Jessica Li

PREAMBLE

1. Helen Xiao

2. ~ 12 hours (theory: 4 hr, computational: 8 hr)

3. macrotrends - S&P 500 Historical Annual Returns
   wikipedia - list of largest daily changes in the S&P 500 Index
   netcials

4. Confirming I have not shared code nor used anyone else's code.

PROBLEM 1

(1) $X \sim N(\mu, 0)$

$$\int \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2}} dx$$

$$y = x - \mu \rightarrow \frac{dy}{dx} = 1 \rightarrow dy = dx$$

$$\int \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = 1 \checkmark$$

$X \sim N(\mu, \sigma^2)$

$$\int \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx = \int \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$z = \frac{x-\mu}{\sigma} \rightarrow \frac{dz}{dx} = \frac{1}{\sigma} \rightarrow \sigma dz = dx$$

$$\int \frac{1}{\cancel{\sigma}\sqrt{2\pi}} e^{-\frac{z^2}{2}} \cancel{\sigma} dz = 1 \checkmark$$

(2)
$$E(e^{tx}) = \int e^{tx} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \int e^{tx} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-x^2+2x\mu-\mu^2}{2\sigma^2}} dx$$

$$= \int \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-x^2+2x(\mu+\sigma^2 t)-\mu^2}{2\sigma^2}} dx$$

$$= \int \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-(\mu+\sigma^2 t))^2 + \cancel{\mu^2} + 2\mu\sigma^2 t + \sigma^4 t^2 - \cancel{\mu^2}}{2\sigma^2}} dx$$

$$= e^{\mu t + \frac{1}{2}\sigma^2 t^2} \underbrace{\int \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-(\mu+\sigma^2 t))^2} dx}_{X \sim N(\mu+\sigma^2 t, \sigma^2)}$$

$$= e^{\mu t + \frac{1}{2}\sigma^2 t^2} \checkmark$$

(3) if X_1 and X_2 are independent, then $m_{X_1}(t) m_{X_2}(t) = m_{X_1+X_2}(t)$

$$m_{X_1}(t) m_{X_2}(t) = e^{\mu_1 t + \frac{1}{2}\sigma_1^2 t^2} \cdot e^{\mu_2 t + \frac{1}{2}\sigma_2^2 t^2}$$

$$= e^{(\mu_1+\mu_2)t + \frac{1}{2}(\sigma_1^2+\sigma_2^2)t^2}$$

$$= m_{X_1+X_2}(t)$$

$\rightarrow$ therefore $X_1 + X_2 \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$

PROBLEM 2

(1) $\mu = \pi(Y_t+1) + (1-\pi)(Y_t-1)$

$= \cancel{\pi Y_t} + \pi + Y_t - 1 - \cancel{\pi Y_t} + \pi$

$\boxed{\mu = 2\pi + Y_t - 1}$

$\sigma^2 = E(x^2) + (E(x))^2$

$= \pi(Y_t+1)^2 + (1-\pi)(Y_t-1)^2 + (2\pi + Y_t - 1)^2$

$\boxed{\sigma^2 = 4\pi - 4\pi^2}$

(2) $E(aX) = aE(X) \rightarrow \boxed{\mu = a(2\pi + Y_t - 1)}$

$Var(aX) = a^2\, Var(X) \rightarrow \boxed{\sigma^2 = a^2(4\pi - 4\pi^2)}$

(3) $E(aY_{t+1} + b(1+r)) = E(aY_{t+1}) + E(b(1+r)) \rightarrow \boxed{\mu = a(2\pi + Y_t - 1) + b + br}$

$Var(aY_{t+1} + b(1+r)) = Var(aY_{t+1}) \rightarrow \boxed{\sigma^2 = a^2(4\pi - 4\pi^2)}$

PROBLEM 3

(1) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$

$AA^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ ✓

$A^{-1}A = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ ✓

(2)

$F = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad F - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \quad |F - \lambda I| = \lambda^2 - \lambda - 1 = 0 \rightarrow \boxed{\lambda = \frac{1 \pm \sqrt{5}}{2}}$

↑ eigenvalues

$\begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

① $\lambda = \frac{1+\sqrt{5}}{2} : \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}$

eigenvectors

② $\lambda = \frac{1-\sqrt{5}}{2} : \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$

(3)

Let $A = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$ and v_i be a $N \times 1$ vector of all 0s except for $v_{i1} = i$, $i \in [1, n]$

Suppose $i = 1$:

$$v_1' A v_1 = [1 \ 0 \ldots 0] \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \lambda_1 > 0$$

For all i, $v_i' A v_i = \lambda_i$. Hence, the matrix A is positive definite if $\lambda_i > 0$, $i \in [1, n]$

(4) $\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}$ where $a, b < 0$ ★ negative diagonal terms

$\begin{bmatrix} var(x) & cov(x,y) \\ cov(x,y) & var(y) \end{bmatrix}$ ★ variance is always positive and so a covariance matrix has positive diagonal terms

PROBLEM 4

(1) worst 5 years: 2008, 2002, 2022, 2001, 2000

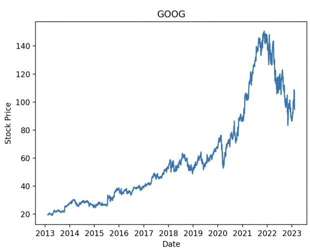
↳ recession, high interest rates, unemployment rates ↑

best 3 years: 1995, 1997, 2013

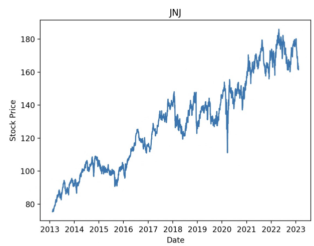
(2) top 3 days w/ worst performance: 10/19/1987: −20.47% → Black Monday → consumer underconfidence → heavy selling

3/16/2020: −11.98%
3/12/2020: −9.51%
} COVID-19 → unemployment + uncertainty ↑

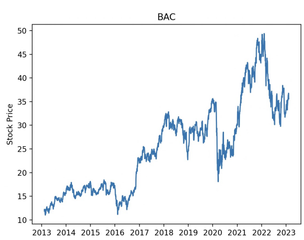
(3)



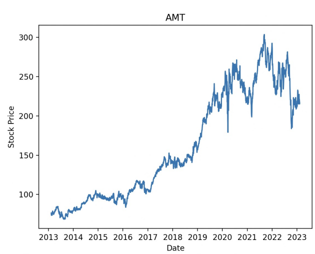
- steady increase up until 2020
- dip in 2020 due to COVID-19
- quick rebound in 2021
- decrease in 2022 due to increase in interest rates + overhiring



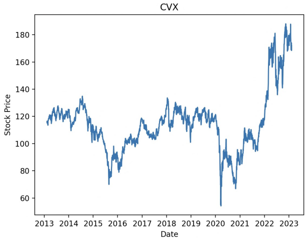
- consistent, upward trend over last 10 years
- dip in 2020 b/c of COVID
- no decrease in 2022 b/c investors like safer assets during times of volatility (ex: healthcare giant JNJ)



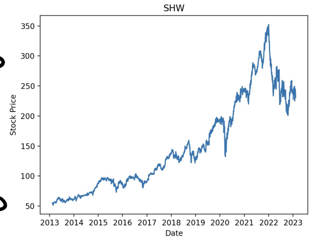
- drop in 2016 b/c low energy prices detered energy industry from borrowing, Brexit, and low interest rates
- drop in 2020 b/c of COVID-19
- decrease in 2022 due to inflation threatening consumer demand



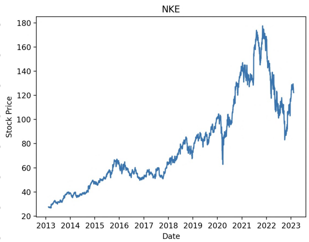
- steady increase until sudden increase in 2019 probably due to expansion + diversific-ation
- dip in 2020 due to COVID-19
- drop in 2022



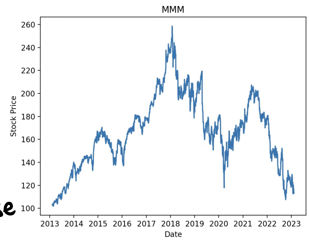
- decrease from 2014-2016 due to low oil prices
- increase until 2018 + stagnant until drop in 2020 due to COVID
- massive increase between 2021-2022



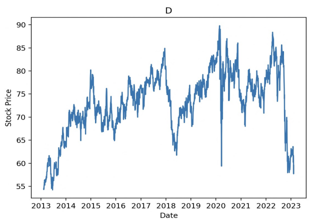
- steady increase up until 2020
- dip in 2020 b/c of COVID-19
- increase until 2022
- drop in 2022 due to decreases in demand for materials



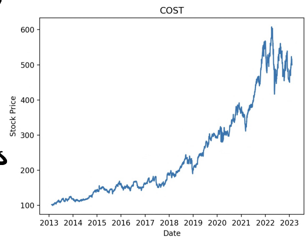
- upward trend from 2013-2020 except for slight dip in 2016
- dip in 2020 due to COVID-19
- tremendous increase until decrease in 2022
- current upswing



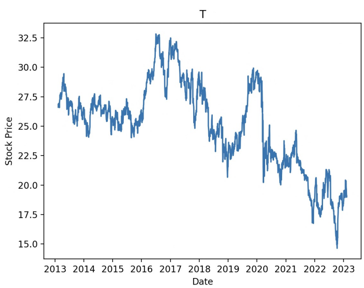
- steep increase until 2015
- volatile between 2015 and 2017
- downward trajectory ending with dip in 2020 due to COVID-19
- increase until mid-2021 where 3M decreased



- steep increases until 2015
- highly volatile from 2015-2023 w/ massive spikes and drops

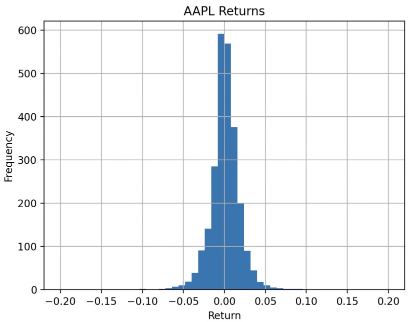


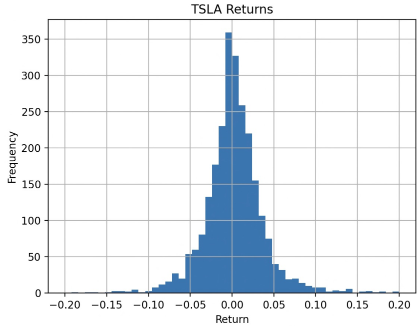
- steady + consistent upward trend
- slight dips in 2019 and 2021 but recovered fast
- slight instability from 2022-present
- overall highly stable



- overall downward trend w/ some increases in 2016 and 2019
- worst drops in 2020 due to COVID-19
- downward probably due to stalls in growth, debt and investor conflict

(4)



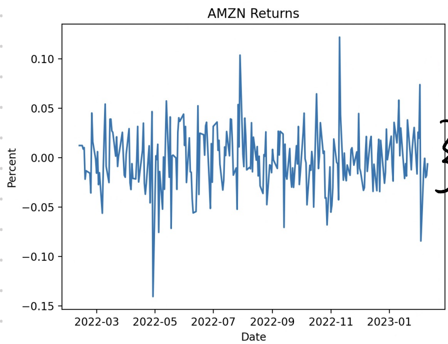


★ biggest drop all time
↳ 9/29/2000
↳ −52%

last 10 years:
- largest drop: 3/20/2020
  ↳ −7.26%
- largest increase: 8/24/2015
  ↳ +8.7%

last 10 years:
- largest drop: 8/24/2022
  ↳ −66.72%
- largest increase: 2/3/2020
  ↳ 15.78%

(5)

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-0.140494377749234 2022-04-29 00:00:00-04:00
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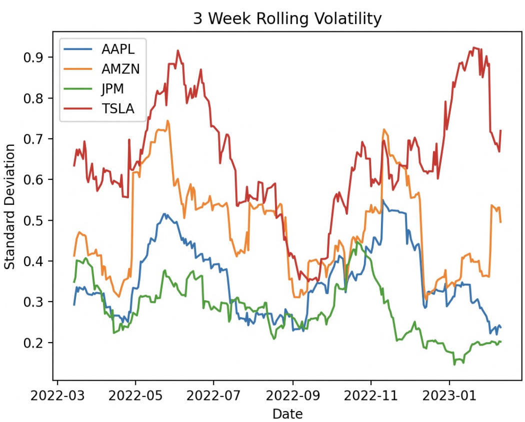
1 drop below 10%: 4/29/2022 → −14.05%

