Conditional Probability and Independence

Independent Events [Ross S3.4]

Definition 7.1: Events E and F are called **independent** if

$$P[EF] = P[E]P[F]$$

Two events that are not independent are said to be **dependent**.

From previous examples, $P[E|F]$ is not necessarily the same as $P[E]$.

But, if E and F are independent (and $P[F] > 0$):

$$P[E|F] = \frac{P[EF]}{P[F]} = \frac{P[E]P[F]}{P[F]} = P[E]$$

Example 7.1: Two 6-sided dice are rolled. Let

$E_1 = \{\text{sum is 6}\}$
$E_2 = \{\text{sum is 7}\}$
$F = \{\text{1st die is 4}\}$
$G = \{\text{2nd die is 3}\}$

Then:

$$\begin{aligned} P[E_1F] &= P[(4,2)] = 1/36, \quad P[E_1]P[F] = 5/36 \times 1/6 \neq 1/36 \\ P[E_2F] &= P[(4,3)] = 1/36, \quad P[E_2]P[F] = 1/6 \times 1/6 = 1/36 \end{aligned}$$

So E_1 and F are not independent, but E_2 and F are independent.

Similarly, E_2 and G are independent.

Example 7.2: Say $EF = \emptyset$ with $P[E] > 0$ and $P[F] > 0$. Are E and F independent?

Solution: No!

$$P[E|F] = \frac{P[EF]}{P[F]} = \frac{P[\emptyset]}{P[F]} = 0$$

but $P[E] > 0$.

Proposition 7.1 *If E and F are independent, then E and F^c are independent*

Why?

$$\begin{aligned} P[E] &= P[EF \cup EF^c] \\ &= P[EF] + P[EF^c] \\ &= P[E]P[F] + P[EF^c] \end{aligned}$$

$$\Rightarrow \quad P[EF^c] = P[E] - P[E]P[F] = P[E](1 - P[F]) = P[E]P[F^c]$$

Example 7.3: If E is independent of F and E is independent of G, is E independent of FG?

Solution: Not necessarily. In Example 7.1:

E_2 is independent of F and E_2 is independent of G

Now $P[E_2] = 6/36$, but $P[E_2|FG] = P[\{\text{sum is } 7\}|(4,3)] = 1$.

Definition 7.2: Events E and F are called conditionally independent given G when

$$P[EF|G] = P[E|G]P[F|G].$$

What does this mean?

$$\begin{aligned} P[E|G]P[F|G] &= P[EF|G] \\ &= \frac{P[EFG]}{P[G]} \\ &= \frac{P[E|FG] \times P[F|G] \times P[G]}{P[G]} \end{aligned}$$

So, this is equivalent to $P[E|FG] = P[E|G]$.

In words: If G is known to have occured, the additional information that F occured does not change the probability of E.

Definition 7.3: The 3 events E, F and G are said to be independent if

$$\begin{aligned} P[EFG] &= P[E]P[F]P[G] \\ P[EF] &= P[E]P[F] \\ P[EG] &= P[E]P[G] \\ P[FG] &= P[F]P[G] \end{aligned}$$

Now, E is independent of any event formed from F and G.

Example 7.4:

$$\begin{aligned} P[E(F \cup G)] &= P[EF \cup EG] \\ &= P[EF] + P[EG] - P[EF \cap EG] \\ &= P[E]P[F] + P[E]P[G] - P[E]P[FG] \end{aligned}$$

$$
\begin{aligned}
&= P[E](P[F] + P[G] - P[FG]) \\
&= P[E]P[F \cup G]
\end{aligned}
$$

Definition 7.4: Events $E_1, E_2, \ldots, E_n$ are said to be independent if

$$P\left[\bigcap_{i \in A} E_i\right] = \prod_{i \in A} P[E_i] \tag{7.1}$$

for every $A \subset \{1, ..., n\}$.

Definition 7.5: An infinite set of events $E_1, E_2, \ldots$ is independent if every finite subset is independent.

Example 7.5: A system has n components. Each component functions/fails independently of any other. Component i has probability p_i of functioning. If at least one component functions, the system functions. What is the probability that the system functions?

Solution: Let $A_i = \{\text{component } i \text{ functions}\}$.

$$
\begin{aligned}
P[\text{system functions}] &= 1 - P[\text{system does not function}] \\
&= 1 - P[\text{all components fail}] \\
&= 1 - P[\cap_i A_i^c] \\
&= 1 - P[A_1^c]P[A_2^c] \cdots P[A_n^c] \qquad \text{[by independence]} \\
&= 1 - (1 - p_1)(1 - p_2) \cdots (1 - p_n)
\end{aligned}
$$

Sometimes each E_i is the outcome of one instance of a sequence of repeated sub-experiments, e.g., $E_i = \{i\text{-th coin toss is heads}\}$.

These sub-experiments are often called **trials** (or **repeated trials**).

Example 7.6: Independent trials that consist of repeatedly rolling a pair of fair dice are performed. The outcome of a roll is the sum of the dice.

What is the prob. of
$F = \{$an outcome of 5 eventually occurs, and there was no 7 before this$\}$?

Solution: Let

$E_n = \{$no 5 or 7 appears on first $n-1$ rolls, and 5 appears on n-th roll$\}$.

Then $E_1, E_2, \ldots$ are mutually exclusive and $F = E_1 \cup E_2 \cup \cdots$.

Now:

$$\begin{aligned} P[\text{roll a 5}] &= 4/36 = 1/9 \\ P[\text{roll a 7}] &= 6/36 \\ P[\text{not roll a 5 or 7}] &= 1 - 10/36 = 13/18 \end{aligned}$$

and

$$\begin{aligned} P[E_n] &= P[\{\text{no 5 or 7 on 1st roll}\} \\ &\qquad \cap \cdots \\ &\qquad \cap \{\text{no 5 or 7 on } n-1 \text{ roll}\} \\ &\qquad \cap \{\text{5 on } n\text{th roll}\}] \\ &= P[\{\text{no 5 or 7 on 1st roll}\}] \\ &\qquad \times \cdots \\ &\qquad \times P[\{\text{no 5 or 7 on } n-1 \text{ roll}\}] \\ &\qquad \times P[\{\text{5 on } n\text{th roll}\}] \\ &= \frac{1}{9}\left(\frac{13}{18}\right)^{n-1} \end{aligned}$$

Then:

$$
\begin{aligned}
P[\cup_{n=1}^{\infty} E_n] &= \sum_{n=1}^{\infty} P[E_n] \\
&= \frac{1}{9} \times \sum_{n=1}^{\infty} \left(\frac{13}{18}\right)^{n-1} \\
&= \frac{1}{9} \times \frac{1}{1 - 13/18}
\end{aligned}
$$