Properties of Expectations

Moment Generating Functions [Ross S7.7]

Definition 36.1: The **moment generating function** (MGF) $M_X(t)$ of a random variable X is

$$
\begin{aligned}
M_X(t) &= E[e^{tX}] \\
&= \begin{cases} \sum_x e^{tx} p_X(x) & \text{discrete case} \\ \int_{-\infty}^{\infty} e^{tx} f_X(x)dx & \text{continuous case} \end{cases}
\end{aligned}
$$

Note: a closely related concept is the **characteristic function** defined as

$$\phi_X(t) = E[e^{itX}] \qquad i = \sqrt{-1}$$

$M_X(t)$ is called moment generating function because we can find the moments $E[X^n]$ from it easily:

$$
\begin{aligned}
M'_X(t) &= \frac{d}{dt} E[e^{tX}] & [f'(t) = \text{derivative of } f(t)] \\
&= E\left[\frac{d}{dt} e^{tX}\right] \\
&= E\left[X e^{tX}\right]
\end{aligned}
$$

$$M_X^{(n)}(t) = E\left[X^n e^{tX}\right] \qquad [f^{(n)}(t) = n\text{th derivative of } f(t)]$$

Hence
$$
\begin{aligned}
M'_X(0) &= E[X] \\
M_X^{(n)}(0) &= E[X^n]
\end{aligned}
$$

Example 36.1: Find $M_X(t)$ if $X \sim \text{Poisson}(\lambda)$. Use this to find $E[X]$, $E[X^2]$ and $Var[X]$.

Solution:

$$
\begin{aligned}
M_X(t) &= E[e^{tX}] \\
&= \sum_{n=0}^{\infty} e^{tn} p_X(n) \\
&= \sum_{n=0}^{\infty} e^{tn} \frac{\lambda^n}{n!} e^{-\lambda} \\
&= e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda e^t)^n}{n!} \\
&= e^{-\lambda} \exp(\lambda e^t) \\
&= \exp(\ \lambda(e^t - 1)\ )
\end{aligned}
$$

$$
\begin{aligned}
M'_X(t) &= \lambda e^t \exp(\ \lambda(e^t-1)\ ) \\
M''_X(t) &= (\lambda e^t)^2 \exp(\ \lambda(e^t-1)\ ) + \lambda e^t \exp(\ \lambda(e^t-1)\ )
\end{aligned}
$$

So

$$
\begin{aligned}
E[X] &= M'_X(0) = \lambda \\
E[X^2] &= M''_X(0) = \lambda^2 + \lambda \\
\Rightarrow Var[X] &= E[X^2] - (E[X])^2 \\
&= \lambda
\end{aligned}
$$

Example 36.2: Find $M_X(t)$ if $X \sim \mathcal{N}(\mu, \sigma^2)$. Use this to find $E[X]$, $E[X^2]$ and $Var[X]$.

Solution: Let $Z = (X-\mu)/\sigma$. Then $Z \sim \mathcal{N}(0,1)$ and:

$$
\begin{aligned}
M_Z(t) &= E[e^{tZ}] \\
&= \int_{-\infty}^{\infty} e^{tz} f_Z(z) dz \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tz} e^{-z^2/2} dz \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{z^2 - 2zt}{2}\right) dz \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{(z-t)^2}{2} + \frac{t^2}{2}\right) dz \\
&= e^{t^2/2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{(z-t)^2}{2}\right) dz \\
&= e^{t^2/2}
\end{aligned}
$$

Since $X = \mu + \sigma Z$:

$$
\begin{aligned}
M_X(t) &= E[e^{tX}] \\
&= E[e^{t(\mu+\sigma Z)}] \\
&= E[e^{t\mu} e^{t\sigma Z}] \\
&= e^{t\mu} E[e^{t\sigma Z}] \\
&= e^{t\mu} M_Z(t\sigma) \\
&= e^{t\mu} e^{\frac{t^2\sigma^2}{2}} \\
&= \exp\left(\frac{t^2\sigma^2}{2} + \mu t\right)
\end{aligned}
$$

$$
\begin{aligned}
M'_X(t) &= (\mu + t\sigma^2) \exp\left(\frac{t^2\sigma^2}{2} + \mu t\right) \\
M''_X(t) &= (\mu + t\sigma^2)^2 \exp\left(\frac{t^2\sigma^2}{2} + \mu t\right) + \sigma^2 \exp\left(\frac{t^2\sigma^2}{2} + \mu t\right)
\end{aligned}
$$

So

$$
\begin{aligned}
E[X] &= M'_X(0) = \mu \\
E[X^2] &= M''_X(0) = \mu^2 + \sigma^2 \\
\Rightarrow Var[X] &= E[X^2] - (E[X])^2 \\
&= \sigma^2
\end{aligned}
$$

MGF of Sum of Independent Random Variables [Ross S7.7]

Let X and Y be independent random variables:

$$
\begin{aligned}
M_{X+Y}(t) &= E\left[e^{t(X+Y)}\right] \\
&= E\left[e^{tX} e^{tY}\right] \\
&= E\left[e^{tX}\right] E\left[e^{tY}\right] & [\text{since } X \text{ and } Y \text{ are independent}] \\
&= M_X(t) M_Y(t)
\end{aligned}
$$

Another useful fact: the distribution of X ($f_X(x)$ or $p_X(k)$) is uniquely determined by $M_X(t)$.

Your textbook has tables of MGF for different distributions.

Example 36.3: Let $X \sim \text{Poisson}(\lambda_1)$ and $Y \sim \text{Poisson}(\lambda_2)$ be independent. What is the distribution of $X+Y$?

Solution:

$$
\begin{aligned}
M_{X+Y}(t) &= M_X(t) M_Y(t) \\
&= \exp(\ \lambda_1(e^t-1)\ ) \exp(\ \lambda_2(e^t-1)\ ) \\
&= \exp(\ (\lambda_1+\lambda_2)(e^t-1)\ )
\end{aligned}
$$

So $X+Y$ is $\text{Poisson}(\lambda_1+\lambda_2)$.

Example 36.4: Let $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ and $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$ be independent. What is the distribution of $X+Y$?

Solution:

$$
\begin{aligned}
M_{X+Y}(t) &= M_X(t) M_Y(t) \\
&= \exp\left(\frac{t^2\sigma_X^2}{2} + \mu_X t\right) \exp\left(\frac{t^2\sigma_Y^2}{2} + \mu_Y t\right) \\
&= \exp\left(\frac{t^2(\sigma_X^2 + \sigma_Y^2)}{2} + (\mu_X + \mu_Y)t\right)
\end{aligned}
$$

So $X+Y$ is $\sim \mathcal{N}(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$

Joint Moment Generating Functions

For random variables $X_1, X_2, \ldots, X_n$, the joint moment generating function is defined as

$$M(t_1, \ldots, t_n) = E\left[e^{t_1X_1 + t_2X_2 + \cdots + t_nX_n}\right]$$

Then

$$M_{X_i}(t) = E[e^{tX_i}] = M(0, 0, \ldots, t, 0, \ldots, 0)$$

The joint MGF uniquely determines the joint pdf.

If $X_1, \ldots, X_n$ are independent then:

$$
\begin{aligned}
M(t_1, t_2, \ldots, t_n) &= E\left[e^{t_1X_1 + t_2X_2 + \cdots + t_nX_n}\right] \\
&= E\left[e^{t_1X_1}\right] E\left[e^{t_2X_2}\right] \cdots E\left[e^{t_nX_n}\right] \\
&= M_{X_1}(t_1) M_{X_2}(t_2) \cdots M_{X_n}(t_n)
\end{aligned}
$$

Since the joint MGF uniquely specifies the joint distribution, then $X_1, \ldots, X_n$ independent is equivalent to

$$M(t_1, t_2, \ldots, t_n) = M_{X_1}(t_1) M_{X_2}(t_2) \cdots M_{X_n}(t_n)$$

Example 36.5: $X \sim \mathcal{N}(\mu, \sigma^2)$ and $Y \sim \mathcal{N}(\mu, \sigma^2)$ are independent. Show that $X+Y$ and $X-Y$ are independent.

Solution:

$$
\begin{aligned}
E\left[e^{t(X+Y)+s(X-Y)}\right] &= E\left[e^{(t+s)X + (t-s)Y}\right] \\
&= E\left[e^{(t+s)X}\right] E\left[e^{(t-s)Y}\right] \\
&= e^{\mu(t+s) + \sigma^2(t+s)^2/2} e^{\mu(t-s) + \sigma^2(t-s)^2/2} \\
&= \underbrace{e^{2\mu t + \sigma^2 t^2}}_{M_{X+Y}(t)} \underbrace{e^{\sigma^2 s^2}}_{M_{X-Y}(t)}
\end{aligned}
$$

The 1st term is the MGF (in t) of $\mathcal{N}(2\mu, 2\sigma^2)$.

The 2nd term is the MGF (in s) for $\mathcal{N}(0, 2\sigma^2)$.