Jointly Distributed Random Variables

Conditional Distributions:

Discrete Case [Ross S6.4]

Recall that for $P[F] > 0$:

$$P[E|F] = \frac{P[EF]}{P[F]}$$

Say $p_Y(y) > 0$. The **conditional pmf** for X given Y is

$$\begin{aligned} p_{X|Y}(x|y) &= P[X = x \,|\, Y = y] \\ &= \frac{P[X = x, Y = y]}{P[Y = y]} \\ &= \frac{p_{XY}(x,y)}{p_Y(y)} \end{aligned}$$

The **conditional cdf** for X given Y is

$$\begin{aligned} F_{X|Y}(x|y) &= P[X \le x \,|\, Y = y] \\ &= \frac{P[X \le x, Y = y]}{P[Y = y]} \\ &= \sum_{a \le x} \frac{P[X = a, Y = y]}{P[Y = y]} \\ &= \sum_{a \le x} p_{X|Y}(a|y) \end{aligned}$$

If X and Y are independent:

$$\begin{aligned} p_{X|Y}(x|y) &= \frac{p_{XY}(x,y)}{p_Y(y)} \\ &= \frac{p_X(x)p_Y(y)}{p_Y(y)} \\ &= p_X(x) \end{aligned}$$

Example 27.1: Let $X \sim \mathsf{Poisson}(\lambda_1)$ and $Y \sim \mathsf{Poisson}(\lambda_2)$ be independent. Find the conditional pmf for X given $X + Y = n$.

Solution:

$$\begin{aligned} P[X = k \,|\, X + Y = n] &= \frac{P[X = k, X + Y = n]}{P[X + Y = n]} \\ &= \frac{P[X = k, Y = n - k]}{P[X + Y = n]} \\ &= \frac{P[X = k]P[Y = n - k]}{P[X + Y = n]} \end{aligned} \tag{27.1}$$

If $k > n$: $P[Y = n - k] = 0 \Rightarrow P[X = k \,|\, X + Y = n] = 0$.

If $k < 0$: $P[X = k] = 0 \Rightarrow P[X = k \,|\, X + Y = n] = 0$.

From Ex. 26.3, $X + Y$ is $\sim \mathsf{Poisson}(\lambda_1 + \lambda_2)$. So for $0 \le k \le n$:

$$\begin{aligned} P[X = k|X + Y = n] &= \frac{\lambda_1^k e^{-\lambda_1}}{k!} \frac{\lambda_2^{n-k} e^{-\lambda_2}}{(n-k)!} \left[\frac{(\lambda_1 + \lambda_2)^n e^{-(\lambda_1 + \lambda_2)}}{n!} \right]^{-1} \\ &= \frac{n!}{k!(n-k)!} \frac{\lambda_1^k \lambda_2^{n-k}}{(\lambda_1 + \lambda_2)^n} \\ &= \binom{n}{k} \left(\frac{\lambda_1}{\lambda_1 + \lambda_2} \right)^k \left(\frac{\lambda_2}{\lambda_1 + \lambda_2} \right)^{n-k} \end{aligned}$$

This is binomial with parameters n and $\lambda_1/(\lambda_1 + \lambda_2)$.

Example 27.2: Let $X_1, X_2, \ldots, X_n$ be iid and $\sim \mathsf{Bernoulli}(p)$.

Say these result in k ones. Show that each of the $\binom{n}{k}$ possible orderings of k ones are then equally likely.

Solution: Let $Z = X_1 + \cdots + X_n$. We are conditioning on $Z = k$.

Let $x_1, x_2, \ldots, x_n$ be binary, and such that $x_1 + x_2 + \cdots + x_n = k$.

$$\begin{aligned} P[X_1 = x_1, \ldots, X_n = x_n | Z = k] &= \frac{P[X_1 = x_1, \ldots, X_n = x_n, Z = k]}{P[Z = k]} \\ &= \frac{P[X_1 = x_1, \ldots, X_n = x_n]}{P[Z = k]} \quad (*) \\ &= \frac{p^k (1-p)^{n-k}}{\binom{n}{k} p^k (1-p)^{n-k}} \\ &= \frac{1}{\binom{n}{k}} \end{aligned}$$

(*) since $\{X_1 = x_1, \ldots, X_n = x_n\} \subset \{Z = k\}$ when $x_1 + \cdots + x_n = k$.

Continuous Case [Ross S6.5]

If X and Y are continuous, for $f_Y(y) > 0$, the **conditional pdf** of X given $Y = y$ is

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

We also define:

$$P[X \in A | Y = y] = \int_A f_{X|Y}(x|y) dx$$

and then

$$\begin{aligned} \int_{-\infty}^{\infty} P[X \in A|Y = y] f_Y(y) dy &= \int_{-\infty}^{\infty} \left[\int_A f_{X|Y}(x|y) dx \right] f_Y(y) dy \\ &= \int_{-\infty}^{\infty} \int_A f_{X|Y}(x|y) f_Y(y) dy dx \\ &= \int_A \int_{-\infty}^{\infty} f_{XY}(x,y) dy dx \\ &= P[X \in A] \end{aligned} \tag{27.2}$$

With $A = (-\infty, a]$, we get the **conditional cdf**

$$F_{X|Y}(a|y) = P[X \le a | Y = y] = \int_{-\infty}^{a} f_{X|Y}(x|y) dx$$

If X and Y are independent and $f_Y(y) > 0$:

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f_{XY}(x,y)}{f_Y(y)} \\ &= \frac{f_X(x) f_Y(y)}{f_Y(y)} \\ &= f_X(x) \end{aligned}$$

Example 27.3: The joint pdf of X and Y is

$$f_{XY}(x,y) = \begin{cases} \frac{e^{-x/y} e^{-y}}{y} & 0 < x < \infty,\ 0 < y < \infty \\ 0 & \text{else} \end{cases}$$

Find $P[X > 1|Y = 1]$.

Solution: For $y > 0$:

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f_{XY}(x,y)}{f_Y(y)} \\ &= \frac{f_{XY}(x,y)}{\int_{-\infty}^{\infty} f_{XY}(x,y) dx} \\ &= \frac{\frac{1}{y} e^{-x/y} e^{-y}}{e^{-y} \int_0^{\infty} \frac{1}{y} e^{-x/y} dx} \\ &= \frac{\frac{1}{y} e^{-x/y} e^{-y}}{e^{-y} \times 1} \\ &= \frac{1}{y} e^{-x/y} \end{aligned}$$

Hence:

$$\begin{aligned} P[X > 1|Y = y] &= \int_1^{\infty} f_{X|Y}(x|y) dx \\ &= \int_1^{\infty} \frac{1}{y} e^{-x/y} dx \\ &= -\ e^{-x/y} \Big|_1^{\infty} \\ &= e^{-1/y} \end{aligned}$$