Lab 6

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November 1, 2023

1.

Mathematical analysis indicates that 4th order Runge-Kutta is the most accurate with an $O(h^4)$ error overall, followed by 2nd order Runge-Kutta with $O(h^2)$ error overall, and finally Euler's method with $O(h)$ error overall. Accuracy should also increase as step size decreases.

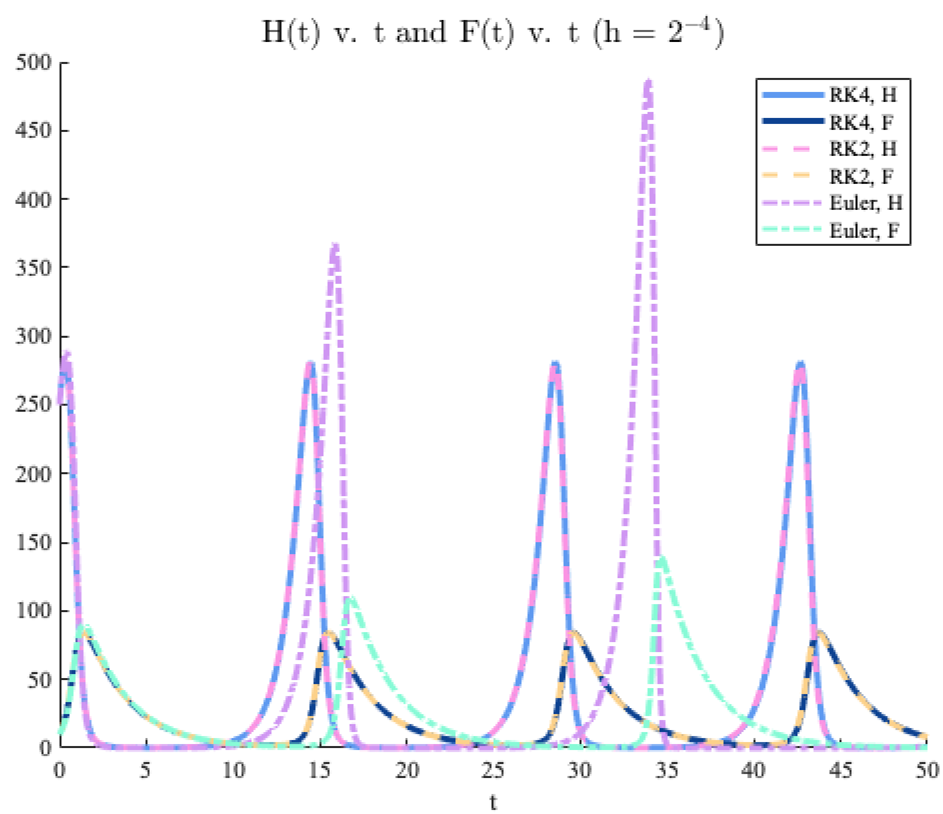


Figure 1: Solutions H(t) and F(t) to the Lotka-Volterra equations with a step size of $h = 2^{-4} = 0.0625$.

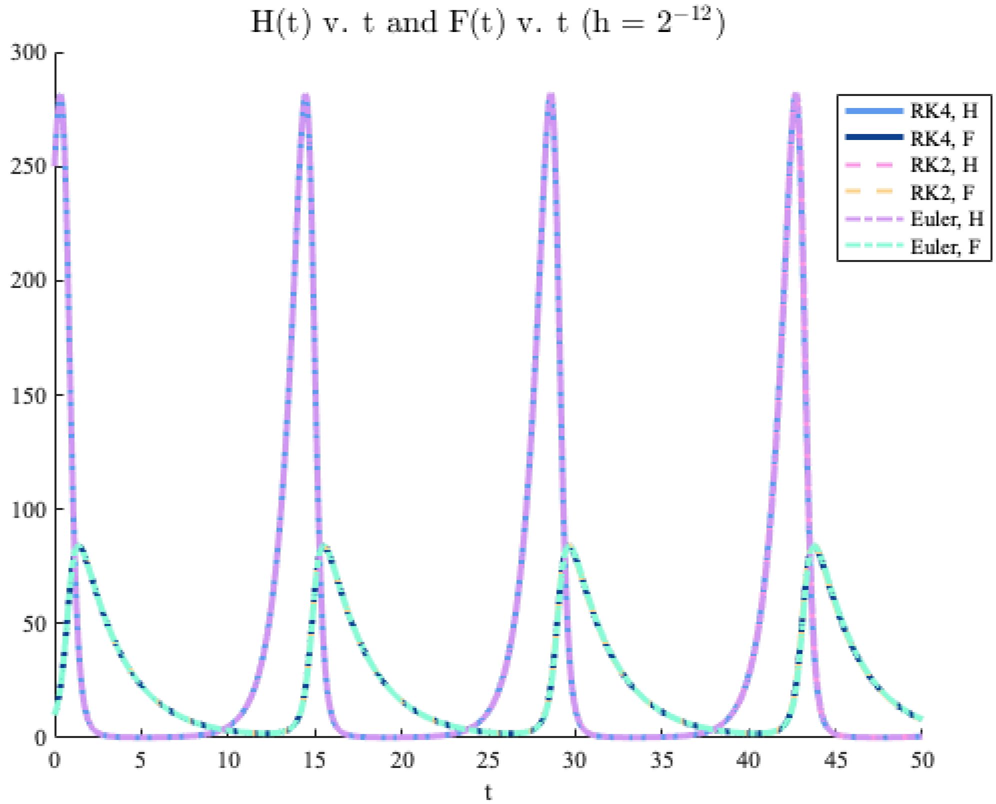


Figure 2: Solutions H(t) and F(t) to the Lotka-Volterra equations with a step size of $2^{-12} \approx 0.000244$.

These expectations agree with what we observe in the graphs of **H(t) v. t** and **F(t) v. t**. As step size decreases from h = 2^{-4} to 2^{-12}, all three methods converge toward the same solutions for H and F, the exact solutions. With a larger value of h, we can see that RK4 and RK2 both produce much better approximations of the exact solutions than Euler's method. For small h, the three methods become visually indistinguishable.

In the graph of **Absolute Error of H and F v. n at t = 50**, where step size is 2^{-n}, I assume that the exact solutions are best approximated by RK4 with the smallest step size used, h = 2^{-12} ≈ 0.000244.

- For all methods, the absolute errors for H(t) and F(t) at t = 50 decrease as step size decreases, in accordance with our expectations.
- Although RK4 and RK2 were hard to distinguish in the graphs of **H(t) v. t** and **F(t) v. t,** the graph of **Absolute Error v. n** allows us to rank these methods based on the magnitudes of their errors. RK4 has the lowest errors and is therefore the most accurate, followed by RK2, and finally Euler.

- Interestingly, all three methods approximate H slightly more accurately than F for any of the step sizes tested, but the errors of both decrease at roughly the same rate for each method.

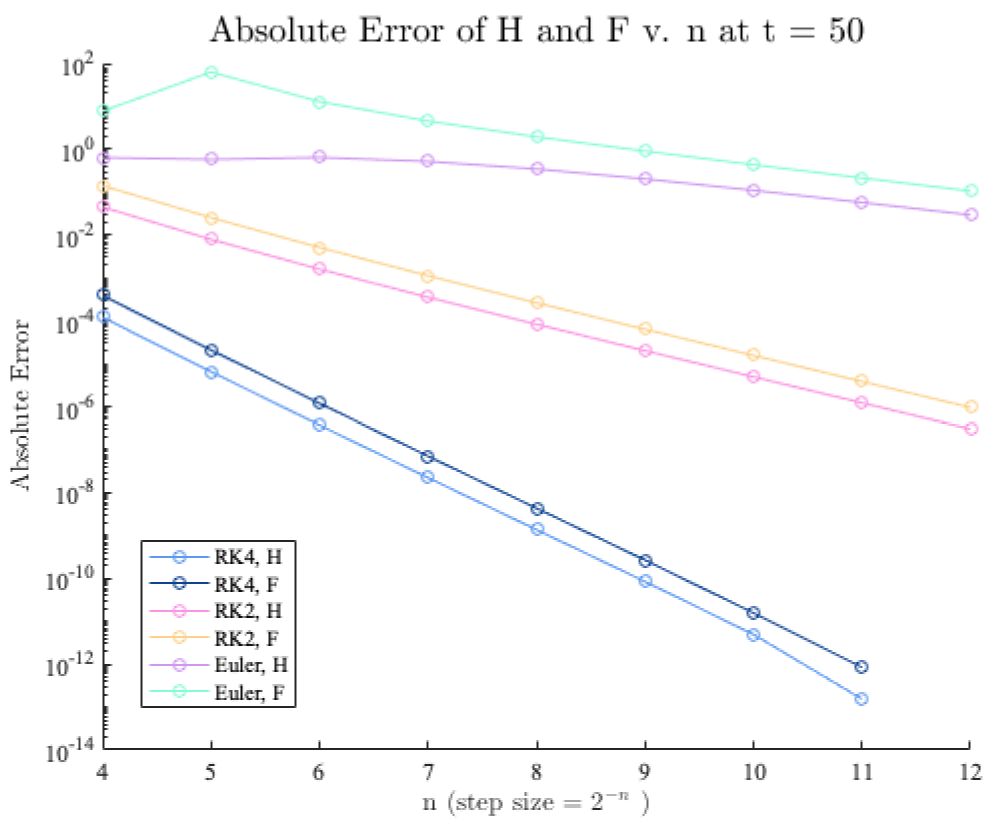


Figure 3: Absolute error of H and F v. n at t = 50, where step size h = 2^{-n}.

2.

Stability depends on the method, ODE, and step size. A table of the stability criteria for four methods and the ODE $y' = -7y$ is provided below. The methods are listed in increasing order of stability. Calculations are provided in the **Additional Figures** section at the end of Question 2.

Method	**Stability criterion**
2-Step Adams-Bashforth (least stable)	$h < 1/7$
2nd Order Runge-Kutta	$h < 2/7$
2-Step Adams Predictor-Corrector	$h < 12/35$
2-Step Adams Moulton (most stable)	$h < 6/7$

To compare the stability of each method, I plotted their numerical solutions for increasing values of h, from 1/8, 1/7, 1/6, 1/3, 1/2, to 1 (see **Additional Figures** at the end of this section).

- With $h = 1/8$, a step size that satisfies all stability criteria, all methods are stable.
- 2-Step Adams-Bashforth (AB2) is the least stable method analytically and the first to diverge from the exact solution. At $h = 1/7$, right at its stability threshold, the errors for this method remain relatively constant long-term; the numerical solution fails to converge. AB2 loses stability for $h > 1/7$. In the graph for $h = 1/6$, AB2 clearly diverges while the other methods still converge toward the analytic solution.
- 2nd Order Runge-Kutta (RK2) is the third most stable method. At $h = 1/3$, just above its stability criterion of 2/7, the errors of the RK2 numerical solution grow exponentially.
- The second most stable method by the mathematical and numerical analyses is the 2-Step Predictor-Corrector (AB2/AM2). At $h = 1/3$, just below its stability criterion of 12/35, the method is stable, but not very accurate. Above its stability criterion at $h = 1/2$, the method clearly diverges.

- Implicit 2-Step Adams-Moulton (AM2) is the most stable method. As indicated by the mathematical analysis and supported by the graphs, AM2 converges for $h < 6/7$ and diverges for $h > 6/7$, demonstrating the highest tolerance for large step sizes.

The graphs below provide a closer look at the stability of the 2-Step Adams Predictor-Corrector. The stability criterion of this method was calculated to be $h < 12/35$.

- The first graph compares the exact solution with the AB2/AM2 solution for three different step sizes, one which violates the stability criterion ($h = 1/2$), another which satisfies ($h = 0.001$), and a third that satisfies but is not very accurate ($h = 1/3$). While the solutions for the two stable step sizes $h = 0.001$ and $h = 1/3$ converge toward the exact solution, the solution for the unstable step size $h = 1/2$ diverges with the errors growing exponentially as t increases.
- The second graph focuses on the two solutions that satisfy the stability criterion ($h = 0.001$ and $h = 1/2$) to enable a more meaningful comparison between them. While the $h = 0.001$ solution closely approximates the analytic curve, the $h = 1/3$ solution oscillates dramatically at lower values of t, but ultimately converges toward the exact values as t increases, demonstrating overall stability. However, the inaccuracy of the $h = 1/3$ solution indicates that stability does not guarantee accuracy.
- The third graph plots the absolute errors of each of the three step sizes as a function of t. Absolute error clearly decreases exponentially as t increases for the smallest and most accurate step size, decreases very slowly for the stable but inaccurate step size, and increases exponentially for the large, unstable step size.

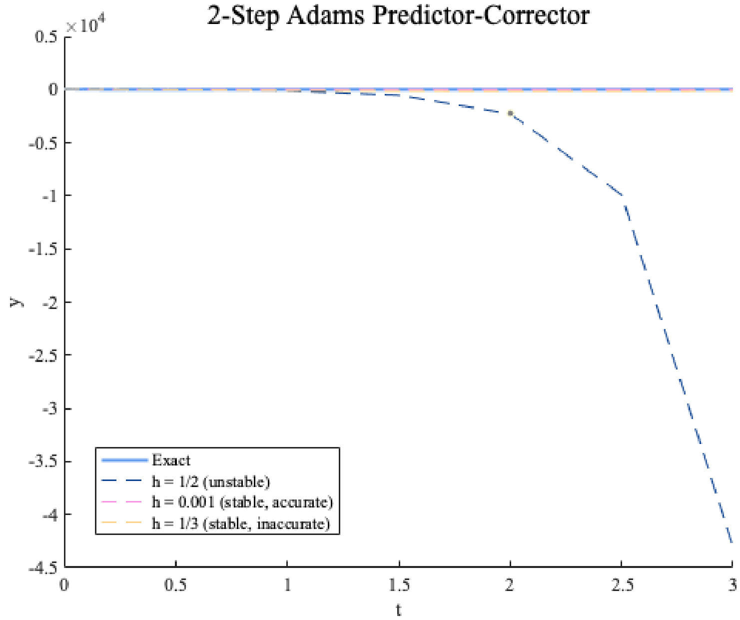


Figure 1: All three step sizes – unstable, stable and accurate, and stable and inaccurate – plotted on the same graph.

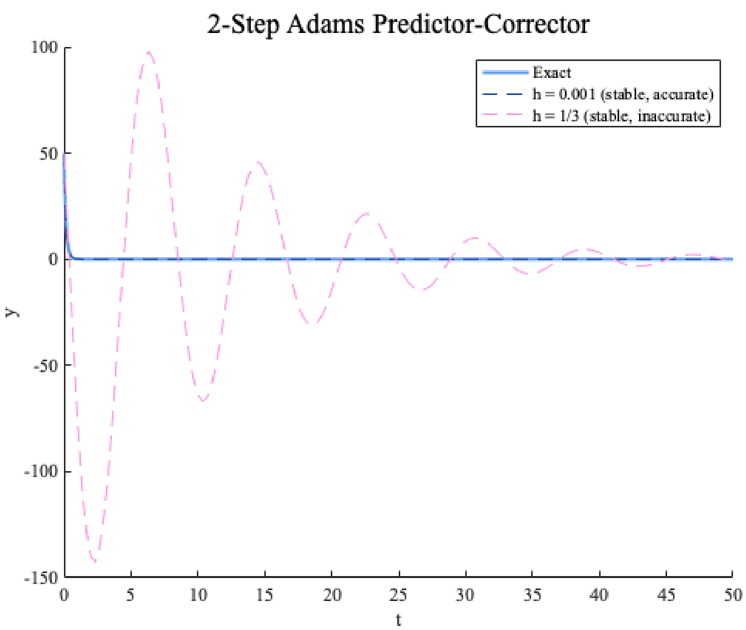


Figure 2: Graph of stable step sizes (h = 0.001 and h = 1/3) only. The 2-Step Adams Predictor-Corrector converges for both, but is more accurate with a step size of h = 0.001.

