Limit Theorems

The Central Limit Theorem (CLT) [Ross 8.3]

Proposition 39.1 *The Central Limit Theorem*

Let $X_1, X_2, \dots$ be a sequence of iid random variables having mean μ and variance σ^2. Then, the distribution of

$$Z_n = \frac{X_1 + X_2 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{X_i - \mu}{\sigma}$$

tends to the standard normal as $n \to \infty$. Specifically,

$$P[Z_n \le a] \to \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-u^2/2} du}_{\Phi(a)} \qquad \text{as } n \to \infty$$

Why is the CLT true?

Let $Y_i = \frac{X_i - \mu}{\sigma}$. Then Y_i are iid with mean 0 and variance 1 and

$$Z_n = \frac{Y_1 + Y_2 + \cdots + Y_n}{\sqrt{n}}$$

We will show that the MGF of Z_n converges to the MGF of $\mathcal{N}(0,1)$, i.e., to $e^{t^2/2}$.

The MGF of $Y_i/\sqrt{n}$

$$E\left[e^{tY_i/\sqrt{n}}\right] = M_Y\left(\frac{t}{\sqrt{n}}\right)$$

So, the MGF of $Z_n = \sum_{i=1}^{n} Y_i/\sqrt{n}$ is

$$M_{Z_n}(t) = \left[M_Y\left(\frac{t}{\sqrt{n}}\right)\right]^n$$

We want to show that

$$\lim_{n\to\infty} \left[M_Y\left(\frac{t}{\sqrt{n}}\right)\right]^n = e^{t^2/2}$$

Define $L(t) = \ln M_Y(t)$. Then

$$L(0) = \ln M_Y(0) = 0 \qquad M_Y(0) = E[e^0] = 1$$

$$L'(0) = \frac{M_Y'(0)}{M_Y(0)} = \frac{E[Y]}{1} = 0 \qquad M_Y'(0) = E[Y] = 0$$

$$L''(0) = \frac{M_Y(0)M_Y''(0) - [M_Y'(0)]^2}{[M_Y(0)]^2} = 1 \qquad M_Y''(0) = E[Y^2] = 1$$

So, for small t, $L(t) = \frac{1}{2}t^2 + O(t^3)$.

Finally,

$$\begin{aligned}
\lim_{n\to\infty} \ln M_{Z_n}(t) &= \lim_{n\to\infty} \ln\left[M_Y(t/\sqrt{n})\right]^n \\
&= \lim_{n\to\infty} n \ln M_Y(t/\sqrt{n}) \\
&= \lim_{n\to\infty} \frac{L(t/\sqrt{n})}{n^{-1}} \\
&= \lim_{n\to\infty} \frac{\frac{1}{2}(t/\sqrt{n})^2}{n^{-1}} + O\left(\frac{(t/\sqrt{n})^3}{n^{-1}}\right) \\
&= \lim_{n\to\infty} \frac{t^2}{2} + O\left(\frac{1}{\sqrt{n}}\right) \\
&= \frac{t^2}{2}
\end{aligned}$$

So $\quad \lim_{n\to\infty} M_{Z_n}(t) = e^{t^2/2}$

The CLT can be used to approximate probabilities:

Example 39.1: An astronomer takes iid measurements $X_1, X_2, \dots$ of the distance of a star.

Each X_i has mean d (the true distance) and variance 4 light-years^2.

How many measurements are needed to be 95% certain that the average of the measurements is within ± 0.5 light-years of the true value d?

Solution: Let

$$Z_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{X_i - d}{\sqrt{4}}$$

By the CLT, when n is large, this is approximately $\mathcal{N}(0,1)$.

$$\begin{aligned}
&P\left[-0.5 \le \left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) - d \le 0.5\right] \\
&= P\left[-0.5 \le \frac{1}{n}\sum_{i=1}^{n} (X_i - d) \le 0.5\right] \\
&= P\left[-0.5 \times \frac{\sqrt{n}}{2} \le \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \frac{X_i - d}{2} \le 0.5 \times \frac{\sqrt{n}}{2}\right] \\
&= P\left[-\frac{\sqrt{n}}{4} \le Z_n \le \frac{\sqrt{n}}{4}\right] \\
&\approx \Phi\left(\frac{\sqrt{n}}{4}\right) - \Phi\left(-\frac{\sqrt{n}}{4}\right) \\
&= 2\Phi\left(\frac{\sqrt{n}}{4}\right) - 1
\end{aligned}$$

For this to be at least 0.95, we need

$$\Phi\left(\frac{\sqrt{n}}{4}\right) \ge 0.975$$

From the $\Phi(.)$ Table [Notes #18], $\sqrt{n}/4 \ge 1.96$.

The smallest integer than makes this true is $n = 62$.

Note: This analysis assumes that with 62 observations, Z_n is well approximated by a Gaussian.

The Chebyshev inequality is not an approximation.

$$E\left[\sum_{i=1}^{n} \frac{X_i}{n}\right] = d \qquad Var\left[\sum_{i=1}^{n} \frac{X_i}{n}\right] = \frac{4}{n}$$

So by Chebyshev:

$$P\left[\left|\sum_{i=1}^{n} \frac{X_i}{n} - d\right| \ge 0.5\right] \le \frac{4/n}{(0.5)^2} = \frac{16}{n}$$

95% confident $\Rightarrow 16/n \le 0.05 \Rightarrow n \ge 320$ measurements are enough.

Example 39.2: Let $X_1, \dots, X_{10}$ be the outcomes of 10 fair dice rolls. Use the CLT to approximate $P[30 \le X_1 + \cdots + X_{10} \le 40]$.

Solution: Here $E[X_i] = \frac{7}{2}$ and $Var[X_i] = \frac{35}{12}$

Then

$$\begin{aligned}
&P[30 \le \sum_{i=1}^{10} X_i \le 40] \\
&= P[29.5 \le \sum_{i=1}^{10} X_i \le 40.5] \\
&= P\left[\frac{1}{\sqrt{10}}\frac{29.5 - 10 \cdot \frac{7}{2}}{\sqrt{35/12}} \le \frac{1}{\sqrt{10}}\sum_{i=1}^{10} \frac{X_i - \frac{7}{2}}{\sqrt{35/12}} \le \frac{1}{\sqrt{10}}\frac{40.5 - 10 \cdot \frac{7}{2}}{\sqrt{35/12}}\right] \\
&\approx P[-1.0184 \le Z \le 1.10184] \qquad Z \sim \mathcal{N}(0,1) \\
&= 2\Phi(1.0184) - 1 \\
&\approx 0.6915
\end{aligned}$$

Strong Law of Large Numbers [Ross S8.4]

We saw earlier the *weak* law of large numbers. This suggests that there is a strong law of large numbers as well (and there is).

Proposition 39.2 *Strong Law of Large Numbers*

Let $X_1, X_2, \dots$ be iid with commom mean $E[X_i] = \mu$. Then

$$P\left[\lim_{n\to\infty} \frac{X_1 + X_2 + \cdots + X_n}{n} = \mu\right] = 1$$