Continuous Random Variables

2) Normal (Gaussian) random variables [Ross 5.4]

Example 18.1: Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Find the distribution of $Z = (X-\mu)/\sigma$.

Solution:

$$Z = \frac{X-\mu}{\sigma} = \frac{1}{\sigma}X - \frac{\mu}{\sigma}$$

Plugging $a = 1/\sigma$ and $b = -\mu/\sigma$ into Proposition 17.1, we get

$$Z \sim \mathcal{N}(\underbrace{a\mu + b}_{0}, \underbrace{a^2\sigma^2}_{1})$$

so $Z \sim \mathcal{N}(0,1)$.

Definition 18.1: $\mathcal{N}(0,1)$ is called a **standard normal** or **standard Gaussian** distribution.

Example 18.2: Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Find $E[X]$ and $Var[X]$. [Var is Hard]

Solution: Let $Z = (X-\mu)/\sigma$.

Then $Z \sim \mathcal{N}(0,1)$ and $X = \sigma Z + \mu$

$$\begin{aligned}
E[Z] &= \int_{-\infty}^{\infty} z f_Z(z) dz \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z e^{-z^2/2} dz \\
&= \frac{-1}{\sqrt{2\pi}} e^{-z^2/2} \Big|_{-\infty}^{\infty} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
Var[Z] &= E[Z^2] - (E[Z])^2 \\
&= E[Z^2] \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-z^2/2} dz && d(uv) = du\, v + u\, dv \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{z}_{u} \cdot \underbrace{z e^{-z^2/2} dz}_{dv} && uv = \int v du + \int u dv \\
&= \frac{1}{\sqrt{2\pi}} \left[uv \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} v du \right] && du = dz \quad v = -e^{-z^2/2} \\
&= \frac{1}{\sqrt{2\pi}} \left[-z e^{-z^2/2} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -e^{-z^2/2} dz \right] \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} dz \\
&= 1
\end{aligned}$$

So
$$\begin{aligned}
E[X] &= E[\sigma Z + \mu] = \sigma E[Z] + \mu = \mu \\
Var[X] &= Var[\sigma Z + \mu] = \sigma^2 Var[Z] = \sigma^2
\end{aligned}$$

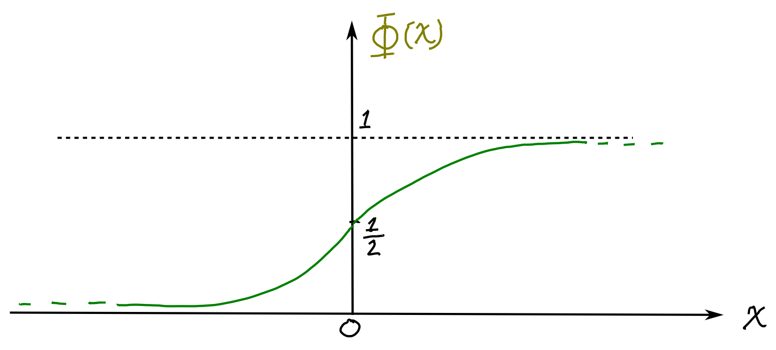
CDF of Normal Random Variables

Definition 18.2: For an $\mathcal{N}(0,1)$ distribution, we define

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du \qquad \text{[CDF of standard normal]}$$

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-u^2/2} du \qquad \text{[Q-function]}$$

Note: $\Phi(x) + Q(x) = 1; \Phi(-x) = Q(x) = 1 - \Phi(x)$.



There is also the "error function":

$$\begin{aligned}
\operatorname{erf}(x) &= \frac{2}{\sqrt{\pi}} \int_0^x e^{-v^2} dv && u = \sqrt{2}v, du = \sqrt{2}dv \\
&= \frac{2}{\sqrt{2\pi}} \int_0^{\sqrt{2}x} e^{-u^2/2} du \\
&= 2\left[\frac{1}{\sqrt{2\pi}} \int_0^{\sqrt{2}x} e^{-u^2/2} du \right] \\
&= 2\left[-\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-u^2/2} du + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\sqrt{2}x} e^{-u^2/2} du \right] \\
&= 2\left[-\frac{1}{2} + \Phi(\sqrt{2}x) \right] \\
&= 2\Phi(\sqrt{2}x) - 1
\end{aligned}$$

Table of $\Phi(x)$:

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976

For Gaussian other than $\mathcal{N}(0,1)$, $\Phi(.)$ can still be used with proper transformation:

Example 18.3: The grades in a course follow an $\mathcal{N}(\mu, \sigma^2)$ distribution. What is the probability that a random student is at least one σ above the mean μ?

Solution: Let $X \sim \mathcal{N}(\mu, \sigma^2)$

$$\begin{aligned}
P[X > \mu + \sigma] &= P\left[\frac{X-\mu}{\sigma} > \frac{\mu+\sigma-\mu}{\sigma} \right] \\
&= P[Z > 1] && \text{where } Z \sim \mathcal{N}(0,1) \\
&= 1 - \Phi(1) \\
&\approx 0.15866
\end{aligned}$$

Although not asked, in addition:

$$\begin{aligned}
P[X > \mu + 2\sigma] &= 1 - \Phi(2) \approx 0.02275 \\
P[X < \mu - \sigma] &= \Phi(-1) = 1 - \Phi(1) \approx 0.15866 \\
P[\mu < X < \mu + \sigma] &= \Phi(1) - \Phi(0) \approx 0.34134
\end{aligned}$$

Example 18.4: In finance, the Value At Risk (VaR) of an investment is the value $v > 0$ such that there is only a 1% chance the investment will lose more than v.

If the profit from an investment is $X \sim \mathcal{N}(\mu, \sigma^2)$, what is its VaR?

Solution: We want $v > 0$ such that $0.01 = P[X < -v]$. Then there is a 1% chance we will lose more than v.

$$\begin{aligned}
0.01 &= P[X < -v] \\
&= P\left[\frac{X-\mu}{\sigma} < \frac{-v-\mu}{\sigma} \right] \\
&= P\left[Z < -\frac{v+\mu}{\sigma} \right] && \text{where } Z \sim \mathcal{N}(0,1)
\end{aligned}$$

Let $a = (v+\mu)/\sigma$. Then:

$$\begin{aligned}
P[Z < -a] = 0.01 &\Leftrightarrow P[Z > a] = 0.01 \\
&\Leftrightarrow P[Z \le a] = 0.99 \\
&\Leftrightarrow \Phi(a) = 0.99 \\
&\Leftrightarrow a = \Phi^{-1}(0.99) = 2.33
\end{aligned}$$

Since $a = (v+\mu)/\sigma = 2.33$, we get

$$v = 2.33\sigma - \mu$$

The normal distribution is used (and mis-used) a lot:

- Central Limit Thm: normal is a good approximation when observation is sum of many small independent components, e.g., thermal noise
- Measurement errors
- A good model for parameter estimation errors under some conditions
- Finance (e.g., Black-Scholes option pricing)
- It is the maximum entropy distribution subject to a specified variance.
- Is the velocity distribution of particles in an ideal gas with $\sigma^2 = kT/m$.