Jointly Distributed Random Variables

Example 25.1: Let X and Y have joint density

$$f_{XY}(x,y) = \begin{cases} 6e^{-2x}e^{-3y} & x>0,\ y>0 \\ 0 & \text{else} \end{cases}$$

Are X and Y independent?

Solution:

Alternate method: Notice that

$$f_{XY}(x,y) = h(x)g(y)$$

where

$$h(x) = \begin{cases} e^{-2x} & x>0 \\ 0 & \text{else} \end{cases} \qquad g(y) = \begin{cases} 6e^{-3y} & y>0 \\ 0 & \text{else} \end{cases}$$

So,
$$\begin{aligned} 1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\ dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x)g(y)\ dxdy \\ &= \underbrace{\int_{-\infty}^{\infty} h(x)\ dx}_{C_1}\underbrace{\int_{-\infty}^{\infty} g(y)\ dy}_{C_2} \\ &= C_1C_2 \end{aligned}$$

Now,
$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dy = \int_{-\infty}^{\infty} h(x)g(y)dy = C_2h(x) \\ f_Y(y) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dx = \int_{-\infty}^{\infty} h(x)g(y)dx = C_1g(y) \end{aligned}$$

Finally,
$$\begin{aligned} f_X(x)f_Y(y) &= C_1C_2h(x)g(y) \\ &= h(x)g(y) \\ &= f_{XY}(x,y) \end{aligned}$$

So if you can factor $f_{XY}(x,y) = h(x)g(y)$, then X and Y are independent!

And if X and Y are independent, then $f_{XY}(x,y)$ can be factored as $f_{XY}(x,y) = f_X(x)f_Y(y)$.

Proposition 25.1 *X and Y are independent if and only if $f_{XY}(x,y) = h(x)g(y)$ for some $h(x)$ and $g(y)$.*

Example 25.2: Let X and Y have joint pdf

$$f_{XY}(x,y) = \begin{cases} 24xy & x>0,\ y>0,\ 0<x+y<1 \\ 0 & \text{else} \end{cases}$$

Are X and Y independent?

Solution:

Example 25.3: Two people decide to meet. Each arrives independently and uniformly between noon and 1pm.

What is the probability that the first to arrive waits longer than 10 min for the second the arrive?

Solution: