Engineering Optimization

Concepts and Applications

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Contents

- Outline of remaining lectures
- Unconstrained problems
  - …

Covered so far …

1. Introduction:
   – Negative null form
   – Applications

2. Optimization problem:
   – Definition
   – Characteristics

3. Problem checking:
   – Boundedness
   – Monotonicity analysis

4. Optimization model:
   – Model simplification
   – Approximation

Upcoming topics

Optimization problem

- Negative null form
- Problem definition
- Problem checking
- Model simplification

Special topics

- Linear / convex problems
- Sensitivity analysis
- Topology optimization

Solution methods

- Unconstrained problems
  - Optimality criteria
  - Optimization algorithms
- Constrained problems
  - Optimality criteria
  - Optimization algorithms

Contents

- Outline of remaining lectures
- Unconstrained problems
  - Transformation methods
  - Existence of solutions, optimality conditions
  - Nature of stationary points
  - Global optimality

Unconstrained Optimization

$$\min_{\mathbf{x}} \quad f(\mathbf{x})$$

$$\underline{\mathbf{x}} \le \mathbf{x} \le \overline{\mathbf{x}}$$

- Why?
  - Elimination of active constraints → unconstrained problem
  - Develop basic understanding useful for constrained optimization
  - Transformation of constrained problems into unconstrained problems
  - Relevant engineering problems (potential energy minimization)

Transforming constrained problem

- Reformulation through *barrier functions*:

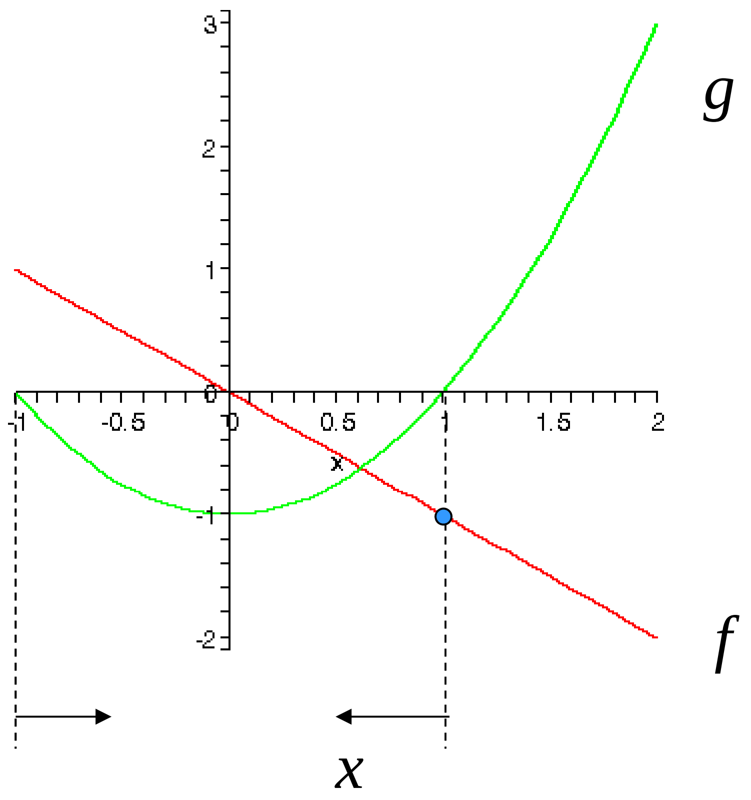
$$f = -x$$

$$g = x^2 - 1 \leq 0$$

Transformation:

