Chapter 6

Continuous-Time Periodic Signal Analysis: Fourier Series

6.1 Fourier Series

Continuous-time periodic signals can be analyzed by Fourier series.

A periodic signal $x(t)$ with period T_0 is defined by:

$$x(t) = x(t + T_0), \qquad -\infty < t < \infty$$

The average power of a periodic signal $x(t)$ with period T_0 is:

$$P_{av} = \frac{1}{T_0} \int_{T_0} |x(t)|^2 \, dt$$

<u>Theorem</u>: A periodic signal $x(t)$ with period T_0 can be expressed by the following (*<u>exponential</u>*) Fourier series:

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

Where $\omega_0 = \frac{2\pi}{T_0}$ fundamental angular frequency

$$D_n = \frac{1}{T_0} \int_{T_0} x(t) e^{-jn\omega_0 t} \, dt \qquad \text{coefficients of the Fourier series}$$

<u>Remark</u>: The component D_n has the angular frequency $\omega_n = n\omega_0$ where n is an integer. In addition, we allow both positive and negative frequencies.

Remark: There are other forms of the Fourier series (see textbook): The *trigonometric form* and the *compact trigonometric form*. However, we use the *exponential form* since it is commonly used and more convenient for analysis.

- **Existence of Fourier Series**

The Fourier series of a periodic signal $x(t)$ with period T_0 exists if the signal satisfies the following Dirichlet conditions:

(a) The signal $x(t)$ must be absolutely integrable over T_0, i.e. $\int_{T_0} |x(t)|\, dt < \infty$.

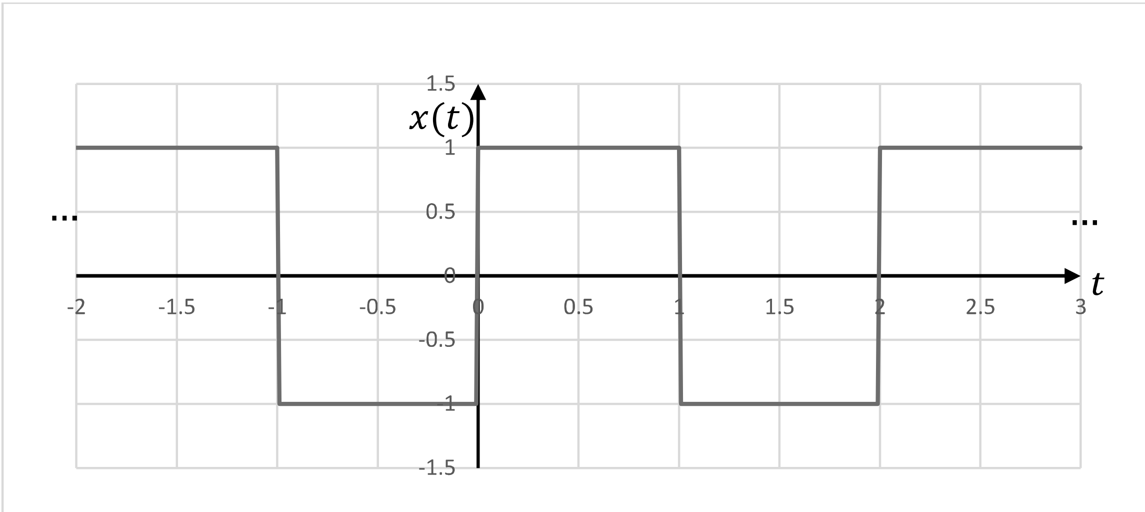
(b) $x(t)$ must have a finite number of finite discontinuities in one period.

(c) $x(t)$ must have a finite number of maxima and minima in one period.

