Signals and Systems

Lecture Notes

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Chapter 4

Transform-Domain Analysis of Continuous-Time Systems

This is also called *frequency-domain analysis*. It is based on Laplace transform.

4.1 The Laplace Transform (LT)

Given a *continuous-time* (CT) *causal* signal $x(t)$, its Laplace transform is defined by:

$$\mathcal{L}\{x(t)\} = X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt$$

where $s = \sigma + j\omega \ \epsilon \ \mathbb{C}$ is a complex parameter. The Laplace transform $X(s)$ exists for a specific set of values of s, called the Region of Conversion (ROC).

Remark: Above definition is called the *unilateral Laplace transform*. In addition to this, the textbook discusses the *bilateral Laplace transform*, which is not needed and not required in this course since our signals are assumed to be always causal [$x(t) = 0$ for $t < 0$]. The unilateral transform is simpler to use than the bilateral one.

Remark: The lower limit of the integral has been chosen to be $t = 0^-$ rather than $t = 0$ to include any impulse functions at $t = 0$ and to allow us to use initial conditions at $t = 0^-$.

Example: Let $x(t) = e^{\lambda t}u(t), \quad \lambda \in \mathbb{R}$ (real)

$$X(s) = \int_{0^-}^{\infty} e^{\lambda t}u(t).e^{-st}dt = \int_{0}^{\infty} e^{(\lambda-s)t}dt$$

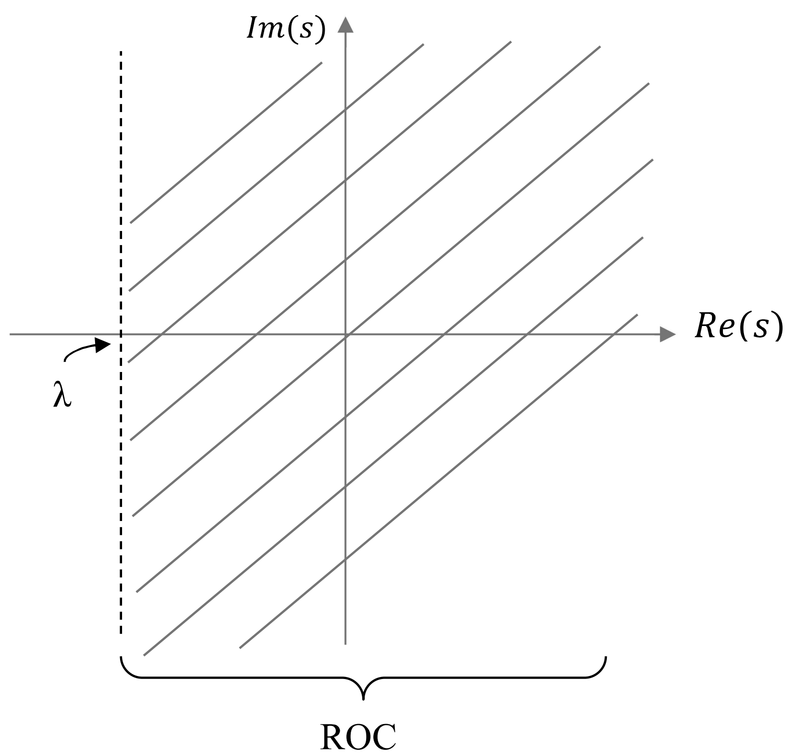
$$= \frac{1}{\lambda - s} e^{(\lambda-s)t}\Big|_0^\infty = \frac{1}{\lambda - s}\left[e^{(\lambda-s)\infty} - 1\right]$$

Notice that

$$e^{(\lambda-s)\infty} = \begin{cases} 0 & \text{if } Re(\lambda - s) < 0 \quad \text{or} \quad Re(s) > \lambda \\ \infty & \quad \text{otherwise} \end{cases}$$

then,

$$X(s) = \frac{1}{s - \lambda} \quad \text{where} \quad Re(s) > \lambda \quad \text{(ROC)}$$



Remark: If $\lambda = 0$, then $x(t) = u(t)$ and $\mathcal{L}\{u(t)\} = \frac{1}{s}$ where ROC: $\text{Re}(s) > 0$.

Example: $x(t) = \delta(t)$

$$\mathcal{L}\{x(t)\} = X(s) = \int_{0^-}^{\infty} \delta(t)e^{-st}dt = 1$$

where ROC is all values of *s*.

Remark: Some signals do not have LT. For example, the signal $x(t) = e^{t^2}$ has no LT.

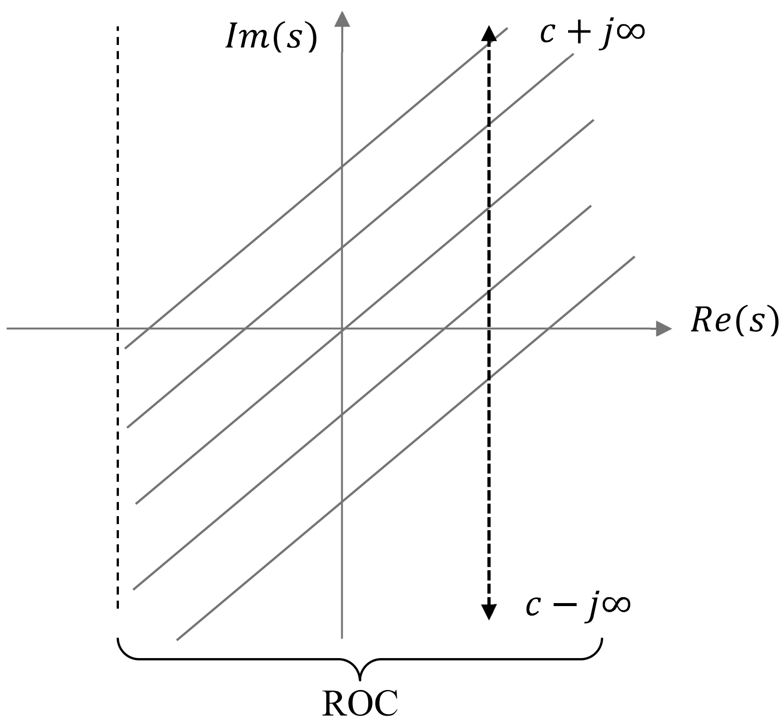
Remark: See Table 4.1 in textbook for useful Laplace transform pairs.

- **Inverse Laplace Transform**

Given $X(s)$, find $x(t)$:

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s)e^{st}ds$$

where $c =$ constant. The path of integration is inside the ROC and along the line $c + j\omega$, where ω changes from $-\infty$ to $+\infty$.



Remark: Application of inverse Laplace transform relation requires the integration in the complex plain. Fortunately, we will avoid this complexity by finding the inverse LT using tables rather than using the integration in the complex plain.

4.2 Properties of the Laplace Transform

(1) **Linearity**

$$\mathcal{L}\{c_1x_1(t) + c_2x_2(t)\} = c_1X_1(s) + c_2X_2(s)$$

(2) **Time-Shifting**

If $x(t) \leftrightarrow X(s)$ and for $t_0 \geq 0$, then:

$$\mathcal{L}\{x(t - t_0)u(t - t_0)\} = X(s)e^{-st_0}$$

(3) **Frequency-Shifting**

$$\mathcal{L}\{x(t)e^{s_0t}\} = X(s - s_0)$$

(4) **Time-Differentiation**

$$\mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = sX(s) - x(0^-)$$

$$\mathcal{L}\left\{\frac{dx^2(t)}{dt^2}\right\} = s^2X(s) - sx(0^-) - \dot{x}(0^-)$$

In general,

$$\mathcal{L}\left\{\frac{dx^n(t)}{dt^n}\right\} = s^nX(s) - s^{n-1}x(0^-) - s^{n-2}\dot{x}(0^-) - \cdots - x^{(n-1)}(0^-)$$

(5) **Frequency-Differentiation**

$$\mathcal{L}\{tx(t)\} = -\frac{d}{ds}X(s)$$

(6) **Time-Integration**

$$\mathcal{L}\left\{\int_{0^-}^{t} x(\tau)d\tau\right\} = \frac{X(s)}{s}$$

$$\mathcal{L}\left\{\int_{-\infty}^{t} x(\tau)d\tau\right\} = \frac{X(s)}{s} + \frac{\int_{-\infty}^{0^-} x(\tau)d\tau}{s}, \quad \text{where } t \geq 0$$

(7) **Frequency Integration**

$$\mathcal{L}\left\{\frac{x(t)}{t}\right\} = \int_{s}^{\infty} X(u)du$$

(8) **Time-Convolution**

If $x_1(t) \leftrightarrow X_1(s)$ and $x_2(t) \leftrightarrow X_2(s)$, then

$$x_1(t) * x_2(t) \leftrightarrow X_1(s)X_2(s)$$

Remark: For LTI systems, we have $y(t) = x(t) * h(t)$ and $Y(s) = X(s)H(s)$. This gives another definition of the *transfer function* of the system:

$$H(s) = \frac{Y(s)}{X(s)}$$

(9) **Frequency-Convolution**

$$x_1(t)x_2(t) \leftrightarrow \frac{1}{2\pi j} X_1(s) * X_2(s)$$

(10) **Initial and Final Values**

If $x(t)$ is continuous at $t = 0^+$, then:

$$x(0^+) = \lim_{s\to\infty} sX(s)$$

provided the limit exists.

