

Conditional Probability and Independence

Baye's Theorem [Ross S3.3]

Law of Total Probability:

Let $E, F \subset S$.

$$\text{Then } E = ES = E(F \cup F^c) = EF \cup EF^c$$

$$\begin{aligned}\text{and } P[E] &= P[EF] + P[EF^c] \\ &= P[E|F]P[F] + P[E|F^c]P[F^c]\end{aligned}$$

Example 6.1: The probability of an insurance claim is

- 0.4 for 30% of persons (type 1),
- 0.5 for 70% of persons (type 2).

What is the probability that a random person has a claim?

Solution:

Let F_1, \dots, F_n partition S .

Then
$$E = ES = E\left(\bigcup_{i=1}^n F_i\right)$$
$$= \bigcup_{i=1}^n (EF_i)$$

So
$$P[E] = P[\bigcup_{i=1}^n (EF_i)]$$
$$= \sum_{i=1}^n P[EF_i]$$
$$= \sum_{i=1}^n P[E|F_i]P[F_i] \quad \text{[Law of total probability]}$$

Example 6.2: You roll a 4-sided die. If result is ≤ 2 , you roll once more and otherwise stop. What is probability that the sum ≥ 4 ?

Solution:

Baye's Theorem and Inference:

Let F_1, F_2, \dots, F_n partition S .

Say we know $P[E|F_j]$. We want to compute $P[F_j|E]$:

$$\begin{aligned} P[F_j|E] &= \frac{P[EF_j]}{P[E]} \\ &= \frac{P[E|F_j]P[F_j]}{P[E|F_1]P[F_1] + P[E|F_2]P[F_2] + \dots + P[E|F_n]P[F_n]} \end{aligned} \tag{6.1}$$

This is Baye's theorem/rule.

Application to inference:

Before any partial information is revealed (i.e., observing E occurs), the probabilities are:

$$P[F_1], P[F_2], \dots, P[F_n] \quad \left. \vphantom{P[F_1], P[F_2], \dots, P[F_n]} \right\} \quad \text{“prior probabilities”}$$

After observing E occur, they are revised as:

$$P[F_1|E], P[F_2|E], \dots, P[F_n|E] \quad \left. \vphantom{P[F_1|E], P[F_2|E], \dots, P[F_n|E]} \right\} \quad \text{“posterior probabilities”}$$

according to (6.1).

Posterior probabilities are key to practical inference (e.g., classification, pattern recognition, detection, etc.)

Example 6.3: A 3-card deck has

- 1) one card with red on both sides
- 2) one card with black on both sides
- 3) one card with red on one side + black on the other.

One side of 1 card is picked at random. It is red. What is the probability that other side is black?

Solution:

Example 6.4: A blood test has 95% prob of detecting a disease when it is present. It has a 1% false positive rate when it is not present. 0.5% of people have the disease.

- a) If a random person tests positive, what is prob. that disease is present?
- b) If a random person tests negative, what is prob. that disease is present?

Solution: