Lab 5

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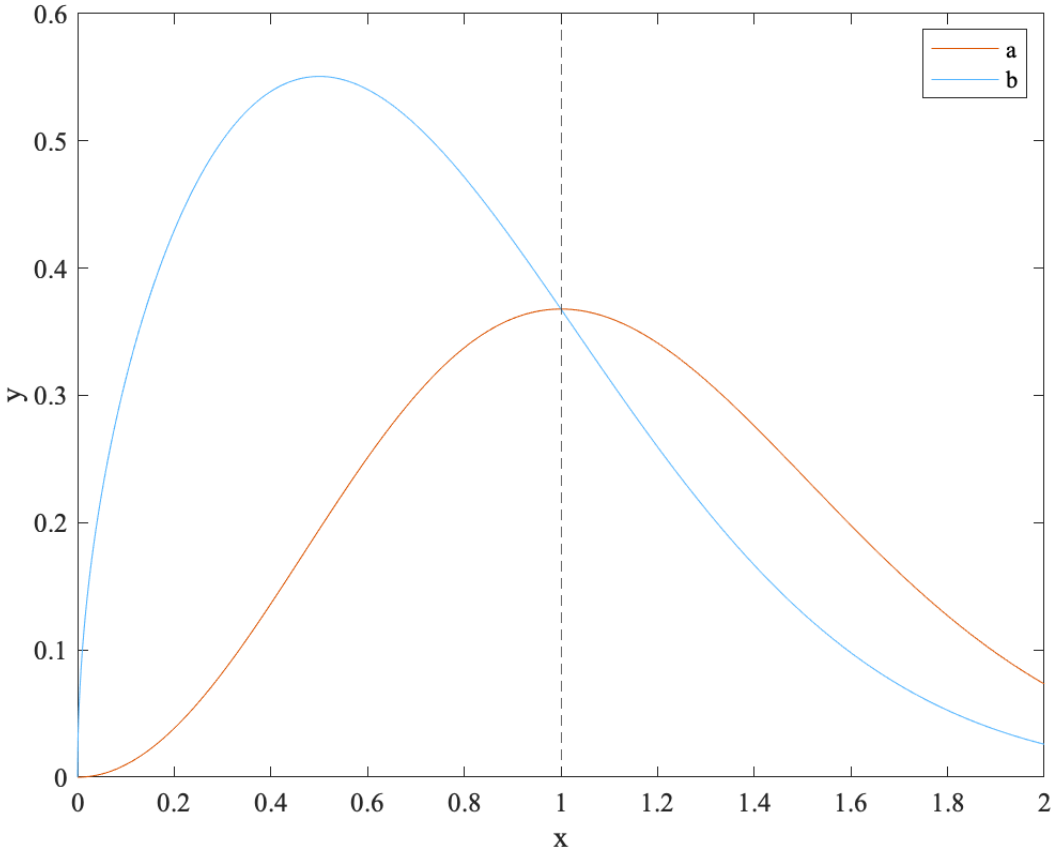
1

Function (a)

$$\int_0^1 x^2 e^{-x^2} dx$$

Function (b)

$$\int_0^1 x^{1/2} e^{-x^2} dx$$



A plot of functions (a) and (b) on the interval [0, 1] is shown above.

Results for function (a)

Method	**Limits**	**Approximation**	**n** (2^n steps)	**Evaluations**
Romberg	[0, 1]	0.18947	6	134
C. Trapezoidal	[0, 1]	0.18947	7	129
Romberg	[1, 2]	0.23325	5	69
C. Trapezoidal	[1, 2]	0.23325	13	8193

Results for function (b)

Method	**Limits**	**Approximation**	**n** (2^n steps)	**Evaluations**
Romberg	[0, 1]	0.45339	18	524306
C. Trapezoidal	[0, 1]	0.45339	20	1048577
Romberg	[1, 2]	0.15316	5	69
C. Trapezoidal	[1, 2]	0.15316	14	16385

Discussion

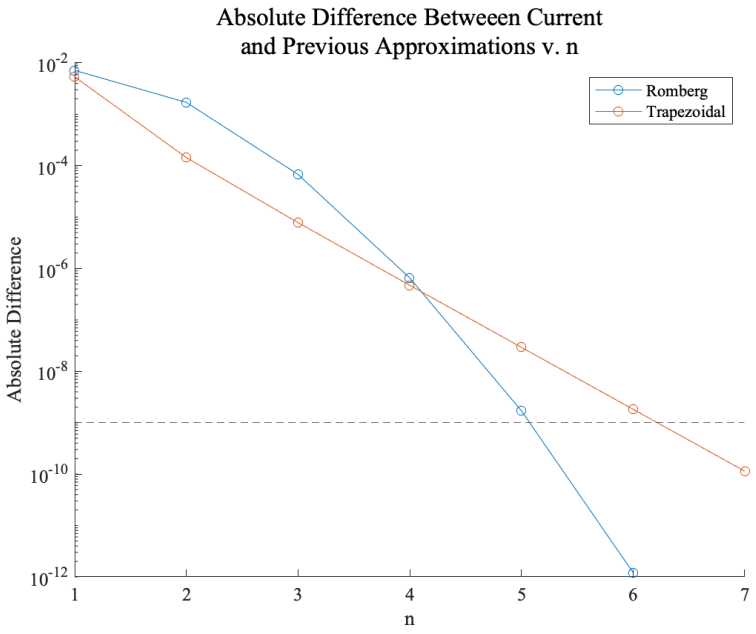
- Both methods, Romberg and Composite Trapezoidal, converged to the same value for functions (a) and (b) regardless of the limits of integration.
- We expect and observe that Romberg integration converges faster than the Composite Trapezoidal rule. The number of subintervals doubles each time n increases by 1. Each doubling should increase the order of the error of the Romberg approximation by a factor of $O(h^2)$, but only multiply the error of the Trapezoidal approximation by 1/4. For all integrals, we see that Trapezoidal requires more subintervals to achieve the same level of accuracy as Romberg. This is particularly clear for integrals of (a) and (b) over [1, 2]. Whereas Trapezoidal only uses 2 and 4 times as many subintervals as Romberg to approximate the integrals over [0, 1], it uses 2^8 and 2^9 times as many subintervals as Romberg to approximate integrals of the same functions over [1, 2].
- The methods generally seem to require more subintervals to approximate regions with large slope changes. For example, Romberg and Composite Trapezoidal used n

= 18 and 20 for function (b) on [0, 1] , a steep curve with a rapidly decreasing slope, but only n = 6 and 7 for function (a) on [0, 1], where the slope gradually transitions from increasing to decreasing. This could be a motivation for using Adaptive Quadrature, which uses variable step sizes to achieve a uniform error across an interval where the behavior of the function varies.

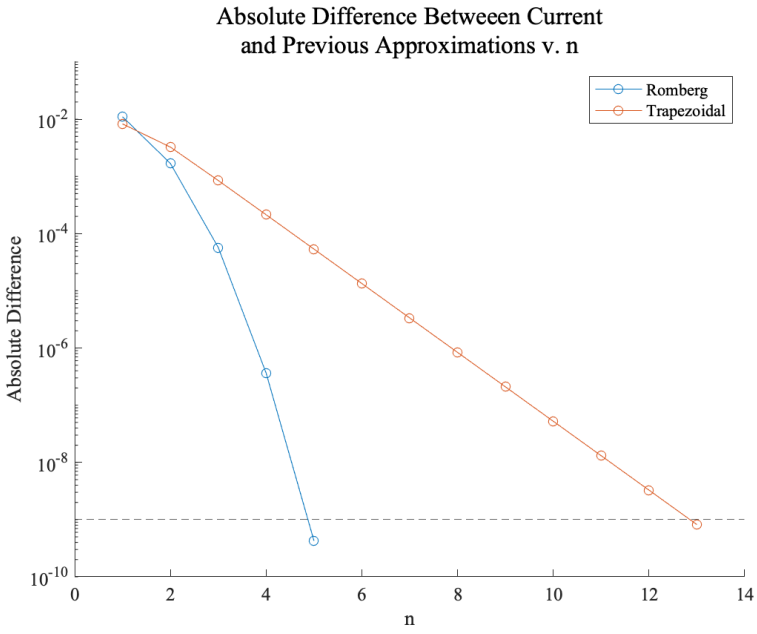
- The number of Composite Trapezoidal function evaluations is $2^n + 1$, or one more than the number of subintervals. The number of Romberg function evaluations is the sum of function evaluations for Composite Trapezoidal with 1 to 2^n subintervals, represented by the first column of the Romberg table R(0, 0) . . . R(n, 0). Computing the number of function evaluations shows that Romberg not only converges with fewer subintervals than Trapezoidal to achieve the same level of accuracy, but is generally more efficient, especially for large n, when the cost of a doubling is high. For example, to approximate the integral of function (a) over [0, 1], Romberg and Trapezoidal use 2^6 and 2^7 steps respectively, amounting to ~130 evaluations each. However, to approximate the integral of (b) over [0, 1], Romberg does 524306 evaluations for 2^{18} steps, about half as many evaluations as Trapezoidal, which 1048577 total for 2^{20} steps.

