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Review

Estimating value at risk of portfolio by conditional copula-GARCH method

Jen-Jsung Huang [a], Kuo-Jung Lee [b,*], Hueimei Liang [c], Wei-Fu Lin [a]

[a] Department of Finance, National Sun Yat-sen University, 70, Lienhai Rd. Kaohsiung 80424, Taiwan, ROC

[b] Department of Commerce Automation and Management, National Pingtung Institute of Commerce, 51, Minsheng E. Rd., Pingtung City 900, Taiwan, ROC

[c] Department of Business Administration, National Sun Yat-sen University, 70, Lienhai Rd. Kaohsiung 80424, Taiwan, ROC

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ABSTRACT

Copula functions represent a methodology that describes the dependence structure of a multi-dimension random variable and has become one of the most significant new tools to handle risk factors in finance, such as Value-at Risk (VaR), which is probably the most widely used risk measure in financial institutions. Combining copula and the forecast function of the GARCH model, this paper proposes a new method, called conditional copula-GARCH, to compute the VaR of portfolios. This work presents an application of the copula-GARCH model in the estimation of a portfolio's VaR, composed of NASDAQ and TAIEX. The empirical results show that, compared with traditional methods, the copula model captures the VaR more successfully. In addition, the Student-t copula describes the dependence structure of the portfolio return series quite well.

Contents

1. Introduction

Value at Risk (VaR) has become the standard measure used by financial analysts to quantify the market risk of an asset or a portfolio (Hotta et al., 2008). VaR is defined as a measure of how the market risk of an asset or asset portfolio is likely to decrease over a certain time period under general conditions. It is typically implemented by securities houses or investment banks to measure the market risk of their asset portfolios (market value at risk), and yet it is actually a very general concept that has broader applications. However, VaR estimation is not difficult to compute if only a single asset in a portfolio is owned, and becomes very difficult due to the complexity of the joint multivariate distribution. Besides, one of the main difficulties in estimating VaR is to model the dependence structure, especially because VaR is concerned with the tail of the distribution (Hotta et al., 2008).

Theoretical research that relied on the VaR as a risk measurement was initiated by Jorion (1997) and Dowd (1998), who applied the VaR approach based on risk management emerging as the industry standard by choice or by regulation. Jorion (2000)

* Corresponding author. Tel.: +886 8 7238700x6167; fax: +886 8 7210801.
E-mail address: kjlee@npic.edu.tw (K.-J. Lee).

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provides an introduction to VaR as well as discussing its estimation. The existing related academic literatures of VaR focus mainly on measuring VaR from different estimation methods to various calculation models. The first classical works in VaR methodology distinguish mainly three traditional estimation concepts, i.e., the historical, Monte-Carlo and variance–covariance approaches.

Computational problems arise when one increases the number of assets in a portfolio. The traditional approaches for estimating VaR assume that the joint distribution is known, such as the most commonly used normality of the joint distribution of the assets return in theoretical and empirical models. The linear correlation assumes, for example, that the variance of the return on a risky asset portfolio depends on the variances of the individual assets and also on the linear correlation between the assets in the portfolio. In reality, the finance asset return distribution has fatter tails than normal distributions. Hence, it is shown in many empirical works that such multivariate distributions do not provide adequate results due to the presence of asymmetry and excess financial data.

Linear correlation has a serious deficiency; namely, it is not invariant under non-linear strictly increasing transformation. Meanwhile the dependence measures derived from copulas can overcome this shortcoming and have broader applications (Nelsen, 1997; Wei and Hu, 2002; Vandenhende and Lambert, 2003). Furthermore, copulas can be used to describe more complex multivariate dependence structures, such as non-linear and tail dependence (Hürlimann, 2004).

Longin and Solnik (2001) and Ang and Chen (2002) found evidence that asset returns are more highly correlated during volatile markets and during market downturns. It is obvious that a stronger dependence exists between big losses than between big gains. One is unable to model such asymmetries with symmetric distributions. The use of linear correlation to model the dependence structure shows many disadvantages, as found by Embrechts et al. (2002). Therefore, the problem raised from normality could lead to an inadequate VaR estimate.

In order to overcome these problems, this paper resorts to the copula theory which allows us to construct a flexible multivariate distribution with different margins and different dependent structures, which allows the joint distribution of the portfolio to be free from any normality and linear correlation. The dependence measures derived from copulas can overcome this shortcoming and have broader applications. Financial markets exist with high (low) volatility accompanied by high (low) volatility, which means heteroskedasticity in econometrics. This is explained and fitted by the well-known GARCH (generalized autoregressive conditional heteroskedastic) model and is widely reported in financial literature, as shown by Engle (1996) for an excellent survey.

Meanwhile, the copula method is based on the Sklar (1959) theorem which describes the copula as an indicator of the dependencies among the variables. It explains the dependent function or connection function which connects the joint distribution and the univariate marginal distribution. Copula in particular has recently become a most significant new tool, it is generally applied in the financial field, such as risk management, portfolio allocation, derivative asset pricing, and so on. In our work we focus on portfolio risk management, especially in estimating VaR.

Patton (2001) constructed the conditional copula by allowing the first and second conditional moments to vary in time. After the methodological expansion of Patton (2001), the conditional copula began to be used in the estimation of VaR. Time variation in the first and second conditional moments is widely discussed in the statistical literature, and so allowing the temporal variation in the conditional dependence in the time series seems to be natural. Rockinger and Jondeau (2001) used the Plackett copula with the GARCH process with innovations modeled by the Student-t asymmetrical generalized distribution of Hansen (1994), and proposed a new measure of conditional dependence. Palaro and Hotta (2006) used a mixed model with the conditional copula and multivariate GARCH to estimate the VaR of a portfolio composed of NASDAQ and S&P500 indices. Jondeau and Rockinger (2006) took normal GARCH based copula for the VaR estimation of a portfolio composed of international equity indices.

This paper combines GARCH and copula to fit the financial data and to present a more adequate model in order to replace the classical joint multivariate normal distribution. The conditional copula-GARCH model, built for computing the VaR of portfolios, should be more reasonable and adequate. The conditional means being full of all past information, and it is mainly due to forecasting and fitting purposes that we estimate the one-day ahead VaR. Our work analyzes a portfolio composed of the NASDAQ and TAIEX indices with daily returns and estimates of the one-day ahead VaR position following the flexible copula model.

This paper is related to Palaro and Hotta (2006) and Ozun and Cifter (2007), in which they discuss the application of conditional copula in estimating the VaR of a portfolio. But unlike the two literatures, we apply various copulas completely with different marginal distribution to estimate VaR of a portfolio with two assets, NASDAQ and TAIEX. In addition, compared with traditional methods (including the historical simulation method, variance–covariance method, exponential weighted moving average method), this paper proves that the Student-t copula-GARCH model captures the VaR of the portfolio more successfully.

The rest of the paper is organized as follows. Section 2 presents marginal models including the GARCH and GJR models. Section 3 presents Sklar's theorem and the copula families. In addition, we introduce the estimation procedures of VaR. Section 4 presents the empirical procedure and results, followed by a conclusion in Section 5.

2. Model for the marginal distribution

GARCH models have become important in the analysis of time series data, particularly in financial applications when the goal is to analyze and forecast volatility. It was first observed by Engle (1982) that although many financial time series, such as stock returns and exchange rates, are unpredictable, there is an apparent clustering in the variability or volatility. This is often referred to as conditional heteroskedasticity, since it is assumed that overall the series is stationary, but the conditional expected value of the variance may be time-dependent. Our marginal model is built on the classical GARCH model and the GJR model, in which the standard innovation is to obey the normal distribution and Student-t distribution respectively.

2.1. GARCH-n and GARCH-t model

Let the returns of a given asset be given by $\{X_t\}t = 1, \ldots, T$. We consider that GARCH(1,1) with standard innovation is a standard normal (GARCH-n) or a standardized Student-t (GARCH-t) distribution respectively, where the model is as follows:

$$\begin{aligned} &x_t = \mu + a_t \\ &a_t = \sigma_t \varepsilon_t \\ &\sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \beta \sigma_{t-1}^2 \\ &\varepsilon_t \sim N(0,\ 1) \quad \text{or} \quad \varepsilon_t \sim t_d. \end{aligned} \tag{1}$$

Here, $\mu = E(x_t) = E(E(x_t|\Omega_{t-1})) = E(\mu_t) = \mu$ is the unconditional mean of series return, $\sigma_t^2 = Var(x_t|\Omega_{t-1}) = Var(a_t|\Omega_{t-1})$ is the conditional variance, $\alpha_0 > 0$, $\alpha_1 \geq 0$, $\beta \geq 0$, and $\alpha_1 + \beta < 1$, where Ω_{t-1} is the information set at $t-1$. In the normal case, $\alpha_1 + \beta < 1$ is sufficient for a stationary covariance, the ergodic process, and implies that the unconditional variance of a_t is finite, whereas its conditional variance σ_t^2 evolves over time. In the case of non-normal distributions, the condition is $\alpha_1 Var(\varepsilon_t)+\beta < 1$. Under slightly weaker conditions, a_t may be ergodic and strictly stationary. Besides, d are the degrees of freedom. The method we estimate for the parameters is MLE. We let $\Omega_{t-1} = \{a_0, a_1, \dots, a_{t-1}\}$. The joint density function can then be written as $f(a_1, \dots, a_t) = f(a_t|\Omega_{t-1})f(a_{t-1}|\Omega_{t-2}) \dots f(a_1|\Omega_0)f(a_0)$. Given data $a_1, \dots, a_t$ the log-likelihood is the following:

$$LLF = \sum_{k=0}^{n-1} f(a_{n-k}|\Omega_{n-k-1}). \tag{2}$$

This can be evaluated using the model volatility equation for any assumed distribution for ε_t. Here, LLF can be maximized numerically to obtain MLE. The method for estimating the parameters above is the MLE method, which is introduced in the following section. We use the observation $(x_1, x_2, \dots, x_t)$ to get the conditional marginal distribution of X_{t+1} defined as the following:

$$\begin{aligned} &P(X_{t+1} \le x|\Omega_t) = P(a_{t+1} \le (x-u)|\Omega_t) \\ &= P(\varepsilon_{t+1} \le \frac{(x-\mu)}{\sqrt{\alpha_0 + \alpha_1 a_t^2 + \beta\sigma_t^2}}|\Omega_t) \\ &= \begin{cases} N\left(\left.\dfrac{(x-\mu)}{\sqrt{\alpha_0 + \alpha_1 a_t^2 + \beta\sigma_t^2}}\right|\Omega_t\right), & \text{if } \varepsilon \sim N(0,1) \\ t_d\left(\left.\dfrac{(x-\mu)}{\sqrt{\alpha_0 + \alpha_1 a_t^2 + \beta\sigma_t^2}}\right|\Omega_t\right), & \text{if } \varepsilon \sim t_d. \end{cases} \end{aligned} \tag{3}$$

2.2. GJR-n and GJR-t model

In the GJR model (see Glosten et al., 1993) the following is the volatility generating process, where GJR-n means the standard innovation is a standard normal distribution and GJR-t means the standard innovation is a standardized Student-t distribution.

$$\begin{aligned} &x_t = \mu + a_t \\ &a_t = \sigma_t\varepsilon_t \\ &\sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \beta\sigma_{t-1}^2 + \gamma s_{t-1} a_{t-1}^2 \\ &\varepsilon_t \sim N(0,1) \quad \text{or} \quad \varepsilon_t \sim t_d \\ &\quad \text{where } s_{t-1} = \begin{cases} 1, & a_{t-1} < 0 \\ 0, & a_{t-1} \ge 0. \end{cases} \end{aligned} \tag{4}$$

Moreover, $\alpha_0 > 0, \alpha_1 \ge 0, \beta \ge 0, \beta+\gamma \ge 0$, and $\alpha_1+\beta+\frac{1}{2}\gamma < 1$, while s_t is a dummy variable which equals one when ε_t is negative and is nil elsewhere.

Unlike the classical GARCH model, the GJR model contains an asymmetric effort. Here, asymmetry is captured by the term multiplying γ. When γ is positive, it means that negative shocks ($\varepsilon < 0$) introduce more volatility than positive shocks of the same size in the subsequent period. The estimation of the parameters above are also introduced in the following section. The conditional marginal distribution of X_{t+1} is almost the same as the GARCH model, which is defined as the following:

$$\begin{aligned} &P(X_{t+1} \le x|\Omega_t) = P\left(\varepsilon_{t+1} \le \left.\frac{(x-\mu)}{\sqrt{\alpha_0 + \alpha_1 a_t^2 + \beta\sigma_t^2 + \gamma \cdot s_t\varepsilon_t^2}}\right|\Omega_t\right) \\ &= \begin{cases} N\left(\left.\dfrac{(x-\mu)}{\sqrt{\alpha_0 + \alpha_1 a_t^2 + \beta\sigma_t^2 + \gamma \cdot s_t\varepsilon_t^2}}\right|\Omega_t\right), & \text{if } \varepsilon \sim N(0,1) \\ t_d\left(\left.\dfrac{(x-\mu)}{\sqrt{\alpha_0 + \alpha_1 a_t^2 + \beta\sigma_t^2 + \gamma \cdot s_t\varepsilon_t^2}}\right|\Omega_t\right), & \text{if } \varepsilon \sim t_d \end{cases} \end{aligned} \tag{5}$$

3. Copula theory and estimation procedures

In statistics literature the idea of a copula arose as early as the 19th century in the context of discussions of non-normality in multivariate cases. Modern theories about copulas can be dated to about forty years ago when Sklar (1959) defined and provided some fundamental properties of a copula.

3.1. Sklar's theorem

Let F denote an n-dimensional distribution function with margins $F_1, F_2, \dots, F_n$, and then there exists a copula representation (canonical decomposition) for all real $(x_1, x_2, \dots, x_n)$ such that:

$$\begin{aligned} F(x_1, \dots, x_n) &= P(X_1 \le x_1, \dots, X_n \le x_n) \\ &= C(P(X_1 \le x_1), \dots, P(X_n \le x_n)) \\ &= C(F_1(x_1), \dots, F_n(x_n)). \end{aligned} \tag{6}$$

When the variables are continuous, Sklar's theorem shows that any multivariate probability distribution function can be represented with a marginal distribution and a dependent structure, which is derived below:

$$\begin{aligned} f(x_1, \dots, x_n) &= \frac{\partial F(x_1, \dots, x_n)}{\partial x_1 \dots \partial x_n} \\ &= \frac{\partial C(u_1, \dots, u_n)}{\partial u_1 \dots \partial u_n} \times \prod_i \frac{\partial F_i(x_i)}{\partial x_i} \\ &= c(\tilde{u}) \times \prod_i f_i(x_i). \end{aligned} \tag{7}$$

Here, $f_i, i = 1, \dots, n$, is the density function of $F_i, i = 1, \dots, n$, and $u_i = F_i(x_i)$ for $i = 1, \dots, n$, $\tilde{u} = (u_1, \dots, u_2)$, and $c(\tilde{u})$ is the copula density function.

If all the margins are continuous, then the copula is unique and is in general otherwise determined uniquely by the ranges of the marginal distribution functions. An important feature of this result is that the marginal distributions do not need to be in any way similar to each other, nor is the choice of copula constrained by the choice of marginal distributions.

3.2. The copula family

The copula family used in our work includes commonly used copulas which are the Gaussian copula, the Student-t copula, and the Archimedean copula family such as the Clayton copula, Rotated-Clayton copula, Frank copula, Plackett copula, Gumbel copula and the Rotated-Gumbel copula. The class of Archimedean copulas was named by Ling (1965), but it was recognized by Schweizer and Sklar (1961) in the study of t-norms. The main reasons why they are of interest are that they are not elliptical copula, and allow us to model a big variety of different dependence structures. We consider in particular the one-parameter Archimedean copulas. This paper investigates the eight kinds of copula and examines whether they suit the financial data or not.

The copula family studied in this paper includes the Gaussian copula, Student-t copula, Clayton copula, Rotated-Clayton copula, Frank copula, Plackett copula, Gumbel copula, and the Rotated-Gumbel copula, which are shown as follows:

(1) Gaussian copula

For notational convenience, we set $u_i \equiv F_i(x_i)$. The Gaussian (or normal) copula is the copula of the multivariate normal distribution which is defined by the following:

$$C_{\text{Gaussian}}(u_1, u_2; \rho) = \Phi_\rho(\Phi^{-1}(u_1), \Phi^{-1}(u_2)), \tag{8}$$

where Φ_ρ is a joint distribution of a multi-dimensional standard normal distribution, with linear correlation coefficient ρ, Φ being the standard normal distribution function.

(2) Student-t copula

The t copula is based on the multivariate t distribution in the same way as the Gaussian copula is derived from the multivariate normal. The Student-t copula is defined by:

$$C_T(u_1, u_2; \rho, d) = t_{d,\rho}(t_d^{-1}(u_1), t_d^{-1}(u_2)). \quad (9)$$

The t copula is fast growing in usage, because the degree of tail dependency can be set by changing the degrees of freedom parameter, d. Large values for, d say $d = 100$, approximate a Gaussian distribution. Conversely, small values for, d say, $d = 3$ increase the tail dependency, until for $d = 1$ you simulate a Cauchy distribution.

(3) Clayton copula and Rotated-Clayton copula

The Clayton family was first proposed by Clayton (1978). The cumulative distribution functions (CDFs) are defined by the following:

$$C_{Clayton}(u_1, u_2; \omega) = (u_1^{-\omega} + u_2^{-\omega} - 1)^{-\frac{1}{\omega}} \quad (10)$$

$$C_{Rotated\text{-}Clayton}(u_1, u_2; \omega) = u_1 + u_2 - 1 + C_{Clayton}(1 - u_1, 1 - u_2; \omega) \quad (11)$$

where $\omega \in [-1, \infty)$.

(4) Plackett copula

Rockinger and Jondeau (2001) used Plackett's copula and a dependence measure to check whether the linear dependence varies with time. They worked with returns of European stock market series, the S&P500 index, and the Nikkei index. One disadvantage of Plackett's copula is that it cannot be easily extended for dimensions larger than two. The CDF is defined by the following:

$$C_{Plackett}(u_1, u_2; \eta) = \frac{1}{2(\eta - 1)}(1 + (\eta - 1)(u_1 + u_2) - \sqrt{(1 + (\eta - 1)(u_1 + u_2))^2 - 4\eta(\eta - 1)u_1 u_2}) \quad (12)$$

where $\eta \in [0, \infty)$.

(5) Frank copula

The Frank family, which appeared in Frank (1979), is discussed at length in Genest (1987). The Frank copula is defined by:

$$C_{Frank}(u_1, u_2; \lambda) = \frac{-1}{\lambda} \log\left(\frac{\lambda(1 - e^{-\lambda}) - (1 - e^{-\lambda u_1})(1 - e^{-\lambda u_2})}{1 - e^{-\lambda}}\right) \quad (13)$$

where $\lambda \in (-\infty, 0) \cup (0, +\infty)$.

(6) Gumbel copula and Rotated-Gumbel copula

The Gumbel family was introduced by Gumbel (1960). Since it has been discussed in Hougaard (1986), it is also known as the Gumbel–Hougaard family. The CDFs are defined by the following:

$$C_{Gumbel}(u_1, u_2; \delta) = \exp\left(-\left((-\log u_1)^{\delta} + (-\log u_2)^{\delta}\right)^{\frac{1}{\delta}}\right) \quad (14)$$

$$C_{Rotated\text{-}Gumbel}(u_1, u_2; \delta) = u_1 + u_2 - 1 + C_{Gumbel}(1 - u_1, 1 - u_2; \delta) \quad (15)$$

where $\delta \in [0, \infty)$.

3.3. Estimation method

This paper uses estimation methods such as the maximum likelihood method and inference function for margins (IFM) method. First, the expression for the log-likelihood function is:

$$l(\theta) = \sum_{t=1}^{T} \ln c(F_1(x_{1t}; \theta_1), F_2(x_{2t}; \theta_2), \ldots, F_n(x_{nt}; \theta_n)) + \sum_{t=1}^{T}\sum_{j=1}^{n} \ln f_j(x_{jt}; \theta_j), \quad (16)$$

where θ is the set of all parameters of both the marginal distribution and copula. Given a set of marginal distributions and a copula, the previous log-likelihood may be written, and by maximization we obtain the maximum likelihood estimator:

$$\hat{\theta}_{MLE} = \arg\max l(\theta). \quad (17)$$

Throughout this section we assume that the usual regularity conditions for asymptotic maximum likelihood theory hold for the multivariate model (i.e., the copula) as well as for all of its margins (i.e., the univariate distribution). Under these regularity conditions, the maximum likelihood estimator exists and it is consistent and asymptotically efficient. It also verifies the properties are asymptotically normal, and we have:

$$\sqrt{T}(\hat{\theta}_{MLE} - \theta_0) \to N(0, \xi^{-1}(\theta_0)), \quad (18)$$

where $\xi(\theta_0)$ is the usual Fisher's information matrix and θ_0 is the usual true value. The covariance matrix of $\hat{\theta}_{MLE}$ (Fisher's information matrix) may be estimated by the inverse of the negative Hessian matrix of the likelihood function. The second estimation method used in this paper is the IFM method. The parameters of IFM are estimated in two stages, and are computationally simpler than the maximum likelihood method.

In the first stage we estimate the marginal parameters θ_1 by performing the estimation of the univarate marginal distributions:

$$\hat{\theta}_1 = \arg\max \theta_1 \sum_{t=1}^{T}\sum_{j=1}^{n} \ln f_j(x_{jt}; \theta_1). \quad (19)$$

In the second stage, given θ_1, we perform the estimation of the copula parameter θ_2 that:

$$\hat{\theta}_2 = \arg\max \theta_2 \sum_{t=1}^{T} \ln c(F_1(x_{1t}), F_2(x_{2t}), \ldots, F_n(x_{nt}); \theta_2, \hat{\theta}_1). \quad (20)$$

The IFM estimator is defined as:

$$\hat{\theta}_{IFM} = (\hat{\theta}_1, \hat{\theta}_2)'. \quad (21)$$

In each stage we use the maximum likelihood method. Like the ML estimator, it verifies the properties of asymptotic normality (Joe and Xu, 1996):

$$\sqrt{T}(\hat{\theta}_{IFM} - \theta_0) \to N(0, V^{-1}(\theta_0)), \quad (22)$$

where $V(\theta_0)$ is the information matrix of Godambe (Cherubini et al., 2004).

3.4. Estimation of VaR

VaR is a concept developed in the field of risk management in finance. It is a measure defining how a portfolio of assets is likely to decrease over a certain time period. We define the VaR of a portfolio at a time t (return from $t - \Delta t$ to t), with a confidence level $(1 - \alpha)$, where $\alpha \in (0, 1)$ is defined as:

$$VaR_t(\alpha) = \inf\{s : F_t(s) \geq \alpha\}, \quad (23)$$

where F_t is the distribution function of the portfolio return $X_{p,t}$ at time t, and we have $P(X_{p,t} \leq VaR_t(\alpha)|\Omega_{t-1}) = \alpha$. This means that we have $100(1-\alpha)\%$ confidence that the loss in the period Δt will not be larger than VaR, where Ω_{t-1} means the information set at time $t - 1$.

We consider our portfolio return $X_{p,t}$ composed by a two-asset return denoted as $X_{1,t}$ and $X_{2,t}$ respectively. The portfolio return is approximately equal to the following:

$$X_{p,t} = wX_{1,t} + (1-w)X_{2,t}, \tag{24}$$

where w and $(1-w)$ are the portfolio weights of asset 1 and asset 2. Thus, the portfolio return is defined as:

$$\begin{aligned} &P(X_{p,t} \le VaR_t(\alpha)|\Omega_{t-1}) \\ &= P(wX_{1,t} + (1-w)X_{2,t} \le VaR_t(\alpha)|\Omega_{t-1}) \\ &= P\left(X_{1,t} \le \frac{VaR_t(\alpha)}{w} - \frac{1-w}{w}X_{2,t}|\Omega_{t-1}\right) = \alpha. \end{aligned} \tag{25}$$

In our work, we arbitrarily consider the two assets' weight to be equal, but this is not a constraint and they can vary freely. It means $w = 1/2$, where the confidence level α is assumed to be equal to 0.05, such that:

$$\begin{aligned} &P(X_{p,t} \le VaR_t(\alpha)|\Omega_{t-1}) \\ &= P\left(\frac{1}{2}X_{1,t} + \frac{1}{2}X_{2,t} \le VaR_t(\alpha)|\Omega_{t-1}\right) \\ &= P\left(X_{1,t} \le \frac{VaR_t(\alpha)}{2} - X_{2,t}|\Omega_{t-1}\right) = 0.05. \end{aligned} \tag{26}$$

Because the portfolio return is continuous, the VaR estimation formula is defined by the following, and Sklar's theorem is introduced here.

$$\begin{aligned} &P(X_{p,t} \le VaR_t|\Omega_{t-1}) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\frac{VaR_t}{2}-x_{2,t}} f(x_{1,t}, x_{2,t}|\Omega_{t-1})\mathrm{d}x_{1,t}\mathrm{d}x_{2,t} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\frac{VaR_t}{2}-x_{2,t}} c(F(x_{1,t}), F(x_{2,t})|\Omega_{t-1})f(x_{1,t}|\Omega_{t-1}) \\ &\quad \times f(x_{2,t}|\Omega_{t-1})\mathrm{d}x_{1,t}\mathrm{d}x_{2,t}. \end{aligned} \tag{27}$$

In addition to the conditional copula-GARCH method, we estimate the VaR by using different classical approaches, such as the historical simulation method, variance–covariance method, exponential weighted moving average (EWMA) method, and univariate GARCH-VaR method, which we present briefly in the following.

Historical simulation assumes that the distribution of the return will reappear. It can be thought of as estimating the distribution of the returns under the empirical distribution of the data. It is common and easy to use and compute. The variance–covariance method assumes the asset to have normality, and the VaR estimation formula is defined by:

$$\sigma_{p,t}^2 = [w_1 w_2] \times \begin{bmatrix} \sigma_{1,t}^2 & \sigma_{12,t} \\ \sigma_{21,t} & \sigma_{2,t}^2 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = w\Sigma_t w'$$

$$VaR_{p,t}(\alpha) = \sigma_{p,t} \cdot Z_\alpha + u_{p,t}, \tag{28}$$

where $u_{p,t}$ and $\sigma_{p,t}^2$ are the return and the variance of the portfolio return in time t, respectively, w_i is the portfolio weight of asset i, and Z_α is the standardized normal inverse with α probability. The EWMA method is generally used in Riskmetrics methodology. Assuming a normality of portfolio return distribution, we let $\sigma_{p,t}^2$ be the variance of the portfolio return in time t, and x_p is the portfolio return. The estimated variance, using data up to a time $t-1$, is given by:

$$\sigma_{p,t|t-1}^2 = (1-\lambda)x_{p,t-1}^2 + \lambda\sigma_{p,t-1|t-2}^2, \tag{29}$$

where $\sigma_{p,t|t-1}^2$ is the smoothed variance considering data up to a time t. The parameter λ is re-estimated at every time t by minimizing the quantity (see Palaro and Hotta, 2006), $\sum_i^{t-1}(\sigma_{p,(i+1)|i}^2 - x_{p,i+1}^2)^2$. For the univariate GARCH-VaR method, we fit the portfolio return series $X_{p,t}$ directly. The model is the same as what we introduced in Section 2, including GARCH and GJR with normal and Student-t innovation, respectively. It is not difficult to estimate $\sigma_t^2|\Omega_{t-1}$, the conditional variance of the portfolio return, by the GARCH or GJR model. The VaR estimates are easy to compute by:

$$VaR_{p,t}(\alpha)|\Omega_{t-1} = (\sigma_{p,t} \cdot Z_\alpha + u_t)|\Omega_{t-1}. \tag{30}$$

Table 1
Descriptive statistic and Engle tests.

Statistics	TWIEX		NASDAQ	
Sample number	1639		1639	
Mean	−0.0029		−0.0276	
Standard deviation	1.5976		1.8083	
Skewness	−0.4355		0.3337	
Excess of Kurtosis	4.7457		4.9834	
Engle-test	Q-statistic	P-value	Q-statistic	P-value
LM(4)	29.6751	(0.0000)	55.9236	(0.0000)
LM(6)	35.7357	(0.0000)	77.1165	(0.0000)
LM(8)	41.6420	(0.0000)	97.1018	(0.0000)
LM(10)	43.2247	(0.0000)	98.4249	(0.0000)

4. Empirical results

4.1. The data

This research aims at examining the performance of the conditional copula- GARCH methodology for the period between July 3, 2000 and May 18, 2007 with 1639 daily observations. We investigate the interactions between two major stock indices, i.e., NASDAQ and TAIEX (Taiwan Stock Exchanged Capitalization Weighted Index). All the data are from The Global Financial Database, sampled at a daily frequency. To eliminate spurious correlation generated by holidays, we eliminate those observations when a holiday occurred at least for one country from the database. The market returns and absolute returns of TWIEX and NASDAQ are shown in Fig. 1.

We define returns x_t as the following:

$$x_t = 100 \times \ln\left(\frac{P_t}{P_{t-1}}\right), \tag{31}$$

where P_t is the value of the index at time t. It is easy to see in Fig. 1 the evidence of the stylized fact known as volatility clustering. Hence, we test whether the squared return is serially correlated or not, which is called ARCH effects and shown in Table 1.

Table 1 provides summary statistics on market returns and statistic tests about the ARCH effects. We can find that TWIEX has a negative skewness (−0.4355) and NASDAQ has a positive skewness (0.3337). The LM(K) statistic clearly indicates that ARCH effects are likely to be found in both TWIEX and NASDAQ market returns. We consider the GARCH and GJR model introduced in the previous section to fit the time series data in order to create i.i.d observations to estimate the copula. Moreover, an excess of kurtosis is significant when higher than 3. It means that the empirical observations of returns display fatter tails than the normal distribution.

4.2. The marginal distribution

Because the return series has volatility clustering, it is necessary to consider marginal distribution for adjusting the empirical return distribution. Thus, we consider the univariate marginal model introduced in Section 2, the classical GARCH model and the GJR model. We fit GARCH and GJR models for the return series TWIEX and NASDAQ as the initial models with normal and Student-t

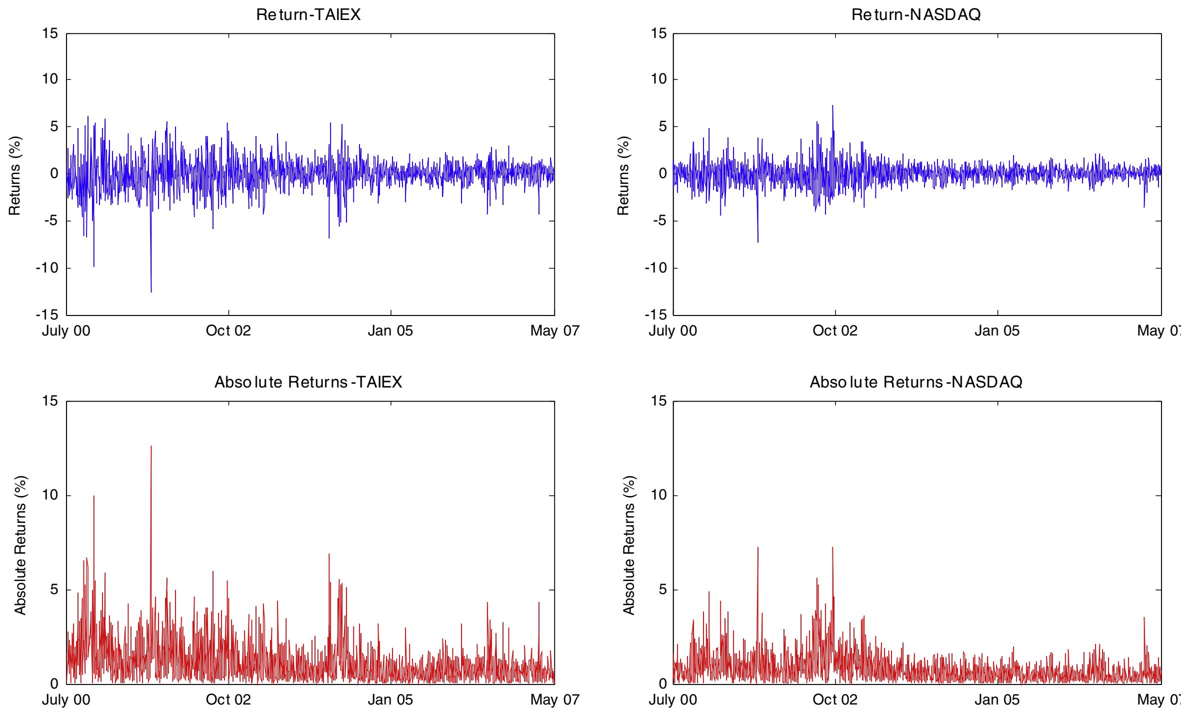

Fig. 1. Daily returns and absolute returns of TWIEX and NASDAQ.

distributions, respectively. Tables 2 and 3 represent the maximum likelihood results, the parameter we estimate, the ARCH and Ljung-box test for model adequacy, and the AIC (Akaike information criterion) and BIC (Bayesian information criterion) criterion for model selection.

Tables 2 and 3 show that the Ljung-Box test applied to the residuals of the GARCH-n, GARCH-t, GJR-n and GJR-t models does not reject the null hypothesis of autocorrelations at lags 1, 3, 5, and 7 at the 5% significance level. The square of the residuals series tested by the Engle-test for all models also does not reject the null hypothesis of ARCH effects at lags 4, 6, 8, and 10 at the 5% significance level. The parameter and the standard deviation we estimate shows that the GARCH or GJR model parameter is significant, which is not equal to 0. Therefore, we consider that all the models are adequate.

There is one important thing we should notice. We distribute the data into two groups, sample-in and sample-out data, in order to test whether the VaR we estimate is adequate or not. The sample-in data contain the first 1000 observations, and the leftover 639 observations are sample-out data for the test. All the marginal model distributions and copula functions we estimate using the sample-in data contain 1000 return observations.

4.3. Copula modeling

After having estimated the parameters of the marginal distribution F_i, we continue to estimate the copula parameters as explained previously. Eight copula functions are applied in our work: Gaussian copula, Student-t copula, and some Archimedean copula. According to the MLE and IFM methods, the selected copula functions will be fitted to these residuals series. The copula modeling result is showed in Table 4. Table 4 shows the results we estimate from the MLE or IFM method. One thing to note is that we estimate the parameter ρ in a Gaussian copula without the MLE or IFM method. We use Kendall's tau ρ_τ transform method to estimate the parameter ρ in Gaussian copula, because it provides a more efficient way of estimating ρ. Kendall's tau transform method is defined as the following:

$$\rho_\tau = \frac{2}{\pi}\arcsin(\rho). \tag{32}$$

It is obvious to find the best fitting copula function which is the Student-t copula, where the AIC and BIC criterion for model selection is used here. Table 4 shows that Student-t copula's AIC and BIC are the smallest, especially with the GARCH-n marginal distribution model. The following fitting copula are Frank and Plackett's copulas, especially with the GARCH-t marginal distribution, which are a better fit than the Gaussian copula. In fact, the Gaussian copula with the GARCH-t marginal distribution model is the well-known distribution, which is the multivariate normal distribution we always assume in the classical method. According to the AIC and BIC values of all kinds of copulas, the Student-t copula is the best fitting function to describe the dependence structure of the bivariate return series.

4.4. Estimation of VaR

This paper initially uses the sample-in data, which contains 1000 return observations, to estimate VaR_{1001} at a time $t = 1001$, and at each new observation we re-estimate VaR, because of the conditional level and the VaR estimation formula. It means that we estimate VaR_{1002} by using observations $t = 2$ to $t = 1001$ and estimate VaR_{1003} by using observations $t = 3$ to $t = 1002$ until the sample-out observations we have updated are used up. Because we have 639 sample-out observations left, there is a total of 639 tests for VaR. The number of violations of the VaR estimation are calculated using various copula functions are presented in Table 5.

The numbers of violations in Table 5 are the numbers of sample observations being located out of the critical value. The mean error shows for each copula function, the average absolute discrepancy per marginal model between the observed and expected number of violations. When the estimation of the number of violations calculated by various copula functions is closer to the expected number of violations (i.e., 32), the values of the mean error are

Table 2
Parameter estimates of GARCH model and statistic test.

	GARCH-*n*				GARCH-*t*			
	TWIEX		NASDAQ		TWIEX		NASDAQ	
Parameter	Value	Std	Value	Std	Value	Std	Value	Std
μ	0.0346	0.0560	0.0203	0.0534	0.0158	0.0507	−0.0020	0.0540
α_0	0.0478	0.0179	0.0090	0.0107	0.0343	0.0218	0.0127	0.0130
α_1	0.0584	0.0116	0.0442	0.0096	0.0513	0.0140	0.0431	0.0112
β	0.9300	0.0117	0.9538	0.0094	0.9395	0.0162	0.9535	0.0119
d					7.2173	1.2457	18.1900	5.8152
LLF	2009.70		2092.20		1982.50		2087.80	
AIC	−4011.40		−4176.40		−3955.00		−4165.50	
BIC	−3991.70		−4156.80		−3930.50		−4141.00	
Lags	P-value	Q-statistic	P-value	Q-statistic.	P-value	Q-statistic	P-value	Q-statistic
Ljung-Box test								
QW(1)	0.1186	2.4351	0.5895	0.2912	0.1194	2.4252	0.5997	0.2755
QW(3)	0.0927	6.4254	0.7857	1.0641	0.1039	6.1640	0.7820	1.0794
QW(5)	0.2605	6.5013	0.8258	2.1652	0.2837	6.2381	0.8206	2.2017
QW(7)	0.3811	7.4758	0.9276	2.4929	0.4189	7.0968	0.9266	2.5064
Engles test								
LM(4)	0.6163	2.6597	0.0744	8.5174	0.6204	2.6361	0.0811	8.3036
LM(6)	0.7881	3.1636	0.1955	8.6303	0.7679	3.3187	0.2097	8.4087
LM(8)	0.8214	4.3789	0.3049	9.4609	0.8069	4.5256	0.3188	9.2852
LM(10)	0.9046	4.7929	0.4272	10.1524	0.8978	4.9000	0.4433	9.9680

Table 3
Parameter estimates of GJR model and statistic test.

	GJR-*n*				GJR-*t*			
	TWIEX		NASDAQ		TWIEX		NASDAQ	
Parameter	Value	Std	Value	Std	Value	Std	Value	Std
μ	0.0027	0.0568	−0.0424	0.0541	−0.0069	0.0506	−0.0451	0.0540
α_0	0.0500	0.0125	0.0267	0.0143	0.0328	0.0141	0.0271	0.0148
α_1	0.0092	0.0098	0.0000	0.0116	0.0039	0.0129	0.0000	0.0139
β	0.9422	0.0101	0.0947	0.0134	0.9568	0.0123	0.9482	0.0143
γ	0.0688	0.0147	0.0946	0.0211	0.0578	0.0167	0.0914	0.0240
d			0.6705	0.1810	7.6814	1.3303	35.9170	24.9580
LLF	2001.30		2075.00		1976.50		2074.00	
AIC	−3992.60		−4140.10		−3941.00		−4136.10	
BIC	−3968.00		−4115.50		−3911.60		−4106.70	
Lags	P-value	Q-statistic	P-value	Q-statistic	P-value	Q-statistic	P-value	Q-statistic
Ljung-Box test								
QW(1)	0.1166	2.4620	0.8767	0.6852	0.1188	2.4334	0.6695	0.1822
QW(3)	0.0707	7.0394	0.7780	2.4902	0.0943	6.3844	0.8732	0.7000
QW(5)	0.2118	7.1210	0.8872	2.9759	0.2616	6.4879	0.7777	2.4932
QW(7)	0.3259	8.0761			0.4077	7.2060	0.8889	2.9574
Engles test								
LM(4)	0.5266	3.1895	0.0516	10.5889	0.5549	3.0177	0.0641	10.4093
LM(6)	0.6857	3.9336	0.1021	10.5840	0.6259	4.3760	0.1086	10.5044
LM(8)	0.7878	4.7124	0.1822	11.3584	0.7338	5.2201	0.1887	11.2355
LM(10)	0.8782	5.1897	0.2689	12.2463	0.8462	5.6186	0.2792	12.0877

small. From the results in Table 5, the Student-t copula shows the minimum mean error with a 95% and 99% level of confidence ($\alpha = 0.05$ and $\alpha = 0.01$). That is, the Student-t copula is the best adequate copula for describing the return distribution of the portfolio. Besides, Fig. 2 shows the VaR plot we estimate using the Student-t copula with a marginal distribution, the GARCH-n model at $\alpha = 0.05$ and $\alpha = 0.01$. VaR is an estimate of investment loss in the worst case scenario with a relatively high level of confidence. In this Figure the VaR of portfolio is located almost below the portfolio returns, and describes the expectation of investment loss well. The portfolio return of VaR with a 99% confidence is surely lower than that with a 95% confidence.

The following is the comparison with traditional VaR estimation. The results we estimate using the traditional methods and copula-GARCH methods introduced previously are shown in Table 6. In this table, t-copula-GARCH-n stands for the Student-t copula with a GARCH-n marginal distribution and t-copula-GJR-n, the Student-t copula with a GJR-n marginal distribution. It is obvious to see the VaR of the historical simulation (HS) method and variance–covariance (VC) method are underestimated and this represents the highest mean error at $\alpha = 0.05$ and $\alpha = 0.01$. The univariate GARCH-VaR method is better than the HS and VC methods, but it still underestimates VaR, and the mean error is still high. The EWMA method of estimating VaR seems to be adequate, but not as good as the method of conditional copula-GARCH. The various VaRs we estimate are shown in Fig. 3 with $\alpha = 0.01$. In this figure we can see that trends of the HS and VC methods are more flat and these methods can not reflect the risk of portfolio returns with time-varying. For the other methods, the difference of VaRs in most regions is not obvious except for the observations from 400

Table 4
Parameter estimates for families of copula and model selection statistic.

Copula	Parameter	GARCH-*n*	GARCH-*t*	GJR-*n*	GJR-*t*
Gaussian	ρ	0.276	0.276	0.274	0.275
	LLF	32.8040	32.6587	33.0453	33.0122
	AIC	−65.6061	−65.3154	−66.0886	−66.0224
	BIC	−65.6011	−65.3105	−66.0837	−66.0175
Student-*t*	ρ	0.376	0.373	0.367	0.365
	d	2.1	2.1	2.1	2.1
	LLF	74.3746	73.4864	70.9018	70.5034
	AIC	−148.7432	−146.9688	−141.7996	−141.0028
	BIC	−148.7334	−146.9590	−141.7898	−140.9930
Clayton	ω	0.169	0.337	0.172	0.306
	LLF	11.2204	21.7353	10.5964	19.2404
	AIC	−22.4388	−43.4686	−21.1909	−38.4788
	BIC	−22.4339	−43.4637	−21.1860	−38.4739
Rotated-clayton	ω	0.291	0.396	0.297	0.376
	LLF	26.6667	32.2645	28.5496	32.1842
	AIC	−53.3314	−64.5271	−57.0973	−64.3664
	BIC	−53.3265	−64.5222	−57.0923	−64.3615
Plackett	η	2.226	2.335	2.165	2.245
	LLF	35.3990	35.6584	34.0739	34.2639
	AIC	−70.7960	−71.3149	−68.1459	−68.5257
	BIC	−70.7911	−71.3100	−68.1410	−69.5208
Frank	λ	1.732	1.836	1.68	1.76
	LLF	37.0299	37.1743	35.9395	36.0821
	AIC	−74.0577	−74.3467	−71.8770	−72.1622
	BIC	−74.0528	−74.3418	−71.8720	−72.1573
Gumbel	δ	1.173	1.226	1.169	1.21
	LLF	28.1070	34.0253	28.4759	32.5314
	AIC	−56.2120	−68.0487	−56.9466	−65.0608
	BIC	−56.2071	−68.0437	−56.9450	−65.0559
Rotated-Gumbel	δ	1.132	1.209	1.135	1.193
	LLF	15.3826	26.7443	15.7769	24.4130
	AIC	−30.9632	−53.4867	−31.5519	−48.8240
	BIC	−30.9583	−53.4817	−31.5470	−48.8191

