# Properties of Expectations

**Conditional Expectation** [Ross S7.5]

Recall that for 2 discrete random variables $X$ and $Y$ with $P[Y = y] > 0$:

$$p_{X|Y}(x|y) = P[X = x|Y = y] = \frac{p_{XY}(x, y)}{p_Y(y)}$$

We can define the **conditional expectation**:

$$E[X|Y = y] = \sum_x x p_{X|Y}(x|y)$$

Similarly, if $X$ and $Y$ are continuous, then provided $f_Y(y) > 0$:

$$f_{X|Y}(x|y) = \frac{f_{XY}(x, y)}{f_Y(y)},$$

and

$$E[X|Y = y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y) dx$$

**Example 33.1:** Say $X$ and $Y$ have joint pdf [see Example 27.3]

$$f_{XY}(x, y) = \begin{cases} \frac{e^{-x/y} e^{-y}}{y} & 0 < x < \infty,\ 0 < y < \infty \\ 0 & \text{else} \end{cases}$$

Find $E[X|Y = y]$.

*Solution:*

*Note:* Conditional expectations satisfy all the properties of ordinary expectation, e.g.,

$$E[g(X) \mid Y = y] = \begin{cases} \sum_x g(x) p_{X|Y}(x|y) & \text{discrete case} \\ \int_{-\infty}^{\infty} g(x) f_{X|Y}(x|y) dx & \text{continuous case} \end{cases}$$

and

$$E\left[\sum_{i=1}^{n} X_i \,\middle|\, Y = y\right] = \sum_{i=1}^{n} E[X_i|Y = y]$$

---

**Computing Expectations by Conditioning**

$E[X|Y = y]$ is a function of $y$, say $g(y)$.
Let $E[X|Y]$ be $g(Y)$, i.e., in Example 33.1:

$$E[X|Y = y] = y$$

So, $$E[X|Y] = Y$$

**Proposition 33.1** $E[X] = E[\, E[X|Y]\,]$, *i.e.,*

$$E[X] = \sum_y E[X|Y = y] p_Y(y) \qquad \textit{[discrete case]}$$

$$E[X] = \int_{-\infty}^{\infty} E[X|Y = y] f_Y(y) dy \qquad \textit{[continuous case]}$$

Why? [Continuous Case]

$$\begin{aligned}\int_{-\infty}^{\infty} E[X|Y = y] f_Y(y) dy &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x f_{X|Y}(x|y) dx\right] f_Y(y) dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{X|Y}(x|y) f_Y(y)\, dx dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{XY}(x, y)\, dx dy \\ &= E[X]\end{aligned}$$

**Example 33.2:** You are in a room with 3 doors.

The 1st door exits the building after 3 min of travel.

The 2nd door returns to where you are after 5 min.

The 3rd door returns to where you are after 7 min.

Each time you enter the room, you are equally likely to pick each of the 3 doors. What is the expected time until you leave the building?

*Solution:*

**Example 33.3:** The number of people that enter a store in a day is random with mean 50.

The amount spent by each person is iid with mean \$8, and independent of the number of people that enter.

What is the expected amount spent in the store in one day? [Hard]

*Solution:*