Signals and Systems

Lecture Notes

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Chapter 3

Time-Domain Analysis of Discrete-Time Systems

3.1 Introduction

Systems whose inputs and outputs are discrete-time signals are called *discrete-time* (DT) systems. For example, digital circuits and digital computers are DT systems.

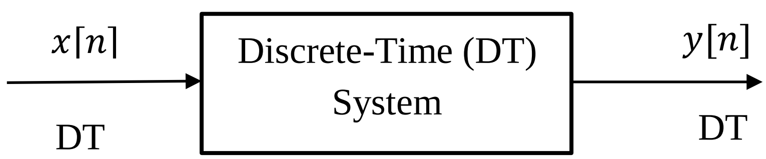

Figure 1

A discrete-time signal is a sequence of numbers. A DT system processes a sequence of numbers or signal $x[n]$ at its input and produces another sequence or signal $y[n]$ at its output, where n is an integer.

Discrete-time signals can be naturally discrete such as census data and stock prices. They are also obtained by sampling continuous-time (CT) signals such as voice or image. There are several advantages in processing CT signals by DT systems. This is achieved by converting the CT signal $x(t)$ to a DT signal $x[n]$ which is processed by the DT system to produce the output signal $y[n]$ which is converted to a CT signal $y(t)$. The conversion of a CT signal to a DT signal is achieved by a system called a *continuous-to-discrete (C/D) converter*, also called an *analog-to-digital (A/D) converter*. The conversion of a DT signal to a CT signal is achieved by a system called a *discrete-to-continuous (D/C) converter*, also called a *digital-to-analog (D/A) converter*.

Figure 2

3.2 Examples of Discrete-Time Systems

Several DT systems are discussed where their system models are derived and their realizations are shown.

Example: (Bank Account)

Let: $y[n] =$ account balance at time n (beginning of month n)

$x[n] =$ deposit (or withdrawal) during month n

$r =$ monthly interest rate

The balance at the beginning of next month $(n+1)$ is:

$$y[n+1] = y[n] + ry[n] + x[n]$$

$$\Rightarrow \quad y[n+1] - (1+r)y[n] = x[n]$$

This *difference equation* (DE) is the system model. It is in *advance form* since the signals are in terms of the current or future time, here the time indices n and $n+1$. The difference equation can be re-written differently by replacing n by $(n-1)$:

$$y[n] - (1+r)y[n-1] = x[n-1]$$

This is the difference equation in *delay form* since the signals are in terms of the current or past time, here the time indices n and $n-1$. As we will see later, the advance form is more convenient for mathematical analysis while the delay form is more convenient for system realization (in software or hardware).

- **Realization Block Diagram**

Difference equations with constant coefficients can be realized using three basic digital circuit elements: a *time delay* (D), a *digital multiplier*, and a *digital adder*. The delay can be implemented using flip-flops.

For example, the last difference equation in delay form can be rewritten:

$$y[n] = ay[n-1] + x[n-1]$$

where $a = 1 + r$

This equation can be realized as shown in Figure 3 below.

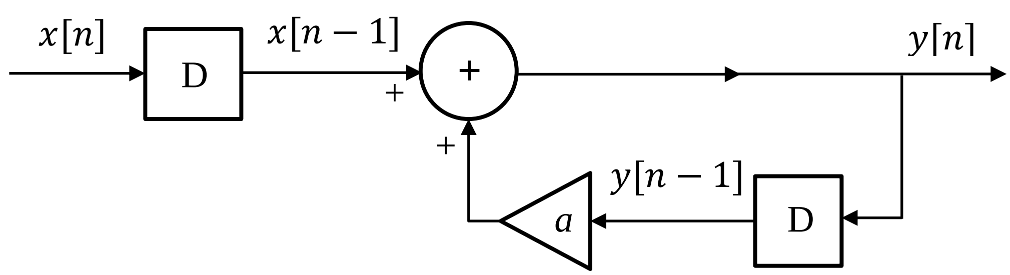

Figure 3

Example: (Digital Differentiator)

The continuous-time differentiation system $y(t) = \frac{dx(t)}{dt}$ can be implemented approximately by a discrete-time differentiator. The CT signals $x(t)$ and $y(t)$ are sampled every T seconds which gives $x[n] = x(nT)$ and $y[n] = y(nT)$.

The differentiation can be written as:

$$y(nT) = \lim_{T \to 0} \frac{x(nT) - x((n-1)T)}{T}$$

If $x(t)$ is slowly changing and T is relatively small, this can be approximated by:

$$y[n] \cong \frac{1}{T}[x[n] - x[n-1]]$$

This difference equation can be realized as shown in Figure 4.

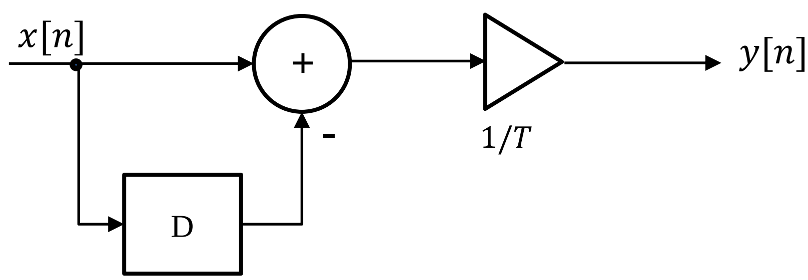

Figure 4

Example: (Digital Integrator)

The CT system described by $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$ can be implemented by a digital integrator. For $t = nT$, where T is the sampling interval:

$$y(nT) = \lim_{T\to 0} \sum_{k=-\infty}^{n} x(kT)\, T$$

For small T:

$$y[n] = T \sum_{k=-\infty}^{n} x[k]$$

This difference equation is in *non-recursive* form since it involve the current value of $y[n]$ only. This form is not easy to implement since it involves the storage of all current and past values of $x[n]$. The last equation can be re-written as:

$$y[n] = \underbrace{T \sum_{k=-\infty}^{n-1} x[k]}_{y[n-1]} + Tx[n]$$

$$\Rightarrow \qquad y[n] = y[n-1] + Tx[n]$$

This is the difference equation in *recursive* form, whose computation and realization are more efficient, as shown in Figure 5.

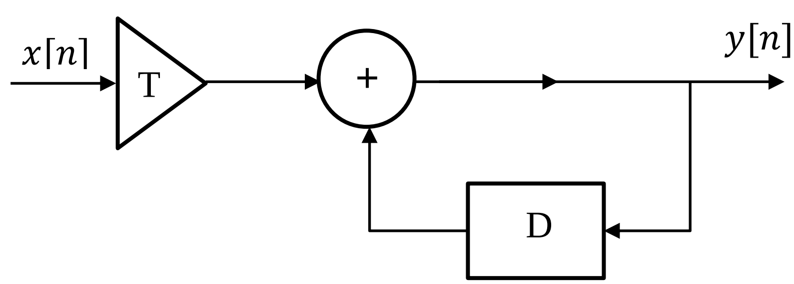

Figure 5

Example: (Differential Equation)

A CT system described by the differential equation $\frac{dy}{dt} + ay(t) = x(t)$ an be approximately implemented by a DT system. For $t = nT$, the equation can be written as:

$$\lim_{T \to 0} \frac{y[n] - y[n-1]}{T} + ay[n] = x[n]$$

For small T:

$$\frac{1 + aT}{T} y[n] - \frac{1}{T} y[n-1] = x[n]$$

or

$$y[n] + \alpha y[n-1] = \beta x[n]$$

where $\alpha = \frac{-1}{1+aT}$ and $\beta = \frac{T}{1+aT}$. This is a first order difference equation with constant coefficients.

Remark: From above examples, we notice that all of them are described by difference equations with constant coefficients. Most practical systems are of this class. Therefore, in this course, this class of systems is considered.

3.3 Linearity, Time-Invariance, and Causality

The properties of *linearity*, *time-invariance*, and *causality* for DT systems are the same as for CT systems.

Examples:

$y[n] = x[n-2]$	LTI, Causal
$y[n] = x[n+2]$	LTI, Non-causal
$y[n] = e^{n+1}x[n]$	Linear, Time-varying, Causal
$y[n] = \sum_{k=0}^{n} x[k]\,,\ n \geq 0$	LTI, Causal

$$y[n+1] + \underbrace{n}_{\substack{\text{time}\\ \text{varying}}} y[n] = x\underbrace{[n+2]}_{\substack{\text{non}\\ \text{causal}}} \underbrace{\cdot}_{\substack{\text{non}-\\ \text{linear}}} x[n]$$

Nonlinear, Time-varying, Non-causal

$$y[n+2] - 2y[n+1] + y[n] = 3x[n+2] + x[n+1] - 2x[n]$$

This last system is a 2nd order difference equation (DE) with constant coefficients which is LTI and causal.

Remark: Systems described by difference equations with constant coefficients are *linear* and time-invariant (LTI).

Remark: (Order of a DE) The order of a DE is the highest time difference N of the output $y[n]$ or the highest time difference M of the input $x[n]$, whichever is higher, when the DE is written in advance form. For example, the system:

$$y[n+2] - y[n+1] + y[n] = x[n+3] + x[n+2]$$

has $N = 2$ and $M = 3$. Therefore, the order of the system is 3.

<u>Remark</u>: (<u>Causality Condition</u>) A system described by a DE with constant coefficients is causal when $M \leq N$. Otherwise, it is non-causal. For example, the system described by the last DE is non-causal since $M = 3 > N = 2$.

<u>Example</u>: (Order of a system)

$$y[n+1] + y[n-1] = x[n+1] + x[n]$$

This is not of order 1 since it is not written in advance form. Replacing n by $n + 1$, we get the *<u>advance form</u>*:

$$\underbrace{y[n+2] + y[n]}_{\text{order } N=2} = \underbrace{x[n+2] + x[n+1]}_{\text{order } M=2}$$

So, the order of the system is $M = N = 2$. This system is LTI and causal.

3.4 Discrete-Time System Equations

A large class of practical LTI systems are described by ***difference equations*** (DEs) with constant coefficients. The general difference equation in advance form is given by:

$$y[n+N] + a_{N-1}y[n+N-1] + \cdots + a_0 y[n] \\ = b_M x[n+M] + b_{M-1}x[n+M-1] + \cdots + b_0 x[n]$$

For a causal system, we must have $M \leq N$. For generality, we set $M = N$.

$$y[n+N] + a_{N-1}y[n+N-1] + \cdots + a_0 y[n] \\ = b_N x[n+N] + b_{N-1}x[n+N-1] + \cdots + b_0 x[n]$$

Remark: Above equation is normalized in the sense that $a_N = 1$.

Remark: Some of the coefficients can be zero. However, if a_0 and b_0 are both zero at the same time, then the time index n can be replaced by $n-1$ which results in a DE with order $(N-1)$ and at least one non-zero new coefficients a_0 and b_0. This means the terms of $y[n]$ or $x[n]$ or both must exist for the equation to be in standard form.

- **Advance Operator Notation ($\boldsymbol{E - Operator}$):**

$$Ex[n] = x[n+1]$$

$$E^2 x[n] = x[n+2]$$

$$\vdots$$

$$E^N x[n] = x[n+N]$$

Using the E-operator, the general difference equation can be rewritten as:

$$(E^N + a_{N-1}E^{N-1} + \cdots + a_1 E + a_0)y[n] \\ = (b_N E^N + b_{N-1}E^{N-1} + \cdots + b_1 E + b_0)x[n]$$

or

$$Q(E)y[n] = P(E)x[n]$$

where

$$Q(E) = E^N + a_{N-1}E^{N-1} + \cdots + a_1 E + a_0$$

$$P(E) = b_N E^N + b_{N-1}E^{N-1} + \cdots + b_1 E + b_0$$

Remark: There are two causes for the system response $y[n]$, namely the existence of initial conditions (ICs) and the application of the input $x[n]$. Since the system is linear, the total system response can be written as:

$$y[n] = y_{zi}[n] + y_{zs}[n]\,, \qquad n \geq 0$$

where

$y_{zi}[n] =$ Zero-input response due to IC's only (with $x[n] = 0$, for all n)

$y_{zs}[n] =$ Zero-state response due to input $x[n]$, $n \geq 0$ (with IC's=0)

Remark: There are two methods for solving the DE (finding $y[n]$):

- Recursive (Iterative) Solution
- Analytic Solution

- **Recursive (Iterative) Solution**

This method is based on rewriting the difference equation in delay form:

$$y[n] = -a_{N-1}y[n-1] - a_{N-2}y[n-2] - \cdots - a_0 y[n-N] + b_N x[n] + b_{N-1}x[n-1] + \cdots + b_0 x[n-N]$$

This equation allows us to compute the system response point-by-point, i.e. $y[0], y[1], y[2], \cdots$, etc. Also, it allows us to compute each of the zero-input response $y_{zi}[n]$ and the zero-state response $y_{zs}[n]$ separately, or the total response $y[n]$ directly. Given the initial conditions $\{y[-1], y[-2], \cdots, y[-N]\}$ with $x[n] = 0$ for $n \geq 0$, the zero-input response $y_{zi}[n]$ is computed. While, given the input $x[n]$ for $n \geq 0$ with ICs=0, the zero-state response $y_{zs}[n]$ is computed. The total response $y[n]$ can be computed when both the input and ICs are applied at the same time.