If $x(t)$ converges as $t \to \infty$, then:

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

provided the limits exist

(11) **Time Scaling**

$$x(at) \quad \leftrightarrow \quad \frac{1}{a} X\left(\frac{s}{a}\right), \quad \text{where} \quad a > 0$$

Remark: Time-Compression $\rightarrow$ Frequency-Expansion

Time-Expansion $\rightarrow$ Frequency-Compression.

4.3 Finding the Inverse Laplace Transform

As mentioned before, finding the inverse LT using its direct relation given before requires integration in the complex plain, which is not easy to use, and therefore we try to avoid it in this course.

Instead, using tables is much simpler. In particular, we will use Table 4.1 for LT pairs and Table 4.2 for LT properties. In addition, since most practical transforms are *rational functions*, we will use *partial-fraction expansion* (PFE) to express the rational function in simpler terms that match the entries in the LT table. This is illustrated in the following examples.

Example: (Strictly Proper Rational, $M < N$)

$$X(s) = \frac{2s+3}{s^2+s-2}$$

(a) Factor denominator and write as the sum of partial fractions:

$$X(s) = \frac{2s+3}{s^2+s-2} = \frac{2s+3}{(s-1)(s+2)} = \frac{c_1}{s-1} + \frac{c_2}{s+2}$$

(b) Find c_1 and c_2 (using cover-up method):

$$c_1 = \left.\frac{2s+3}{(s+2)}\right|_{s=1} = \frac{2+3}{3} = \frac{5}{3}$$

$$c_2 = \left.\frac{2s+3}{(s-1)}\right|_{s=-2} = \frac{2(-2)+3}{(-2-1)} = \frac{1}{3}$$

(c) Use LT table pair

$$e^{\lambda t}u(t) \;\leftrightarrow\; \frac{1}{s-\lambda}$$

This gives:

$$x(t) = \left[\frac{5}{3}e^{t} + \frac{1}{3}e^{-2t}\right]u(t)$$

Example: (Proper Rational, $M = N$)

$$X(s) = \frac{2s^2 + 4s - 1}{s^2 + s - 2}$$

(a) Isolate the constant part (by long division) and factor denominator.

$$X(s) = 2 + \underbrace{\frac{2s + 3}{(s-1)(s+2)}}_{\text{strictly proper rational}}$$

(b) Apply partial-fraction expansion on the strictly proper rational.

$$X(s) = 2 + \frac{5/3}{s-1} + \frac{1/3}{s+2}$$

(c) Use tables

$$\delta(t) \;\leftrightarrow\; 1$$

$$\Rightarrow \quad x(t) = 2\delta(t) + \frac{5}{3}e^{t} + \frac{1}{3}e^{-2t}, \quad t \geq 0$$

<u>Remark</u>: In last example, there is no need for detailed long division. The first division step is obviously 2. Just write:

$$X(s) = 2 + \frac{c_1}{s-1} + \frac{c_2}{s+2}$$

then find c_1 and c_2 by cover-up method.

<u>Remark</u>: For <u>*improper rational*</u> ($M > N$), the system is <u>unstable</u> and <u>amplifies noise</u>.

Example: (repeated roots)

$$H(s) = \frac{s^2 + 4}{(s-1)(s-2)^2}$$

(a) Apply PFE

$$H(s) = \frac{c_1}{s-1} + \frac{c_2}{s-2} + \frac{c_3}{(s-2)^2}$$

(b) Find c_1 and c_3 by cover-up

$$c_1 = \left.\frac{s^2+4}{(s-2)^2}\right|_{s=1} = \frac{1+4}{(1-2)^2} = 5$$

$$c_3 = \left.\frac{s^2+4}{(s-1)}\right|_{s=2} = \frac{(2)^2+4}{(2-1)} = 8$$

To find c_2:

$$(s-2)^2 H(s) = c_1 \frac{(s-2)^2}{(s-1)} + c_2(s-2) + c_3$$

$$\Rightarrow \quad \frac{d}{ds}[(s-2)^2 H(s)]_{s=2} = 0 + c_2 + 0$$

$$c_2 = \frac{d}{ds}\left[\frac{s^2+4}{(s-1)}\right]_{s=2} = \left.\frac{2s(s-1) - 1(s^2+4)}{(s-1)^2}\right|_{s=2} = \frac{4-(4+4)}{1} = -4$$

(c) Use tables:

$$te^{\lambda t}u(t) \;\leftrightarrow\; \frac{1}{(s-\lambda)^2}$$

$$\Rightarrow \quad h(t) = (5e^{t} - 4e^{2t} + 8te^{2t})u(t)$$

Remark: (Easier way to find c_2)

After finding c_1 and c_3

$$H(s) = \frac{s^2+4}{(s-1)(s-2)^2} = \frac{5}{s-1} + \frac{c_2}{s+2} + \frac{8}{(s+2)^2}$$

Use any convenient value, say $s = 0$, that does not make any of the denominators zero, then solve for c_2:

$$\frac{0+4}{(0-1)(0-2)^2} = \frac{5}{0-1} + \frac{c_2}{0+2} + \frac{8}{(0+2)^2}$$

$$\Rightarrow \quad c_2 = -4$$

4.4 Finding System Response Using Laplace Transform

Finding the response for LTI systems is relatively easy by using Laplace transform. The following example shows how to find the zero-input response $y_{zi}(t)$, the zero-state response $y_{zs}(t)$, and the total response $y(t)$, simultaneously.

Example: Find $y_{zi}(t)$, $y_{zs}(t)$, and $y(t)$, $t \geq 0$ for the system:

$$(D+3)y = (D-1)x$$

$$y(0^-) = 1, \qquad x(t) = u(t)$$

Solution:

$$\mathcal{L}[(D+3)y] = \mathcal{L}[(D-1)x]$$

$$\mathcal{L}[Dy] + 3\mathcal{L}[y] = \mathcal{L}[Dx] - \mathcal{L}[x]$$

$$\overbrace{sY(s) - y(0^-)}^{\mathcal{L}[Dy]} + 3Y(s) = \overbrace{sX(s) - x(0^-)}^{\mathcal{L}[Dx]} - X(s)$$

$$\Rightarrow \quad (s+3)Y(s) = (s-1)X(s) + y(0^-)$$

$$\Rightarrow \quad Y(s) = \underbrace{\frac{s-1}{s+3}X(s)}_{Y_{zs}(s)} + \underbrace{\frac{y(0^-)}{s+3}}_{Y_{zi}(s)}$$

We have $X(s) = \mathcal{L}[u(t)] = \frac{1}{s}$

$$\Rightarrow \quad Y(s) = \left[\frac{-1/3}{s} + \frac{-4/(-3)}{s+3}\right] + \left[\frac{1}{s+3}\right]$$

$$\Rightarrow \quad y(t) = \mathcal{L}^{-1}[Y(s)] = \underbrace{\left[-\frac{1}{3} + \frac{4}{3}e^{-3t}\right]u(t)}_{y_{zs}(t)} + \underbrace{[e^{-3t}]u(t)}_{y_{zi}(t)}$$

- **Transfer Function (Revisited)**

For LTI systems, the relation between the input and the output is given by the convolution $y(t) = h(t) * x(t)$. Taking the Laplace transform of this relation gives $Y(s) = H(s)X(s)$, which can be solved easily for the transfer function:

$$H(s) = \frac{Y(s)}{X(s)}$$

For LTI systems described by the differential equation $Q(D)y(t) = P(D)x(t)$, or:

$$\begin{aligned}(D^N + a_{N-1}D^{N-1} + \cdots + a_1 D + a_0)y(t)\\ = (b_N D^N + b_{N-1}D^{N-1} + \cdots + b_1 D + b_0)x(t)\end{aligned}$$

Assume all ICs are zero, i.e. $y(0^-) = \dot{y}(0^-) = \cdots = y^{(N-1)}(0^-) = 0$, then:

$$y(t) \leftrightarrow Y(s)$$

$$Dy(t) \leftrightarrow sY(s)$$

$$D^2y(t) \leftrightarrow s^2Y(s)$$

$$\vdots$$

$$D^Ny(t) \leftrightarrow s^NY(s)$$

Similar for $x(t)$.

