Delft University of Technology

Networked and Distributed Control

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Exercise 1: the rise of the Medici

The MATLAB code for this exercise can be found in `Question1.m`.

The weight matrix for the families is given in table 1.

Table 1: The weight matrix for the families.

	Castellani	Peruzzi	Strozzi	Ridolfi	Barbadori	Bischeri	Tornabuoni	Medici	Accaiaiuoli	Guadagni	Albizzi	Salviati	Lamberteschi	Gimori	Pazzi
Castellani	0	1	1	0	1	0	0	0	0	0	0	0	0	0	0
Peruzzi	1	0	1	0	0	1	0	0	0	0	0	0	0	0	0
Strozzi	1	1	0	1	0	1	0	0	0	0	0	0	0	0	0
Ridolfi	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0
Barbadori	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0
Bischeri	0	1	1	0	0	0	0	0	0	1	0	0	0	0	0
Tornabuoni	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0
Medici	0	0	0	1	1	0	1	0	1	0	1	1	0	0	0
Accaiaiuoli	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
Guadagni	0	0	0	0	0	1	1	0	0	0	1	0	1	0	0
Albizzi	0	0	0	0	0	0	0	1	0	1	0	0	0	1	0
Salviati	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1
Lamberteschi	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
Gimori	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
Pazzi	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0

a) The Bonacich centrality is calculated by normalizing the leading left eigenvector (corresponding to eigevalue 1) of $P^\top$. The Bonacich centrality for the fifteen families can be found in table 2 on the left.

b) The closeness centrality is calculated by:

$$\pi_i = \frac{n}{\sum_{j \in \mathcal{V}} \operatorname{dist}(i,j)}, \quad i \in \mathcal{V} \tag{1}$$

The closeness centrality for the fifteen families can be found in table 2 in the middle.

Table 2: The Bonacich (left), closeness (middle) and betweenness (right) centrality of the different families.

Family name	Bonacich centrality
Castellani	0.0789
Peruzzi	0.0789
Strozzi	0.1053
Ridolfi	0.0526
Barbadori	0.0526
Bischeri	0.0789
Tornabuoni	0.0526
Medici	0.1579
Accaiaiuoli	0.0263
Guadagni	0.1053
Albizzi	0.0789
Salviati	0.0526
Lamberteschi	0.0263
Ginori	0.0263
Pazzi	0.0263

Family name	Closeness centrality
Castellani	0.4167
Peruzzi	0.3947
Strozzi	0.4545
Ridolfi	0.4839
Barbadori	0.4688
Bischeri	0.4286
Tornabuoni	0.4839
Medici	0.6000
Accaiaiuoli	0.3947
Guadagni	0.4839
Albizzi	0.5172
Salviati	0.4167
Lamberteschi	0.3409
Ginori	0.3571
Pazzi	0.3061

Family name	Betweenness centrality
Castellani	0.0444
Peruzzi	0.0178
Strozzi	0.0800
Ridolfi	0.0785
Barbadori	0.0800
Bischeri	0.0993
Tornabuoni	0.0622
Medici	0.4519
Accaiaiuoli	0.0000
Guadagni	0.2148
Albizzi	0.1778
Salviati	0.1156
Lamberteschi	0.0000
Ginori	0.0000
Pazzi	0.0000

c) The calculated decay centralities for $\delta = 0.25, 0.5, 0.75$ are given in table 3. The decay centrality represents the number of all outgoing paths of a certain node when $\delta \to 1$ and it represents how close in terms of distance the node is with respect to other nodes when $\delta \to 0$

Table 3: The decay centralities, for different δ-values, of the different families.

Family name	$\delta = 0.25$	$\delta = 0.5$	$\delta = 0.75$
Castellani	1.0273	3.0625	6.996
Peruzzi	1.0127	2.9688	6.8115
Strozzi	1.2734	3.5000	7.4297
Ridolfi	1.0508	3.4375	7.5820
Barbadori	1.0039	3.3125	7.4414
Bischeri	1.1182	3.2813	7.1982
Tornabuoni	1.0156	3.3750	7.5469
Medici	1.8594	4.6250	8.5781
Accaiaiuoli	0.6523	2.5625	6.6211
Guadagni	1.3320	3.6875	7.6758
Albizzi	1.2500	3.7500	7.8750
Salviati	0.8867	2.9375	6.9492
Lamberteschi	0.5205	2.0938	5.9443
Ginori	0.5000	2.1250	6.0938
Pazzi	0.4092	1.17188	5.3994

d) The betweenness centrality is calculated with:

$$\pi_k = \frac{1}{n^2} \sum_{i,j \in \mathcal{V}} g_{ij}^{(k)}, \quad k \in \mathcal{V} \tag{2}$$

The betweenness centralities are given in table 2 on the right.

e) From the different types of centralities, it is clearly noted that the Medici family is of great importance in the network of Italian families. The betweenness centrality becomes 0 for families that are at the edges of the network, meaning that they are only connected to 1 other family in the network which are Acaiaiuoli, Lamberteschi, Ginori and Pazzi. It can be noted that the Bonacich centrality makes a better distinction between the central families and the others. This is because the Bonacich centrality normalizes the out-degree, thus not letting the nodes contribute to the centralities of their neighbours in the same way.

Exercise 2: Graphs and Opinions

a) The graph is **balanced** since for all nodes, the in-degree equals the out-degree. The graph is **not regular** since node *1* has a self-loop. The graph is **aperiodic** since the graph contains a self-loop at node 1 and the common demoninator for all cycles is thus 1. The shortest path from any node to any other node is maximum three, so the **diameter of the graph is 3**.

b) The entries of the weight matrix (W_{ij}) are equal to 1 if there is a link between the node i and j and 0 elsewhere. The out-degree vector w is the sum of the rows of the weight matrix W. The normalized weight matrix P is equal to $D^{-1}W$, where D has the entries of w on its diagonal. The Laplacian L is equal to $D - W$, where D has again the entries of w on its diagonal.

$$W = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \qquad w = \begin{bmatrix} 3 \\ 3 \\ 2 \\ 2 \\ 3 \\ 2 \end{bmatrix} \qquad P = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} 2 & -1 & 0 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 2 & 0 & 0 & -1 \\ -1 & 0 & 0 & 2 & -1 & 0 \\ 0 & -1 & 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 0 & -1 & 2 \end{bmatrix}$$

c) The Bonacich centrality vector π is the eigenvector that belongs to the eigenvalue 1 of the vector $P^\top$. Thus, π is found by solving $(T^\top - I)\pi = 0$. For balanced graphs, the node centrality is the normalized degree, $\pi_i = \frac{w_i}{n\bar{w}}$. Where the average degree is calculated as

$$\bar{w} = \frac{1}{n}\mathbb{1}^\top W \mathbb{1} = \frac{1}{6}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 3 & 3 & 2 & 2 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{15}{6}$$

Thus the Bonacich centrality vector becomes:

$$\pi_i = \frac{w_i}{15} = \begin{bmatrix} \frac{3}{15} & \frac{3}{15} & \frac{2}{15} & \frac{2}{15} & \frac{3}{15} & \frac{2}{15} \end{bmatrix}^\top$$

d) The degree centrality of node 1 and node 5 are both equal to 3 since they have both 3 edges. The closeness centrality of nodes 1 and 2 is calculated with equation 1 of question 1 and are $\frac{6}{9}$ and $\frac{6}{7}$, respectively. The betweenness centrality of nodes 2 and 3 is calculated with equation 2 of question 1 and are $6\frac{2}{3}$ and $1\frac{2}{3}$

e) Since $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ is a connected, aperiodic graph, the DeGroot opinion dynamics converge to a consensus vector $\lim_{t\to+\infty} x(t) = \bar{x}\mathbb{1}$ as t grows large, for every initial opinion vector $x(0) \in \mathbb{R}^n$

f) If the opinion vector $x(t)$ converges, its limit for $t \to \infty$ is $x = \bar{x}\mathbb{1}$ with the consensus value $\bar{x} = \pi' x(0)$. Hence to maximize (minimize) the opinion vector, the assigned values should be highest (lowest) for the nodes with highest (lowest) Bonacich centrality π_i. Thus, the value assignment $x(0)^+$ that maximizes the limit and the value assignment $x(0)^-$ that minimizes the limit are given by

$$x(0)^+ = \begin{bmatrix} 2 & 2 & 1 & 1 & 2 & 1 \end{bmatrix}^\top \quad x(0)^- = \begin{bmatrix} 1 & 1 & 2 & 2 & 1 & 2 \end{bmatrix}^\top$$

g) The set of regular agents is $\mathcal{R} = \{2, 3, 4, 5\}$ and the set of stubborn agents is $\mathcal{S} = \{1, 6\}$. $Q \in \mathbb{R}^{\mathcal{R}\times\mathcal{R}}$, $B \in \mathbb{R}^{\mathcal{R}\times\mathcal{S}}$, $y \in \mathbb{R}^{\mathcal{R}}$ and $u \in \mathbb{R}^{\mathcal{S}}$ are given below. For any initial opinion vector $x(0) \in \mathbb{R}^n$, the asymptotic opinion value for all the remaining nodes are given by

$$y = (I-Q)^{-1}Bu = \begin{bmatrix} \frac{2143}{5000} \\ \frac{7143}{10000} \\ \frac{2857}{10000} \\ \frac{2857}{5000} \end{bmatrix} = \begin{bmatrix} 0.4286 \\ 0.7143 \\ 0.2857 \\ 0.5714 \end{bmatrix} = \begin{bmatrix} y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 \end{bmatrix} \quad B = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \quad u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

h) The condensation graphs of $\mathcal{G}_\mathcal{A}$ and $\mathcal{G}_\mathcal{B}$ are shown in figure 1 below.

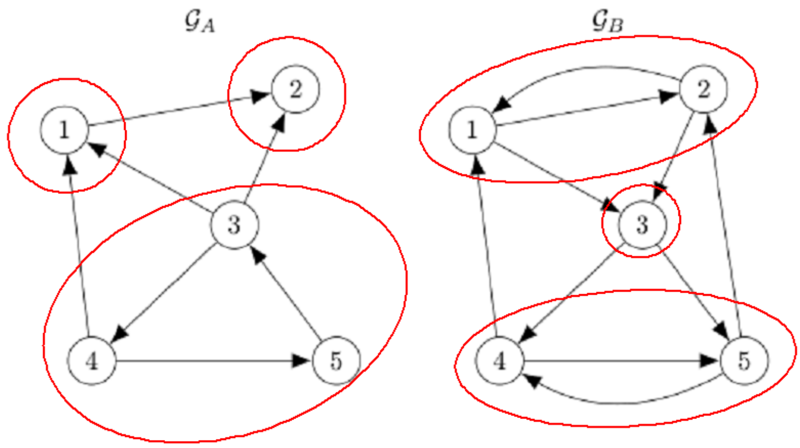

Figure 1: Condensation graphs with connected components encircled in red.

Exercise 3: Influence on Twitter

The MATLAB code for this exercise can be found in `Question3.m`.

a) The nodes with the highest PageRank centrality are node 2, 1, 112, 9 and 26 respectively.

b) Node 15 and 16 are chosen to be stubborn nodes because they do belong to the group of nodes with the highest PageRank centralities. In figure 2, the opinion of nodes 255 till 264 is shown. These nodes are chosen to show that some of them converge to 1 (the opinion of node 15) and some of them converge to a value between 1 and 0 and are thus not influenced by the stubborn nodes.

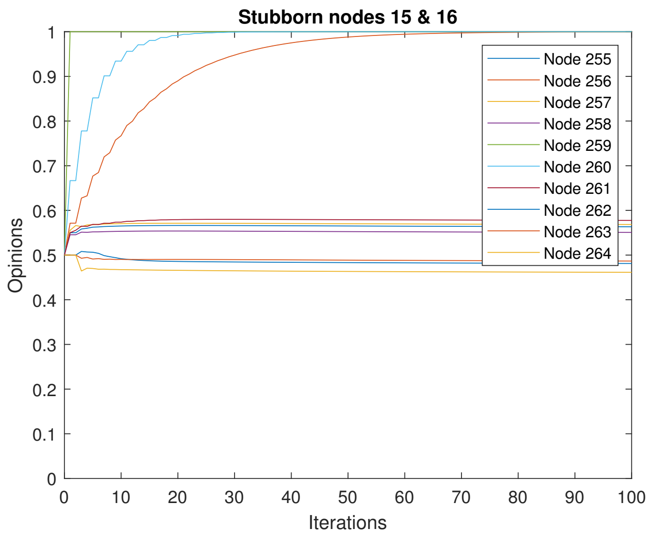

Figure 2: Opinions over time with two stubborn nodes with a low PageRank centrality.

c) In figure 3, the opinion of nodes 255 till 264 is shown again. Node 1, which has one of the highest PageRank centralities, is chosen to be a stubborn node instead of node 15. As one can see, all the nodes converge to the opinion of node 1.

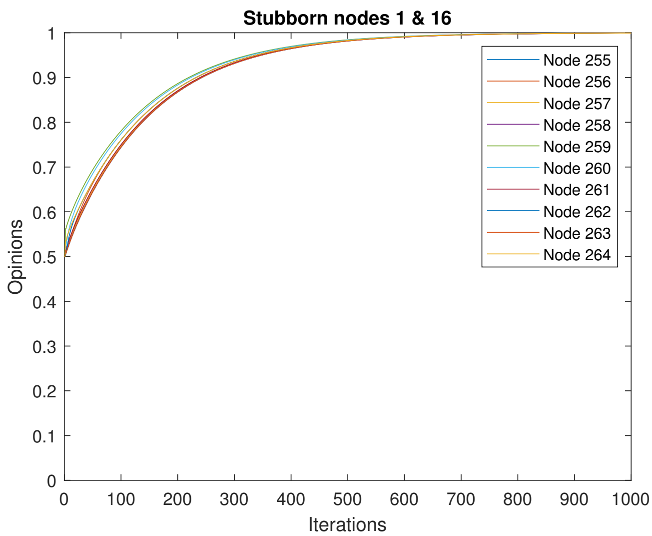

Figure 3: Opinions over time with two stubborn nodes, where one of the stubborn nodes has a high PageRank centrality.

