Continuous Random Variables

Distribution of a function of a random variable [Ross S5.7]

Given a random variable X and $Y = g(X)$, want to find pdf of Y.

Two-step approach: first, calculate

$$F_Y(y) = P[g(X) \leq y]. \tag{20.1}$$

Second, differentiate to get

$$f_Y(y) = \frac{d}{dy} F_Y(y) \tag{20.2}$$

Example 20.1: Let $X \sim U(0,1)$ and $Y = \sqrt{X}$. Find $F_Y(y)$ and $f_Y(y)$.

Solution: For $y \geq 0$:

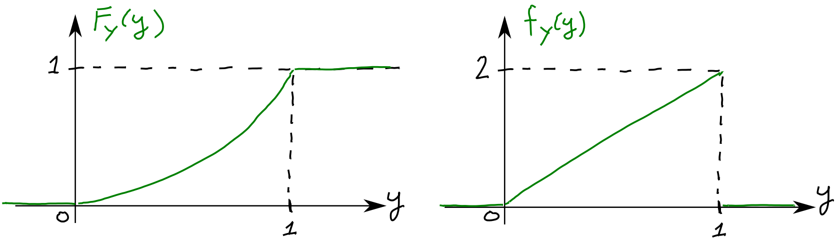
$$\begin{aligned}
F_Y(y) &= P[Y \leq y] \\
&= P[\sqrt{X} \leq y] \\
&= P[X \leq y^2] \\
&= F_X(y^2) \\
&= \begin{cases} y^2 & \text{for } 0 \leq y \leq 1 \\ 1 & \text{for } 1 < y \end{cases}
\end{aligned}$$

Since Y cannot be negative, $F_Y(y) = 0$ for $y < 0$. Hence

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ y^2 & 0 \leq y \leq 1 \\ 1 & 1 < y \end{cases} \tag{20.3}$$

Differentiating (20.3), we get

$$
\begin{aligned}
f_Y(y) &= \frac{d}{dy} F_Y(y) \\
&= \begin{cases} 0 & y < 0 \\ 2y & 0 \le y \le 1 \\ 0 & 1 < y \end{cases}
\end{aligned}
$$



Example 20.2: Let $Y = X^2$. What is $f_Y(y)$ in terms of $f_X(x)$?

Solution: For $y \ge 0$:

$$
\begin{aligned}
F_Y(y) &= P[Y \le y] \\
&= P[X^2 \le y] \\
&= P[-\sqrt{y} \le X \le \sqrt{y}] \\
&= F_X(\sqrt{y}) - F_X(-\sqrt{y})
\end{aligned}
$$

$$
\begin{aligned}
f_Y(y) &= \frac{d}{dy} F_Y(y) \\
&= \frac{d}{dy} F_X(\sqrt{y}) - \frac{d}{dy} F_X(-\sqrt{y}) \\
&= f_X(\sqrt{y}) \frac{1}{2\sqrt{y}} - f_X(-\sqrt{y}) \frac{-1}{2\sqrt{y}} \\
&= \frac{1}{2\sqrt{y}} \left(f_X(\sqrt{y}) + f_X(-\sqrt{y}) \right)
\end{aligned}
$$

For $y < 0$: $\quad F_Y(y) = P[X^2 \leq y] = 0 \Rightarrow f_Y(y) = 0$

Example 20.3: Let $Y = aX + b$. What is $f_Y(y)$ in terms of $f_X(x)$?

Solution: If $a > 0$

$$
\begin{aligned}
F_Y(y) &= P[Y \leq y] \\
&= P[aX + b \leq y] \\
&= P\left[X \leq \frac{y-b}{a}\right] \\
&= F_X\left(\frac{y-b}{a}\right)
\end{aligned}
$$

$$
\begin{aligned}
f_Y(y) &= \frac{d}{dy} F_Y(y) \\
&= \frac{d}{dy} F_X\left(\frac{y-b}{a}\right) \\
&= \frac{1}{a} f_X\left(\frac{y-b}{a}\right)
\end{aligned}
$$

If $a < 0$:

$$\begin{aligned} F_Y(y) &= P[Y \leq y] \\ &= P[aX + b \leq y] \\ &= P\left[X \geq \frac{y-b}{a}\right] && \text{[We divided by } a < 0\text{]} \\ &= 1 - P\left[X < \frac{y-b}{a}\right] \\ &= 1 - P\left[X \leq \frac{y-b}{a}\right] \\ &= 1 - F_X\left(\frac{y-b}{a}\right) \end{aligned}$$

$$\begin{aligned} f_Y(y) &= \frac{d}{dy} F_Y(y) \\ &= \frac{d}{dy}\left(1 - F_X\left(\frac{y-b}{a}\right)\right) \\ &= -\frac{1}{a} f_X\left(\frac{y-b}{a}\right) \end{aligned}$$

Since $a < 0$ in this second case, both cases can be combined:

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

which makes sense since the density of probability can't be negative!

Proposition 20.1 *Let X be a continuous random variable with pdf $f_X(x)$. Let $g(x)$ be differentiable and either strictly increasing or strictly decreasing.*

Then $Y = g(X)$ *has pdf*

$$f_Y(y) = \begin{cases} f_X\left(g^{-1}(y)\right)\left|\frac{d}{dy}g^{-1}(y)\right| & \textit{if } y = g(x) \textit{ for some } x \\ 0 & \textit{else} \end{cases}$$

Why? Only consider the case that $g(x)$ is strictly increasing.

Say $y = g(x)$ for some x. Then

$$\begin{aligned} F_Y(y) &= P[g(X) \leq y] \\ &= P[X \leq g^{-1}(y)] \\ &= F_X(g^{-1}(y)) \end{aligned}$$

So
$$\begin{aligned} f_Y(y) &= \frac{d}{dy}F_Y(y) \\ &= \frac{d}{dy}F_X(g^{-1}(y)) \\ &= f_X(g^{-1}(y))\frac{d}{dy}g^{-1}(y) \end{aligned}$$

If there is no x such that $y = g(x)$, then either:

- y is less than all possible values $g(x)$
- y is greater than all possible values $g(x)$

Then, $P[g(X) \leq y]$ is either 0 or 1.

Either way, $f_Y(y) = 0$.