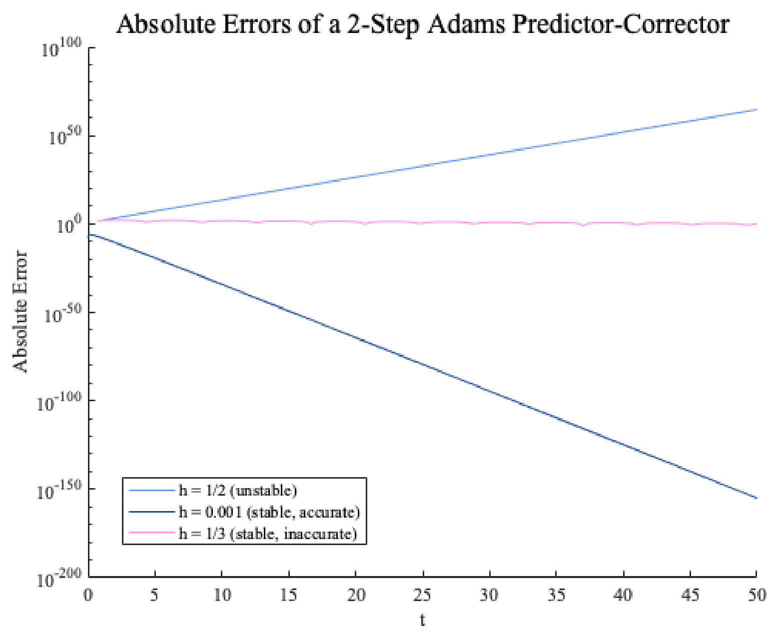


Figure 3: Absolute errors confirm the expected relative stability of each step size. Above the stability criterion of 12/35 for this method and ODE, the errors for h = 1/2 grow exponentially. Far below the stability criterion, the errors for h = 0.001 decrease exponentially, while just below 12/35, the errors for h = 1/3 still decrease, but much more slowly.

Additional Figures

Series 1: Derivations of the stability criteria for AB2, RK2, AB2/AM2, and AM2.

Adams-Bashforth 2-step

$$w_{i+1} = w_i + \frac{h}{2}(3f_i - f_{i-1}) \qquad f_i = -7w_i$$

$$w_{i+1} = w_i + \frac{h}{2}(3(-7w_i) - (-7w_{i-1}))$$

$$w_{i+1} = \left(1 - \frac{21}{2}h\right)w_i + \frac{7}{2}hw_{i-1}$$

$$w_{i+1} - \left(1-\frac{21}{2}h\right)w_i - \frac{7}{2}hw_i = 0$$

$$\varepsilon_{i+1} - \left(1-\frac{21}{2}h\right)\varepsilon_i - \frac{7}{2}h\varepsilon_i = 0$$

assume $\varepsilon_i \sim \lambda^i$: $\qquad \lambda^2 - \left(1-\frac{21}{2}h\right)\lambda - \frac{7}{2}h = 0$

Given $A\lambda^2 + B\lambda + C$, $|\lambda_{1,2}| < 1$ if $\frac{C}{A} < 1$ and $|B| < A + C$.

1) $\frac{C}{A} < 1$

$-\frac{7}{2}h < 1 \rightsquigarrow h > -\frac{2}{7}$ always true, since $h > 0$.

2) $|B| < A + C$

$\cancel{1} - \frac{21}{2}h < \cancel{1} - \frac{7}{2}h$

always true

$\frac{21}{2}h - 1 < 1 - \frac{7}{2}h$

$\frac{28}{2}h < 2 \rightsquigarrow \boxed{h < \frac{1}{7}}$

Runge-Kutta 2nd order

$$w_{i+1} = w_i + hf\left(w_i + \frac{h}{2}f_i,\ t_i + \frac{h}{2}\right)$$

$$w_{i+1} = w_i + hf\left(w_i + \frac{h}{2}(-7w_i),\ t_i + \frac{h}{2}\right)$$

$$w_{i+1} = w_i + h\cdot -7\left(1 - \frac{7}{2}h\right)w_i = \left(1 - 7h + \frac{49}{2}h^2\right)w_i$$

know $\lim_{i\to\infty} w_{i+1} = 0$ (and $\varepsilon_{i+1} \approx \left(1 - 7h + \frac{49}{2}h^2\right)\varepsilon_i$) so $\cancel{1} - 7h + \frac{49}{2}h^2 < \cancel{1}$

$$-7\cancel{h} + \frac{49}{2}h^{\cancel{2}} < 0 \rightsquigarrow \boxed{h < \frac{2}{7}}$$

2-Step Adams-Bashforth/Adams-Moulton Predictor-Corrector $\qquad f_i = -7w_i$

Adams-Bashforth 2-step: $w_{i+1} = w_i + \frac{h}{2}(3f_i - f_{i-1})$ $\qquad$ predictor (explicit)

