Lab 4

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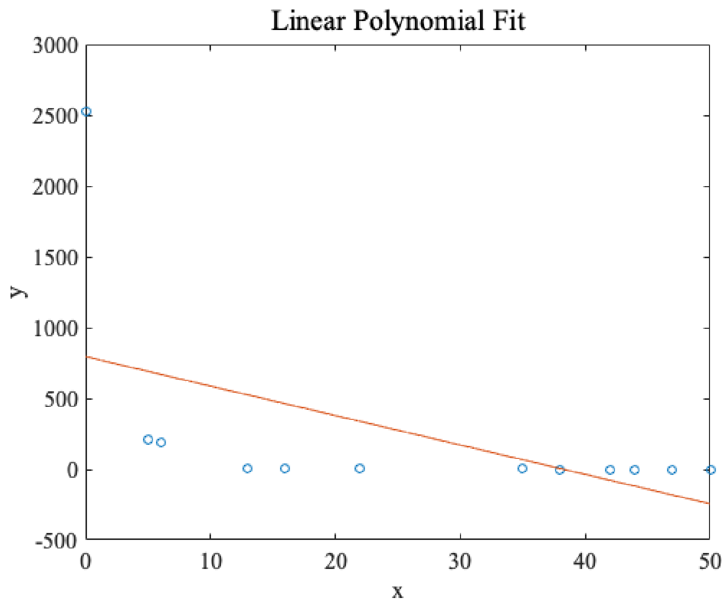
1.

Linear Polynomial Fit

$P_1(x) = 794.5747 - 20.7683x$

Sum of absolute errors = 4.7095e+03



Cubic Polynomial Fit

For a set of m points, we assume that the approximating polynomial is

$P_3(x) = a_0 + a_1x + a_2x^2 + a_3x^3$

We want to find $[a_0, a_1, a_2, a_3]$ that minimizes the sum of squared errors E. Setting $\delta E/\delta a_j = 0$ for $j = 0, 1, 2, 3$ results in the normal equations:

$$a_0 \sum_{i=1}^{m} x_i^0 + a_1 \sum_{i=1}^{m} x_i^1 + a_2 \sum_{i=1}^{m} x_i^2 + a_3 \sum_{i=1}^{m} x_i^3 = \sum_{i=1}^{m} y_i x_i^0$$

$$a_0 \sum_{i=1}^{m} x_i^{1} + a_1 \sum_{i=1}^{m} x_i^{2} + a_2 \sum_{i=1}^{m} x_i^{3} + a_3 \sum_{i=1}^{m} x_i^{4} = \sum_{i=1}^{m} y_i x_i^{1}$$

$$a_0 \sum_{i=1}^{m} x_i^{2} + a_1 \sum_{i=1}^{m} x_i^{3} + a_2 \sum_{i=1}^{m} x_i^{4} + a_3 \sum_{i=1}^{m} x_i^{5} = \sum_{i=1}^{m} y_i x_i^{2}$$

$$a_0 \sum_{i=1}^{m} x_i^{3} + a_1 \sum_{i=1}^{m} x_i^{4} + a_2 \sum_{i=1}^{m} x_i^{5} + a_3 \sum_{i=1}^{m} x_i^{6} = \sum_{i=1}^{m} y_i x_i^{3}$$

These form a linear system of equations that can be written in the form $Xa = b$, where $a = [a_0\ a_1\ a_2\ a_3]^T$ is the unknown vector of coefficients. We can calculate an element in row k and column j of X by evaluating

$$X_{kj} = \sum_{i=1}^{m} x_i^{k+j} \qquad k = 0, 1, 2, 3, \quad j = 0, 1, 2, 3$$

For this problem, $X =$

12	318	12048	503124
318	12048	503124	21905508
12048	503124	21905508	975793908
503124	21905508	975793908	44133809148

Every kth element of b can also be calculated:

$$b_k = \sum_{i=1}^{m} y_i x_i^{k} \qquad k = 0, 1, 2, 3$$

For this problem, $b =$

2924.868475
2241.99921
13034.03714
82067.32308

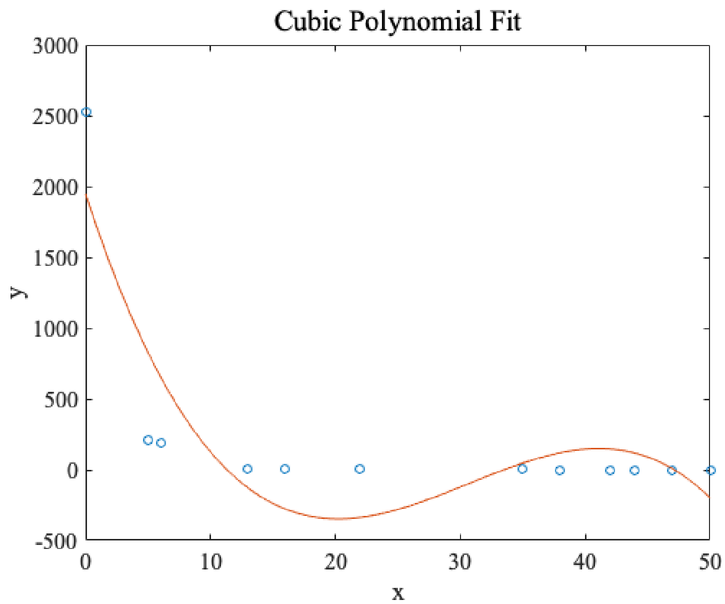
To solve the system, $a = X^{-1}b =$

1.9493
-0.2717
0.0100
-0.0001

Therefore, the cubic approximation for this data set is

$P_3(x) = 1.949 - 0.2717x + 0.0100x^2 - 0.0001x^3$

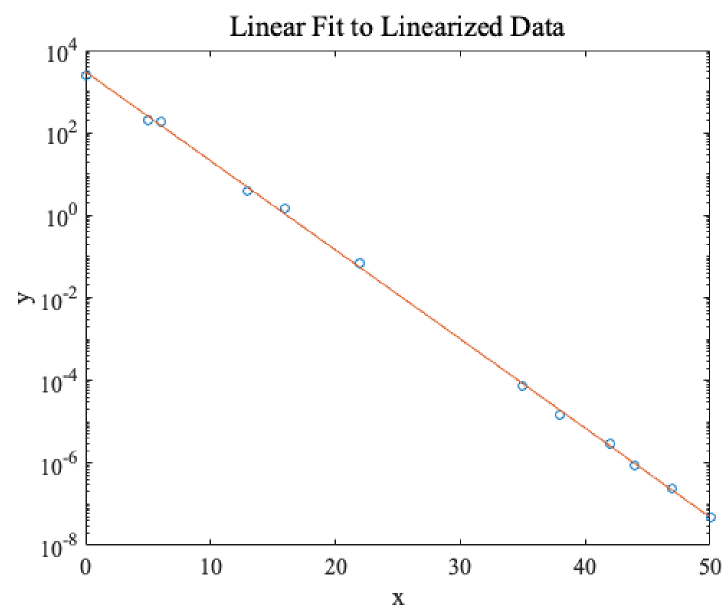
Sum of absolute errors = 3.0460e+03



Linear Fit to Linearized Data

$y = 3013.8 * e^{-0.49710x}$

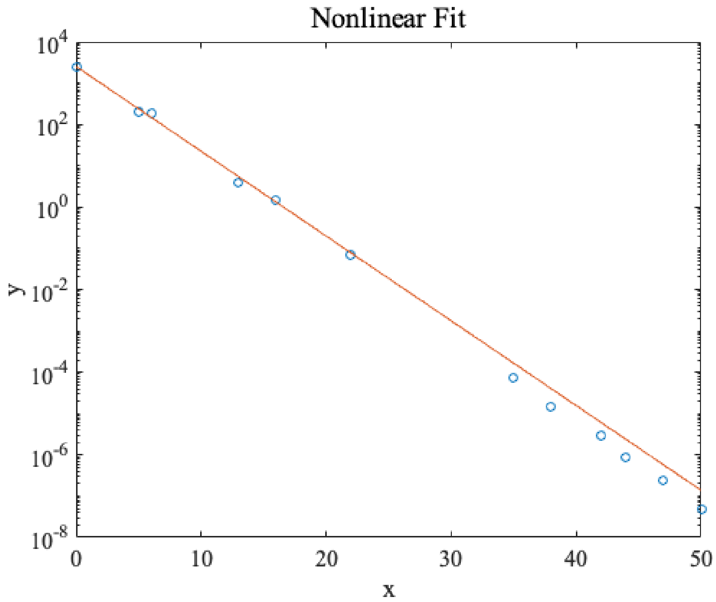
Sum of absolute errors = 567.9959



Nonlinear Fit

$y = 2523.3 * e^{-0.47253x}$

Sum of absolute errors = 69.9917

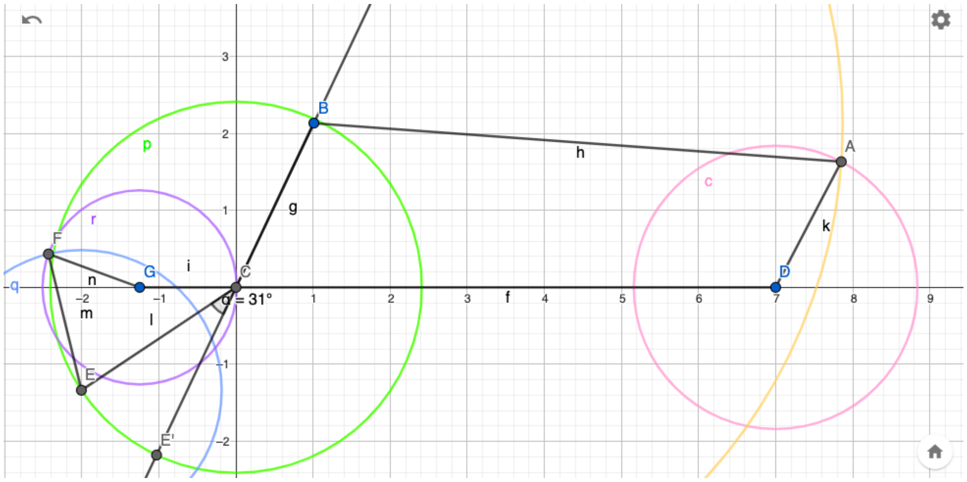


- I performed a nonlinear fit using Newton's method to find the roots *a* and *b* of the nonlinear system $\delta E/\delta a = 0$ and $\delta E/\delta b = 0$, assuming that the approximating function had the exponential form $y = be^{ax}$.
- The nonlinear and linearized approach gave similar values for *a*, but different values for *b*. The nonlinear approximation gave *a* = -0.47253 and *b* = 2523.3, while the linearized approximation gave *a* = -0.49710 and *b* = 3013.8. Furthermore, the sum of absolute errors for the nonlinear approximation was 69.9917, much lower compared to 567.9959 for the linearized solution.
- We should expect the linearized solution to differ from the nonlinear solution because the linearized solution minimizes the sum of squared errors of *log(y),* not *y*, whereas the original, nonlinear approximation minimizes the sum of squared errors of *y*. In other words, the two solve different minimization problems, and we expect the nonlinear method to be more accurate. The linearized approach tends to put more weight on smaller values of *y* because *log* reduces the residuals more on larger *y* than on smaller *y*.
- Although the nonlinear method is a better fit to the data in terms of minimizing the sum of squared errors, we can argue that the linearized approach is more favorable in practice because it gives a similar estimate of *a*, the power in the exponent, which influences the approximation much more than *b*, the coefficient in front, while being more computationally efficient than Newton's method.

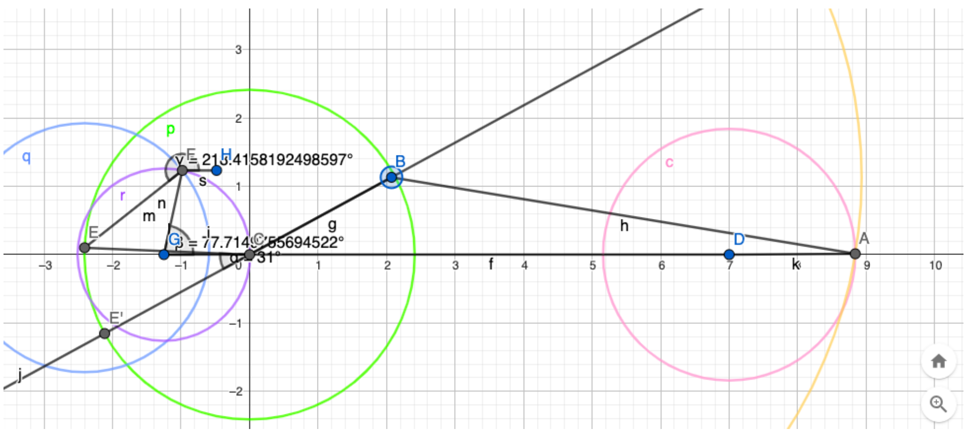
- Overall, the nonlinear fit gave the lowest sum of squared errors, followed by the linear fit to the linearized data, then the cubic polynomial fit, and finally the linear polynomial fit.