Additional Figures

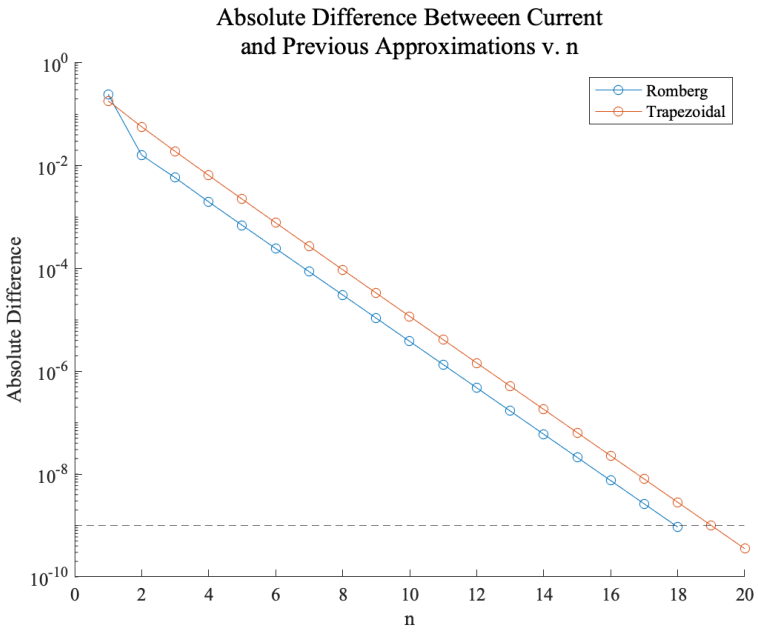
1: Absolute differences between the current and previous approximations of the integral as a function of n (2^n subintervals). Horizontal dashed line y = 10^{-9} indicates the stopping condition.



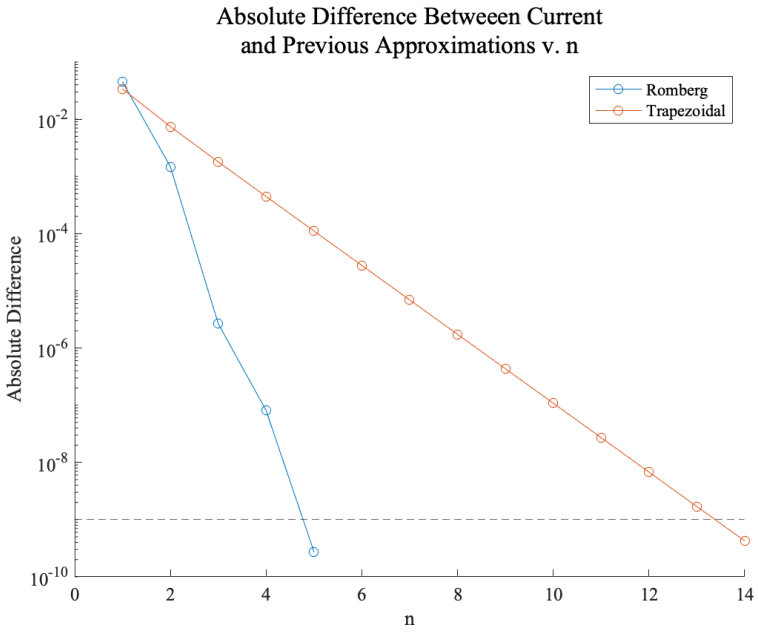
1A: Function (a) with limits of integration [0, 1]



1B: Function (a) with limits of integration [1, 2]



1C: Function (b) with limits of integration [0, 1]



1C: Function (b) with limits of integration [1, 2]

2

Gaussian approximations

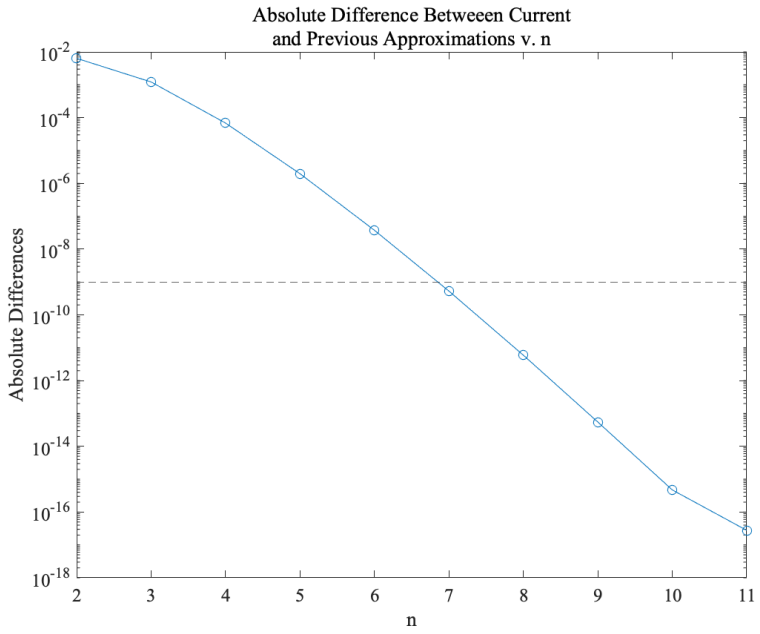
n	(a) over [0, 1]	(a) over [1, 2]	(b) over [0, 1]	(b) over [1, 2]
1	0.1947	0.2371	0.5507	0.1291
2	0.1883	0.2344	0.4582	0.1541
3	0.1895	0.2332	0.4559	0.1532
4	0.1895	0.2333	0.4545	0.1532
5	0.1895	0.2333	0.454	0.1532
6	0.1895	0.2333	0.4538	0.1532
7	0.1895	0.2333	0.4536	0.1532
8	0.1895	0.2333	0.4536	0.1532
9	0.1895	0.2333	0.4535	0.1532
10	0.1895	0.2333	0.4535	0.1532
11	0.1895	0.2333	0.4535	0.1532

