Continuous Random Variables

Expectation [Ross 5.2]

Definition 16.1: For a continuous random variable X,

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx.$$

Example 16.1: Find $E[X]$ if

$$f_X(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{else} \end{cases}$$

Solution:

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x f_X(x) dx. \\ &= \int_0^1 2x^2 dx \\ &= \frac{2}{3} \end{aligned}$$

Example 16.2: Let X have pdf

$$f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{else} \end{cases}$$

Find $E[e^X]$.

Solution: Let $Y = e^X$. Find $f_Y(y)$ by first determining $F_Y(y)$.

Since X ranges from 0 to 1, $Y = e^X$ ranges from 1 to e. So, for $1 \leq y \leq e$:

$$\begin{aligned}
F_Y(y) &= P[Y \leq y] \\
&= P[e^X \leq y] \\
&= P[X \leq \ln y] \\
&= \int_0^{\ln y} f_X(x)\, dx \\
&= \ln y
\end{aligned}$$

Then $$\begin{aligned}
f_Y(y) &= \frac{d}{dy} F_Y(y) \\
&= \frac{1}{y}
\end{aligned}$$

for $1 \leq y \leq e$.

Y cannot take values outside this interval, so outside this interval $f_Y(y) = 0$.

Finally $$\begin{aligned}
E[Y] &= \int_{-\infty}^{\infty} y f_Y(y) dy \\
&= \int_1^e y \times \frac{1}{y} dy \\
&= e - 1
\end{aligned}$$

Proposition 16.1 *For a continuous random variable X,*

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

Example 16.3: Solve Example 16.2 using Proposition 16.1.

Solution:

$$\begin{aligned}
E[e^X] &= \int_{-\infty}^{\infty} e^x f_X(x)dx \\
&= \int_0^1 e^x dx \\
&= e - 1
\end{aligned}$$

Proposition 16.2 *If X is a non-negative random variable, then*

$$E[X] = \int_0^{\infty} P[X > x]dx$$

Why?

$$\begin{aligned}
\int_0^{\infty} P[Y > y]dy &= \int_0^{\infty} \left[\int_y^{\infty} f_Y(u)du\right] dy \\
&= \int_0^{\infty} \int_y^{\infty} f_Y(u)dudy \\
&= \int_0^{\infty} \int_0^{u} f_Y(u)dydu \\
&= \int_0^{\infty} \left[\int_0^{u} dy\right] f_Y(u)du \\
&= \int_0^{\infty} uf_Y(u)du \\
&= E[Y]
\end{aligned}$$

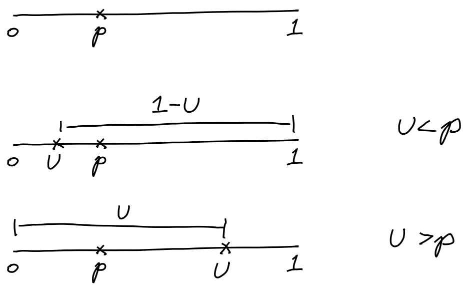
Example 16.4: A point p on a stick of length 1, where $0 \leq p \leq 1$ is fixed.

Let the stick be broken at U, where

$$f_U(u) = \begin{cases} 1 & 0 \leq u \leq 1 \\ 0 & \text{else} \end{cases}$$

Determine the expected length of the piece that contains p.

Solution:



Let $L(U)$ denote the length of the substick that contains p. Then

$$L(U) = \begin{cases} 1 - U & U < p \\ U & U > p \end{cases}$$

$$\begin{aligned} E[L(U)] &= \int_0^1 L(u) f_U(u) du \\ &= \int_0^p L(u) f_U(u) du + \int_p^1 L(u) f_U(u) du \\ &= \int_0^p (1-u) du + \int_p^1 u du \\ &= \frac{1}{2} + p(1-p) \end{aligned}$$

Proposition 16.3 *For a continuous random variable X,*

$$E[aX+b] = aE[X]+b$$

Why?

$$\begin{aligned} E[aX+b] &= \int_{-\infty}^{\infty} (ax+b) f_X(x)dx \\ &= a\int_{-\infty}^{\infty} x f_X(x)dx + b\int_{-\infty}^{\infty} f_X(x)dx \\ &= aE[X]+b \end{aligned}$$

Definition 16.2: For a continuous random variable X,

$$Var[X] = E[(X - E[X])^2]$$

Again, $Var[X] = E[X^2] - (E[X])^2$

Also, $Var[aX+b] = a^2 Var[X]$.

Example 16.5: Find $Var[X]$ in Example 16.1

Solution:

$$\begin{aligned} Var[X] &= E[X^2] - (E[X])^2 \\ &= E[X^2] - (2/3)^2 \qquad\qquad \text{[from Example 16.1]} \end{aligned}$$

$$\begin{aligned} E[X^2] &= \int_{-\infty}^{\infty} x^2 f_X(x) dx \\ &= \int_0^1 x^2 \times 2x \; dx \\ &= 1/2 \end{aligned}$$

$$Var[X] = 1/2 - (2/3)^2$$