Jointly Distributed Random Variables

Examples [Ross S6.1]

Example 23.1: The joint pdf of X and Y is given by

$$f_{XY}(x,y) = \begin{cases} 2e^{-x}e^{-2y} & x > 0 \text{ and } y > 0 \\ 0 & \text{else} \end{cases}$$

Compute

a) $P[X > 1, Y < 1]$

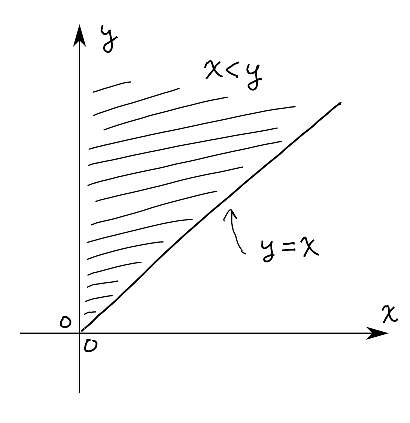
b) $P[X < Y]$

c) $P[X < a]$ (assume $a > 0$)

Solution:

a)
$$\begin{aligned} P[X > 1, Y < 1] &= P[X \in (1, \infty), Y \in (-\infty, 1)] \\ &= \int_{-\infty}^{1} \int_{1}^{\infty} f_{XY}(x,y)dxdy \\ &= \int_{0}^{1} \int_{1}^{\infty} 2e^{-x}e^{-2y}dxdy \\ &= \int_{0}^{1} \left[-2e^{-x}e^{-2y}\right]_{x=1}^{x=\infty} dy \\ &= \int_{0}^{1} 2e^{-1}e^{-2y}dy \\ &= \left[-e^{-1}e^{-2y}\right]_{y=0}^{y=1} \\ &= e^{-1} - e^{-3} \end{aligned}$$

b)
$$\begin{aligned} P[X < Y] &= \iint\limits_{x<y} f_{XY}(x,y)dxdy \\ &= \iint\limits_{\substack{x<y \\ x>0 \\ y>0}} 2e^{-x}e^{-2y}dxdy \end{aligned}$$



So
$$\begin{aligned} P[X < Y] &= \int_{0}^{\infty} \int_{0}^{y} 2e^{-x}e^{-2y}dxdy \\ &= \int_{0}^{\infty} \left[-2e^{-x}e^{-2y}\right]_{x=0}^{x=y} dy \\ &= \int_{0}^{\infty} 2e^{-2y} - 2e^{-3y}dy \\ &= \left[-e^{-2y} + \frac{2}{3}e^{-3y}\right]_{0}^{\infty} \\ &= 1 - \frac{2}{3} \\ &= \frac{1}{3} \end{aligned}$$

c)
$$\begin{aligned} P[X < a] &= P[X \in (-\infty, a), Y \in (-\infty, \infty)] \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{a} f_{XY}(x,y)dxdy \\ &= \int_{0}^{\infty} \int_{0}^{a} 2e^{-x}e^{-2y}dxdy \\ &= \int_{0}^{\infty} \left[-2e^{-x}e^{-2y}\right]_{x=0}^{x=a} dy \\ &= \int_{0}^{\infty} 2(1-e^{-a})e^{-2y}dy \\ &= -(1-e^{-a})e^{-2y}\Big|_{y=0}^{y=\infty} \\ &= (1-e^{-a}) \end{aligned}$$

Example 23.2: Given $R > 0$, consider the joint pdf

$$f_{XY}(x,y) = \begin{cases} c & \text{if } x^2 + y^2 \le R^2 \\ 0 & \text{else} \end{cases}$$

for some $c > 0$.

a) Find c.

b) Find the marginal pdf of X.

c) Let $D = \sqrt{X^2 + Y^2}$ be the distance of the pair (X, Y) from the origin. Find $P[D \le a]$.

d) Find $E[D]$.

Note: This is the uniform distribution on a disk of radius R.

