Lab 8

Jessie Li // November 15, 2023

A.

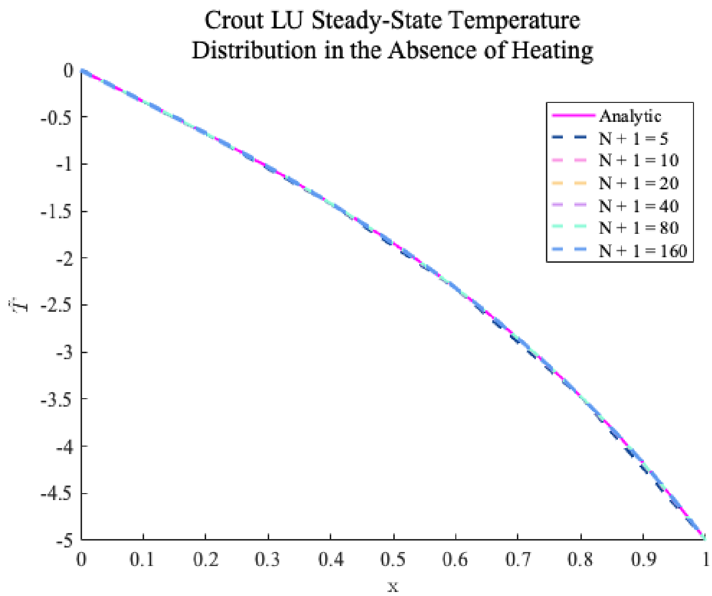
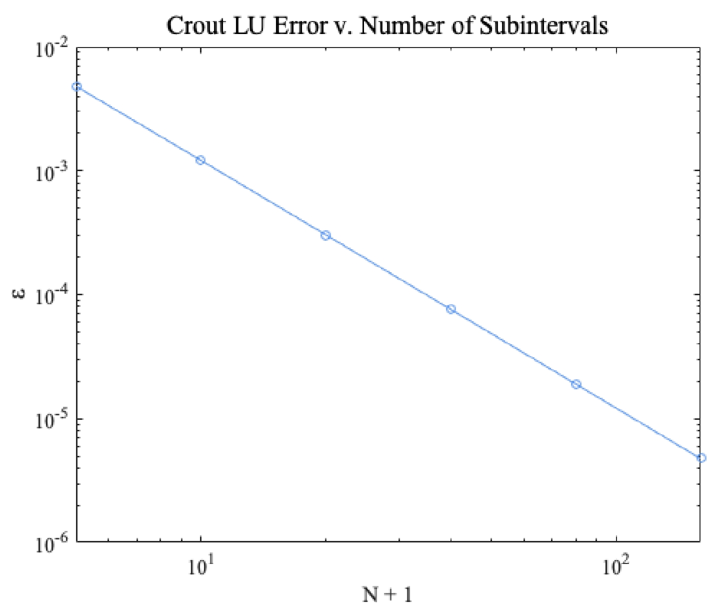


Figure 1: Steady-state temperature distribution in the absence of heating, using Crout LU decomposition to solve a system of finite difference equations for N interior nodes.



N+1	Max. Error
5	0.048
10	0.0012
20	3.0349e-04
40	7.5931e-05
80	1.8998e-05
160	4.7496e-06

Figure 2: Maximum absolute error $\varepsilon = \max|\overline{T}_i - \overline{T}_i^a|$ as a function of $1/\Delta x$ on a log-log scale.

The Crout LU decomposition results agree with the left division results from Lab 7. Error versus the inverse of step size still confirms that this method is $O(h^2)$, since halving step step size reduces error by a factor of 4.

B.