Example: A system is described by $(E + 0.5)y[n] = x[n]$ with $y[-1] = 1$ and $x[n] = n.u[n],\ n \geq 0$. Find the total response $y[n]$?

Solution:

Rewrite the difference equation in delay form:

$$y[n] = -\frac{1}{2}y[n-1] + x[n-1]$$

Compute $y[n]$ point-by-point by iteration:

$$n = 0 \quad \Rightarrow \quad y[0] = -\frac{1}{2}y[-1] + \overbrace{x[-1]}^{0} = -\frac{1}{2} - 0 = -\frac{1}{2}$$

$$n = 1 \quad \Rightarrow \quad y[1] = -\frac{1}{2}y[0] + x[0] = -\frac{1}{2}\left(-\frac{1}{2}\right) + 0 = \frac{1}{4}$$

$$n = 2 \quad \Rightarrow \quad y[2] = -\frac{1}{2}y[1] + x[1] = -\frac{1}{2}\left(\frac{1}{4}\right) + 1 = \frac{7}{8}$$

$$n = 3 \quad \Rightarrow \quad y[3] = -\frac{1}{2}y[2] + x[2] = -\frac{1}{2}\left(\frac{7}{8}\right) + 2 = \frac{25}{16}$$

$$\vdots$$

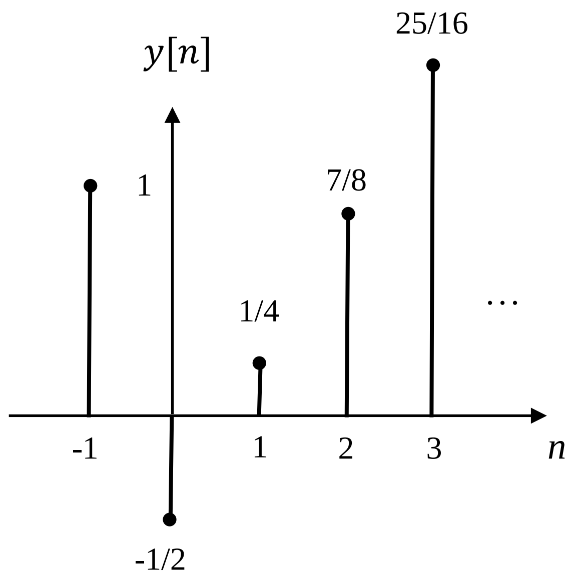

Figure 6

Remark: Iterative solution is useful for computer computation or for finding few initial values $\{y[0], y[1], \cdots\}$. It is difficult to apply for large values of n, say finding $y[1000]$ by hand computation without a computer. The other major disadvantage is that it does not reveal the behavior of the system, such as the stability of the system.

- **Analytic Solution (Closed-Form Solution)**

Given an LTI system described by the difference equation $Q(E)y[n] = P(E)x[n]$ with initial conditions $\{y[-1], y[-2], \cdots, y[-N]\}$ and an input $x[n],\ n \geq 0$, the system response $y[n]$ can be decomposed into two components by linearity:

$$y[n] = y_{zi}[n] + y_{zs}[n], \quad n \geq 0$$

where

$y_{zi}[n] =$ Zero-input response due to ICs only (with $x[n] = 0$, for all n)

$y_{zs}[n] =$ Zero-state response due to the input $x[n],\ n \geq 0$ (with ICs=0)

Remark: The analytic solution for DT systems is very similar to CT systems. Each of the system response components are discussed in the following sections.

3.5 Zero-Input Response

Given the DT system $Q(E)y[n] = P(E)x[n]$, the zero-input response $y_{zi}[n]$ is the system response due to non-zero initial conditions with $x[n] = 0$ for all n, i.e.

$$Q(E)y_{zi}[n] = 0, \quad n \geq 0$$

or

$$(E^N + a_{N-1}E^{N-1} + \cdots + a_1 E + a_0)y_{zi}[n] = 0$$

Assume the solution:

$$y_{zi}[n] = Cr^n$$

where C and r are constants. Applying the *E*-operator:

$$Ey_{zi}[n] = Cr^{n+1} = r\, Cr^n$$

$$E^2 y_{zi}[n] = Cr^{n+2} = r^2\, Cr^n$$

$$\vdots$$

$$E^N y_{zi}[n] = Cr^{n+N} = r^N\, Cr^n$$

Substituting in above DE:

$$(r^N + a_{N-1}r^{N-1} + \cdots + a_1 r + a_0)\, \underbrace{C}_{\neq 0}\, \underbrace{r^n}_{\neq 0} = 0$$

$$\Rightarrow \quad \underbrace{r^N + a_{N-1}r^{N-1} + \cdots + a_1 r + a_0}_{Q(r)=\text{charectristic polynomial}} = 0$$

This is the *characteristic equation* that can be factored in the product form:

$$Q(r) = (r - r_1)(r - r_2) \cdots (r - r_N) = 0$$

where $\{r_1,\ r_2,\ \cdots\ ,r_N\}$ are the *characteristic values*, *characteristic roots*, *eigenvalues*, or *natural frequencies* of the system.

