Continuous Random Variables

Distribution of a function of a random variable [Ross S5.7]

Given a random variable X and $Y = g(X)$, want to find pdf of Y.

Two-step approach: first, calculate

$$F_Y(y) = P[g(X) \le y]. \tag{20.1}$$

Second, differentiate to get

$$f_Y(y) = \frac{d}{dy} F_Y(y) \tag{20.2}$$

Example 20.1: Let $X \sim U(0,1)$ and $Y = \sqrt{X}$. Find $F_Y(y)$ and $f_Y(y)$.

Solution:

Example 20.2: Let $Y = X^2$. What is $f_Y(y)$ in terms of $f_X(x)$?

Solution:

Example 20.3: Let $Y = aX + b$. What is $f_Y(y)$ in terms of $f_X(x)$?

Solution:

Proposition 20.1 *Let X be a continuous random variable with pdf $f_X(x)$. Let $g(x)$ be differentiable and either strictly increasing or strictly decreasing. Then $Y = g(X)$ has pdf*

$$f_Y(y) = \begin{cases} f_X\left(g^{-1}(y)\right)\left|\frac{d}{dy}g^{-1}(y)\right| & \textit{if } y = g(x) \textit{ for some } x \\ 0 & \textit{else} \end{cases}$$

Why? Only consider the case that $g(x)$ is strictly increasing.

Say $y = g(x)$ for some x. Then

$$
\begin{aligned}
F_Y(y) &= P[g(X) \leq y] \\
&= P[X \leq g^{-1}(y)] \\
&= F_X(g^{-1}(y))
\end{aligned}
$$

So
$$
\begin{aligned}
f_Y(y) &= \frac{d}{dy} F_Y(y) \\
&= \frac{d}{dy} F_X(g^{-1}(y)) \\
&= f_X(g^{-1}(y)) \frac{d}{dy} g^{-1}(y)
\end{aligned}
$$

If there is no x such that $y = g(x)$, then either:

- y is less than all possible values $g(x)$
- y is greater than all possible values $g(x)$

Then, $P[g(X) \leq y]$ is either 0 or 1.

Either way, $f_Y(y) = 0$.