Adams-Moulton 2-step: $w_{i+1} = \frac{h}{12}(5f_{i+1} + 8f_i - f_{i-1})$ $\qquad$ corrector (implicit).

$$w_{i+1} = w_i + \frac{h}{12}\left[5(-7w_{i+1}) + 8(-7w_i) - (-7w_{i-1})\right]$$

$$w_{i+1} = w_i + \frac{h}{12}\left[-35w_{i+1} - 56w_i + 7w_{i-1}\right] = w_i + \frac{h}{12}\left[-35\overbrace{\left(w_i + \frac{3h}{2}(-7w_i) - \frac{h}{2}(-7w_{i-1})\right)}^{AB2} - 56w_i + 7w_{i-1}\right]$$

$$w_{i+1} = w_i + \frac{h}{12}\left[-35w_i + \frac{735}{2}hw_i - \frac{245}{2}hw_{i-1} - 56w_i + 7w_{i-1}\right]$$

$$w_{i+1} = \left(1 - \frac{91}{12}h + \frac{735}{24}h^2\right)w_i + \left(\frac{7}{12}h - \frac{245}{24}h^2\right)w_{i-1}$$

$$\varepsilon_{i+1} - \left(1 - \frac{91}{12}h + \frac{735}{24}h^2\right)\varepsilon_i - \left(\frac{7}{12}h - \frac{245}{24}h^2\right)\varepsilon_{i-1} = 0$$ $\qquad$ assume $\varepsilon_i \sim \lambda^i$

$$\lambda^2 - \underbrace{\left(1 - \frac{91}{12}h + \frac{735}{24}h^2\right)}_{B}\lambda - \underbrace{\left(\frac{7}{12}h - \frac{245}{24}h^2\right)}_{C} = 0$$

$A=1$

1) $\frac{C}{A} < 1$

$$\frac{245}{24}h^2 - \frac{7}{12}h < 1$$

$$\frac{245}{24}h^2 - \frac{7}{12}h - 1 < 0. \rightsquigarrow 245h^2 - 14h - 24 < 0.$$

$$\lambda_{1,2} = \frac{14 \pm \sqrt{196 + 4(245)(24)}}{245(2)}, \quad \text{positive root } \lambda = \frac{14+154}{490} = \frac{12}{35}$$

$$\boxed{h < \frac{12}{35}}$$

2) $|B| < A + C$

$$\cancel{1} - \frac{91}{12}h + \frac{735}{24}h^2 < \cancel{1} - \frac{7}{12}h + \frac{245}{24}h^2$$

$$\frac{490}{24}h^{\cancel{2}} < \frac{84}{12}\cancel{h}$$

$$h < \frac{12}{35}$$

$$-1 + \frac{91}{12}h - \frac{735}{24}h^2 < 1 - \frac{7}{12}h + \frac{245}{24}h^2$$

$$\frac{980}{24}h^2 - \frac{98}{12}h + 2 > 0$$

$980h^2 - 196h + 48 > 0$ $\qquad$ always true.

$$\frac{+196 \pm \sqrt{(196)^2 - 4\cdot 980\cdot 48}}{2(980)}$$ $\qquad$ $b^2 - 4ac < 0$.

2-step Adams-Moulton

$$w_{i+1} = w_i + \frac{h}{12}(5f_{i+1} + 8f_i - f_{i-1}) \qquad f_i = -7w_i$$

$$w_{i+1} = w_i + \frac{h}{12}\left(5(-7w_{i+1}) + 8(-7w_i) - (-7w_{i-1})\right) = w_i - \frac{35}{12}hw_{i+1} - \frac{56}{12}hw_i + \frac{7}{12}hw_{i-1}$$

$$\left(1 + \frac{35}{12}h\right)w_{i+1} = \left(1 - \frac{56}{12}h\right)w_i + \frac{7}{12}hw_{i-1}$$

$$\left(1 + \frac{35}{12}h\right)\varepsilon_{i+1} - \left(1 - \frac{56}{12}h\right)\varepsilon_i - \frac{7}{12}h\varepsilon_{i-1} = 0 \xrightarrow{\text{assume } \varepsilon_j \sim \lambda^i} \left(1 + \frac{35}{12}h\right)\lambda^2 - \left(1 - \frac{56}{12}h\right)\lambda - \frac{7}{12}h = 0.$$

1) $\frac{C}{A} < 1$

$$\frac{-7/12}{1 + \frac{35}{12}h} < 1 \rightsquigarrow 0 < 1 + \frac{42}{12}h$$

always true.

2) $|B| < A + C$

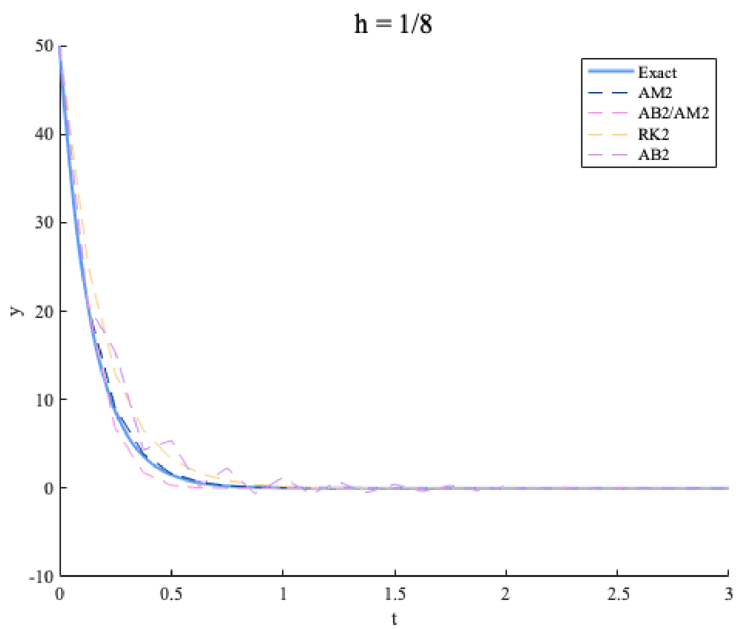
$$\cancel{1} - \frac{56}{12}h < 1 + \frac{35}{12}h - \frac{7}{12}h = \cancel{1} + \frac{28}{12}h$$

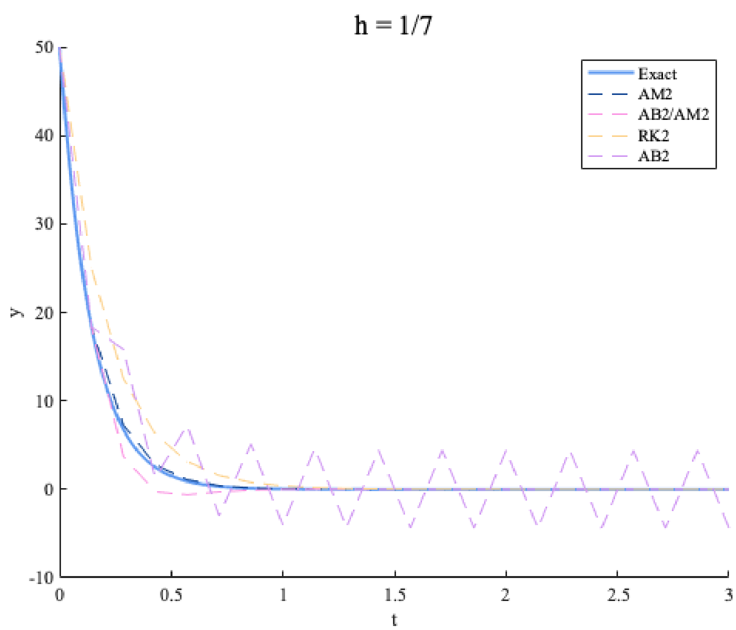
always true.

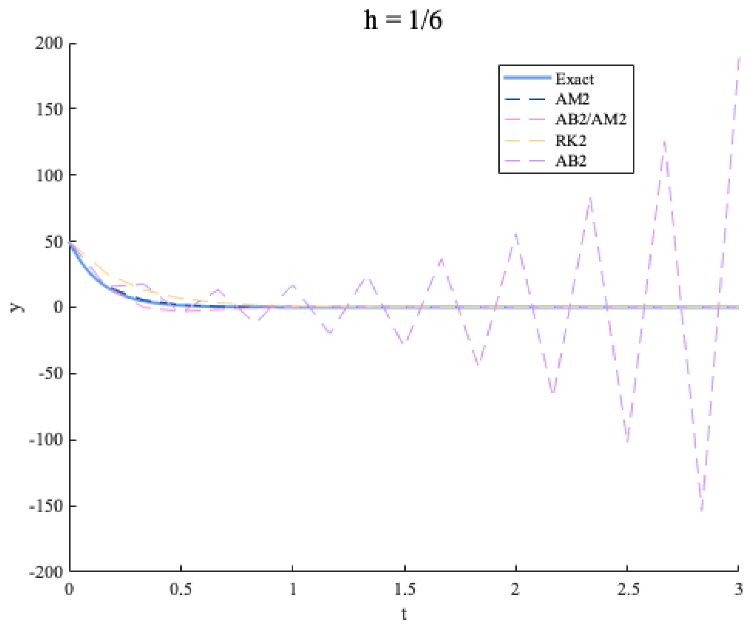
$$\frac{56}{12}h - 1 < 1 + \frac{28}{12}h$$

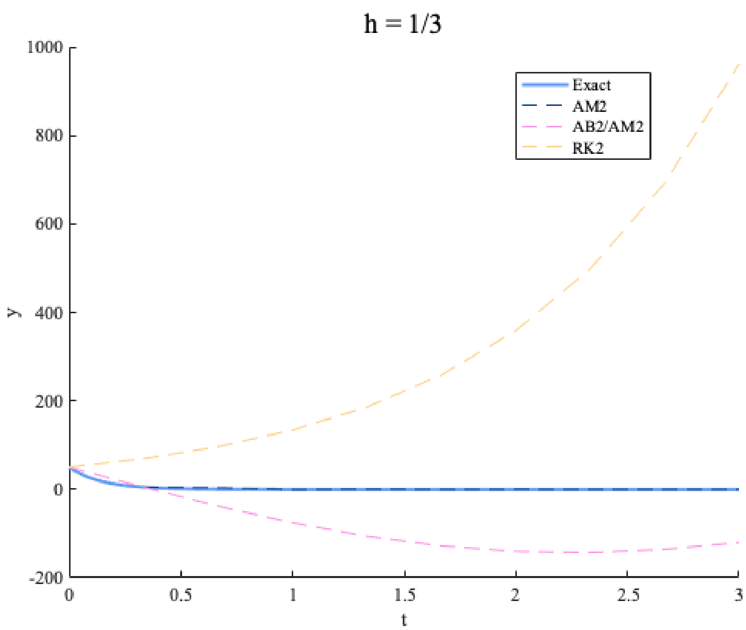
$$\frac{28}{12}h < 2 \rightsquigarrow h < \frac{24}{28} \rightsquigarrow \boxed{h < \frac{6}{7}}$$

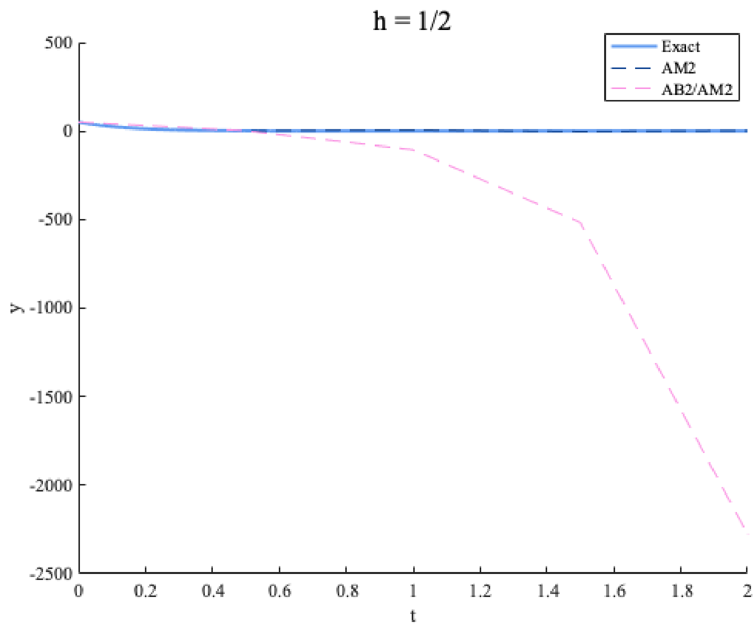
Series 2: Numerical solutions generated by AB2, RK2, AB2/AM2, and AM2 for various h.

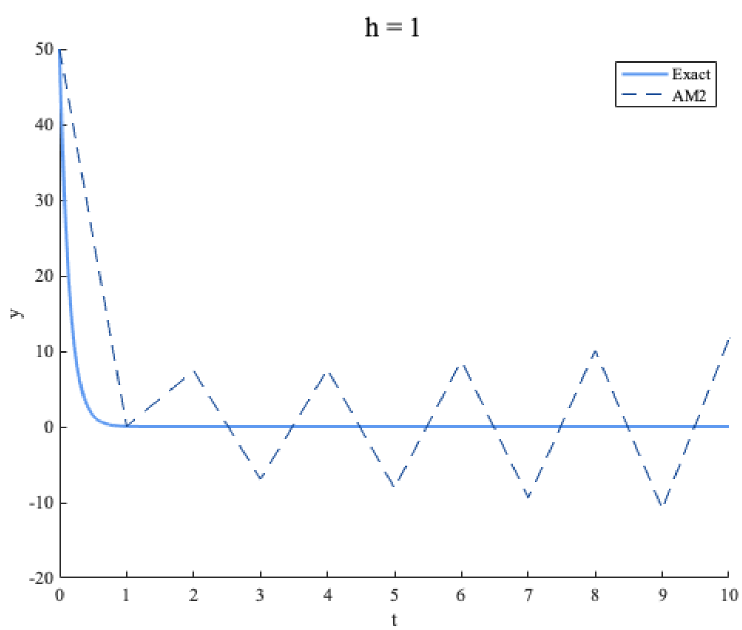





h = 1/6
Exact
AM2
AB2/AM2
RK2
AB2
y
t
200
150
100
50
0
-50
-100
-150
-200
0
0.5
1
1.5
2
2.5
3


h = 1/3
Exact
AM2
AB2/AM2
RK2
y
t
1000
800
600
400
200
0
-200
0
0.5
1
1.5
2
2.5
3


h = 1/2
Exact
AM2
AB2/AM2
500
0
-500
-1000
-1500
-2000
-2500
y
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
t


h = 1
Exact
AM2
50
40
30
20
10
0
-10
-20
y
0
1
2
3
4
5
6
7
8
9
10
t

3.

I used a fourth-order single-step method, the 4th Order Runge-Kutta method to start the 4-Step Adams Predictor-Corrector and chose a step size of h = 0.001. In the graphs of current (y') versus voltage (y), the orbits spiral outward from the origin as t increases from 0 to 100. As amplitude *a* increases from 0.5 to 5.5, the spirals loosen and approach a rectangular shape. A simpler, lower order method with the same step size h = 0.001 such as RK2 gives a similar solution as the higher order 4-Step Adams Predictor-Corrector.