- **Case 1 (Distinct Roots)**

When the roots $\{r_1, r_2, \cdots, r_N\}$ are distinct, the *characteristic modes* of the system are $\{r_1^n,\ r_2^n,\ \cdots\ ,\ r_N^n\}$. The system has N possible solutions, and the sum is the general solution:

$$y_{zi}[n] = C_1 r_1^n + C_2 r_2^n + \cdots + C_N r_N^n = \sum_{i=1}^{N} C_i r_i^n$$

- **Case 2 (Repeated Roots)**

When a root r_i is repeated m times, the corresponding characteristic modes are $\{r_i^n,\ nr_i^n,\ n^2 r_i^n,\ \cdots\ ,\ n^{m-1} r_i^n\}$, and the general solution for this root is:

$$y_{zi}(t) = \sum_{k=1}^{m} C_k n^{k-1}\, r_i^n$$

- **Case 3 (Complex Roots)**

For real systems (real coefficients of DE), complex roots occur in pairs of conjugate; i.e. if $r_1 = \alpha + j\beta = |r|e^{j\Omega}$ is a root of the system, then $r_2 = r_1^* = \alpha - j\beta = |r|e^{-j\Omega}$ is also a root. For these two distinct roots, the solution is:

$$y_{zi}(t) = C_1 r_1^n + C_2 r_2^n = C_1 |r|^n e^{j\Omega \mathrm{n}} + C_2 |r|^n e^{-j\Omega \mathrm{n}}$$

Also, for real systems, $y_{zi}[n]$ must be real, for which the constants C_1 and C_2 must be complex conjugates, i.e. if $C_1 = \frac{C}{2} e^{j\theta}$, then $C_2 = C_1^* = \frac{C}{2} e^{-j\theta}$. And, above complex form of the solution can be written in the real form:

$$y_{zi}[n] = \frac{C}{2}|r|^n\left[e^{j(\Omega n+\theta)} + e^{-j(\Omega n+\theta)}\right]$$

$$\Rightarrow \quad y_{zi}[n] = C|r|^n \cos(\Omega n + \theta), \quad n \geq 0 \qquad \text{(Real Form)}$$

where C and θ are constants found from ICs.

<u>Example</u>: A system is described by:

$$y[n+2] + 0.3y[n+1] - 0.1y[n] = x[n+2] + 2x[n]$$

with initial conditions $y[-1] = 1$, $y[-2] = -7$. Find $y_{zi}[n]$?

<u>Solution:</u>

Characteristic equation: $Q(r) = r^2 + 0.3r - 0.1 = (r-0.2)(r+0.5) = 0$

Roots: $r_1 = 0.2$, $r_2 = -0.5$

Modes: $(0.2)^n$, $(-0.5)^n$

$$\Rightarrow \quad y_{zi}[n] = [C_1(0.2)^n + C_2(-0.5)^n]u[n]$$

To find the constants C_1 and C_2 we compute $y[0]$ and $y[1]$ from the system equation, iteratively:

$$n = -2 \;\; \Rightarrow \;\; y_{zi}[0] = -0.3\,\underbrace{y[-1]}_{1} + 0.1\,\underbrace{y[-2]}_{-7} + \underbrace{x[0]}_{0} + 2\,\underbrace{x[-2]}_{0} = -1$$

$$n = -1 \;\; \Rightarrow \;\; y_{zi}[1] = -0.3\,\underbrace{y[0]}_{-1} + 0.1\,\underbrace{y[-1]}_{1} + \underbrace{x[1]}_{0} + 2\,\underbrace{x[-1]}_{0} = 0.4$$

Substituting in $y_{zi}[n]$:

$$y_{zi}[0] = C_1 + C_2 = -1$$

$$y_{zi}[1] = C_1(0.2) + C_2(-0.5) = 0.4$$

The solution of these two linear equations is: $C_1 = -\frac{1}{7},\ C_2 = -\frac{6}{7}$

$$\Rightarrow \quad y_{zi}[n] = \left[-\frac{1}{7}(0.2)^n - \frac{6}{7}(-0.5)^n\right]u[n]$$

<u>Remark</u>: The initial conditions $\boldsymbol{y[-1]}$ and $\boldsymbol{y[-2]}$ are given on the total response. However, the zero-state response $\boldsymbol{y_{zs}[n] = 0}$ for $\boldsymbol{n < 0}$ since $\boldsymbol{x[n] = 0}$ for $\boldsymbol{n < 0}$. Therefore, we have $\boldsymbol{y[-1] = y_{zi}[-1]}$ and $\boldsymbol{y[-2] = y_{zi}[-2]}$. So, the constants $\boldsymbol{C_1}$ and $\boldsymbol{C_2}$ can be found from $\boldsymbol{y_{zi}[n]}$ equation by setting $\boldsymbol{n = -1}$ and $\boldsymbol{n = -2}$, and ignoring $\boldsymbol{u[n]}$:

$$\left\{\begin{matrix} y_{zi}[-1] = C_1(0.2)^{-1} + C_2(-0.5)^{-1} = y[-1] = 1 \\ y_{zi}[-2] = C_1(0.2)^{-2} + C_2(-0.5)^{-2} = y[-2] = -7 \end{matrix}\right\}$$

Solution of these equations gives $\left\{C_1 = -\frac{1}{7},\ C_2 = -\frac{6}{7}\right\}$ which are the same as we found before. This method of finding C_1 and C_2 is used in the textbook.

<u>Example</u>: A system is described by:

$$y[n+2] - 4y[n+1] + 4y[n] = 3x[n+1]$$

with ICs: $y[-1] = 1,\ \ y[-2] = 0$. Find $y_{zi}[n]$?

<u>Solution:</u>

Characteristic equation: $\qquad Q(r) = r^2 - 4r + 4 = (r-2)^2 = 0$

Roots: $\quad r_1 = r_2 = 2$ (repeated)

$$\Rightarrow \quad y_{zi}[n] = [C_1(2)^n + C_2 n(2)^n]u[n]$$

Compute C_1 and C_2:

$$y[-1] = 1 = y_{zi}[-1] = C_1(2)^{-1} + C_2(-1)(2)^{-1}$$

$$y[-2] = 0 = y_{zi}[-2] = C_1(2)^{-2} + C_2(-2)(2)^{-2}$$

Solving these two equations gives: $\quad C_1 = 4, \quad C_2 = 2$

$$\Rightarrow \quad y_{zi}[n] = [4(2)^n + 2n(2)^n]u[n]$$

<u>Example</u>: A system is described by:

$$y[n+2] + 4y[n] = 2x[n]$$

with ICs $y[-1] = 0$, $y[-2] = 1$. Find $y_{zi}[n]$?

<u>Solution</u>:

Characteristic equation: $\quad Q(r) = r^2 + 4 = (r - j2)(r + j2) = 0$

Roots: $\quad r_1 = j2 = 2e^{j\frac{\pi}{2}}, \quad r_2 = -j2 = 2e^{-j\frac{\pi}{2}}$

$$\Rightarrow \quad y_{zi}[n] = C(2)^n \cos\left(\frac{\pi}{2}n + \theta\right)u[n]$$

To find C and θ, compute $y[0]$ and $y[1]$ first from the system equation by iteration:

$$n = -2 \quad \Rightarrow \quad y[0] = -4\,\underbrace{y[-2]}_{1} + 2\,\underbrace{x[-2]}_{0} = -4$$

$$n = -1 \quad \Rightarrow \quad y[1] = -4\,\underbrace{y[-1]}_{0} + 2\,\underbrace{x[-1]}_{0} = 0$$

Substituting in $y_{zi}[n]$:

$$y[0] = y_{zi}[0] = C(2)^0 \cos\left(\frac{\pi}{2}0 + \theta\right) = C\cos(\theta) = -4$$

$$y[1] = y_{zi}[1] = C(2)^1 \cos\left(\frac{\pi}{2}(1) + \theta\right) = 2C \underbrace{\cos\left(\frac{\pi}{2} + \theta\right)}_{-\sin(\theta)} = 0$$

Solving last two equations gives: $\theta = 0, \quad C = -4$

$$\Rightarrow \quad y_{zi}[n] = \left[-4(2)^n \cos\left(\frac{\pi}{2}n\right)\right] u[n]$$

<u>Remark</u>: Another possible solution for the constants: $\theta = \pi, \; C = +4$.

<u>Remark</u>: It is possible to compute the constants C and θ from the equations $y[-1] = y_{zi}[-1] = 0$ and $y[-2] = y_{zi}[-2] = 1$ directly without computing $y[0]$ and $y[1]$ by iteration.

3.6 The Unit Impulse Response

Consider an LTI and causal system described by the difference equation:

$$Q(E)y[n] = P(E)x[n]$$

$$(E^N + a_{N-1}E^{N-1} + \cdots + a_1E + a_0)y[n]$$
$$= (b_NE^N + b_{N-1}E^{N-1} + \cdots + b_1E + b_0)x[n]$$

The <u>*unit impulse response*</u> $h[n]$ is the sytem response when the input is $x[n] = \delta[n]$ with all initial conditions are zero (system at rest), that is $y[n] = h[n]$ when $x[n] = \delta[n]$:

$$Q(E)h[n] = P(E)\delta[n]$$

subject to ICs: $h[-1] = h[-2] = \cdots = h[-N] = 0$

<u>Remark</u>: Since the system is causal, $h[n] = 0$ for $n < 0$.

<u>Remark</u>: Remember $\delta[n] = 1$ for $n = 0$ and $\delta[n] = 0$ for $n \neq 0$.

The solution for $h[n]$ can be found by <u>*iteration*</u> or in <u>*closed-form*</u>.

<u>Example</u>: (Iterative Solution)

$$y[n+2] + 0.3y[n+1] - 0.1y[n] = x[n+2] + 2x[n]$$

Find $h[n]$?

<u>Solution</u>:

Write the system equation in delay form and set $x[n] = \delta[n]$ and $y[n] = h[n]$:

$$h[n] = -0.3h[n-1] + 0.1h[n-2] + \delta[n] + 2\delta[n-2]$$

$$h[0] = -0.3\,\underbrace{h[-1]}_{0} + 0.1\,\underbrace{h[-2]}_{0} + \underbrace{\delta[0]}_{1} + 2\,\underbrace{\delta[-2]}_{0} = 1$$

$$h[1] = -0.3\,\underbrace{h[0]}_{1} + 0.1\,\underbrace{h[-1]}_{0} + \underbrace{\delta[1]}_{0} + 2\,\underbrace{\delta[-1]}_{0} = -0.3$$

$$h[2] = -0.3\,\underbrace{h[1]}_{-0.3} + 0.1\,\underbrace{h[0]}_{1} + \underbrace{\delta[2]}_{0} + 2\,\underbrace{\delta[0]}_{1} = 2.19$$

$$\vdots$$

<u>Remark</u>: Iterative solution is tedious and does not reveal the behavior of $h[n]$.

- **Closed-Form Solution of $\boldsymbol{h[n]}$**

We have $x[n] = \delta[n] \Rightarrow y[n] = h[n]$. Since $x[n] = 0$ for $n > 0$, then $h[n]$ contains characteristic modes for $n > 0$. At $n = 0$, $h[n]$ may have $\delta[n]$ component, i.e.

$$h[n] = A_0\delta[n] + y_c[n]u[n]$$

where $y_c[n]$ is the linear combination of the characteristic modes of the system with N unknown coefficients $\{c_1, c_2, \cdots, c_N\}$ in $y_c[n]$.

The unknowns $\{A_0, c_1, c_2, \cdots, c_N\}$ can be determined from $h[0], h[1], \cdots, h[N]$ which can be computed from the ICs $h[-1] = h[-2] = \cdots = h[-N] = 0$ iteratively.

<u>Remark</u>: It is shown in the textbook that $A_0 = \frac{b_0}{a_0}$ where a_0 is the coefficient of $y[n]$ and b_0 is the coefficient of $x[n]$ in the system difference equation. Therefore,

$$h[n] = \frac{b_0}{a_0}\delta[n] + y_c[n]u[n]$$

Example: Find the impulse response $h[n]$ for the system:

$$y[n+2] - 2y[n+1] + y[n] = 3x[n+2] + 2x[n]$$

Solution:

Characteristic equation: $Q(r) = r^2 - 2r + 1 = (r-1)^2 = 0$

Roots: $r_1 = r_2 = 1$ (repeated)

Characteristic modes: $(1)^n = 1, \quad n(1)^n = n$

$$\Rightarrow \quad h[n] = A_0\delta[n] + (c_1 + c_2 n)u[n], \;\; n \geq 0$$

Find new ICs iteratively using $h[-1] = \cdots = h[-N] = 0$. From the system equation with $x[n] = \delta[n] \;\rightarrow\; y[n] = h[n]$:

$$h[n] = 2h[n-1] - h[n-2] + 3\delta[n] + 2\delta[n-2]$$

$$h[0] = 2\underbrace{h[-1]}_{0} - \underbrace{h[-2]}_{0} + 3\underbrace{\delta[0]}_{1} + 2\underbrace{\delta[-2]}_{0} = 3$$

$$h[1] = 2\underbrace{h[0]}_{3} - \underbrace{h[-1]}_{0} + 3\underbrace{\delta[1]}_{0} + 2\underbrace{\delta[-1]}_{0} = 6$$

$$h[2] = 2\underbrace{h[1]}_{6} - \underbrace{h[0]}_{3} + 3\underbrace{\delta[2]}_{0} + 2\underbrace{\delta[0]}_{1} = 11$$

Substituting in $h[n]$ to solve for $\{A_0, c_1, c_2\}$,

$$\left.\begin{aligned} h[0] &= 3 = A_0 + c_1 \\ h[1] &= 6 = c_1 + c_2 \\ h[2] &= 11 = c_1 + 2c_2 \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} c_1 &= 1 \\ c_2 &= 5 \\ A_0 &= 2 = \frac{b_0}{a_0} \end{aligned}\right.$$

$$\Rightarrow \quad h[n] = 2\delta[n] + [1 + 5n]u[n]$$

Remark: (Special Case)

If $a_0 = 0$, then the coefficient $A_0 = \frac{b_0}{a_0}$ in the relation for $h[n]$ does not apply. In this case, $h[n]$ is given by:

$$h[n] = A_0\delta[n] + A_1\delta[n-1] + y_c[n]u[n]$$

Similarly, if $a_0 = a_1 = 0$, then:

$$h[n] = A_0\delta[n] + A_1\delta[n-1] + A_2\delta[n-2] + y_c[n]u[n]$$

and so on. (See textbook section 3.12)

3.7 Zero-State Response

Assume an LTI system S: $y[n] = S\{x[n]\}$

The impulse response is: $h[n] = S\{\delta[n]\}$

By time-invariance: $h[n-m] = S\{\delta[n-m]\}$

By sifting/sampling property: $x[n] = \sum_{m=-\infty}^{\infty} x[m]\delta[n-m]$

Substituting in $y[n]$:

$$y[n] = S\left\{\sum_{m=-\infty}^{\infty} x[m]\delta[n-m]\right\}$$

By linearity of S and the summation operations:

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]\,\underbrace{S\{\delta[n-m]\}}_{h[n-m]}$$

This can be rewritten as:

$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

This is the *convolution sum* for DT systems, similar to the convolution integral for CT systems. It is valid for all n ($-\infty < n < +\infty$).

Remark: Properties of the convolution sum are similar to the convolution integral.

Remark: For causal systems: $h[n] = 0$ for $n < 0$

$$\Rightarrow \quad h[n-m] = 0 \quad \text{for} \quad n-m < 0 \quad \text{or} \quad m > n$$

And, if the input $x[n]$ is a causal signal: $x[m] = 0 \quad \text{for} \quad m < 0$

Then, the convolution sum can be rewritten as:

$$y[n] = x[n] * h[n] = \sum_{m=0}^{n} x[m]h[n-m]$$

Therefore, the output $y[n]$ is a causal signal, too.

Example: Let $x_1[n] = r_1^n u[n]$ and $x_2[n] = r_2^n u[n]$, where $r_2 \neq r_1$. Find the convolution $x_1 * x_2$?

Solution:

$$x_1 * x_2 = \sum_{m=-\infty}^{\infty} x_1[m]x_2[n-m] = \sum_{m=-\infty}^{\infty} r_1^m \underbrace{u[m]}_{=\begin{cases}0, & m<0\\ 1, & m\geq 0\end{cases}} . r_2^{n-m} \underbrace{u[n-m]}_{=\begin{cases}0, & m>n\\ 1, & m\leq n\end{cases}}$$

For $n < 0$: $x_1 * x_2 = 0$

For $n \geq 0$:

$$x_1 * x_2 = \sum_{m=0}^{n} r_1^m . r_2^{n-m} = r_2^n \underbrace{\sum_{m=0}^{n} \left(\frac{r_1}{r_2}\right)^m}_{\text{geometric sum}}$$

Using the geometric sum formula $\sum_{m=0}^{n} z^m = \frac{z^{n+1}-1}{z-1}, \quad z \neq 1$ gives:

$$x_1 * x_2 = r_2^n \frac{\left(\frac{r_1}{r_2}\right)^{n+1} - 1}{\left(\frac{r_1}{r_2}\right) - 1} = \frac{1}{r_1 - r_2}({r_1}^{n+1} - {r_2}^{n+1}), \quad n \geq 0$$

$$\Rightarrow \quad x_1 * x_2 = \frac{1}{r_1 - r_2}({r_1}^{n+1} - {r_2}^{n+1})u[n]$$

Example: An LTI system is described by:

$$y[n+2] - 2y[n+1] + y[n] = 3x[n+2] + 2x[n]$$

Find the output $y[n]$, $n \geq 0$ when $y[-1] = -1$, $y[-2] = 1$, and $x[n] = u[n]$.

Solution:

(1) Find the zero-input response $y_{zi}[n]$ (due to ICs):

$$Q(r) = r^2 - 2r + 1 = (r-1)^2 = 0$$

Roots: $r_1 = r_2 = 1$ (repeated twice)

Modes: $(1)^n = 1, \quad n(1)^n = n$

$$\Rightarrow \quad y_{zi}[n] = (C_1 + C_2 n)\,, \quad n \geq 0$$

Use IC's to find the constants C_1 and C_2:

$$y[-1] = y_{zi}[-1] = C_1 - C_2 = -1 \qquad (1)$$

$$y[-2] = y_{zi}[-2] = C_1 - 2C_2 = 1 \qquad (2)$$

Solving (1) and (2) gives: $C_1 = -3$ and $C_2 = -2$

$$\Longrightarrow \quad y_{zi}[n] = (-3 - 2n)u[n]$$

(2) Find the impulse response $h[n]$:

This was found for the same system in a previous example:

$$h[n] = 2\delta[n] + (1 + 5n)u[n]$$

(3) Find the zero-state response $y_{zs}[n]$ (due to $x[n] = u[n]$):

For $n \geq 0$: $y_{zs}[n] = h * x = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$

$$\Rightarrow \quad y_{zs}[n] = \sum_{m=-\infty}^{\infty} \left\{ 2\delta[m] + (1+5m) \underbrace{u[m]}_{=\begin{cases}0, & m<0 \\ 1, & m\geq 0\end{cases}} \right\} \underbrace{u[n-m]}_{=\begin{cases}0, & m>n \\ 1, & m\leq n\end{cases}}$$

$$= 2 + \underbrace{\sum_{m=0}^{n} 1}_{n+1} + 5 \underbrace{\sum_{m=0}^{n} m}_{n(n+1)/2}$$

$$\Rightarrow \quad y_{zs}[n] = \left\{ 2 + (n+1) + \frac{5}{2} n(n+1) \right\} u[n]$$

(4) Total Response: $y[n] = y_{zi}[n] + y_{zs}[n], \; n \geq 0$

$$y[n] = (-3-2n)u[n] + \left\{ 2 + (n+1) + \frac{5}{2} n(n+1) \right\} u[n]$$

$$\Rightarrow \quad y[n] = \left(\frac{3}{2} n + \frac{5}{2} n^2 \right) u[n]$$

Remark: Similar to the convolution integral, the convolution sum can be performed in several methods to help in its computation.