Substituting:

$$\begin{aligned}\overbrace{(s^N + a_{N-1}s^{N-1} + \cdots + a_1 s + a_0)}^{Q(s)}Y(s)\\ = \underbrace{(b_N s^N + b_{N-1}s^{N-1} + \cdots + b_1 s + b_0)}_{P(s)}X(s)\end{aligned}$$

$$\Rightarrow \quad H(s) = \frac{Y(s)}{X(s)} = \frac{P(s)}{Q(s)} = \frac{b_N s^N + b_{N-1}s^{N-1} + \cdots + b_1 s + b_0}{s^N + a_{N-1}s^{N-1} + \cdots + a_1 s + a_0}$$

<u>Remark</u>: The Transfer function $H(s)$ can be found by inspection from the differential equation by replacing D by s in $Q(D)$ and $P(D)$ and taking the ratio.

<u>Remark</u>: The roots of $P(s) = 0$ are called the *<u>zeros</u>* of $H(s)$ and the roots of $Q(s) = 0$ are called the *<u>poles</u>* of $H(s)$. System <u>stability</u> is usually stated in terms of the <u>poles</u> of $H(s)$ instead of the roots of $Q(\lambda)$. If there is no terms cancellation between $P(s)$ and $Q(s)$, the <u>asymptotic stability</u> is determined. Otherwise, <u>BIBO stability</u> is determined.

<u>Example</u>: Find $h(t)$ and $y(t)$ for the system $(D^2 + 3D + 2)y = (D - 5)x$ with $x(t) = u(t)$

<u>Solution</u>: By inspection:

$$H(s) = \frac{P(s)}{Q(s)} = \frac{s-5}{s^2 + 3s + 2} = \frac{s-5}{(s+1)(s+2)} = \frac{-6}{s+1} + \frac{7}{s+2}$$

$$\Rightarrow \quad h(t) = [-6e^{-t} + 7e^{-2t}]u(t)$$

$$Y(s) = H(s)X(s) = \frac{s-5}{(s+1)(s+2)}\left(\frac{1}{s}\right) = \frac{6}{s+1} + \frac{-7/2}{s+2} + \frac{-5/2}{s}$$

$$\Rightarrow \quad y(t) = \left(-\frac{5}{2} + 6e^{-t} - \frac{7}{2}e^{-2t}\right)u(t)$$

<u>Remark</u>: Above system has one zero at $s = 5$ and two poles at $s = -1$ and $s = -2$. Since there is no cancelation of terms between $P(s)$ and $Q(s)$ and the two poles are in left-half plain, then the system is asymptotically stable.

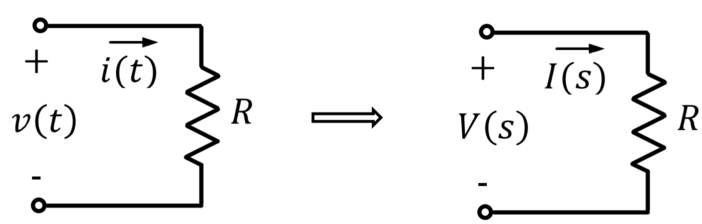
4.5 Electric Circuit Analysis Using Laplace Transform

In general, electric circuits with resistors, inductors, and capacitors are modeled by differential equations. Instead of solving the differential equations in time-domain, Laplace transform can be used to simplify the analysis since it changes the differentiation to a multiplication by s. In addition, it accommodates initial conditions and any kind of input voltage or current sources, not necessarily sinusoids as is seen in previous electric circuits course. Moreover, Laplace transform allows us to obtain the zero-input response, the zero-state response, transient response, or steady-state response.

The procedure for circuit analysis using LT is simple. First, the circuit is transformed from the time-domain into the transform-domain. Second, the circuit is solved in the transform-domain, which is easy since any differentiations or integrations are eliminated. Finally, the results are transformed back to the –time-domain.

Here, we review how the basic circuit elements (resistor, inductor, and capacitor) are transformed from the time-domain into transform-domain using LT.

Resistor:



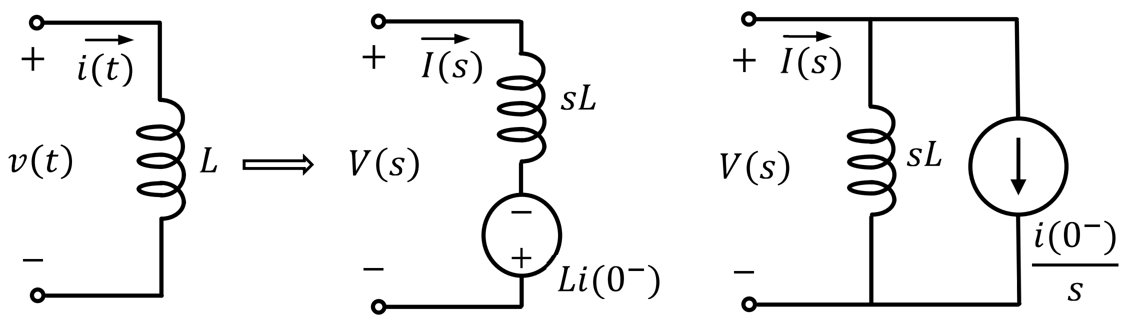
$v(t) = Ri(t)$ $\qquad\qquad$ $V(s) = RI(s)$

Inductor:

$$v(t) = L\frac{di(t)}{dt} \quad \Rightarrow \quad V(s) = L[sI(s) - i(0^-)]$$

$$= sLI(s) - Li(0^-)$$

$$= sL\left[I(s) - \frac{i(0^-)}{s}\right]$$

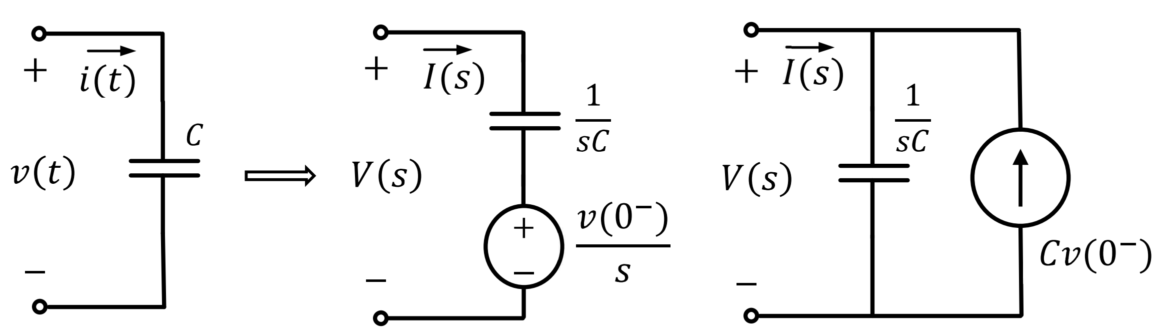


Capacitor:

