Lab 7

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Parameters

Newton's Method

- Tolerance = 1×10^{-14}
- Maximum number of iterations = 20
- Initial guess for $y'(a)$, $u_0 = -1.5$

4-Step Adams Predictor-Corrector

- Number of subintervals, $n = 5000$ (step size, $h = 0.01$)

Results

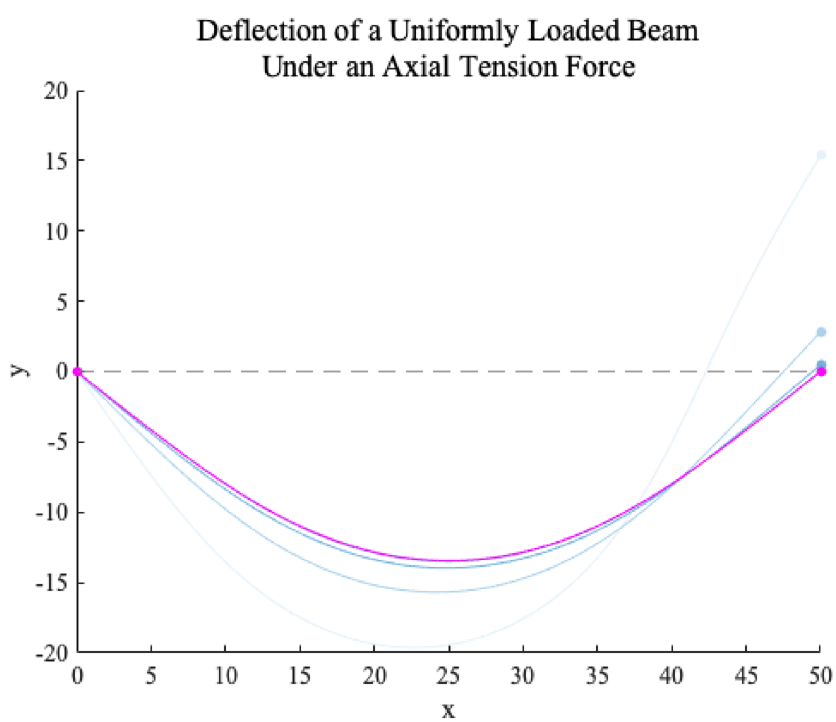


Figure 1: 4-Step Adams Predictor-Corrector solutions *y(x)* to the nonlinear boundary value problem describing the deflection of a uniformly loaded, long rectangular beam under an axial tension force. Iterations 1–7 of Newton's method are shown in progressively darker shades of blue. The final solution, after 8 iterations, is shown in magenta.

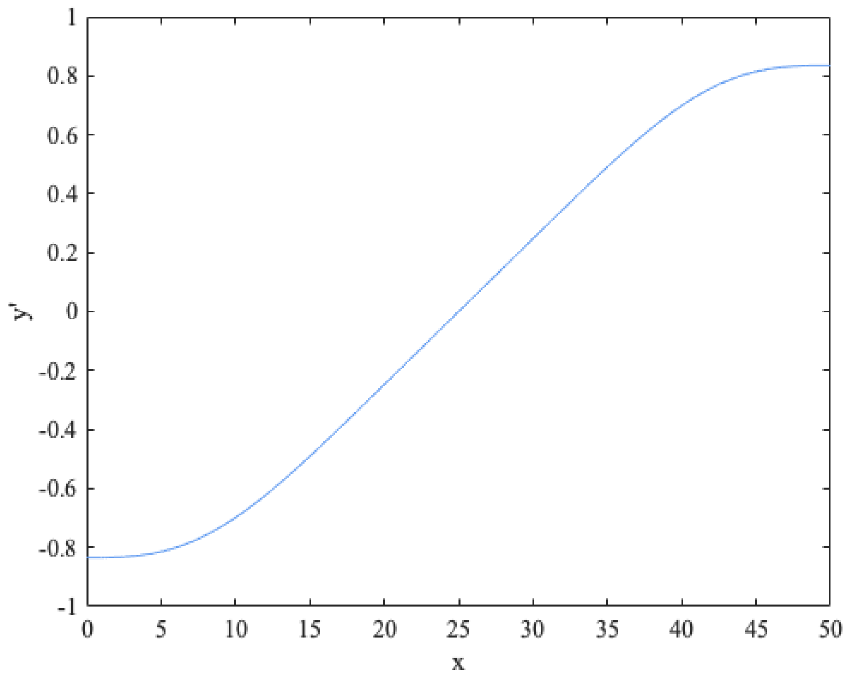


Figure 2: Final 4-Step Adams Predictor-Corrector solution for *y'(x)*.

