Random Variables (rv)

Mean and Variance of Poisson [Ross S4.7]

Intuition: Say $X \sim \mathsf{Binomial}(n, p)$ with $\lambda = np$, n large, and p small

Then:

$$\begin{aligned} E[X] &= np = \lambda \\ Var[X] &= np(1-p) \\ &= \lambda(1-p) \\ &\approx \lambda \end{aligned}$$

Exact: Let $X \sim \mathsf{Poisson}(\lambda)$. Then

$$\begin{aligned} E[X] &= \sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} e^{-\lambda} \\ &= \sum_{k=1}^{\infty} k \frac{\lambda^k}{k!} e^{-\lambda} \\ &= \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} e^{-\lambda} \\ &= \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda} \\ &= \lambda \sum_{\ell=0}^{\infty} \frac{\lambda^{\ell}}{\ell!} e^{-\lambda} && \ell = k-1 \\ &= \lambda \end{aligned}$$

$$\begin{aligned} E[X^2] &= \sum_{k=0}^{\infty} k^2 \frac{\lambda^k}{k!} e^{-\lambda} \\ &= \sum_{k=1}^{\infty} k^2 \frac{\lambda^k}{k!} e^{-\lambda} \\ &= \sum_{k=1}^{\infty} \frac{k\lambda^k}{(k-1)!} e^{-\lambda} \\ &= \sum_{\ell=0}^{\infty} \frac{(\ell+1)\lambda^{\ell+1}}{\ell!} e^{-\lambda} && \ell = k-1 \\ &= \lambda \left(\underbrace{\sum_{\ell=0}^{\infty} \frac{\ell\, \lambda^{\ell}}{\ell!} e^{-\lambda}}_{\lambda} + \underbrace{\sum_{\ell=0}^{\infty} \frac{\lambda^{\ell}}{\ell!} e^{-\lambda}}_{1} \right) \\ &= \lambda(1+\lambda) \end{aligned}$$

So
$$\begin{aligned} Var[X] &= E[X^2] - (E[X])^2 \\ &= \lambda(1+\lambda) - (\lambda)^2 \\ &= \lambda \end{aligned}$$

Example 13.1: A radioactive substance with a large # of atoms emits 3.2 alpha particles per second on average. What is the probability that no more than 2 alpha particles are emitted in a 1 second interval?

Solution: If X = # of emitted particles in 1 second, then X is Poisson with $E[X] = 3.2 = \lambda$

$$\begin{aligned} P[X \le 2] &= P[X=0] + P[X=1] + P[X=2] \\ &= e^{-3.2} + 3.2e^{-3.2} + \frac{(3.2)^2}{2!} e^{-3.2} \\ &\approx 0.3799 \end{aligned}$$

D) The geometric random variable [Ross 4.8.1]

Consider an infinite sequence of independent Bernoulli(p) trials.

Let X be trial # of first outcome that is a 1.

X is called **geometric** with parameter p, denoted $X \sim \mathsf{Geometric}(p)$

$$\begin{aligned} p_X(k) &= P[(k-1) \text{ zeros followed by a one}] && \text{for } k = 1, 2, \ldots \\ &= \begin{cases} (1-p)^{k-1}p & k \ge 1 \\ 0 & \text{else} \end{cases} \end{aligned}$$

Example 13.2: A bag contains 2 white balls and 3 black balls. Balls are randomly drawn until a black ball is drawn. The selected ball is replaced before the next draw.

a) What is the probability that exactly n draws are needed?

b) What is the probability that at least k draws are needed?

Solution: In each draw, the probability of getting a black ball is $3/5 = 0.6$.

If X = # of draws until a black ball, then $X \sim \mathsf{Geometric}(p)$ with $p = 0.6$.

a)

$$\begin{aligned} P[X = n] &= \left(1 - \frac{3}{5}\right)^{n-1} \times \frac{3}{5} \\ &= \left(\frac{2}{5}\right)^{n-1} \times \frac{3}{5} \end{aligned}$$

b)

$$\begin{aligned} P[X \ge k] &= \sum_{n=k}^{\infty} P[X = n] \\ &= \frac{3}{5} \times \sum_{n=k}^{\infty} \left(\frac{2}{5}\right)^{n-1} \\ &= \frac{3}{5} \times \left(\frac{2}{5}\right)^{k-1} \sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^{n} \\ &= \frac{3}{5} \times \left(\frac{2}{5}\right)^{k-1} \frac{1}{1 - 2/5} \\ &= \left(\frac{2}{5}\right)^{k-1} && (= P[\text{first } k-1 \text{ draws are white}]) \end{aligned}$$

Mean and Variance

If $X \sim \mathsf{Geometric}(p)$, then:

$$\begin{aligned} E[X] &= \sum_{k=1}^{\infty} k(1-p)^{k-1}p \\ &= \cdots && \text{[see Ross example 4.8b]} \\ &= \frac{1}{p} \end{aligned}$$

$$\begin{aligned} E[X^2] &= \sum_{k=1}^{\infty} k^2(1-p)^{k-1}p \\ &= \cdots && \text{[see Ross example 4.8c]} \\ &= \frac{2-p}{p^2} \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} Var[X] &= E[X^2] - (E[X])^2 \\ &= \frac{2-p}{p^2} - \left(\frac{1}{p}\right)^2 \\ &= \frac{1-p}{p^2} \end{aligned}$$