Assignment 1

Networked and Distributed Control Systems (SC42100)
TU Delft, 3mE, DCSC
Spring 2020

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(Special thanks to **Giacomo Como** and **Gustav Nilsson**!)

Due: June 1, 2020

- **All answers should be clearly motivated.**
  Providing end results of calculations only is not sufficient.
- Please submit your Assignment as a **typeset PDF file**.
- Please also send your **Matlab code** written to solve the **Matlab Problems** (namely, Problems 1, 3 and 5).
  Comment your code well. Clarity is more important than efficiency.
  Your code should be written in a general way: if a question is slightly modified, getting the new correct answer should only require slight modifications in your code as well.
- For all problems that are **not Matlab problems**, ***using Matlab or any other tool is forbidden***: **pen and paper** are enough!
- Submission should be via e-mail to the instructor Gabriel Gleizer (`G.Gleizer@tudelft.nl`) and cc'd to Giulia Giordano (`G.Giordano@tudelft.nl`).
- **Submission deadline: 9:00 AM, June 1, 2020**.

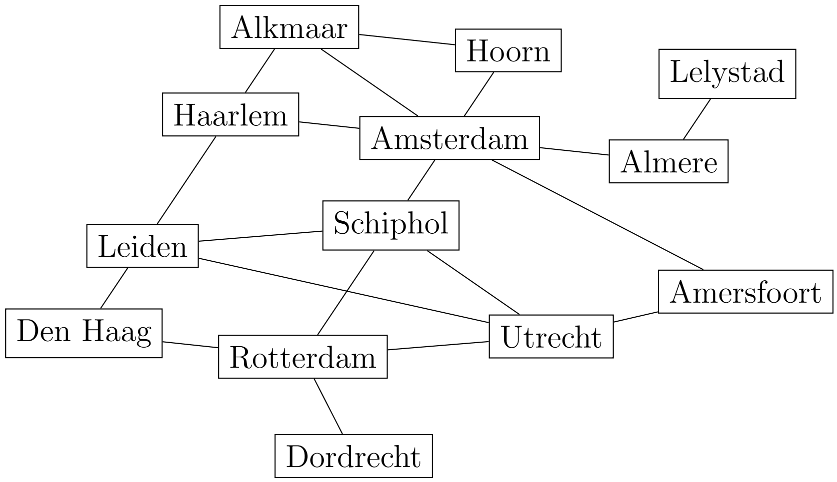

Figure 1: Simplified railway map of the Randstad.

1 City connectivity [MATLAB Problem]

The graph in Figure 1 illustrates the railway connection between some of the major cities in the Randstad. Implement algorithms that compute different centrality measures in this network, and determine the following centrality measures for the cities Amsterdam, Schiphol, Rotterdam and Utrecht.

a) The Bonacich centrality;

b) The closeness centrality;

c) The decay centrality, defined as

$$\pi_i = \sum_{j \neq i} \delta^{\mathrm{dist}(i,j)},$$

for $\delta = 0.25, 0.5, 0.75$.
What does the decay centrality represent when $\delta \to 0$ and when $\delta \to 1$?

d) The betweenness centrality;

e) Comment your results briefly.

2 Graphs and Opinions

First, consider the unweighted directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ in Figure 2.

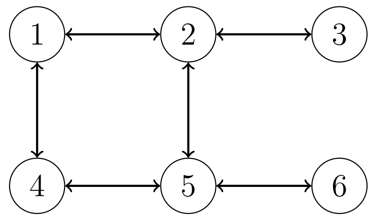

Figure 2: The graph for Problem 2, first part (the links have all unitary weights).

a) Is $\mathcal{G}$ balanced? Is it regular? Is it aperiodic? What is its diameter?

b) Compute the weight matrix W, the out-degree vector w, the normalized weight matrix P and the Laplacian matrix L.

c) Determine the normalised Bonacich centrality vector π for the graph $\mathcal{G}$.

d) Compare the following values (determine which is the largest, or if they are equal): the degree centrality of nodes 1 and 5; the closeness centrality of nodes 1 and 2; the betweenness centrality of nodes 2 and 3.

Now consider the discrete-time De Groot opinion dynamics on $\mathcal{G}$, of the form

$$x(t+1) = Px(t), \qquad t = 0, 1, 2... \tag{1}$$

where P is the normalised weight matrix of $\mathcal{G}$ and $x(t)$ is the opinion vector.

e) Does the opinion vector $x(t)$ converge to consensus as t grows large? Why?

f) Which assignment of the values $\{1, 1, 1, 2, 2, 2\}$ to the nodes (each value in the set has to be used once and only once) as initial opinions in the De Groot opinion dynamics (1) maximizes $\lim_{t \to +\infty} \sum_i x_i(t)$ and which one minimizes it? Motivate your answer and specify the obtained maximum and minimum value for the limit above.

g) Let node 1 be stubborn with value 0 and node 6 be stubborn with value 1. Compute the asymptotic opinion value for all the remaining nodes.

Then, consider the two unweighted graphs $\mathcal{G}_A$ and $\mathcal{G}_B$ in Figure 2.

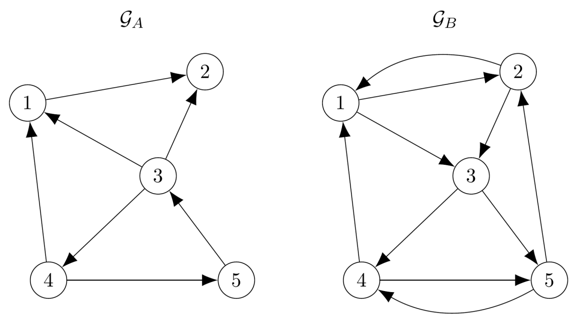

Figure 3: The graphs for Problem 2, second part.

h) Determine the number of connected components and draw the condensation graph of both $\mathcal{G}_A$ and $\mathcal{G}_B$.

3 Influence on Twitter [MATLAB Problem]

You will work on a subgraph extracted from the Twitter network: data obtained by crawling the network from a subset of one user's followers, then the followers' followers and so on (crawling stopped after approximately 6000 users. As you might discover, this crawling method has its draw-backs when determining the PageRank...). In order to anonymize the data, the user IDs were converted to sequential integers, where 1 is the user for which the crawling process started. In addition, a small part of the original data was altered.

The file `twitter.mat` contains a three-column list with links, where the first two columns represent the tail and the head of the link, and the third column the weight of the link. Here a link (i, j) represents that i follows j. The adjacency matrix can be loaded into Matlab using the following code:

```
load -ascii twitter.mat
W = spconvert(twitter);
```

The file `users.mat` contains, on the ith row, the Twitter user id of node i.

a) The matrix W is not square. Explain why and how to solve it without altering the graph.

b) Compute ***iteratively*** the PageRank, find the five most central nodes.

c) Simulate the discrete-time consensus algorithm with two stubborn nodes, one with value 0 and one with value 1. Plot how the opinions change over time for 10 nodes of your choice.

d) Investigate how the the choice of nodes with respect to their PageRank affects the opinion distribution. Plot the opinion distribution for a couple of choices of the stubborn nodes.

4 Flows, flows, flows

Consider the network in Figure 4 with specified link capacities, and throughput $\lambda = \mu$.

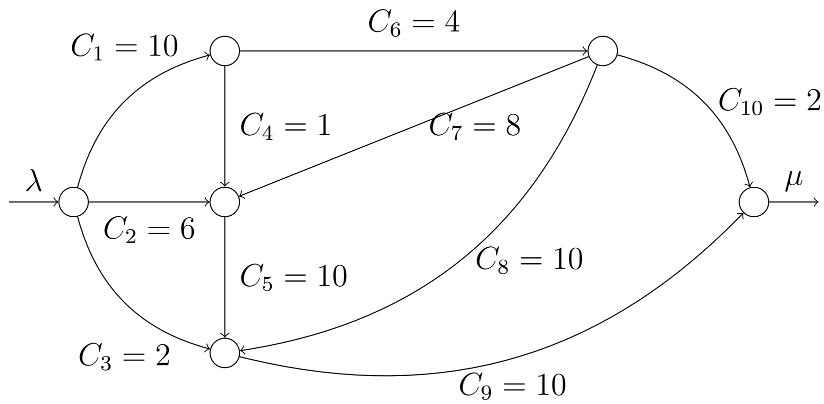

Figure 4: The network for Problem 4, first part.

a) Determine the min-cut and the corresponding min-cut capacity.

b) Design a possible flow choice with throughput $\lambda = \mu = 11$ that is compatible with the assigned maximum capacities.

c) What is the minimum total capacity that has to be removed from the network so that no feasible flow of throughput $\lambda = \mu = 8$ can exist? How should it be removed?

d) You are given 2 extra units of capacity. How should you distribute them in order to increase the min-cut capacity as much as possible?