(See example 3.21 in textbook for using the convolution Table 3.1).

(See example 3.23 in textbook for graphical method).

(See example 3.24 in textbook for sliding-tape method).

- **Unit Step Response**

Assume an LTI system.

Impulse response: $h[n] = S\{\delta[n]\}$

Step response: $g[n] = S\{u[n]\}$

then

$$g[n] = \sum_{m=-\infty}^{n} h[m]$$

$$h[n] = g[n] - g[n-1]$$

Proof: From

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]\, x[n-m]$$

$$\Rightarrow \quad g[n] = \sum_{m=-\infty}^{\infty} h[m]\, \underbrace{u[n-m]}_{=\begin{cases}1, & m\le n\\ 0, & m>n\end{cases}} = \sum_{m=-\infty}^{n} h[m]$$

Also, $g[n] - g[n-1] = \sum_{m=-\infty}^{n} h[m] - \sum_{m=-\infty}^{n-1} h[m]$

$= \sum_{m=-\infty}^{n-1} h[m] + h[n] - \sum_{m=-\infty}^{n-1} h[m] = h[n]$

- **System Response to the Everlasting Exponential**

Assume an LTI system with an impulse response $h[n]$.

Let the input be the everlasting exponential $x[n] = z^n, \ -\infty < n < \infty$ where z is a complex constant. The output is given by:

$$y[n] = h[n] * x[n] = \sum_{m=-\infty}^{\infty} h[m]\, x[n-m]$$

$$= \sum_{m=-\infty}^{\infty} h[m]\, z^{n-m} = z^n \underbrace{\sum_{m=-\infty}^{\infty} h[m]\, z^{-m}}_{H(z) \ \text{function of } z}$$

$$\Rightarrow \quad y[n] = H(z) z^n$$

The function $H(z)$ is called the <u>transfer function</u> of the system. For a given value of z, $H(z)$ is a constant, not a function of time n. Therefore, the output $y[n]$ is also an everlasting exponential, i.e.

Everlasting Exponential input $\Longrightarrow$ Everlasting Exponential output

– For the case of an LTI system described by a difference equation:

$$Q(E)y[n] = P(E)x[n]$$

The transfer function is:

$$H(z) = \frac{P(z)}{Q(z)}$$

The proof is similar to the case of CT systems (See textbook).

3.8 Interconnection of Systems

Similar to CT systems.

- **Parallel Connection:**

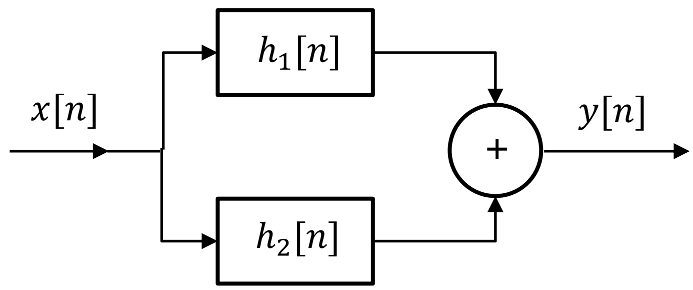

Figure 7

Equivalent impulse response: $h[n] = h_1[n] + h_2[n]$

- **Cascade Connection:**

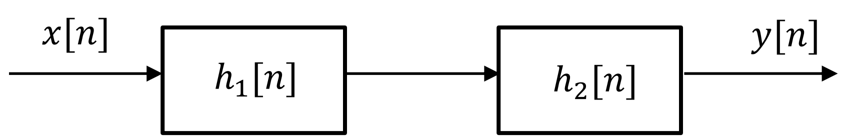

Figure 8

Equivalent impulse response: $h[n] = h_1[n] * h_2[n]$

3.9 DT System Stability

Almost similar to CT systems.

- **Internal (Asymptotic) Stability**

Assume an LTI system is in *zero-state* or *rest state* [ICs = 0, $x[n] = 0$, $y[n] = 0$] which is a stable state. Due to the application of small disturbances (non-zero ICs), we say:

(1) The system is <u>*asymptotically stable*</u> (AS) if all system modes $\to 0$ as $t \to \infty$, i.e. system goes back to its rest state.

(2) The system is <u>*unstable*</u> if at least one system mode $\to \infty$ as $t \to \infty$.

(3) The system is <u>*marginally stable*</u> (MS) if some modes are bounded, while the remaining modes $\to 0$ as $t \to \infty$.

For an LTI system described by $Q(E)y[n] = P(E)x[n]$ with N roots:

$$r_1,\ r_2,\ \cdots\ ,\ r_N \qquad \text{(Complex in general)}$$

The system modes are of the form:

$$r_i^n \quad \text{(Distinct)} \qquad \text{or} \qquad n^k r_i^n \quad \text{(Repeated)},\ k = 1,2,\ldots$$

Let $r_i = |r_i|e^{j\beta}$, and for the distinct roots:

$$\lim_{n\to\infty} |r_i^n| = \lim_{n\to\infty} |r_i|^n \underbrace{\left|e^{j\beta n}\right|}_{=1} = \lim_{n\to\infty} |r_i|^n = \begin{cases} 0 & \text{if} \quad |r_i| < 1 \\ \infty & \text{if} \quad |r_i| > 1 \end{cases}$$

This is also valid for the repeated modes $n^k r_i^n$. Therefore, we conclude:

System is <u>asymptotically stable</u> (AS) if $\lvert r_i \rvert < 1$ for all roots

System is <u>unstable</u> if $\lvert r_i \rvert > 1$ for at least one root

For the case $|r_i| = 1$ and unrepeated root:

$$\lim_{n\to\infty} |r_i^n| = \lim_{n\to\infty} |r_i|^n = \lim_{n\to\infty} (1)^n = 1$$

For the case $|r_i| = 1$ and repeated root:

$$\lim_{n\to\infty} \left|n^k r_i^n\right| = \lim_{n\to\infty} |n|^k |r_i|^n = \lim_{n\to\infty} |n|^k (1)^n = \lim_{n\to\infty} |n|^k \;\rightarrow\; \infty$$

Therefore, we conclude:

System is marginally stable (MS) if for some roots $|r_i| = 1$ and unrepeated, while the remaining roots are inside the unit circle, i.e. $|r_i| < 1$.