Solution:

a)
$$\begin{aligned} 1 &= \iint\limits_{\mathbb{R}^2} f_{XY}(x,y)dxdy \\ &= \iint\limits_{x^2+y^2 \le R^2} c\ dxdy \\ &= c \iint\limits_{x^2+y^2 \le R^2} 1\ dxdy \\ &= c \times \pi R^2 \end{aligned}$$

So, $c = 1/\pi R^2$.

b)
$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dy \\ &= \int_{y:x^2+y^2 \le R^2} c\ dy \\ &= \int_{y:y^2 \le R^2 - x^2} c\ dy && (23.1) \\ &= \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} c\ dy && \text{assuming } x^2 \le R^2 \\ &= c\sqrt{R^2 - x^2} \end{aligned}$$

If $x^2 > R^2$, then the set of y in (23.1) is empty and the integral is 0. So

$$f_X(x) = \begin{cases} \frac{2}{\pi R^2}\sqrt{R^2 - x^2} & x^2 \le R^2 \\ 0 & \text{else} \end{cases}$$

c) Assuming $0 \le a \le R$:

$$\begin{aligned} P[D \le a] &= P[X^2 + Y^2 \le a^2] \\ &= \iint\limits_{x^2+y^2 \le a^2} f_{XY}(x,y)\ dxdy \\ &= \iint\limits_{x^2+y^2 \le a^2} c\ dxdy && \text{since } a^2 \le R^2 \\ &= c \times \pi a^2 \\ &= \frac{a^2}{R^2} \end{aligned}$$

If $a > R$, since $X^2 + Y^2$ cannot be larger than R^2, then $P[D \le a] = 1$. Formally:

$$\begin{aligned} P[D \le a] &= P[X^2 + Y^2 \le a^2] \\ &= \iint\limits_{x^2+y^2 \le a^2} f_{XY}(x,y)\ dxdy \\ &= \iint\limits_{x^2+y^2 \le R^2} c\ dxdy + \iint\limits_{R^2 < x^2+y^2 \le a^2} 0\ dxdy \\ &= c \times \pi R^2 \\ &= 1 \end{aligned}$$

If $a < 0$, since D can't be negative, $P[D \le a] = 0$.

d) The pdf of D for $0 \le a \le R$ is

$$f_D(a) = \frac{d}{da}\frac{a^2}{R^2} = \frac{2a}{R^2}$$

and 0 otherwise. Therefore

$$\begin{aligned} E[D] &= \int_{-\infty}^{\infty} a f_D(a)da \\ &= \int_{0}^{R} \frac{2a^2}{R^2}da \\ &= \frac{2R}{3} \end{aligned}$$

Example 23.3: The joint pdf of X and Y is

$$f_{XY}(x,y) = \begin{cases} e^{-(x+y)} & x > 0 \text{ and } y > 0 \\ 0 & \text{else} \end{cases}$$

Find the pdf of $Z = X/Y$.

Solution:

X and Y only take +ve values $\Rightarrow X/Y$ only takes +ve values.

Assume $a > 0$:

$$\begin{aligned} F_Z(a) &= P\left[\frac{X}{Y} \le a\right] \\ &= P\left[X \le aY\right] \\ &= \iint\limits_{x \le ay} f_{XY}(x,y)dxdy \\ &= \int_{0}^{\infty} \int_{0}^{ay} e^{-x}e^{-y}\ dxdy \\ &= \int_{0}^{\infty} (1 - e^{-ay})e^{-y}\ dy \\ &= \int_{0}^{\infty} e^{-y} - e^{-(1+a)y}\ dy \\ &= 1 - \frac{1}{1+a} \end{aligned}$$

and $F_Z(a) = 0$ for $a \le 0$.

Therefore
$$\begin{aligned} f_Z(a) &= \frac{d}{da}F_Z(a) \\ &= \begin{cases} \frac{1}{(1+a)^2} & a > 0 \\ 0 & \text{else} \end{cases} \end{aligned}$$