Remark: When Fourier series exists, it converges as follows:

$$\sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} = \begin{cases} x(t), & \text{if } x(t) \text{ is continuous at } t. \\ \frac{1}{2}[x(t^-) + x(t^+)], & \text{if } x(t) \text{ is discontinuous at } t. \end{cases}$$

Example: Find the Fourier series for the following square wave periodic signal:



Solution: $T_0 = 2, \quad \omega_0 = \frac{2\pi}{T_0} = \pi$

$$D_n = \frac{1}{T_0}\int_{T_0} x(t)e^{-jn\omega_0 t}dt = \frac{1}{2}[\int_0^1 (1)e^{-jn\pi t}dt + \int_1^2 (-1)e^{-jn\pi t}\, dt]$$

$$= \frac{1}{2(-jn\pi)}\left[\left(e^{-jn\pi} - 1\right) - \left(e^{-j2n\pi} - e^{-jn\pi}\right)\right]$$

$$= \frac{1}{-j2n\pi}[(-1)^n - 1 - 1 + (-1)^n]$$

$$\Rightarrow \quad D_n = \frac{1}{jn\pi}[1 - (-1)^n], \qquad n \neq 0$$

For $n = 0$: $\quad D_0 = \frac{1}{T_0}\int_0^{T_0} x(t)dt = 0$

Fourier Series:

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\pi t}$$

Remark: From Dirichlet:

$$\sum_{n=-\infty}^{\infty} D_n e^{jn\pi t} = \begin{cases} x(t) = 1, & t \in (0,1) \\ x(t) = -1, & t \in (1,2) \\ \frac{1}{2}[x(1^-) + x(1^+)] = \frac{1}{2}[1-1] = 0, & t = 1 \end{cases}$$

- **Fourier Spectra**

Plots that represent the frequency content of $x(t)$. They represent D_n as a function of ω $\left(\text{or } f = \frac{\omega}{2\pi}\right)$. In general, D_n is complex:

$$D_n = |D_n| e^{j\angle D_n}$$

where

$|D_n|$ versus ω is the amplitude spectra

$\angle D_n$ versus ω is the phase spectra

The spectra exists for specific values of the frequency: $\omega = n\omega_0\,, n =$ integer, i.e. $\omega = 0, \pm\omega_0, \pm 2\omega_0, \ldots$ This is also seen from:

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{\underbrace{jn\omega_0}_{\omega} t}$$

Remark (Symmetry Property): If $x(t)$ is real, then

$$D_n^* = D_{-n}$$

$\Rightarrow$ $|D_n|$ is an even function

$\angle D_n$ is an odd function

Example: Spectra of the square wave (from previous example):

$$D_n = \begin{cases} \frac{1}{jn\pi}[1-(-1)^n], & n \neq 0 \\ 0, & n = 0 \end{cases}$$

Alternatively:

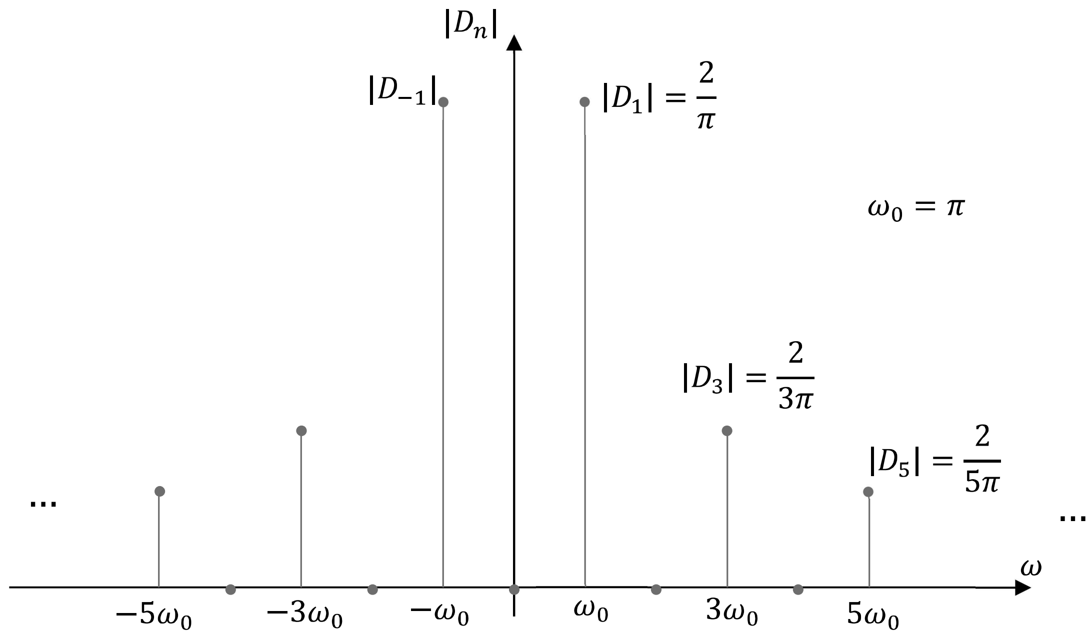
$$D_n = \begin{cases} \frac{2}{jn\pi}, & n\ odd \\ 0, & n\ even \end{cases}$$