System is unstable if for at least one root $|r_i| = 1$ and repeated.

Summary: (Stability and location of roots of $\boldsymbol{Q(r)}$ in the complex plane)

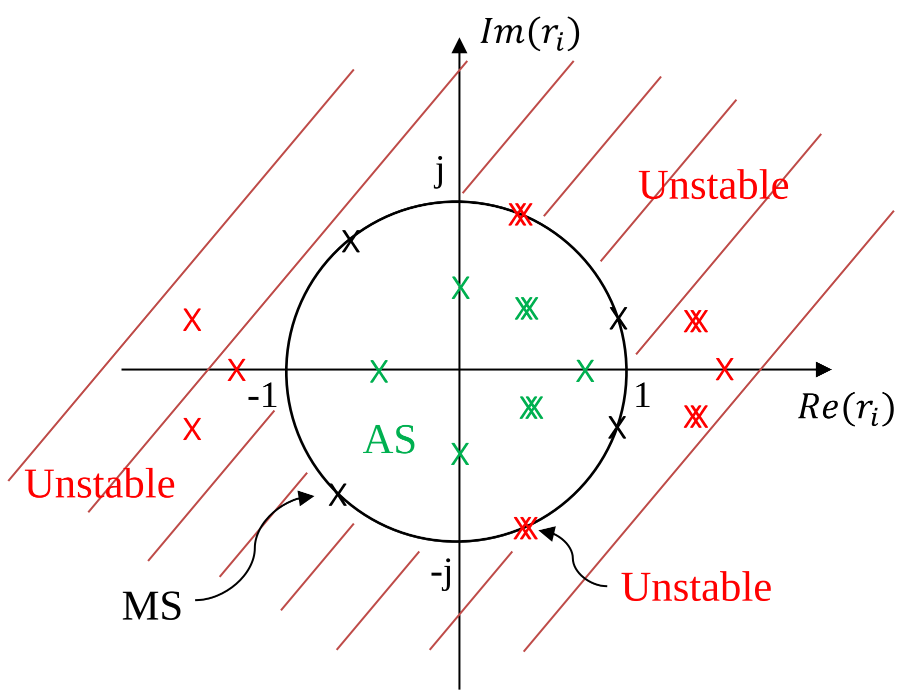

Figure 9

Example:

$$y[n+2] - 0.6y[n+1] - 0.16y[n] = 5x[n]$$

$$\Longrightarrow \; Q(r) = r^2 - 0.6r - 0.16 = (r+0.2)(r-0.8)$$

Roots: $r_1 = -0.2, \quad r_2 = 0.8$

$\Longrightarrow$ System is AS since $|r_1| < 1$ and $|r_2| < 1$ (inside the unit circle)

Example:

$$y[n+2] + 1.5y[n+1] - y[n] = 2x[n]$$

$$\Longrightarrow \; Q(r) = r^2 + 1.5r - 1 = (r-0.5)(r+2)$$

Roots: $r_1 = 0.5, \quad r_2 = -2$

$\Longrightarrow$ System is unstable due to r_2 (outside the unit circle)

Example: (Oscillator)

$$y[n+2] - y[n+1] + y[n] = 3x[n+1]$$

$$Q(r) = r^2 - r + 1 = \left(r - \frac{1}{2} - j\frac{\sqrt{3}}{2}\right)\left(r - \frac{1}{2} + j\frac{\sqrt{3}}{2}\right)$$

Roots: $r_1, r_2 = \frac{1}{2} \pm j\frac{\sqrt{3}}{2} = 1e^{\pm j\frac{\pi}{3}}$ (complex on the unit circle)

$\Longrightarrow$ System is MS since $|r_1| = |r_2| = 1$ (unrepeated)

- **External Bounded-Input Bounded-Output (BIBO) Stability**

BIBO stability for DT systems is similar to CT systems.

Definition: A system is *BIBO stable* if every bounded input $x[n]$ produces bounded output $y[n]$. If even one bounded input produces an unbounded output, the system is *BIBO unstable*.

For LTI system, and from the convolution sum:

$$y[n] = h[n] * x[n] = \sum_{m=-\infty}^{\infty} h[m]\, x[n-m]$$

$$|y[n]| = \left| \sum_{m=-\infty}^{\infty} h[m]\, x[n-m] \right| \leq \sum_{m=-\infty}^{\infty} |h[m]|\, |x[n-m]|$$

If $x[n]$ is bounded, i.e. $|x[n-m]| \leq K_1 < \infty$, where K_1 is a constant, then:

$$|y[n]| \leq K_1 \sum_{m=-\infty}^{\infty} |h[m]|$$

The output is bounded iff $h[n]$ is *absolutely summable*, i.e.:

$$\sum_{m=-\infty}^{\infty} |h[m]| \leq K_2 < \infty$$

where K_2 is a constant. We conclude:

An LTI system is BIBO stable iff its impulse response $h[n]$ is absolutely summable.

Remark: The relationships between asymptotic stability and BIBO stability for DT systems are similar to CT systems:

AS system $\Rightarrow$ BIBO stable
MS (with no hidden MS modes) $\Rightarrow$ BIBO unstable
BIBO stable $\not\Rightarrow$? (cannot say, system can be AS, MS or even unstable).

<u>Example</u>:

$$h[n] = \{(-0.2)^n + (0.8)^n\}u[n]$$

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} |\{(-0.2)^n + (0.8)^n\}u[n]|$$

$$= \sum_{n=0}^{\infty} |(-0.2)^n + (0.8)^n| \leq \sum_{n=0}^{\infty} |(-0.2)^n| + \sum_{n=0}^{\infty} |(0.8)^n|$$

$$= \sum_{n=0}^{\infty} (0.2)^n + \sum_{n=0}^{\infty} (0.8)^n = \frac{1}{1-0.2} + \frac{1}{1-0.8} = 6.25$$

Since $h[n]$ is absolutely summable $\Rightarrow$ System is BIBO stable.

<u>Example</u>: (Oscillator)

$$h[n] = 6\cos\left(\frac{\pi}{3}n\right)u[n] \;\Rightarrow\; \sum_{n=-\infty}^{\infty} |h[n]| \to \infty$$

$\Rightarrow$ System is BIBO unstable.

Bad input: $x[n] = c\cos\left(\frac{\pi}{3}n\right)u[n]$

(External resonance: The output is unbounded even though the input is bounded)