Jointly Distributed Random Variables

Example 25.1: Let X and Y have joint density

$$f_{XY}(x,y) = \begin{cases} 6e^{-2x}e^{-3y} & x > 0,\ y > 0 \\ 0 & \text{else} \end{cases}$$

Are X and Y independent?

Solution: Compute the marginals $f_X(x)$ and $f_Y(y)$, and see if

$$f_{XY}(x,y) = f_X(x)f_Y(y)$$

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy \\ &= \begin{cases} \int_0^{\infty} 6e^{-2x}e^{-3y}\,dy & x > 0 \\ 0 & x \leq 0 \end{cases} \\ &= \begin{cases} 2e^{-2x} & x > 0 \\ 0 & x \leq 0 \end{cases} \end{aligned}$$

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dx \\ &= \begin{cases} 3e^{-3y} & y > 0 \\ 0 & y \leq 0 \end{cases} \end{aligned}$$

Since $f_{XY}(x,y) = f_X(x)f_Y(y)$, then X and Y are independent.

Alternate method: Notice that

$$f_{XY}(x,y) = h(x)g(y)$$

where

$$h(x) = \begin{cases} e^{-2x} & x > 0 \\ 0 & \text{else} \end{cases} \qquad g(y) = \begin{cases} 6e^{-3y} & y > 0 \\ 0 & \text{else} \end{cases}$$

So,
$$\begin{aligned} 1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\ dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x)g(y)\ dxdy \\ &= \underbrace{\int_{-\infty}^{\infty} h(x)\ dx}_{C_1}\underbrace{\int_{-\infty}^{\infty} g(y)\ dy}_{C_2} \\ &= C_1C_2 \end{aligned}$$

Now,
$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dy = \int_{-\infty}^{\infty} h(x)g(y)dy = C_2h(x) \\ f_Y(y) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dx = \int_{-\infty}^{\infty} h(x)g(y)dx = C_1g(y) \end{aligned}$$

Finally,
$$\begin{aligned} f_X(x)f_Y(y) &= C_1C_2h(x)g(y) \\ &= h(x)g(y) \\ &= f_{XY}(x,y) \end{aligned}$$

So if you can factor $f_{XY}(x,y) = h(x)g(y)$, then X and Y are independent!

And if X and Y are independent, then $f_{XY}(x,y)$ can be factored as $f_{XY}(x,y) = f_X(x)f_Y(y)$.

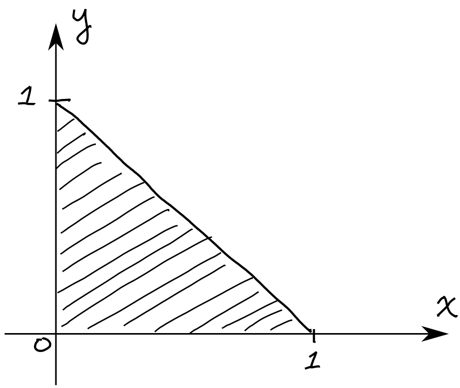
Proposition 25.1 *X and Y are independent if and only if $f_{XY}(x, y) = h(x)g(y)$ for some $h(x)$ and $g(y)$.*

Example 25.2: Let X and Y have joint pdf

$$f_{XY}(x, y) = \begin{cases} 24xy & x > 0,\ y > 0,\ 0 < x + y < 1 \\ 0 & \text{else} \end{cases}$$

Are X and Y independent?

Solution: No. Below is the region where $f_{XY}(x, y) > 0$.



This region cannot be where $h(x)g(y) > 0$ for any choice of $h(x)$ and $g(y)$.

Example 25.3: Two people decide to meet. Each arrives independently and uniformly between noon and 1pm.

What is the probability that the first to arrive waits longer than 10 min for the second the arrive?

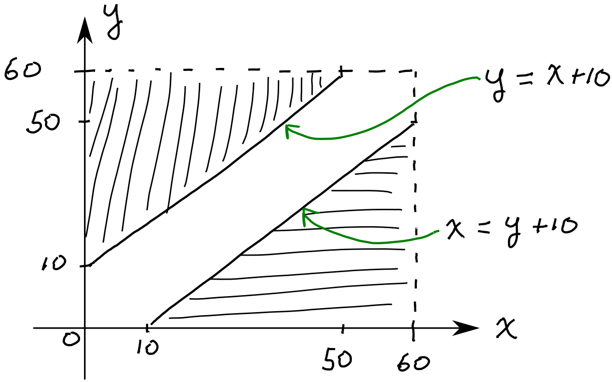
Solution: Let X and Y be times at which they arrive in minutes past noon.

Time of first to arrive is $\min(X, Y)$

Time of last to arrive is $\max(X, Y)$

We want $P[\underbrace{\{\max(X, Y) > \min(X, Y) + 10\}}_{E}]$

and $E = \{Y > X + 10\} \cup \{X > Y + 10\}$



$$P[E] = 2P[Y > X + 10]$$

$$= 2 \iint_{y > x+10} f_{XY}(x, y)dxdy$$

$$= 2 \iint_{y > x+10} f_X(x)f_Y(y)dxdy$$

$$= 2 \iint_{\substack{y > x+10 \\ 0 < x < 60 \\ 0 < y < 60}} \left(\frac{1}{60}\right)^2 dydx$$

$$= \frac{1}{1800} \int_0^{50} \int_{x+10}^{60} dydx$$

$$
\begin{aligned}
&= \frac{1}{1800} \int_0^{50} (50 - x)dx \\
&= \frac{25}{36}
\end{aligned}
$$