Jointly Distributed Random Variables

Multiple Joint Random Variables [Ross S6.1]

The joint CDF of random variables $X_1, X_2, \dots, X_n$ is

$$F_{X_1,X_2,\dots,X_n}(a_1, a_2, \dots, a_n) = P[X_1 \le a_1, X_2 \le a_2, \dots, X_n \le a_n]$$

If $X_1, X_2, \dots, X_n$ are discrete, their joint PMF is:

$$p_{X_1,X_2,\dots,X_n}(a_1, a_2, \dots, a_n) = P[X_1 = a_1, X_2 = a_2, \dots, X_n = a_n]$$

Also

1) $$\begin{aligned} p_{X_2,\dots,X_n}(a_2, \dots, a_n) &= P[X_2 = a_2, \dots, X_n = a_n] \\ &= \sum_{a_1} P[X_1 = a_1, X_2 = a_2, \dots, X_n = a_n] \\ &= \sum_{a_1} p_{X_1,X_2,\dots,X_n}(a_1, a_2, \dots, a_n) \qquad \text{[marginalization]} \end{aligned}$$

2) $$\sum_{a_1,a_2,\dots,a_n} p_{X_1,X_2,\dots,X_n}(a_1, a_2, \dots, a_n) = 1$$

$X_1, \dots, X_n$ are continuous rv's if there is a non-negative $f_{X_1,\dots,X_n}(x_1, \dots, x_n)$ such that for all $C \subset \mathbb{R}^n$:

$$P[(X_1, \dots, X_n) \in C] = \int \cdots \int_C f_{X_1,\dots,X_n}(x_1, \dots, x_n)\, dx_1 \cdots dx_n$$

So,

$$\begin{aligned} P[X_1 \in A_1, \dots, X_n \in A_n] &= P[(X_1, \dots, X_n) \in A_1 \times \cdots \times A_n] \\ &= \int \cdots \int_{A_1 \times \cdots \times A_n} f_{X_1,\dots,X_n}(x_1, \dots, x_n)\, dx_1 \cdots dx_n \\ &= \int_{A_n} \cdots \int_{A_1} f_{X_1,\dots,X_n}(x_1, \dots, x_n)\, dx_1 \cdots dx_n \end{aligned}$$

Also

1) $$\begin{aligned} P[X_2 \in A_2, \dots, X_n \in A_n] &= P[X_1 \in (-\infty, \infty), X_2 \in A_2, \dots, X_n \in A_n] \\ &= \int_{A_n} \cdots \int_{A_2} \int_{-\infty}^{\infty} f_{X_1,X_2,\dots,X_n}(x_1, x_2, \dots, x_n)\, dx_1 dx_2 \cdots dx_n \end{aligned}$$

So $$f_{X_2,\dots,X_n}(x_2, \dots, x_n) = \int_{-\infty}^{\infty} f_{X_1,X_2,\dots,X_n}(x_1, x_2, \dots, x_n)\, dx_1 \qquad \text{[marginalization]}$$

2) $$\begin{aligned} 1 &= P[X_1 \in (-\infty, \infty), \dots, X_n \in (-\infty, \infty)] \\ &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{X_1,\dots,X_n}(x_1, \dots, x_n)\, dx_1 \cdots dx_n \end{aligned}$$

Example 24.1: Let X, Y and Z have the joint pdf

$$f_{XYZ}(x, y, z) = \begin{cases} c & x^2 + y^2 + z^2 \le R^2 \\ 0 & \text{else} \end{cases}$$

for some $c > 0$.

Note: this pdf is a uniform distribution on a sphere of radius R.

a) Find c.

b) What is the marginal pdf $f_{XY}(x, y)$?

Solution:

a) We can find c from

$$\begin{aligned} 1 &= \iiint_{\mathbb{R}^3} f_{XYZ}(x, y, z)\, dxdydz \\ &= \iiint_{x^2+y^2+z^2 \le R^2} c\, dxdydz \\ &= \frac{4}{3}\pi R^3 \times c \end{aligned}$$

So, $c = \frac{3}{4\pi R^3}$.

b) We marginalize out the random variable Z:

$$\begin{aligned} f_{XY}(x, y) &= \int_{-\infty}^{\infty} f_{XYZ}(x, y, z)\, dz \\ &= \int_{z: x^2+y^2+z^2 \le R^2} c\, dz \\ &= \begin{cases} 0 & x^2 + y^2 > R^2 \\ \int_{-a}^{a} c\, dz & x^2 + y^2 \le R^2 \end{cases} \qquad \text{where } a = \sqrt{R^2 - (x^2 + y^2)} \\ &= \begin{cases} 0 & x^2 + y^2 > R^2 \\ 2ac & x^2 + y^2 \le R^2 \end{cases} \\ &= \begin{cases} 0 & x^2 + y^2 > R^2 \\ \frac{3}{2\pi R^3}\sqrt{R^2 - (x^2 + y^2)} & x^2 + y^2 \le R^2 \end{cases} \end{aligned}$$

Independent Random Variables [Ross S6.2]

Two events E and F are independent when $P[EF] = P[E]P[F]$.

