Delft University of Technology

Networked and Distributed Control Systems
SC42100

Assignment 1

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1 City connectivity

For this question the centrality is determined for the cities Amsterdam, Schiphol, Rotterdam and Utrecht, the entries in the centrality vectors below correspond to the cities in this order.

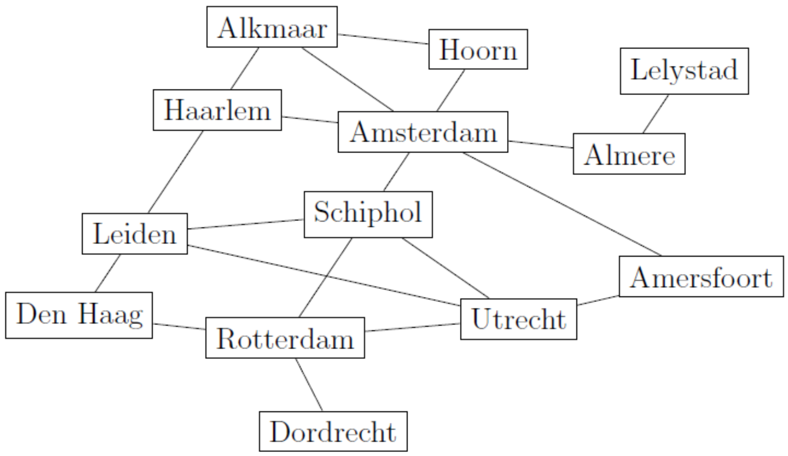

Figure 1: Simplified railway map of the Randstad

a) The Bonacich centrality

Since the normalized weight matrix P is a stochastic matrix, the largest-in-modulus eigenvalue of $P^\top$ is 1. This eigenvalue corresponds to an eigenvector that is then normalized to obtain the Bonacich centrality.

$$\begin{aligned} \pi &= P^\top \pi \\ \pi &= \begin{bmatrix} 0.1579 & 0.1053 & 0.1053 & 0.1053 \end{bmatrix} \end{aligned} \tag{1}$$

Amsterdam has 6 direct connections to surrounding cities, while Schiphol, Rotterdam and Utrecht each only have 4 connections, so the centrality measure for these cities has a lower value than Amsterdam.

b) The closeness centrality

The closeness centrality for node i is calculated by the inverse average distance from i to all other nodes j in $\mathcal{V}$.

$$\text{closeness}(i) = \frac{n}{\sum_{j \in \mathcal{V}} \text{dist}(i, j)}, \quad i \in \mathcal{V} \tag{2}$$

$$\pi = \begin{bmatrix} 0.650 & 0.619 & 0.500 & 0.520 \end{bmatrix}$$

A higher value means the city is closer to the other cities, which explains why Amsterdam has the highest value and Schiphol has a slightly lower value because this city is near Amsterdam.

c) The decay centrality

The decay centrality shows how close this city is to cities around it, where cities far away have little influence on the centrality measure. This is defined as follows

$$\text{decay}(i) = \sum_{j \neq i} \delta^{\text{dist}(i,j)} \tag{3}$$

$$\begin{aligned}
\delta = 0.25 : \quad & \pi_{0.25} = \begin{bmatrix} 1.7813 & 1.4531 & 1.2539 & 1.3008 \end{bmatrix} \\
\delta = 0.5 \ : \quad & \pi_{0.5} \ = \begin{bmatrix} 4.2500 & 3.8750 & 3.3125 & 3.4375 \end{bmatrix} \\
\delta = 0.75 : \quad & \pi_{0.75} = \begin{bmatrix} 7.5938 & 7.3594 & 6.6914 & 6.8320 \end{bmatrix}
\end{aligned}$$

Again we can see that Amsterdam has the highest value for the centrality measure since this city is centralized in the Randstad with many cities closeby.
When $\delta \to 0$, the 'decay' occurs instantly, resulting in a decay centrality of 0 for all nodes since it can not reach any other node, while $\delta \to 1$ means no decay at all, resulting in a decay centrality of $n-1$ for all nodes.

d) The betweenness centrality

The betweenness centrality of node k is found by taking the fraction of minimum-distance paths from nodes i to j that pass through node k.

$$\text{betweenness}(k) = \frac{1}{n^2} \sum_{i,j \in \mathcal{V}} g_{ij}^{(k)}, \quad k \in \mathcal{V} \tag{4}$$

$$\pi = \begin{bmatrix} 0.4043 & 0.2130 & 0.1607 & 0.0671 \end{bmatrix}$$

From fig. 1 it can be seen that Amsterdam is in a central location, meaning that many shortest paths between different cities might pass through this city which is confirmed by the relatively high value for the betweenness centrality of Amsterdam. For example all paths from Almere or Lelystad to any other city must pass through Amsterdam. Utrecht however has a relatively low betweenness centrality, which could be explained by the fact that this city is more at the edge of the Randstad, causing that the shortest paths only have to pass through this city for some cities going to or coming from Amersfoort.

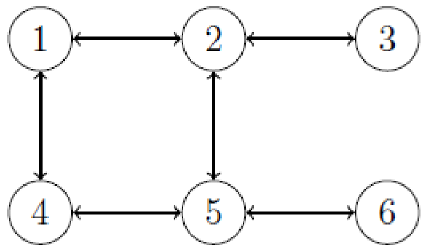

Figure 2: Directed unweighted graph

2 Graphs and Opinions

For this exercise the graph in fig. 2 is considered.

a) Properties

- This graph is balanced, since the in-degrees are equal to the out-degrees for every node in the graph.
- The graph is not regular, since the average degree $\bar{w} = 2$ is not equal to the out-degree of all nodes i, for example $(w_2 = 3) \neq (\bar{w} = 2)$.
- The graph is not aperiodic, because the graph has no self-loops.
- The diameter (defined as the max distance of any node i to any node j) is 3.

b) Matrices

The weight matrix W and the out-degree vector w of this matrix is given as:

$$W = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad w = W\mathbb{1} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

The normalized weight matrix P is computed by $P = D^{-1}W$, where $D = diag(w)$, so P is:

$$P = D^{-1}W = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

And the Laplacian matrix L is computed as follows:

$$L = D - W = \begin{bmatrix} 2 & -1 & 0 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 2 & -1 & 0 \\ 0 & -1 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

c) Bonacich centrality

The normalized Bonacich centrality is given as:

$$\pi = P^\top \pi \tag{5}$$

The eigenvector that corresponds to the largest-in-modulus eigenvalue 1 is:

$$\pi = \begin{bmatrix} \frac{1}{6} & \frac{1}{4} & \frac{1}{12} & \frac{1}{6} & \frac{1}{4} & \frac{1}{12} \end{bmatrix}^\top \tag{6}$$

d) Centralities

- The degree centrality of node 1 is 2, because it has 2 incoming links. The degree centrality of node 5 is 3. So node 5 is more central than node 2, looking at the degree centrality.
- The closeness centrality of node 1 is $\frac{6}{1+2+1+2+3} = \frac{2}{3}$. The closeness centrality of node 2 is $\frac{6}{1+1+2+1+2} = \frac{6}{7}$. So node 2 is more central considering the closeness centrality, which we already assumed after visual inspection of the graph.
- The betweenness centrality of node 2 is $\frac{1}{6^2} \cdot 10 = \frac{10}{36}$. And the betweenness centrality of node 3 is 0, because no path is crossing node 3.

e) Convergence

Assume that all nodes have a self loop. In that case, the graph is connected and aperiodic, so the opinion vector $x(t)$ converges to consensus as t goes to infinity.

f) Initial value assignment

Since the graph is connected and aperiodic, the DeGroot opinion dynamics can be used to find a consensus vector where the nodes converge to:

$$\lim_{t \to \infty} x(t) = (\bar{x})\mathbb{1} \tag{7}$$

with consensus value

$$\bar{x} = \pi^\top x(0) \tag{8}$$

where the Bonacich vector is the normalized vector of the vector given in eq. (6), which is:

$$\pi = \begin{bmatrix} \frac{2}{2\sqrt{7}} & \frac{3}{2\sqrt{7}} & \frac{1}{2\sqrt{7}} & \frac{2}{2\sqrt{7}} & \frac{3}{2\sqrt{7}} & \frac{1}{2\sqrt{7}} \end{bmatrix}^\top$$

To maximize this, x can be initialized as: $x(0) = \begin{bmatrix} 2 & 2 & 1 & 1 & 2 & 1 \end{bmatrix}^\top$, which results in a maximum value of $\bar{x} = \frac{20}{2\sqrt{7}}$. To get the smallest consensus value $x(0) = \begin{bmatrix} 1 & 1 & 2 & 2 & 1 & 2 \end{bmatrix}^\top$, which gives the consensus value $\bar{x} = \frac{16}{2\sqrt{7}}$.

g) Stubborn nodes

Node 1 is a sink with value 0 and node 6 is a sink with value 1. The partitioned matrix looks like:

$$P = \left[\begin{array}{c|c} Q & B \\ \hline 0 & I \end{array}\right] = \left[\begin{array}{cccc|cc} \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \tag{9}$$

The regular agents' opinions vector $y(t)$ converge to:

$$y = (I - Q)^{-1} B u = \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} & 0 & -\frac{1}{4} \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{4} & 0 & -\frac{1}{4} & \frac{3}{4} \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 0 \\ \frac{1}{3} & 0 \\ 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/2 \end{bmatrix} \tag{10}$$

So nodes 2, 3 and 4 converge to $\frac{1}{4}$ and node 5 converges to $\frac{1}{2}$.

h) Connected components

For the last part of this question, we consider the graph in fig. 3.

The number of connected components of graph A is 3 (node 1, node 2, nodes 3, 4 and 5) and of graph B is 1. In fig. 4 the graphs are divided into a condensation graph.

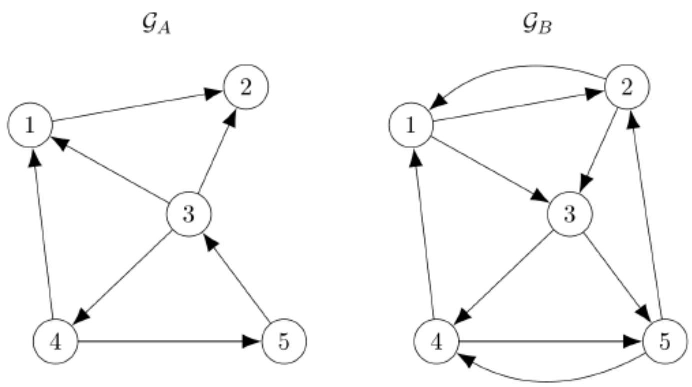

Figure 3: Unweighted graphs

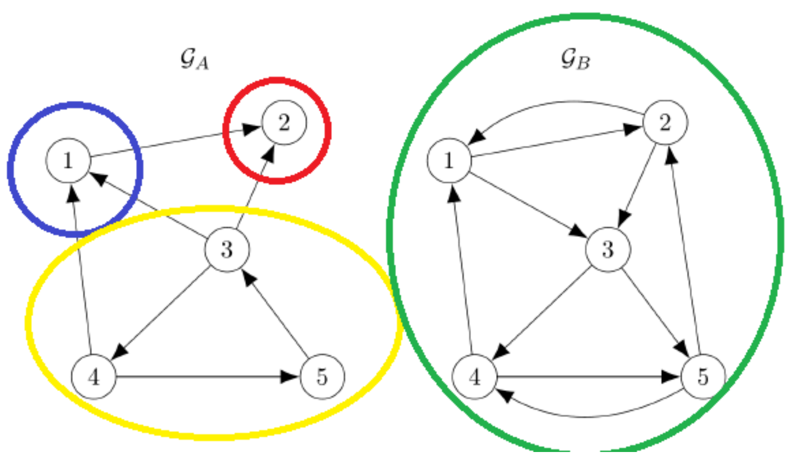

Figure 4: Condensation graph

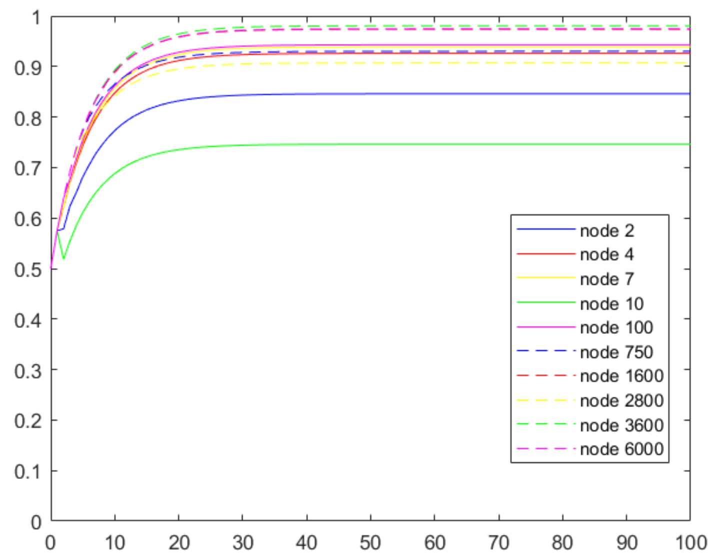

Figure 5: Convergence of twitter graph

3 Influence on Twitter

a) Non-square matrix W

The adjacency matrix is not square and without changing the graph, this can be solved by adding zeros to make the adjacency matrix square, so this does not add any links.

b) Iterative PageRank

The PageRank is computed iteratively using:

$$y(t+1) = (1-\beta)P^{\top}y(t) + \beta\mu \tag{11}$$

The PageRank centrality gives the relative importance of webpages, so the higher the number, the more central the node is in terms of PageRank. For the graph in Twitter the following nodes are most central 2, 1, 112, 9 and 26.

The first node is the most central node, after that, all other nodes converge to the same consensus vector, so are all equally central.

c) Adding stubborn nodes

In fig. 5 the convergence of ten of the nodes is illustrated, where the node 120 is a stubborn node with value 0 and 4000 is a stubborn node with value 1. Since there are two different distinct opinions, the nodes will never converge all to the same consensus value, but depending on the distance to the stubborn nodes, they converges to a different value between $[0, 1]$. Since also the first node is a stubborn

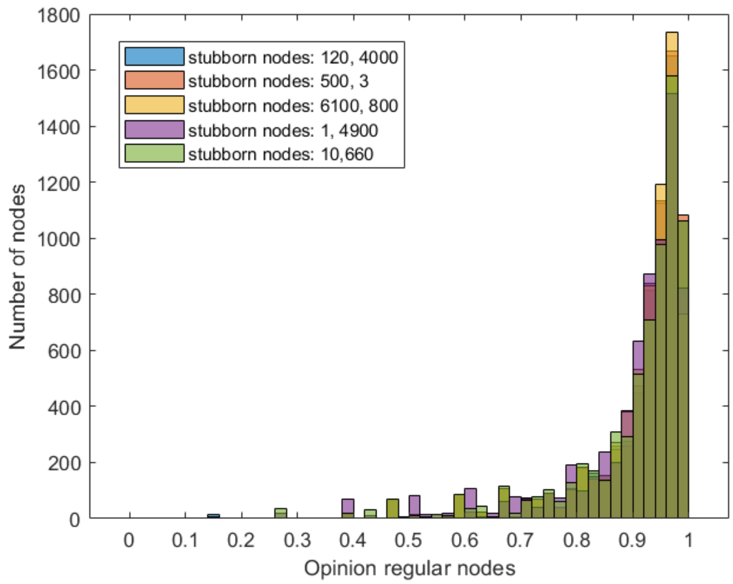

Figure 6: Caption

d) Change in opinion

By changing the stubborn nodes, the opinions of the regular nodes change as well, in fig. 6 the distribution for multiple combinations of stubborn nodes is plotted. The first stubborn node has value 0 and the second value has value 1.

4 Flows, flows, flows

a) Min-cut and min-cut capacity

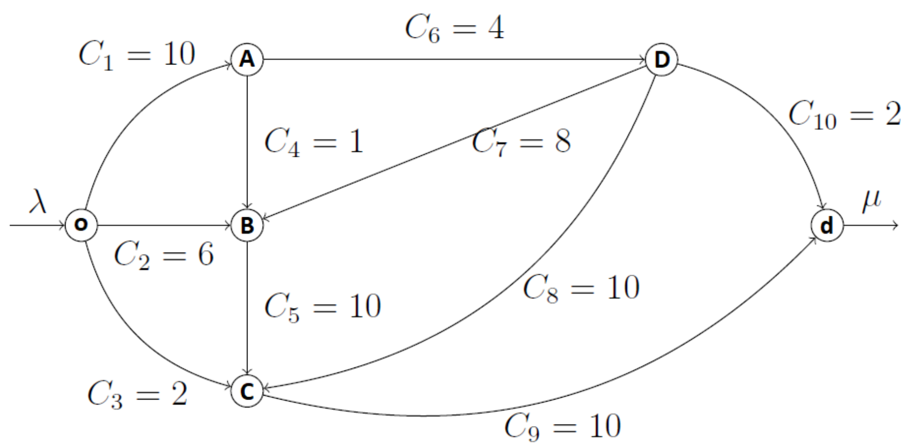

Figure 7: Network

o-d cuts	with capacities
$\mathcal{U}_1 = \{o\}$	$\mathcal{C}_{\mathcal{U}_1} = \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3 = 18$
$\mathcal{U}_2 = \{o, A\}$	$\mathcal{C}_{\mathcal{U}_2} = \mathcal{C}_2 + \mathcal{C}_3 + \mathcal{C}_4 + \mathcal{C}_6 = 13$
$\mathcal{U}_3 = \{o, B\}$	$\mathcal{C}_{\mathcal{U}_3} = \mathcal{C}_1 + \mathcal{C}_3 + \mathcal{C}_5 = 22$
$\mathcal{U}_4 = \{o, C\}$	$\mathcal{C}_{\mathcal{U}_4} = \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_9 = 26$
$\mathcal{U}_5 = \{o, D\}$	$\mathcal{C}_{\mathcal{U}_5} = \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3 + \mathcal{C}_7 + \mathcal{C}_8 + \mathcal{C}_9 + \mathcal{C}_{10} = 48$
$\mathcal{U}_6 = \{o, A, B\}$	$\mathcal{C}_{\mathcal{U}_6} = \mathcal{C}_3 + \mathcal{C}_5 + \mathcal{C}_6 = 16$
$\mathcal{U}_7 = \{o, B, C\}$	$\mathcal{C}_{\mathcal{U}_7} = \mathcal{C}_1 + \mathcal{C}_9 = 20$
$\mathcal{U}_8 = \{o, A, C\}$	$\mathcal{C}_{\mathcal{U}_8} = \mathcal{C}_2 + \mathcal{C}_4 + \mathcal{C}_6 + \mathcal{C}_9 = 21$
$\mathcal{U}_9 = \{o, A, D\}$	$\mathcal{C}_{\mathcal{U}_9} = \mathcal{C}_2 + \mathcal{C}_3 + \mathcal{C}_4 + \mathcal{C}_7 + \mathcal{C}_8 + \mathcal{C}_{10} = 29$
$\mathcal{U}_{10} = \{o, B, D\}$	$\mathcal{C}_{\mathcal{U}_{10}} = \mathcal{C}_1 + \mathcal{C}_3 + \mathcal{C}_5 + \mathcal{C}_8 + \mathcal{C}_{10} = 34$
$\mathcal{U}_{11} = \{o, C, D\}$	$\mathcal{C}_{\mathcal{U}_{11}} = \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_7 + \mathcal{C}_9 + \mathcal{C}_{10} = 36$
$\mathcal{U}_{12} = \{o, A, B, C\}$	$\mathcal{C}_{\mathcal{U}_{12}} = \mathcal{C}_6 + \mathcal{C}_9 = 14$
$\mathcal{U}_{13} = \{o, A, B, D\}$	$\mathcal{C}_{\mathcal{U}_{13}} = \mathcal{C}_3 + \mathcal{C}_5 + \mathcal{C}_8 + \mathcal{C}_{10} = 24$
$\mathcal{U}_{14} = \{o, B, C, D\}$	$\mathcal{C}_{\mathcal{U}_{14}} = \mathcal{C}_1 + \mathcal{C}_9 + \mathcal{C}_{10} = 22$
$\mathcal{U}_{15} = \{o, A, C, D\}$	$\mathcal{C}_{\mathcal{U}_{15}} = \mathcal{C}_2 + \mathcal{C}_4 + \mathcal{C}_7 + \mathcal{C}_9 + \mathcal{C}_{10} = 27$
$\mathcal{U}_{16} = \{o, A, B, C, D\}$	$\mathcal{C}_{\mathcal{U}_{16}} = \mathcal{C}_9 + \mathcal{C}_{10} = 12$

The o-d cut with the minimum capacity is $\mathcal{U}_{16}$, with min-cut capacity $C^*_{o,d} = 12$.

