## Random Variables (rv)

**Mean and Variance of Poisson** [Ross S4.7]

Intuition: Say $X \sim \text{Binomial}(n,p)$ with $\lambda = np$, $n$ large, and $p$ small

Then:

$$\begin{aligned}
E[X] &= np = \lambda \\
Var[X] &= np(1-p) \\
&= \lambda(1-p) \\
&\approx \lambda
\end{aligned}$$

Exact: Let $X \sim \text{Poisson}(\lambda)$. Then

$$\begin{aligned}
E[X] &= \sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} e^{-\lambda} \\
&= \sum_{k=1}^{\infty} k \frac{\lambda^k}{k!} e^{-\lambda} \\
&= \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} e^{-\lambda} \\
&= \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda} \\
&= \lambda \sum_{\ell=0}^{\infty} \frac{\lambda^\ell}{\ell!} e^{-\lambda} && \ell = k-1 \\
&= \lambda
\end{aligned}$$

$$\begin{aligned}
E[X^2] &= \sum_{k=0}^{\infty} k^2 \frac{\lambda^k}{k!} e^{-\lambda} \\
&= \sum_{k=1}^{\infty} k^2 \frac{\lambda^k}{k!} e^{-\lambda} \\
&= \sum_{k=1}^{\infty} \frac{k\lambda^k}{(k-1)!} e^{-\lambda} \\
&= \sum_{\ell=0}^{\infty} \frac{(\ell+1)\lambda^{\ell+1}}{\ell!} e^{-\lambda} && \ell = k-1 \\
&= \lambda \left( \underbrace{\sum_{\ell=0}^{\infty} \frac{\ell\,\lambda^\ell}{\ell!} e^{-\lambda}}_{\lambda} + \underbrace{\sum_{\ell=0}^{\infty} \frac{\lambda^\ell}{\ell!} e^{-\lambda}}_{1} \right) \\
&= \lambda(1+\lambda)
\end{aligned}$$

So
$$\begin{aligned}
Var[X] &= E[X^2] - (E[X])^2 \\
&= \lambda(1+\lambda) - (\lambda)^2 \\
&= \lambda
\end{aligned}$$

**Example 13.1:** A radioactive substance with a large # of atoms emits 3.2 alpha particles per second on average. What is the probability that no more than 2 alpha particles are emitted in a 1 second interval?

*Solution:*

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**D) The geometric random variable** [Ross 4.8.1]

Consider an infinite sequence of independent Bernoulli($p$) trials.

Let $X$ be trial # of first outcome that is a 1.

$X$ is called **geometric** with parameter $p$, denoted $X \sim \text{Geometric}(p)$

$$\begin{aligned}
p_X(k) &= P[(k-1) \text{ zeros followed by a one}] && \text{for } k = 1, 2, \ldots \\
&= \begin{cases} (1-p)^{k-1}p & k \geq 1 \\ 0 & \text{else} \end{cases}
\end{aligned}$$

**Example 13.2:** A bag contains 2 white balls and 3 black balls. Balls are randomly drawn until a black ball is drawn. The selected ball is replaced before the next draw.

a) What is the probability that exactly $n$ draws are needed?

b) What is the probability that at least $k$ draws are needed?

*Solution:*

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**Mean and Variance**

If $X \sim \text{Geometric}(p)$, then:

$$\begin{aligned}
E[X] &= \sum_{k=1}^{\infty} k(1-p)^{k-1}p \\
&= \cdots && \text{[see Ross example 4.8b]} \\
&= \frac{1}{p}
\end{aligned}$$

$$\begin{aligned}
E[X^2] &= \sum_{k=1}^{\infty} k^2(1-p)^{k-1}p \\
&= \cdots && \text{[see Ross example 4.8c]} \\
&= \frac{2-p}{p^2}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad Var[X] &= E[X^2] - (E[X])^2 \\
&= \frac{2-p}{p^2} - \left(\frac{1}{p}\right)^2 \\
&= \frac{1-p}{p^2}
\end{aligned}$$