$$\tilde{f} = f - \frac{1}{r}\ln(-g)$$

$$\tilde{f} = -x - \frac{1}{r}\ln\left(1 - x^2\right)$$



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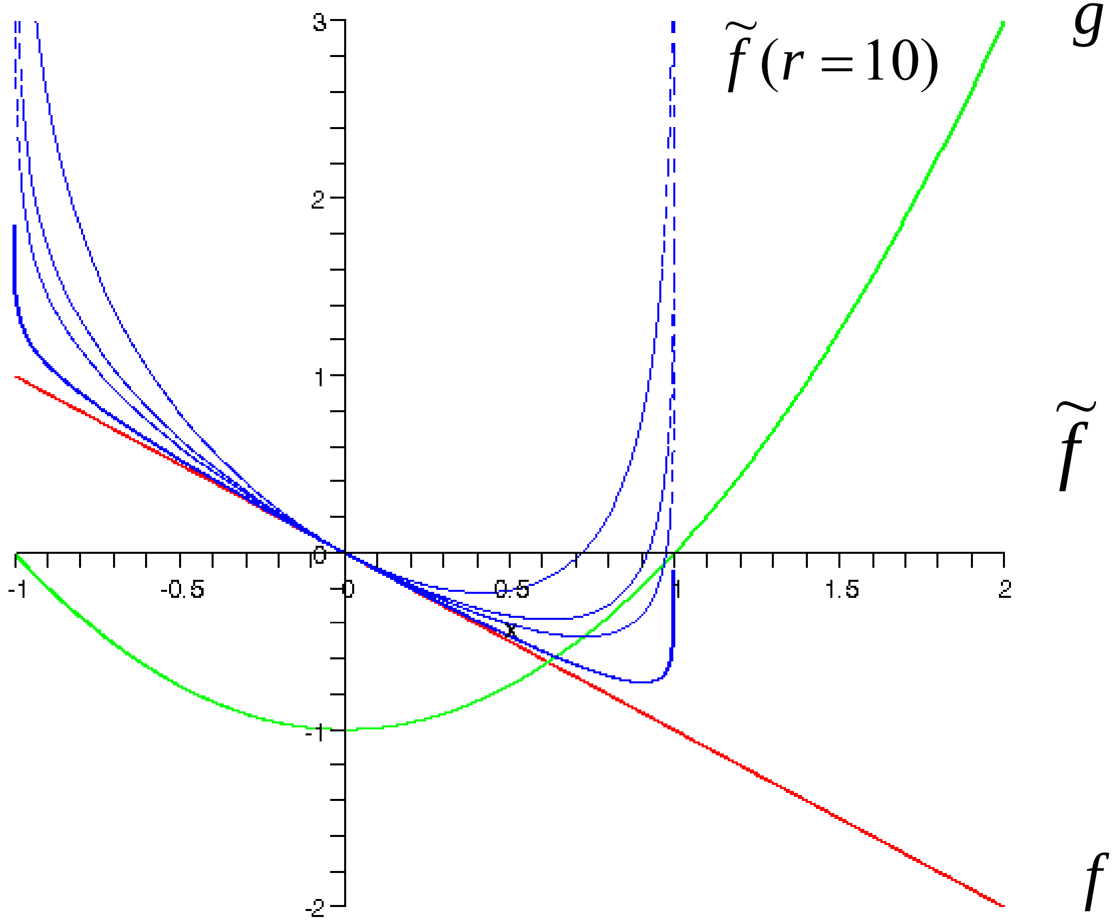
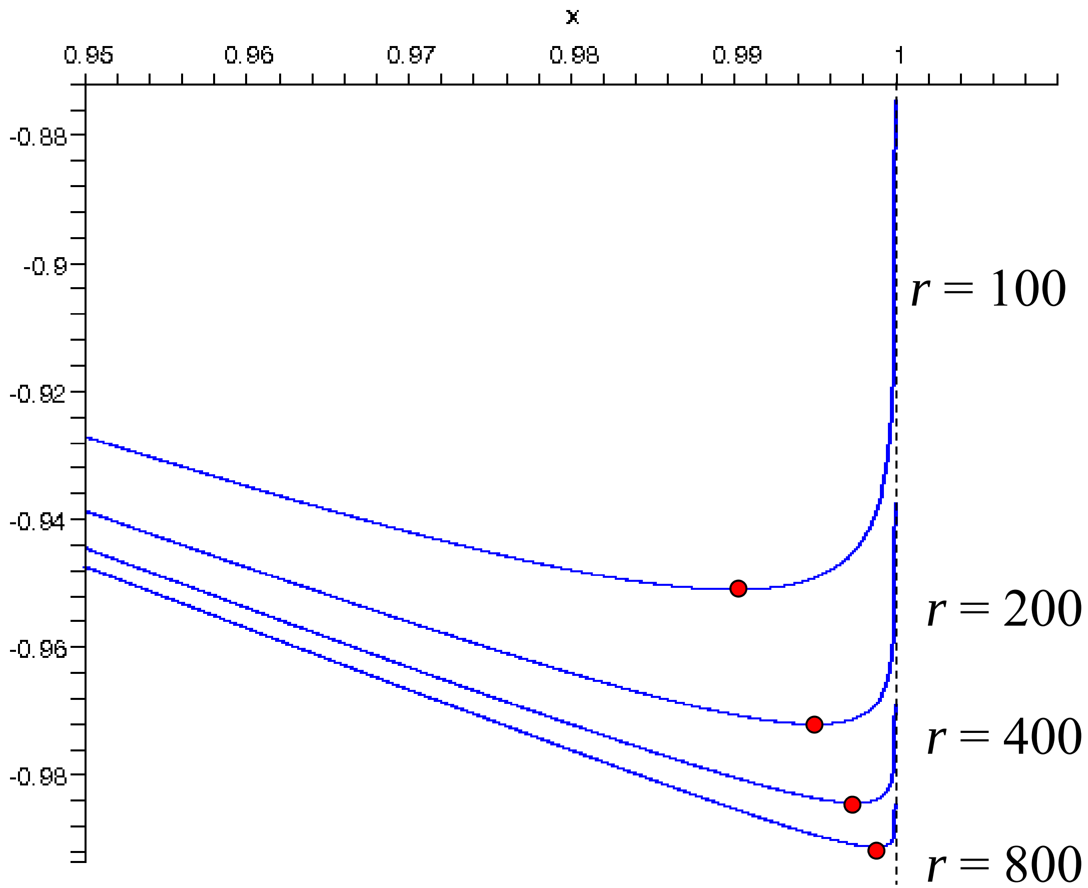
Transformed problem



$$\tilde{f} = f - \frac{1}{r}\ln(-g)$$

Transformed problem

- Barrier functions result in feasible, *interior* optimum:



Penalization

- Alternative reformulation: *penalty functions*

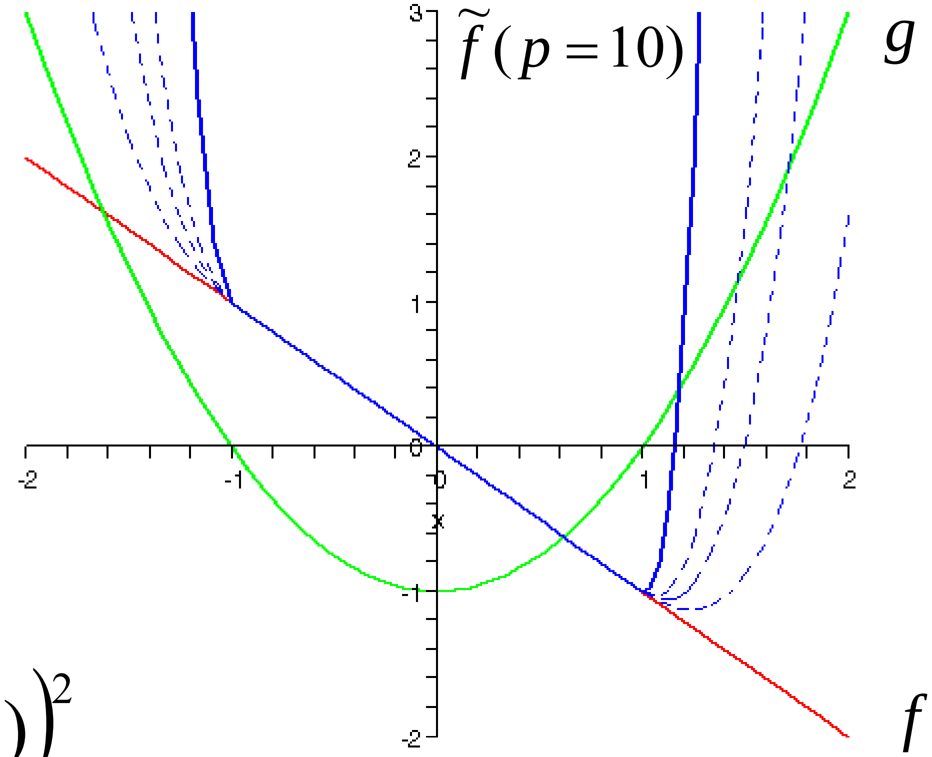
$$f = -x$$

$$g = x^2 - 1 \le 0$$

Transformation:

$$\tilde{f} = f + p\left(\max(0, g)\right)^2$$

$$\tilde{f} = -x + p\left(\max(0, x^2 - 1)\right)^2$$



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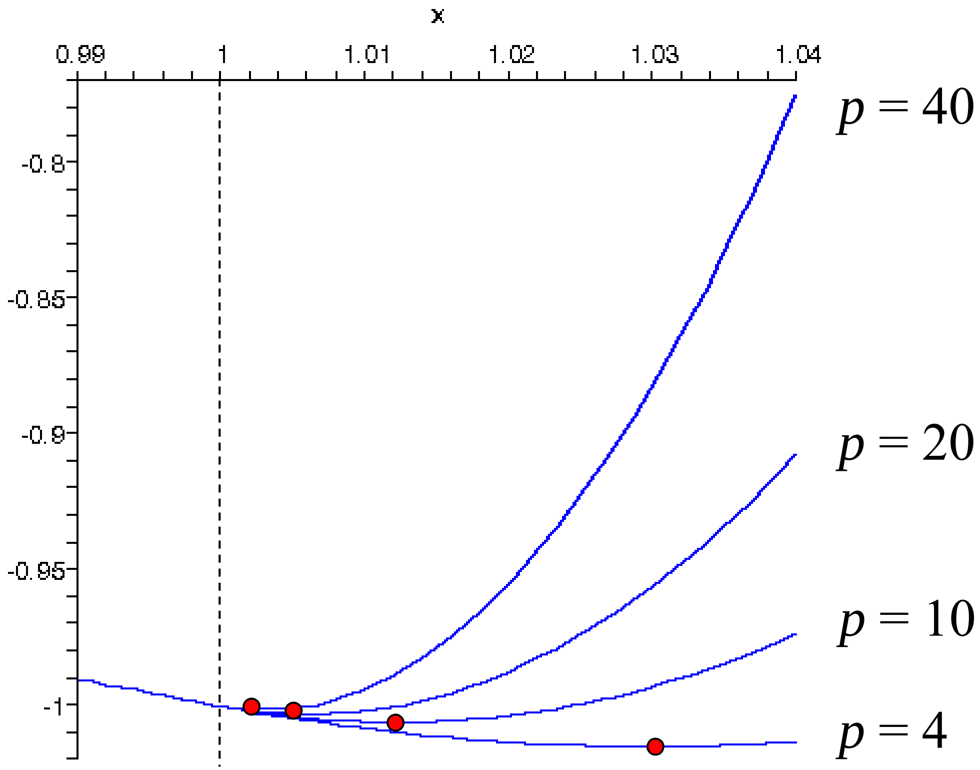
Penalization (2)

- Penalty functions result in infeasible, *exterior* optimum:



Problem transformation summary

	Barrier function	Penalty function
Need feasible starting point?	Yes	No
Nature of optimum	Interior (feasible)	Exterior (infeasible)
Type of constraints	g	g, h

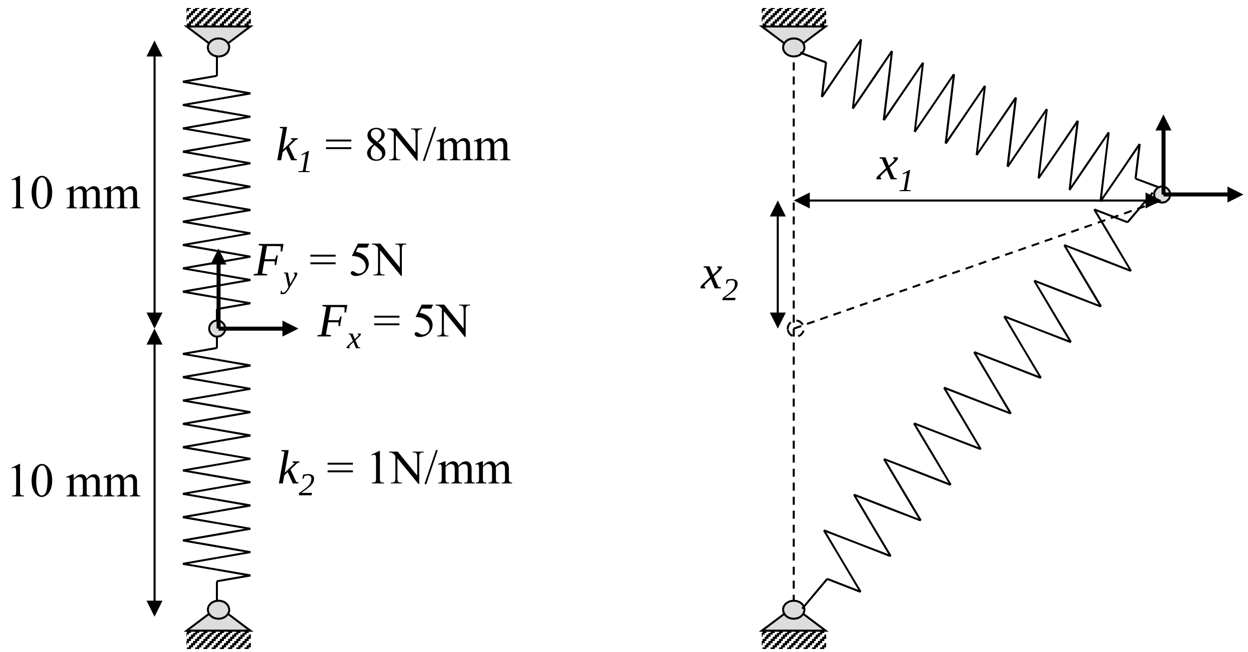
Generally constants r, p are iteratively increased to converge.

Unconstrained Optimization

- Why?
  - Elimination of active constraints → unconstrained problem
  - Develop basic understanding useful for constrained optimization
  - Transformation of constrained problems into unconstrained problems
  - Relevant engineering problems (potential energy minimization)

Unconstrained engineering problem

- Example: displacement of loaded structure



- Equilibrium: minimum potential energy

Unconstrained engineering problem

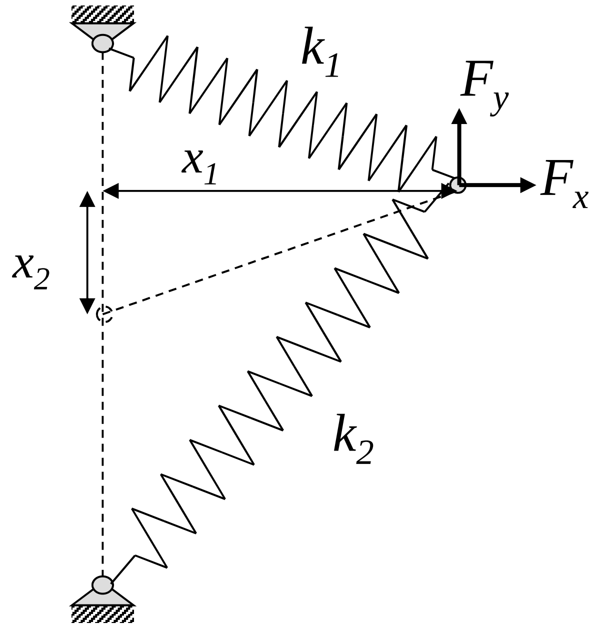
- Potential energy:

$$\Pi = \frac{1}{2}k_1 u_1{}^2 + \frac{1}{2}k_2 u_2{}^2 - F_x x_1 - F_y x_2$$

$$= 4\left(\sqrt{x_1{}^2 + (10 - x_2)^2} - 10\right)^2$$

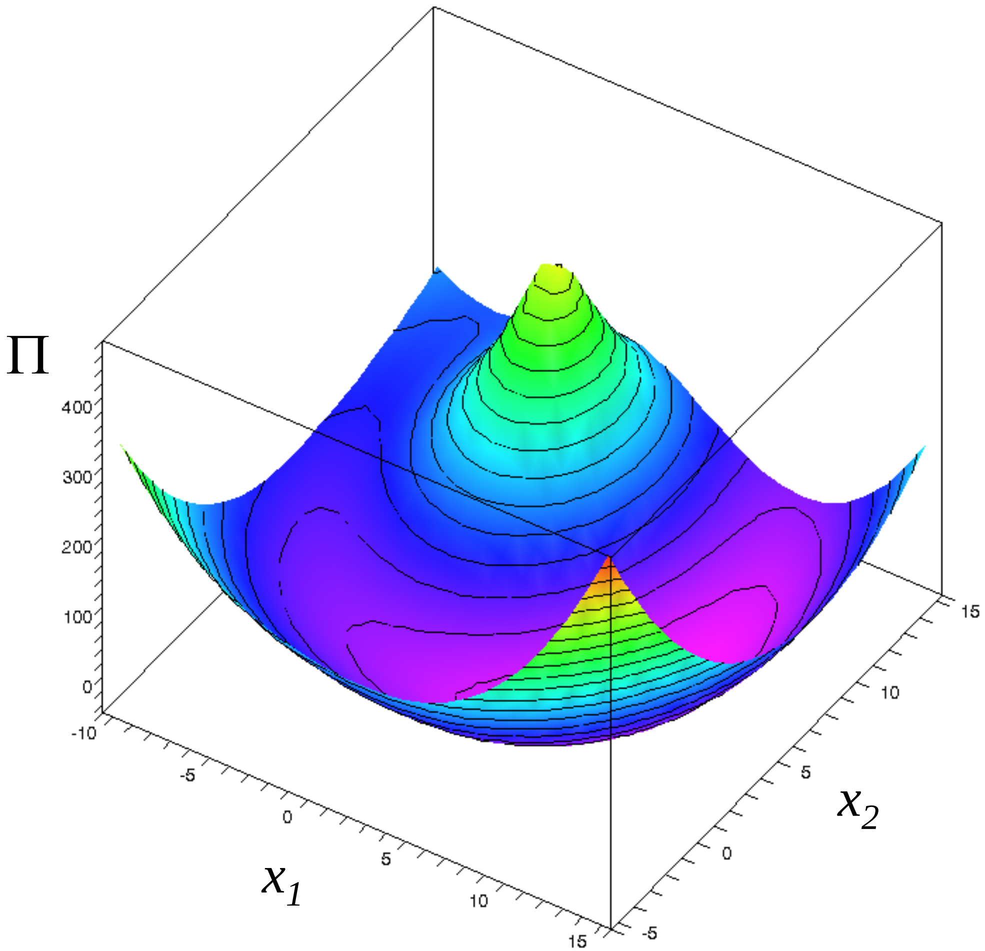
$$+ 0.5\left(\sqrt{x_1{}^2 + (10 + x_2)^2} - 10\right)^2$$

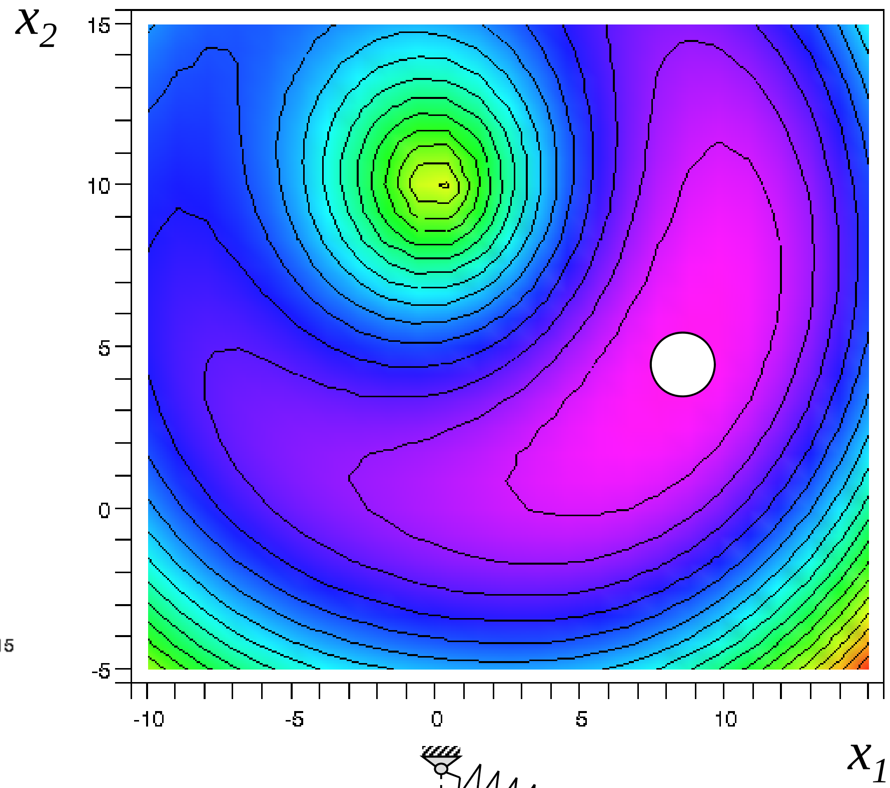
$$- 5x_1 - 5x_2$$

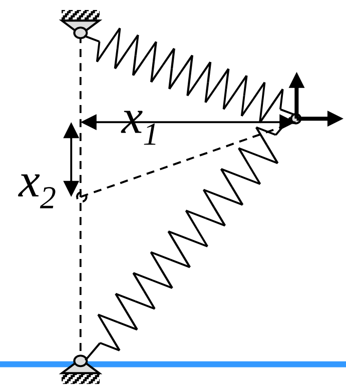


- Equilibrium: $\min_{x_1, x_2} \Pi$

Unconstrained engineering problem







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Significance of Optimality Conditions

Question:

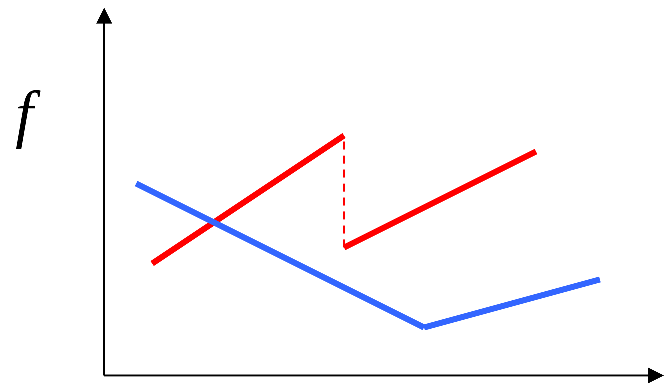
Given a point $\mathbf{x}^*$, how can we determine if it is a minimizer?

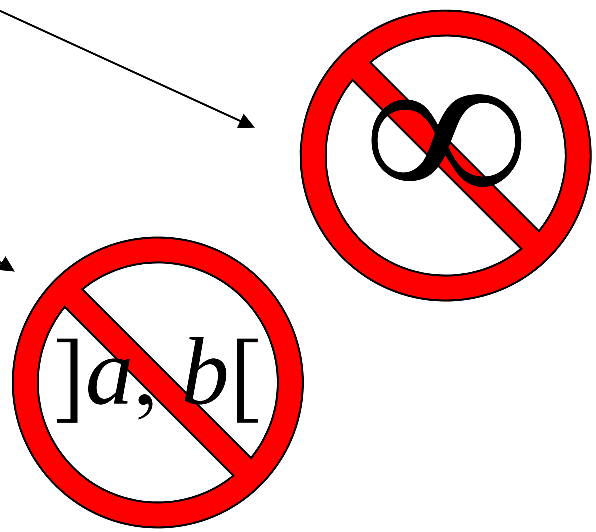
i.e. *How to recognize/identify an optimal point?*

Answer: Check **optimality conditions** *at point* $\mathbf{x}^*$.

Theory for solving unconstrained optimization problems

- Assumptions:
  - Objective *continuous* and *differentiable* (C^1)
  - Domain *closed* and *bounded* (compact)



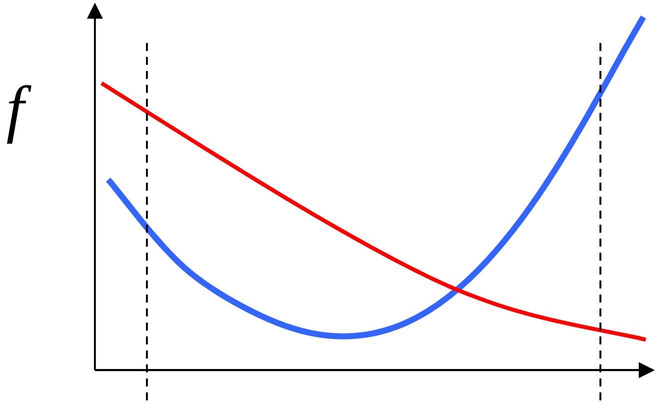


Existence of minima

- Weierstrass Theorem:

  *"A **continuous** function on a **compact** set has a maximum and a minimum in that set"*

- *Sufficient* condition for existence!
- *Interior* optima only exist for *non-monotonic* functions



- Compact set: use $\leq$, not $<$ in constraints!

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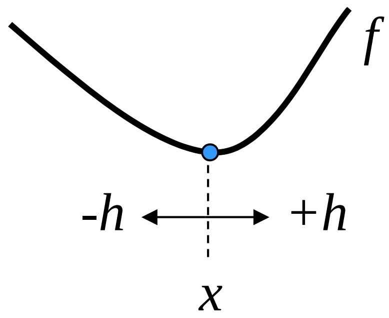
One-dimensional case

- Calculus: conditions for local minimum of f
  - Derivative zero: $f' = 0$ (necessary)
  - Second derivative positive: $f'' > 0$ (sufficient)
- Interpretation through Taylor series:



$$f(x+h) = f + f'h + o(h^2)$$

$$\left.\begin{aligned} \Delta f &= f(x+h) - f = f'h + o(h^2) \geq 0 \\ \Delta f &= f(x-h) - f = -f'h + o(h^2) \geq 0 \end{aligned}\right\} \Rightarrow \quad f' = 0$$

One-dimensional case (2)