Some of the o-d cuts are unnecessary as the sets consist of parts that are not connected like $\mathcal{U}_5$, which will always have a greater capacity than $\mathcal{U}_1$ for example.

b) Flow choice with $\lambda = \mu = 11$

First we find the link-path incidence matrix $A \in \{0,1\}^{\mathcal{E}\times\mathcal{P}}$ and node-link incidence matrix $B \in \{-1,0,+1\}^{\mathcal{V}\times\mathcal{E}}$, where $\mathcal{V}$ is the set of nodes, $\mathcal{E}$ is the set of links and $\mathcal{P}$ is the set of all o-d paths in $\mathcal{G}$.

$$\mathcal{P} = \{(o,A,D,d),(o,A,B,C,d),(o,B,C,d),(o,C,d),(o,A,D,B,C,d),(o,A,D,C,d)\}$$

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} +1 & +1 & +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & +1 & 0 & +1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & +1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & 0 & -1 & +1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & +1 & +1 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix}$$

with network flow $f \in \mathbb{R}^{\mathcal{E}}$ and exogenous net flow $\nu \in \mathbb{R}^{\mathcal{V}}$ such that

$$\begin{aligned} Bf &= \nu \\ f &\geq 0 \end{aligned}$$

A nonnegative vector $z \in \mathbb{R}^{\mathcal{P}}$ is introduced, with aggregate flow $z_p \geq 0$ on o-d path p. This is designed to be equal to $z = \begin{bmatrix} 2 & 1 & 4 & 2 & 0 & 2 \end{bmatrix}^\top$ which satisfies the throughput $\tau = \lambda = \mu = 11 = \mathbb{1}^\top z$. This results in a network flow that satisfies the link capacities, $0 \leq f_{e_k} \leq C_{e_k}$.

$$f = Az = \begin{bmatrix} 5 \\ 4 \\ 2 \\ 1 \\ 5 \\ 4 \\ 0 \\ 2 \\ 9 \\ 2 \end{bmatrix}, \quad \text{with} \quad \nu = Bf = BAz = \tau(\delta^{(o)} - \delta^{(d)}) = \begin{bmatrix} 11 \\ 0 \\ 0 \\ 0 \\ 0 \\ -11 \end{bmatrix}$$

c) Minimum total capacity removed so that no feasible flow of $\lambda = \mu = 8$ can exist

As found in Question 4a, the min-cut capacity with the current link capacities is $C^*_{o,d} = 12$ with bottleneck $\mathcal{U}_{16}$. To decrease the maximal possible throughput to below $\lambda = \mu = 8$, this min-cut capacity has to be decreased to 7, which means a minimum total capacity of 5 has to be removed from the network so that no feasible flow of throughput $\lambda = \mu = 8$ can exist, as proven by the max-flow min-cut theorem. This capacity is removed from the link capacities in cut $\mathcal{U}_{16}$, such that $C_9 + C_{10} = 7$ and $C_9, C_{10} \geq 0$.

d) Distribute 2 extra units of capacity to increase min-cut capacity

The current min-cut capacity corresponds to cut $\mathcal{U}_{16}$, so the link capacity of C_9 or C_{10} is increased by 1 unit. Now, there are two o-d cuts with a minimal capacity of 13: $\mathcal{U}_2$ and $\mathcal{U}_{16}$. With only 1 unit of capacity left, it is not possible to increase the min-cut capacity to 14, since the new minimal capacity cuts do not have any overlap in links. Therefore, 1 unit of capacity can be added to any link as this cannot increase the min-cut capacity any more, and the other unit of capacity is added to either C_9 or C_{10} to increase the min-cut capacity to $C^*_{o,d} = 13$ with bottlenecks $\mathcal{U}_2$ and $\mathcal{U}_{16}$.

e) User optimum flow vector

The graph has nodes $\mathcal{V} = \{o, a, b, d\}$, links $\mathcal{E} = \{e_1, e_2, e_3, e_4, e_5\}$ and o-d paths $\mathcal{P} = \{(o, a, d), (o, b, d), (o, a, b, d)\}$. The link-path incidence matrix and node-link incidence matrix are

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} +1 & +1 & 0 & 0 & 0 \\ -1 & 0 & +1 & +1 & 0 \\ 0 & -1 & -1 & 0 & +1 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix}$$

To compute the Wardrop equilibrium, the total delay on each o-d path $p \in \mathcal{P}$ is calculated from

$$\text{total delay on path } p = \sum_{e \in \mathcal{E}} A_{ep} d_e(f_e^{(0)}) \tag{12}$$

with the Wardrop equilibrium flow vector $f^{(0)} = Az$ and the given delay functions on the links.

$$\begin{aligned}
f_1 &= z_{p^{(1)}} + z_{p^{(3)}} && \rightarrow \quad d_1(f_1) = z_{p^{(1)}} + z_{p^{(3)}} + 1 \\
f_2 &= z_{p^{(2)}} && \rightarrow \quad d_2(f_2) = 5z_{p^{(2)}} + 1 \\
f_3 &= z_{p^{(3)}} && \rightarrow \quad d_3(f_3) = 1 \\
f_4 &= z_{p^{(1)}} && \rightarrow \quad d_4(f_4) = 5z_{p^{(1)}} + 1 \\
f_5 &= z_{p^{(2)}} + z_{p^{(3)}} && \rightarrow \quad d_5(f_5) = z_{p^{(2)}} + z_{p^{(3)}} + 1
\end{aligned}$$

$$
\begin{aligned}
&\text{total delay on path } p^{(1)} = d_1(f_1) + d_4(f_4) && = 6z_{p^{(1)}} + z_{p^{(3)}} + 2\\
&\text{total delay on path } p^{(2)} = d_2(f_2) + d_5(f_5) && = 6z_{p^{(2)}} + z_{p^{(3)}} + 2\\
&\text{total delay on path } p^{(3)} = d_1(f_1) + d_3(f_3) + d_5(f_5) && = z_{p^{(1)}} + z_{p^{(2)}} + 2z_{p^{(3)}} + 3
\end{aligned}
$$

