Properties of Expectations

Conditional Expectation [Ross S7.5]

Recall that for 2 discrete random variables X and Y with $P[Y = y] > 0$:

$$\begin{aligned} p_{X|Y}(x|y) &= P[X = x|Y = y] \\ &= \frac{p_{XY}(x, y)}{p_Y(y)} \end{aligned}$$

We can define the **conditional expectation**:

$$E[X|Y = y] = \sum_x x p_{X|Y}(x|y)$$

Similarly, if X and Y are continuous, then provided $f_Y(y) > 0$:

$$f_{X|Y}(x|y) = \frac{f_{XY}(x, y)}{f_Y(y)},$$

and

$$E[X|Y = y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y) dx$$

Example 33.1: Say X and Y have joint pdf [see Example 27.3]

$$f_{XY}(x, y) = \begin{cases} \frac{e^{-x/y} e^{-y}}{y} & 0 < x < \infty, \ 0 < y < \infty \\ 0 & \text{else} \end{cases}$$

Find $E[X|Y = y]$.

Solution: From Example 27.3, for $x > 0, y > 0$

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f_{XY}(x,y)}{f_Y(y)} \\ &= \frac{1}{y} e^{-x/y} \end{aligned}$$

So, $\quad E[X|Y = y] = \int_0^\infty \frac{x}{y} e^{-x/y} dx = y$

Note: Conditional expectations satisfy all the properties of ordinary expectation, e.g.,

$$E[g(X) \mid Y = y] = \begin{cases} \sum_x g(x) p_{X|Y}(x|y) & \text{discrete case} \\ \\ \int_{-\infty}^{\infty} g(x) f_{X|Y}(x|y) dx & \text{continuous case} \end{cases}$$

and

$$E\left[\sum_{i=1}^n X_i \,\middle|\, Y = y \right] = \sum_{i=1}^n E[X_i | Y = y]$$

Computing Expectations by Conditioning

$E[X|Y = y]$ is a function of y, say $g(y)$.
Let $E[X|Y]$ be $g(Y)$, i.e., in Example 33.1:

$$E[X|Y = y] = y$$

So, $\qquad E[X|Y] = Y$

Proposition 33.1 $E[X] = E[\ E[X|Y]\ ]$, *i.e.,*

$$E[X] = \sum_y E[X|Y = y]p_Y(y) \qquad \textit{[discrete case]}$$

$$E[X] = \int_{-\infty}^{\infty} E[X|Y = y]f_Y(y)dy \qquad \textit{[continuous case]}$$

Why? [Continuous Case]

$$\begin{aligned}
\int_{-\infty}^{\infty} E[X|Y = y]f_Y(y)dy &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x f_{X|Y}(x|y)dx\right] f_Y(y)dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{X|Y}(x|y)f_Y(y)\ dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{XY}(x,y)\ dxdy \\
&= E[X]
\end{aligned}$$

Example 33.2: You are in a room with 3 doors.

The 1st door exits the building after 3 min of travel.

The 2nd door returns to where you are after 5 min.

The 3rd door returns to where you are after 7 min.

Each time you enter the room, you are equally likely to pick each of the 3 doors. What is the expected time until you leave the building?

Solution: Let X = time to leave building, and Y = door choice.

$$
\begin{aligned}
E[X] = E[X|Y=1]P[Y=1] \\
+ E[X|Y=2]P[Y=2] \\
+ E[X|Y=3]P[Y=3] \\
= \frac{1}{3}(E[X|Y=1] + E[X|Y=2] + E[X|Y=3])
\end{aligned}
$$

Also, $E[X|Y=1] = 3$

$E[X|Y=2] = 5 + E[X]$

$E[X|Y=3] = 7 + E[X]$

Combining, $E[X] = \frac{1}{3}(3 + 5 + E[X] + 7 + E[X])$

$\Rightarrow E[X] = 15$

Example 33.3: The number of people that enter a store in a day is random with mean 50.

The amount spent by each person is iid with mean $8, and independent of the number of people that enter.

What is the expected amount spent in the store in one day? [Hard]

Solution:

Let N = # customers that enter store in one day.

Let X_i = amount spent by ith customer.

Total amount spent is $Y = \sum_{i=1}^{N} X_i$.

$$E\left[\sum_{i=1}^{N} X_i\right] = E\left[E\left[\sum_{i=1}^{N} X_i \middle| N\right]\right]$$

and $$E\left[\sum_{i=1}^{N} X_i \middle| N = n\right] = E\left[\sum_{i=1}^{n} X_i \middle| N = n\right]$$
$$= E\left[\sum_{i=1}^{n} X_i\right]$$
$$= \sum_{i=1}^{n} E\left[X_i\right]$$
$$= nE[X_1]$$

so $$E\left[\sum_{i=1}^{N} X_i \middle| N\right] = NE[X_1]$$

Therefore $$E\left[\sum_{i=1}^{N} X_i\right] = E\left[\ NE[X_1]\ \right]$$
$$= E[N]E[X_1]$$
$$= 50 \times 8$$