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PREAMBLE

1. Helen Xiao, Iñaki Arango, Alex Cai

2. 9 hours (THEORY: 5, COMPUTATIONAL: 4)

3. ChatGPT

4. I have not shared my code with anyone and have not used anyone's code.

PROBLEM 1

$B_t \sim N(0,t) \rightarrow \sqrt{2\lambda}\, B_t \sim \sqrt{2\lambda}\, N(0,t)$

PDF: $u(\lambda,t,x) = \frac{1}{2\sqrt{\pi\lambda t}} e^{-\frac{x^2}{4\lambda t}}$

NEED TO SHOW: $\underset{①}{\frac{d}{dt} u(\lambda,t,x)} = \underset{②}{\lambda \frac{d^2}{dx^2} u(\lambda,t,x)}$

① $\frac{d}{dt} u(\lambda,t,x) = \frac{1}{2\sqrt{\pi\lambda}}\left[-\frac{1}{2}t^{-\frac{3}{2}} e^{\frac{-x^2}{4\lambda t}} + t^{-\frac{1}{2}} e^{\frac{-x^2}{4\lambda t}}\left(-\frac{x^2}{4\lambda}(-1)t^{-2}\right)\right]$

$= \underline{\frac{1}{2\sqrt{\pi\lambda t}} e^{-\frac{x^2}{4\lambda t}}\left[-\frac{1}{2t} + \frac{x^2}{4\lambda t^2}\right]}$

② $\frac{d}{dx} u(\lambda,t,x) = \frac{1}{2\sqrt{\pi\lambda t}} e^{-\frac{x^2}{4\lambda t}}\left(-\frac{x}{2\lambda t}\right)$

$\frac{d^2}{dx^2} u(\lambda,t,x) = \frac{1}{2\sqrt{\pi\lambda t}}\left[-\frac{1}{2\lambda t} e^{-\frac{x^2}{4\lambda t}} + e^{-\frac{x^2}{4\lambda t}}\left(-\frac{x}{2\lambda t}\right)\left(-\frac{x}{2\lambda t}\right)\right]$

$= \frac{1}{2\sqrt{\pi\lambda t}} e^{-\frac{x^2}{4\lambda t}}\left[-\frac{1}{2\lambda t} + \frac{x^2}{4\lambda^2 t^2}\right]$

$\lambda \frac{d^2}{dx^2} u(\lambda,t,x) = \underline{\frac{1}{2\sqrt{\pi\lambda t}} e^{-\frac{x^2}{4\lambda t}}\left[-\frac{1}{2t} + \frac{x^2}{4\lambda t^2}\right]}$

Since $\frac{d}{dt} u(\lambda,t,x) = \lambda \frac{d^2}{dx^2} u(\lambda,t,x)$, the PDF of $\sqrt{2\lambda}\, B_t$ satisfies the diffusion equation.

PROBLEM 2

THEOREM: $\langle B \rangle_T = T$

PROOF: $\langle B \rangle_T = \sum_{i=1}^{n} (B_{t_i} - B_{t_{i-1}})^2$

$$E[\langle B \rangle_T] = \sum_{i=1}^{n} E(B_{t_i} - B_{t_{i-1}})^2$$

$$= \sum_{i=1}^{n} t_i - t_{i-1}$$

$$= T$$

★ $B_{t_i} - B_{t_{i-1}} \sim N(0, t_i - t_{i-1})$
$\overset{d}{=} \sqrt{t_i - t_{i-1}}\, N(0,1)$

<u>NEED TO SHOW</u>: $Var(\langle B \rangle_T) \to 0$

$$Var(\langle B \rangle_T) = E\left[\langle B \rangle_T - E[\langle B \rangle_T]\right]^2$$

$$= E\left[\sum_{i=1}^{n}\left[(B_{t_i} - B_{t_{i-1}})^2 - (t_i - t_{i-1})\right]\right]^2$$

$$= \sum_{i=1}^{n} E\left[(B_{t_i} - B_{t_{i-1}})^2 - (t_i - t_{i-1})\right]^2 + \underbrace{2\sum_{i<j} E\left[(B_{t_i} - B_{t_{i-1}})^2 - (t_i - t_{i-1})\right]\left[(B_{t_j} - B_{t_{j-1}})^2 - (t_j - t_{j-1})\right]}$$

$$= \sum_{i<j} E\left[(B_{t_i} - B_{t_{i-1}})^2 (B_{t_j} - B_{t_{j-1}})^2\right] - E\left[(B_{t_i} - B_{t_{i-1}})^2 - (t_j - t_{j-1})\right] - E\left[(B_{t_j} - B_{t_{j-1}})^2 - (t_i - t_{i-1})\right] + E\left[(t_i - t_{i-1})(t_j - t_{j-1})\right]$$

$$= (t_i - t_{i-1})(t_j - t_{j-1}) - (t_j - t_{j-1})(t_i - t_{i-1}) - (t_j - t_{j-1})(t_i - t_{i-1}) + (t_i - t_{i-1})(t_j - t_{j-1})$$

$$= 0$$

$$= \sum_{i=1}^{n}\left[E\left[(B_{t_i} - B_{t_{i-1}})^4\right] - 2E\left[(B_{t_i} - B_{t_{i-1}})^2\right](t_i - t_{i-1}) + (t_i - t_{i-1})^2\right]$$

$$= \sum_{i=1}^{n}\left[3(t_i - t_{i-1})^2 - 2(t_i - t_{i-1})^2 + (t_i - t_{i-1})^2\right]$$

$$= 2\sum_{i=1}^{n} (t_i - t_{i-1})^2$$

$$= 2\sum_{i=1}^{n} \frac{1}{n^2} \to 0 \text{ as } n \to \infty \quad ✓$$

PROBLEM 3

(1) $\frac{dp}{dt} = p\left(1 - \frac{p}{k}\right) = \frac{p(k-p)}{k}$

$\int \frac{1}{p(k-p)}\,dp = \int \frac{1}{k}\,dt$

$\frac{1}{k}\int \frac{1}{p} + \frac{1}{k-p}\,dp = \frac{t}{k} + c$

$\ln p - \ln(k-p) = t + c$

$\ln\left(\frac{p}{k-p}\right) = t + c$

$\frac{p}{k-p} = Ae^t \qquad * \; t=0 \rightarrow A = \frac{P_0}{k-P_0}$

$p = kAe^t - pAe^t$

$p(t) = \frac{kAe^t}{1+Ae^t} \rightarrow p(t) = \frac{P_0 k e^t}{(k-P_0)+P_0 e^t}$

SIMULATION:

$p(t) = \frac{5(100)e^t}{(100-5)+5e^t} = \frac{500e^t}{95+5e^t}$

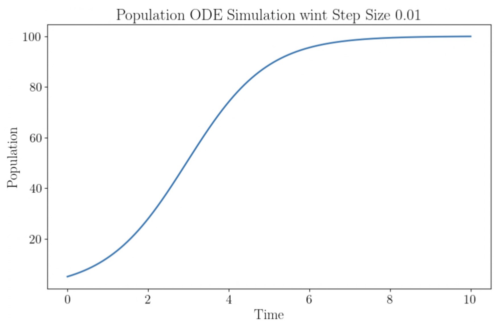
SOLUTION FOR $t=1$: $p(t) = \frac{500e}{95+5e} \approx \underline{\underline{12.516}}$

```python
def dpdt(p, K):
    return p * (1 - p/K)

def euler(p0, K, t_max, step_size):
    t = [0]
    p = [p0]
    for i in range (int(t_max/step_size)):
        t.append(t[i] + step_size)
        p_new = p[i] + step_size * dpdt(p[i], K)
        p.append(p_new)
    return t, p

p0 = 5
K = 100
t_max = 10
step_size = 0.01

t, p = euler(p0, K, t_max, step_size)
t[100], p[100]
```

✓ 0.0s

```
(1.0000000000000007, 12.470668118998324)
```

↳ very close to 12.516




```python
step_size = 1
t, p = euler(p0, K, t_max, step_size)
t[1], p[1]
```

✓ 0.0s

```
(1, 9.75)
```

We can see that larger step sizes are less accurate in approximating solutions.

(2) $\frac{dx}{dt} = \frac{1}{1-t}$

$\int 1\,dx = \int \frac{1}{1-t}\,dt$

$x = -\ln|1-t| + c$

$\begin{cases} x_0 = 0 \\ 0 = -\ln|1-0| + c \\ c = 0 \end{cases}$

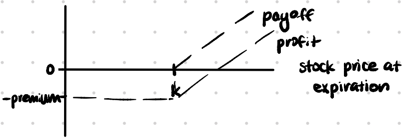
$\boxed{x = -\ln|1-t|}$

↳ this blows up in finite time since x becomes undefined when $t=1$

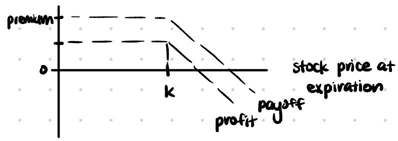
PROBLEM 4

(1)

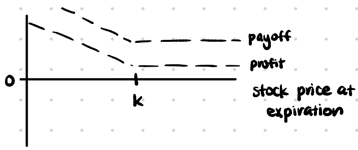
LONG CALL



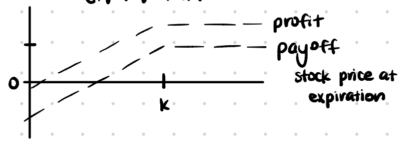
SHORT CALL



LONG PUT



SHORT PUT



(2) PUT-CALL PARITY : $\underbrace{C}_{\text{call price}} + \underbrace{Ke^{r(T-t)}}_{\text{PV of strike price}} = \underbrace{P}_{\text{put price}} + \underbrace{S}_{\text{spot price}}$

① find price of an asset + prices of call option + put option w/ same strike price + expiration

② verify put-call parity identify

$$2.5 + 155e^{0.05(0.5)} = 4.5 + 150 \rightarrow 161.42 \neq 154.5$$

(3) SPOT PRICE: \$169.4

STRIKE K		
	call option	put option
deep in the money	140	190
out of the money	180	150

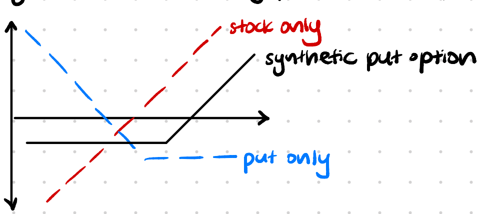
(4) buy a call option at k_1 (in the money): $k_1 < S$

2 call options (at the money): sell at S

buy a call option at k_2 (out of the money): $k_2 > S$

$\boxed{\text{payoff function: } (S_T - K_1) + 2(S - S_T) + 0}$

(5) buy stock + at-the-money put option to protect against depreciation in stock price



PROBLEM 5

$$d(f(t,B_t)) = \partial_t f(t,B_t)dt + \partial_x f(t,B_t)dB_t + \frac{1}{2}\partial^2_{xx} f(t,B_t)dt$$

(1) $X_t = B_t^4 \to f(x) = x^4$

$$\boxed{d(B_T^4) = 4B_t^3 dB_t + 6B_t^2 dt}$$

(2) $X_t = tB_t^3 \to f(t,x) = tx^3$

$$\boxed{d(TB_T^3) = B_t^3 dt + 3_t B_t^2 dB_t + 3tB_t dt}$$

(3) $X_t = t\cos(B_t) \to f(t,x) = t\cos(x)$

$$\boxed{d(t\cos B_T) = \cos(B_t)dt - t\sin B_t dB_t - \frac{t}{2}\cos B_t dt}$$

PROBLEM 6

ITÔ'S FORMULA 2.0:

$$d(f(t,B_t)) = \partial_t f(t,B_t)dt + \underbrace{\partial_x f(t,B_t)dB_t}_{\text{martingale}} + \frac{1}{2}\partial^2_{xx} f(t,B_t)dt$$

$\star$ if $\partial_t f = -\frac{1}{2}\partial^2_{xx} f$, $d(f(t,B_t))$ = martingale

(1) $m_t = e^{(B_t - \frac{t}{2})}$

① $\partial_t m_t = e^{(B_t-\frac{t}{2})}(-\frac{1}{2}) = \underline{-\frac{1}{2}e^{(B_t-\frac{t}{2})}}$

② $\partial_{B_t} m_t = \partial^2_{B_tB_t} m_t = e^{(B_t-\frac{t}{2})}$

$-\frac{1}{2}\partial^2_{B_tB_t} m_t = \underline{-\frac{1}{2}e^{(B_t-\frac{t}{2})}}$

since $\partial_t m_t = -\frac{1}{2}\partial^2_{B_tB_t} m_t$, m_t is a martingale.