Discussion

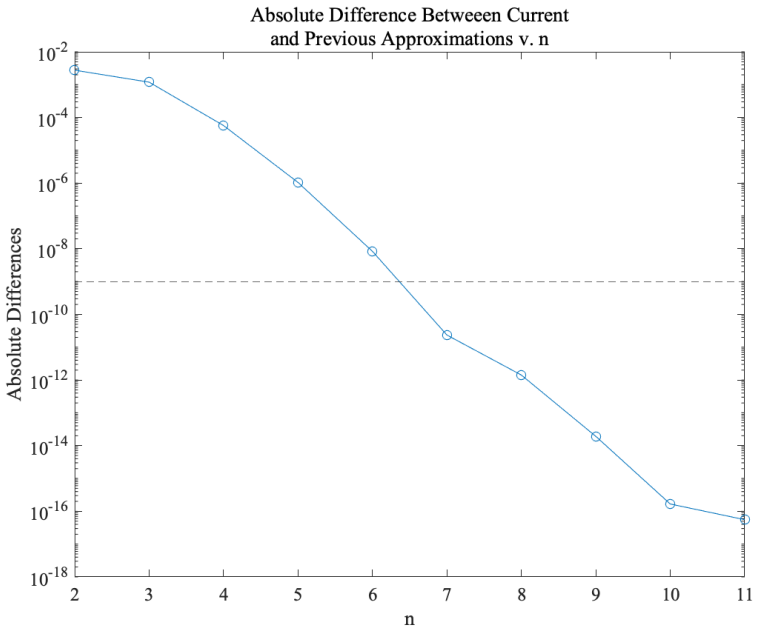
- The number of function evaluations is equal to n, the number of points used in the Gaussian approximation.
- As n increases, the Gaussian approximation converges to the same value as the Romberg and Composite Trapezoidal methods for each integral, but requires far less function evaluations to achieve the same level of accuracy (see graphs under Additional Figures). For the Gaussian approximation with n points to agree with the Gaussian approximation with n - 1 points to within 10^{-9}, 7 points (function evaluations) are enough to approximate the integral of (a) over [0, 1], (a) over [1, 2], and (b) over [1, 2]. Based on the error graph, more than 11, but presumably still much less than the 524306 required by Romberg, function evaluations are required to approximate (b) over [0, 1].
- As with Romberg and Composite Trapezoidal, approximating the integral of (b) over [0, 1] seems to be the most computationally expensive out of the four integrals.

Additional Figures

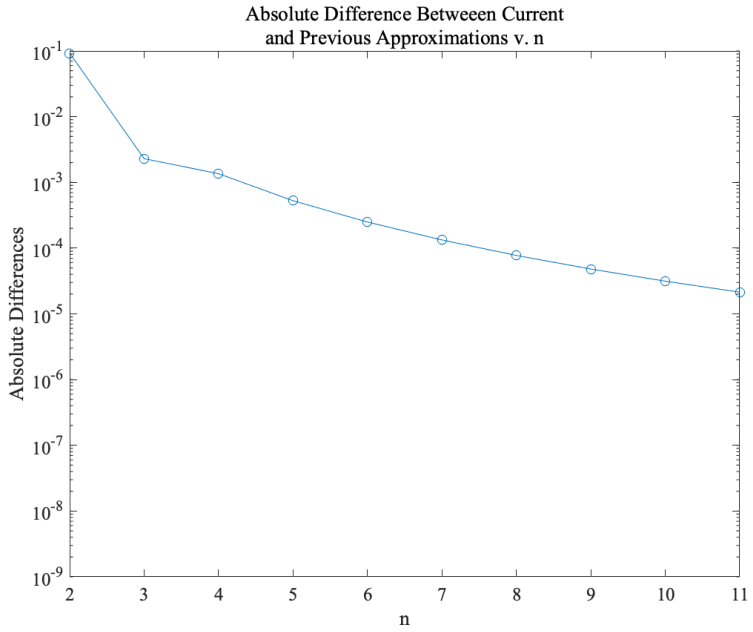
2: Absolute differences between the current and previous approximations of the integral as a function of n (2^n subintervals). Horizontal dashed line y = 10^{-9} indicates the stopping condition from Question 1.



