Properties of Expectations

Expectation of Sums of Random Variables [Ross S7.2]

Recall that the mean value of X is

$$E[X] = \begin{cases} \sum_x x p_X(x) & X \text{ is discrete} \\ \int_{-\infty}^{\infty} x f_X(x) dx & X \text{ is continuous} \end{cases}$$

Proposition 30.1 *Let X and Y be two random variables. Let $g(x,y)$ be a function. Then*

$$E[g(X,Y)] = \begin{cases} \sum_y \sum_x g(x,y) p_{XY}(x,y) & X, Y \text{ are discrete} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{XY}(x,y) dxdy & X, Y \text{ are continuous} \end{cases}$$

Why?

[Only show for continuous case and $g(x,y)$ is non-negative]

Recall from Proposition 16.2:

$$E[Z] = \int_0^{\infty} P[Z > t] dt$$

So

$$\begin{aligned} E[g(X,Y)] &= \int_0^{\infty} P[g(X,Y) > t] dt \\ &= \int_0^{\infty} \iint_{(x,y):g(x,y)>t} f_{XY}(x,y) dxdy \, dt \\ &= \iint_{\mathbb{R}^2} \int_0^{g(x,y)} f_{XY}(x,y) dt \, dxdy \\ &= \iint_{\mathbb{R}^2} g(x,y) f_{XY}(x,y) dxdy \end{aligned}$$

Example 30.1: The positions $X \sim U(0,L)$ and $Y \sim U(0,L)$ of two persons on a road are independent. What is the mean distance between them?

Solution: Here,

$$f_{XY}(x,y) = \begin{cases} \frac{1}{L^2} & 0 < x < L, 0 < y < L \\ 0 & \text{else} \end{cases}$$

We want

$$\begin{aligned} E[|X-Y|] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x-y| f_{XY}(x,y) dxdy \\ &= \frac{1}{L^2}\int_0^L \int_0^L |x-y| dxdy \end{aligned}$$

Now $$\begin{aligned} \int_0^L |x-y| dx &= \int_0^y (y-x) dx + \int_y^L (x-y) dx \\ &= \frac{L^2}{2} + y^2 - yL \end{aligned}$$

So $$\begin{aligned} E[|X-Y|] &= \frac{1}{L^2}\int_0^L \frac{L^2}{2} + y^2 - yL \, dy \\ &= \frac{L}{3} \end{aligned}$$

Example 30.2: Show $E[X+Y] = E[X] + E[Y]$.

Solution: [Continuous case only, discrete is similar]

With $g(x,y) = x + y$:

$$\begin{aligned} E[X+Y] &= E[g(X,Y)] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{XY}(x,y) dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y) f_{XY}(x,y) dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{XY}(x,y) dxdy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{XY}(x,y) dxdy \\ &= E[X] + E[Y] \end{aligned}$$

Note: by induction, $E[X_1 + \cdots + X_n] = E[X_1] + \cdots + E[X_n]$.

Example 30.3: Let $X_1, X_2, \ldots, X_n$ be iid with (common) mean μ. The quantity

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

is called the **sample mean**. What is $E[\bar{X}]$?

Solution:

$$\begin{aligned} E[\bar{X}] &= E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] \\ &= \frac{1}{n} E\left[\sum_{i=1}^{n} X_i\right] \\ &= \frac{1}{n}\sum_{i=1}^{n} E[X_i] \\ &= \frac{1}{n}\sum_{i=1}^{n} \mu \\ &= \mu \end{aligned}$$

Example 30.4: 10 friends play a game. Each has a ball and picks one of 10 targets randomly, independently of the others. Each has probability p of hitting their chosen target.

What is the expected number of targets not hit?

Solution:

Let $X_i = 1$ if target i is not hit, and 0 otherwise; X = # targets not hit.

Each person independently hits target i with probability $p/10$.

So $$\begin{aligned} P[X_i = 1] &= \left(1 - \frac{p}{10}\right)^{10} \\ E[X_i] &= 1 \times P[X_i = 1] + 0 \times P[X_i = 0] \\ &= \left(1 - \frac{p}{10}\right)^{10} \end{aligned}$$

$$\begin{aligned} E[X] &= E[X_1 + \cdots + X_{10}] \\ &= E[X_1] + \cdots + E[X_{10}] \\ &= 10\left(1 - \frac{p}{10}\right)^{10} \end{aligned}$$