# Properties of Expectations

## **Conditional Expectation** [Ross S7.5]

**Example 34.1:** Recall that $X$ and $Y$ are jointly (bivariate) Gaussian (normal) with parameters:

$$\mu_X, \mu_Y, \sigma_X > 0, \sigma_Y > 0, -1 < \rho < 1$$

when $f_{XY}(x,y)$ is given by

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \tag{28.1}$$
$$\times \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - 2\rho\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}\right]\right\}$$

We now show that $\rho$ is the correlation between $X$ and $Y$.

From Notes #28:

$$\begin{aligned} E[X] &= \mu_X \\ E[Y] &= \mu_Y \\ Var[X] &= \sigma_X^2 \\ Var[Y] &= \sigma_Y^2 \end{aligned}$$

Therefore $\quad \rho(X,Y) = \dfrac{Cov[X,Y]}{\sigma_X\sigma_Y}$

$$= \frac{E[XY] - \mu_X\mu_Y}{\sigma_X\sigma_Y}$$

To determine $E[XY]$, recall from Notes #28 that $f_{X|Y}(x|y)$ is Gaussian pdf where $X$ has mean

$$\mu_X + \rho \frac{\sigma_X}{\sigma_Y}(y - \mu_Y).$$

So

$$E[X|Y = y] = \mu_X + \rho \frac{\sigma_X}{\sigma_Y}(y - \mu_Y)$$

Now, $$E[XY] = E[\, E[XY|Y]\, ]$$

and
$$\begin{aligned} E[XY|Y = y] &= E[Xy|Y = y] \\ &= yE[X|Y = y] \\ &= y\left(\mu_X + \rho \frac{\sigma_X}{\sigma_Y}(y - \mu_Y)\right) \\ &= \mu_X y + \rho \frac{\sigma_X}{\sigma_Y}(y^2 - \mu_Y y) \end{aligned}$$

$$\Rightarrow E[XY|Y] = \mu_X Y + \rho \frac{\sigma_X}{\sigma_Y}(Y^2 - \mu_Y Y)$$

Therefore
$$\begin{aligned} E[XY] &= E[\, E[XY|Y]\, ] \\ &= E\left[\mu_X Y + \rho \frac{\sigma_X}{\sigma_Y}(Y^2 - \mu_Y Y)\right] \\ &= \mu_X E[Y] + \rho \frac{\sigma_X}{\sigma_Y}(E[Y^2] - \mu_Y E[Y]) \\ &= \mu_X \mu_Y + \rho \frac{\sigma_X}{\sigma_Y}(E[Y^2] - \mu_Y^2) \end{aligned}$$

$$
\begin{aligned}
&= \mu_X \mu_Y + \rho \frac{\sigma_X}{\sigma_Y} Var[Y] \\
&= \mu_X \mu_Y + \rho \frac{\sigma_X}{\sigma_Y} \sigma_Y^2 \\
&= \mu_X \mu_Y + \rho \sigma_X \sigma_Y
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \rho(X, Y) &= \frac{E[XY] - \mu_X \mu_Y}{\sigma_X \sigma_Y} \\
&= \frac{\rho \sigma_X \sigma_Y}{\sigma_X \sigma_Y} \\
&= \rho
\end{aligned}
$$

---

**Computing Probabilities by Conditioning**

We can use conditioning to compute probabilities:

Let $A$ be an event.

Let random variable $Y \in \{y_1, y_2, \ldots\}$ and $B_i = \{Y = y_i\}$.

Then $B_1, B_2, \ldots$ partition the sample space $S$. So by law of total probability:

$$
\begin{aligned}
P[A] &= P[A|B_1]P[B_1] + P[A|B_2]P[B_2] + \cdots \\
&= P[A|Y = y_1]P[Y = y_1] + P[A|Y = y_2]P[Y = y_2] + \cdots \\
&= \sum_n P[A|Y = y_n]P[Y = y_n]
\end{aligned}
$$

Similarly, if $Y$ is continuous:

$$
P[A] = \int_{-\infty}^{\infty} P[A \mid Y = y] f_Y(y) dy
$$

**Example 34.2:** Say $X$ and $Y$ are independent random variables with densities $f_X(x)$ and $f_Y(y)$.

Find $P[X < Y]$.

*Solution:* Method 1:

$$\begin{aligned}P[X < Y] &= \iint\limits_{x<y} f_X(x)f_Y(y)dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{y} f_X(x)f_Y(y)dxdy \\ &= \int_{-\infty}^{\infty} F_X(y)f_Y(y)dxdy\end{aligned}$$

Method 2:

$$\begin{aligned}P[X < Y] &= \int_{-\infty}^{\infty} P[X < Y \mid Y = y]f_Y(y)dy \\ &= \int_{-\infty}^{\infty} P[X < y \mid Y = y]f_Y(y)dy \\ &= \int_{-\infty}^{\infty} P[X < y]f_Y(y)dy \\ &= \int_{-\infty}^{\infty} F_X(y)f_Y(y)dy\end{aligned}$$

**Example 34.3:** Say $X$ and $Y$ are independent with densities $f_X(x)$ and $f_Y(y)$. Find the cdf and pdf of $X + Y$ by conditioning on $Y$.

*Solution:*

$$P[X + Y \leq a] = \int_{-\infty}^{\infty} P[X + Y \leq a \mid Y = y]f_Y(y)dy$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} P[X + y \leq a \mid Y = y] f_Y(y) dy \\
&= \int_{-\infty}^{\infty} P[X + y \leq a] f_Y(y) dy \\
&= \int_{-\infty}^{\infty} P[X \leq a - y] f_Y(y) dy \\
&= \int_{-\infty}^{\infty} F_X(a - y) f_Y(y) dy
\end{aligned}
$$

and, taking derivatives:

$$
\begin{aligned}
f_{X+Y}(a) &= \frac{d}{da} P[X + Y \leq a] \\
&= \frac{d}{da} \int_{-\infty}^{\infty} F_X(a - y) f_Y(y) dy \\
&= \int_{-\infty}^{\infty} \frac{d}{da} F_X(a - y) f_Y(y) dy \\
&= \int_{-\infty}^{\infty} f_X(a - y) f_Y(y) dy
\end{aligned}
$$