Properties of Expectations

Covariance, Variance of Sums [Ross S7.4]

Proposition 31.1 *If X and Y are independent, then for any functions $g(x)$ and $h(y)$:*

$$\begin{aligned} &i) \quad E[g(X)h(Y)] = E[g(X)]E[h(Y)] \\ &ii) \quad g(X) \text{ and } h(Y) \text{ are independent.} \end{aligned}$$

Why?

$$\begin{aligned} i) \quad E[g(X)h(Y)] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)f_{XY}(x,y)dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)f_X(x)f_Y(y)dxdy \\ &= \int_{-\infty}^{\infty} g(x)f_X(x)dx \int_{-\infty}^{\infty} h(y)f_Y(y)dy \\ &= E[h(Y)]E[g(X)] \end{aligned}$$

ii) Let $A = \{x \mid g(x) \le a\}$ and $B = \{y \mid h(y) \le b\}$. Then:

$$\begin{aligned} &P[g(X) \le a, h(Y) \le b] \\ &\qquad = P[X \in A, Y \in B] \\ &\qquad = P[X \in A]\ P[Y \in B] && \text{since } X \text{ and } Y \text{ are independent} \\ &\qquad = P[g(X) \le a]\ P[h(Y) \le b] \end{aligned}$$

For a single random variable X, its mean and variance give us some information about X.

For two random variables X and Y, **covariance** (and **correlation**) will give us information about the relationship between the pair X and Y.

Definition 31.1: The **covariance** between X and Y, denoted $Cov[X,Y]$, is defined as

$$Cov[X,Y] = E\,[\;(X - E[X])(Y - E[Y])\;]$$

Just as $Var[X] = E[X^2] - (E[X])^2$, we also have:

$$\begin{aligned}
Cov[X,Y] &= E\,[\;(X - E[X])(Y - E[Y])\;] \\
&= E\,[\;XY - E[X]Y - E[Y]X + E[X]E[Y])\;] \\
&= E[XY] + E[\;-E[X]Y\;] + E[\;-E[Y]X\;] + E[\;E[X]E[Y]\;] \\
&= E[XY] - E[X]E[Y] - E[Y]E[X] + E[X]E[Y] \\
&= E[XY] - E[X]E[Y]
\end{aligned}$$

Note: If X and Y are independent, then $E[XY] = E[X]E[Y]$
so $Cov[X,Y] = 0$.

Example 31.1: Does $Cov[X,Y] = 0$ imply X and Y are independent?

Solution: No!

Let $P[X=0] = P[X=1] = P[X=-1] = 1/3$.

Let $Y = \begin{cases} 0 & X \neq 0 \\ 1 & X = 0 \end{cases}$

a) X and Y are not independent since

$$\frac{1}{3} = P[X=0, Y=0] \neq P[X=0]P[Y=0] = \frac{1}{3}\frac{2}{3}$$

b) Since $XY = 0$

$$Cov[X,Y] = \underbrace{E[XY]}_{=0} - \underbrace{E[X]}_{=0} E[Y] = 0$$

Proposition 31.2

i) $Cov[X,Y] = Cov[Y,X]$

ii) $Cov[X,X] = Var[X]$

iii) $Cov[aX,Y] = aCov[X,Y] = Cov[X,aY]$

iv) $Cov\left[\sum_{i=1}^{n} X_i,\ \sum_{j=1}^{m} Y_j\right] = \sum_{i=1}^{n}\sum_{j=1}^{m} Cov[X_i,Y_j]$

Why? i), ii) and iii) follow from

$$Cov[X,Y] = E[XY] - E[X]E[Y]$$

For iv), let $\quad U = \sum_{i=1}^{n} X_i \qquad V = \sum_{j=1}^{m} Y_i$

$$E[X_i] = \mu_i \qquad E[Y_j] = \nu_j$$

Then: $\quad E\left[U\right] = \sum_{i=1}^{n} \mu_i \qquad E\left[V\right] = \sum_{j=1}^{m} \nu_j$