(2) $E[m_{t+1}|F_t] = E\left[e^{(B_{t+1}-\frac{t+1}{2})}|F_t\right]$

$= E[e^{B_{t+1}}|F_t]\,e^{-(\frac{t+1}{2})}$

$= E[e^{B_{t+1}-B_t+B_t}|F_t]\,e^{-(\frac{t+1}{2})}$

$= E[e^{B_{t+1}-B_t}|F_t]\,E[e^{B_t}|F_t]\,e^{-(\frac{t+1}{2})}$

$= e^{\frac{1}{2}}e^{B_t}\,e^{-(\frac{t+1}{2})}$

$= e^{B_t-\frac{t}{2}}$

$= m_t$ ✓

PROBLEM 7

$m_t = \int_0^t f(B_u)\, dB_u$

THEOREM:

$Var(m_t) = E[m_t^2] = \int_0^t E[f^2(B_u)]\, du$

(1) $X_t = \int_0^t u\, dB_u$

$$Var(X_t) = \int_0^t E[u^2]\, du = \int_0^t u^2\, du = \frac{u^3}{3}\Big|_0^t = \boxed{\frac{t^3}{3}}$$

(2) $X_t = \int_0^t \cos(u)\, dB_u$

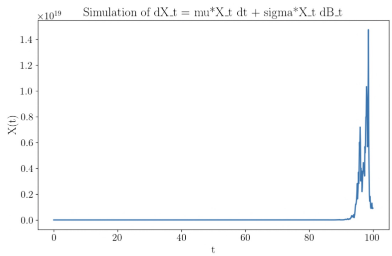
$$Var(X_t) = \int_0^t E[\cos^2(u)]\, du = \int_0^t \cos^2(u)\, du = \frac{u}{2} + \frac{1}{4}\sin(2u)\Big|_0^t = \boxed{\frac{t}{2} + \frac{1}{4}\sin(2t)}$$

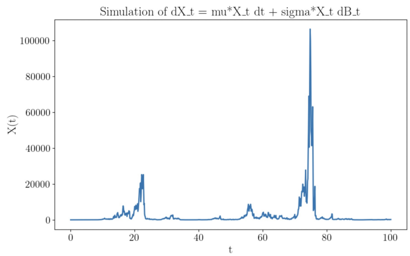
(3) $X_t = \int_0^t B_s^2\, dB_s$

$$Var(X_t) = \int_0^t E[B_s^4]\, ds = \int_0^t 3s^2\, ds = s^3\Big|_0^t = \boxed{t^3}$$

PROBLEM 8

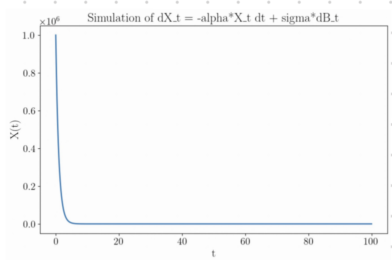
(1)

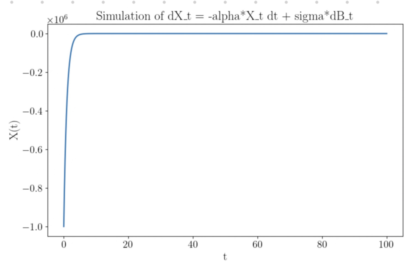




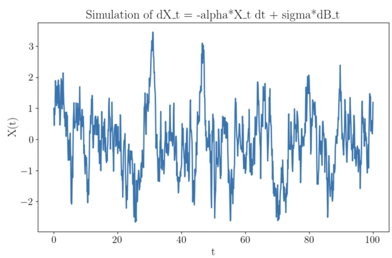
as $t \to \infty$, $X_t \to \infty$

(2)





X_t reverts to its long term mean of 0 for all X_0.



* histogram in jupyter notebook pdf

PROBLEM 9

(1) $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

$V(x) = -\frac{x^2}{2}$

SDE : $dX_t = -X_t d_t + \sqrt{2}\, dB_t$

↳ 8.2!!

(2) $f_X(x) \propto e^{-\frac{1}{2}(x-3)^2}$

$\boxed{\mu=3 \quad \sigma=1}$

$V(x) = -\frac{1}{2}(x-3)^2$

$V'(x) = -x+3$

SDE : $dX_t = -x+3+\sqrt{2}\, dB_t$

★ simulation in jupyter notebook pdf

(3) $f_X(x) \propto \frac{1}{2}e^{-\frac{1}{2}(x-3)^2} + \frac{1}{2}e^{-\frac{1}{2}(x+3)^2}$

★ simulation in jupyter notebook pdf