Sample Spaces with Equally Likely Outcomes [Ross S2.5]

Say $S = \{1, 2, \ldots N\}$.

Then $\quad 1 = P[S] = P[1] + P[2] + \cdots + P[N]. \qquad (4.1)$

If each outcome is equally likely:

$$P[1] = P[2] = \cdots = P[N] \qquad (4.2)$$

Combining (4.1) and (4.2):

$$P[1] = P[2] = \cdots = P[N] = 1/N \qquad (4.3)$$

Then, for any subset $E \subset S$:

$$P[E] = P\left[\bigcup_{i \in E} \{i\}\right] = \sum_{i \in E} P[i] = \sum_{i \in E} 1/N = |E|/N = |E|/|S|.$$

Example 4.1: If 2 dice are rolled, what is the probability that the sum is 9? Assume equaly likely outcomes.

Solution:

$\{\text{sum} = 9\} = \{(3,6), (4,5), (5,4), (6,3)\}$

$\Rightarrow \qquad \text{prob} = 4/36$

Example 4.2: An urn has 7 white balls and 5 black balls.

If we draw 3 balls at random, what is the probability that 1 is white and 2 are black?

Solution: Put a unique mark on each ball. Then there are $12 \times 11 \times 10 = 1320$ outcomes.

Case 1: 1st ball is white; there are $7 \times 5 \times (5-1) = 140$ ways.

Case 2: 2nd ball is white; there are $5 \times 7 \times (5-1) = 140$ ways.

Case 3: 3rd ball is white; there are $5 \times (5-1) \times 7 = 140$ ways.

$\Rightarrow \qquad \text{prob} = \dfrac{3 \times 140}{1320} = 7/22$

These problems all boil down to counting combinations. I'll assume you learned counting in ECE108 and skip the topic, except for the next problem which is a nice application of the inclusion/exclusion principle.

Example 4.3: Matching Problem

Each of n persons throws their hat into the center of a room and picks a hat at random.

What is the probability that no person selects their own hat? [Hard]

Solution: There are $n \times (n-1) \times \cdots \times 1 = n!$ possible hat assignments.

Let $E_i = \{\text{person } i \text{ selects hat } \# \, i\}$.

$$\begin{aligned}
&P[E_1 \cup E_2 \cup \cdots \cup E_n] \\
&\quad = P[E_1] + P[E_2] + \cdots P[E_n] \\
&\quad\quad - \sum_{i_1 < i_2} P[E_{i_1} E_{i_2}] \\
&\quad\quad\quad \vdots \\
&\quad\quad + (-1)^{m+1} \sum_{i_1 < \cdots < i_m} P[E_{i_1} E_{i_2} \cdots E_{i_m}] \\
&\quad\quad\quad \vdots \\
&\quad\quad + (-1)^{n+1} P[E_1 E_2 \cdots E_n]
\end{aligned}$$

Now, $E_{i_1} E_{i_2} \cdots E_{i_m}$ means persons $i_1, i_2, \ldots, i_m$ have their own hat. This leaves $(n-m)$ people with an unknown hat arrangement. There are $(n-m)!$ ways to arrange these.

$$\Rightarrow \qquad P[E_{i_1} E_{i_2} \cdots E_{i_m}] = \frac{(n-m)!}{n!}$$

Also, $\displaystyle\sum_{i_1 < \cdots < i_m} P[E_{i_1} E_{i_2} \cdots E_{i_m}]$ has $\binom{n}{m}$ terms in the sum.

So,
$$\begin{aligned}
\sum_{i_1 < \cdots < i_m} P[E_{i_1} E_{i_2} \cdots E_{i_m}] &= \binom{n}{m} \frac{(n-m)!}{n!} \\
&= \frac{n!}{(n-m)!m!} \frac{(n-m)!}{n!} \\
&= \frac{1}{m!}
\end{aligned}$$

$$\Rightarrow \qquad P[E_1 \cup E_2 \cup \cdots \cup E_n] = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{(-1)^{n+1}}{n!}$$

$$\begin{aligned}
P[E_1^c E_2^c \cdots E_n^c] &= 1 - P[E_1 \cup E_2 \cup \cdots \cup E_n] \\
&= 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}
\end{aligned}$$

This is a truncation of the Taylor series for e^{-1}.

When n is large, this is ≈ 0.369.