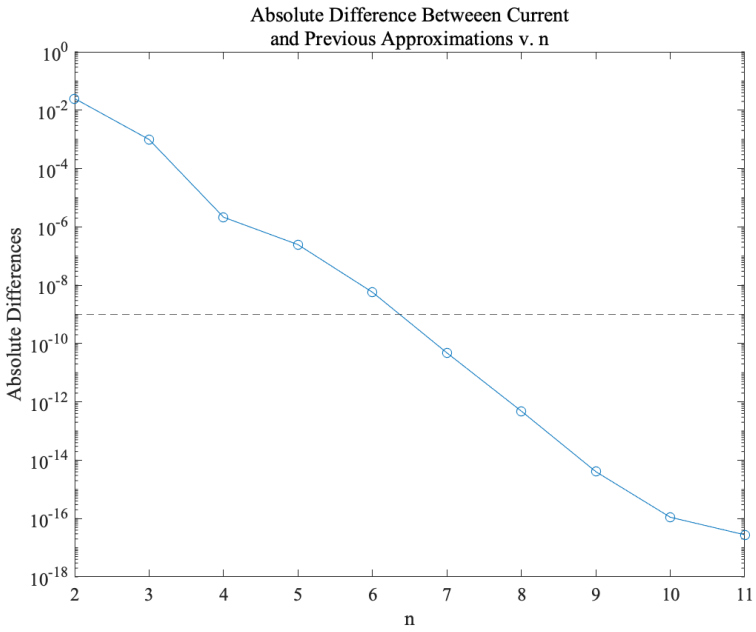
2A: Function (a) with limits of integration [0, 1]



2B: Function (a) with limits of integration [1, 2]



2C: Function (b) with limits of integration [0, 1]



2D: Function (b) with limits of integration [1, 2]

Each figure in the **Graphs** section (below **Discussion**) consists of three graphs:

- **y(t) v. t** – Numerical solutions for n = 3 to 12. The exact analytical solution is shown in blue.
- **Absolute Error v. t** – Plot of the difference between each of the numerical solutions (n = 3 to 12) and the exact solution.
- **Absolute Error at y(2) v. t** – Plot of absolute value of the error at t = 2 versus 1/Δt for the method.

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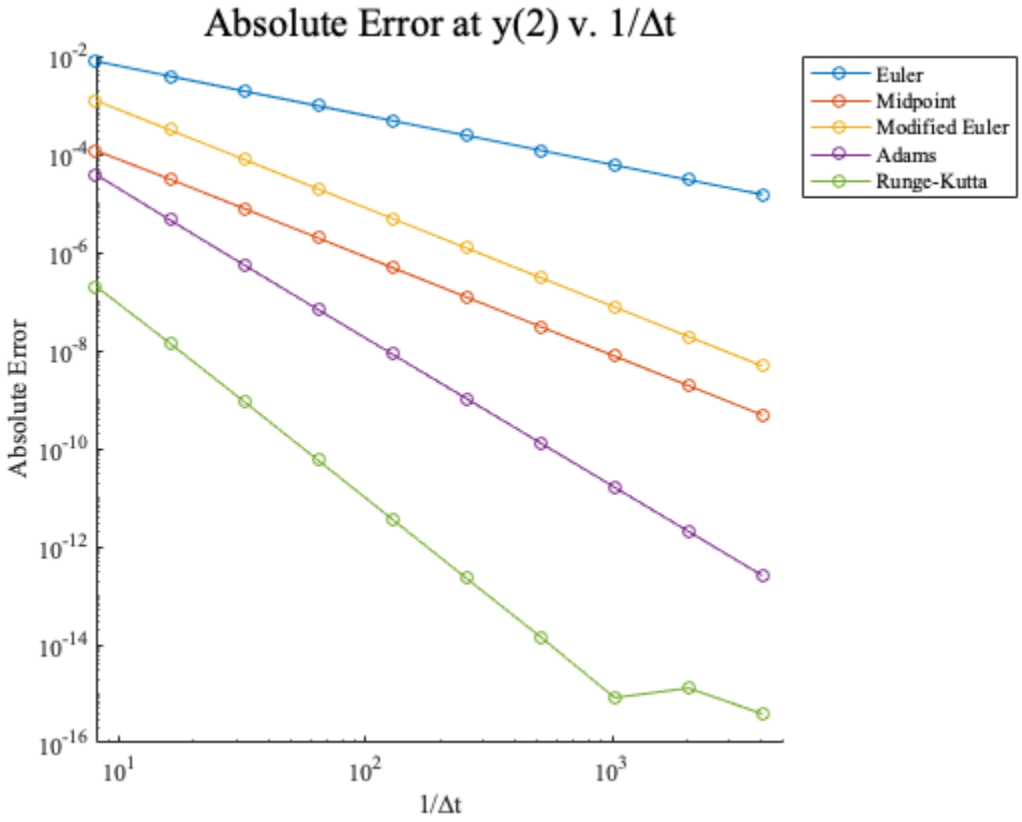
Absolute Error at y(2) v. t

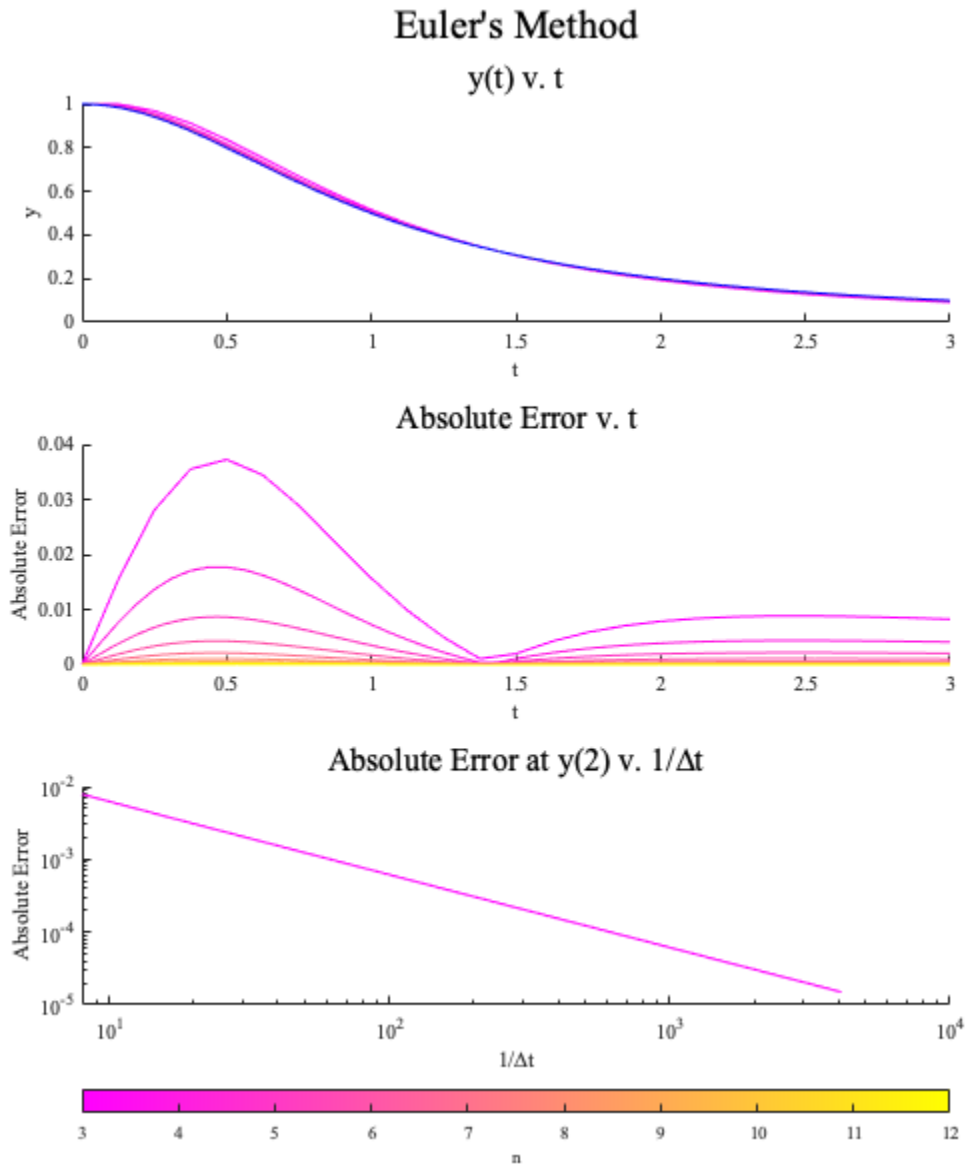


All of the **Absolute Error at y(2) v. t** graphs for each method combined into a single plot.

