

Properties of Expectations

Moment Generating Functions [Ross S7.7]

Definition 36.1: The **moment generating function** (MGF) $M_X(t)$ of a random variable X is

$$\begin{aligned} M_X(t) &= E[e^{tX}] \\ &= \begin{cases} \sum_x e^{tx} p_X(x) & \text{discrete case} \\ \int_{-\infty}^{\infty} e^{tx} f_X(x) dx & \text{continuous case} \end{cases} \end{aligned}$$

Note: a closely related concept is the **characteristic function** defined as

$$\phi_X(t) = E[e^{itX}] \quad i = \sqrt{-1}$$

$M_X(t)$ is called moment generating function because we can find the moments $E[X^n]$ from it easily:

$$\begin{aligned} M'_X(t) &= \frac{d}{dt} E[e^{tX}] & [f'(t) = \text{derivative of } f(t)] \\ &= E \left[\frac{d}{dt} e^{tX} \right] \\ &= E [X e^{tX}] \end{aligned}$$

$$M_X^{(n)}(t) = E [X^n e^{tX}] \quad [f^{(n)}(t) = n\text{th derivative of } f(t)]$$

Hence

$$\begin{aligned} M'_X(0) &= E[X] \\ M_X^{(n)}(0) &= E[X^n] \end{aligned}$$

Example 36.1: Find $M_X(t)$ if $X \sim \text{Poisson}(\lambda)$. Use this to find $E[X]$, $E[X^2]$ and $\text{Var}[X]$.

Solution:

$$\begin{aligned}M_X(t) &= E[e^{tX}] \\&= \sum_{n=0}^{\infty} e^{tn} p_X(n) \\&= \sum_{n=0}^{\infty} e^{tn} \frac{\lambda^n}{n!} e^{-\lambda} \\&= e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda e^t)^n}{n!} \\&= e^{-\lambda} \exp(\lambda e^t) \\&= \exp(\lambda(e^t - 1))\end{aligned}$$

$$M'_X(t) = \lambda e^t \exp(\lambda(e^t - 1))$$

$$M''_X(t) = (\lambda e^t)^2 \exp(\lambda(e^t - 1)) + \lambda e^t \exp(\lambda(e^t - 1))$$

So

$$E[X] = M'_X(0) = \lambda$$

$$E[X^2] = M''_X(0) = \lambda^2 + \lambda$$

$$\begin{aligned}\Rightarrow \text{Var}[X] &= E[X^2] - (E[X])^2 \\&= \lambda\end{aligned}$$

Example 36.2: Find $M_X(t)$ if $X \sim \mathcal{N}(\mu, \sigma^2)$. Use this to find $E[X]$, $E[X^2]$ and $Var[X]$.

Solution: Let $Z = (X - \mu)/\sigma$. Then $Z \sim \mathcal{N}(0, 1)$ and:

$$\begin{aligned}M_Z(t) &= E[e^{tZ}] \\&= \int_{-\infty}^{\infty} e^{tz} f_Z(z) dz \\&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tz} e^{-z^2/2} dz \\&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{z^2 - 2zt}{2}\right) dz \\&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{(z-t)^2}{2} + \frac{t^2}{2}\right) dz \\&= e^{t^2/2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{(z-t)^2}{2}\right) dz \\&= e^{t^2/2}\end{aligned}$$

Since $X = \mu + \sigma Z$:

$$\begin{aligned}M_X(t) &= E[e^{tX}] \\&= E[e^{t(\mu + \sigma Z)}] \\&= E[e^{t\mu} e^{t\sigma Z}] \\&= e^{t\mu} E[e^{t\sigma Z}] \\&= e^{t\mu} M_Z(t\sigma) \\&= e^{t\mu} e^{\frac{t^2 \sigma^2}{2}} \\&= \exp\left(\frac{t^2 \sigma^2}{2} + \mu t\right)\end{aligned}$$

$$M'_X(t) = (\mu + t\sigma^2) \exp\left(\frac{t^2\sigma^2}{2} + \mu t\right)$$

$$M''_X(t) = (\mu + t\sigma^2)^2 \exp\left(\frac{t^2\sigma^2}{2} + \mu t\right) + \sigma^2 \exp\left(\frac{t^2\sigma^2}{2} + \mu t\right)$$

So

$$E[X] = M'_X(0) = \mu$$

$$E[X^2] = M''_X(0) = \mu^2 + \sigma^2$$

$$\begin{aligned}\Rightarrow \text{Var}[X] &= E[X^2] - (E[X])^2 \\ &= \sigma^2\end{aligned}$$

MGF of Sum of Independent Random Variables [Ross S7.7]

Let X and Y be independent random variables:

$$\begin{aligned}M_{X+Y}(t) &= E\left[e^{t(X+Y)}\right] \\ &= E\left[e^{tX}e^{tY}\right] \\ &= E\left[e^{tX}\right] E\left[e^{tY}\right] \quad \text{[since } X \text{ and } Y \text{ are independent]} \\ &= M_X(t)M_Y(t)\end{aligned}$$

Another useful fact: the distribution of X ($f_X(x)$ or $p_X(k)$) is uniquely determined by $M_X(t)$.

Your textbook has tables of MGF for different distributions.

Example 36.3: Let $X \sim \text{Poisson}(\lambda_1)$ and $Y \sim \text{Poisson}(\lambda_2)$ be independent. What is the distribution of $X + Y$?

Solution:

$$\begin{aligned}M_{X+Y}(t) &= M_X(t)M_Y(t) \\&= \exp(\lambda_1(e^t - 1)) \exp(\lambda_2(e^t - 1)) \\&= \exp((\lambda_1 + \lambda_2)(e^t - 1))\end{aligned}$$

So $X + Y$ is $\text{Poisson}(\lambda_1 + \lambda_2)$.

Example 36.4: Let $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ and $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$ be independent. What is the distribution of $X + Y$?

Solution:

$$\begin{aligned}M_{X+Y}(t) &= M_X(t)M_Y(t) \\&= \exp\left(\frac{t^2\sigma_X^2}{2} + \mu_X t\right) \exp\left(\frac{t^2\sigma_Y^2}{2} + \mu_Y t\right) \\&= \exp\left(\frac{t^2(\sigma_X^2 + \sigma_Y^2)}{2} + (\mu_X + \mu_Y)t\right)\end{aligned}$$

So $X + Y$ is $\sim \mathcal{N}(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$

Joint Moment Generating Functions

For random variables X_1, X_2, \dots, X_n , the joint moment generating function is defined as

$$M(t_1, \dots, t_n) = E[e^{t_1 X_1 + t_2 X_2 + \dots + t_n X_n}]$$

Then

$$M_{X_i}(t) = E[e^{tX_i}] = M(0, 0, \dots, t, 0, \dots, 0)$$

The joint MGF uniquely determines the joint pdf.

If X_1, \dots, X_n are independent then:

$$\begin{aligned} M(t_1, t_2, \dots, t_n) &= E \left[e^{t_1 X_1 + t_2 X_2 + \dots + t_n X_n} \right] \\ &= E \left[e^{t_1 X_1} \right] E \left[e^{t_2 X_2} \right] \dots E \left[e^{t_n X_n} \right] \\ &= M_{X_1}(t_1) M_{X_2}(t_2) \dots M_{X_n}(t_n) \end{aligned}$$

Since the joint MGF uniquely specifies the joint distribution, then X_1, \dots, X_n independent is equivalent to

$$M(t_1, t_2, \dots, t_n) = M_{X_1}(t_1) M_{X_2}(t_2) \dots M_{X_n}(t_n)$$

Example 36.5: $X \sim \mathcal{N}(\mu, \sigma^2)$ and $Y \sim \mathcal{N}(\mu, \sigma^2)$ are independent. Show that $X + Y$ and $X - Y$ are independent.

Solution:

$$\begin{aligned} E \left[e^{t(X+Y)+s(X-Y)} \right] &= E \left[e^{(t+s)X+(t-s)Y} \right] \\ &= E \left[e^{(t+s)X} \right] E \left[e^{(t-s)Y} \right] \\ &= e^{\mu(t+s)+\sigma^2(t+s)^2/2} e^{\mu(t-s)+\sigma^2(t-s)^2/2} \\ &= \underbrace{e^{2\mu t + \sigma^2 t^2}}_{M_{X+Y}(t)} \underbrace{e^{\sigma^2 s^2}}_{M_{X-Y}(s)} \end{aligned}$$

The 1st term is the MGF (in t) of $\mathcal{N}(2\mu, 2\sigma^2)$.

The 2nd term is the MGF (in s) for $\mathcal{N}(0, 2\sigma^2)$.