Jointly Distributed Random Variables

Conditional Distributions:

Discrete Case [Ross S6.4]

Recall that for $P[F] > 0$:

$$P[E|F] = \frac{P[EF]}{P[F]}$$

Say $p_Y(y) > 0$. The **conditional pmf** for X given Y is

$$\begin{aligned} p_{X|Y}(x|y) &= P[X = x \,|\, Y = y] \\ &= \frac{P[X = x, Y = y]}{P[Y = y]} \\ &= \frac{p_{XY}(x, y)}{p_Y(y)} \end{aligned}$$

The **conditional cdf** for X given Y is

$$\begin{aligned} F_{X|Y}(x|y) &= P[X \le x \,|\, Y = y] \\ &= \frac{P[X \le x, Y = y]}{P[Y = y]} \\ &= \sum_{a \le x} \frac{P[X = a, Y = y]}{P[Y = y]} \\ &= \sum_{a \le x} p_{X|Y}(a|y) \end{aligned}$$

If X and Y are independent:

$$\begin{aligned} p_{X|Y}(x|y) &= \frac{p_{XY}(x, y)}{p_Y(y)} \\ &= \frac{p_X(x)p_Y(y)}{p_Y(y)} \\ &= p_X(x) \end{aligned}$$

Example 27.1: Let $X \sim \mathsf{Poisson}(\lambda_1)$ and $Y \sim \mathsf{Poisson}(\lambda_2)$ be independent. Find the conditional pmf for X given $X + Y = n$.

Solution:

Example 27.2: Let $X_1, X_2, \ldots, X_n$ be iid and $\sim \mathsf{Bernoulli}(p)$.

Say these result in k ones. Show that each of the $\binom{n}{k}$ possible orderings of k ones are then equally likely.

Solution:

Continuous Case [Ross S6.5]

If X and Y are continuous, for $f_Y(y) > 0$, the **conditional pdf** of X given $Y = y$ is

$$f_{X|Y}(x|y) = \frac{f_{XY}(x, y)}{f_Y(y)}$$

We also define:

$$P[X \in A | Y = y] = \int_A f_{X|Y}(x|y)dx$$

and then

$$\begin{aligned} \int_{-\infty}^{\infty} P[X \in A | Y = y] f_Y(y) dy &= \int_{-\infty}^{\infty} \left[\int_A f_{X|Y}(x|y) dx \right] f_Y(y) dy \\ &= \int_{-\infty}^{\infty} \int_A f_{X|Y}(x|y) f_Y(y) dy dx \\ &= \int_A \int_{-\infty}^{\infty} f_{XY}(x, y) dy dx \\ &= P[X \in A] \end{aligned} \tag{27.2}$$

With $A = (-\infty, a]$, we get the **conditional cdf**

$$F_{X|Y}(a|y) = P[X \le a | Y = y] = \int_{-\infty}^{a} f_{X|Y}(x|y)dx$$

If X and Y are independent and $f_Y(y) > 0$:

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f_{XY}(x, y)}{f_Y(y)} \\ &= \frac{f_X(x) f_Y(y)}{f_Y(y)} \\ &= f_X(x) \end{aligned}$$

Example 27.3: The joint pdf of X and Y is

$$f_{XY}(x, y) = \begin{cases} \frac{e^{-x/y} e^{-y}}{y} & 0 < x < \infty,\ 0 < y < \infty \\ 0 & \text{else} \end{cases}$$

Find $P[X > 1 | Y = 1]$.

Solution: