Chapter 7

Fourier Transform Analysis

7.1 Fourier Transform

The Fourier series is used for the analysis of periodic signals, while the Fourier Transform can be used for both periodic and aperiodic signals.

Definition: The Fourier Transform (FT) of a continuous-time signal $x(t)$ is defined by:

$$\mathcal{F}\{x(t)\} = X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt\,, \quad -\infty < \omega < \infty$$

and the inverse FT is:

$$\mathcal{F}^{-1}\{X(\omega)\} = x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\, d\omega\,, \quad -\infty < t < \infty$$

Remark: The following notation is used for the signal and its Fourier transform:

$$x(t) \Leftrightarrow X(\omega)$$

Remark: In general, the Fourier transform $X(\omega)$ is a complex function of the continuous variable ω:

$$X(\omega) = \underbrace{|X(\omega)|}_{magnitude}\; e^{\overbrace{j\angle X(\omega)}^{phase}}$$

Remark: If $x(t)$ is real, then: $X^*(\omega) = X(-\omega)$

$$\Rightarrow \begin{cases} |X(\omega)| & even\ function \\ \angle X(\omega) & odd\ function \end{cases}$$

Remark: If $x(t)$ is causal and absolutely integrable, then:

$$X(\omega) = \underbrace{\mathcal{L}\{x(t)\}_{s=j\omega}}_{Laplace\ Transform}$$

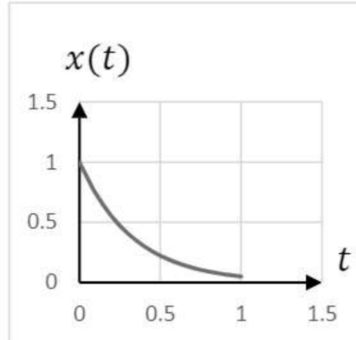
- **Existence of FT**

If $x(t)$ satiefies Dirichlet's conditions (see FS), then $X(\omega)$ is guaranteed to exist. However, there are signals that violate Dirichlet's conditions, but still have FT, such as $x(t) = \frac{sinat}{t}$.

Remark (Convergence of FT): When $X(\omega)$ exists, then:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\, d\omega = \begin{cases} x(t)\,, \quad if\ \ x(t)\ \ is\ continuous\ at\ t \\ \frac{1}{2}[x(t^-) + x(t^+)]\,,\ \ if\ x(t)\ is\ discontinuous\ \ at\ t \end{cases}$$

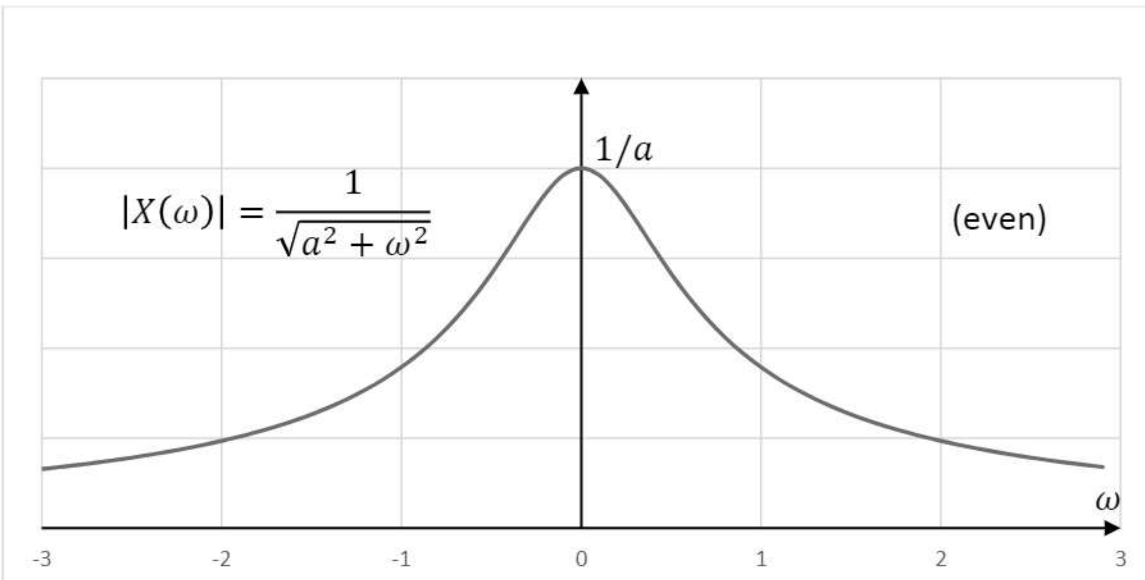
Example: $x(t) = e^{-at}u(t), \quad a > 0$

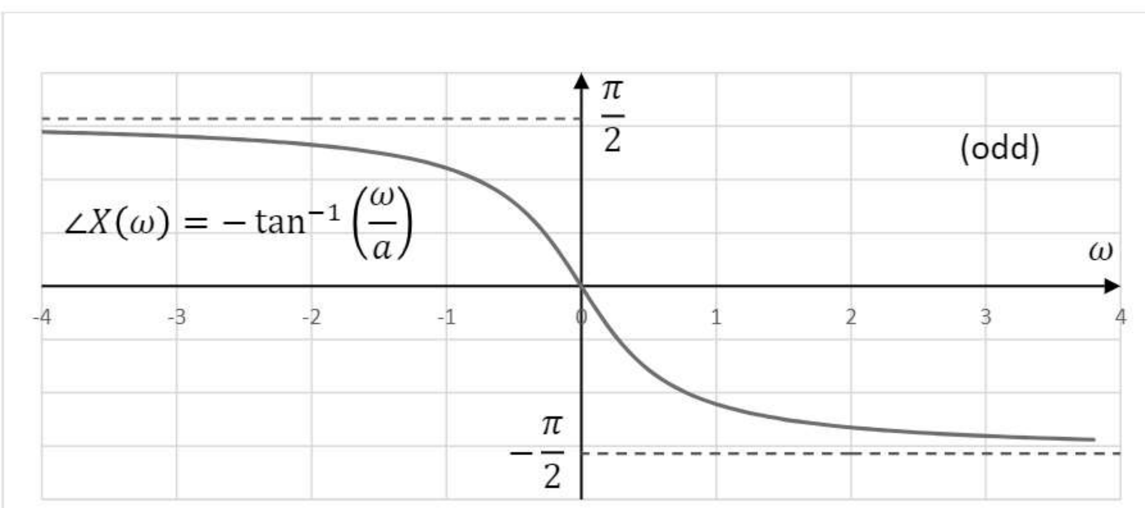


$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt$$

$$= \int_0^{\infty} e^{-(a+j\omega)t}\, dt = \frac{1}{-(a+j\omega)}[0-1]$$

$$\Longrightarrow \quad X(\omega) = \frac{1}{a+j\omega}$$

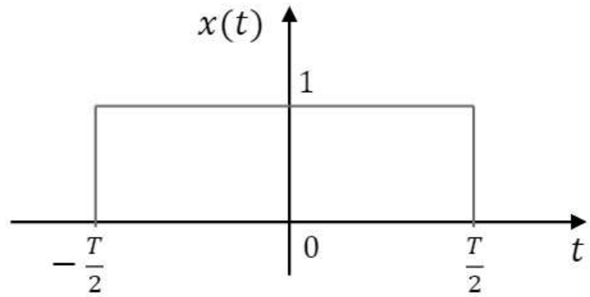




7.2 Transforms of Some Useful Signals

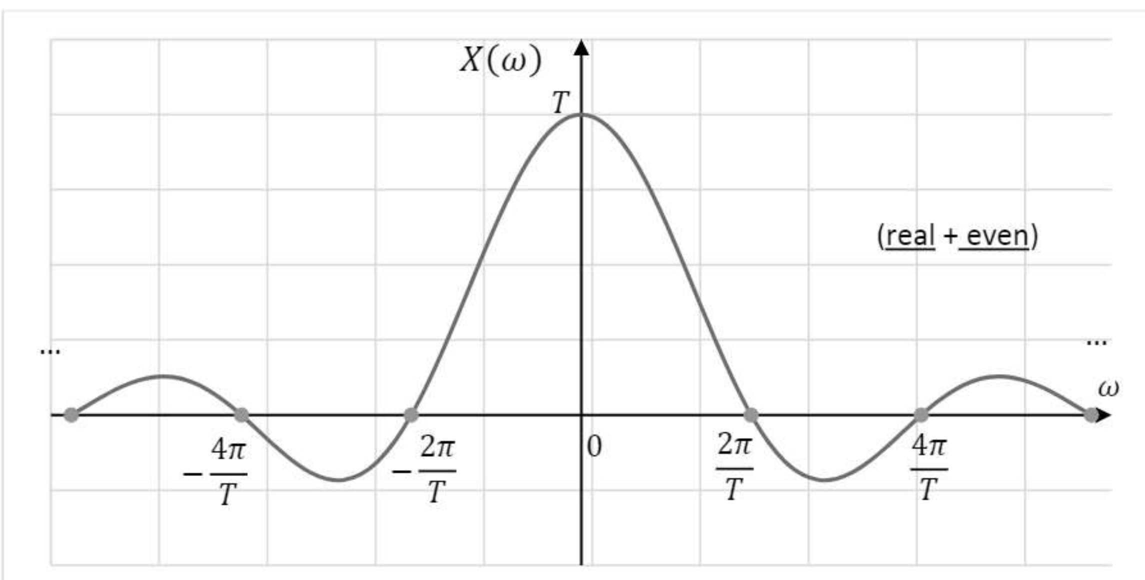
- Gate Function:

$$x(t) = rect\left(\frac{t}{T}\right) = \begin{cases} 1\,, & |t| < \frac{T}{2} \\ \frac{1}{2}\,, & |t| = \frac{T}{2} \\ 0\,, & else \end{cases}$$



$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-j\omega t}\,dt = \frac{1}{-j\omega}\left[e^{-j\omega\left(\frac{T}{2}\right)} - e^{j\omega\left(\frac{T}{2}\right)}\right]$$

$$= T\,\frac{sin\left(\frac{\omega T}{2}\right)}{\frac{\omega T}{2}} \triangleq T\,sinc\left(\frac{\omega T}{2}\right)$$

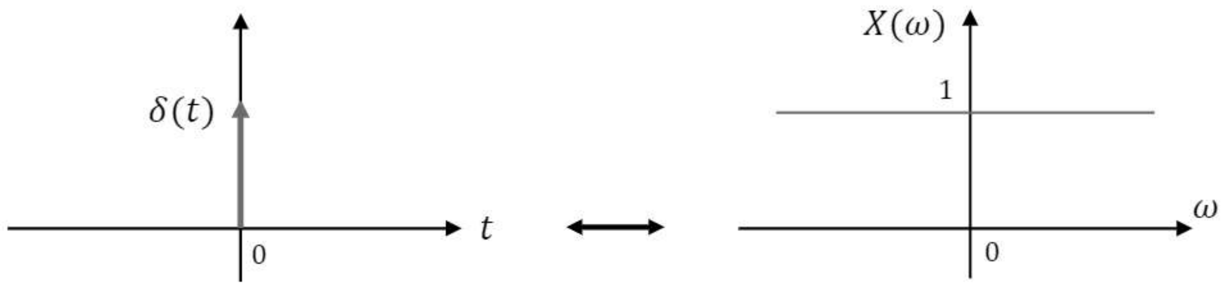


Remark: (Definition of the $sinc(\alpha)$ function)

$$sinc(\alpha) = \frac{\sin(\alpha)}{\alpha} \quad \text{where} \quad sinc(\alpha) = \begin{cases} 1, & \text{when } \alpha = 0 \\ 0, & \text{when } \alpha = \pm\pi, \pm 2\pi, \ldots \end{cases}$$

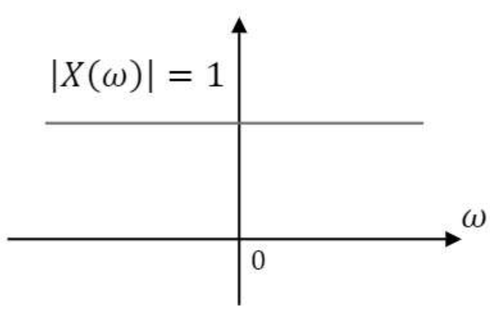
- Impulse Function:

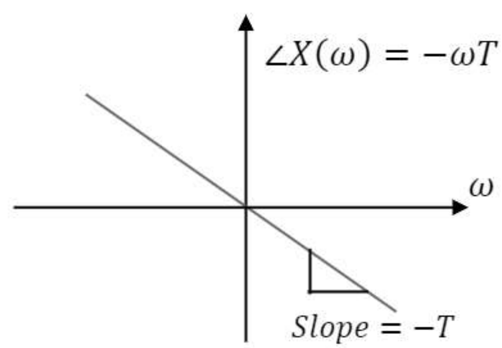
$$x(t) = \delta(t) \quad \Rightarrow \quad X(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t}\, dt = 1$$



- Shifted Impulse:

$$x(t) = \delta(t-T) \quad \Rightarrow \quad X(\omega) = \int_{-\infty}^{\infty} \delta(t-T) e^{-j\omega t}\, dt = e^{-j\omega T}$$

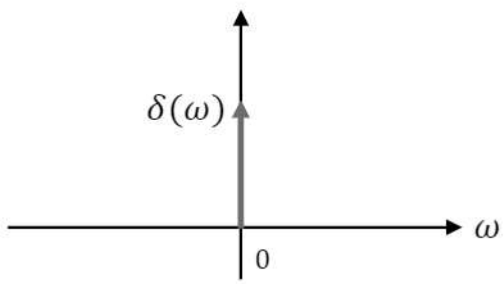


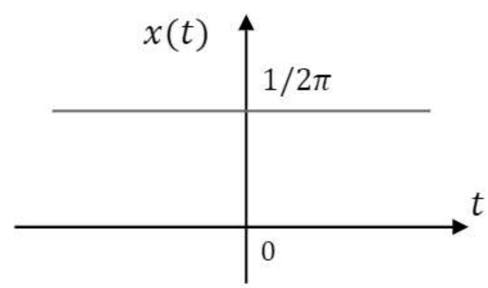


- <u>Constant (DC) Function</u>: (Impulse in frequency domain)

Start from frequency domain:

$$X(\omega) = \delta(\omega) \;\; \Rightarrow \;\; x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t}\, d\omega = \frac{1}{2\pi}$$





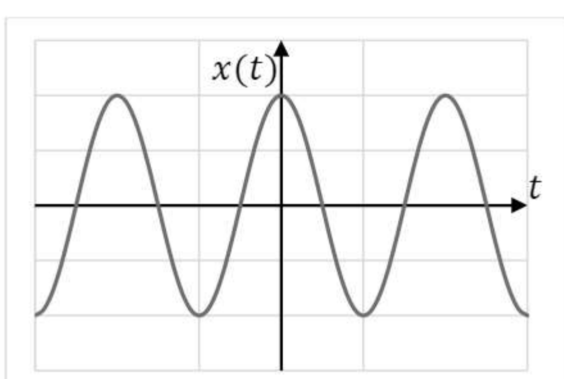
- <u>Shifted Impulse in Frequency Domain:</u>

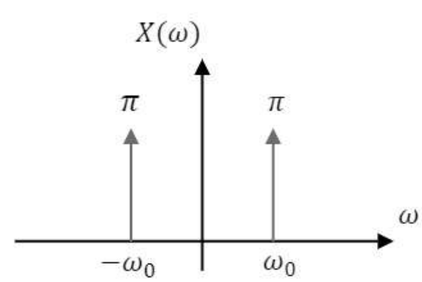
$$X(\omega) = \delta(\omega - \omega_0) \;\; \Rightarrow \;\; x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t}\, d\omega = \frac{1}{2\pi} e^{j\omega_0 t}$$

Alternatively: $\quad x(t) = e^{j\omega_0 t} \;\; \leftrightarrow \;\; X(\omega) = 2\pi\delta(\omega - \omega_0)$

This gives:

$$x(t) = \cos(\omega_0 t) \;\; \leftrightarrow \;\; X(\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$





- <u>Fourier Transform of a Periodic Signal</u>

Since $x(t) = e^{j\omega_0 t} \quad \leftrightarrow \quad X(\omega) = 2\pi\delta(\omega - \omega_0)$

then

$$x(t) = e^{jn\omega_0 t} \quad \leftrightarrow \quad X(\omega) = 2\pi\delta(\omega - n\omega_0)$$

And by linearity:

$$x(t) = D_n e^{jn\omega_0 t} \quad \leftrightarrow \quad X(\omega) = D_n 2\pi\delta(\omega - n\omega_0)$$

Also,

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \quad \leftrightarrow \quad \mathrm{X}(\omega) = \sum_{n=-\infty}^{\infty} D_n 2\pi\delta(\omega - n\omega_0)$$

(Fourier Series) (Fourier Transform)

7.3 Properties of Fourier Transform

(1) Linearity: Let $x_1(t) \leftrightarrow X_1(\omega)$ and $x_2(t) \leftrightarrow X_2(\omega)$**,** then:

$$\mathcal{F}\{c_1x_1(t) + c_2x_2(t)\} = c_1X_1(\omega) + c_2X_2(\omega)$$

for any real or complex constants c_1 and c_2.

(2) Conjugation: If $x(t) \leftrightarrow X(\omega)$, then:

$$x^*(t) \leftrightarrow X^*(-\omega)$$

Proof:

$$\mathcal{F}\{x^*(t)\} = \int_{-\infty}^{\infty} x^*(t)e^{-j\omega t}\, dt = \left[\int_{-\infty}^{\infty} x(t)e^{j\omega t}\, dt\right]^* = X^*(-\omega)$$

(3) Real Signals: If $x(t)$ is real, then by previous property:

$$X^*(\omega) = X(-\omega)$$

(4) Duality: If $x(t) \leftrightarrow X(\omega)$, then:

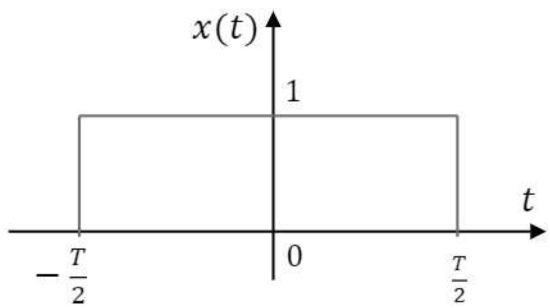
$$X(t) \leftrightarrow 2\pi x(-\omega)$$

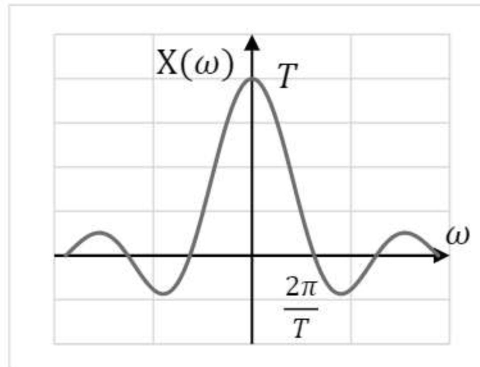
Proof:

$$\int_{-\infty}^{\infty} \qquad du = \int_{-\infty}^{\infty} X(t)e^{-j\omega t}dt = \mathcal{F}\{X(t)\}$$

Example: From previous example, we found:

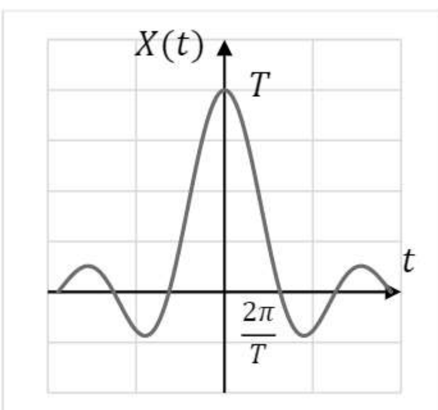
$$x(t) = rect\left(\frac{t}{T}\right) \;\leftrightarrow\; X(\omega) = T\,sinc\left(\frac{\omega T}{2}\right)$$

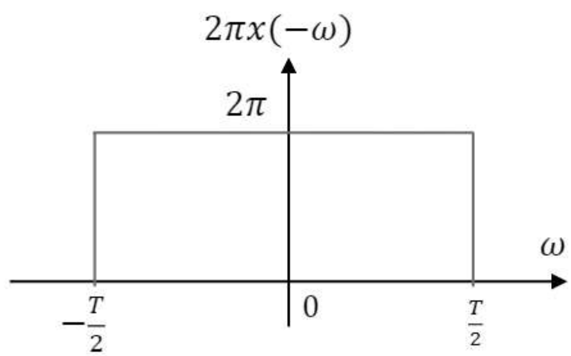




then by duality:

$$X(t) = T\,sinc\left(\frac{tT}{2}\right) \;\leftrightarrow\; 2\pi x(-\omega) = 2\pi\,rect\left(\frac{\omega}{T}\right)$$





(5) <u>Time and Frequency Scaling</u>: If $x(t) \leftrightarrow X(\omega)$, then:

$$x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

<u>Proof</u>: For a positive real a:

$$\mathcal{F}\{x(at)\} = \int_{-\infty}^{\infty} x(at)e^{-j\omega t}\,dt = \frac{1}{a}\int_{-\infty}^{\infty} x(u)e^{-j\left(\frac{\omega}{a}\right)u}\,du = \frac{1}{a}X\left(\frac{\omega}{a}\right)$$

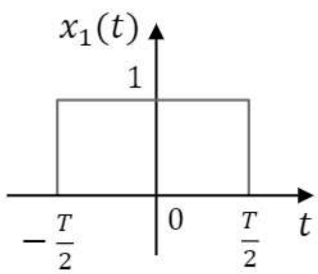
Similarly, for $a < 0$: $\quad x(at) \leftrightarrow \frac{1}{-a}X\left(\frac{\omega}{a}\right)$

Therefore,

$$x(at) \leftrightarrow \frac{1}{|a|}X\left(\frac{\omega}{a}\right)$$

Remark: Setting $a = -1$, we get: $\quad x(-t) \leftrightarrow X(-\omega)$

Example:



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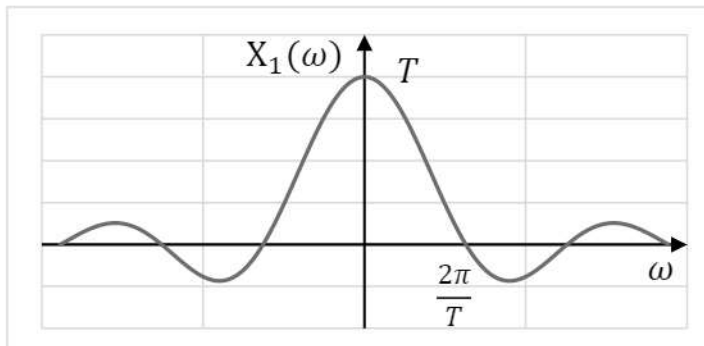

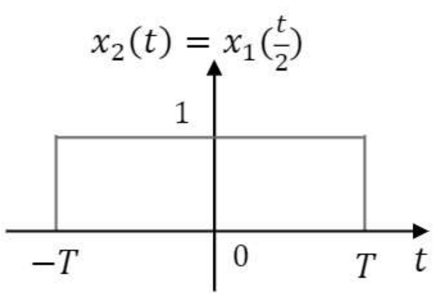

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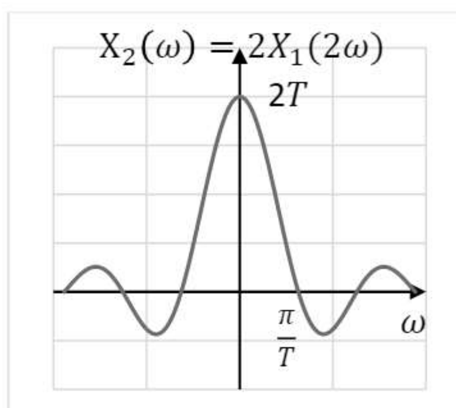