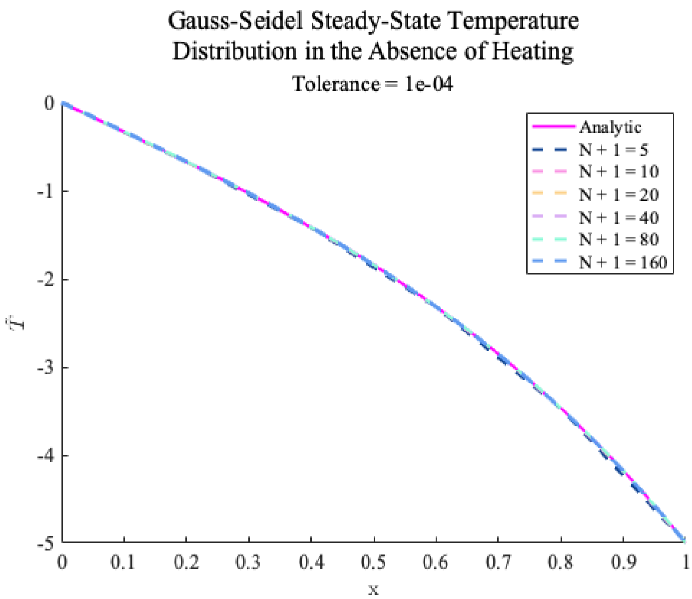


Figure 1: Steady-state temperature distribution using a Gauss-Seidel iteration to solve the same system of finite difference equations. For the initial guess, I added a small random offset to the analytic solution for each N (otherwise, most runs would converge in 1 iteration of Gauss-Seidel). I used the L_∞ relative norm of at least 10^{-4} as the stopping criterion. Although the graph above shows the solution specifically for a tolerance of 10^{-4}, tolerances of 10^{-5} and 10^{-6} gave similar results, but required more iterations. The number of iterations also increased as N increased. For a tolerance of 10^{-4}, the solution for N + 1 = 5 subintervals converged after 11 iterations, while N + 1 = 160 converged after 194. With the same initial guesses, the solutions for N + 1 = 5 and N + 1 = 160 with a tolerance of 10^{-6} required 19 and 1486 iterations respectively to converge.

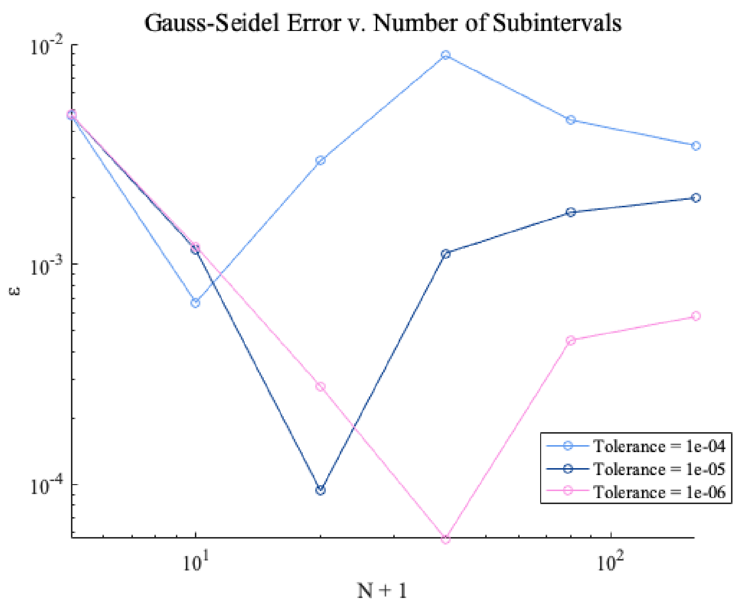


Figure 2: The Gauss-Seidel error behavior differs from the behavior of the error associated with the solution from the Crout LU decomposition. Plotting the error versus N + 1 for various tolerances suggests that the maximum absolute error between the numerical and analytic solutions is generally smaller for lower tolerances. Whereas the Crout LU error consistently decreases as N increases, the Gauss-Seidel error first decreases to some minimum between the smallest and largest values of N, then increases. This means that unlike Crout LU, where the largest N (smallest step size) clearly led to the most accurate solution, the most accurate Gauss-Seidel solution for all tolerances is for an intermediate value of N, which seems to increase as tolerance decreases.

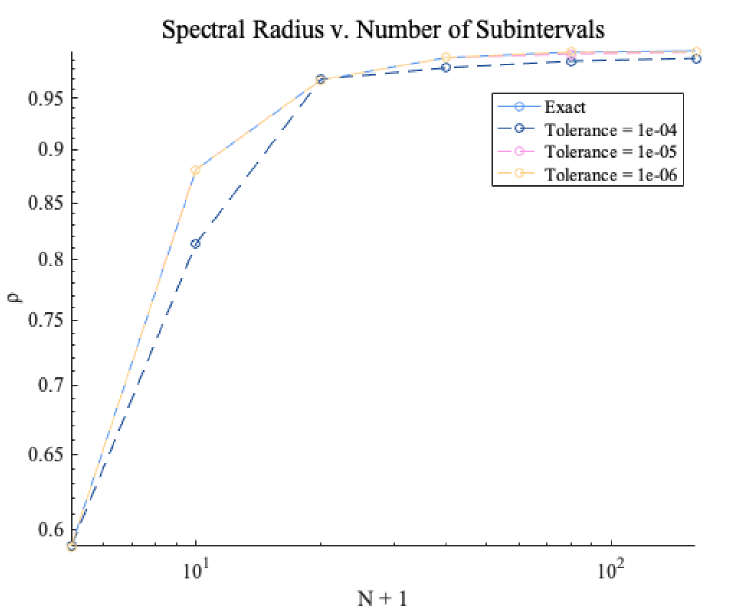
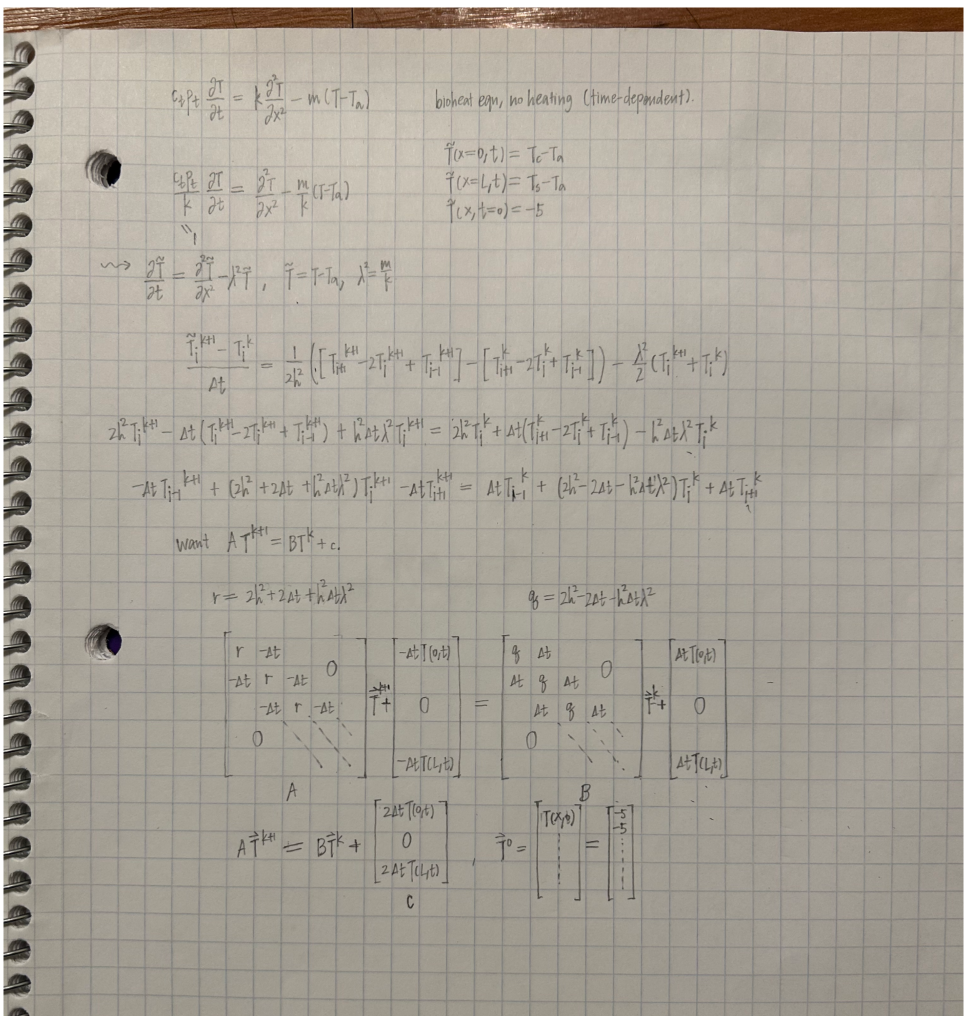


Figure 3: Estimates of the spectral radius of A, the matrix describing the system of finite difference equations, for various N and tolerances. These estimates were computed as $||\delta^{k+1}||_2 / ||\delta^k||_2$, where $\delta^k = x^k - x^{k-1}$. As N increases, the spectral radius approaches but never reaches 1 (since the spectral radius must be less than 1 for the iteration to converge). This is consistent with the Gauss-Seidel iteration converging slower for larger values of N. Based on the graph, lowering the tolerance seems to result in more accurate estimates of the spectral radius. Tolerances of 10^{-5} and 10^{-6} produce estimates that visually overlap with the exact spectral radii, whereas the largest tolerance, 10^{-4}, deviates but still follows the same trend.

C.



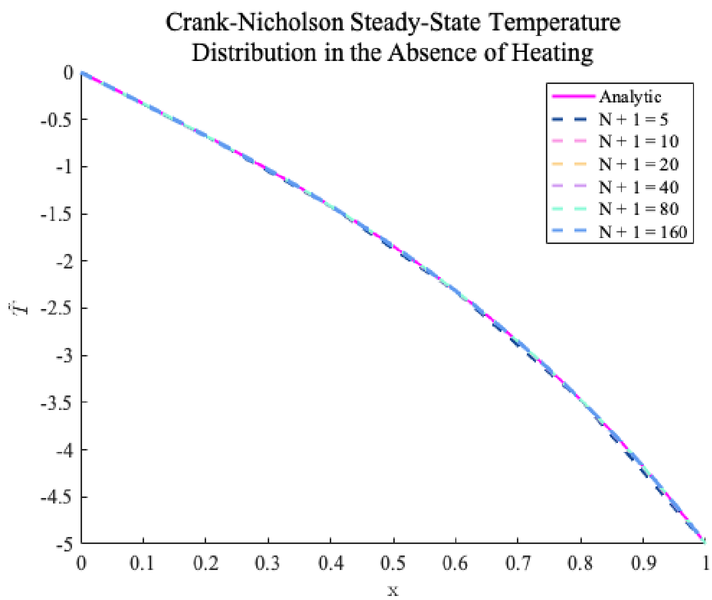


Figure 1: The transient or steady-state solution to the bioheat equation with the time derivative term, using the Crank-Nicholson time-stepping algorithm. To ensure stability, I used $\Delta t = 1/(4n) << h/2$. I used an L_∞ absolute norm between the current previous solutions with a tolerance of 10^{-8} as my stopping criterion. Larger values of N not only required a smaller Δt, but also more time-steps to reach steady-state. The solution for N + 1 = 5 required 27 iterations with $\Delta t = 1/16$, fewer iterations but a larger step size than the solution for N + 1 = 160, which used 803 iterations and $\Delta t = 1/640$.

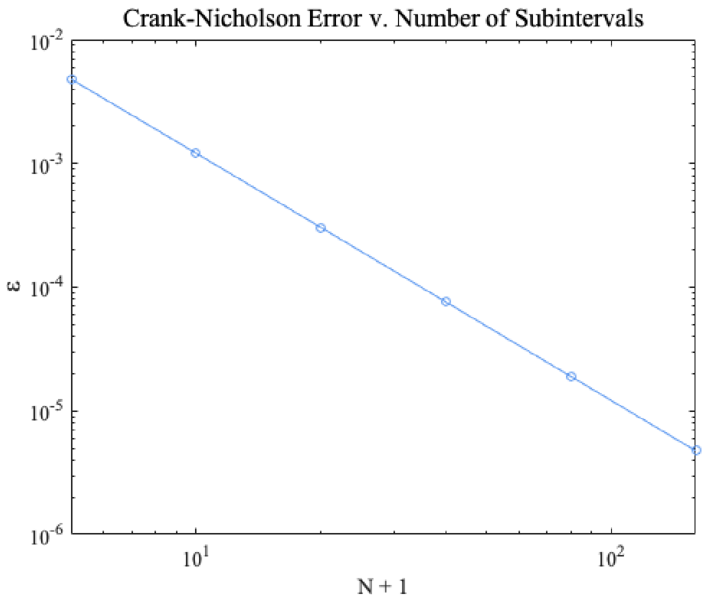


Figure 2: The error behavior of the steady-state Crank-Nicholson solution agrees with the error behavior of the finite difference solution using Crout LU decomposition.

Additional Figures

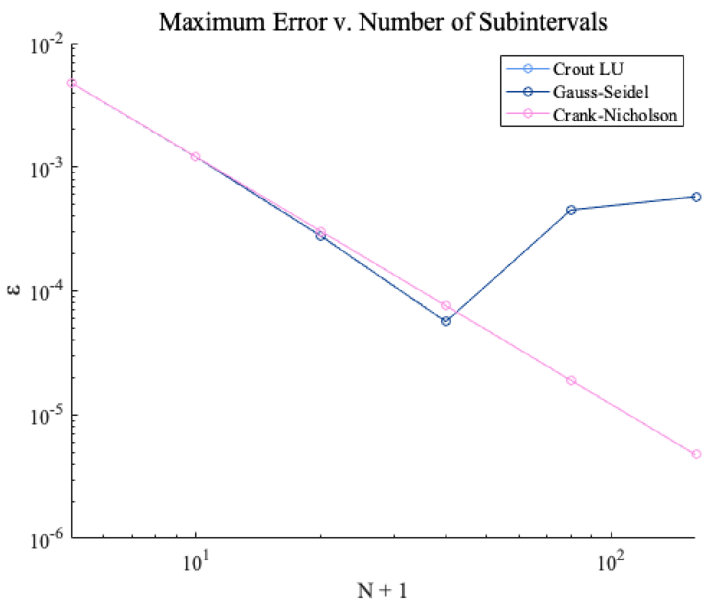


Figure 1: The Crout LU and Crank-Nicholson errors overlap. The dark blue curve is the error associated with the Gauss-Seidel iteration with a tolerance of 10^{-6}, which clearly differs from Crout LU and Crank-Nicholson. For larger N, Crout LU and Crank-Nicholson lead to more accurate solutions than Gauss-Seidel, but this is somewhat expected. Crout LU and Crank-Nicholson both solve an approximating system exactly, whereas the Gauss-Seidel solution is an approximation of an approximating system.