Homework Problem Set 2

Networked and Distributed Control Systems (SC42100)

TU Delft, 3ME, DCSC, Spring 2020

- All answers should be clearly motivated. Only end results of calculations are not sufficient.
- Please also send your Matlab code used for Problem 3.
- Please submit your homework assignments as a pdf file (preferably typeset, but good quality scans of hand-written notes with Matlab printouts of your plots are also acceptable, although not preferred).
- Submit via BrightSpace with a confirmation email sent to Tamas Keviczky (`T.Keviczky@tudelft.nl`) and cc'd to Gabriel Gleizer(`G.Gleizer@tudelft.nl`).
- Hand-in deadline (via BrightSpace): 9:00, June 19, 2020

Problem 1

(a) Consider the following convex optimization problem with a complicating constraint:

$$\begin{aligned} \underset{\theta_1,\theta_2}{\text{minimize}} \quad & f_1(\theta_1) + f_2(\theta_2) \\ \text{subject to} \quad & \theta_1 \in \Theta_1, \quad \theta_2 \in \Theta_2 \\ & h_1(\theta_1) + h_2(\theta_2) \leq 0 \end{aligned}$$

where Θ_1, Θ_2 are convex sets and all the functions are convex.

Apply primal decomposition to the problem and show the resulting two subproblems. What is the role of the master problem?

(b) In order to solve the master problem in question (a), the two subproblems are solved independently to obtain subgradients. It turns out that we can find a subgradient for the optimal value of each subproblem from an optimal dual variable associated with the coupling constraint. This leads to the second question:

Let $p(z)$ be the optimal value of the convex optimization problem:

$$\begin{aligned} \underset{\theta}{\text{minimize}} \quad & f(\theta) \\ \text{subject to} \quad & \theta \in \Theta \\ & h(\theta) \leq z \end{aligned}$$

Let λ^* be an optimal dual variable associated with the constraint $h(\theta) \leq z$. Show that $-\lambda^*$ is a subgradient of p at z.

Problem 2

Consider the combined consensus/projected incremental subgradient method for N agents shown in the lecture slides:

$$\theta_{k+1}^i = \mathcal{P}_\Theta \left[\sum_{j=1}^{N} [W^\varphi]_{ij} \left(\theta_k^j - \alpha_k g^j(\theta_k^j) \right) \right], \quad i = 1, \ldots, N$$

Assume that the communication graph used for consensus is strongly connected, balanced, and the matrix W is doubly stochastic. Show that as $\varphi \to \infty$ (i.e., the agents reach consensus in each iteration of the algorithm), the combined consensus / projected incremental subgradient method becomes a standard subgradient method.

Problem 3

Similarly to what has been seen in the lectures, we would like to solve a multi-aircraft coordination problem, where each aircraft i can be described by a linear time-invariant dynamical system

$$x_i(t+1) = Ax_i(t) + Bu_i(t), \quad x_i(0) = x_{i,0}, \quad i = 1, \ldots, 4, \quad t = 0, \ldots, T_{\text{final}} - 1.$$

The objective is to find a decomposition-based algorithm for coordinating towards a common target state at T_{final}, i.e., to ensure that

$$x_i(T_{\text{final}}) = x_f, \quad \forall i$$

while satisfying a limitation on the total control energy

$$\sum_t u_i(t)^T u_i(t) \leq u_{\max}^2, \quad \forall i.$$

The objective function to be minimized is the following quadratic function

$$\sum_i \sum_t x_i(t)^T x_i(t) + u_i(t)^T u_i(t).$$

The optimization variables are the (private) controls u_i of the individual aircraft, as well as the (public) common terminal state x_f.

As a minimum requirement, derive a solution based on dual decomposition and illustrate it with the use of Matlab. Show a plot of the resulting aircraft states. A Matlab file is attached (`aircraft.m`) with the matrices A_i and B_i along with the initial states, control limit, and horizon length for a four-aircraft example. For extra points (and a good learning experience), you can also try out primal decomposition, prox decomposition, ADMM, or any other decomposition method you can find.

Remark: Not everyone might have access to solvers that are able to solve the above quadratically constrained quadratic programs (QCQP). If you do (e.g., by using Yalmip and a suitable solver such as Gurobi), then you can proceed with the original version of the problem. For those of you who are having difficulty with this, you can substitute the original quadratic local control input energy constraints with the following linear constraints:

$$|u_i(t)| \leq \frac{u_{\max}}{T_{\text{final}}}, \quad \forall i, t.$$

Problem 4

Consider the positive definite quadratic function partitioned into two sets of variables

$$V(u) = \frac{1}{2}u^T H u + c^T u + d$$

$$V(u_1, u_2) = \frac{1}{2}\begin{pmatrix} u_1^T & u_2^T \end{pmatrix}\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} c_1^T & c_2^T \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + d$$

in which $H > 0$. Imagine we wish to optimize this function by first optimizing over the u_1 variables holding u_2 fixed and then optimizing over the u_2 variables holding u_1 fixed as shown in Figure 1.

Let us see if this procedure, while not necessarily efficient, is guaranteed to converge to the optimum.

(a) Given an initial point (u_1^p, u_2^p), show that the next iteration is

$$\begin{aligned} u_1^{p+1} &= -H_{11}^{-1}\left(H_{12}u_2^p + c_1\right) \\ u_2^{p+1} &= -H_{22}^{-1}\left(H_{21}u_1^p + c_2\right) \end{aligned}$$

The procedure can be summarized as

$$u^{p+1} = Au^p + b$$

in which the iteration matrix A and constant b are given by

$$A = \begin{pmatrix} 0 & -H_{11}^{-1}H_{12} \\ -H_{22}^{-1}H_{21} & 0 \end{pmatrix}, \quad b = \begin{pmatrix} -H_{11}^{-1}c_1 \\ -H_{22}^{-1}c_2 \end{pmatrix}$$

(b) Establish that the optimization procedure converges by showing the iteration matrix is stable

$$|\operatorname{eig}(A)| < 1$$

(c) Given that the iteration converges, show that it produces the same solution as

$$u^* = -H^{-1}c$$

Problem 5

Consider again the iteration defined in Problem 4.

(a) Prove that the cost function is monotonically decreasing when optimizing one variable at a time

$$V(u^{p+1}) < V(u^p) \quad \forall u^p \neq -H^{-1}c$$

(b) Show that the following expression gives the size of the decrease

$$V(u^{p+1}) - V(u^p) = -\frac{1}{2}(u^p - u^*)^T P (u^p - u^*)$$

in which

$$P = HD^{-1}\tilde{H}D^{-1}H, \quad \tilde{H} = D - N, \quad D = \begin{pmatrix} H_{11} & 0 \\ 0 & H_{22} \end{pmatrix}, \quad N = \begin{pmatrix} 0 & H_{12} \\ H_{21} & 0 \end{pmatrix}$$

and $u^* = -H^{-1}c$ is the optimum.

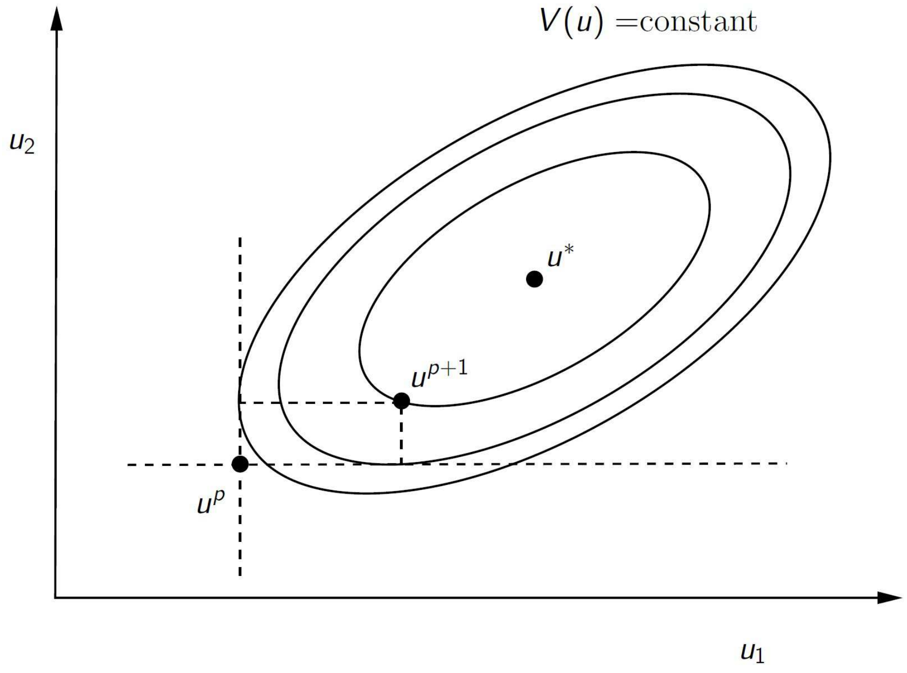

Figure 1: Optimizing a quadratic function in one set of variables at a time.

Hint: to simplify the algebra, first change coordinates and move the origin of the coordinate system to u^*.

Useful facts:

- If H is a positive definite, symmetric matrix partitioned in the following way, then

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} > 0 \quad \Rightarrow \quad \bar{H} = \begin{pmatrix} H_{11} & -H_{12} \\ -H_{21} & H_{22} \end{pmatrix} > 0.$$

- For any Q real symmetric and R real matrices:

$$Q > 0 \text{ and } R \text{ nonsingular} \quad \Rightarrow \quad R^T Q R > 0.$$