Continuous Random Variables

[Ross S5.1]

We saw random variables where the set of possible outcomes was discrete. In some cases, a random variable can take a continuum of values:

$X =$ time at which a train arrives

$Y =$ voltage across a resistor

$Z =$ rainfall measured in mm

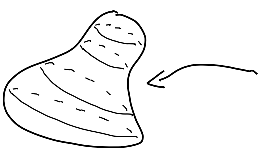
Definition 15.1: We say X is a continuous random variable if there is a non-negative function $f_X(x)$ such that

$$P[X \in B] = \int_B f_X(x)\,dx = \int_B f_X(u)\,du$$

$f_X(x)$ is called **probability density function** (pdf).

[Textbook omits subscript X on $f_X(x)$...]

This is similar to mass density: if I know $\rho(x)$, the **density of mass** in kg/m^3 at every point $x \in \mathbb{R}^3$, then the mass inside any volume V is:



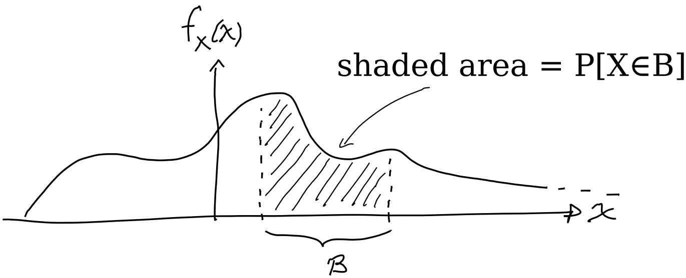
3D volume $V \subset \mathbb{R}^3$

density $\rho(\underline{x})$ at $\underline{x} \in \mathbb{R}^3$

$$m(V) = \iiint_V \rho(\underline{x})\, d\underline{x}$$

$f_X(x)$ is similar, except it measures the *density of probability*, not mass:

$$P[X \in B] = \int_B f_X(x)\, dx$$



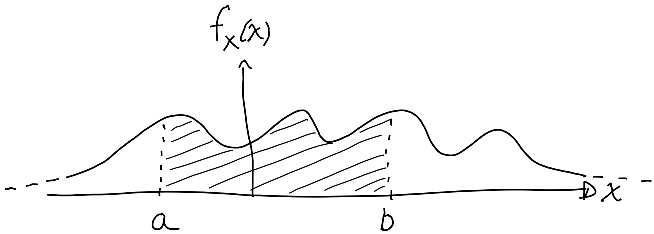
Since X must take some value:

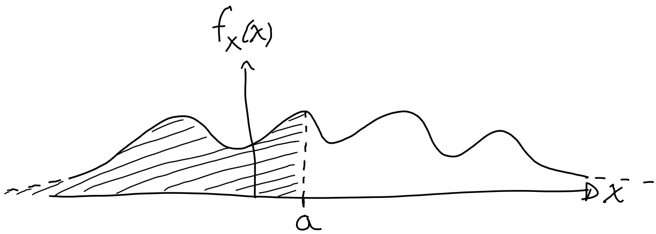
$$1 = P[X \in (-\infty, \infty)] = \int_{-\infty}^{\infty} f_X(x)\, dx. \tag{15.1}$$

Note: Say X has units of kg. Since dx has units of kg, $f_X(x)$ has units of kg^{-1}.

Once we know $f_X(x)$, all probability statements about X can be answered:

1) $P[\ X \in [a,b]\ ] = \int_a^b f_X(x) dx$





2) $P[X = a] = P[\ X \in [a, a]\ ] = \int_a^a f_X(x)dx = 0$

3) $F_X(a) = P[X \leq a] = P[\ X \in (-\infty, a]\ ] = \int_{-\infty}^a f_X(x)dx$

4) $f_X(a) = \frac{d}{da} F_X(a)$

Example 15.1: The lifetime of a motor in months is a random variable with pdf

$$f_X(x) = \begin{cases} \lambda e^{-x/100} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

for some constant λ. What is the probability that it functions for

a) between 50 and 150 months?

b) fewer than 100 months?

Solution: a) The pdf of X must integrate to 1:

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} f_X(x)dx \\
&= \lambda \int_0^{\infty} e^{-x/100}dx \\
&= \lambda \left[-100e^{-x/100}\right]_0^{\infty} \\
&= \lambda(0 - (-100))
\end{aligned}$$

So, $\quad \lambda = 1/100$

$$\begin{aligned}
P[50 < X < 150] &= \int_{50}^{150} f_X(x)dx \\
&= \int_{50}^{150} \frac{1}{100}e^{-x/100}dx \\
&= e^{-1/2} - e^{-3/2} \\
&\approx 0.383
\end{aligned}$$

b)

$$\begin{aligned}
P[X < 100] &= \int_{-\infty}^{100} f_X(x)dx \\
&= \int_0^{100} \frac{1}{100}e^{-x/100}dx \\
&= 1 - e^{-1} \\
&\approx 0.632
\end{aligned}$$

Example 15.2: Let X have pdf $f_X(x)$, and $Y = 2X$. Find $f_Y(y)$.

Solution:

$$
\begin{aligned}
F_Y(a) &= P[Y \leq a] \\
&= P[2X \leq a] \\
&= P[X \leq \frac{a}{2}] \\
&= F_X\left(\frac{a}{2}\right)
\end{aligned}
$$

and $\quad f_Y(a) = \dfrac{d}{da} F_X\left(\dfrac{a}{2}\right) = f_X\left(\dfrac{a}{2}\right) \times \dfrac{1}{2}$

Note: $\quad \displaystyle\int_{-\infty}^{\infty} f_Y(u)\, du = \int_{-\infty}^{\infty} \frac{1}{2} f_X\left(\frac{u}{2}\right)\, du \quad$ let $v = u/2 \rightarrow dv = du/2$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} f_X(v)\, dv \\
&= 1
\end{aligned}
$$