Random Variables (rv)

Examples [Ross S4.5]

Example 11.1: [Friendship Paradox]

There are n people named $1, 2, \ldots, n$.

Person i has $f(i)$ friends. Let $m = \sum_{i=1}^{n} f(i)$.

Let X be a random person, equally likely to be any of the n people.

Let $Z = f(X)$, i.e., Z is # of friends of random person X.

Then

$$E[Z] = \sum_{i=1}^{n} f(i) \underbrace{P[X = i]}_{1/n} = \frac{m}{n} \qquad \text{[by Prop. 10.1]}$$

$$E[Z^2] = \sum_{i=1}^{n} (f(i))^2 \, P[X = i] = \frac{1}{n} \sum_{i=1}^{n} (f(i))^2$$

Now, each person writes the names of their friends on a sheet of paper (one sheet per friend).

There are m sheets, and one sheet is drawn at random, each sheet being equally likely to be chosen.

Let

$$Y = \text{ name of friend on drawn sheet}$$
$$W = f(Y)$$

Now

$$P[Y = i] = \frac{f(i)}{m} \qquad \left[\text{as opposed to } \frac{1}{n} \right]$$

$$\begin{aligned}
E[W] &= E[f(Y)] \\
&= \sum_i f(i) \, P[Y = i] \\
&= \sum_i f(i) \times \frac{f(i)}{m} \\
&= \frac{n}{m} \times \frac{1}{n} \sum_i (f(i))^2 \\
&= \frac{E[Z^2]}{E[Z]} \\
&\geq E[Z] \qquad \text{[since } E[Z^2] \geq (E[Z])^2]
\end{aligned}$$

So:

$$\text{(expected \# of friends of random person } = E[Z])$$
$$\leq \text{(expected \# of friends of random friend } = E[W])$$

═══════════════════

Example 11.2: There are n days in a year.

Persons $1, 2$ and 3 are independently born on day r with probability p_r, for $r = 1, 2, \ldots, n$.

Let $A_{i,j} = \{\text{persons } i \text{ and } j \text{ born on same day}\}$

a) Find $P[A_{1,3}]$

b) Find $P[A_{1,3} \mid A_{1,2}]$

Solution:

———————————

Remark 11.1: We had $E[aX + b] = aE[X] + b$. What about $Var[aX + b]$?

$$Var[aX + b] = E\left[(aX + b - E[aX + b])^2\right]$$
$$= E\left[(aX + b - aE[X] - b)^2\right]$$
$$= E\left[(aX - aE[X])^2\right]$$
$$= E\left[a^2(X - E[X])^2\right]$$

$$
\begin{aligned}
&= E\left[a^2 Y\right] && \text{where } Y = (X - E[X])^2 \\
&= a^2 E\left[Y\right] \\
&= a^2 E\left[(X - E[X])^2\right] \\
&= a^2\, Var[X]
\end{aligned}
$$

———————————————

Remark 11.2: If X has units of, say, kg, then:

- $E[X] = \mu_X$ has units of kg,

- $Var[X] = \sigma_X^2$ has units of kg^2.

We also define $SD[X] = \sqrt{Var[X]} = \sigma_X$, called **standard deviation**.

$SD[X]$ has units of kg again.