Chapter 5

Discrete-Time Systems Analysis in Transform Domain

The Laplace transform is used for *continuous-time* systems, while Z-transform is used for *discrete-time* (DT) systems.

5.1 The Z-Transform

The Z-transform of a DT signal $x[n]$ is defined by:

$$\mathcal{Z}\{x[n]\} = X[z] = \sum_{n=0}^{\infty} x[n]z^{-n}$$

The Z-transform exists for values of $z \in \mathbb{C}$ for which the sum converges. These values of z are called the Region of Convergence (ROC).

Remark: $X[z]$ is in general a complex valued function. Also, above definition is called a *single-sided* or *unilateral* Z-transform. The *bilateral* Z-transform, defined in the textbook, is not needed and not required in this course since our signals are assumed to be causal $\{x[n] = 0 \text{ for } n < 0\}$.

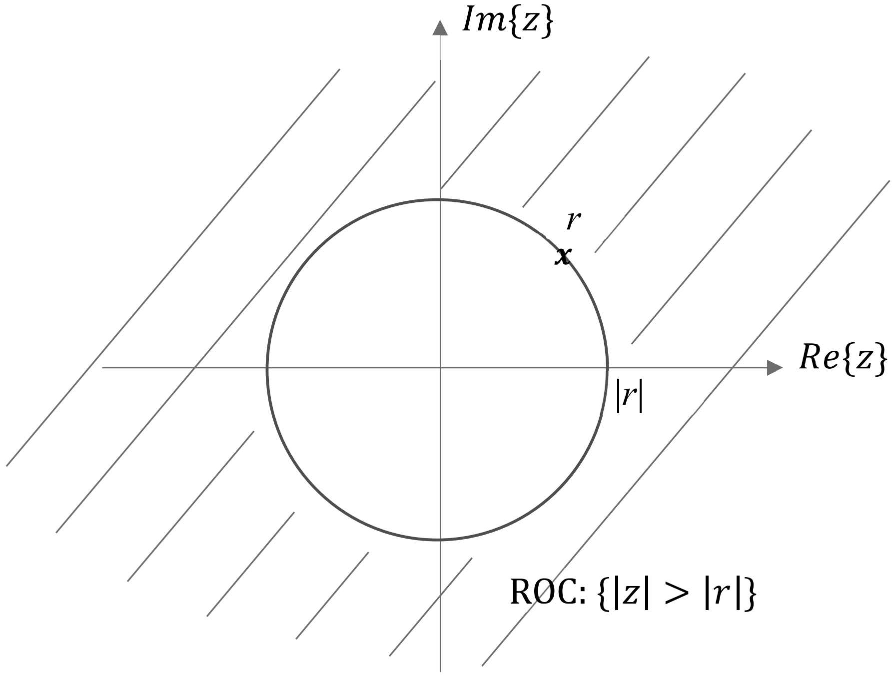
Example: $x[n] = r^n u[n]$, r = complex in general

$$\Rightarrow \quad X[z] = \sum_{n=0}^{\infty} x[n]z^{-n} = \sum_{n=0}^{\infty} r^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{r}{z}\right)^n$$

Using the geometric sum:

$$\sum_{n=0}^{\infty} \alpha^n = 1 + \alpha + \alpha^2 + \cdots = \frac{1}{1-\alpha}, \qquad \text{provided } |\alpha| < 1$$

$$\Rightarrow \quad X[z] = \frac{1}{1-\frac{r}{z}} = \frac{z}{z-r}\,, \quad \text{provided } \left|\frac{r}{z}\right| < 1 \text{ or } |z| > |r| \quad \text{(ROC)}$$



More Examples:

1) $x[n] = \delta[n] \quad \Rightarrow \quad X[z] = \sum_{n=0}^{\infty} \delta[n] z^{-n} = 1$

2) $x[n] = \delta[n-m] \quad \Rightarrow \quad X[z] = \begin{cases} 0, & m < 0 \\ z^{-m}, & m \geq 0 \end{cases}$

3) $x[n] = u[n] \quad \Rightarrow \quad X[z] = \sum_{n=0}^{\infty} z^{-n} = \frac{z}{z-1}$

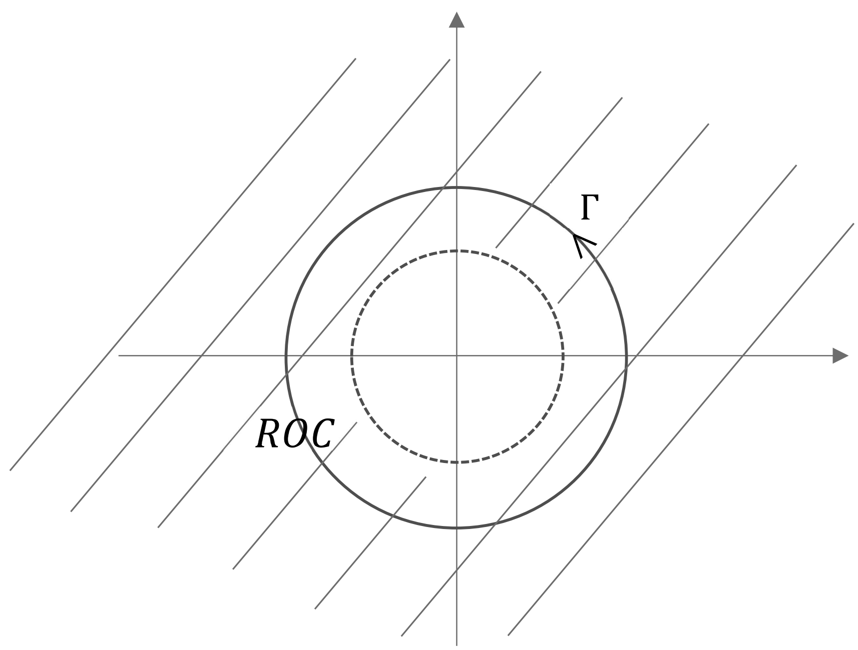
4) $x[n] = (-1)^n u[n] \quad \Rightarrow \quad X[z] = \sum_{n=0}^{\infty} (-z)^{-n} = \frac{z}{z+1}$

Remark: Z-transform does not exist always. For example, $x[n] = (n)^n$ has no ZT.

- **<u>Inverse ZT</u>:** Mathematically given by:

$$x[n] = \mathcal{Z}^{-1}\{X[z]\} = \frac{1}{2\pi j}\oint_{\Gamma} X[z]z^{n-1}\,dz \quad \text{for } n \geq 0$$

where Γ = a simple "closed contour" in ROC.



Remark: Above complex integral is difficult to evaluate and rarely used. Instead, the inverse of ZT is mainly found by using tables of well-known ZT pairs.

Remark: The functions we study in this course are mainly *rational*. For example, the transfer function for a LTI system described by a difference equation with constant coefficients is always rational.

- **<u>Finding Inverse of Z-Transform</u>**

For *<u>rational</u>* functions, two methods are mainly used:

(1) Apply partial-fraction expansion (PFE) and use tables.

(2) Expand in power series.

<u>Example</u>: (Using PFE and Tables)

$$X[z] = \frac{z-1}{(z-3)(z+4)}$$

Apply PFE on $\frac{X[z]}{z}$ instead of $X[z]$:

$$\Rightarrow \quad \frac{X[z]}{z} = \frac{z-1}{z(z-3)(z+4)} = \frac{\frac{1}{12}}{z} + \frac{\frac{2}{21}}{z-3} + \frac{-\frac{5}{28}}{z+4}$$

$$\Rightarrow \quad X[z] = \frac{1}{12} + \frac{2}{21} \cdot \frac{z}{z-3} - \frac{5}{28} \cdot \frac{z}{z+4}$$

Using ZT table 5.1 in textbook:

$$x[n] = \frac{1}{12}\delta[n] + \frac{2}{21}(3)^n u[n] - \frac{5}{28}(-4)^n u[n]$$

<u>Example</u>: (Using Power Series Expansion)

Find $x[0]$ and $x[1]$ if

$$X[z] = \frac{z^3 - z^2}{z^3 + 6z^2 + 11z + 6}$$

Solution**:**

By long-division:

$$
\begin{array}{r l}
 & \quad 1 - 7z^{-1} + \cdots \\
z^3 + 6z^2 + 11z + 6) & \; z^3 - z^2 \\
 & \underline{- \quad z^3 + 6z^2 + 11z + 6} \\
 & \qquad - 7z^2 - 11z - 6 \\
 & \underline{- \qquad - 7z^2 - 42z - 77 - 42z^{-1}} \\
 & \qquad\qquad\quad + 31z + 71 + 42z^{-1}
\end{array}
$$

Since $\quad X[z] = \sum_{n=0}^{\infty} x[n] z^{-n} = x[0] + x[1]z^{-1} + \cdots$

then $\quad x[0] = 1$ and $x[1] = -7$

Remark: Power series expansion is useful for finding the first few samples of $x[n]$. However, it is difficult for finding the general form of $x[n]$ for all n.

Example: (Finding $x[n]$ for special non-rational $X[z]$)

Let $X[z] = e^{\frac{1}{z}}$

Using the expansion $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ gives:

$$X[z] = e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{(1/z)^n}{n!} = \sum_{n=0}^{\infty} \left(\frac{1}{n!}\right) z^{-n}$$

$$\Rightarrow \quad x[n] = \frac{1}{n!} u[n]$$

5.2 Properties of the Z-Transform

1) <u>Linearity:</u> $\mathcal{Z}\{c_1 x_1[n] + c_2 x_2[n]\} = c_1 X_1[z] + c_2 X_2[z]$

2) <u>Right-Shift</u>: For $m > 0$, two shifts:

(a) $$\mathcal{Z}\{x[n-m]u[n-m]\} = z^{-m} X[z]$$

<u>Proof:</u>

$$\mathcal{Z}\{x[n-m]u[n-m]\} = \sum_{n=0}^{\infty} x[n-m]u[n-m]z^{-n}$$

$$= \sum_{n=m}^{\infty} x\Big[\underbrace{n-m}_{i}\Big] z^{-n} = \sum_{i=0}^{\infty} x[i] z^{-(i+m)}$$

$$= z^{-m} \sum_{i=0}^{\infty} x[i] z^{-i} = z^{-m} X[z]$$

(b) $$\mathcal{Z}\{x[n-m]u[n]\} = z^{-m} X[z] + \textstyle\sum_{n=1}^{m} x[-n] z^{(n-m)}$$

For $m = 1$: $\mathcal{Z}\{x[n-1]u[n]\} = z^{-1} X[z] + x[-1]$

For $m = 2$: $\mathcal{Z}\{x[n-2]u[n]\} = z^{-2} X[z] + z^{-1} x[-1] + x[-2]$

3) <u>Left-Shift</u>: For $m > 0$:

$$\mathcal{Z}\{x[n+m]u[n]\} = z^m X[z] - \sum_{n=0}^{m-1} x[n] z^{-n+m}$$

4) **Convolution**:

$$x_1[n] * x_2[n] = \sum_{m=-\infty}^{\infty} x_1[m]x_2[n-m]$$

$$\Rightarrow \quad \mathcal{Z}\{x_1[n] * x_2[n]\} = X_1[z] \cdot X_2[z]$$

5) **Multiplication by n**:

$$\mathcal{Z}\{nx[n]u[n]\} = -z\frac{d}{dz}X[z]$$

6) **Multiplication by r^n:**

$$\mathcal{Z}\{r^n x[n]u[n]\} = X\left[\frac{z}{r}\right]$$

7) **Initial Value:**

For a causal signal $x[n]$:

$$x[0] = \lim_{z\to\infty} X[z]$$

8) **Final Value**:

$$\lim_{n\to\infty} x[n] = \lim_{z\to 1}(z-1)X[z]$$

provided the limit on the left side exists.

Example: (Finding ZT using table and properties)

Find ZT of $x[n] = n\,u[n]$.

Solution:

We have from table: $\mathcal{Z}\{u[n]\} = \frac{z}{z-1}$

Using the property of ZT for the multiplication by n:

$$\mathcal{Z}\{n\,u[n]\} = -z\frac{d}{dz}\left[\frac{z}{z-1}\right] = -z\left[\frac{(z-1)-z}{(z-1)^2}\right] = \frac{z}{(z-1)^2}$$

5.3 DT System Responses Using ZT

- Given an LTI system described by a difference equation (DE):

$$Q(E)y[n] = P(E)x[n]$$

To find $y_{zi}[n]$, $y_{zs}[n]$, and $y[n]$, convert into Z-domain, find $Y[z]$, then convert back into time domain.

<u>Example</u>:

Given $y[n+1] - 2y[n] = x[n+1] - 3x[n]$ where $y[-1] = 1$ and $x[n] = 4^n u[n]$. Find $y_{zi}[n]$, $y_{zs}[n]$ and $y[n]$?

<u>Solution:</u>

Re-write DE in delay form:

$$y[n] - 2y[n-1] = x[n] - 3x[n-1]$$

Take ZT of DE using the following right-shift property:

$$\mathcal{Z}\{y[n-1]\} = \mathcal{Z}\{y[n-1]u[n]\} = z^{-1}Y[z] + y[-1]$$

$$\mathcal{Z}\{x[n-1]\} = \mathcal{Z}\{x[n-1]u[n]\} = z^{-1}X[z] + \underbrace{x[-1]}_{0}$$

$$\Rightarrow \; Y[z] - 2\left[z^{-1}Y[z] + y[-1]\right] = X[z] - 3\left[z^{-1}X[z] + \underbrace{x[-1]}_{0}\right]$$

$$\Rightarrow \; (1 - 2z^{-1})Y[z] = (1 - 3z^{-1})X[z] + 2y[-1]$$

$$\Rightarrow \; Y[z] = \underbrace{\frac{1 - 3z^{-1}}{1 - 2z^{-1}}X[z]}_{Y_{zs}[z]} + \underbrace{\frac{2y[-1]}{1 - 2z^{-1}}}_{Y_{zi}[z]} = \underbrace{\frac{1 - 3z^{-1}}{1 - 2z^{-1}} \cdot \frac{z}{z-4}}_{use\ PFE} + \frac{2}{1 - 2z^{-1}}$$

$$\Rightarrow \quad Y[z] = \left[\frac{1}{2}\cdot\frac{z}{z-2} + \frac{1}{2}\cdot\frac{z}{z-4}\right] + (2)\frac{z}{z-2}$$

Take $\mathcal{Z}^{-1}$ using tables:

$$y[n] = \underbrace{\frac{1}{2}[(2)^n + (4)^n]u[n]}_{y_{zs}[n]} + \underbrace{(2)(2)^n u[n]}_{y_{zi}[n]}$$

- **<u>Transfer Function</u>**

For LTI system:

$$y[n] = h[n] * x[n] \quad \Longrightarrow \quad Y[z] = H[z]\cdot X[z]$$

The transfer function is:

$$H[z] = \mathcal{Z}\{h[n]\} = \frac{Y[z]}{X[z]}$$

For LTI system described by DE:

$$Q(E)y[n] = P(E)x[n]$$

Taking the ZT:

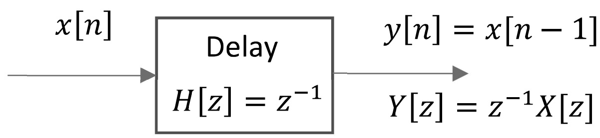
$$Q[z]Y[z] = P[z]X[z]$$

$$\Rightarrow \quad H[z] = \frac{Y[z]}{X[z]} = \frac{P[z]}{Q[z]}$$

<u>Remark</u>: $H[z]$ can be found by inspection from DE. Also, poles of $H[z]$ are the same as the roots of $Q[z]$. So, system stability can be stated in terms of <u>the poles</u> of $H[z]$.

5.4 DT System Realization

Similar to the CT systems, except integrators are replaced by delay elements.



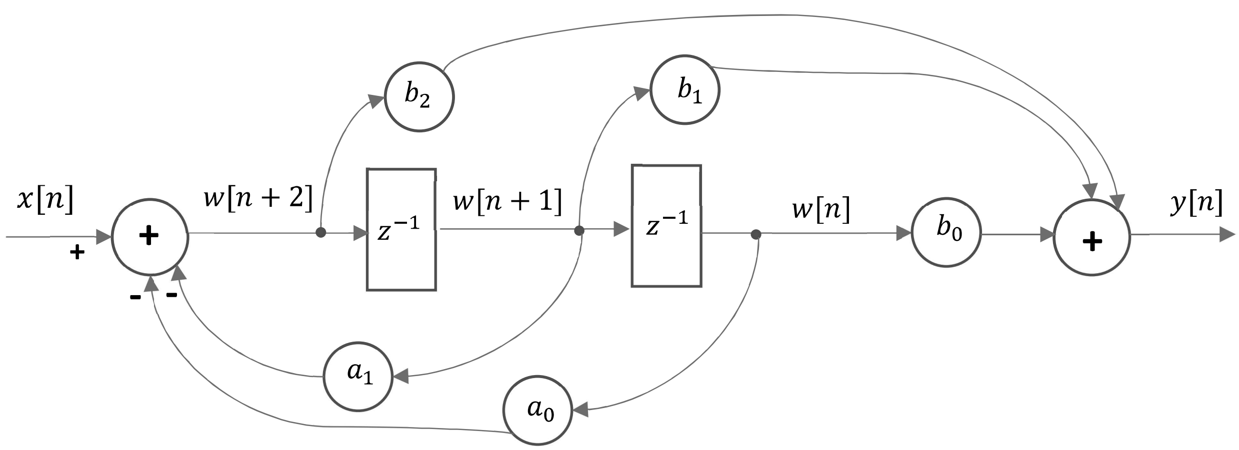
<u>Example</u>: (Direct Form)

$$H[z] = \frac{b_2 z^2 + b_1 z + b_0}{z^2 + a_1 z + a_0}$$

$$\Rightarrow \quad Y[z] = (b_2 z^2 + b_1 z + b_0) \underbrace{\left[\frac{1}{z^2 + a_1 z + a_0}\right] X[z]}_{W[z]}$$

$$\Rightarrow \quad w[n+2] + a_1 w[n+1] + a_0 w[n] = x[n]$$

and $$y[n] = b_2 w[n+2] + b_1 w[n+1] + b_0 w[n]$$



5.5 Frequency Response of DT Systems

We found before that for LTI systems: If the input is the everlasting exponential $x[n] = z_0^n, \quad -\infty < n < \infty$, then the output is also an everlasting exponential:

$$y[n] = H[z_0]\, z_0^n, \qquad -\infty < n < \infty$$

where $H[z_0]$ is the transfer function evaluated at $z = z_0$.

Assume the LTI system is asymptotically stable, and let $z_0 = e^{j\Omega}$, then

$$x[n] = e^{j\Omega n} \;\Rightarrow\; y[n] = H\left[e^{j\Omega}\right] e^{j\Omega n}$$

The Function $H\left[e^{j\Omega}\right] = H[z]\big|_{z=e^{j\Omega}}$ is called the Frequency Response. It is in general a complex function of the angular frequency Ω, which can be written as:

$$H\left[e^{j\Omega}\right] = \underbrace{\left|H[e^{j\Omega}]\right|}_{magnitude} \cdot e^{\underbrace{j\angle H[e^{j\Omega}]}_{phase\ or\ angle}}$$

Remark: The function $H[e^{j\Omega}]$ is periodic with period 2π since (for m integer);

$$H\left[e^{j(\Omega+2\pi m)}\right] = H\left[e^{j\Omega} \cdot \underbrace{e^{j2\pi m}}_{=1}\right] = H[e^{j\Omega}]$$

- **Response due to an Everlasting Sinusoid**

From: $\qquad x[n] = e^{j\Omega n} \;\Rightarrow\; y[n] = H[e^{j\Omega}] \cdot e^{j\Omega n}$