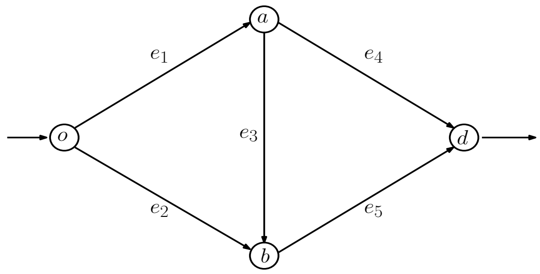

Figure 5: The network for Problem 4, second part.

Now, consider network flows on the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ in Figure 5 with unitary throughput from node o to node d.

Assume the link capacities are all infinite and let the delay functions on the links be given by

$$d_1(x) = d_5(x) = x + 1\,, \qquad d_3(x) = 1\,, \qquad d_2(x) = d_4(x) = 5x + 1\,.$$

e) Compute the user optimum (i.e., the Wardrop equilibrium) flow vector.

f) Compute the social optimum flow vector, i.e., the flow vector that minimizes the average delay from o to d.

g) Compute the price of anarchy.

h) Find a vector of tolls on the links that reduce the price of anarchy to 1.

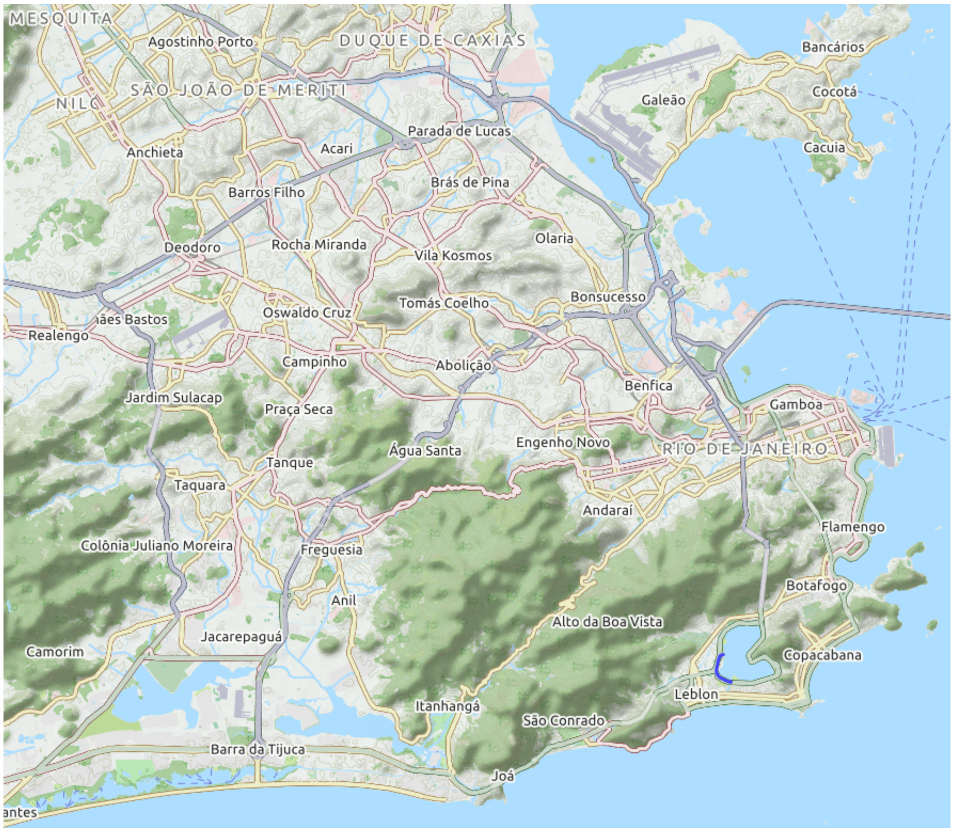

Figure 6: Main roads and highways of Rio de Janeiro.

5 Traffic in Rio [MATLAB Problem]

Consider traffic flows on the road network in Rio de Janeiro, see Figure 6. To simplify the problem, an approximate highway map is given in Figure 7, covering part of the real network.