Convergence of Newton's Method

Newton's method converged after 8 iterations to u = −0.83467. The error of each iteration was calculated as the absolute difference between $w(b)$ and $y(b) = 0$, where $w(b)$ is the approximated value of y at $x = b$ found using a 4-Step Adams Predictor-Corrector with step size $h = 0.01$ and initial conditions $y(a) = 0$ and $y'(a) = u_k$, with u_k being the value of u at the start of iteration k of Newton's method. The initial guess for u was −1.5. Tolerance was set to 1×10^{-14}.

Iteration	u	$w(b)$	$\lvert w(b) - y(b) \rvert$
1	−1.5 (= u_0)	15.3772	15.3772
2	−1.03842	2.8287	2.8287
3	−0.87892	0.4886	0.4886
4	−0.83746	0.0289	0.0289
5	−0.83468	1.2603e−04	1.2603e−04
6	−0.83467	2.4351e−09	2.4351e−09
7	−0.83467	−4.6747e−14	4.6747e−14
8	−0.83467	1.8058e−15	1.8058e−15

Table 1: Value of $u = y'(a)$ before each iteration of Newton's method. The last two columns show $w(b)$ and the error of each iteration for each u.

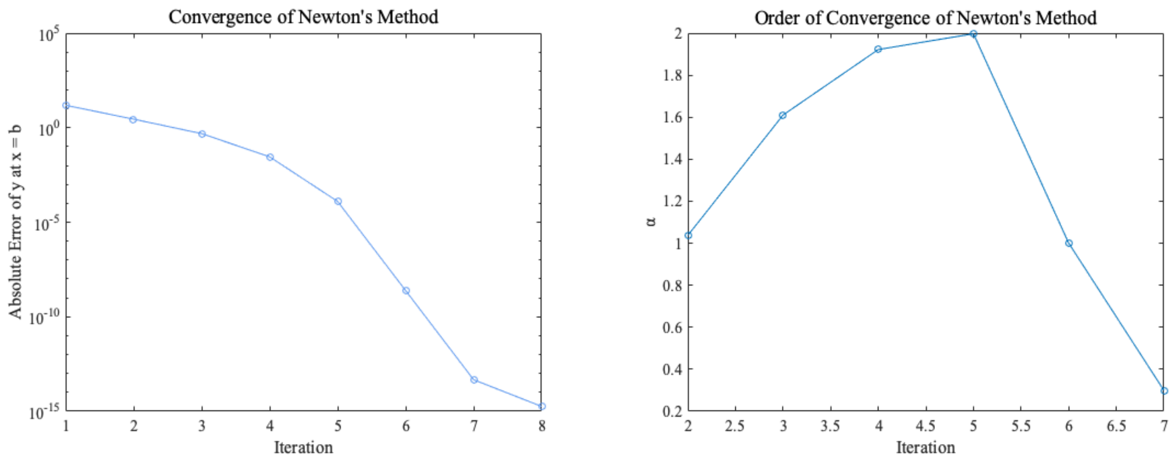


Figure 3: Absolute errors $E = |w(b) - y(b)|$ of Newton's method versus iteration (left) and the order of convergence α versus iteration (right). To calculate α, I took the ratio of the difference in log error of the next and current iteration versus the current and previous iteration, $[\log(E_{k+1}) - \log(E_k)] / [\log(E_k) - \log(E_{k-1})]$.

Assuming that the fixed point function $p(u) = u - [z_1(b, u) - y(b)] \ / \ [\delta z_1(b, u)/\delta u]$ has a simple root, Newton's method should converge quadratically (α = 2). Newton's method should converge faster than the Secant method (α ≈ 1.618), although it requires more function evaluations (many which come from solving a second IVP).

The plots above are somewhat inconsistent with this expectation. Based on the graphs, the order of convergence starts at α ≈ 1, increases to α ≈ 2 by iteration 6, then falls to about 0.3 by the last iteration. Some of the error might be attributed to the $O(h^5)$ truncation error associated with the 4-Step Adams Predictor Corrector. Furthermore, since MATLAB is limited to 16 digits of precision, rounding could also be inflating the errors (particularly in the last few iterations where errors are already small).

Truncation Error of the 4-Step Adams Predictor-Corrector

To study the truncation error associated with the 4-Step Adams Predictor-Corrector, I used the final value of $u = -0.83467$ obtained from Newton's method and plotted the solution $y(x)$ for various step sizes h = [10, 1, 0.1, 0.01, 0.001, 0.0001] and the errors, assuming the solution for the smallest step size 0.0001 is close enough to exact.

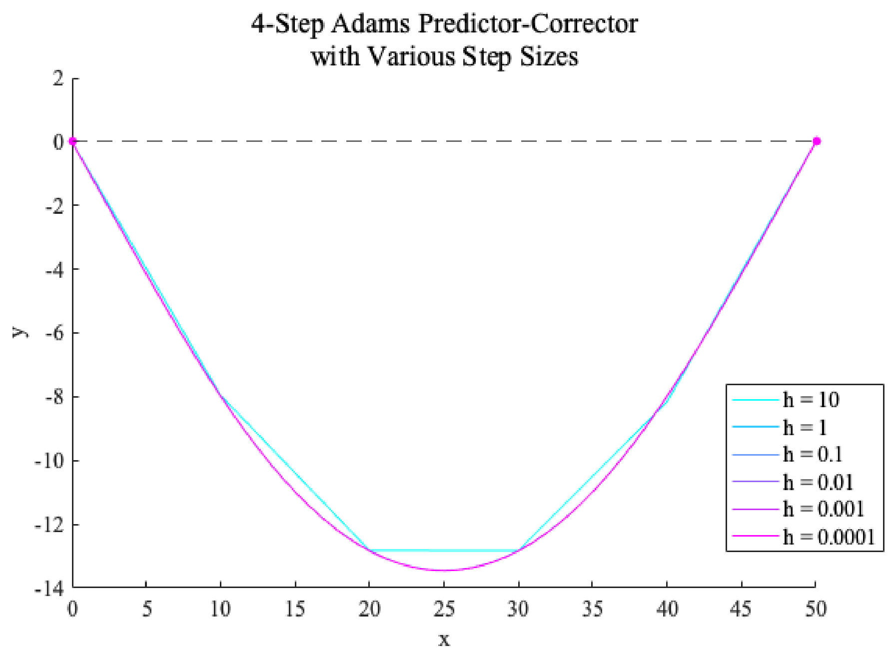
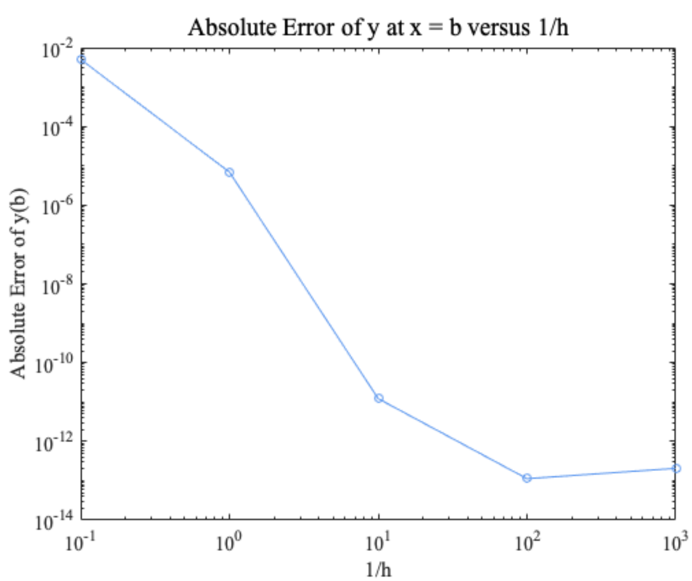


Figure 4: 4-Step Adams Predictor-Corrector solution for *y(x)* for various *h*. The most accurate solution, with *h* = 0.0001, is magenta.



h	Error in y(b)
10	0.0051
1	6.9147e-06
0.1	1.2185e-11
0.01	1.2194e-13
0.001	2.0664e-13

Figure 5: Absolute error of y at $x = b$ versus $1/h$ for $h =$ [10, 1, 0.1, 0.01, 0.001]. The solution for the smallest step size, $h = 0.0001$ was assumed to be “exact” for this comparison.

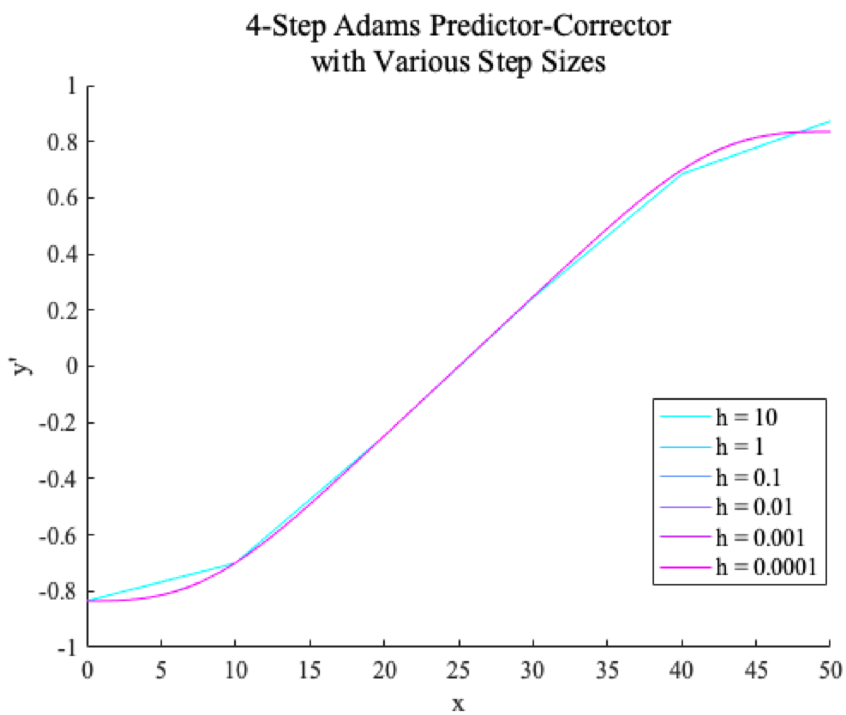
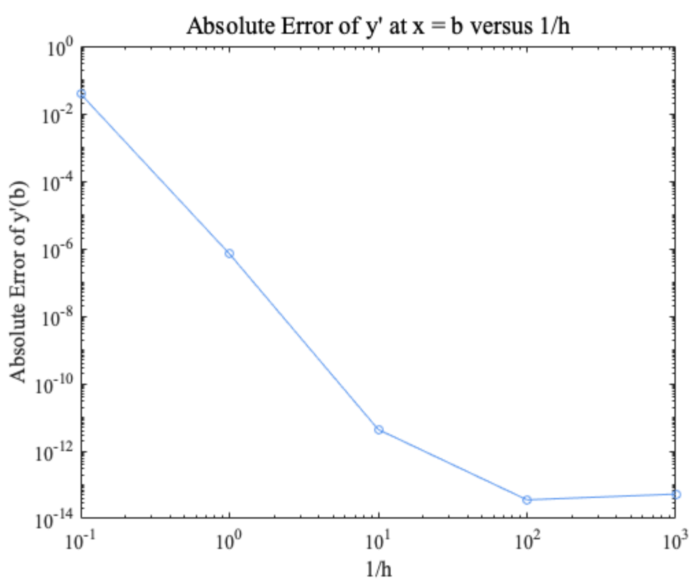


Figure 6: 4-Step Adams Predictor-Corrector solution for *y'(x)* for various *h*.



h	Error in y'(b)
10	0.0385
1	7.0459e-07
0.1	4.3814e-12
0.01	3.6082e-14
0.001	5.3291e-14

Figure 7: Absolute error of *y'* at $x = b$ versus $1/h$ for $h =$ [10, 1, 0.1, 0.01, 0.001]. The calculation of absolute error compares the estimate for *y'(b)* from each of these step sizes against *y'(b)* from the smallest step size, $h = 0.0001$.

When h is relatively large (h = 10 and h = 1), reducing h by a factor of 10 clearly leads to a reduction in error by a factor of ~10^5 in the solutions for both *y(x)* and *y'(x)*, confirming that the truncation error of the 4-Step Adams Predictor-Corrector method is $O(h^5)$.

Further reducing the step size beyond h = 0.1 seems to have a lower-order effect. For *y(x)* and *y'(x)*, decreasing the step size from 0.1 to 0.01 only improves the error by a factor of ~10^2. Further decreasing h from 0.01 to 0.001 leads to a slight (likely negligible) increase in error, possibly suggesting that the solutions for h = 0.01 are within the limit of MATLAB's precision. If this is true, step sizes below 0.01 are unlikely to improve the accuracy of either solution.

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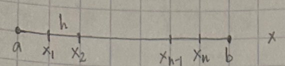
a.

Deriving the System of Finite Difference Equations

Centered second order finite difference approximation.



$$y'' = \frac{f_{i+1} - 2f_i + f_{i-1}}{h^2} + O(h^2)$$

$$\frac{d^2\tilde{T}}{dx^2} - \lambda^2\tilde{T} = 0$$

steady state temp distr. w/out heating

boundary conditions: $\tilde{T}(x=0) = T_c - T_a$, $\tilde{T}(x=L) = T_s - T_a$

$$\underbrace{\frac{\tilde{T}_{i+1} - 2\tilde{T}_i + \tilde{T}_{i-1}}{h^2}}_{\substack{\text{Second cent. diff approx} \\ \frac{d^2\tilde{T}}{dx^2}}} - \lambda^2\tilde{T}_i = 0$$

$$= \tilde{T}_{i+1} - 2\tilde{T}_i + T_{i-1} - \lambda^2 h^2 \tilde{T}_i = 0$$

$$= \tilde{T}_{i+1} - (2+\lambda^2 h^2)\tilde{T}_i + T_{i-1} = 0$$

$$\underbrace{\begin{bmatrix} -(2+\lambda^2h^2) & 1 & & & 0 \\ 1 & -(2+\lambda^2h^2) & 1 & & \\ & 1 & -(2+\lambda^2h^2) & 1 & \\ & & \ddots & \ddots & \ddots \\ 0 & & & & \end{bmatrix}}_{\substack{n \times n \text{ tridiagonal matrix} \\ A}} \underbrace{\begin{bmatrix} \tilde{T}_1 \\ \tilde{T}_2 \\ \vdots \\ \tilde{T}_n \end{bmatrix}}_{\tilde{T}_{noheat}} = \underbrace{\begin{bmatrix} -\tilde{T}(x=0) \\ 0 \\ \vdots \\ 0 \\ -\tilde{T}(x=L) \end{bmatrix}}_{b_{noheat}} \rightsquigarrow \tilde{T}_{noheat} = A \backslash b_{noheat}$$

with heating, same A, different b:

$$\underbrace{\begin{bmatrix} -(2+\lambda^2h^2) & 1 & & \\ 1 & -(2+\lambda^2h^2) & 1 & \\ & 1 & \ddots & \ddots \\ & & \ddots & \ddots \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \tilde{T}_1 \\ \vdots \\ \tilde{T}_n \end{bmatrix}}_{\tilde{T}_{heat}} = \underbrace{\begin{bmatrix} -h^2 \cdot \frac{\sigma E_0}{k} e^{\gamma(L-x_1)} - \tilde{T}(x=0) \\ \vdots \\ -h^2 \cdot \frac{\sigma E_0}{k} e^{\gamma(L-x_i)} \\ \vdots \\ -h^2 \cdot \frac{\sigma E_0}{k} e^{\gamma(L-x_n)} - \tilde{T}(x=L) \end{bmatrix}}_{b_{heat}} \rightsquigarrow \tilde{T}_{heat} = A \backslash b_{heat}$$