Magnitude:

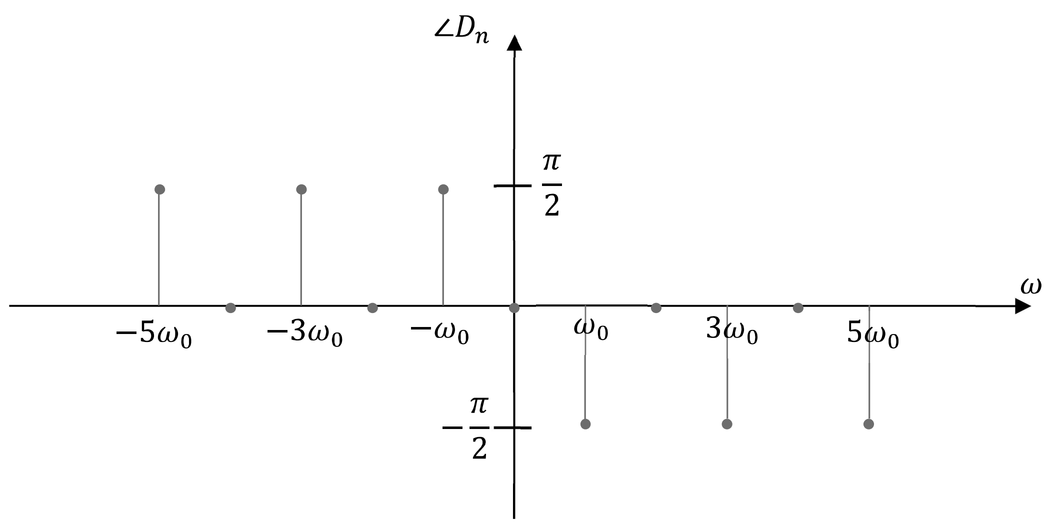
$$|D_n| = \begin{cases} \left|\frac{2}{n\pi}\right|, & n\ odd \\ 0, & n\ even \end{cases}$$

Phase:

$$\angle D_n = \begin{cases} -\frac{\pi}{2}, & n\ postive\ odd \\ \frac{\pi}{2}, & n\ negative\ odd \\ 0, & else \end{cases}$$



$|D_n|$
$|D_{-1}|$
$|D_1| = \frac{2}{\pi}$
$\omega_0 = \pi$
$|D_3| = \frac{2}{3\pi}$
$|D_5| = \frac{2}{5\pi}$
...
...
ω
$-5\omega_0$
$-3\omega_0$
$-\omega_0$
ω_0
$3\omega_0$
$5\omega_0$



$\angle D_n$
$\frac{\pi}{2}$
ω
$-5\omega_0$
$-3\omega_0$
$-\omega_0$
ω_0
$3\omega_0$
$5\omega_0$
$-\frac{\pi}{2}$

Remark: Passing a periodic signal $x(t)$ through an ideal low-pass filter results in a partial sum:

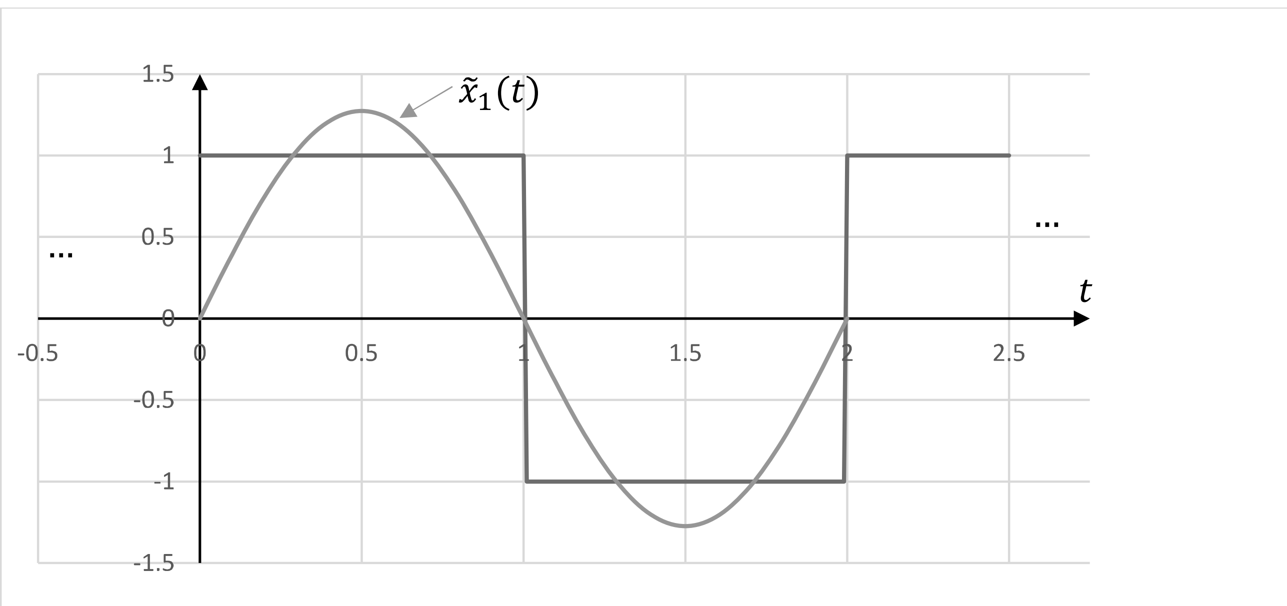
$$\tilde{x}_i(t) = \sum_{n=-i}^{i} D_n e^{jn\omega_0 t}$$

Where i=integer that corresponds to the highest frequency that passes through the filter, $\omega_i = i\omega_0$.

For the previous example: $(\omega_0 = \pi)$

$$\tilde{x}_1(t) = D_{-1}e^{-j\pi t} + D_1 e^{j\pi t} = \frac{2}{-j\pi}e^{-j\pi t} + \frac{2}{j\pi}e^{j\pi t}$$

$$= \frac{4}{\pi}\left[\frac{1}{2j}\left(e^{j\pi t} - e^{-j\pi t}\right)\right] = \frac{4}{\pi}\sin(\pi t)$$



$$\tilde{x}_3(t) = D_{-3}e^{-j3\pi t} + D_{-1}e^{-j\pi t} + D_1 e^{j\pi t} + D_3 e^{j3\pi t}$$

$$= \frac{4}{\pi}\sin(\pi t) + \frac{4}{3\pi}\sin(3\pi t)$$

- **Parseval's Theorem**

Given a periodic signal $x(t)$ with period T_0 that satisfies Dirichlet conditions, the average power of $x(t)$ is:

$$P_{av} = \frac{1}{T_0}\int_{T_0} |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} |D_n|^2$$

Proof:

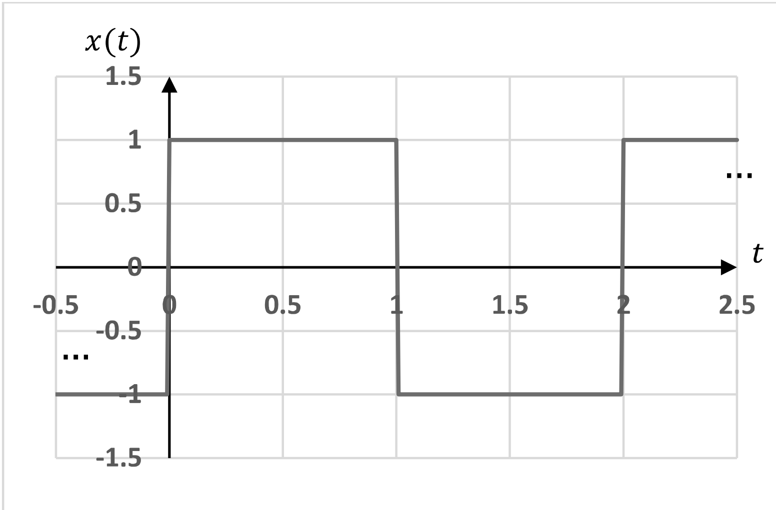
$$P_{av} = \frac{1}{T_0}\int_{T_0} |x(t)|^2 dt = \frac{1}{T_0}\int_{T_0} x(t) \cdot x^*(t)\, dt$$