At the user optimum, the total delay on each path is equal, for flows with the least amount of delay. By making use of the fact that $z_{p^{(1)}} + z_{p^{(2)}} + z_{p^{(3)}} = \tau = 1$, we find that

$$
\begin{aligned}
6z_{p^{(1)}} + z_{p^{(3)}} + 2 &= 6z_{p^{(2)}} + z_{p^{(3)}} + 2 && \rightarrow z_{p^{(1)}} = z_{p^{(2)}}\\
6z_{p^{(1)}} + z_{p^{(3)}} + 2 &= z_{p^{(1)}} + z_{p^{(2)}} + 2z_{p^{(3)}} + 3 &&\\
5z_{p^{(1)}} &= 1 - z_{p^{(1)}} + 3 && \rightarrow z_{p^{(1)}} = z_{p^{(2)}} = \frac{1}{3}\\
z_{p^{(3)}} &= 1 - z_{p^{(1)}} + z_{p^{(2)}} && \rightarrow z_{p^{(3)}} = \frac{1}{3}
\end{aligned}
$$

$$
f^{(0)} = Az = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}^\top
$$

f) Social optimum flow vector

The flow vector that minimizes the average delay from o to d is found by solving the following optimization problem

$$
\begin{aligned}
\min_f \sum_{e\in\mathcal{E}} & f_e^{(0)} d_e(f_e^{(0)})\\
s.t. \quad & f \geq 0\\
& Bf = \nu
\end{aligned}
$$

where

$$
\nu = \tau(\delta^{(o)} - \delta^{(d)}) = \begin{bmatrix} 1\\ 0\\ 0\\ -1 \end{bmatrix}
$$

From this we obtain the following optimization problem:

$$
\begin{aligned}
\min_f \ & f_1(f_1+1) + f_2(5f_2+1) + f_3 + f_4(5f_4+1) + f_5(f_5+1)\\
s.t. \quad & f \geq 0\\
& f_1 + f_2 = 1\\
& -f_1 + f_3 + f_4 = 0\\
& -f_2 - f_3 + f_5 = 0\\
& -f_4 - f_5 = -1
\end{aligned}
$$

By substituting the equality constraints into the objective function this is

$$\min_{f\geq 0} y(f), \quad \text{with} \quad y(f) = 2f_1^2 - 3f_1 + 2f_4^2 - f_4$$

The minimum of this objective function is found by setting the partial derivatives equal to 0.

$$\begin{aligned}
\frac{\partial y}{\partial f_1} &= 4f_1 - 3 = 0 & \rightarrow f_1 &= \frac{3}{4} \\
\frac{\partial y}{\partial f_1} &= 4f_1 - 1 = 0 & \rightarrow f_4 &= \frac{1}{4} \\
& & \rightarrow f_2 = 1 - f_1 &= \frac{1}{4} \\
& & \rightarrow f_5 = 1 - f_4 &= \frac{3}{4} \\
& & \rightarrow f_3 = f_1 - f_4 &= \frac{1}{2}
\end{aligned}$$

This results in the social optimum flow vector

$$f^* = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \frac{3}{4} \end{bmatrix}^\top$$

g) Price of anarchy

The price of anarchy with the Wardrop equilibrium $f^{(0)}$ is

$$\text{PoA}(0) = \frac{\sum_{e\in\mathcal{E}} f_e^{(0)} d_e(f_e^{(0)})}{\min\limits_{\substack{f\geq 0 \\ Bf=\nu}} \sum_{e\in\mathcal{E}} f_e d_e(f_e)} = \frac{\frac{10}{9} + \frac{8}{9} + \frac{1}{3} + \frac{8}{9} + \frac{10}{9}}{\frac{21}{16} + \frac{9}{16} + \frac{1}{2} + \frac{9}{16} + \frac{21}{16}} = \frac{4\frac{1}{3}}{4\frac{1}{4}} = \frac{52}{51}$$

h) Vector of tolls on the links that reduce the price of anarchy to 1

To find a toll vector ω that gives PoA(ω)=1, we want $f^{(\omega^*)} = f^*$. This is achieved with

$$\omega_e = f_e^* d_e'(f_e^*) \tag{13}$$

which results in a vector of tolls on the links of

$$\omega = \begin{bmatrix} \frac{3}{4} & \frac{5}{4} & 0 & \frac{5}{4} & \frac{3}{4} \end{bmatrix}^\top$$

5 Traffic in Rio

a) Shortest path

The shortest path from node 1 (GIG) to node 13 (Barra da Tijuca) is

$$1 - 2 - 3 - 4 - 6 - 12 - 13$$

with path length 21.

b) Maximum flow

The maximum flow between node 1 and 13 is 5400.

c) External inflow/outflow

The external inflow/outflow ν for each node can be computed with

$$\nu = \lambda - \mu = Bf$$

where B is the node-link incidence matrix from `traffic.mat`, f is the flow vector from `flow.mat`, λ is the external inflow and μ is the external outflow. The external flow at each node is given in the following table, where a negative value for ν means an external flow out of the network in this node.