Results

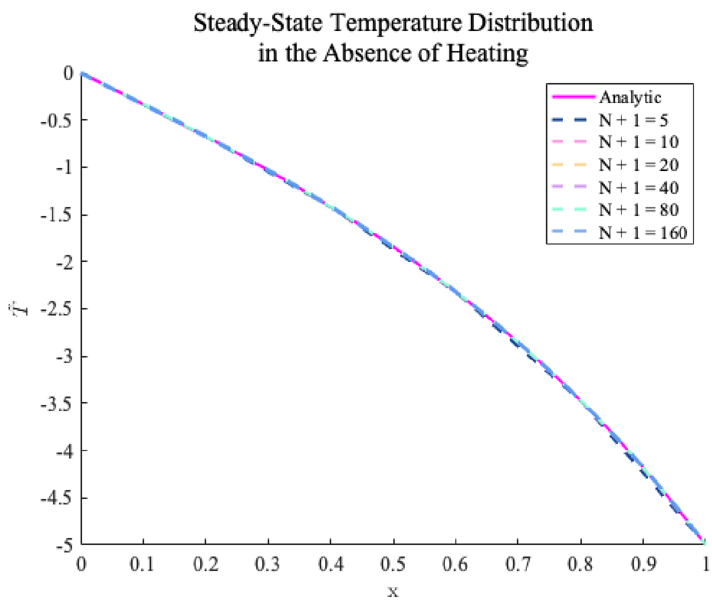
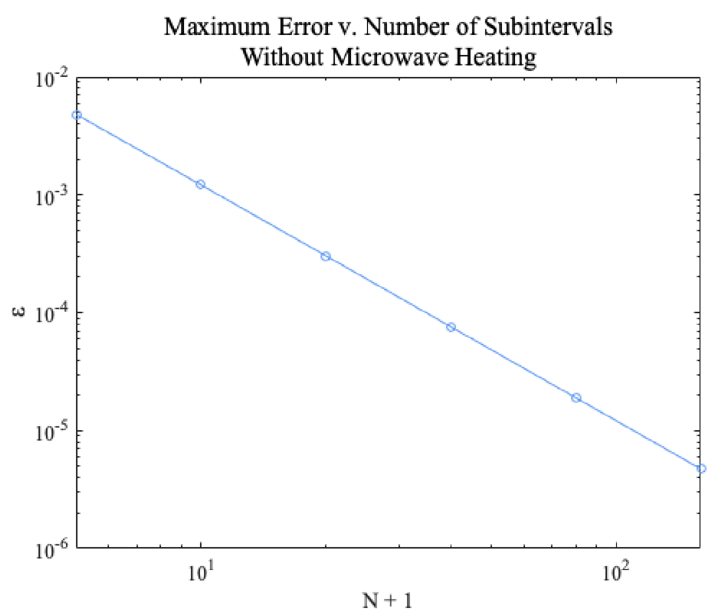


Figure 1: Finite difference solution for the steady-state temperature $\overline{T}$ (in the absence of heating) as a function of x for N interior nodes. Found by solving the set of finite difference equations derived from centered second-order finite difference approximations on equally-spaced nodes.



N+1	Max. Error
5	0.048
10	0.0012
20	3.0349e-04
40	7.5931e-05
80	1.8998e-05
160	4.7696e-06

Figure 2: Maximum absolute error $\varepsilon = \max|\overline{T}_i - \overline{T}_i^a|$ as a function of $1/\Delta x$ on a log-log scale.

Mathematical analysis shows that second centered difference approximations have an O(h^2) truncation error. Plotting error versus the inverse of step size confirms that this method is O(h^2), since halving step step size (doubling $N + 1$, the number of subintervals), multiplies the error by ~1/4.

b.

