

# Conditional Probability and Independence

## Baye's Theorem [Ross S3.3]

Law of Total Probability:

Let  $E, F \subset S$ .

$$\text{Then } E \cap S = E(F \cup F^c) = EF \cup EF^c$$

$$\begin{aligned}\text{and } P[E] &= P[EF] + P[EF^c] \\ &= P[E|F]P[F] + P[E|F^c]P[F^c]\end{aligned}$$

**Example 6.1:** The probability of an insurance claim is

- 0.4 for 30% of persons (type 1),
- 0.5 for 70% of persons (type 2).

What is the probability that a random person has a claim?

*Solution:*

Let  $F = \{\text{type 1 person}\}$

$F^c = \{\text{type 2 person}\}$

$E = \{\text{there is a claim}\}$

Then

$$\begin{aligned}P[E] &= P[E|F]P[F] + P[E|F^c]P[F^c] \\ &= 0.4 \times 0.3 + 0.5 \times 0.7 = 0.47\end{aligned}$$

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Let  $F_1, \dots, F_n$  partition  $S$ .

Then 
$$E = ES = E\left(\bigcup_{i=1}^n F_i\right) \\ = \bigcup_{i=1}^n (EF_i)$$

So 
$$P[E] = P[\cup_{i=1}^n (EF_i)] \\ = \sum_{i=1}^n P[EF_i] \\ = \sum_{i=1}^n P[E|F_i]P[F_i] \quad \text{[Law of total probability]}$$

**Example 6.2:** You roll a 4-sided die. If result is  $\leq 2$ , you roll once more and otherwise stop. What is probability that the sum  $\geq 4$ ?

*Solution:* Let  $F_i = \{\text{first roll} = i\}$ ,  $E = \{\text{sum} \geq 4\}$ .

$$P[F_i] = 1/4$$

$$P[E|F_1] = P[\text{second roll is 3 or 4}] = 1/2$$

$$P[E|F_2] = P[\text{second roll is 2, 3 or 4}] = 3/4$$

$$P[E|F_3] = 0$$

$$P[E|F_4] = 1$$

$$\begin{aligned} \Rightarrow P[E] &= P[E|F_0]P[F_0] + P[E|F_1]P[F_1] + P[E|F_2]P[F_2] + P[E|F_3]P[F_3] \\ &= 1/2 \times 1/4 + 3/4 \times 1/4 + 0 \times 1/4 + 1 \times 1/4 \\ &= 9/16. \end{aligned}$$

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## Baye's Theorem and Inference:

Let  $F_1, F_2, \dots, F_n$  partition  $S$ .

Say we know  $P[E|F_j]$ . We want to compute  $P[F_j|E]$ :

$$\begin{aligned} P[F_j|E] &= \frac{P[EF_j]}{P[E]} \\ &= \frac{P[E|F_j]P[F_j]}{P[E|F_1]P[F_1] + P[E|F_2]P[F_2] + \dots + P[E|F_n]P[F_n]} \end{aligned} \tag{6.1}$$

This is Baye's theorem/rule.

*Application to inference:*

Before any partial information is revealed (i.e., observing  $E$  occurs), the probabilities are:

$$P[F_1], P[F_2], \dots, P[F_n] \quad \left. \vphantom{P[F_1], P[F_2], \dots, P[F_n]} \right\} \quad \text{“prior probabilities”}$$

After observing  $E$  occur, they are revised as:

$$P[F_1|E], P[F_2|E], \dots, P[F_n|E] \quad \left. \vphantom{P[F_1|E], P[F_2|E], \dots, P[F_n|E]} \right\} \quad \text{“posterior probabilities”}$$

according to (6.1).

Posterior probabilities are key to practical inference (e.g., classification, pattern recognition, detection, etc.)

**Example 6.3:** A 3-card deck has

- 1) one card with red on both sides
- 2) one card with black on both sides
- 3) one card with red on one side + black on the other.

One side of 1 card is picked at random. It is red. What is the probability that other side is black?

*Solution:* Let  $S = \{RR, RB, BB\}$ ,  $R = \{\text{side shown is red}\}$ .

$$\begin{aligned}
 P[RB|R] &= \frac{P[RB \cap R]}{P[R]} \\
 &= \frac{P[R|RB]P[RB]}{P[R|RR]P[RR] + P[R|RB]P[RB] + P[R|BB]P[BB]} \\
 &= \frac{\frac{1}{2} \frac{1}{3}}{1 \times \frac{1}{3} + \frac{1}{2} \frac{1}{3} + 0 \times \frac{1}{3}} \\
 &= 1/3
 \end{aligned}$$

Why?

If you see red, there are 3 different ways this could happen (one side of RB + two sides of RR). But only 1 results in black on the other side.

**Example 6.4:** A blood test has 95% prob of detecting a disease when it is present. It has a 1% false positive rate when it is not present. 0.5% of people have the disease.

- a) If a random person tests positive, what is prob. that disease is present?
- b) If a random person tests negative, what is prob. that disease is present?

*Solution:*  $E = \{\text{positive result}\}$ ,  $F = \{\text{desease present}\}$

$$\begin{aligned}\text{a)} \quad P[F|E] &= \frac{P[EF]}{P[E]} \\&= \frac{P[E|F]P[F]}{P[E|F]P[F] + P[E|F^c]P[F^c]} \\&= \frac{0.95 \times 0.005}{0.95 \times 0.005 + 0.01 \times 0.995} \\&\approx 0.323\end{aligned}$$

$$\begin{aligned}\text{b)} \quad P[F|E^c] &= \frac{P[E^cF]}{P[E^c]} \\&= \frac{P[E^c|F]P[F]}{P[E^c|F]P[F] + P[E^c|F^c]P[F^c]} \\&= \frac{0.05 \times 0.005}{0.05 \times 0.005 + 0.99 \times 0.995} \\&\approx 0.000254\end{aligned}$$