2.

i.

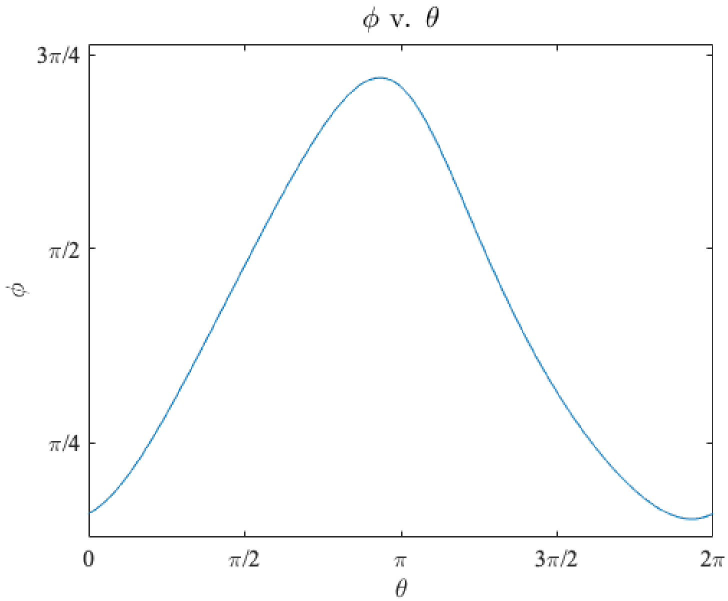


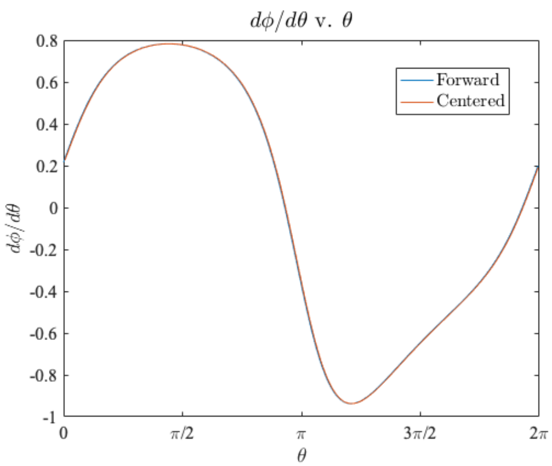
Sketch of the washing machine created in Geogebra.

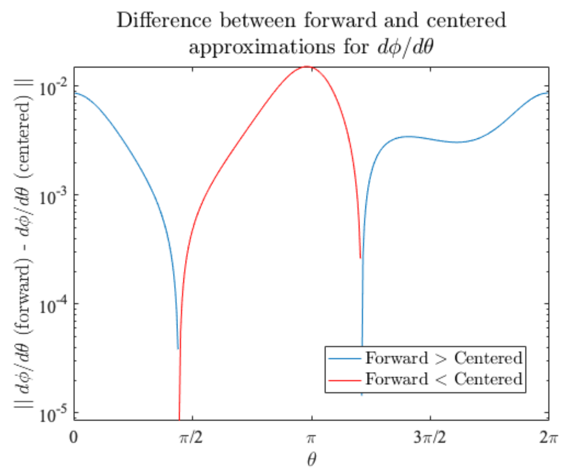


When $\theta = 0$ degrees, $\phi \approx 28.5$ degrees and $\theta_3 \approx 208$ degrees. These serve as the initial approximations for Newton's method.

* In the graphs below, θ and φ are in radians.
* $d\varphi/d\theta$ is unitless.