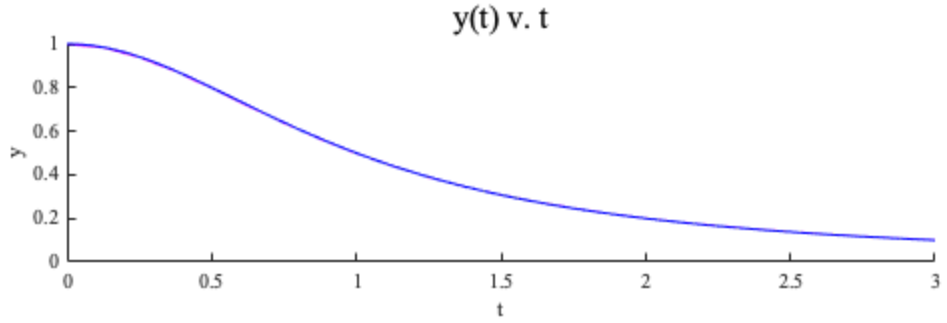
Discussion

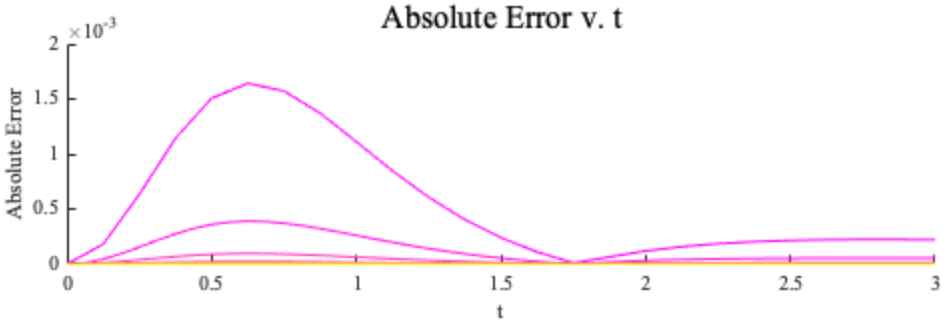
- Based on the combined **Absolute Error at y(2) v. t** graph above, Runge-Kutta Order 4 leads to the most accurate approximation of y(t), followed by the two-step Adams-Bashforth & Adams-Moulton Predictor Corrector (with a Runge-Kutta Order 4 initial step), the Midpoint Rule, Modified Euler's method, and finally Euler's method.
- This is consistent with what we expect:
    - Euler's Method should be the least accurate, with $O(h^2)$ error per step and $O(h)$ error overall.
    - Midpoint Rule (Runge-Kutta Order 2) and Modified Euler's method should both have $O(h^3)$ error per step and $O(h^2)$ error overall. Values of y(t) calculated with either method should be closer to the exact solution than approximations with Euler's method, which we observe to be true. Here, the Midpoint Rule performs slightly better than Modified Euler's method, but their absolute errors are relatively close in magnitude and parallel, which suggests that the two methods are comparable.
    - Adams-Bashforth & Adams-Moulton should be $O(h^4)$ per step and $O(h^3)$ overall, more accurate than the Midpoint Rule and Modified Euler's method. Consistent with this expectation, the errors for the Adams Predictor Corrector in the graph are smaller in magnitude than the errors for the Midpoint Rule, Modified Euler's method, and Euler's method.
    - As the highest-order method, Runge-Kutta Order 4, with $O(h^5)$ error per step and $O(h^4)$ error overall, should lead to the most accurate approximations. Our results confirm this: the Runge-Kutta Order 4 errors are the smalles among the four methods.

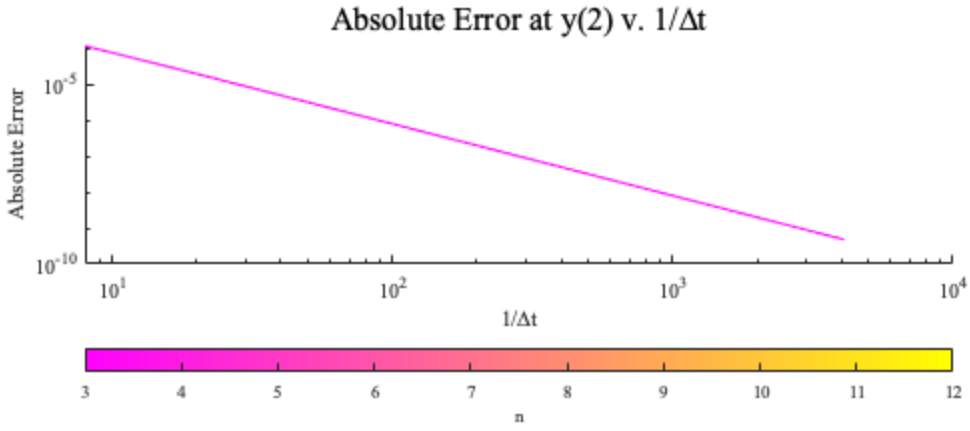
Graphs


Euler's Method
y(t) v. t
y
t
Absolute Error v. t
Absolute Error
t
Absolute Error at y(2) v. 1/Δt
Absolute Error
1/Δt
n

