Conditional Probability and Independence

Conditional Probability [Ross S3.1, S3.2]

Conditional probability is one of the most important concepts in this course.

- it is a tool to compute probabilities,
- it lets us update probabilities when partial information is revealed.

Example 5.1: Say we toss two dice.

What is the probability that the sum is 9?

Solution: This event is $E = \{(3,6),(4,5),(5,4),(6,3)\}$.

So $P[E] = 4/36$.

Example 5.2: Say I roll 1st die (but not 2nd) and get a 4.

What is the probability that the sum will be 9?

Solution: All possible outcomes given this new information are:

$$F = \{(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)\}.$$

The other 30 cases are inconsistent with the 1st die roll
$\Rightarrow$ they now have probability = 0.

The 6 cases in F had the same probability before the 1st die was rolled.

They should now be equally likely after the outcome of 1st die roll, i.e., each has probability 1/6.

After the 1st die roll was revealed (i.e., after F was revealed to occur):

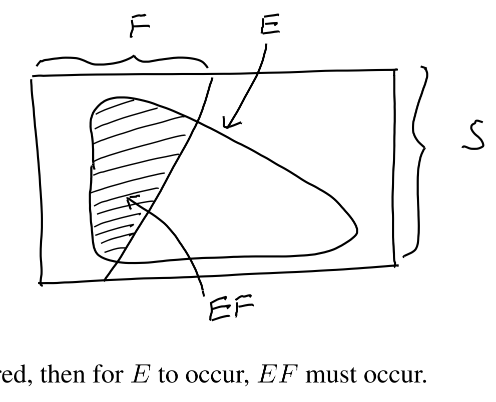
$$\{\text{sum} = 9\} = EF = \{(4,5)\}$$

and this has probability 1/6.

We say that **the probability of E given F has occured** is 1/6, or

$$P[E \mid F] = 1/6.$$

Let's generalize: let's not assume the elements of S are equally likely:



If F has occured, then for E to occur, EF must occur.

If F has occured, our sample space S is reduced to F.

So if F has occured, probabilities should be computed relative to F:

Definition 5.1: If $P[F] > 0$, then

$$P[E \mid F] = \frac{P[EF]}{P[F]}.$$

Example 5.3: A coin is flipped twice. What is the probability of two heads if

a) first flip is heads?

b) at least one flip is heads?

Solution: a) $S = \{hh, ht, th, tt\}$, $E = \{hh\}$ and $F = \{ht, hh\}$. So

$$P[E|F] = \frac{P[EF]}{P[F]} = \frac{P[hh]}{P[F]} = \frac{1/4}{1/2} = 1/2.$$

b) Here, $F = \{hh, ht, th\}$. So

$$P[E|F] = \frac{P[EF]}{P[F]} = \frac{1/4}{3/4} = 1/3.$$

Example 5.4: Two 4-sided dice are rolled. Let

$$E = \{\text{ max of both rolls is 3}\}$$
$$F = \{\text{ min of both rolls is 2}\}$$

What is $P[E \mid F]$?

Solution:

$$E = \{(1,3),(2,3),(3,3),(3,2),(3,1)\}$$
$$F = \{(4,2),(3,2),(2,2),(2,3),(2,4)\}$$

$$P[E|F] = \frac{P[EF]}{P[F]} = \frac{2/16}{5/16} = 2/5.$$

Conditional Probability satisfies the axioms of probability:

For fixed F with $P[F] > 0$, the function $P[\cdot|F]$ satisfies all the same axioms as $P[\cdot]$:

[A1] $P[E|F] = P[EF]/P[F] \geq 0$ since $P[EF] \geq 0$
$P[E|F] = P[EF]/P[F] \leq 1$ since $EF \subset F$

[A2] $P[S|F] = P[SF]/P[F] = P[F]/P[F] = 1$.

[A3] Let $E_1 \cap E_2 = 0$. Then $E_1F \cap E_2F = 0$.

$$\begin{aligned} P[E_1 \cup E_2|F] &= P[(E_1 \cup E_2)F]/P[F] \\ &= P[E_1F \cup E_2F]/P[F] \\ &= P[E_1F]/P[F] + P[E_2F]/P[F] \\ &= P[E_1|F] + P[E_2|F] \end{aligned}$$

Multiplication Rule:

Since F_1F_2 is a set, we also write $P[E|F_1F_2] = P[EF_1F_2]/P[F_1F_2]$, etc.

Now

$$\begin{aligned} P[E_1E_2\cdots E_n] &= P[E_1] \times \frac{P[E_1E_2]}{P[E_1]} \times \frac{P[E_1E_2E_3]}{P[E_1E_2]} \times \cdots \\ &\quad \cdots \times \frac{P[E_1E_2 \ldots E_{n-1}]}{P[E_1E_2 \ldots E_{n-2}]} \times \frac{P[E_1E_2 \ldots E_n]}{P[E_1E_2 \ldots E_{n-1}]} \\ &= P[E_1] \times P[E_2|E_1] \times P[E_3|E_1E_2] \times \cdots \\ &\quad \cdots \times P[E_n|E_1E_2\cdots E_{n-1}] \end{aligned}$$

Example 5.5: 3 grad and 12 ugrad students are randomly divided into 3 groups of 5. What is the prob that each group has exactly 1 grad student?

Solution:

Let $E_1 = \{\text{grad \#1 is in a group}\}$,
$E_2 = \{\text{grad \#1 and \#2 are in different groups}\}$
$E_3 = \{\text{all grads are in different groups}\}$

Then $E_3 = E_1E_2E_3$ and

$$\begin{aligned} P[E_3] &= P[E_1E_2E_3] \\ &= P[E_1]P[E_2|E_1]P[E_3|E_1E_2] \end{aligned}$$

with

$$P[E_1] = 1$$
$$P[E_2|E_1] = 10/14$$
$$P[E_3|E_1E_2] = 5/13$$