$$= \frac{1}{T_0}\int_{T_0} [\sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}][\sum_{m=-\infty}^{\infty} D_m^* e^{-jm\omega_0 t}]\, dt$$

$$= \frac{1}{T_0}\sum_{n}\sum_{m} D_n D_m^* \underbrace{\int_{T_0} e^{j(n-m)\omega_0 t}\, dt}_{=\begin{cases} T_0, & if\ m=n \\ 0, & else \end{cases}}$$

$$\Rightarrow \quad P_{av} = \sum_{n=-\infty}^{\infty} |D_n|^2$$

- **<u>Example</u>:** Square wave



In time-domain:

$$P_{av} = \frac{1}{T_0}\int_0^{T_0} |x(t)|^2\, dt = \frac{1}{2}\int_0^{1} (1)^2\ dt + \frac{1}{2}\int_1^{2} (-1)^2\, dt = 1$$

In frequency-domain:

$$D_n = \begin{cases} \dfrac{2}{jn\pi}, & n\ odd \\ 0, & n\ even \end{cases}$$

$$P_{av} = \sum_{n=-\infty}^{\infty} |D_n|^2 = \sum_{\substack{n=-\infty \\ n\ odd}}^{\infty} \frac{4}{n^2\pi^2} = \frac{8}{\pi^2} \sum_{\substack{n\geq 1 \\ n\ odd}} \frac{1}{n^2}$$

<u>Remark</u>: Last formula can be used to approximate the constant *pi*(π) by a truncated sum. Setting $P_{av} = 1$ and solving for π, we get the sum:

$$\Rightarrow \quad \pi = \left[8 \sum_{\substack{n\geq 1 \\ n\ odd}} \frac{1}{n^2} \right]^{\frac{1}{2}}$$

Remark: Passing $x(t)$ through an ideal low-pass filter results in a partial sum for the average power:

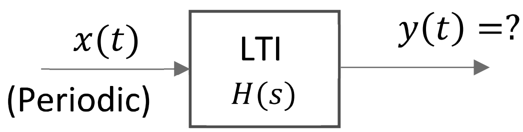
$$\tilde{P}_{av;i} = \sum_{n=-i}^{i} |D_n|^2$$

- **Example:** Square Wave

$\boldsymbol{i}$	$\%average\ Power\ \left(\frac{\tilde{P}_{av;i}}{P_{av}} \times 100\%\right)$
1	81%
3	90%
5	93.3
…	…

6.2 Response of LTI Systems to Periodic Signals

Assume an LTI system with transfer function $H(s)$ and a periodic input:



By Fourier series theorem: $x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$, $\omega_0 = \frac{2\pi}{T_0}$

Since the system is LTI:

$$x(t) = e^{st} \quad \Rightarrow \quad y(t) = H(s)e^{st}$$

(everlasting expo.) (everlasting expo.)

For $s = jn\omega_0$:

$$x(t) = e^{jn\omega_0 t} \quad \Rightarrow \quad y(t) = H(jn\omega_0)e^{jn\omega_0 t}$$

Also, by linearity, multiply by D_n and take the sum:

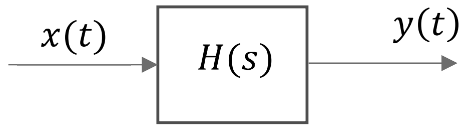
$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \quad \Rightarrow \quad y(t) = \sum_{n=-\infty}^{\infty} \underbrace{D_n H(jn\omega_0)}_{\widehat{D}_n} e^{jn\omega_0 t}$$

(Fourier series of $x(t)$) (Fourier series of y(t))

Therefore, if the input $x(t)$ is periodic with period T_0 and Fourier coefficients D_n, then the output $y(t)$ is also periodic with the same period T_0 and Fourier coefficients $\widehat{D}_n$ given by:

$$\boxed{\widehat{D}_n = D_n \cdot H(jn\omega_0)}$$

- **<u>Example</u>:** Consider the LTI system $H(s) = \frac{1}{s+1}$ shown below:



Let $x(t)$ be the square wave, as before, for which $\omega_0 = \frac{2\pi}{T_0} = \pi$ and Fourier series coefficients:

$$D_n = \begin{cases} \dfrac{2}{jn\pi}, & n \; odd \\ 0, & n \; even \end{cases}$$

Then, $y(t)$ is also periodic with $\omega_0 = \pi$ and $\widehat{D}_n$ given by:

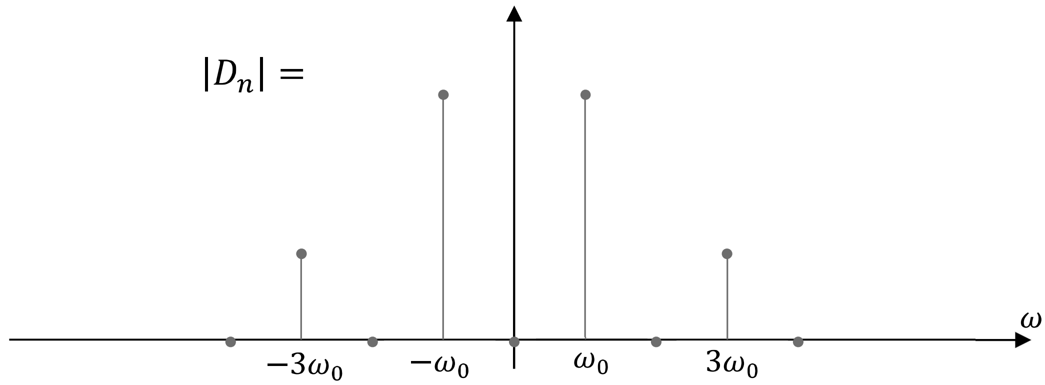
$$\widehat{D}_n = D_n H(jn\omega_0) = \begin{cases} \dfrac{2}{jn\pi} \cdot \dfrac{1}{jn\pi + 1}, & n \; odd \\ 0, & n \; even \end{cases}$$

Spectra of $y(t)$:

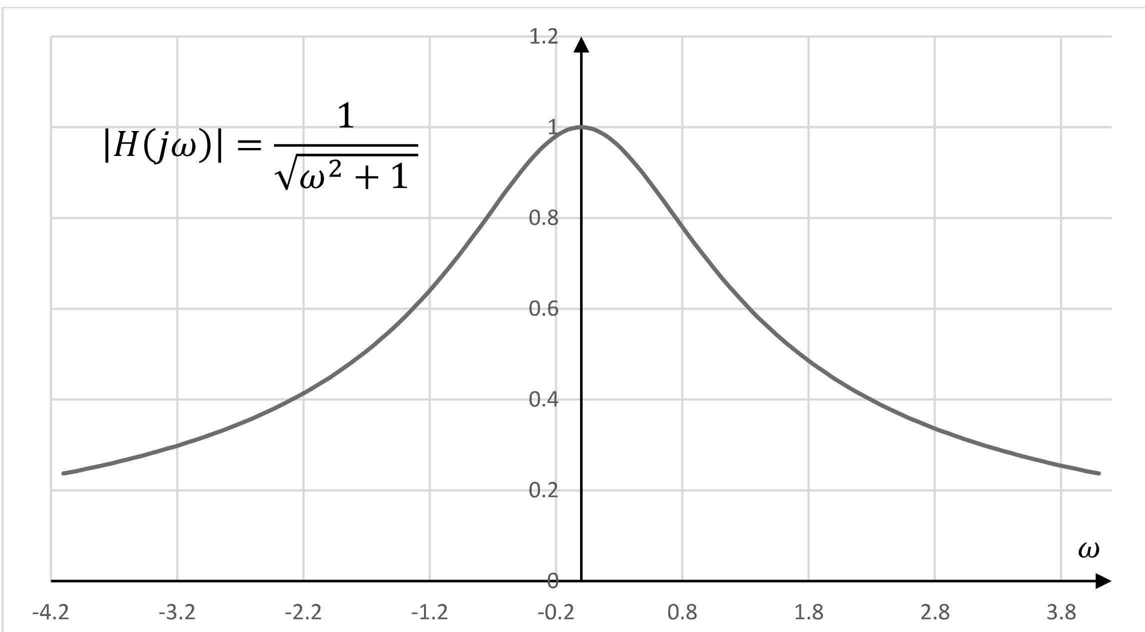
$$\left|\widehat{D}_n\right| = |D_n| \cdot |H(jn\omega_0)|$$

$$\angle\widehat{D}_n = \angle D_n + \angle H(jn\omega_0)$$

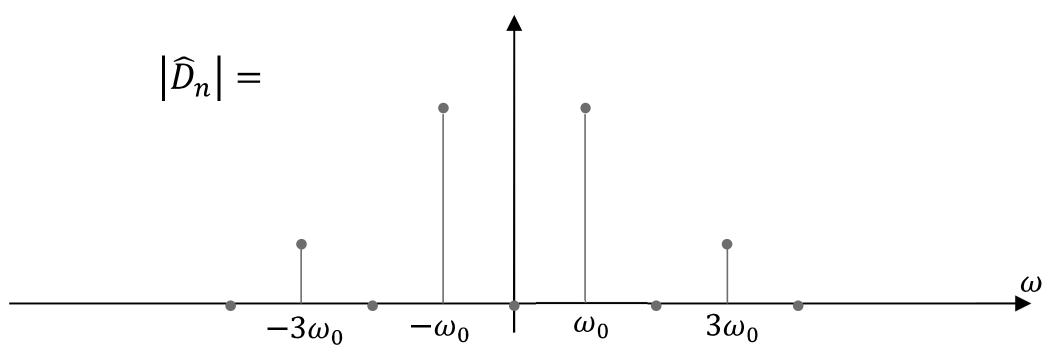
<u>Remark</u>: Magnitude plots are multiplied, while phase plots are added.



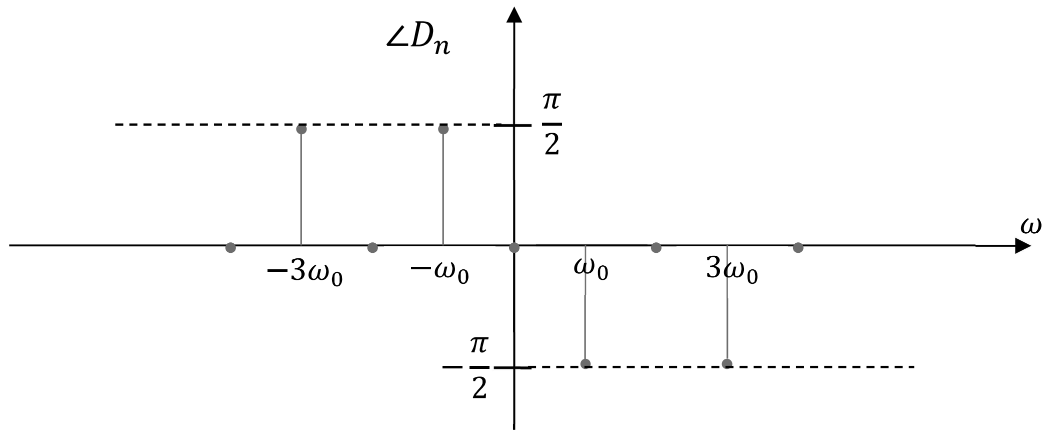
$|D_n| =$
ω
$-3\omega_0$
$-\omega_0$
ω_0
$3\omega_0$