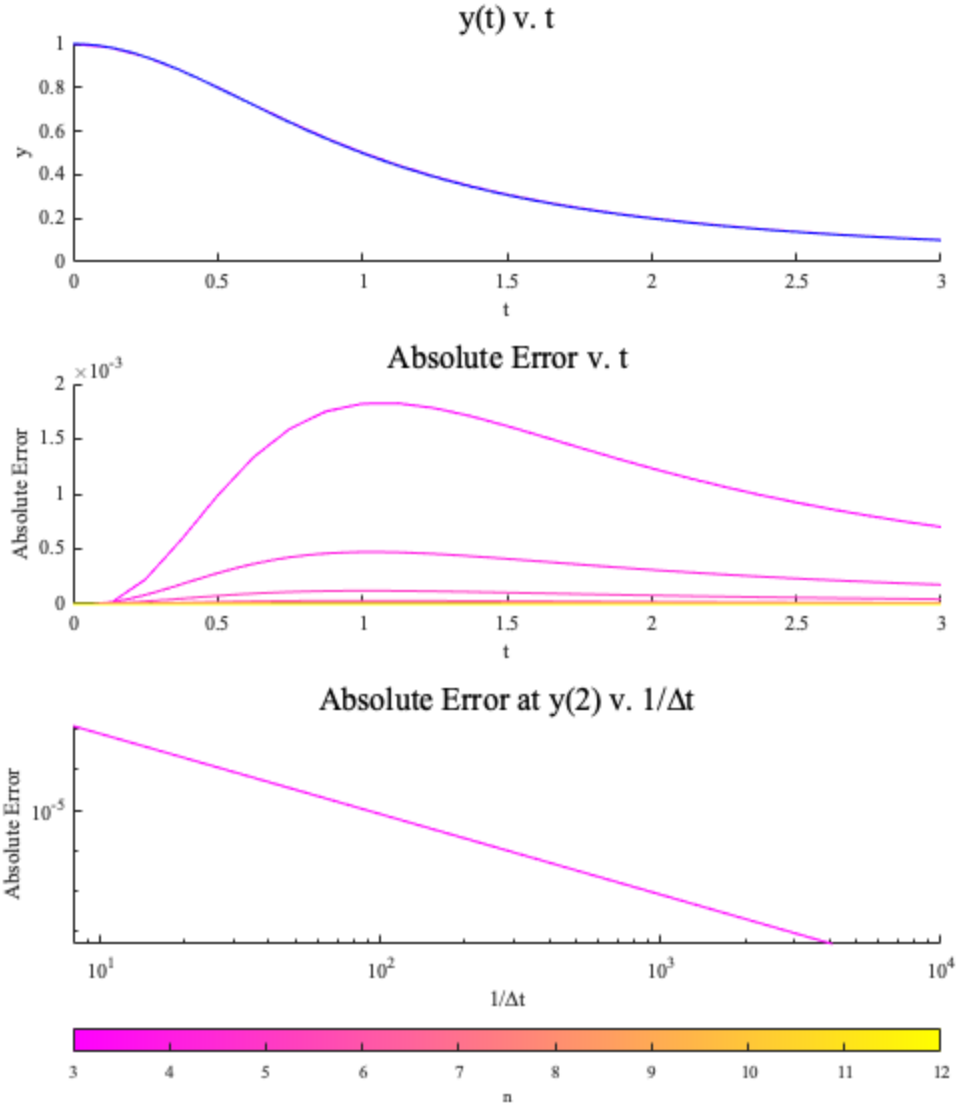
Midpoint Rule



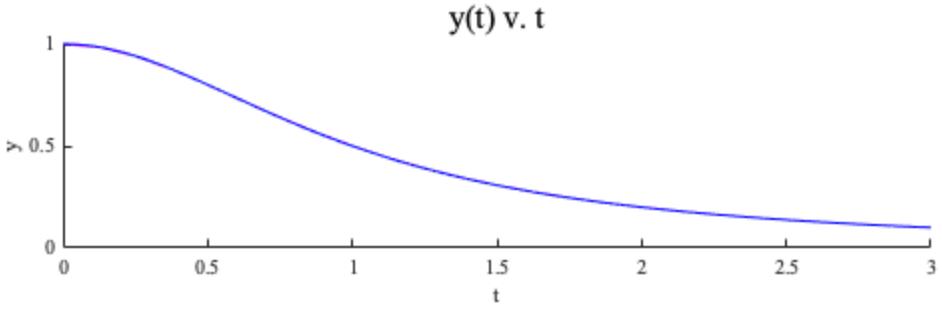


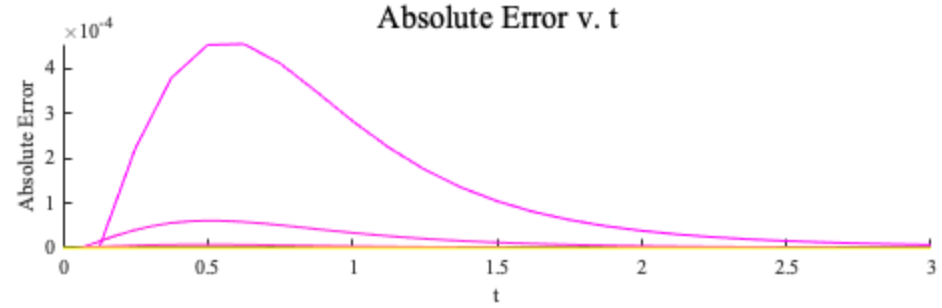


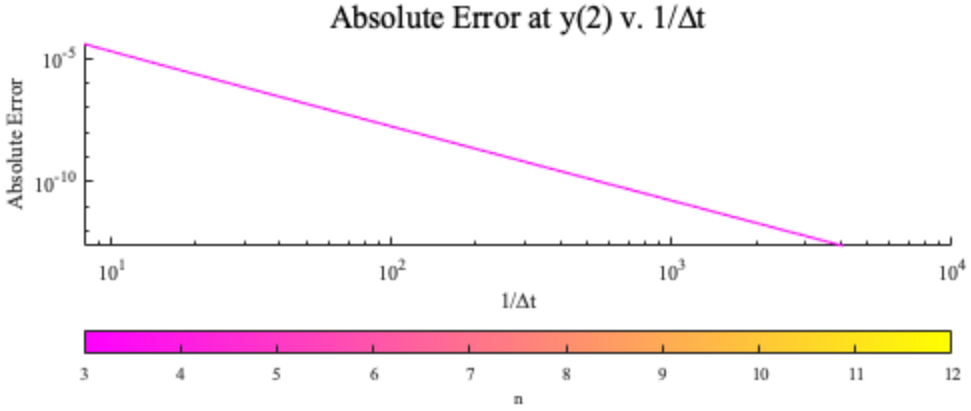
Modified Euler's Method



Adams-Bashforth & Adams-Moulton Predictor Corrector







Runge-Kutta 4th Order Method