

Conditional Probability and Independence

Conditional Probability [Ross S3.1, S3.2]

Conditional probability is one of the most important concepts in this course.

- it is a tool to compute probabilities,
- it lets us update probabilities when partial information is revealed.

Example 5.1: Say we toss two dice.

What is the probability that the sum is 9?

Solution: This event is $E = \{(3, 6), (4, 5), (5, 4), (6, 3)\}$.

So $P[E] = 4/36$.

Example 5.2: Say I roll 1st die (but not 2nd) and get a 4.

What is the probability that the sum will be 9?

Solution: All possible outcomes given this new information are:

$$F = \{(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6)\}.$$

The other 30 cases are inconsistent with the 1st die roll

\Rightarrow they now have probability = 0.

The 6 cases in F had the same probability before the 1st die was rolled.

They should now be equally likely after the outcome of 1st die roll, i.e., each has probability $1/6$.

After the 1st die roll was revealed (i.e., after F was revealed to occur):

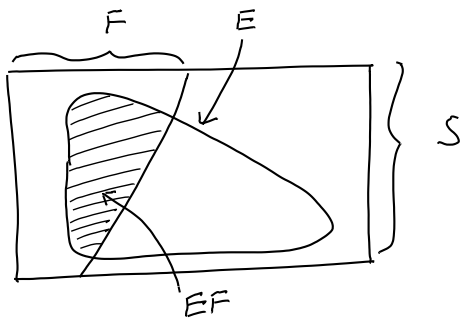
$$\{\text{sum} = 9\} = EF = \{(4, 5)\}$$

and this has probability $1/6$.

We say that **the probability of E given F has occurred** is $1/6$, or

$$P[E \mid F] = 1/6.$$

Let's generalize: let's not assume the elements of S are equally likely:



If F has occurred, then for E to occur, EF must occur.

If F has occurred, our sample space S is reduced to F .

So if F has occurred, probabilities should be computed relative to F :

Definition 5.1: If $P[F] > 0$, then

$$P[E \mid F] = \frac{P[EF]}{P[F]}.$$

Example 5.3: A coin is flipped twice. What is the probability of two heads if

a) first flip is heads?

b) at least one flip is heads?

Solution: a) $S = \{hh, ht, th, tt\}$, $E = \{hh\}$ and $F = \{ht, hh\}$. So

$$P[E|F] = \frac{P[EF]}{P[F]} = \frac{P[hh]}{P[F]} = \frac{1/4}{1/2} = 1/2.$$

b) Here, $F = \{hh, ht, th\}$. So

$$P[E|F] = \frac{P[EF]}{P[F]} = \frac{1/4}{3/4} = 1/3.$$

Example 5.4: Two 4-sided dice are rolled. Let

$$E = \{ \text{max of both rolls is 3} \}$$

$$F = \{ \text{min of both rolls is 2} \}$$

What is $P[E | F]$?

Solution:

$$E = \{(1, 3), (2, 3), (3, 3), (3, 2), (3, 1)\}$$

$$F = \{(4, 2), (3, 2), (2, 2), (2, 3), (2, 4)\}$$

$$P[E|F] = \frac{P[EF]}{P[F]} = \frac{2/16}{5/16} = 2/5.$$

Conditional Probability satisfies the axioms of probability:

For fixed F with $P[F] > 0$, the function $P[\cdot|F]$ satisfies all the same axioms as $P[\cdot]$:

$$\begin{aligned} \text{[A1]} \quad P[E|F] &= P[EF]/P[F] \geq 0 && \text{since } P[EF] \geq 0 \\ P[E|F] &= P[EF]/P[F] \leq 1 && \text{since } EF \subset F \end{aligned}$$

$$\text{[A2]} \quad P[S|F] = P[SF]/P[F] = P[F]/P[F] = 1.$$

$$\text{[A3]} \quad \text{Let } E_1 \cap E_2 = \emptyset. \text{ Then } E_1 F \cap E_2 F = \emptyset.$$

$$\begin{aligned} P[E_1 \cup E_2|F] &= P[(E_1 \cup E_2)F]/P[F] \\ &= P[E_1 F \cup E_2 F]/P[F] \\ &= P[E_1 F]/P[F] + P[E_2 F]/P[F] \\ &= P[E_1|F] + P[E_2|F] \end{aligned}$$

Multiplication Rule:

Since $F_1 F_2$ is a set, we also write $P[E|F_1 F_2] = P[EF_1 F_2]/P[F_1 F_2]$, etc.

Now

$$\begin{aligned} P[E_1 E_2 \cdots E_n] &= P[E_1] \times \frac{P[E_1 E_2]}{P[E_1]} \times \frac{P[E_1 E_2 E_3]}{P[E_1 E_2]} \times \cdots \\ &\quad \cdots \times \frac{P[E_1 E_2 \cdots E_{n-1}]}{P[E_1 E_2 \cdots E_{n-2}]} \times \frac{P[E_1 E_2 \cdots E_n]}{P[E_1 E_2 \cdots E_{n-1}]} \\ &= P[E_1] \times P[E_2|E_1] \times P[E_3|E_1 E_2] \times \cdots \\ &\quad \cdots \times P[E_n|E_1 E_2 \cdots E_{n-1}] \end{aligned}$$

Example 5.5: 3 grad and 12 ugrad students are randomly divided into 3 groups of 5. What is the prob that each group has exactly 1 grad student?

Solution:

Let $E_1 = \{\text{grad \#1 is in a group}\},$

$E_2 = \{\text{grad \#1 and \#2 are in different groups}\}$

$E_3 = \{\text{all grads are in different groups}\}$

Then $E_3 = E_1 E_2 E_3$ and

$$\begin{aligned} P[E_3] &= P[E_1 E_2 E_3] \\ &= P[E_1]P[E_2|E_1]P[E_3|E_1 E_2] \end{aligned}$$

with

$$P[E_1] = 1$$

$$P[E_2|E_1] = 10/14$$

$$P[E_3|E_1 E_2] = 5/13$$