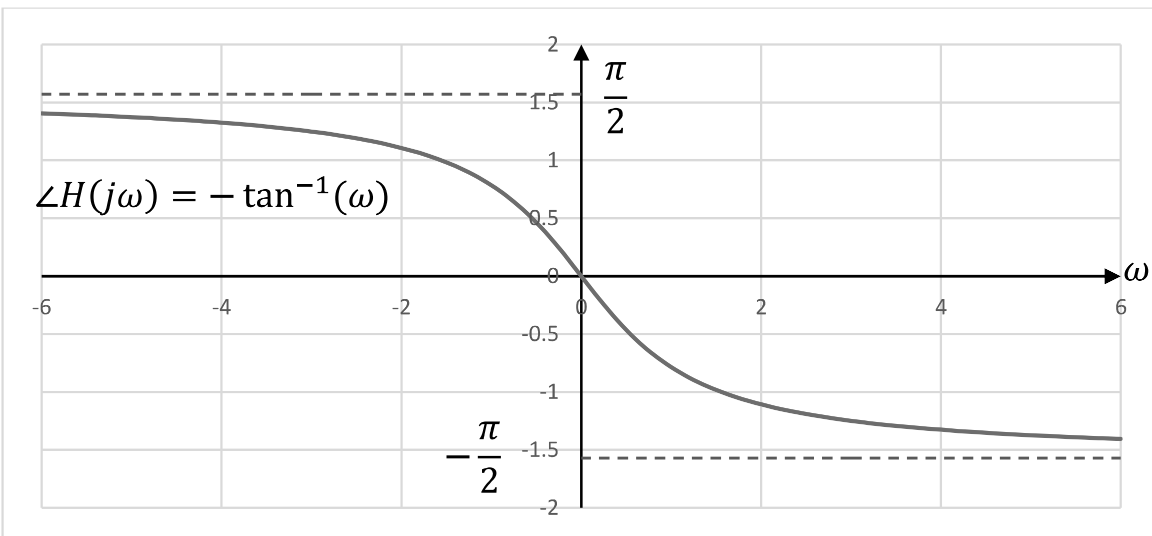
$|H(j\omega)| = \dfrac{1}{\sqrt{\omega^2+1}}$
1.2
1
0.8
0.6
0.4
0.2
0
ω
-4.2
-3.2
-2.2
-1.2
-0.2
0.8
1.8
2.8
3.8



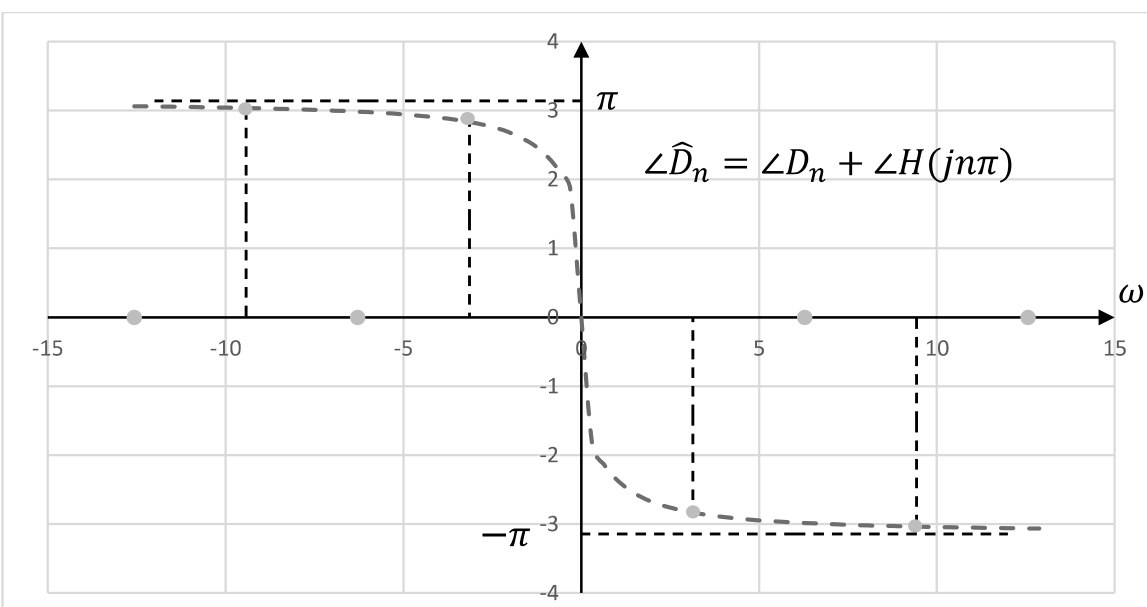
$|\widehat{D}_n| =$
ω
$-3\omega_0$
$-\omega_0$
ω_0
$3\omega_0$



$\angle D_n$
$\frac{\pi}{2}$
ω
$-3\omega_0$
$-\omega_0$
ω_0
$3\omega_0$
$-\frac{\pi}{2}$



$\frac{\pi}{2}$
$\angle H(j\omega) = -\tan^{-1}(\omega)$
ω
$-\frac{\pi}{2}$



π
$\angle \widehat{D}_n = \angle D_n + \angle H(jn\pi)$
ω
$-\pi$