Remark: Time Compression $\leftrightarrow$ Frequency Expansion

Time Expansion $\leftrightarrow$ Frequency Compression

(6) Time-Shifting: If $x(t) \leftrightarrow X(\omega)$, then:

$$x(t - t_0) \;\leftrightarrow\; X(\omega)\, e^{-j\omega t_0}$$

Proof:

$$\mathcal{F}\{x(t - t_0)\} = \int_{-\infty}^{\infty} x(t - t_0) e^{-j\omega t}\, dt$$

Let $t - t_0 = u \;\Rightarrow\; dt = du,$ and substitute:

$$\mathcal{F}\{x(t - t_0)\} = \int_{-\infty}^{\infty} x(u) e^{-j\omega(u + t_0)}\, du = e^{-j\omega t_0} \underbrace{\int_{-\infty}^{\infty} x(u) e^{-j\omega u}\, du}_{X(\omega)}$$

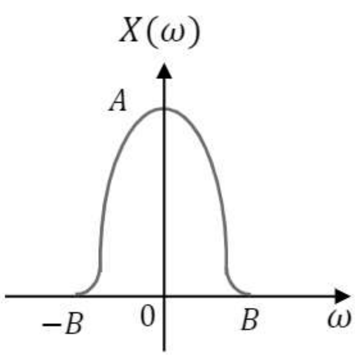
(7) Frequency-Shifting: If $x(t) \leftrightarrow X(\omega)$, then:

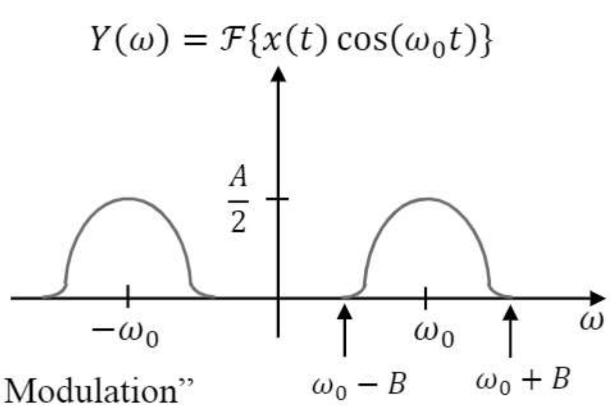
$$x(t)\, e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0)$$

Proof: Similar to previous property.

Remark: Since $\cos(\omega_0 t) = \frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right]$, then:

$$x(t)\cos(\omega_0 t) \;\leftrightarrow\; \frac{1}{2}[X(\omega - \omega_0) + X(\omega + \omega_0)]$$





"Amplitude Modulation"

(8) Convolution: If $x_1(t) \leftrightarrow X_1(\omega)$ and $x_2(t) \leftrightarrow X_2(\omega)$, then:

$$x_1(t) * x_2(t) \quad \leftrightarrow \quad X_1(\omega) \cdot X_2(\omega)$$

and

$$x_1(t) \cdot x_2(t) \quad \leftrightarrow \quad \frac{1}{2\pi} X_1(\omega) * X_2(\omega)$$

(9) Time Differentiation: If $x(t) \leftrightarrow X(\omega)$, then:

$$\frac{dx(t)}{dt} \leftrightarrow j\omega X(\omega)$$

and, in general:

$$\frac{d^n x(t)}{dt^n} \leftrightarrow (j\omega)^n X(\omega)$$

(10) Time Integration: If $x(t) \leftrightarrow X(\omega)$, then:

$$\int_{-\infty}^{t} x(\tau) d\tau \quad \leftrightarrow \quad \frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$$

Example:

$$\mathcal{F}\{u(t)\} = \mathcal{F}\{\int_{-\infty}^{t} \delta(\tau)\, d\tau\} = \frac{1}{j\omega} + \pi\delta(\omega)$$

where $\mathcal{F}\{\delta(t)\} = 1$.

7.4 Signal Energy (Parseval's Theorem)

If $x(t) \leftrightarrow X(\omega)$, the energy of the signal is:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

<u>Proof:</u>

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} x(t) x^*(t) dt$$

$$= \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \, d\omega \right] x^*(t) \, dt$$

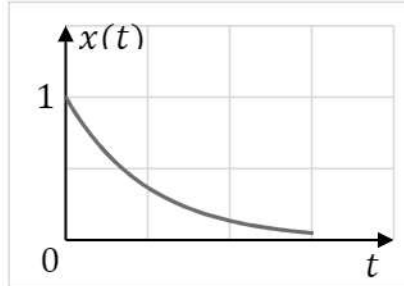
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \left[\int_{-\infty}^{\infty} x^*(t) e^{j\omega t} \, dt \right] d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \left[\int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt \right]^* d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot X^*(\omega) \, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

Example:

$$x(t) = e^{-t}u(t)$$



In the time-domain:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \ dt = \int_{0}^{\infty} e^{-2t} \, dt = \frac{1}{2}$$

In the frequency-domain, we have:

$$\mathcal{F}\{x(t)\} = X(\omega) = \frac{1}{1 + j\omega}$$

By Parseval's theorem:

$$E_x = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \ d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1 + \omega^2} \ d\omega$$

$$= \frac{2}{2\pi} \int_{0}^{\infty} \frac{1}{1 + \omega^2} \ d\omega = \frac{1}{\pi} \tan^{-1}(\omega) \Big|_0^{\infty}$$

$$= \frac{1}{\pi} \left(\frac{\pi}{2} - 0 \right) = \frac{1}{2}$$

7.5 Response of LTI Systems

For an LTI system, the zero-state response is given by $y(t) = h(t) * x(t)$, where $h(t)$ is the impulse response.