So, $\quad Cov\left[U,\ V\right] = E\left[\left(\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \mu_i\right)\left(\sum_{j=1}^{m} Y_j - \sum_{j=1}^{m} \nu_j\right)\right]$

$$
\begin{aligned}
&= E\left[\sum_{i=1}^{n}(X_i - \mu_i)\sum_{j=1}^{m}(Y_j - \nu_j)\right] \\
&= E\left[\sum_{i=1}^{n}\sum_{j=1}^{m}(X_i - \mu_i)(Y_j - \nu_j)\right] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} E\left[(X_i - \mu_i)(Y_j - \nu_j)\right] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} Cov[X_i, Y_j]
\end{aligned}
$$

Now, say we pick $n = m$ and $Y_j = X_j$ in part iv) of Proposition 31.2:

$$
\begin{aligned}
Var\left[\sum_{i=1}^{n} X_i\right] &= Cov\left[\sum_{i=1}^{n} X_i,\ \sum_{j=1}^{n} X_j\right] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} Cov[X_i, X_j] \\
&= \sum_{i=1}^{n}\left(Cov[X_i, X_i] + \sum_{\substack{j=1 \\ j\neq i}}^{n} Cov[X_i, X_j]\right) \\
&= \sum_{i=1}^{n} Var[X_i] + \sum_{\substack{i,j \\ j\neq i}} Cov[X_i, X_j] \\
&= \sum_{i=1}^{n} Var[X_i] + 2\sum_{\substack{i,j \\ i<j}} Cov[X_i, X_j]
\end{aligned}
$$

In the special case that each pair X_i, X_j are independent when $i \neq j$, then:

$$Var\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} Var[X_i]$$

Example 31.2: Recall (from Example 30.3) that $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is called the **sample mean**. Let

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

be the **sample variance**.

Let $X_1, \ldots, X_n$ be iid with (common) mean μ and variance σ^2.

Find a) $Var[\bar{X}]$ and b) $E[S^2]$. [b) is hard]

Solution: a)

$$\begin{aligned} Var[\bar{X}] &= Var\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] \\ &= \frac{1}{n^2} Var\left[\sum_{i=1}^{n} X_i\right] \\ &= \frac{1}{n^2}\sum_{i=1}^{n} Var\left[X_i\right] \\ &= \frac{\sigma^2}{n} \end{aligned}$$

b) First some algebra:

$$(n-1)S^2 = \sum_{i=1}^{n}(X_i - \mu + \mu - \bar{X})^2$$

$$= \sum_{i=1}^{n}(X_i-\mu)^2 + \sum_{i=1}^{n}(\bar{X}-\mu)^2 - 2\sum_{i=1}^{n}(\bar{X}-\mu)(X_i-\mu)$$

$$= \sum_{i=1}^{n}(X_i-\mu)^2 + \sum_{i=1}^{n}(\bar{X}-\mu)^2 - 2(\bar{X}-\mu)\underbrace{\sum_{i=1}^{n}(X_i-\mu)}_{n(\bar{X}-\mu)}$$

$$= \sum_{i=1}^{n}(X_i-\mu)^2 - n(\bar{X}-\mu)^2$$

Hence
$$\begin{aligned}(n-1)E[S^2] &= E\left[\sum_{i=1}^{n}(X_i-\mu)^2\right] - nE[(\bar{X}-\mu)^2] \\ &= \sum_{i=1}^{n} E[(X_i-\mu)^2] - nVar[\bar{X}] \\ &= n\sigma^2 - n\frac{\sigma^2}{n} \\ &= (n-1)\sigma^2\end{aligned}$$

Example 31.3: Compute the variance of $X \sim \mathsf{Binomial}(n,p)$.

Solution: $X = X_1 + \cdots + X_n$ where $X_1, ..., X_n$ are iid and $\sim \mathsf{Bernoulli}(p)$.

$$\begin{aligned}Var[X] &= Var[X_1 + \cdots + X_n] \\ &= \underbrace{Var[X_1]}_{p(1-p)} + \cdots + \underbrace{Var[X_n]}_{p(1-p)} \qquad \text{[since } X_i \text{ are independent]} \\ &= np(1-p)\end{aligned}$$