Jointly Distributed Random Variables

Joint Distribution of Functions of Random Variables [Ross S6.7]

Let X and Y have joint pdf $f_{XY}(x,y)$.

In some examples we computed the distribution of $Z = g(X,Y)$, e.g.

- in Example 23.2 we computed the cdf of $D = \sqrt{X^2+Y^2}$
- in Example 23.3 we computed the pdf of $Z = X/Y$.

Now, consider

$$\begin{aligned} Y_1 &= g_1(X_1, X_2) \\ Y_2 &= g_2(X_1, X_2) \end{aligned}$$

and we want the joint pdf of Y_1 and Y_2.

We make the following assumptions on g_1 and g_2:

- The system of equations

$$\begin{aligned} y_1 &= g_1(x_1, x_2) \\ y_2 &= g_2(x_1, x_2) \end{aligned}$$

  can be uniquely solved for x_1 and x_2 in terms of y_1 and y_2:

$$\begin{aligned} x_1 &= h_1(y_1, y_2) \\ x_2 &= h_2(y_1, y_2). \end{aligned}$$

- g_1 and g_2 have continuous partial derivates such that the determinant

$$J(x_1,x_2) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix} = \frac{\partial g_1}{\partial x_1}\frac{\partial g_2}{\partial x_2} - \frac{\partial g_1}{\partial x_2}\frac{\partial g_2}{\partial x_1} \neq 0$$

Under these conditions, the pdf of Y_1 and Y_2 can be shown to be:

$$f_{Y_1Y_2}(y_1,y_2) = f_{X_1X_2}(x_1,x_2)\,|J(x_1,x_2)|^{-1} \tag{29.1}$$

where

$$\begin{aligned} x_1 &= h_1(y_1, y_2) \\ x_2 &= h_2(y_1, y_2). \end{aligned}$$

Example 29.1: Let

$$\begin{aligned} Y_1 &= X_1 + X_2 \\ Y_2 &= X_1 - X_2 \end{aligned}$$

Find the joint pdf $f_{Y_1Y_2}(y_1,y_2)$ in terms of $f_{X_1X_2}(x_1,x_2)$.

Solution: Solving

$$\begin{aligned} y_1 &= x_1 + x_2 \\ y_2 &= x_1 - x_2 \end{aligned}$$

we get

$$\begin{aligned} x_1 &= \frac{1}{2}y_1 + \frac{1}{2}y_2 \\ x_2 &= \frac{1}{2}y_1 - \frac{1}{2}y_2 \end{aligned}$$

Also

$$J(x_1,x_2) = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \neq 0$$

Hence, from (29.6)

$$f_{Y_1Y_2}(y_1,y_2) = \frac{1}{2} f_{X_1X_2}\left(\frac{1}{2}y_1 + \frac{1}{2}y_2, \frac{1}{2}y_1 - \frac{1}{2}y_2\right) \tag{29.2}$$

Example 29.2: Let R and Θ be two random variables with joint pdf $f_{R\Theta}(r,\theta)$. Consider the change of variables

$$\begin{aligned} X &= R\cos\Theta \\ Y &= R\sin\Theta. \end{aligned}$$

Find $f_{R\Theta}(r,\theta)$ in terms of $f_{XY}(x,y)$. [Hard]

Note: This is Problem T9.1; see also textbook Example 6.7b for a different tedious approach.

Solution: We have the system of equations

$$\begin{aligned} x &= g_1(r,\theta) = r\cos\theta \\ y &= g_2(r,\theta) = r\sin\theta \end{aligned}$$

which is solved by

$$\begin{aligned} r &= h_1(x,y) = \sqrt{x^2+y^2} \\ \theta &= h_2(x,y) = \begin{cases} \tan^{-1}(y/x) & x>0, y>0 \\ \tan^{-1}(y/x)+\pi & x<0 \\ \tan^{-1}(y/x)+2\pi & x>0, y<0 \end{cases} \end{aligned}$$

where $h_2(x,y)$ is the angle of the vector (x,y).

Computing the Jacobian determinant

$$\begin{aligned} J(r,\theta) &= \begin{vmatrix} \frac{\partial g_1}{\partial r} & \frac{\partial g_1}{\partial \theta} \\ \frac{\partial g_2}{\partial r} & \frac{\partial g_2}{\partial \theta} \end{vmatrix} \\ &= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} \\ &= r\cos^2\theta + r\sin^2\theta \\ &= r \end{aligned}$$

So, the pdf $f_{XY}(x,y)$ and $f_{R\Theta}(r,\theta)$ are related by

$$\begin{aligned} f_{XY}(x,y) &= f_{R\Theta}(r,\theta)\,|J(r,\theta)|^{-1} && (29.3) \\ &= f_{R\Theta}(r,\theta)/r && (29.4) \end{aligned}$$

or equivalently:

$$\begin{aligned} f_{R\Theta}(r,\theta) &= f_{XY}(x,y)r && (29.5) \\ &= f_{XY}(r\cos\theta, r\sin\theta)r && (29.6) \end{aligned}$$

So, to compute the probability that $(R,\Theta) \in A$:

$$\begin{aligned} P[(R,\Theta)\in A] &= \iint_{(r,\theta)\in A} f_{R\Theta}(r,\theta)\,dr\,d\theta \\ &= \iint_{(r,\theta)\in A} f_{XY}(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta \end{aligned}$$