Table 1: External inflow and outflow in each node

Node	1	2	3	4	5	6	7	8	9	10	11	12	13
λ	3329	12983	9693	12394	5141	17739	4552	10057	3717	3332	5961	9031	0
μ	0	3329	5141	12394	5141	7842	7435	13564	3717	10057	8500	9049	11760
ν	3329	9654	4552	0	0	9897	-2883	-3507	0	-6725	-2539	-18	-11760

d) Social optimum

Optimization problem:

$$\begin{aligned} \min_f \sum_{e \in \mathcal{E}} & \frac{l_e C_e}{1 - f_e/C_e} - l_e C_e \\ \text{s.t.} \quad & Bf = \lambda - \mu \\ & 0 \leq f \leq C \end{aligned}$$

with

$$\lambda = \begin{bmatrix} 3329 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{and} \quad \mu = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 3329 \end{bmatrix}$$

The flows of the social optimum f^* are shown in Table 2.

e) Wardrop equilibrium

Optimization problem:

$$\min_f \sum_{e\in\mathcal{E}} \int_0^{f_e} d_e(x)dx = \min_f \sum_{e\in\mathcal{E}} -C_e l_e \ln\left(1-\frac{f_e}{C_e}\right)$$
$$\text{s.t.} \quad Bf = \lambda - \mu$$
$$0 \le f \le C$$

The flows of the Wardrop equilibrium f^W are shown in Table 2.

f) Wardrop equilibrium with tolls

Optimization problem:

$$\min_f \sum_{e\in\mathcal{E}} -C_e l_e \ln\left(1-\frac{f_e}{C_e}\right) + \omega_e f_e$$
$$\text{s.t.} \quad Bf = \lambda - \mu$$
$$0 \le f \le C$$

where ω_e is the toll on link e, computed as $\omega_e = f_e^* d_e'(f_e^*)$.
The flows of the new Wardrop equilibrium $f^{(\omega^*)}$ are shown in Table 2. It can be observed that the flow vector at the new Wardrop equilibrium $f^{(\omega^*)}$ equal to the social optimum f^*, which means that the Price of Anarchy is reduced to 1 by introducing this vector of tolls.

g) System optimum and Wardrop equilibrium with cost as additional delay

Optimization problem for system optimum:

$$\min_f \sum_{e\in\mathcal{E}} \frac{l_e C_e}{1-f_e/C_e} - l_e C_e - f_e l_e$$
$$\text{s.t.} \quad Bf = \lambda - \mu$$
$$0 \le f \le C$$

Optimization problem for Wardrop equilibrium with tolls:

$$\min_f \sum_{e\in\mathcal{E}} -C_e l_e \ln\left(1-\frac{f_e}{C_e}\right) - f_e l_e + \omega_e f_e$$
$$\text{s.t.} \quad Bf = \lambda - \mu$$
$$0 \le f \le C$$

The flows of the system optimum f^* and the Wardrop equilibrium with tolls $f^{(\omega^*)}$, and the toll vector ω are shown in Table 2. The Wardrop equilibrium for the network with tolls is equal to the system optimum f^*.

Table 2: Flows and tolls on the links (rounded to nearest integer)

Link	$f^*_{(5d)}$	$f^W_{(5e)}$	$f^{(\omega^*)}_{(5f)}$	$f^*_{(5g)}$	$f^{(\omega^*)}_{(5g)}$	$\omega_{(5f)}$	$\omega_{(5g)}$
1	3329	3329	3329	3329	3329	21	21
2	2536	3253	2536	1891	1891	1	1
3	793	76	793	1438	1438	0	1
4	1320	2478	1321	837	837	0	0
5	1216	774	1216	1054	1054	0	0
6	69	0	69	1204	1204	0	1
7	2045	2555	2045	1071	1071	6	2
8	0	0	0	243	243	0	1
9	0	0	0	0	0	0	0
10	1216	774	1216	1054	1054	0	0
11	0	0	0	231	231	0	1
12	1216	774	1216	824	824	1	1
13	0	0	0	231	231	0	0
14	21	0	21	404	404	0	4
15	1216	774	1216	1054	1054	4	3
16	48	0	48	557	557	0	3
17	2045	2555	2045	1314	1314	5	2
18	0	0	0	77	77	0	1
19	1237	774	1237	1535	1535	1	1
20	2092	2555	2092	1794	1794	3	2

6 Drugs

a)

The system for this assignment is shown in fig. 8. This is an open compartmental system. The states of the systems can be written as follows:

$$\begin{aligned} \dot{x}_D(t) &= \lambda + k_{SD}x_S(t) - (k_{DU} + k_{DM})x_D(t) \\ \dot{x}_M(t) &= k_{DM}X_D(t) - k_{MU}x_M(t) \\ \dot{x}_S(t) &= k_{DS}x_D(t) - k_{SD}x_S(t) \end{aligned} \tag{14}$$

This can be described as a continuous-time system:

$$\dot{\mathbf{x}} = -L^\top \mathbf{x} + \boldsymbol{\lambda} \tag{15}$$

Consider the urine as a sink in the graph and the drug input a source. The adjacency matrix W for $i, j \in \{D, M, S, I, U\}$ can be written as (with I is the input node and U is the urine):

$$W = \begin{bmatrix} 0 & k_{DM} & k_{DS} & 0 & k_{DU} \\ 0 & 0 & 0 & 0 & k_{MU} \\ k_{SD} & 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{16}$$