Assuming the system is asymptotically stable and taking FT:

$$Y(\omega) = H(\omega) \cdot X(\omega)$$

where $H(\omega) = \mathcal{F}\{h(t)\} = \frac{Y(\omega)}{X(\omega)}$ is the *frequency response* of the system, a complex function in general, which can be written as:

$$H(\omega) = \underbrace{|H(\omega)|}_{magnitude} \; e^{\overbrace{j\angle H(\omega)}^{phase}}$$

Therefore,

$$|Y(\omega)| = |H(\omega)| \cdot |X(\omega)|$$

$$\angle Y(\omega) = \angle H(\omega) + \angle X(\omega)$$

Remark: For the case of an LTI system described by the differential equation (DE) $Q(D)y(t) = P(D)x(t)$, the frequency response $H(\omega) = \frac{Y(\omega)}{X(\omega)}$ can be found by inspection of the DE. This can be shown as follows:

$$\left(\sum_{i=0}^{N} a_i D^i\right) y(t) = \left(\sum_{i=0}^{N} b_i D^i\right) x(t)$$

Assume the system to be asymptotically stable (i.e. no poles on the $j\omega$ axis), and using the differentiation property of FT:

$$\mathcal{F}\left\{\sum_{i=0}^{N} a_i D^i y(t)\right\} = \mathcal{F}\left\{\sum_{i=0}^{N} b_i D^i x(t)\right\}$$

$$\Rightarrow \sum_{i=0}^{N} a_i \underbrace{\mathcal{F}\{D^i y(t)\}}_{(j\omega)^i Y(\omega)} = \sum_{i=0}^{N} b_i \underbrace{\mathcal{F}\{D^i x(t)\}}_{(j\omega)^i X(\omega)}$$

So, the (D) operator is replaced by $(j\omega)$, $y(t)$ by $Y(\omega)$, and $x(t)$ by $X(\omega)$, i.e.

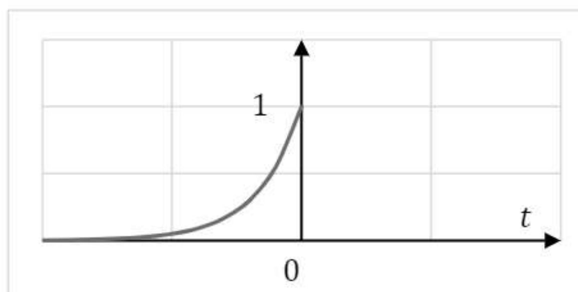
$$Q(j\omega) \cdot Y(\omega) = P(j\omega) \cdot X(\omega)$$

Therefore,

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{P(j\omega)}{Q(j\omega)}$$

Example:

Given the system $(D+2)y(t) = x(t)$, find its response to the input $x(t) = e^{3t}u(-t)$?



Solution:

Notice that the single-sided Laplace Transform will not work here, since $x(t)$ is non-causal signal. Using Fourier Transform:

$$X(\omega) = \int_{-\infty}^{0} e^{3t} e^{-j\omega t}\, dt = \frac{1}{3 - j\omega}$$

By inspection from DE:

$$H(\omega) = \frac{P(j\omega)}{Q(j\omega)} = \frac{1}{j\omega + 2}$$

$$\Rightarrow \quad Y(\omega) = H(\omega) \cdot X(\omega) = \frac{1}{2 + j\omega} \cdot \frac{1}{3 - j\omega}$$

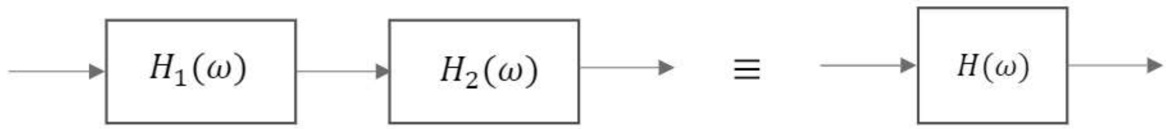
$$= \frac{1/5}{2 + j\omega} + \frac{1/5}{3 - j\omega} \qquad \text{(By PFE)}$$

Taking inverse FT:

$$y(t) = \frac{1}{5} e^{-2t} u(t) + \frac{1}{5} e^{3t} u(-t)$$

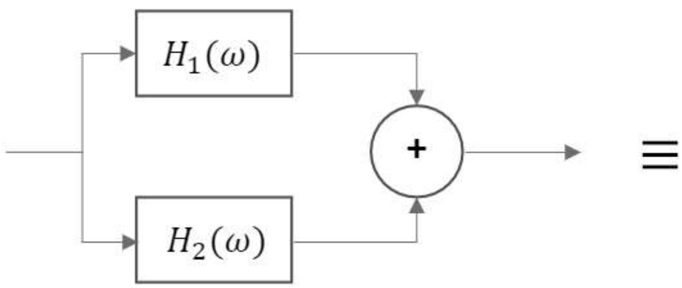
- **Connections of LTI Systems**

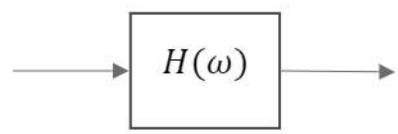
<u>Cascade</u>:



$$H(\omega) = H_1(\omega) \cdot H_2(\omega)$$

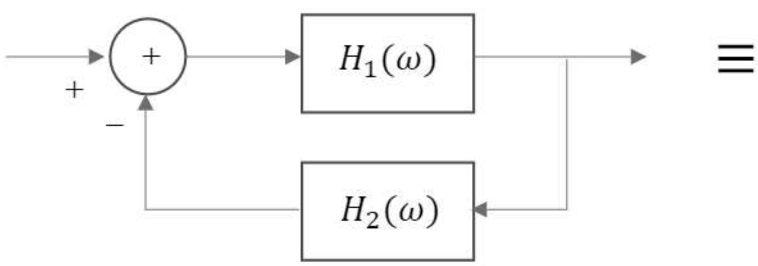
<u>Parallel:</u>

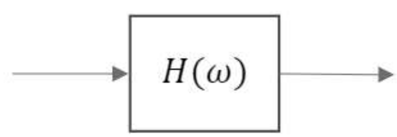




$$H(\omega) = H_1(\omega) + H_2(\omega)$$

<u>Feedback:</u>

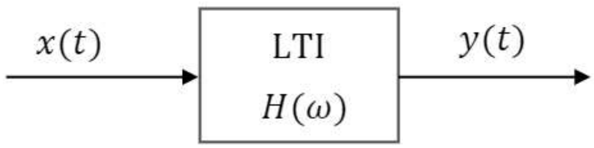




$$H(\omega) = \frac{H_1(\omega)}{1 + H_1(\omega)H_2(\omega)}$$

- **Distortionless System**

Consider an LTI system with a frequency response $H(\omega)$:



Definition: A system is *distortionless* if it satisfies:

$$y(t) = K \cdot x(t - t_0)$$

where K and t_0 are constants, i.e. the distortionless system can delay the signal $x(t)$ and/or scale it.

