

# Conditional Probability and Independence

## Independent Events [Ross S3.4]

**Definition 7.1:** Events  $E$  and  $F$  are called **independent** if

$$P[EF] = P[E]P[F]$$

Two events that are not independent are said to be **dependent**.

From previous examples,  $P[E|F]$  is not necessarily the same as  $P[E]$ .

But, if  $E$  and  $F$  are independent (and  $P[F] > 0$ ):

$$P[E|F] = \frac{P[EF]}{P[F]} = \frac{P[E]P[F]}{P[F]} = P[E]$$

**Example 7.1:** Two 6-sided dice are rolled. Let

$$E_1 = \{\text{sum is 6}\}$$

$$E_2 = \{\text{sum is 7}\}$$

$$F = \{\text{1st die is 4}\}$$

$$G = \{\text{2nd die is 3}\}$$

Then:

$$P[E_1 F] = P[(4, 2)] = 1/36, \quad P[E_1]P[F] = 5/36 \times 1/6 \neq 1/36$$

$$P[E_2 F] = P[(4, 3)] = 1/36, \quad P[E_2]P[F] = 1/6 \times 1/6 = 1/36$$

So  $E_1$  and  $F$  are not independent, but  $E_2$  and  $F$  are independent.

Similarly,  $E_2$  and  $G$  are independent.

**Example 7.2:** Say  $EF = \emptyset$  with  $P[E] > 0$  and  $P[F] > 0$ . Are  $E$  and  $F$  independent?

*Solution:*

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**Proposition 7.1** *If  $E$  and  $F$  are independent, then  $E$  and  $F^c$  are independent*

Why?

$$\begin{aligned}P[E] &= P[EF \cup EF^c] \\&= P[EF] + P[EF^c] \\&= P[E]P[F] + P[EF^c]\end{aligned}$$

$$\Rightarrow P[EF^c] = P[E] - P[E]P[F] = P[E](1 - P[F]) = P[E]P[F^c]$$

**Example 7.3:** If  $E$  is independent of  $F$  and  $E$  is independent of  $G$ , is  $E$  independent of  $FG$ ?

*Solution:*

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**Definition 7.2:** Events  $E$  and  $F$  are called conditionally independent given  $G$  when

$$P[EF|G] = P[E|G]P[F|G].$$

What does this mean?

$$\begin{aligned} P[E|G]P[F|G] &= P[EF|G] \\ &= \frac{P[EF|G]}{P[G]} \\ &= \frac{P[E|FG] \times P[F|G] \times P[G]}{P[G]} \end{aligned}$$

So, this is equivalent to  $P[E|FG] = P[E|G]$ .

In words: If  $G$  is known to have occurred, the additional information that  $F$  occurred does not change the probability of  $E$ .

**Definition 7.3:** The 3 events  $E$ ,  $F$  and  $G$  are said to be independent if

$$\begin{aligned} P[EF|G] &= P[E|G]P[F|G] \\ P[EF] &= P[E]P[F] \\ P[EG] &= P[E]P[G] \\ P[FG] &= P[F]P[G] \end{aligned}$$

Now,  $E$  is independent of any event formed from  $F$  and  $G$ .

**Example 7.4:**

$$\begin{aligned} P[E(F \cup G)] &= P[EF \cup EG] \\ &= P[EF] + P[EG] - P[EF \cap EG] \\ &= P[E]P[F] + P[E]P[G] - P[E]P[FG] \end{aligned}$$

$$\begin{aligned}
&= P[E](P[F] + P[G] - P[FG]) \\
&= P[E]P[F \cup G]
\end{aligned}$$

**Definition 7.4:** Events  $E_1, E_2, \dots, E_n$  are said to be independent if

$$P \left[ \bigcap_{i \in A} E_i \right] = \prod_{i \in A} P[E_i] \quad (7.1)$$

for every  $A \subset \{1, \dots, n\}$ .

**Definition 7.5:** An infinite set of events  $E_1, E_2, \dots$  is independent if every finite subset is independent.

**Example 7.5:** A system has  $n$  components. Each component functions/fails independently of any other. Component  $i$  has probability  $p_i$  of functioning. If at least one component functions, the system functions. What is the probability that the system functions?

*Solution:*

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Sometimes each  $E_i$  is the outcome of one instance of a sequence of repeated sub-experiments, e.g.,  $E_i = \{i\text{-th coin toss is heads}\}$ .

These sub-experiments are often called **trials** (or **repeated trials**).

**Example 7.6:** Independent trials that consist of repeatedly rolling a pair of fair dice are performed. The outcome of a roll is the sum of the dice.

What is the prob. of

$F = \{\text{an outcome of 5 eventually occurs, and there was no 7 before this}\}?$

*Solution:*