Exercise 4: Flows, flows flows

a) The min-cut, the cut with the minimum capacity that makes the sink not reachable from the source, is the cutting of the edges C_6 and C_{10}, which partitions the network in two subset $\mathcal{U}$ and $\mathcal{V}$, which can be seen in figure 4. The corresponding min-cut capacity is given by $C_{\mathcal{U}} = C_{o,d*} = C_6 + C_9 = 4 + 10 = 14$.

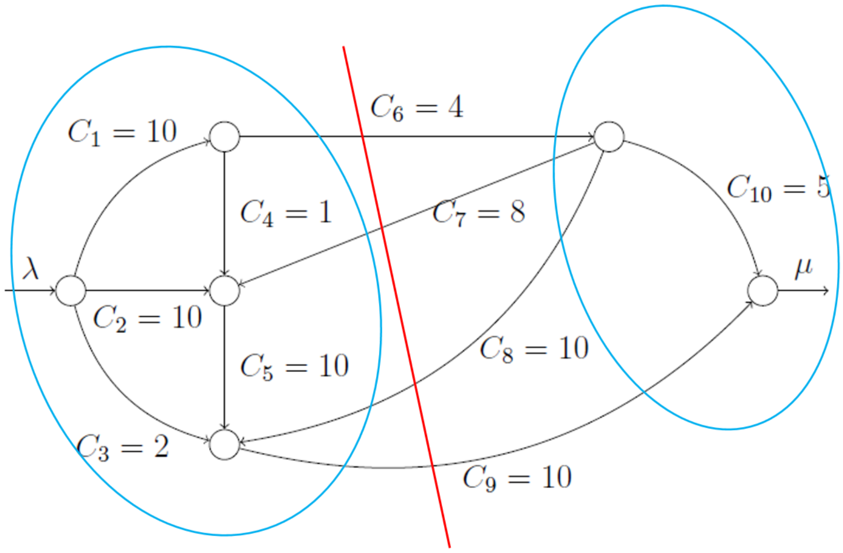

Figure 4: The min-cut in red and corresponding partitions in blue, with $\mathcal{U}$ on the left and $\mathcal{V}$ on the right.

b) In figure 5 below, the designed flow choice is shown, with throughput $\lambda = \mu = 11$ which is compatible with the assigned maximum capacities.

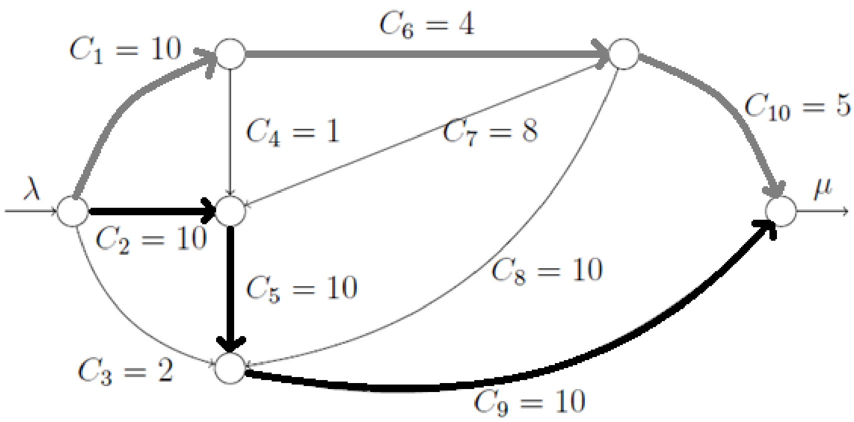

Figure 5: The designed flow choice, with a flow $f = 10$ along the black path and $f = 1$ along the grey path.

c) The minimum total capacity that has to be removed from the network such that there is no feasible flow of throughput $\lambda = \mu = 8$ can exist, is $C_{min} = 7$. This needs to be removed at edge e_9, such that $C_9 = 3$. In this way, the min-cut capacity is $C_{o,d*} = 7$, which equals the largest feasible flow of throughput $\lambda = \mu = 7$. Hence, the flow of throughput $\lambda = \mu = 8$ is **not** feasible in this way.

d) The 2 extra units of capacity should be distributed both to edge e_9, such that $C_9 = 12$. Then, the min-cut capacity amounts $C_{o,d*} = 16$. The 2 extra units can also be distributed such that edges e_6 and e_9 both get 1 extra unit, resulting in the same increased min-cut capacity of $C_{o,d*} = 16$.

e) For the given network, there are three paths from o to a, which are stored in matrix A below. The three paths constitute to the network flow f, which forms the associated delays d, when filled in for the delay functions on the links $d_1(x) = d_5(x) = x + 1$, $d_3(x) = 1$, $d_2(x) = d_4(x) = 5x + 1$.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad f = Az = \begin{bmatrix} z_{p^{(1)}} + z_{p^{(3)}} \\ z_{p^{(2)}} \\ z_{p^{(3)}} \\ z_{p^{(1)}} \\ z_{p^{(2)}} + z_{p^{(3)}} \end{bmatrix} \qquad d = \begin{bmatrix} z_{p^{(1)}} + z_{p^{(3)}} + 1 \\ 5z_{p^{(2)}} + 1 \\ 1 \\ 5z_{p^{(1)}} + 1 \\ z_{p^{(2)}} + z_{p^{(3)}} + 1 \end{bmatrix}$$

The total delays per path t_i are

$$\begin{aligned} t_1 &= d_1 + d_4 = z_{p^{(1)}} + z_{p^{(3)}} + 1 + 5z_{p^{(1)}} + 1 = 6z_{p^{(1)}} + z_{p^{(3)}} + 2 \\ t_2 &= d_2 + d_5 = 5z_{p^{(2)}} + 1 + z_{p^{(2)}} + z_{p^{(3)}} + 1 = 6z_{p^{(2)}} + z_{p^{(3)}} + 2 \\ t_3 &= d_1 + d_3 + d_5 = z_{p^{(1)}} + z_{p^{(3)}} + 1 + 1 + z_{p^{(2)}} + z_{p^{(3)}} + 1 = z_{p^{(1)}} + z_{p^{(2)}} + 2z_{p^{(3)}} + 3 \end{aligned}$$

For the Wardrop equilibrium, the following equality holds: $t_1 = t_2 = t_3$, meaning that the total delay on each path is equal. By setting $t_1 = t_2$, it concluded that for the Wardrop equilibrium, $z_1 = z_2$:

$$6z_{p^{(1)}} + z_{p^{(3)}} + 2 = 6z_{p^{(2)}} + z_{p^{(3)}} + 2$$

Plugging this in the equation for t_3 and setting $t_3 = t_1$:

$$2z_{p^{(1)}} + 2z_{p^{(3)}} + 3 = 6z_{p^{(1)}} + z_{p^{(3)}} + 2$$

Using the conservation of throughput $\tau = z_{p^{(1)}} + z_{p^{(2)}} + z_{p^{(3)}} = 1$ to obtain $z_{p^{(3)}} = 1 - z_{p^{(1)}} - z_{p^{(2)}} = 1 - 2z_{p^{(1)}}$ and plugging this in in the equation above, the following is obtained

$$\begin{aligned} 2z_{p^{(1)}} + 2 - 4z_{p^{(1)}} + 3 &= 6z_{p^{(1)}} + 1 - 2z_{p^{(1)}} + 2 \\ -2z_{p^{(1)}} + 5 &= 4z_{p^{(1)}} + 3 \\ 6z_{p^{(1)}} &= 2 \\ z_{p^{(1)}} &= \frac{1}{3} \end{aligned}$$

Resulting in the Wardrop path flow $z^{(0)}$ and Wardrop equilibrium flow vector $f^{(0)}$ as given below

$$z^{(0)} = \begin{bmatrix} z^{(0)}_{p^{(1)}} \\ z^{(0)}_{p^{(2)}} \\ z^{(0)}_{p^{(3)}} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} \qquad f^{(0)} = \begin{bmatrix} f^{(0)}_1 \\ f^{(0)}_2 \\ f^{(0)}_3 \\ f^{(0)}_4 \\ f^{(0)}_5 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$$

Which gives total delay per path

$$t_1 = t_2 = t_3 = 4\frac{1}{3}$$

f) The social optimum flow vector is computed by minimizing the total delay over all paths

$$\begin{aligned} t_{total} &= t_1 + t_2 + t_3 = 6z_{p^{(1)}} + z_{p^{(3)}} + 2 + 6z_{p^{(2)}} + z_{p^{(3)}} + 2 + z_{p^{(1)}} + z_{p^{(1)}} + 2z_{p^{(3)}} + 3 \\ &= 7z_{p^{(1)}} + 7z_{p^{(2)}} + 4z_{p^{(3)}} + 7 \\ &= 7(1 - z_{p^{(3)}}) + 4z_{p^{(3)}} + 7 \\ &= 14 - 3z_{p^{(3)}} \end{aligned}$$

By inspecting this equation, it is clearly observed that the total delay is minimized when $z_{p^{(3)}} = 1$. Therefore, $z_{p^{(1)}} = z_{p^{(2)}} = 0$, resulting in the social optimum path flow z^* and social optimum flow vector f^* as given below

$$z^* = \begin{bmatrix} z^*_{p^{(1)}} \\ z^*_{p^{(2)}} \\ z^*_{p^{(3)}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad f^* = \begin{bmatrix} f^*_1 \\ f^*_2 \\ f^*_3 \\ f^*_4 \\ f^*_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Resulting in the following delays per path and an average delay per path of $t_{average} = 3\frac{2}{3}$

$$\begin{aligned} t_1 &= 3 \\ t_2 &= 3 \\ t_3 &= 5 \end{aligned}$$

g) The Price of Anarchy associated to the Wardrop equilibrium $f^{(0)}$ is given by:

$$\text{PoA}(0) = \frac{\sum_{e\in\mathcal{E}} f_e^{(0)} d_e\left(f_e^{(0)}\right)}{\min_{f\geq 0} \sum_{e\in\mathcal{E}} f_e d_e\left(f_e\right)} = \frac{4\frac{1}{3}}{3\frac{2}{3}} = 1\frac{2}{11}$$

h) To make the Price of Anarchy equal to 1, the total delay due to the Wardrop equilibrium should become equal to the delay of the social equilibrium. The flow vector of the Wardrop equilibrium should become the flow vector of the social optimum ($z^{(0)} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^\top$), hence the tolls should be placed on links e_2 and e_4. The matrix with delays becomes:

$$d = \begin{bmatrix} z_{p^{(1)}} + z_{p^{(3)}} + 1 \\ 5z_{p^{(2)}} + 1 + \omega_2 \\ 1 \\ 5z_{p^{(1)}} + 1 + \omega_4 \\ z_{p^{(2)}} + z_{p^{(3)}} + 1 \end{bmatrix}$$

And the total delays per path t_i:

$$\begin{aligned} t_1 &= d_1 + d_4 = z_{p^{(1)}} + z_{p^{(3)}} + 1 + 5z_{p^{(1)}} + 1 = 6z_{p^{(1)}} + z_{p^{(3)}} + 2 + \omega_4 \\ t_2 &= d_2 + d_5 = 5z_{p^{(2)}} + 1 + z_{p^{(2)}} + z_{p^{(3)}} + 1 = 6z_{p^{(2)}} + z_{p^{(3)}} + 2 + \omega_2 \\ t_3 &= d_1 + d_3 + d_5 = z_{p^{(1)}} + z_{p^{(3)}} + 1 + 1 + z_{p^{(2)}} + z_{p^{(3)}} + 1 = z_{p^{(1)}} + z_{p^{(2)}} + 2z_{p^{(3)}} + 3 \end{aligned} \tag{3}$$

We know that $z_{p^{(1)}} = z_{p^{(2)}} = 0$ and $z_{p^{(3)}} = 1$. From the first and third equation of 3, we get:

$$\begin{aligned} 2z_{p^{(3)}} + 3 &= z_{p^{(3)}} + 2 + \omega_4 \\ z_{p^{(3)}} &= \omega_4 - 1 \end{aligned}$$

From this we can conclude that $\omega_4 = 2$ since $z_{p^{(3)}}$ should be equal to one. In the same way we get $\omega_2 = 2$. This results in the desired flow vector and a Price of Anarchy of 1.

Exercise 5: Traffic in LA

The MATLAB code for this exercise can be found in `Question5.m`.

a) The shortest path between 1 and 17 with respect to the traveling time is calculated with `graphshortestpath` in MATLAB and is 0.5530 and show in figure 6.

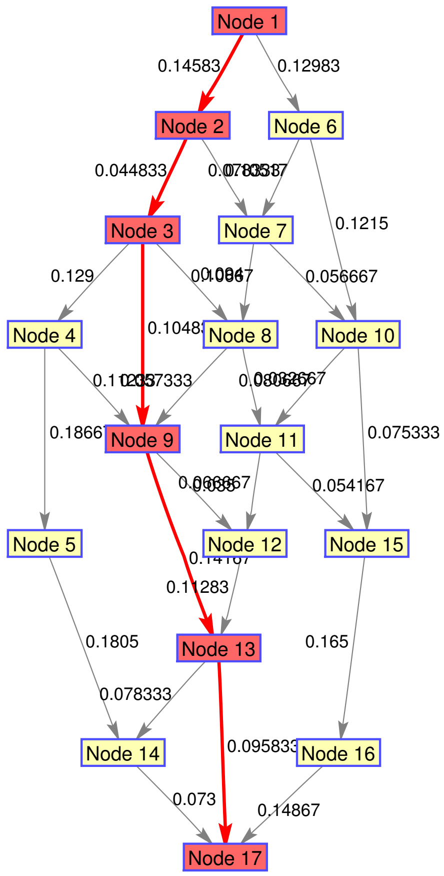

Figure 6: Shortest path with respect to the travel time.

b) The maximum flow between node 1 and 17 is calculated with `graphmaxflow` in MATLAB and is 16806.

c) The external inflow/outflow is calculated with $v = Bf$:

$$v_{1-9} = \begin{bmatrix}16806 & 8570 & 19448 & 4957 & -746 & 4768 & 413 & -2 & -5671\end{bmatrix}^\top$$
$$v_{10-17} = \begin{bmatrix}1169 & -5 & -7131 & -380 & -7412 & -7810 & -3430 & -23544\end{bmatrix}^\top$$

In exercise d, e and f, the total delay is calculated by adding the delays of all the links. The delay for the links is calculated with:

$$d_e(f_e) = \frac{l_e}{1 - \frac{f_e}{C_e}}$$

where f_e the optimum flow of the link for that question. All the coming minimization problems are minimized subjected to the following constraints:

$$Bf = \lambda - \mu$$
$$0 \leq f \leq C$$

d) The total delay of the network with the social optimum is 2.5944×10^4. The flow of each link is given in table 4. The social optimum was found by minimizing:

$$\min_{f_e} \sum_{e \in \mathcal{E}} \frac{l_e C_e}{1 - \frac{f_e}{C_e}} - l_e C_e \tag{4}$$

e) The total delay of the network with the Wardrop equilibrium is 2.6293×10^4. The flow of each link is given in table 4. The Wardrop equilibrium was found by minimizing:

$$\min_{f_e} \sum_{e \in \mathcal{E}} \int_0^{f_e} d_e(x)dx = \min_{f_e} \sum_{e \in \mathcal{E}} -l_e C_e \log(C_e - f_e)$$

f) The total delay of the network with tolls on the links and the Wardrop equilibrium is 2.5944×10^4. This is the same as the delay of the social equilibrium, which makes sense since the tolls were chosen (as in the reader) such that the flows of the social optimum are the same as the flows of the Wardrop equilibrium. The flows and delays are given in table 4. The Wardrop equilibrium with tolls was found by minimizing:

$$\min_{f_e} \sum_{e \in \mathcal{E}} \int_0^{f_e} d_e(x) + \omega_e dx = \min_{f_e} \sum_{e \in \mathcal{E}} -l_e C_e \log(C_e - f_e) + \omega_e f_e$$

with:

$$\omega_e = f_e^* d_e'(f_e^*) = \frac{f_e^* C_e l_e}{(C_e - f_e^*)^2}$$

where f_e^* is the optimal flow of the social optimum of question 5e (eq. 4).

g) The total delay of the network with a social optimum is 2.5986×10^4. The flow of each link is given in table 4. The tolls were calculated with the toll function in the previous question:

$$\omega_e = f_e^* d_e'(f_e^*) \tag{5}$$

since this toll will result in the same social optimum and Wardrop equilibrium. f_e^* is the optimal flow of the social optimum of this question and was found by minimizing:

$$\min_{f_e} \sum_{e \in \mathcal{E}} \frac{l_e C_e}{1 - \frac{f_e}{C_e}} - l_e C_e - l_e f_e$$

The Wardrop equilibrium was found by minimizing:

$$\min_{f_e} \sum_{e \in \mathcal{E}} \int_0^{f_e} d_e(x) - l_e dx = \min_{f_e} \sum_{e \in \mathcal{E}} -l_e C_e \log(C_e - f_e) - l_e f_e$$

The Wardrop equilibrium with tolls was found by minimizing:

$$\min_{f_e} \sum_{e \in \mathcal{E}} \int_0^{f_e} d_e(x) - l_e + \omega_e dx = \min_{f_e} \sum_{e \in \mathcal{E}} -l_e C_e \log(C_e - f_e) - l_e f_e + \omega_e f_e$$

with ω_e given in equation 5. The tolls and flows for the Wardrop equilibrium with and without toll are also given in table 4.

Table 4: The flows and tolls for all the edges in the network for the different optima.

	Question d, e and f				Question g			
Optimum	**Social**	**Wardrop**	**With tolls**	**Tolls**	**Social**	**Wardrop**	**With tolls**	**Tolls**
Edge 1	6642	6716	6642	1.9223	6653	6782	6653	1.9458
Edge 2	6059	6716	6059	0.1851	5775	6005	5775	0.1527
Edge 3	3132	2367	3132	0.0517	3420	3266	3420	0.0597
Edge 4	3132	2367	3132	0.1052	3420	3266	3420	0.1238
Edge 5	10164	10090	10164	1.4406	10153	10024	10153	1.4302
Edge 6	4638	4645	4638	0.4692	4643	4666	4643	0.4714
Edge 7	3006	2804	3006	0.1076	3106	3029	3106	0.1163
Edge 8	2543	2284	2543	0.0569	2662	2641	2662	0.0632
Edge 9	3132	3418	3132	0.2788	3009	3039	3009	0.2483
Edge 10	583	0	583	0.0062	878	777	878	0.0098
Edge 11	0	177	0	0	0	240	0	0
Edge 12	2927	4171	2927	0.0754	2355	2498	2355	0.0506
Edge 13	0	0	0	0	0	0	0	0
Edge 14	3132	2367	3132	0.1267	3420	3266	3420	0.1512
Edge 15	5525	5445	5525	0.4818	5510	5359	5510	0.4763
Edge 16	2854	2353	2854	0.0821	3044	2882	3044	0.0952
Edge 17	4886	4933	4886	0.0686	4882	4891	4882	0.0684
Edge 18	2215	1842	2215	0.0174	2415	2413	2415	0.0198
Edge 19	464	697	464	0.0015	444	628	444	0.0014
Edge 20	2338	3036	2338	0.0139	2008	2101	2008	0.0110
Edge 21	3318	3050	3318	0.0659	3487	3510	3487	0.0738
Edge 22	5656	6087	5656	0.2633	5495	5610	5495	0.2405
Edge 23	2373	2587	2373	0.0670	2204	2263	2204	0.0576
Edge 24	0	0	0	0	0	0	0	0
Edge 25	6414	6919	6414	0.4098	6301	6386	6301	0.3804
Edge 26	5505	4954	5505	0.2873	5624	5529	5623	0.3139
Edge 27	4886	4933	4886	0.1916	4882	4891	4882	0.1911
Edge 28	4886	4933	4886	0.5278	4882	4891	4882	0.5258
Total delay	25944	26293	25944		25986	26004	25986	

Exercise 6: The little chemist

a) The flow dynamics are described as:

$$\dot{x} = -L^\top x + \lambda$$

where $L = D - W$, $x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_0 \end{bmatrix}^\top$ and the external inflow λ is given as $\begin{bmatrix} \lambda & 0 & 0 & 0 & 0 \end{bmatrix}^\top$. x_0 is a dummy node to model the system and is at the head of the arrow with rate k_{20}.

$$W = \begin{bmatrix} 0 & 0 & 0 & k_{14} & 0 \\ k_{21} & 0 & 0 & 0 & k_{20} \\ 0 & k_{32} & 0 & 0 & 0 \\ 0 & 0 & k_{43} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} k_{14} & 0 & 0 & 0 & 0 \\ 0 & k_{21}+k_{20} & 0 & 0 & 0 \\ 0 & 0 & k_{32} & 0 & 0 \\ 0 & 0 & 0 & k_{43} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} k_{14} & 0 & 0 & -k_{14} & 0 \\ -k_{21} & k_{21}+k_{20} & 0 & 0 & -k_{20} \\ 0 & -k_{32} & k_{32} & 0 & 0 \\ 0 & 0 & -k_{43} & k_{43} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

b) Since we have an open compartmental model, we can find the limit densities of each stage by focusing on the subsystem $y = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^\top$ since this part only contains the regular nodes and does not contain the sink x_0. The subsystem converges to:

$$\lim_{t \to +\infty} y(t) = (M^\top)^{-1} \lambda$$

with $\lambda = \begin{bmatrix} \lambda & 0 & 0 & 0 \end{bmatrix}^\top$ and M is the block of L with all the regular nodes:

$$M = \begin{bmatrix} k_{14} & 0 & 0 & -k_{14} \\ -k_{21} & k_{21}+k_{20} & 0 & 0 \\ 0 & -k_{32} & k_{32} & 0 \\ 0 & 0 & -k_{43} & k_{43} \end{bmatrix}$$

This results in:

$$\lim_{t \to +\infty} y(t) = \begin{bmatrix} 20 & 10 & 20 & 20 \end{bmatrix}^\top$$