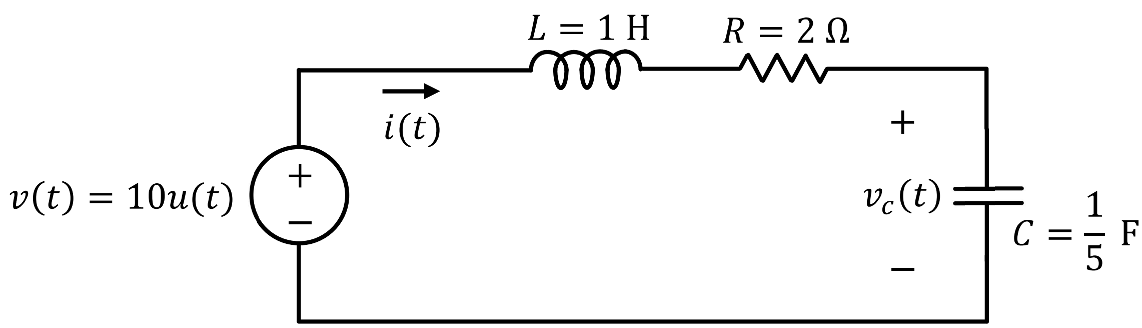
$$i(t) = C\frac{dv(t)}{dt} \quad \Rightarrow \quad I(s) = C[sV(s) - v(0^-)]$$

$$\Rightarrow \quad V(s) = \frac{1}{sC}I(s) + \frac{v(0^-)}{s}$$

$$= \frac{1}{sC}[I(s) + Cv(0^-)]$$

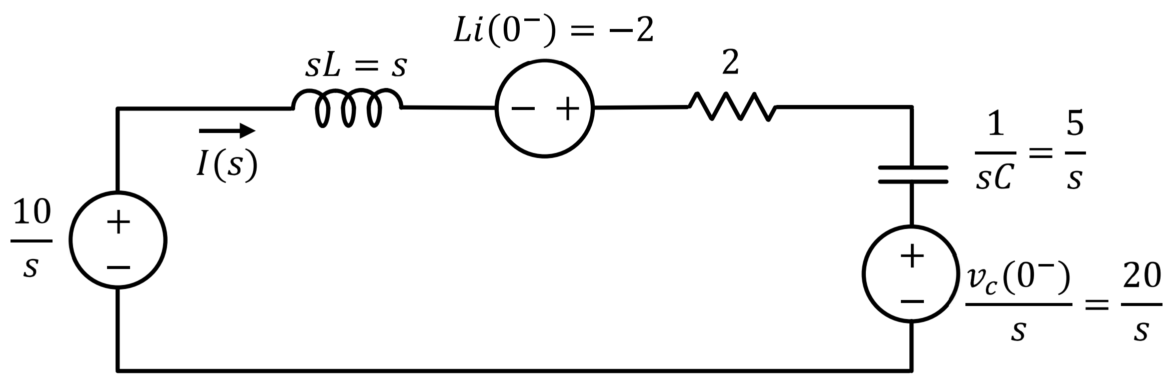


Example: For the circuit shown below, find the current $i(t)$, where $i(0^-) = -2$ A, $v_c(0^-) = 20$ V, and $v(t) = 10u(t)$ V.



Solution:

Transform the circuit into Laplace-domain:



KVL: $$\frac{10}{s} - 2 - \frac{20}{s} = \left(s + 2 + \frac{5}{s}\right) I(s)$$

$$\Rightarrow \quad -(10 + 2s) = (s^2 + 2s + 5)I(s)$$

$$\Rightarrow \quad I(s) = \frac{-(10 + 2s)}{s^2 + 2s + 5}$$

Notice that the denominator has complex roots. The PFE of $I(s)$ has the form:

$$I(s) = \frac{c_1}{s - \lambda_1} + \frac{c_2}{s - \lambda_2}$$

This form has all constants $\{\lambda_1, \lambda_2, c_1, c_2\}$ complex, which makes computation long and tedious. Instead of using PFE, we notice from LT table the following two pairs:

$$\mathcal{L}[e^{-at}\cos(bt).u(t)] = \frac{s+a}{(s+a)^2 + b^2}$$

$$\mathcal{L}[e^{-at}\sin(bt).u(t)] = \frac{b}{(s+a)^2 + b^2}$$

Therefore, we complete the squares in the denominator and rewrite the numerator such that it matches the two LT pairs, as follows:

$$I(s) = \frac{-2(s+1) - 8}{(s+1)^2 + 4} = -2\frac{s+1}{(s+1)^2 + (2)^2} - 4\frac{2}{(s+1)^2 + (2)^2}$$

Using above two LT pairs gives:

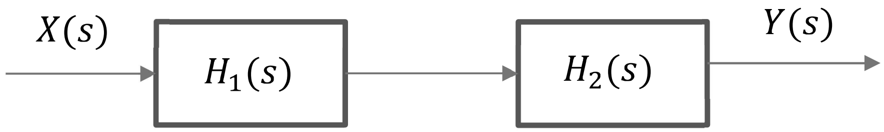
$$i(t) = [-2e^{-t}\cos(2t) - 4e^{-t}\sin(2t)]u(t)$$

Remark: Laplace transform can be used to analyze any linear electric or electronic circuits using the same procedure as given above. Circuits with operational amplifiers (Op-Amps) are included in the assignment problems.

4.6 Block Diagram

Large systems are divided into subsystems for easy analysis and design. For LTI systems represented by block diagrams, the use of Laplace transform makes the analysis easy and convenient.

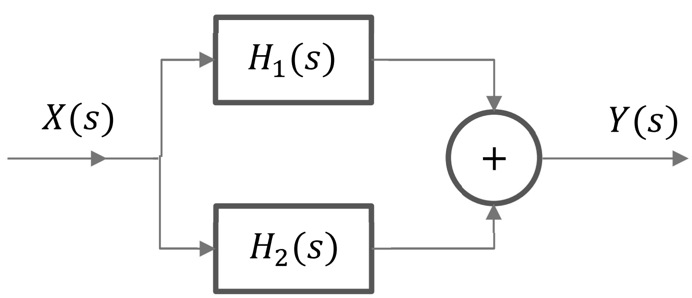
- **Cascade (Series) Connection:**



Equivalent impulse response (time-domain): $h(t) = h_1(t) * h_2(t)$

Equivalent transfer function (Laplace-domain): $H(s) = H_1(s)H_2(s)$

- **Parallel Connection:**

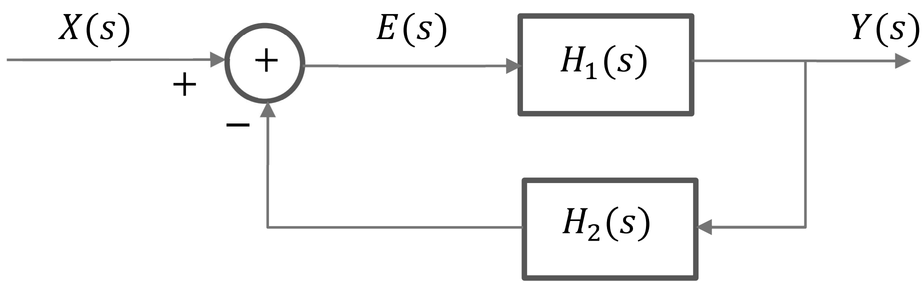


Equivalent impulse response (time-domain): $h(t) = h_1(t) + h_2(t)$

Equivalent transfer function (Laplace-domain): $H(s) = H_1(s) + H_2(s)$

- **Feedback Connection**:

This connection is easier to analyze in transform-domain instead of time-domain.



The equivalent transfer function can be derived as follows:

$$E(s) = X(s) - H_2(s)Y(s)$$

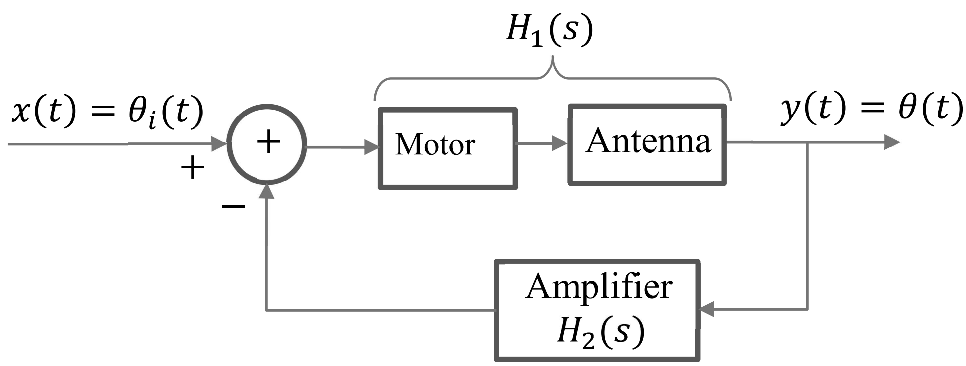
$$Y(s) = H_1(s)E(s) = H_1(s)X(s) - H_1(s)H_2(s)Y(s)$$

$$\Rightarrow \quad [1 + H_1(s).H_2(s)]Y(s) = H_1(s)X(s)$$

$$\Rightarrow \quad H(s) = \frac{Y(s)}{X(s)} = \frac{H_1(s)}{1 + H_1(s)H_2(s)}$$

Example: (Tracking Radar Antenna)

This example shows how to stabilize an unstable system by using a simple feedback, which is commonly used in control systems.



Motor and Antenna: $H_1(s) = \frac{1}{s(s+2)}$ $\begin{cases} MS \text{ stable (pole at 0)} \\ BIBO \text{ unstable} \end{cases}$

Amplifier: $H_2(s) = K$ (constant)

Then, for the overall system with feedback:

$$H(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)} = \frac{\frac{1}{s(s+2)}}{1 + \frac{K}{s(s+2)}} = \frac{1}{s^2 + 2s + K}$$

Let's study the location of roots (poles) as K changes (This subject is studied in more details in control systems course):

For $K = 0$ (no feedback): Poles: $p_1 = 0,\ p_2 = -2$

$\Rightarrow$ System is BIBO unstable

For $0 < K < 1$: $H(s)$ has two real, negative, and distinct poles.

$\Rightarrow$ System is stable and response decays exponentially.

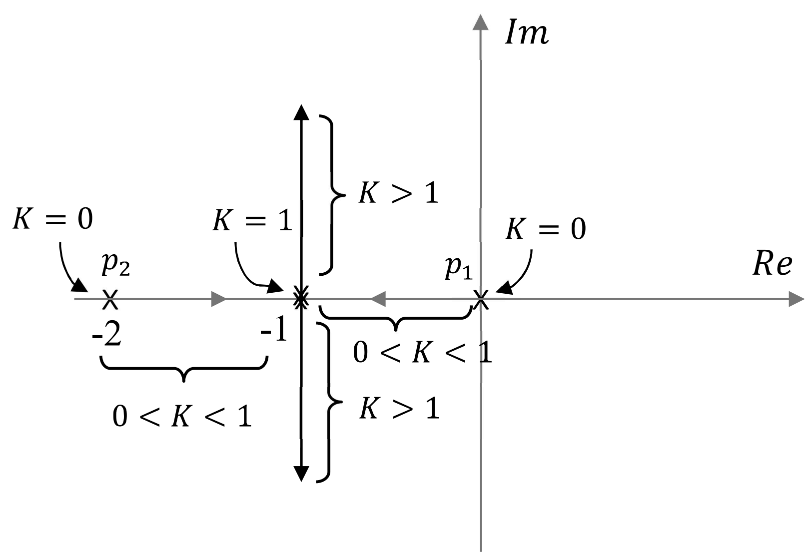
For $K = 1$: $H(s)$ has double poles: $p_1 = p_2 = -1$

⇒ System is stable, and response decays exponentially, but about to oscillate.

For $K > 1$: $H(s)$ has two distinct imaginary poles.

⇒ System is stable, but response has oscillations, which is not desired in certain systems such as this one.

<u>Root Locus</u>



4.7 System Realization

In many situations, the design of a system from its specifications results in obtaining its transfer function, which needs to be realized by practical know subsystems. Any rational transfer function with constant coefficients can be realized by using subsystems of *integrators*, *differentiators*, *scalar multipliers*, and *adders/ subtractors*. Differentiators are avoided in the realization due to practical reasons (stability and noise) as we will see later.

There are more than one way to implement each of the three basic elements, namely the integrator, the scalar multiplier, and the adder/subtractor. A popular way to implement all these three elements is using Op-Amp circuits, which are discussed in the courses of electric circuits and electronic circuits. For example, an op-amp amplifier is an implementation of the scalar multiplier.

The transfer function of the integrator is given by $H(s) = \frac{1}{s}$. This can be shown from the time-integration property of LT with zero initial value:

$$Y(s) = \mathcal{L}\{y(t)\} = \mathcal{L}\left\{\int_{0^-}^{t} x(\tau)d\tau\right\} = \frac{X(s)}{s} \quad \Rightarrow \quad H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s}$$

In addition, using these three basic building blocks, there are more than one possible realization for a given transfer function. Here, we discuss few of the well-known realizations.

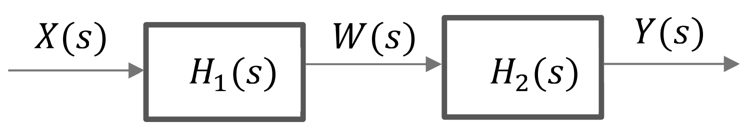
- **Canonical Direct Form Realization:**

Given a general rational transfer function:

$$H(s) = \frac{b_N s^N + b_{N-1}s^{N-1} + \cdots + b_1 s + b_0}{s^N + a_{N-1}s^{N-1} + \cdots + a_1 s + a_0}$$

The realization can be derived by first rewriting the transfer function as a cascade of two systems:

$$H(s) = \underbrace{\left[\frac{1}{s^N + a_{N-1}s^{N-1} + \cdots + a_1 s + a_0}\right]}_{H_1(s)} \cdot \underbrace{\left[b_N s^N + b_{N-1}s^{N-1} + \cdots + b_1 s + b_0\right]}_{H_2(s)}$$



Second, the output of $H_1(s)$ can be written as follows:

$$W(s) = H_1(s)X(s) = \frac{1}{s^N + a_{N-1}s^{N-1} + \cdots + a_1 s + a_0} X(s)$$

or

$$s^N W(s) + a_{N-1}s^{N-1}W(s) + \cdots + a_1 sW(s) + a_0 W(s) = X(s)$$

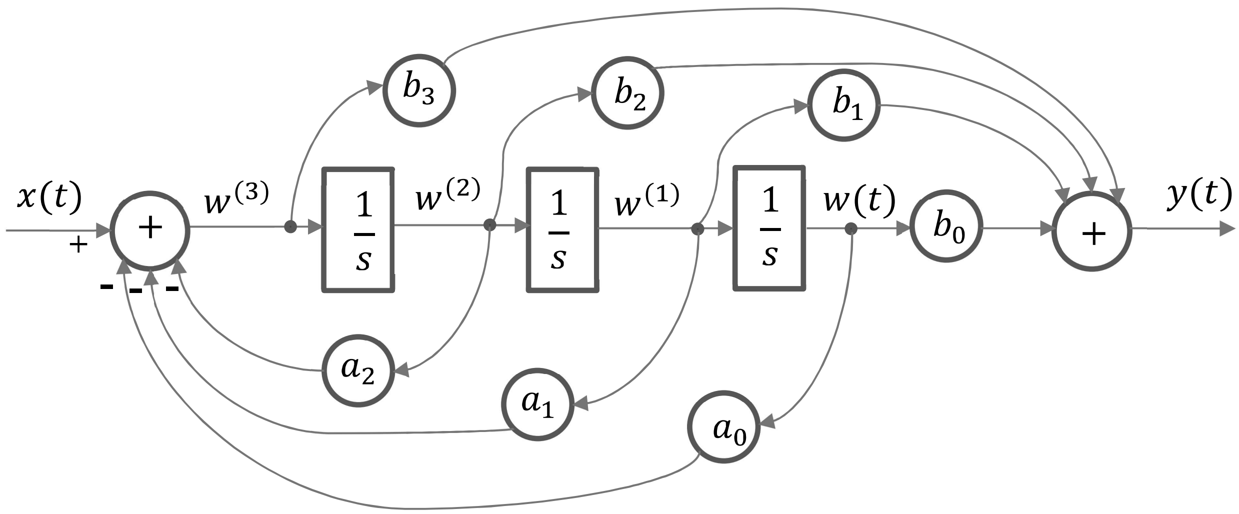
$$\Rightarrow \quad w^{(N)}(t) = -a_{N-1}w^{(N-1)}(t) - \cdots - a_1 w^{(1)}(t) - a_0 w(t) + x(t) \qquad (1)$$