↳ no Amazon is not suddenly worth 10% less

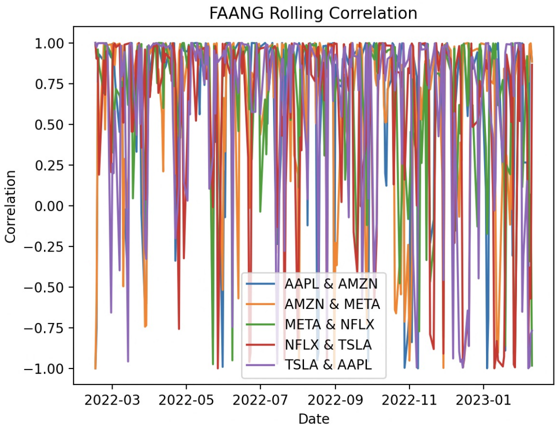
↳ "normal" fluctuation: 3%

(6)



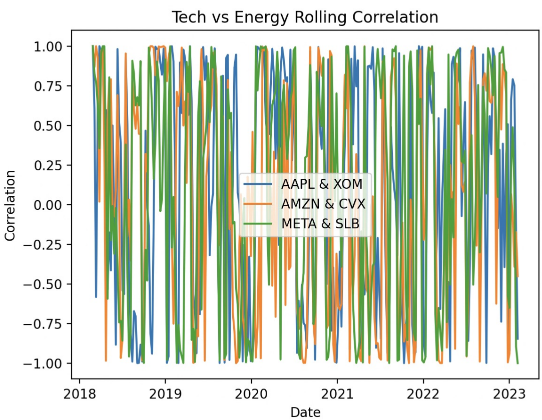
- JPM is the least volatile out of the 4
- this makes sence since the past year has been quite unstable for the tech sector
- most to least volatile: TSLA → AMZN → AAPL → JPM

(7)

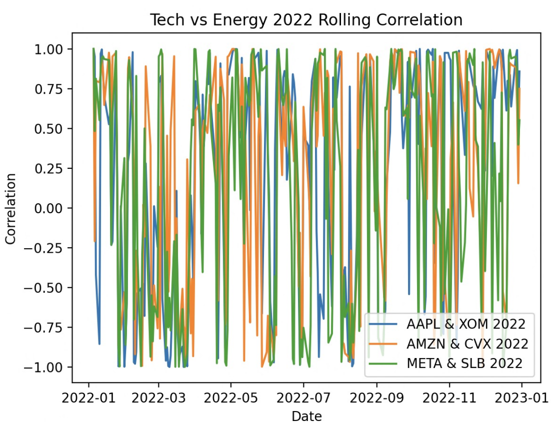


YES, it matches my intuition as the graph is heavily concentrated between 0.75 and 1. This implies that the FAANG companies are really correlated which makes sense since they are in the same sector and are influenced by the same factors.

(8)

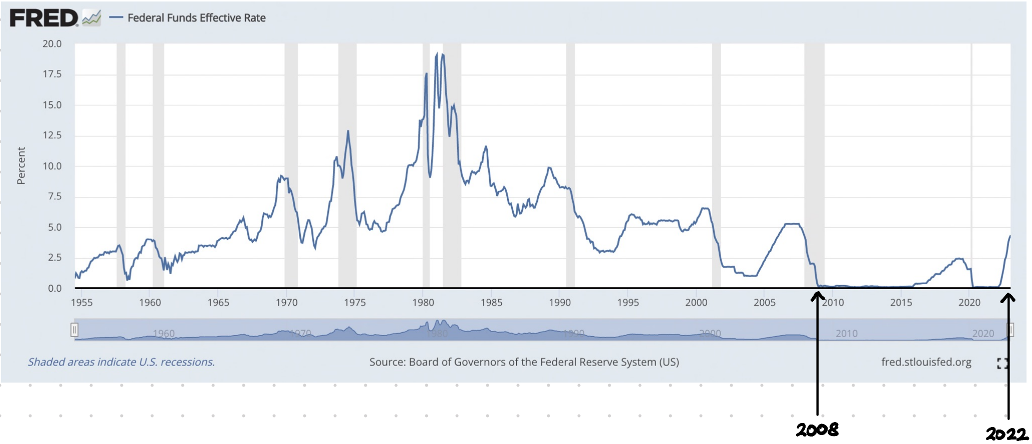


relatively evenly distributed across all correlations which implies little to no correlation between tech + energy stocks



In 2022, tech and energy stocks are slightly more correlated than usual as seen by some bunching of the graph between 0.5 and 1. This may be b/c of strong changes in inflation + interest rates which affect stocks across all markets.

(9)



2008: interest rates are low b/c the FED wanted to incentivize borrowing in order to restore liquidity in the banking sector and stimulate economic activity

2022: interest rates are high b/c the FED wanted to combat high inflation rates by decentivizing consumer spending