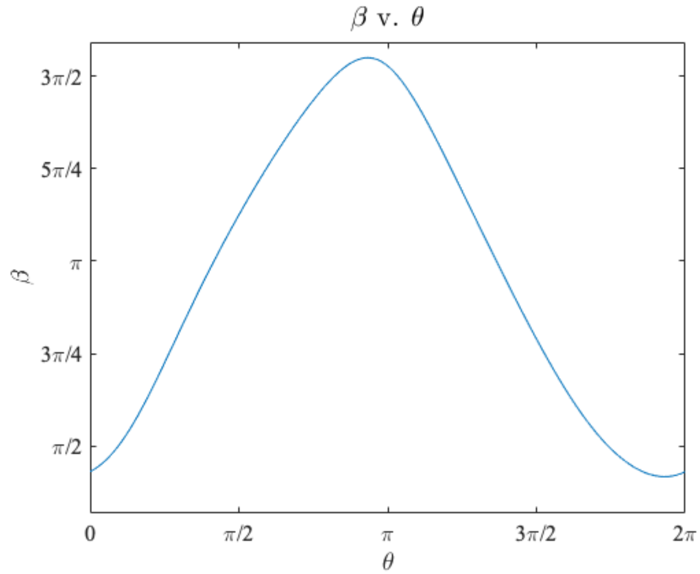


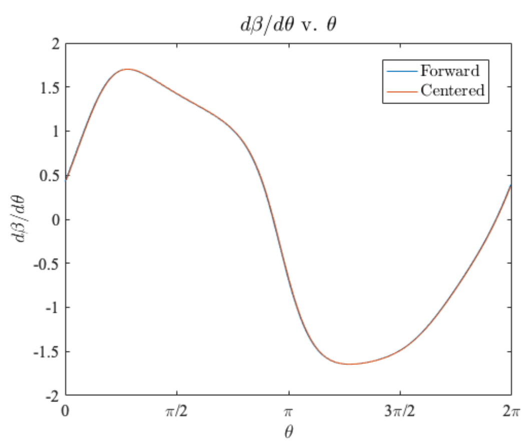
- $\varphi(\theta)$ is possibly sinusoidal. The graph of φ v. θ resembles $sin(\theta)$ shifted right by about $\pi/2$ and upward by π, then scaled vertically by a factor of $\pi/4$.
- Visually, the forward and centered difference approximations are nearly indistinguishable in the graph of $d\varphi/d\theta$ v. θ. However, plotting the absolute differences between the two curves on a log scale reveals that the two approximations are not perfectly aligned. In particular, the centered approximation is greater than the forward approximation between $\theta \approx \pi/2$ radians and $\theta \approx 5\pi/4$ radians, and smaller outside of this range. At $\theta \approx \pi/2$ radians and $\theta \approx 5\pi/4$ radians, near the maximum and minimum of $d\varphi/d\theta$, the difference goes to zero. The maximum differences between the two approximations occur near the critical points of $\varphi(\theta)$ (around $\theta \approx 0, \pi, 2\pi$), and is on the order of 10^{-2}.

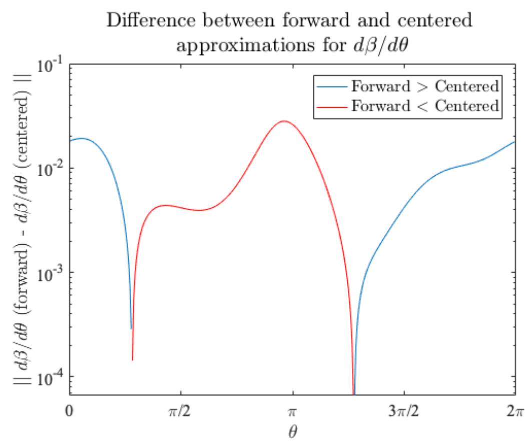
- In general, we expect the centered difference solution to be more accurate. From class, we performed mathematical analyses to show that the centered difference solution has $O(h^2)$ error, which, assuming h is small, is smaller than the $O(h)$ error associated with the forward difference solution.

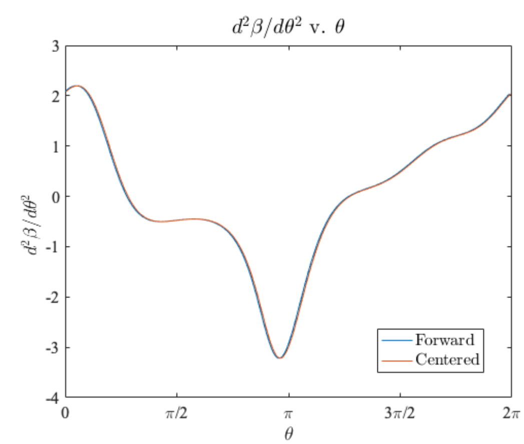
ii.

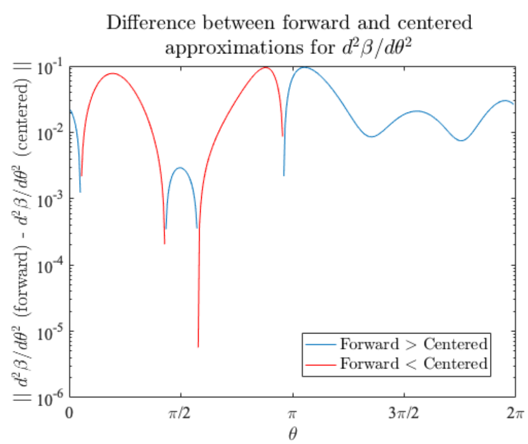
* In the graphs below, θ and β are in radians and t is in seconds.
* $d\beta/d\theta$ is unitless and $d\beta/dt$ has units of radians per second.
* $d^2\beta/d\theta^2$ has units of 1/radians and $d^2\beta/dt^2$ has units of radians per second squared.

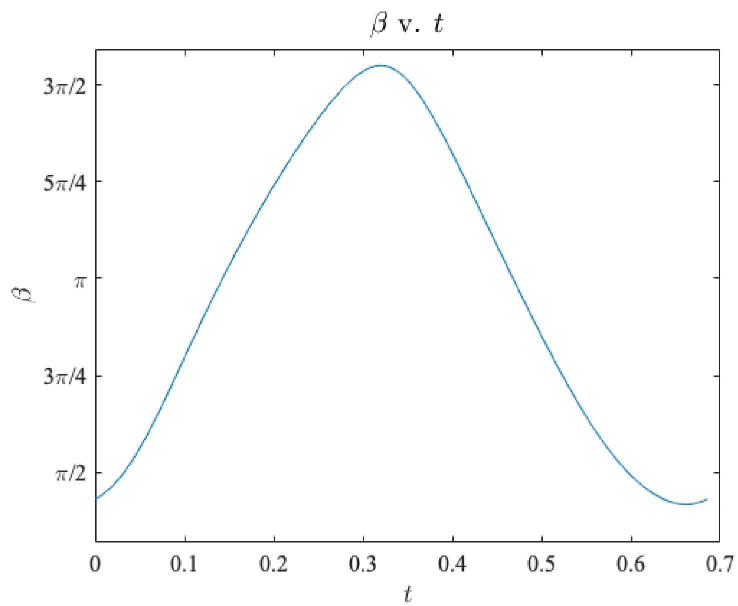


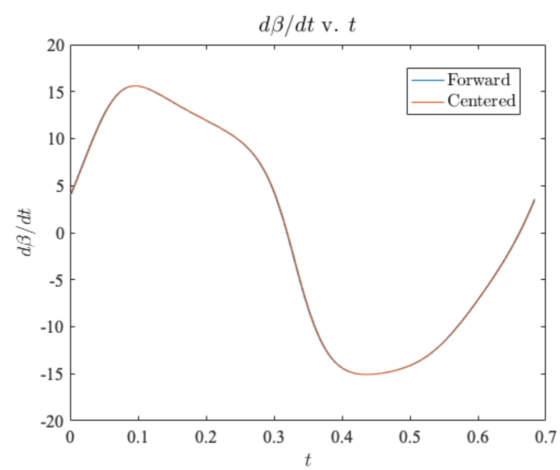


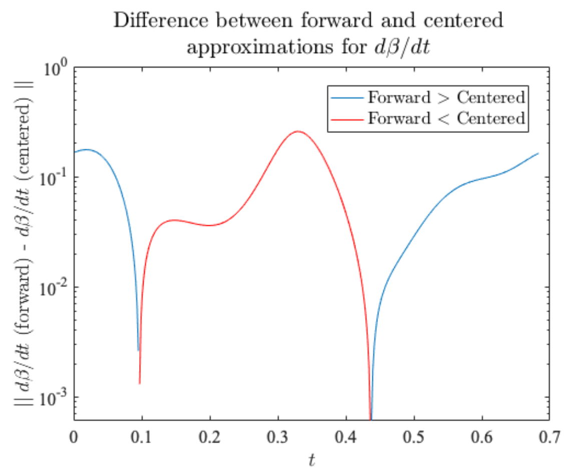


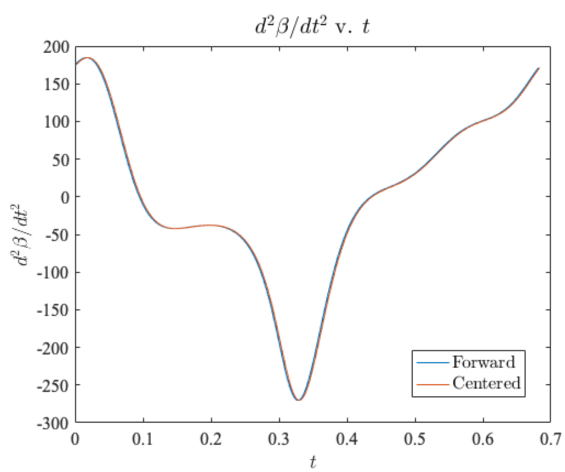


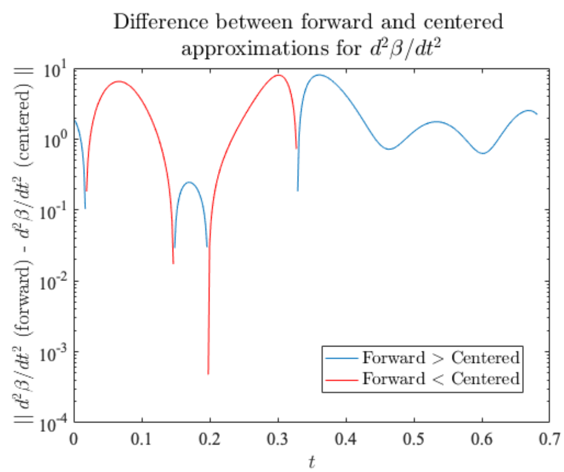












- Graphs of β v. θ and $d\beta/d\theta$ v. θ resemble φ v. θ and $d\varphi/d\theta$ from part (i) in terms of shape, but the y scales are slightly different. The maximum of β is about $3\pi/2$, roughly double the maximum of φ.

- The observations made in part (i) for both graphs φ v. θ and $d\varphi/d\theta$ mostly apply here to β v. θ and $d\beta/d\theta$ v. θ respectively. Similar to part (i), the forward and centered approximations of $d\beta/d\theta$ are most different at the critical points of $\beta(\theta)$, around $\theta \approx 0$, π, and 2π. Here, the maximum difference is on the order of 10^{-1}, slightly larger than the maximum in part (i). Also similar to part (i), the two estimates are about the same at the critical points of $d\beta/d\theta$, around $\theta \approx$ π/4 and 5π/4. Between these critical points, the centered approximation yields higher values of $d\beta/d\theta$ than the forward approximation.
- Although φ v. θ and β v. θ both look sinusoidal, the graph of $d^2\beta/d\theta^2$ suggests that the relationship between both variables and θ might be more complex than a simple transformation of $sin(\theta)$.