The node-link incidence matrix, B, for this traffic network is given in the file `traffic.mat`. The rows of B are associated with the nodes of the network and the columns of B with the links. The ith column of B has a 1 in the row corresponding to the **tail** node of link l_i and a -1 in the row that corresponds to the **head** node of link l_i. Each node represents an intersection between highways.

Each link e where $e = l_1, \dots l_{20}$ has a maximum flow capacity C_e. The capacities in vehicles per hour are given as a vector in the file `capacities.mat`, where C_{l_k} is given by entry k. The flows are estimated from measured data, and capacities using simple formulae from traffic engineering. Moreover, each link has a minimum traveling time l_e, which the drivers experience when the road is empty. In the same manner as for the capacities, the minimum traveling times are given as a vector in the file `traveltime.mat`. These values are simply retrieved from Google Maps estimates out of peak hours. For each link, we

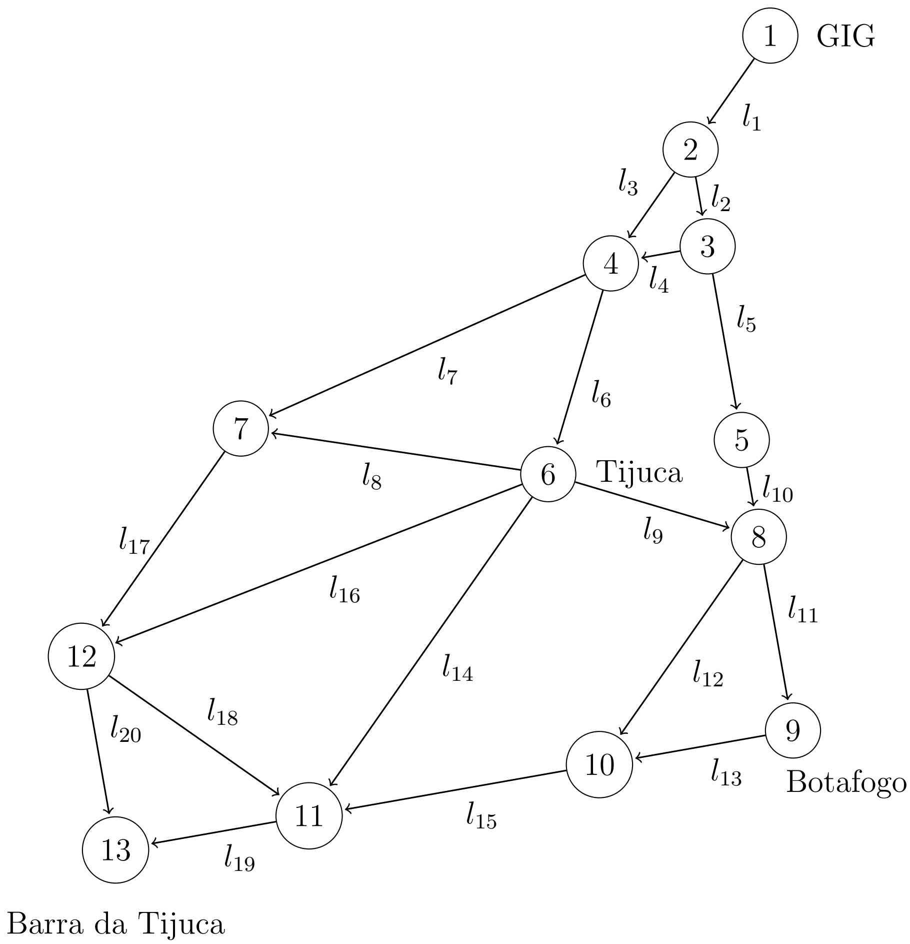

Figure 7: Some possible paths from GIG airport (node 1) to Barra da Tijuca (node 13).

introduce the delay function

$$d_e(f_e) = \frac{l_e}{1 - f_e/C_e}.$$

a) Find the shortest path between node 1 and 13, with respect to traveling time in an empty network. (Hint: use the MATLAB function `graphshortestpath`).

b) Find the maximum flow between node 1 and 13. (Hint: use the MATLAB function `graphmaxflow`)

c) For the flow vector in `flow.mat`, compute the external inflow/outflow at each node.