Finally, the output of $H_2(s)$ can be written as follows:

$$Y(s) = H_2(s)W(s) = [b_N s^N + b_{N-1}s^{N-1} + \cdots + b_1 s + b_0]W(s)$$

$$\Rightarrow \quad y(t) = b_N w^{(N)}(t) + b_{N-1}w^{(N-1)}(t) + \cdots + b_1 w^{(1)}(t) + b_0 w(t) \qquad (2)$$

Equations (1) and (2) are used to realize the transfer function. This is shown for the case of a transfer function of third order ($N = 3$):



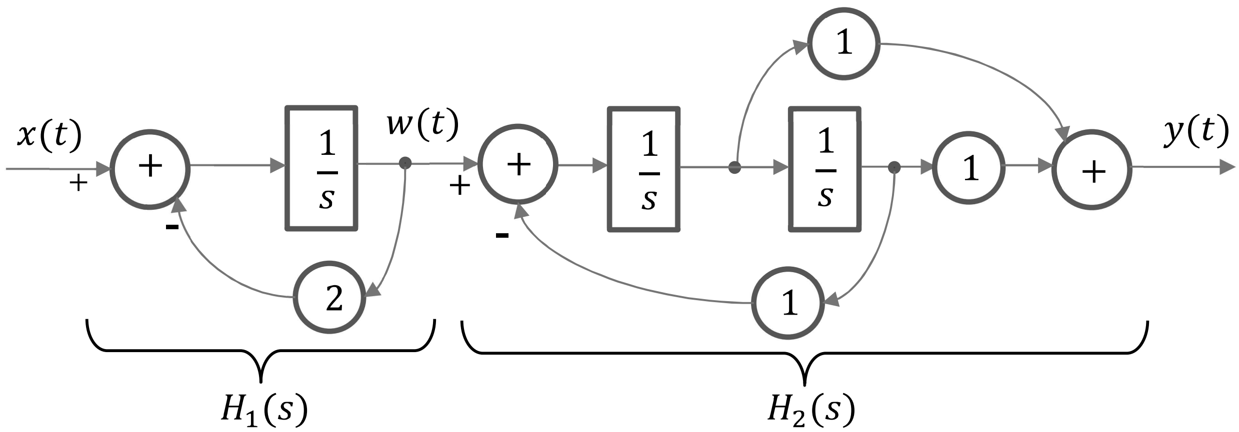
Remark: For large order N ($N > 2$)), direct-form realization is avoided since it has large error at the output due to its sensitivity to coefficient variations and propagation of computational errors.

Remark: A realization is called *canonical* when the number of integrators equal to the order N of the transfer function. See textbook for non-canonical realizations, which are not used in practice. They are usually used for derivation purposes.

- **Cascade (Product) Form**

The transfer function $H(s)$ is written as a product of simpler terms, then each term is realize by a "Direct Form", as shown in the following example:

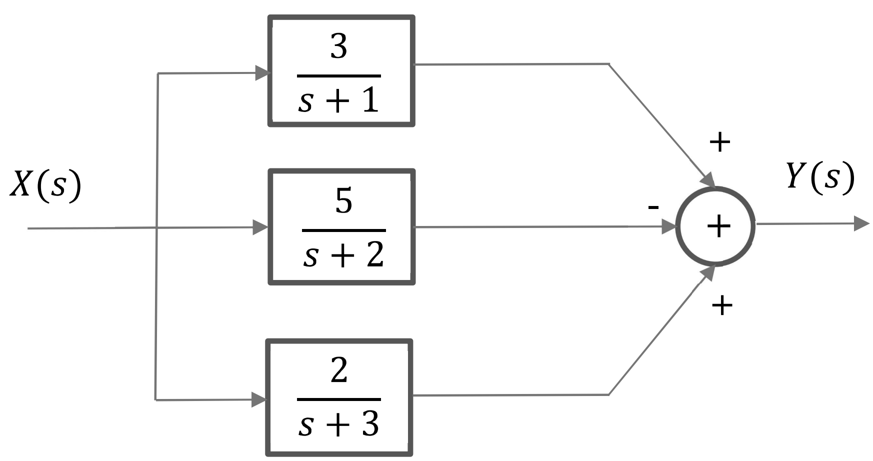
$$H(s) = \frac{s+1}{s^3+2s^2+s+2} = \underbrace{\left[\frac{1}{s+2}\right]}_{H_1(s)} \cdot \underbrace{\left[\frac{s+1}{s^2+1}\right]}_{H_2(s)}$$



- **Parallel (Sum) Form**

The transfer function $H(s)$ is written as a sum of simpler terms using partial-fraction expansion, then each term is realize by a "Direct Form", as shown in the following example:

$$H(s) = \frac{s+7}{(s+1)(s+2)(s+3)} = \frac{3}{s+1} - \frac{5}{s+2} + \frac{2}{s+3}$$



Remark: Cascade and parallel forms have less sensitivity errors than a large order direct-form realization.

4.8 Frequency Response

We found earlier, in chapter 2, that for LTI systems if the input $x(t)$ is an everlasting exponential, then the output $y(t)$ is also an everlasting exponential:

$$x(t) = e^{st} \quad \Longrightarrow \quad y(t) = H(s)e^{st}\,, \qquad -\infty < t < \infty$$

Assume the system is asymptotically stable [i.e. poles of $H(s)$ are in open LHP], and set $s = j\omega$, then:

$$x(t) = e^{j\omega t} \quad \Longrightarrow \quad y(t) = H(j\omega)e^{j\omega t}\,, \qquad -\infty < t < \infty$$

The function $H(j\omega)$ is called the *frequency response* of the system. It is a complex function that can be written in polar form:

$$H(j\omega) = \underbrace{|H(j\omega)|}_{\text{Amplitude Response}} \; e^{\,j\,\overbrace{\angle H(j\omega)}^{\text{Phase Response}}}$$

- **LTI System Response to a Sinusoid**

Taking the real part of the input and the output gives:

$$Re\{x(t)\} = Re\{\mathrm{e}^{\mathrm{j\omega t}}\} = \cos(\omega \mathrm{t})$$

$$\Longrightarrow \quad Re\{y(t)\} = Re\{H(j\omega)e^{j\omega t}\} = Re\{|H(j\omega)|e^{j\angle H(j\omega)}e^{j\omega t}\}$$

$$= |H(j\omega)|.\mathrm{Re}\{e^{j(\omega t + \angle H(j\omega))}\} = |H(j\omega)|.\cos(\omega t + \angle H(j\omega))$$

So,

$$x(t) = \cos(\omega t) \quad \Longrightarrow \quad y(t) = |H(j\omega)|\cos(\omega t + \angle H(j\omega)), \qquad -\infty < t < \infty$$

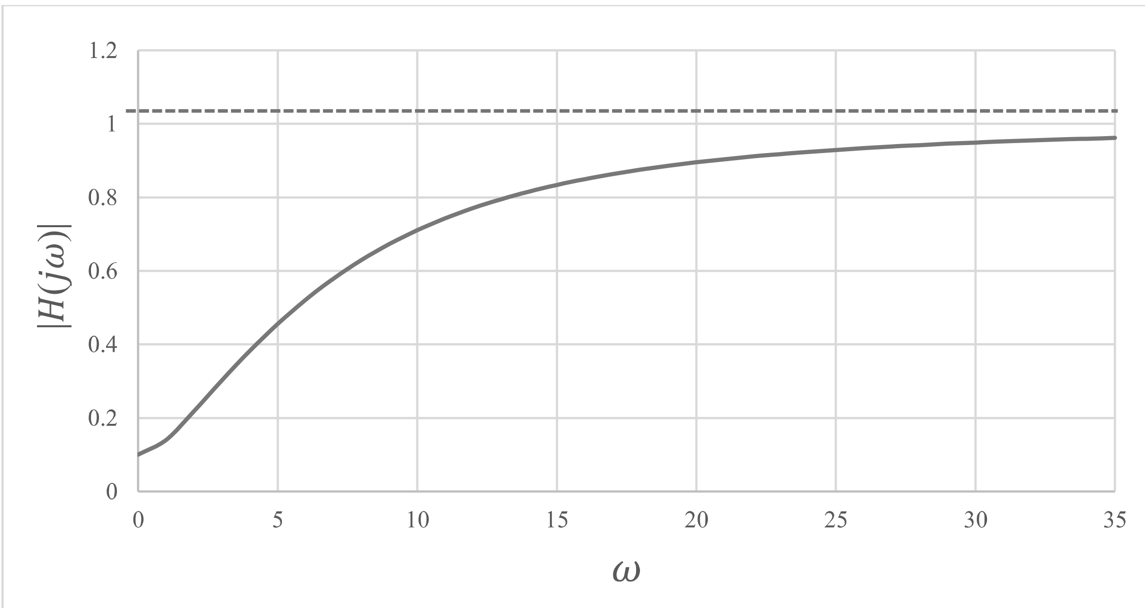
Remark: An *everlasting sinusoid input* results in an *everlasting sinusoid output*.