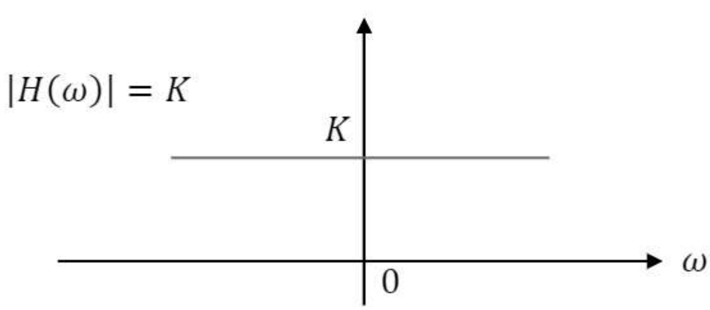
The frequency response of the distortionless system can be found by taking FT:

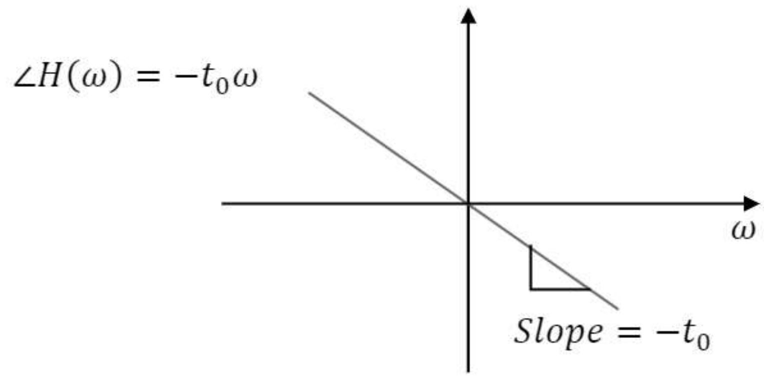
$$Y(\omega) = \mathcal{F}\{K\,x(t - t_0)\} = Ke^{-j\omega t_0}X(\omega)$$

$$\Rightarrow \quad H(\omega) = Ke^{-j\omega t_0}$$

$$\Rightarrow \quad h(t) = K\delta(t - t_0)$$

The magnitude and phase of $H(\omega)$:





Remark:

Practical systems always have some distortion. For practical systems where distortion needs to be minimized, such as amplifiers and communication systems, the magnitude of the frequency response $|H(\omega)|$ is made as close to constant as possible and the phase $\angle H(\omega)$ is made as linear as possible.

Remark:

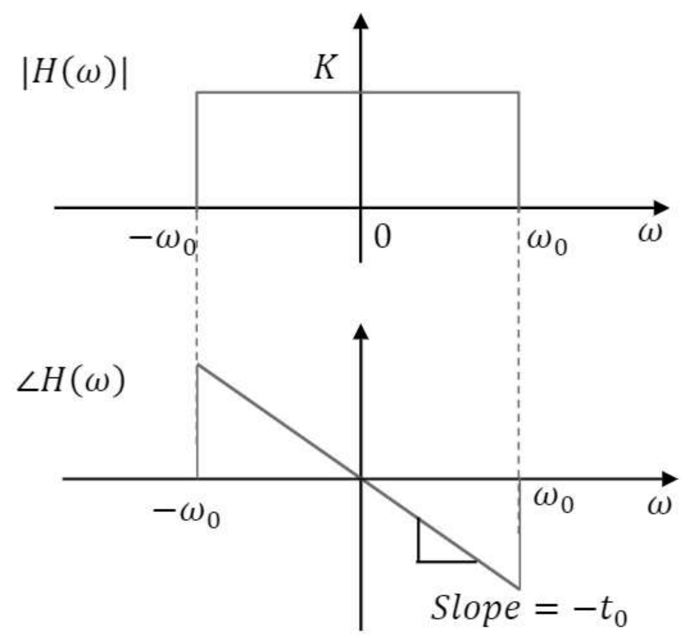
Human ear is sensitive to amplitude distortion, but relatively less sensitive to phase distortion, while human eye is more sensitive to phase distortion.

7.6 Ideal and Practical Filters

<u>Definition:</u> The ideal filter is a system that passes certain band (or bands) of frequencies without distortion and completely rejects the other bands.

<u>Ideal Low-Pass Filter (LPF)</u>

Motivated by the "distortionless system", the frequency response of the ideal LPF is shown:



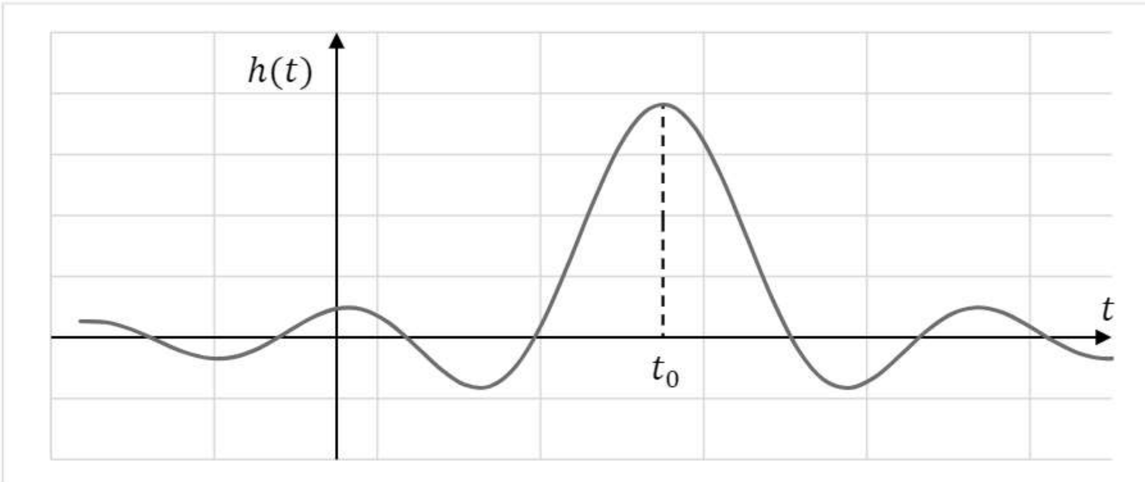
The impulse response of the ideal LPF can be found by taking the inverse FT:

$$h(t) = \mathcal{F}^{-1}\{H(\omega)\} = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} Ke^{-j\omega t_0} \cdot e^{j\omega t}\, d\omega$$

$$= \frac{K}{2\pi}\int_{-\omega_0}^{\omega_0} e^{j\omega(t-t_0)}\, d\omega = \frac{K}{2\pi}\cdot\frac{1}{j(t-t_0)}\left[e^{j\omega_0(t-t_0)} - e^{-j\omega_0(t-t_0)}\right]$$

$$= \frac{K\omega_0}{\pi}\cdot\frac{\sin[\omega_0(t-t_0)]}{\omega_0(t-t_0)}$$

$$\Rightarrow \quad h(t) = \frac{K\omega_0}{\pi} \cdot sinc[\omega_0(t - t_0)]$$



Observation:

Since $h(t) \neq 0$ for $t < 0$, the ideal filter is a non-causal system, which is physically unrealizable.

Remark: Practical or Realizable filters are approximation of the ideal filters. The approximation is performed in time-domain or in frequency-domain:

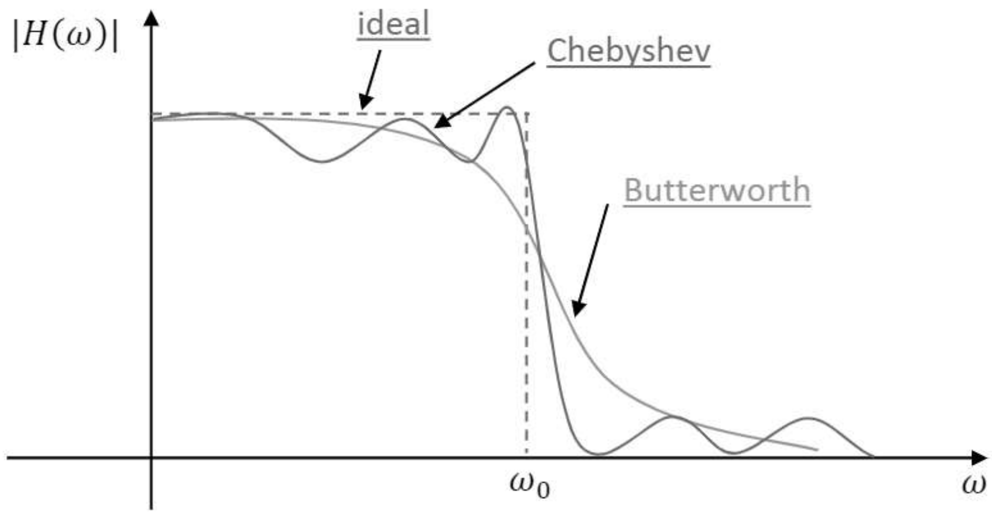
(1) In time-domain, and for an acceptable delay t_0, truncate $h(t)$ to make it causal:

$$h_c(t) = h(t) \cdot u(t)$$

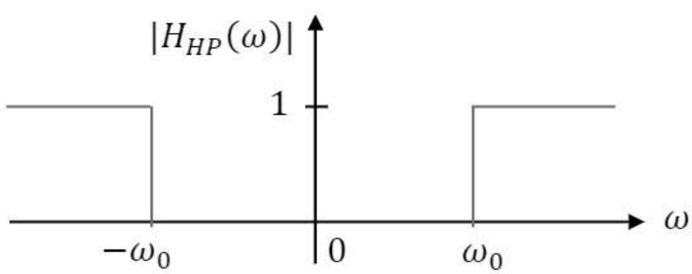
Find $H_c(\omega) = \mathcal{F}\{h_c(t)\}$. Then, find an approximate rational function:

$$H_c(\omega) \simeq \frac{P(j\omega)}{Q(j\omega)}$$

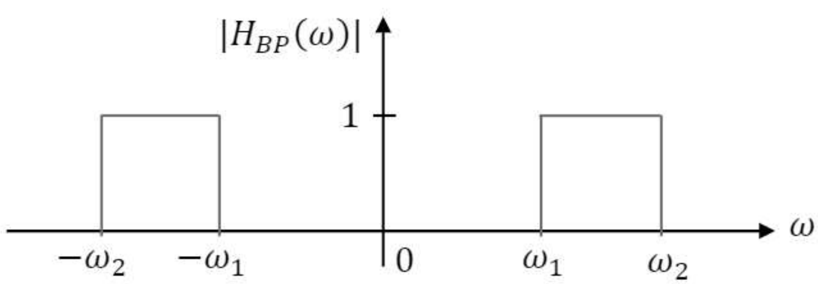
(2) In <u>frequency-domain</u>, well-known approximate realizable filters with specified magnitude response are used to design the filter, such as: <u>Butterworth Filters</u> and <u>Chebyshev Filters</u>. These filters will be studied in later courses.



<u>Ideal High-Pass Filter (HPF):</u> The magnitude frequency response for $K = 1$ is:



<u>Ideal Band-Pass Filter (BPF):</u>



Remark: Even though ideal filters are not realizable, they are used theoretically to simplify the analysis of systems, as we have seen before.

Remark: The condition $h(t) = 0$ for $t < 0$ for a realizable system has an equivalent in the frequency-domain given by the well-known Paley-Wiener criterion on the frequency response:

$$\int_{-\infty}^{\infty} \frac{|\ln |H(\omega)||}{1+\omega^2}\, d\omega < \infty$$

This implies that $H(\omega)$ can be zero at discrete frequencies, but not over a range of frequencies.