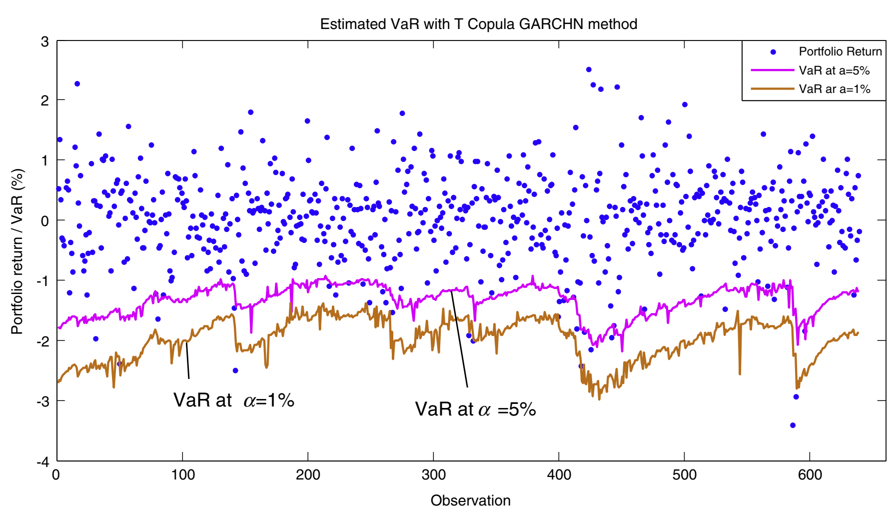

Fig. 2. Estimated VaR using Student-*t* copula with GARCH-*n* model.

to 500. In these regions, the methods of EWMA and GARCH reflect too much. The *t*-copula-GARCH-*n* method therefore captures the extremes most successfully compared to the other methods.

5. Conclusion

The total losses of any portfolios should be estimated correctly in order to allocate economic capital for the investments in a proper way. The correlation among the price or volatility behaviors of the financial assets within a portfolio is a crucial dimension for the proper estimation of the VaR amount. However, restrictions on the joint distributions of the financial assets within the portfolio might decrease the performance of the VaR estimation. The joint distribution of the portfolio should be free from any normality assumptions especially if the portfolio is composed of assets from markets where there exists high volatility and non-linearities in the returns.

This paper describes a model for estimating portfolio VaR by the conditional copula-GARCH model, in which the empirical evidence shows that this method can be quite robust in estimating VaR. Copula-GARCH models allow for a very flexible joint distribution by splitting the marginal behaviors from the dependence relation. In contrast, most traditional approaches for the estimating VaR, such as variance–covariance, and the Monte Carlo approaches,

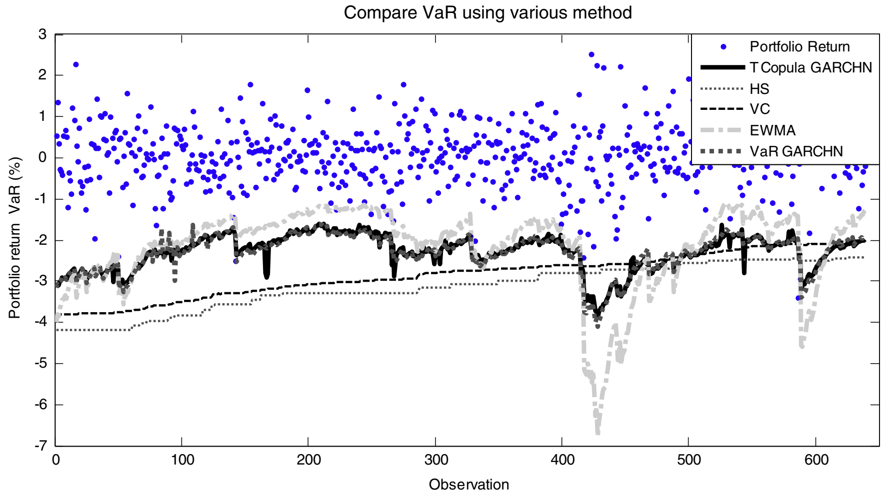

Fig. 3. Comparison of VaR using various methods at $\alpha = 0.01$.

Table 5
Number of violations of the VaR estimation.

At $\alpha = 0.05$					
Trading days	639	Expected no. of violations			32
Copula	GARCH-n	GARCH-t	GJR-n	GJR-t	Mean error
Gaussian	31	23	28	20	6.5
Student-t	32	32	33	30	0.75
Clayton	32	27	33	23	3.75
Rotated-Clayton	34	33	37	42	4.5
Plackett	31	28	31	24	3.5
Frank	31	27	30	24	4
Gumbel	32	30	33	28	1.75
Rotated-Gumbel	31	27	30	23	4.25
At $\alpha = 0.01$					
		Expected no. of violations			7
Gaussian	6	3	9	6	2
Student-t	8	8	10	8	1.5
Clayton	20	4	26	6	9
Rotated-Clayton	9	5	13	8	2.75
Plackett	5	7	10	9	1.75
Frank	6	7	10	9	1.5
Gumbel	8	7	10	9	1.5
Rotated-Gumbel	1	7	9	8	2.25

Table 6
Number of violations of VaR estimation.

Trading days	639		
α	5%	1%	
Expected no. of violations	32	7	Mean error
t-copula-GARCH-n	32	8	1
t-copula-GJR-n	33	10	4
HS	9	2	28
VC	9	2	28
EWMA	28	10	7
GARCH-n	16	4	19
GARCH-t	16	4	19
GJR-n	14	4	21
GJR-t	15	4	20

require the knowledge of joint distributions. This paper estimates different copulas with different univariate marginal distributions, and traditional methods to compare the results. The Student-t copula describes the dependence structure of the portfolio return series quite well, in which we choose it by the AIC and BIC of the model criterion, producing the best results of the reliable VaR limit. The comparison of the performance of the copula method to that of the traditional method shows that the copula model captures the VaR most successfully. The copula method has the feature of flexibility in distribution, which is more appropriate in studying highly volatile financial markets, and which there is lack of in traditional methods.

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