For the following subproblems you can use CVX, a Matlab-based convex optimization tool `cvxr.com/cvx/download`. Make sure you add the cvx-package to your path in MATLAB. The flow optimization problem

$$
\begin{aligned}
\text{minimize } & \sum_{e=1}^{M} f_e^2 \\
\text{subject to } & Bf = \lambda - \mu \\
& 0 \leq f \leq C
\end{aligned}
$$

can be written for CVX in Matlab as

```
cvx_begin
    variable f(M)
    minimize  sum(f.*f)
    subject to
            B*f == lambda - mu
            0 ≤ f ≤ c
cvx_end
```

Consult the CVX Users' Guide online for help if needed.

For the following points, we assume that all net inflows are zero except for the one at node 1, where we keep the computed one from part **c)**. We also assume that all of the net inflow at node 1 leaves the network at node 13.

d) Using CVX, find the social optimum f^* with respect to the delays, i.e, minimize

$$\sum_{e\in\mathcal{E}} f_e d_e(f_e) = \sum_{e\in\mathcal{E}} \frac{f_e l_e}{1 - f_e/C_e} = \sum_{e\in\mathcal{E}} \frac{l_e C_e}{1 - f_e/C_e} - l_e C_e$$

subject to the constraints on the flows. (Hint: Use the CVX-function `inv_pos`)

e) Using CVX, find the Wardrop equilibrium f^W.
(Hint: Use the cost function $\sum_{e\in\mathcal{E}} \int_0^{f_e} d_e(x)dx$.)

f) Introduce tolls, such that the toll on link e is $\omega_e = f_e^* d_e'(f_e^*)$, where f_e^* is the flow at the system optimum. Now the delay on link e is given by $d_e(f_e) + \omega_e$. Use CVX to compute the new Wardrop equilibrium. What do you observe?

g) Instead of the total delay, let the cost be the total additional delay with respect to free flow, i.e.,

$$c_e(f_e) = f_e(d_e(f_e) - l_e).$$

Compute the system optimum f^* for the costs above and construct tolls such that the Wardrop equilibrium coincides with f^*. Verify your result with CVX.

6 Drugs

A simplified compartmental model for how a drug propagates in the human body is shown in Figure 8. Let x_D, x_M and x_S be the concentration of the drug in each of the three compartments, and let k_{ij} the flow rate from compartment i to compartment j, with $i, j \in \{D, M, S\}$.

a) Describe the flow dynamics as a continuous-time compartmental system.

b) Find the limit concentrations for each compartment when $\lambda = 2$, $k_{DS} = 0.6$, $k_{SD} = 0.3$, $k_{DM} = 0.2$, $k_{MU} = 0.1$ and $k_{DU} = 0.4$.

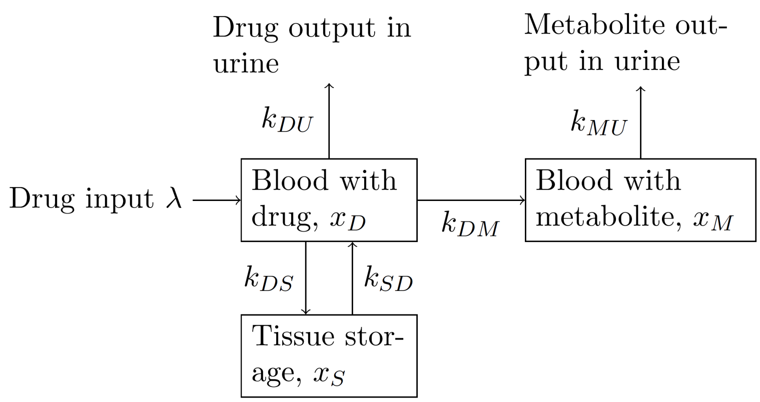

Figure 8: Compartmental model for drug propagation in human body.

7 Playing with graphs 1/2

Prove that for a k-regular undirected graph (i.e., each node has degree k), where k is odd, the total number of undirected edges, m, must be divisible by k. Is the same condition necessary when k is even? Why?

8 Playing with graphs 2/2

Denoting by $\mathcal{C}(n)$ the complete, undirected, unweighted graph with n nodes, determine the corresponding adjacency matrix $A_{\mathcal{C}(n)}$, along with its spectrum, and the corresponding Laplacian matrix $L_{\mathcal{C}(n)}$, along with its spectrum.