Jointly Distributed Random Variables

Joint Distribution of Functions of Random Variables [Ross S6.7]

Let X and Y have joint pdf $f_{XY}(x, y)$.

In some examples we computed the distribution of $Z = g(X, Y)$, e.g.

- in Example 23.2 we computed the cdf of $D = \sqrt{X^2 + Y^2}$
- in Example 23.3 we computed the pdf of $Z = X/Y$.

Now, consider

$$\begin{aligned} Y_1 &= g_1(X_1, X_2) \\ Y_2 &= g_2(X_1, X_2) \end{aligned}$$

and we want the joint pdf of Y_1 and Y_2.

We make the following assumptions on g_1 and g_2:

- The system of equations

  $$\begin{aligned} y_1 &= g_1(x_1, x_2) \\ y_2 &= g_2(x_1, x_2) \end{aligned}$$

  can be uniquely solved for x_1 and x_2 in terms of y_1 and y_2:

  $$\begin{aligned} x_1 &= h_1(y_1, y_2) \\ x_2 &= h_2(y_1, y_2). \end{aligned}$$

- g_1 and g_2 have continuous partial derivates such that the determinant

$$J(x_1, x_2) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix} = \frac{\partial g_1}{\partial x_1}\frac{\partial g_2}{\partial x_2} - \frac{\partial g_1}{\partial x_2}\frac{\partial g_2}{\partial x_1} \neq 0$$

Under these conditions, the pdf of Y_1 and Y_2 can be shown to be:

$$f_{Y_1Y_2}(y_1, y_2) = f_{X_1X_2}(x_1, x_2)\ |J(x_1, x_2)|^{-1} \tag{29.1}$$

where

$$\begin{aligned} x_1 &= h_1(y_1, y_2) \\ x_2 &= h_2(y_1, y_2). \end{aligned}$$

Example 29.1: Let

$$\begin{aligned} Y_1 &= X_1 + X_2 \\ Y_2 &= X_1 - X_2 \end{aligned}$$

Find the joint pdf $f_{Y_1Y_2}(y_1, y_2)$ in terms of $f_{X_1X_2}(x_1, x_2)$.

Solution: Solving

$$\begin{aligned} y_1 &= x_1 + x_2 \\ y_2 &= x_1 - x_2 \end{aligned}$$

we get

$$\begin{aligned} x_1 &= \frac{1}{2}y_1 + \frac{1}{2}y_2 \\ x_2 &= \frac{1}{2}y_1 - \frac{1}{2}y_2 \end{aligned}$$

Also

$$J(x_1, x_2) = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \neq 0$$

Hence, from (29.6)

$$f_{Y_1Y_2}(y_1, y_2) = \frac{1}{2} f_{X_1X_2}\left(\frac{1}{2}y_1 + \frac{1}{2}y_2, \frac{1}{2}y_1 - \frac{1}{2}y_2\right) \tag{29.2}$$

Example 29.2: Let R and Θ be two random variables with joint pdf $f_{R\Theta}(r, \theta)$. Consider the change of variables

$$X = R\cos\Theta$$
$$Y = R\sin\Theta.$$

Find $f_{R\Theta}(r, \theta)$ in terms of $f_{XY}(x, y)$. [Hard]

Note: This is Problem T9.1; see also textbook Example 6.7b for a different tedious approach.

Solution: We have the system of equations

$$x = g_1(r, \theta) = r\cos\theta$$
$$y = g_2(r, \theta) = r\sin\theta$$

which is solved by

$$r = h_1(x, y) = \sqrt{x^2 + y^2}$$

$$\theta = h_2(x, y) = \begin{cases} \tan^{-1}(y/x) & x > 0, y > 0 \\ \tan^{-1}(y/x) + \pi & x < 0 \\ \tan^{-1}(y/x) + 2\pi & x > 0, y < 0 \end{cases}$$

where $h_2(x, y)$ is the angle of the vector (x, y).

Computing the Jacobian determinant

$$
\begin{aligned}
J(r, \theta) &= \begin{vmatrix} \frac{\partial g_1}{\partial r} & \frac{\partial g_1}{\partial \theta} \\ \frac{\partial g_2}{\partial r} & \frac{\partial g_2}{\partial \theta} \end{vmatrix} \\
&= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} \\
&= r\cos^2\theta + r\sin^2\theta \\
&= r
\end{aligned}
$$

So, the pdf $f_{XY}(x, y)$ and $f_{R\Theta}(r, \theta)$ are related by

$$
\begin{aligned}
f_{XY}(x, y) &= f_{R\Theta}(r, \theta)\ |J(r, \theta)|^{-1} && (29.3) \\
&= f_{R\Theta}(r, \theta)/r && (29.4)
\end{aligned}
$$

or equivalently:

$$
\begin{aligned}
f_{R\Theta}(r, \theta) &= f_{XY}(x, y)r && (29.5) \\
&= f_{XY}(r\cos\theta, r\sin\theta)r && (29.6)
\end{aligned}
$$

So, to compute the probability that $(R, \Theta) \in A$:

$$
\begin{aligned}
P[(R, \Theta) \in A] &= \iint_{(r,\theta)\in A} f_{R\Theta}(r, \theta) dr d\theta \\
&= \iint_{(r,\theta)\in A} f_{XY}(r\cos\theta, r\sin\theta) r dr d\theta
\end{aligned}
$$