

Axioms (or Laws) of Probability [Ross S2.3, S2.4]

We wish to assign to each event E a probability, denoted $P[E]$ (or $P(E)$).

How do we determine it?

Frequentist approach: Let $n(E)$ be number of occurrences of E in n repeated experiments. Then define

$$P[E] = \lim_{n \rightarrow \infty} \frac{n(E)}{n}. \quad (3.1)$$

Does this limit exist? In what sense?

Modern Approach: Instead, assume that certain rules (axioms) must hold.

$$[A1] \quad 0 \leq P[E] \leq 1$$

$$[A2] \quad P[S] = 1$$

[A3] If E_1, E_2, \dots are disjoint (i.e., mutually exclusive), then

$$P[E_1 \cup E_2 \cup \dots] = \sum_{i=1}^{\infty} P[E_i]$$

Consequences of axioms:

Corollary 3.1 $P[\emptyset] = 0$.

Why? Let $E_1 = S, E_2 = \emptyset, E_3 = \emptyset, \dots$

Then E_1, E_2, E_3, \dots are disjoint.

Hence,

$$\begin{aligned}P[E_1 \cup E_2 \cup E_3 \cup \cdots] &= P[E_1] + P[E_2] + P[E_3] + \cdots \\&= P[S] + P[\emptyset] + P[\emptyset] + \cdots \\&= 1 + P[\emptyset] + P[\emptyset] + \cdots\end{aligned}$$

But this sum must be ≤ 1 , so $P[\emptyset] = 0$.

Corollary 3.2 *Say E_1, E_2, \dots, E_n are disjoint. Then*

$$P[\cup_{i=1}^n E_i] = \sum_{i=1}^n P[E_i]$$

Why? Take $\emptyset = E_{n+1} = E_{n+2} = \cdots$. Then

$$\begin{aligned}P[\cup_{i=1}^n E_i] &= P[\cup_{i=1}^{\infty} E_i] \\&= \sum_{i=1}^{\infty} P[E_i] \\&= \sum_{i=1}^n P[E_i] + \sum_{i=n+1}^{\infty} P[E_i] \\&= \sum_{i=1}^n P[E_i]\end{aligned}$$

Example 3.1: If each of roulette's 38 possible outcomes are equally likely, then

- 1) $P[00] = P[0] = P[1] \cdots = P[36]$
- 2) $1 = P[\{00, 0, 1, \cdots, 36\}] = P[00] + P[0] + \cdots + P[36]$

Hence,

$$P[00] = P[0] = \cdots = P[36] = 1/38$$

So,

$$\begin{aligned} P[\text{even}] &= P[\{2, 4, \dots, 36\}] \\ &= P[2] + P[4] + \cdots + P[36] \\ &= 18/38 = 9/19 \end{aligned}$$

Corollary 3.3 $P[E^c] = 1 - P[E]$

Why? E and E^c are disjoint, and $E \cup E^c = S$.

$$\Rightarrow 1 = P[S] = P[E \cup E^c] = P[E] + P[E^c]$$

So $P[E^c] = 1 - P[E]$

Corollary 3.4 If $E \subset F$ then $P[E] \leq P[F]$.

Why? Since $E \subset F$, then

- $F = SF = (E \cup E^c)F = EF \cup E^cF = E \cup E^cF$
- E and E^cF are disjoint

Then

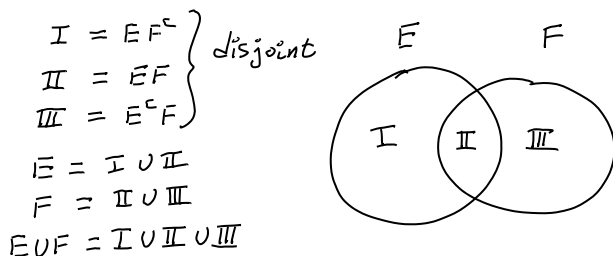
$$\begin{aligned} P[F] &= P[E] + \underbrace{P[E^cF]}_{\geq 0} \\ \Rightarrow P[F] &\geq P[E] \end{aligned}$$

Example 3.2: In roulette, $\text{odd} \subset \text{even}^c$, so

$$\underbrace{P[\text{odd}]}_{9/19} \leq \underbrace{P[\text{even}^c]}_{10/19}$$

Corollary 3.5 $P[E \cup F] = P[E] + P[F] - P[E \cap F]$

Why?



$$\begin{aligned} P[E] + P[F] &= P[I \cup II] + P[II \cup III] \\ &= P[I] + P[II] + P[II] + P[III] \\ &= P[I \cup II \cup III] + P[II] \\ &= P[E \cup F] + P[EF] \end{aligned}$$

Example 3.3: After 5 years, a car may need

- i) new brakes with prob. 0.5
- ii) new tires with prob. 0.4
- iii) both with prob. 0.3

What is probability it needs neither?

Solution: $B = \{\text{needs brakes}\}, T = \{\text{needs tires}\}$

$$P[B] = 0.5 \quad P[T] = 0.4 \quad P[BT] = 0.3$$

$$\begin{aligned} P[\{\text{needs neither}\}] &= P[B^c T^c] \\ &= P[(B \cup T)^c] \\ &= 1 - P[B \cup T] \\ &= 1 - (P[B] + P[T] - P[BT]) \\ &= 1 - (0.5 + 0.4 - 0.3) \\ &= 0.4 \end{aligned}$$

Can we generalize the $P[E \cup F]$ idea of Corollary 3.5? Yes!

$$\begin{aligned} P[E \cup F \cup G] &= P[(E \cup F) \cup G] \\ &= P[(E \cup F)] + P[G] - P[(E \cup F)G] \\ &= P[E] + P[F] - P[EF] + P[G] - P[EG \cup FG] \\ &= P[E] + P[F] + P[G] - P[EF] \\ &\quad - (P[EG] + P[FG] - P[EGFG]) \\ &= P[E] + P[F] + P[G] \\ &\quad - P[EF] - P[EG] - P[FG] \\ &\quad + P[EFG] \end{aligned}$$

Proposition 3.1 *Inclusion/Exclusion Principle*

$$P[E_1 \cup E_2 \cup \dots \cup E_n]$$

$$= P[E_1] + P[E_2] + \dots + P[E_n]$$

include all events

$$- \sum_{1 \leq i_1 < i_2 \leq n} P[E_{i_1} E_{i_2}]$$

exclude intersections of pairs

$$\begin{aligned}
& + \sum_{1 \leq i_1 < i_2 < i_3 \leq n} P[E_{i_1} E_{i_2} E_{i_3}] && \text{include triple intersections} \\
& \vdots && \vdots \\
& + (-1)^{r+1} \sum_{1 \leq i_1 < \dots < i_r \leq n} P[E_{i_1} E_{i_2} \dots E_{i_r}] && (\text{in/ex})\text{clude } r\text{-way intersections} \\
& \vdots && \vdots \\
& + (-1)^{n+1} P[E_1 E_2 \dots E_n] && (\text{in/ex})\text{clude } n\text{-way intersection}
\end{aligned}$$

Proof: See textbook.