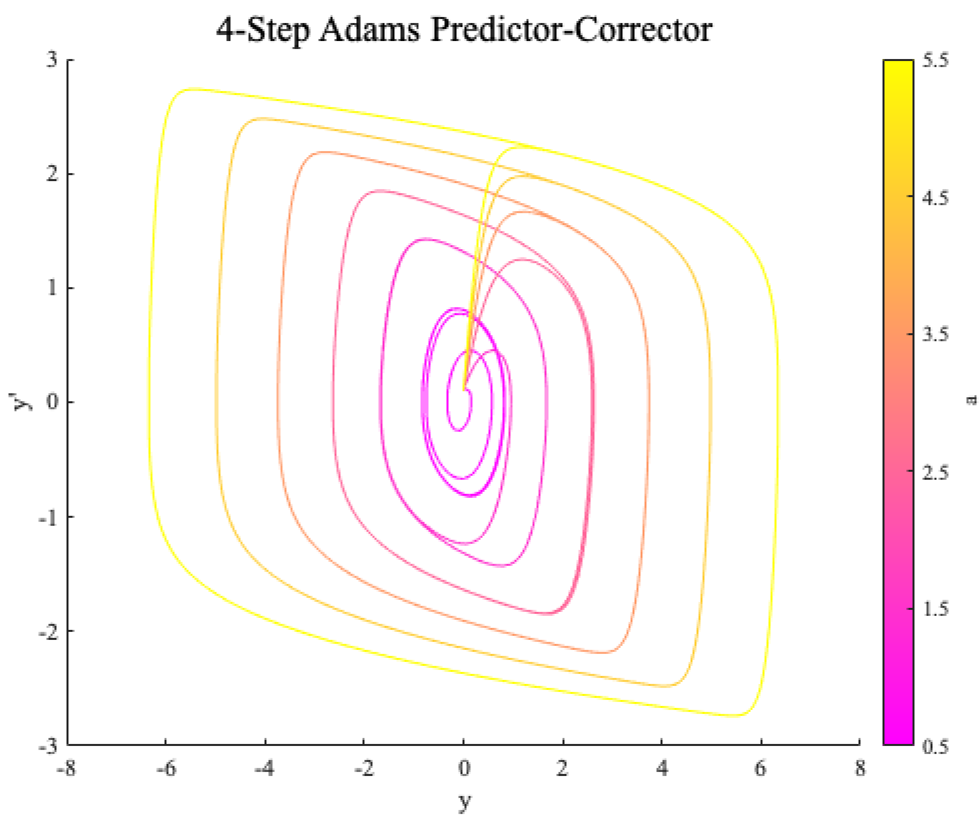


Figure 1: Solutions generated by a 4-Step Adams Predictor-Corrector for various *a* with h = 0.001.

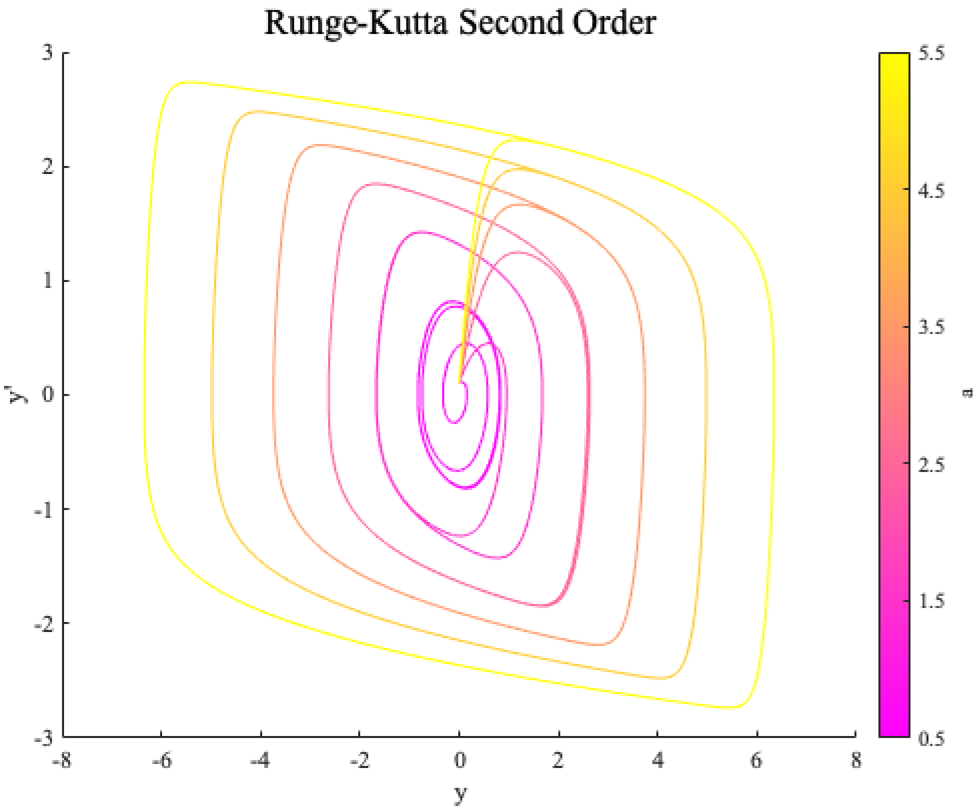


Figure 2: Solutions generated by 2nd-Order Runge-Kutta for various *a* with h = 0.001.