From this matrix, the diagonal matrix D can be computed:

$$D = \begin{bmatrix} k_{DU} + k_{DS} + k_{DU} & 0 & 0 & 0 & 0 \\ 0 & k_{MU} & 0 & 0 & 0 \\ 0 & 0 & k_{SD} & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{17}$$

L can be computed using the adjacency matrix W for $i, j \in \{D, M, S, I, U\}$:

$$L = D - W = \begin{bmatrix} k_{DM} + k_{DS} + k_{DU} & -k_{DM} & -k_{DS} & 0 & -k_{DU} \\ 0 & k_{MU} & 0 & 0 & -k_{MU} \\ -k_{SD} & 0 & k_{SD} & 0 & 0 \\ -\lambda & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{18}$$

The inflow vector is:

$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{19}$$

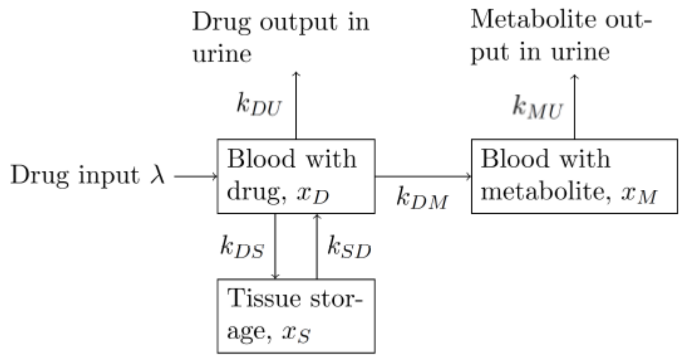

Figure 8: Compartmental model for drug propagation in human body

The drug inside the model can be modeled as:

$$\dot{y} = -M^\top y + \lambda \tag{20}$$

where M is the $\mathcal{R} \times \mathcal{R}$ block of L and $\boldsymbol{\lambda}$ is only the first $\mathcal{R}$ elements, so only considering the regular nodes:

$$\dot{x} = -M^\top x + \lambda = -\begin{bmatrix} k_{DM} + k_{DS} + k_{DU} & -k_{DM} & -k_{DS} \\ 0 & k_{MU} & 0 \\ -k_{SD} & 0 & k_{SD} \end{bmatrix}^\top x + \begin{bmatrix} \lambda \\ 0 \\ 0 \end{bmatrix} \tag{21}$$

This system converges to:

$$\bar{x} = \lim_{t \to \infty} x(t) = (M^\top)^{-1} \boldsymbol{\lambda} \tag{22}$$

b)

For the numerical case, where $\lambda = 2$, $k_{DS} = 0.6$, $k_{SD} = 0.3$, $k_{DM} = 0.2$, $k_{MU} = 0.1$ and $k_{DU} = 0.4$ the system converges to:

$$\lim_{t \to \infty} y(t) = \left(\begin{bmatrix} 1.2 & -0.2 & -0.6 \\ 0 & 0.1 & 0 \\ -0.3 & 0 & 0.3 \end{bmatrix}^\top \right)^{-1} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 10/3 \\ 20/3 \\ 20/3 \end{bmatrix} \tag{23}$$

7 Playing with graphs 1/2

Proposition A k-regular undirected graph, where k is odd, has a total number of undirected edges m, that is divisible by k.

Proof Every node has an degree of k. To be able to make a k-regular graph with k odd, the number of nodes has to be even. By multiplying the number of nodes n with the degree k, we get the total number of edges twice. Dividing the number by 2, gives the total number of edges m, so

$$m = \frac{n \cdot k}{2} \tag{24}$$

This can be rewritten as:

$$\frac{m}{k} = \frac{n}{2} \tag{25}$$

So to make sure m is divisible by k, the number of nodes need to be even, which is also a necessity for a k-regular graph with k odd.

The sometimes holds for k is even, but in the case of k is even the number of nodes does not necessary have to be even and in that case, this equation does not hold.

8 Playing with graphs 2/2

Consider a graph $\mathcal{C}(n)$ as a complete, undirected, unweighted graph with n nodes.

The corresponding adjacency matrix $A_{\mathcal{C}(n)}$ is then given as:

$$A_{\mathcal{C}(n)} = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 0 & 1 & \cdots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 0 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 0 \end{bmatrix} \tag{26}$$

Its spectrum is $\lambda_1 = n, \lambda_2 = -1, \cdots, \lambda_{n-1} = -1, \lambda_n = -1$.

$$\begin{bmatrix} n-1 \\ n-1 \\ \vdots \\ n-1 \\ n-1 \end{bmatrix} \tag{27}$$

The corresponding Laplacian matrix $L_{\mathcal{C}}(n)$ is:

$$L_{\mathcal{C}}(n) = \begin{bmatrix} n-1 & -1 & -1 & \cdots & -1 & -1 & -1 \\ -1 & n-1 & -1 & \cdots & -1 & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & -1 & n-1 & -1 \\ -1 & -1 & -1 & \cdots & -1 & -1 & n-1 \end{bmatrix} \tag{28}$$

and its spectrum is $\lambda_1 = n, \lambda_2 = n, \cdots, \lambda_{n-1} = n, \lambda_n = 0$.