Take the real part: $\qquad x[n] = Re\left\{e^{j\Omega n}\right\} = \cos(\Omega n)$

$$\Rightarrow \quad y[n] = Re\left\{\underbrace{\left|H\left[e^{j\Omega}\right]\right| \cdot e^{j\angle H\left[e^{j\Omega}\right]}}_{H[e^{j\Omega}]} \cdot e^{j\Omega n}\right\}$$

or $$y[n] = |H[e^{j\Omega}]| \cdot Re\left\{e^{j[\Omega n + \angle H[e^{j\Omega}]]}\right\}$$

$$\Rightarrow \quad y[n] = \left|H\left[e^{j\Omega}\right]\right| \cos\left[\Omega n + \angle H\left[e^{j\Omega}\right]\right]$$

<u>Example</u>: An LTI system has T.F: $H[z] = \frac{z}{z-0.8}$

Rewrite: $H[z] = \frac{1}{1-0.8z^{-1}}$

Frequency Response:

$$H\left[e^{j\Omega}\right] = \frac{1}{1-0.8e^{-j\Omega}} = \frac{1}{(1-0.8\cos(\Omega)) + j0.8\sin(\Omega)}$$

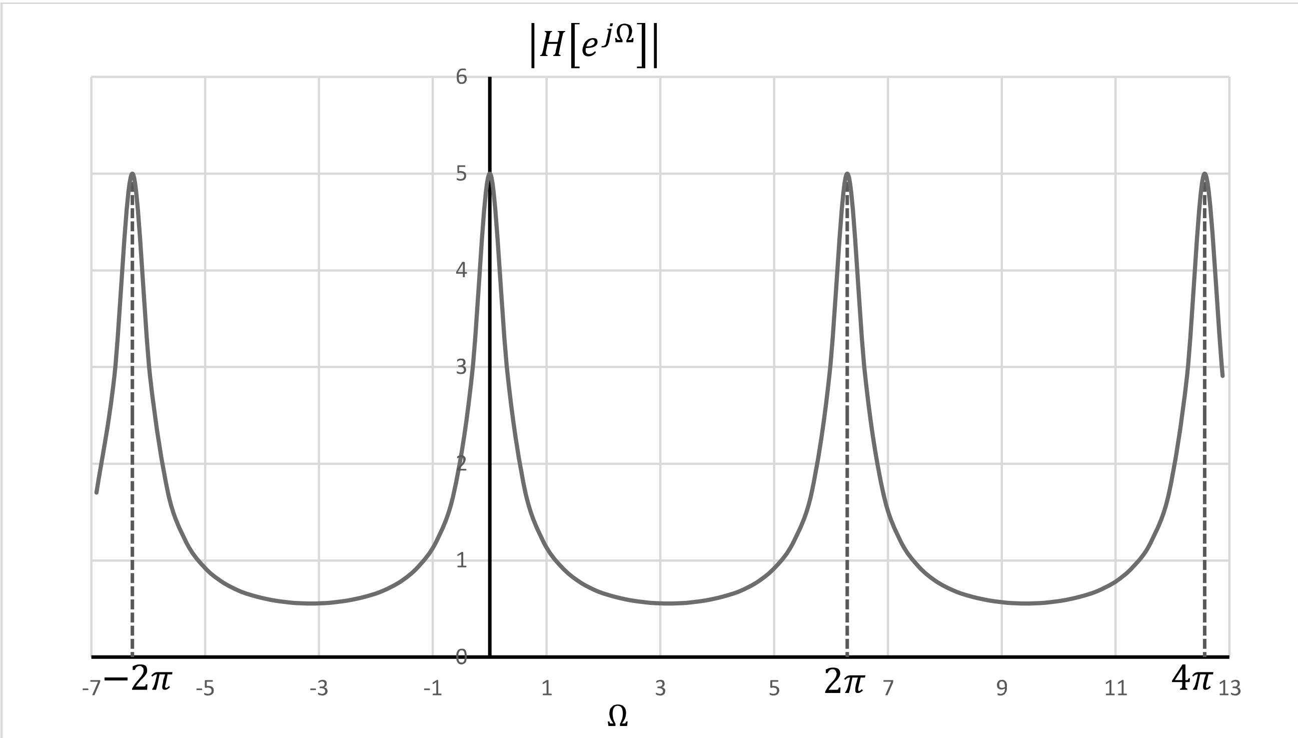
Magnitude:

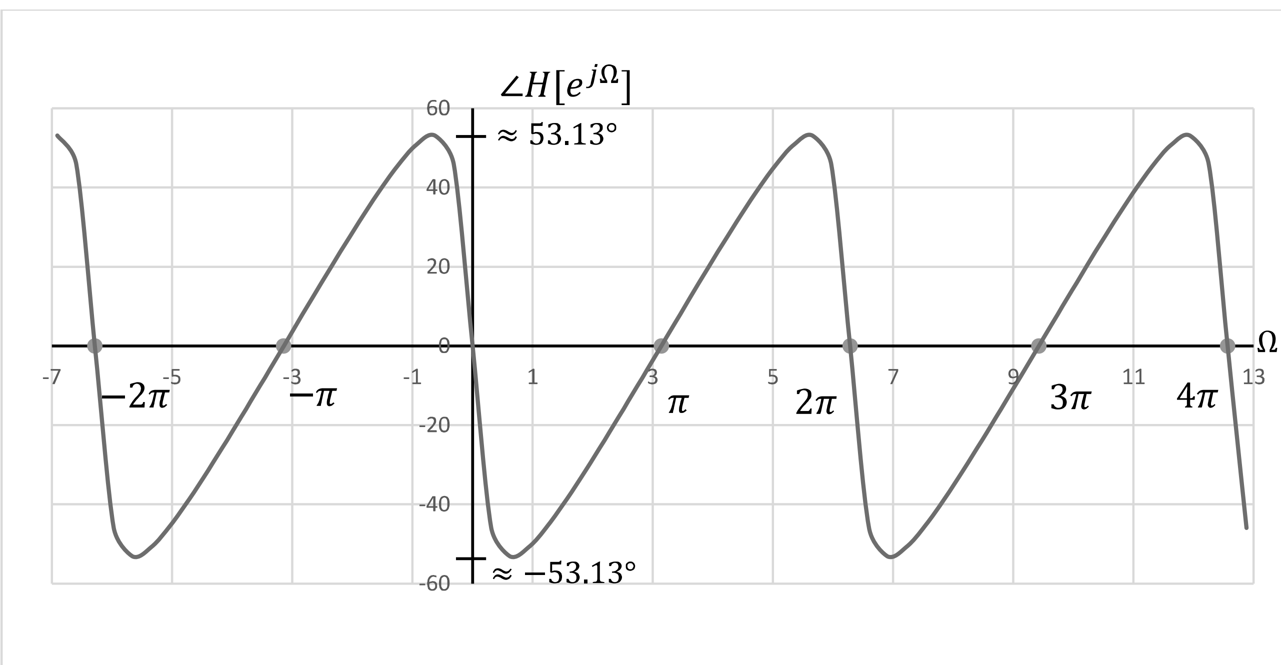
$$\left|H\left[e^{j\Omega}\right]\right| = \frac{1}{\sqrt{(1-0.8\cos(\Omega))^2 + (0.8\sin(\Omega))^2}} = \frac{1}{\sqrt{1.64 - 1.6\cos(\Omega)}}$$

Phase:

$$\angle H\left[e^{j\Omega}\right] = -\tan^{-1}\left(\frac{0.8\sin(\Omega)}{1-0.8\cos(\Omega)}\right)$$

Plots of magnitude and phase:





Example: Find the response of the above system to the input:

$$x[n] = 2\cos(3n)\,, \qquad -\infty < n < \infty$$

Solution:

$$\Omega = 3 \quad \Rightarrow \quad \left|H\left[e^{j3}\right]\right| \simeq 0.557\,, \quad \angle H\left[e^{j3}\right] \simeq -0.063$$

$$\Rightarrow \; y[n] = 2(0.557)\cos(3n - 0.063)\,, \qquad -\infty < n < \infty$$

- **Uniqueness of Frequencies of DT Signals**

The signal $x_0[n] = \cos(\Omega_0 n)$ has an angular (digital) frequency Ω_0 .

The signal $x_1[n] = \cos(\Omega_1 n)$ is identical to $x_0[n] = \cos(\Omega_0 n)$ if $\Omega_1 = \Omega_0 + 2\pi m$ where m=integer. This is because:

$$\cos(\Omega_1 n) = \cos[(\Omega_0 + 2\pi m)n] = \cos[\Omega_0 n + 2\pi mn] = cos(\Omega_0 n)$$

i.e.

The digital frequencies $\Omega_1 = \Omega_0 + 2\pi m$, *m*=integer, are identical to the single frequency Ω_0 .

Remark: Frequencies are unique for the band $-\pi \le \Omega_0 < \pi$ or $0 \le \Omega_0 < 2\pi$. This is called the *fundamental band*. Any frequency Ω outside this range is identical to a frequency in the fundamental band, where $\Omega_0 = \Omega - 2\pi m$, $m =$ integer.

Remark (Further Reduction in the Fundamental Band):

The signal $x_0[n] = \cos[(\pi + \Omega_0)n]$ is identical to the signal $x_1[n] = \cos[(\pi - \Omega_0)n]$. This can be shown as follows:

$$\cos[(\pi \pm \Omega_0)n] = \cos(\Omega_0 n) \cdot \cos(\pi n) \mp \sin(\Omega_0 n) \cdot \underbrace{\sin(\pi n)}_{=0}$$

$$= \cos(\Omega_0 n) \cdot \cos(\pi n)$$

i.e.

$$\boxed{\text{The frequency at } (\pi + \Omega_0) \text{ is identical to the frequency at } (\pi - \Omega_0).}$$

Therefore,

$$\boxed{\text{Digital frequencies are unique for } 0 \le \Omega < \pi \text{ (Baseband).}}$$

- **<u>Sampling a Continuous-Time (CT) Sinusoid</u>**

The CT signal $x(t) = \cos(\omega_0 t)$ has the <u>*frequency*</u> $f_0 = \frac{\omega_0}{2\pi}$ and the <u>*period*</u> $T_0 = \frac{2\pi}{\omega_0} = \frac{1}{f_0}$. Let us sample $x(t)$ every T_s seconds, i.e.

$$t = nT_s \quad \Rightarrow \quad x(nT_s) = x[n] = \cos(\underbrace{\omega_0 T_s}_{\Omega_0} n)$$

So, digital and analog frequencies are related by: $\Omega_0 = \omega_0 T_s$

Since, Ω_0 is unique for $0 \le \Omega_0 < \pi$, then:

$$\Omega_0 = \omega_0 T_s < \pi \quad \Longrightarrow \quad T_s < \frac{\pi}{\omega_0} = \frac{\pi}{2\pi/T_0} < \frac{T_0}{2}$$

$$\boxed{\Longrightarrow \quad f_s = \frac{1}{T_s} > 2f_0 \qquad \text{where } f_s = sampling\ frequency}$$

<u>Remark</u>: This condition is a special case of the well-known <u>*sampling theorem*</u>.

Remark: If a signal $x(t)$ has a band of frequencies and the highest frequency is f_h, then the sampling frequency must be $f_s > 2f_h$ for the exact reconstruction of $x(t)$ from its samples $x[n]$.

Remark (Frequency Aliasing): If, on the other hand, the sampling condition is not satisfied, i.e. $f_s < 2f_0$ or $f_0 > \frac{f_s}{2}$, then the original signal $x(t)$ will be reconstructed with a different frequency. For example, if $f_0 = \frac{f_s}{2} + \Delta f$, then the reconstructed signal $x_r(t) = \cos(2\pi f_1 t)$ will have the frequency $f_1 = \frac{f_s}{2} - \Delta f$, instead of f_0. This phenomenon is called *frequency aliasing*.