Remark: The everlasting response is sometimes called the *steady-state response*.

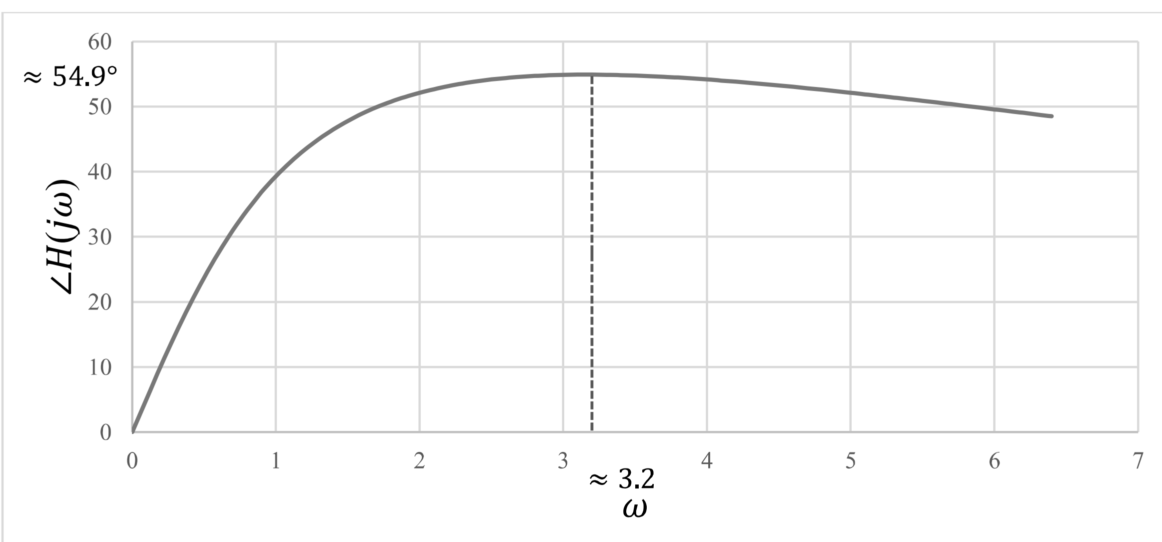
Example: A system has the transfer function: $H(s) = \frac{s+1}{s+10}$

Frequency Response: $H(j\omega) = \frac{j\omega+1}{j\omega+10}$

Magnitude: $|H(j\omega)| = \left[\frac{\omega^2+1}{\omega^2+100}\right]^{1/2}$



Phase: $\angle H(j\omega) = \tan^{-1}(\omega) - \tan^{-1}\left(\frac{\omega}{10}\right)$



For the input $x(t) = 2\sin(3t), \quad -\infty < t < \infty$, the output $y(t)$ is an everlasting sinusoid that has the same frequency $\omega = 3$ and can be found as follows:

$$|H(j3)| = \left[\frac{9+1}{9+100}\right]^{1/2} \approx 0.3$$

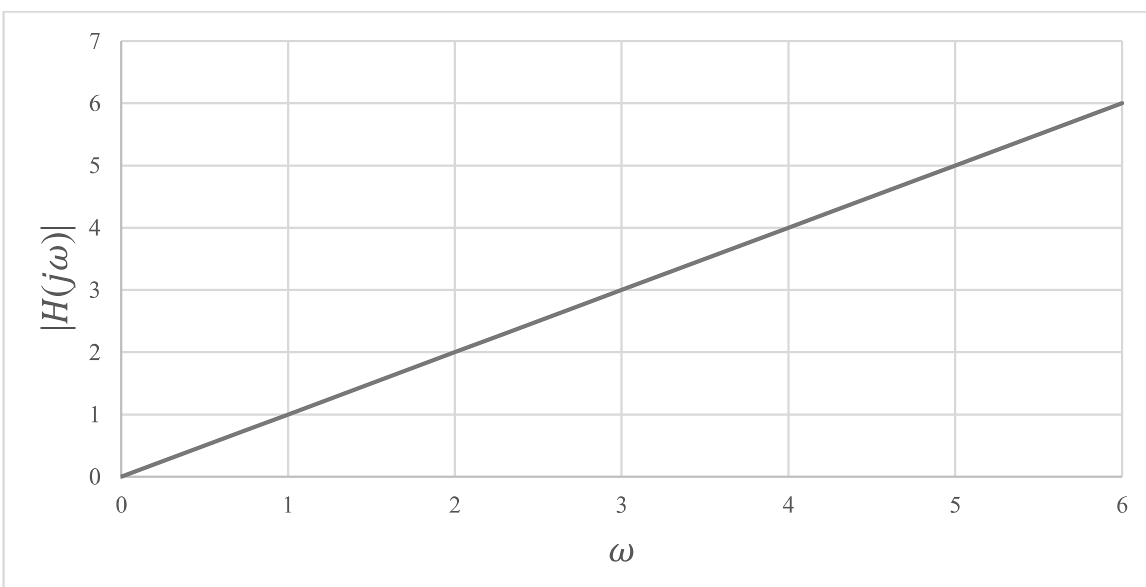
$$\angle H(j3) = \tan^{-1}(3) - \tan^{-1}\left(\frac{3}{10}\right) \approx 54.87^{\text{o}}$$

$$\Longrightarrow \quad y(t) = 2(0.3)\sin(3t + 54.87^{\text{o}})\ , \quad -\infty < t < \infty$$

<u>Example</u>: (Frequency Response of an Ideal Differentiator)

$$y(t) = \frac{dx(t)}{dt} \quad \Longrightarrow \quad Y(s) = sX(s) \quad \Longrightarrow \quad H(s) = \frac{Y(s)}{X(s)} = s$$

Frequency response: $H(j\omega) = j\omega = \omega e^{j\frac{\pi}{2}} \;\Rightarrow\; |H(j\omega)| = \omega$ and $\angle H(j\omega) = \frac{\pi}{2}$

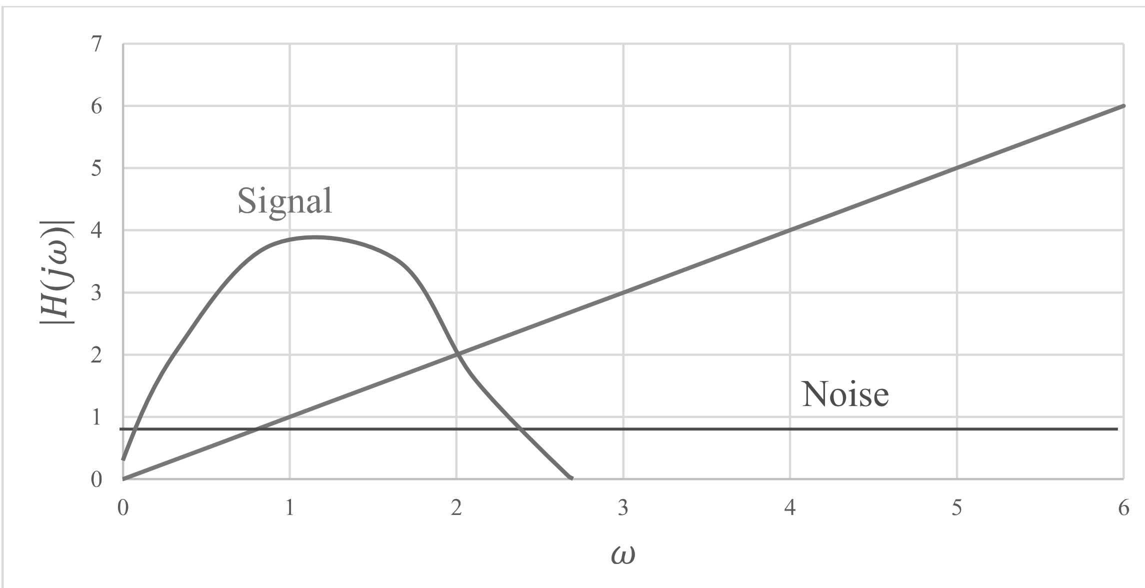


The magnitude of the output: $|Y(j\omega)| = \omega|X(j\omega)|$

However, in presence of <u>noise</u>:

$$|X(j\omega)| \;\Rightarrow\; |X(j\omega)| + \underbrace{|N(j\omega)|}_{\text{noise}}$$

$$\Rightarrow \quad |Y(j\omega)| = \omega|X(j\omega)| + \underbrace{\omega|N(j\omega)|}_{\substack{\text{can be very large for large } \omega. \\ \text{(a practical problem)}}}$$