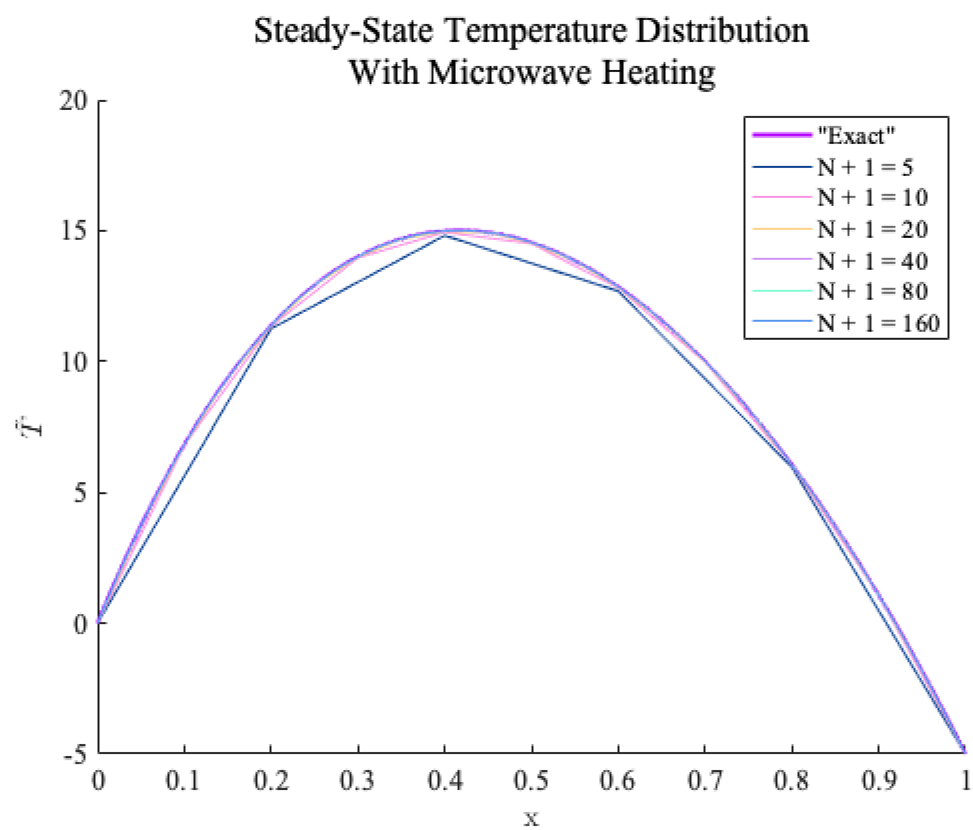
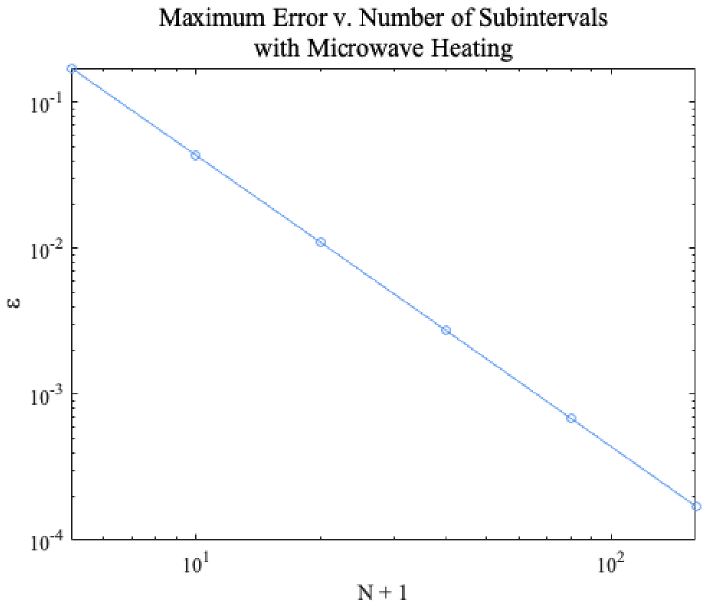


Figure 1: Finite difference solution for the steady-state temperature $\overline{T}$ with heating as a function of x for N interior nodes. The "exact" solution, found by solving the system of finite difference equations for $N + 1 = 16000$, is shown in purple. As $N + 1$ increases (step size decreases), the solutions approach the "exact" one.



N+1	Max. Error
5	0.1724
10	0.0435
20	0.0110
40	0.0027
80	6.8697e-04
160	1.7178e-04

Figure 2: Maximum absolute error relative to $\overline{T}^{\text{“exact”}}$, the “exact” approximation (with $N + 1 = 16000$) as a function of $1/\Delta x$ on a log-log scale.

With heating, the truncation error for the same finite difference method is also $O(h^2)$; error decreases proportional to the square of step size.