In words: Knowing that E has occured does not change the probability of F occuring.

Definition 24.1: The random variables X and Y are **independent** if

$$P[X \in A, Y \in B] = P[X \in A]P[Y \in B] \qquad \forall A, B \subset \mathbb{R} \qquad (24.1)$$

In words: Knowing the outcome of X does not change the probabilities of the outcomes of Y.

Say X and Y are independent. Choosing $A = (-\infty, x]$ and $B = (-\infty, y]$:

$$\begin{aligned} F_{XY}(x, y) &= P[X \in A, Y \in B] \\ &= P[X \in A]P[Y \in B] && \text{by independence} \\ &= F_X(x)F_Y(y) && \forall a, b \in \mathbb{R} \qquad (24.2) \end{aligned}$$

So (24.1) implies (24.2).

It can be shown that if (24.2) holds, then (24.1) holds.

Hence (24.1) and (24.2) are equivalent.

Discrete Case:

If X and Y are discrete, then X and Y independent is also equivalent to

$$p_{XY}(x, y) = p_X(x)p_Y(y) \qquad \forall x, y \qquad (24.3)$$

Why?

i) Choosing $A = \{x\}$ and $B = \{y\}$ in (24.1) yields (24.3):

$$\begin{aligned} p_{XY}(x, y) &= P[X \in A, Y \in B] \\ &= P[X \in A]P[Y \in B] && \text{[using (24.1)]} \\ &= p_X(x)p_Y(y) \end{aligned}$$

ii) (24.3) implies (24.1):

$$\begin{aligned} P[X \in A, Y \in B] &= \sum_{x \in A, y \in B} p_{XY}(x, y) \\ &= \sum_{x \in A, y \in B} p_X(x)p_Y(y) && \text{[using (24.3)]} \\ &= \sum_{x \in A} p_X(x) \sum_{y \in B} p_Y(y) \\ &= P[X \in A]P[Y \in B] \end{aligned}$$

Continuous Case:

If X and Y are continuous, then X and Y independent is also equivalent to

$$f_{XY}(x, y) = f_X(x)f_Y(y) \qquad \forall x, y \qquad (24.4)$$

Why?

i) (24.2) implies (24.4):

$$\begin{aligned} f_{XY}(x, y) &= \frac{\partial^2}{\partial x \partial y} F_{XY}(x, y) \\ &= \frac{\partial^2}{\partial x \partial y} F_X(x)F_Y(y) && \text{[using (24.2)]} \\ &= f_X(x)f_Y(y) \end{aligned}$$

ii) (24.4) implies (24.2):

$$\begin{aligned} F_{XY}(x, y) &= \int_{-\infty}^{y} \int_{-\infty}^{x} f_{XY}(u, v)\, dudv \\ &= \int_{-\infty}^{y} \int_{-\infty}^{x} f_X(u)f_Y(v)\, dudv \\ &= \int_{-\infty}^{x} f_X(u)\, du \int_{-\infty}^{y} f_Y(v)\, dv \\ &= F_X(x)F_Y(y) \end{aligned}$$

Summary:

The discrete rv's X and Y are independent is equivalent to all three:

$$\begin{aligned} P[X \in A, Y \in B] &= P[X \in A]P[Y \in B] && \forall A, B \subset \mathbb{R} && (24.1) \\ F_{XY}(x, y) &= F_X(x)F_Y(y) && \forall x, y \in \mathbb{R} && (24.2) \\ p_{XY}(x, y) &= p_X(x)p_Y(y) && \forall x, y \in \mathbb{R} && (24.3) \end{aligned}$$

The continuous rv's X and Y are independent is equivalent to all three:

$$\begin{aligned} P[X \in A, Y \in B] &= P[X \in A]P[Y \in B] && \forall A, B \subset \mathbb{R} && (24.1) \\ F_{XY}(x, y) &= F_X(x)F_Y(y) && \forall x, y \in \mathbb{R} && (24.2) \\ f_{XY}(x, y) &= f_X(x)f_Y(y) && \forall x, y \in \mathbb{R} && (24.4) \end{aligned}$$

The concept of independence can be extended to more than 2 variables:

Definition 24.2: Random variables $X_1, \dots, X_n$ are independent if for any sets $A_1, \dots, A_n$:

$$P[X_1 \in A_1, \dots, X_n \in A_n] = P[X_1 \in A_1] \times \cdots \times P[X_n \in A_n]$$

Again, this is equivalent to

$$F_{X_1,\dots,X_n}(a_1, \dots, a_n) = F_{X_1}(a_1) \times \cdots \times F_{X_n}(a_n)$$

for all $a_1, \dots, a_n$.

An infinite collection of random variables is independent if every finite subset are independent.