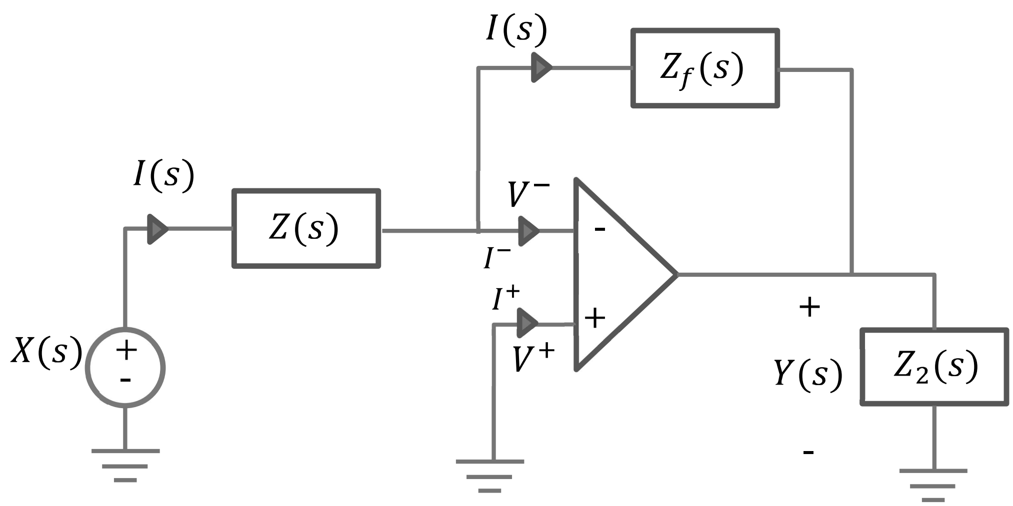
This is the reason differentiator elements are avoided in the realization of systems, in addition to being BIBO unstable.

<u>Remark</u>: In contrast to the enhancement of noise by the differentiator, the integrator suppresses noise. See Example 4.28 in textbook for more examples on frequency response.

<u>Appendix 4A</u>: System Realization Using Operational Amplifier Circuits

Operational amplifier circuits can be used to realize and simulate LTI systems. Here we show examples of op-amp realizations for the basic elements, the scalar multiplier, the integrators, and the adder/subtractor.

- **Basic General Inverting Op-Amp Circuit**



Assume zero initial conditions and "ideal" op-amp ($V^- = V^+$ and $I^- = I^+ = 0$):

$$I(s) = \frac{X(s) - \overbrace{V^-}^{=0}}{Z(s)} = \frac{\overbrace{V^-}^{=0} - Y(s)}{Z_f(s)} \quad \Longrightarrow \quad Y(s) = -\frac{Z_f(s)}{Z(s)} X(s)$$

$$\Longrightarrow \quad H(s) = \frac{Y(s)}{X(s)} = -\frac{Z_f(s)}{Z(s)} \qquad \text{(Transfer Function)}$$

<u>Remark</u>: Using above circuit, various rational transfer functions can be realized by choosing appropriate impedances $Z_f(s)$ and $Z(s)$.

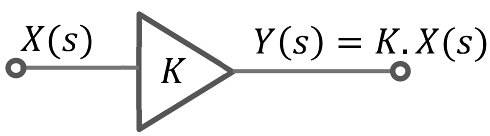
<u>Remark</u>: Non-inverting amplifier configuration can also be used to realize system.

- **Scalar Multiplier (Amplifier)**

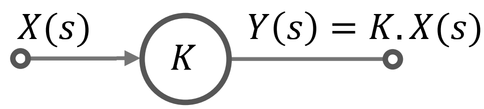
Take $Z_f(s) = R_f, \quad Z(s) = R$

$$\Rightarrow \quad Y(s) = \underbrace{-\frac{R_f}{R}}_{K<0} X(s)$$

Block diagram symbol:



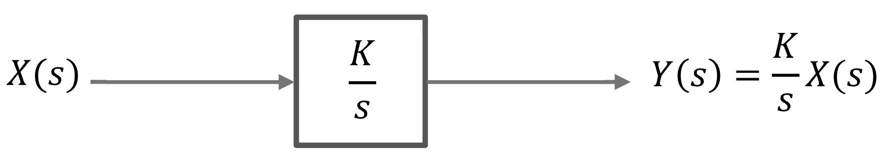
or



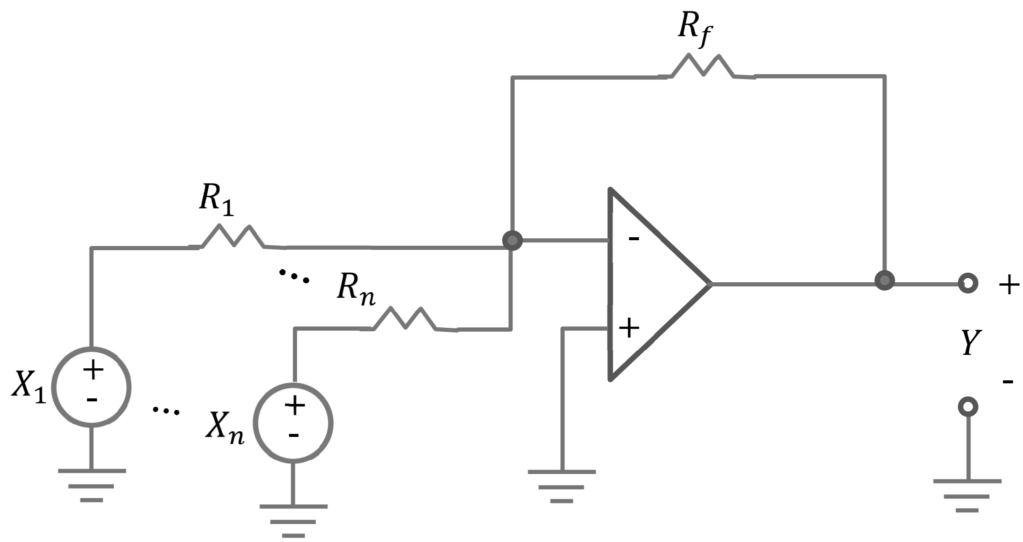
- **Integrator Amplifier**

Take $Z_f(s) = \frac{1}{sC} \quad (capacitor), \quad Z(s) = R$

$$\Rightarrow \quad Y(s) = \underbrace{\left(\frac{-1}{RC}\right)}_{K} . \underbrace{\frac{X(s)}{s}}_{\text{integerator}}$$

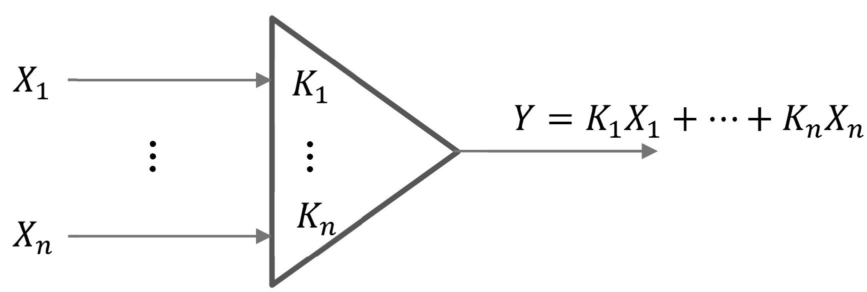


- **Adder/Subtractor (Summing Amplifier)**



$$Y(s) = -\frac{R_f}{R_1}X_1(s) - \cdots - \frac{R_f}{R_n}X_n(s)$$

Block diagram symbol:



or