Properties of Expectations

Expectation of Sums of Random Variables [Ross S7.2]

Recall that the mean value of X is

$$E[X] = \begin{cases} \sum_x x p_X(x) & X \text{ is discrete} \\ \\ \int_{-\infty}^{\infty} x f_X(x) dx & X \text{ is continuous} \end{cases}$$

Proposition 30.1 *Let X and Y be two random variables. Let $g(x,y)$ be a function. Then*

$$E[g(X,Y)] = \begin{cases} \sum_y \sum_x g(x,y) p_{XY}(x,y) & X, Y \textit{ are discrete} \\ \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{XY}(x,y) dx dy & X, Y \textit{ are continuous} \end{cases}$$

Why?

[Only show for continuous case and $g(x,y)$ is non-negative]

Recall from Proposition 16.2:

$$E[Z] = \int_0^{\infty} P[Z > t] dt$$

So

$$\begin{aligned} E[g(X,Y)] &= \int_0^{\infty} P[g(X,Y) > t] dt \\ &= \int_0^{\infty} \iint_{(x,y):g(x,y)>t} f_{XY}(x,y) dx dy \, dt \\ &= \iint_{\mathbb{R}^2} \int_0^{g(x,y)} f_{XY}(x,y) dt \, dx dy \\ &= \iint_{\mathbb{R}^2} g(x,y) f_{XY}(x,y) dx dy \end{aligned}$$

Example 30.1: The positions $X \sim U(0,L)$ and $Y \sim U(0,L)$ of two persons on a road are independent. What is the mean distance between them?

Solution:

Example 30.2: Show $E[X+Y] = E[X] + E[Y]$.

Solution: [Continuous case only, discrete is similar]

Note: by induction, $E[X_1 + \cdots + X_n] = E[X_1] + \cdots + E[X_n]$.

Example 30.3: Let $X_1, X_2, \ldots, X_n$ be iid with (common) mean μ. The quantity

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

is called the **sample mean**. What is $E[\bar{X}]$?

Solution:

Example 30.4: 10 friends play a game. Each has a ball and picks one of 10 targets randomly, independently of the others. Each has probability p of hitting their chosen target.

What is the expected number of targets not hit?

Solution: