \end{aligned}$$

Moments of Binomial

Let $X \sim \mathsf{Binomial}(n, p)$. Then

$$\left.\begin{aligned} E[X] &= np \\ E[X^2] &= n(n-1)p^2 + np \end{aligned}\right] \text{Will prove these later}$$

So

$$\begin{aligned} Var[X] &= E[X^2] - (E[X])^2 \\ &= n(n-1)p^2 + np - (np)^2 \\ &= np(1-p) \end{aligned}$$

Poisson Random Variable [Ross S4.7]

C) We say X is **Poisson** with parameter $\lambda > 0$, denoted $X \sim \mathsf{Poisson}(\lambda)$, if

$$p_X(k) = \begin{cases} \frac{\lambda^k}{k!} e^{-\lambda} & \text{for } k = 0, 1, 2, \dots \\ 0 & \text{else} \end{cases}$$

Note: In Example 9.1 we saw that $\sum_{k=1}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} = 1$.

The Poisson random variable is an approximation of the binomial random variable when:

- n is large
- $\lambda = np$ is moderate

i.e.: $\mathsf{Poisson}(\lambda)$ is $\mathsf{Binomial}(n, \lambda/n)$ when $n \to \infty$.

Why? Let $X \sim \mathsf{Binomial}(n, p)$ with $p = \lambda/n$:

$$\begin{aligned} p_X(k) &= \frac{n!}{(n-k)!\,k!}\, p^k (1-p)^{n-k} \\ &= \frac{n!}{(n-k)!\,k!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \\ &= \frac{n(n-1)\cdots(n-k+1)}{n^k}\, \frac{\lambda^k}{k!}\, \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^k} \end{aligned}$$

If $n \to \infty$:

$$\begin{aligned} &\frac{n}{n} \times \frac{n-1}{n} \times \cdots \times \frac{n-k+1}{n} \to 1 \\ &\left(1 - \frac{\lambda}{n}\right)^k \to 1 \\ &\left(1 - \frac{\lambda}{n}\right)^n \to e^{-\lambda} \end{aligned}$$

$$\Rightarrow p_X(k) \to \frac{\lambda^k}{k!} e^{-\lambda}$$

Example 12.2: Say $n = 100, p = 0.01$. Then $\lambda = 1$.

Then

$$\begin{aligned} p_X(5) &= \frac{100!}{95!\,5!}(0.01)^5(0.99)^{95} \\ &\approx 0.00290 \end{aligned}$$

and

$$\frac{1^5}{5!} e^{-1} \approx 0.00306$$

If we repeat with $n = 1000, p = 0.001$ so $\lambda = 1$ again:

Then

$$\begin{aligned} p_X(5) &= \frac{1000!}{995!\,5!}(0.001)^5(0.999)^{995} \\ &\approx 0.00305 \end{aligned}$$

Poisson should be a good approximation for:

- # of typos on a page
- # of oranges sold in a day at a store
- # of alpha particles emitted by a radioactive substance in 1 second
- # of dead pixels in an LCD display