Limit Theorems

The Central Limit Theorem (CLT) [Ross 8.3]

Proposition 39.1 *The Central Limit Theorem*

Let $X_1, X_2, \ldots$ be a sequence of iid random variables having mean μ and variance σ^2. Then, the distribution of

$$\begin{aligned} Z_n &= \frac{X_1 + X_2 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{X_i - \mu}{\sigma} \end{aligned}$$

tends to the standard normal as $n \to \infty$. Specifically,

$$P[Z_n \leq a] \to \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-u^2/2} du}_{\Phi(a)} \qquad \text{as } n \to \infty$$

Why is the CLT true?

Let $Y_i = \frac{X_i - \mu}{\sigma}$. Then Y_i are iid with mean 0 and variance 1 and

$$Z_n = \frac{Y_1 + Y_2 + \cdots + Y_n}{\sqrt{n}}$$

We will show that the MGF of Z_n converges to the MGF of $\mathcal{N}(0,1)$, i.e., to $e^{t^2/2}$.

The MGF of $Y_i/\sqrt{n}$ is

$$E\left[e^{tY_i/\sqrt{n}}\right] = M_Y\left(\frac{t}{\sqrt{n}}\right)$$

So, the MGF of $Z_n = \sum_{i=1}^n Y_i/\sqrt{n}$ is

$$M_{Z_n}(t) = \left[M_Y\left(\frac{t}{\sqrt{n}}\right)\right]^n$$

We want to show that

$$\lim_{n\to\infty}\left[M_Y\left(\frac{t}{\sqrt{n}}\right)\right]^n = e^{t^2/2}$$

Define $L(t) = \ln M_Y(t)$. Then

$$\begin{aligned}
L(0) &= \ln M_Y(0) = 0 & M_Y(0) &= E[e^0] = 1 \\
L'(0) &= \frac{M_Y'(0)}{M_Y(0)} = \frac{E[Y]}{1} = 0 & M_Y'(0) &= E[Y] = 0 \\
L''(0) &= \frac{M_Y(0)M_Y''(0) - [M_Y'(0)]^2}{[M_Y(0)]^2} & M_Y''(0) &= E[Y^2] = 1 \\
&= 1
\end{aligned}$$

So, for small t, $L(t) = \frac{1}{2}t^2 + O(t^3)$.

Finally,

$$\begin{aligned}
\lim_{n\to\infty} \ln M_{Z_n}(t) &= \lim_{n\to\infty} \ln\left[M_Y(t/\sqrt{n})\right]^n \\
&= \lim_{n\to\infty} n \ln M_Y(t/\sqrt{n}) \\
&= \lim_{n\to\infty} \frac{L(t/\sqrt{n})}{n^{-1}} \\
&= \lim_{n\to\infty} \frac{\frac{1}{2}(t/\sqrt{n})^2}{n^{-1}} + O\left(\frac{(t/\sqrt{n})^3}{n^{-1}}\right) \\
&= \lim_{n\to\infty} \frac{t^2}{2} + O\left(\frac{1}{\sqrt{n}}\right)
\end{aligned}$$

$$= \frac{t^2}{2}$$

So $\lim_{n\to\infty} M_{Z_n}(t) = e^{t^2/2}$

The CLT can be used to approximate probabilities:

Example 39.1: An astronomer takes iid measurements $X_1, X_2, \ldots$ of the distance of a star.

Each X_i has mean d (the true distance) and variance 4 light-years^2.

How many measurements are needed to be 95% certain that the average of the measurements is within ± 0.5 light-years of the true value d?

Solution: Let

$$Z_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{X_i - d}{\sqrt{4}}$$

By the CLT, when n is large, this is approximately $\mathcal{N}(0,1)$.

$$\begin{aligned}
&P\left[-0.5 \le \left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) - d \le 0.5\right] \\
&\quad = P\left[-0.5 \le \frac{1}{n}\sum_{i=1}^{n}(X_i - d) \le 0.5\right] \\
&\quad = P\left[-0.5 \times \frac{\sqrt{n}}{2} \le \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{X_i - d}{2} \le 0.5 \times \frac{\sqrt{n}}{2}\right] \\
&\quad = P\left[-\frac{\sqrt{n}}{4} \le Z_n \le \frac{\sqrt{n}}{4}\right]
\end{aligned}$$

$$\approx \Phi\left(\frac{\sqrt{n}}{4}\right) - \Phi\left(-\frac{\sqrt{n}}{4}\right)$$
$$= 2\Phi\left(\frac{\sqrt{n}}{4}\right) - 1$$

For this to be at least 0.95, we need

$$\Phi\left(\frac{\sqrt{n}}{4}\right) \geq 0.975$$

From the $\Phi(.)$ Table [Notes #18], $\sqrt{n}/4 \geq 1.96$.

The smallest integer than makes this true is $n = 62$.

Note: This analysis assumes that with 62 observations, Z_n is well approximated by a Gaussian.

The Chebyshev inequality is not an approximation.

$$E\left[\sum_{i=1}^{n} \frac{X_i}{n}\right] = d \qquad Var\left[\sum_{i=1}^{n} \frac{X_i}{n}\right] = \frac{4}{n}$$

So by Chebyshev:

$$P\left[\left|\sum_{i=1}^{n} \frac{X_i}{n} - d\right| \geq 0.5\right] \leq \frac{4/n}{(0.5)^2} = \frac{16}{n}$$

95% confident $\Rightarrow 16/n \leq 0.05 \Rightarrow n \geq 320$ measurements are enough.

Example 39.2: Let $X_1, \ldots, X_{10}$ be the outcomes of 10 fair dice rolls. Use the CLT to approximate $P[30 \leq X_1 + \cdots + X_{10} \leq 40]$.

Solution: Here $E[X_i] = \frac{7}{2}$ and $Var[X_i] = \frac{35}{12}$

Then

$$
\begin{aligned}
&P[30 \le \sum_{i=1}^{10} X_i \le 40] \\
&\quad = P[29.5 \le \sum_{i=1}^{10} X_i \le 40.5] \\
&\quad = P\left[\frac{1}{\sqrt{10}} \frac{29.5 - 10 \cdot \frac{7}{2}}{\sqrt{35/12}} \le \frac{1}{\sqrt{10}} \sum_{i=1}^{10} \frac{X_i - \frac{7}{2}}{\sqrt{35/12}} \le \frac{1}{\sqrt{10}} \frac{40.5 - 10 \cdot \frac{7}{2}}{\sqrt{35/12}}\right] \\
&\quad \approx P\left[-1.0184 \le Z \le 1.10184\right] \qquad\qquad Z \sim \mathcal{N}(0,1) \\
&\quad = 2\Phi(1.0184) - 1 \\
&\quad \approx 0.6915
\end{aligned}
$$

Strong Law of Large Numbers [Ross S8.4]

We saw earlier the *weak* law of large numbers. This suggests that there is a strong law of large numbers as well (and there is).

Proposition 39.2 *Strong Law of Large Numbers*

Let $X_1, X_2, \ldots$ be iid with commom mean $E[X_i] = \mu$. Then

$$
P\left[\lim_{n\to\infty} \frac{X_1 + X_2 + \cdots + X_n}{n} = \mu\right] = 1
$$