- As with the first differences, we expect the second centered difference approximations to be more accurate than the second forward difference approximations, with $O(h^2)$ error compared to $O(h)$.
- The difference between the second forward and centered approximations goes to zero near the critical points of $d^2\beta/d\theta^2$. Some of the more obvious ones occur around $\theta \approx \pi/2$ and $\theta = \pi$. In general, the second derivative approximations do not seem to differ any more than the first derivative approximations.
- Converting from θ to t does not seem to change the shape of the graphs. Graphs of β v. θ and β v. t have the same shape; $d\beta/d\theta$ v. θ and $d\beta/dt$ v. t have the same shape; $d^2\beta/d\theta^2$ v. θ and $d^2\beta/dt^2$ v. t have the same shape; and the graphs of differences between the forward and centered approximations also have the same shape.
- Although shape was preserved, the x and y scales changed. In all graphs, the t axis (in seconds) is compressed by a factor of w = 550 / 60 radians per second relative to the θ axis. Additionally, since $d\beta/dt = w * d\beta/d\theta$, the y axis on the graph of $d\beta/dt$ v. t is stretched vertically by a factor of 550/60 relative to $d\beta/d\theta$ v. θ. Similarly, the graph of $d^2\beta/dt^2$ v. t is also vertically stretched, but by a factor of $(550/60)^2$ relative to $d^2\beta/d\theta^2$ v. θ since $d^2\beta/dt^2 = w^2 * d^2\beta/d\theta^2$. The differences between the forward and centered approximations for the first and second derivatives with t as the independent variable are larger compared to those with θ, but this is expected given the scaling of $d\beta/d\theta$ and $d^2\beta/d\theta^2$ by w and w^2 respectively.