Continuous Random Variables

C) Exponential Random Variable [Ross S5.5]

A random variable X with pdf

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{else} \end{cases}$$

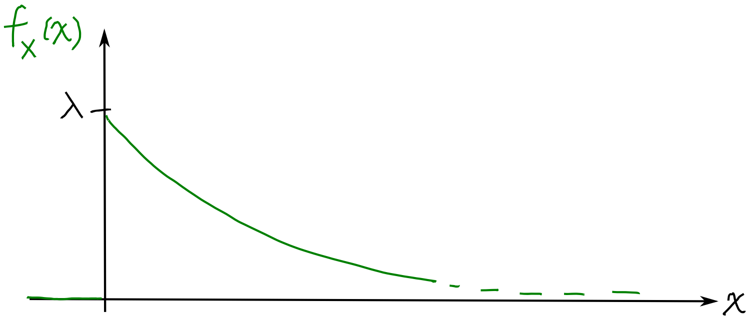
is called **exponential** with **rate parameter** $\lambda > 0$ and denoted $X \sim \mathsf{Exp}(\lambda)$.

Note: If X has units of min then λ has units min^{-1}.

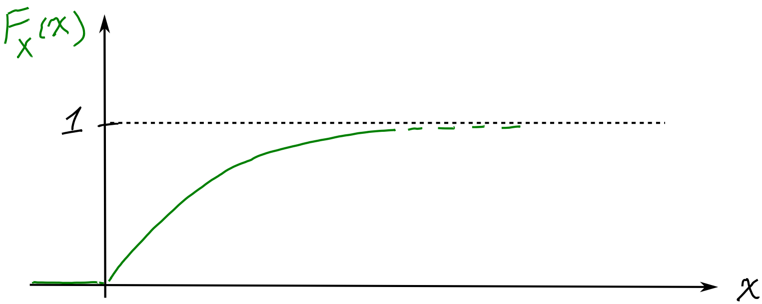
$$\begin{aligned} F_X(a) &= \int_{-\infty}^{a} f_X(u)du \\ &= \begin{cases} 1 - e^{-\lambda a} & a \geq 0 \\ 0 & a < 0 \end{cases} \end{aligned}$$

Example 19.1: For $X \sim \mathsf{Exp}(\lambda)$, what are $E[X]$ and $Var[X]$?

Solution:



$f_X(x)$
λ
x



$F_X(x)$
1
x

Example 19.2: The time someone uses an ATM machine is an exponential random variable with $\lambda = 1/3 \text{ min}^{-1}$. Someone arrives at the ATM just before you. What is the probability that you wait

a) more than 3 min,

b) between 3 and 6 min?

Solution:

Definition 19.1: A non-negative random variable X is called **memoryless** if for all $s > 0$ and all $t > 0$

$$P[X > s + t \mid X > t] = P[X > s]$$

In words: The probability of waiting s seconds more given you have already waited t seconds is the same as waiting s seconds from the start. In other words, no matter how long you have waited, time to wait still has the same distribution.

Example 19.3: Does $\mathsf{Exp}(\lambda)$ have the memoryless property?

Solution: Let $X \sim \mathsf{Exp}(\lambda)$. Then

$$\begin{aligned} P[X > s + t \mid X > t] &= \frac{P[X > s + t, X > t]}{P[X > t]} \\ &= \frac{P[X > s + t]}{P[X > t]} \\ &= \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}} \\ &= e^{-\lambda s} \\ &= P[X > s] \end{aligned}$$

Yes, $\mathsf{Exp}(\lambda)$ has the memoryless property.

Example 19.4: Persons A and B are each being served by a teller. Person C arrives, and waits for one of the two tellers. All service times are exponential with the same parameter λ. What is the probability that C is the last to leave?

Solution:

Example 19.5: A car battery has a lifetime that is exponentially distributed with mean $10,000$ km.

a) What is the probability of completing a 5000 km trip without replacing the battery?

b) What can we say if lifetime is not exponential?

Solution:

The exponential distribution can be used:

- to model service times in queuing systems
- time between radioactive decays
- credit risk modeling in finance
- is maximum entropy distribution on $[0, \infty]$ subject to a specified mean.