- Condition for local minimum: $f''(x) > 0$

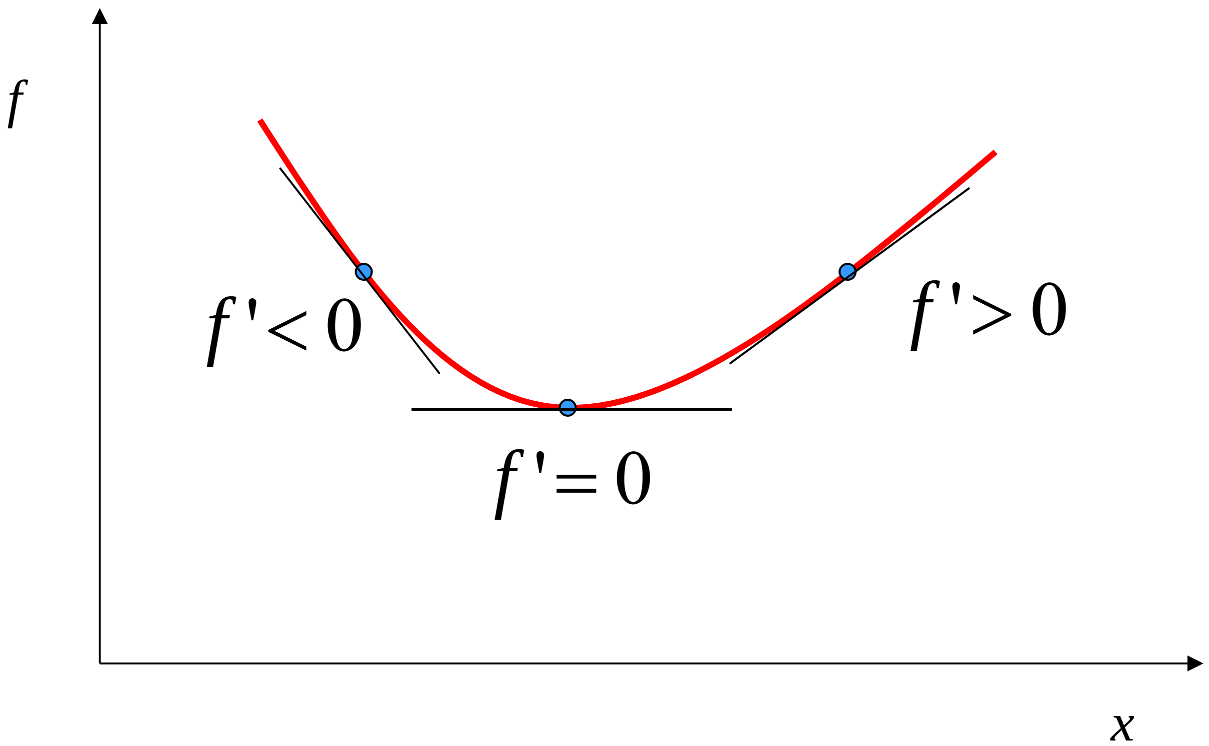
$$\left.\begin{aligned} \Delta f &= f(x+h) - f = \frac{1}{2} f'' h^2 + o(h^3) > 0 \\ \Delta f &= f(x-h) - f = \frac{1}{2} f'' h^2 + o(h^3) > 0 \end{aligned}\right\} \Rightarrow \quad f'' > 0$$

- Other possibilities:

$f''(x) < 0$	maximum
$f''(x) = 0$	? Check higher order derivatives

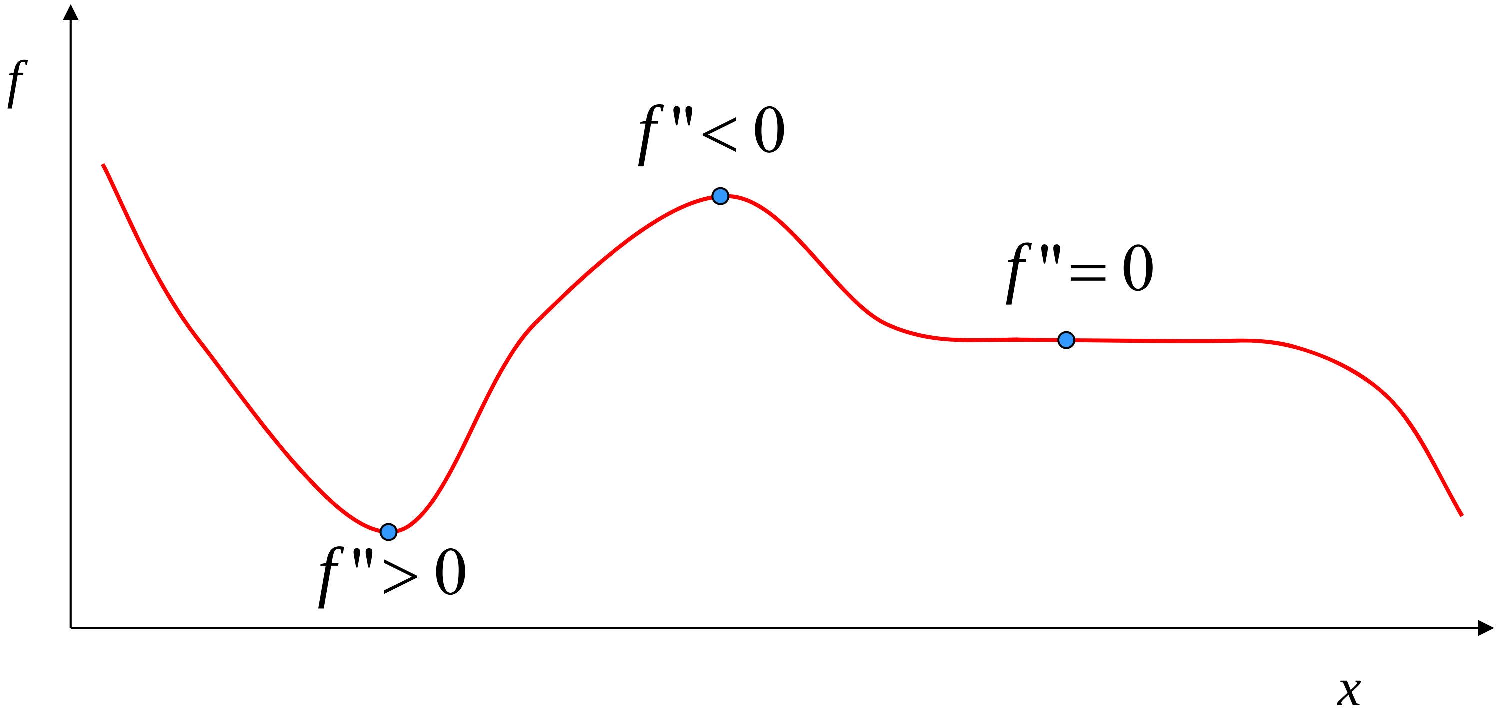
Geometrical interpretation

- Positive f'' $\Leftrightarrow$ f' locally increasing



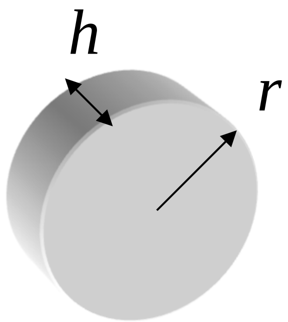
One-dimensional case (3)

- Possible situations for stationary points ($f' = 0$):



Example

- Aspirin pill revisited: "longest-lasting candy"
  - *Maximize* dissolving time → minimize surface area



$$\Rightarrow \quad \min_{r,h} \; 2\pi r^2 + 2\pi r h$$

$$\text{s.t.} \quad \pi r^2 h = 1$$

- Equality constraint active → eliminate h

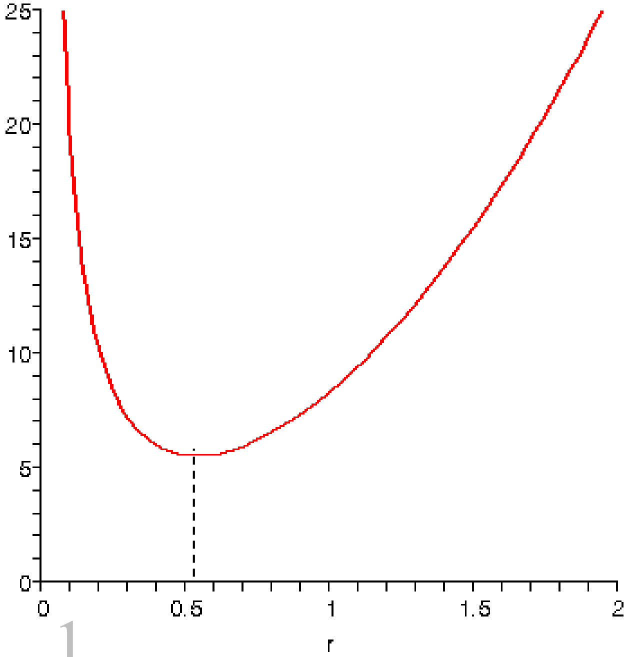
$$h = \frac{1}{\pi r^2} \quad \Rightarrow \quad \min_{r} \; 2\pi r^2 + \frac{2}{r}$$

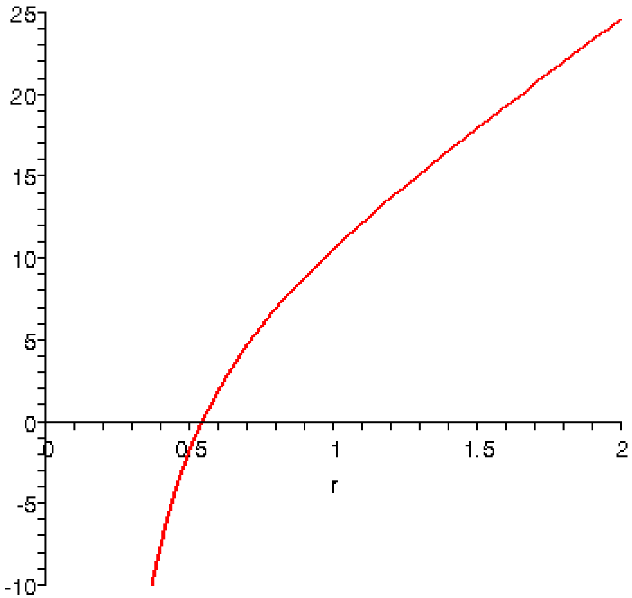
Example (2)

$$f = 2\pi r^2 + \frac{2}{r}$$

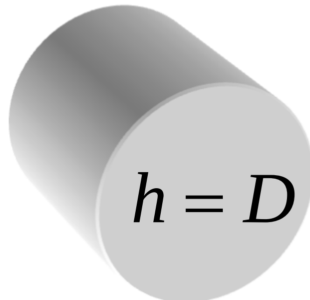
$$f' = 4\pi r - \frac{2}{r^2} = 0$$

$$f'' = 4\pi + \frac{4}{r^3}$$





$$> 0 \quad \forall r > 0$$

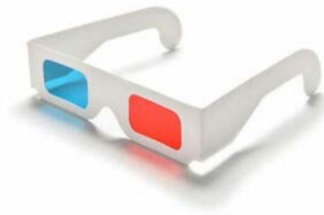


$$h = \frac{1}{\pi r^2}$$

$$r = (2\pi)^{-\frac{1}{3}} \approx 0.542$$

$$h = \frac{(2\pi)^{\frac{2}{3}}}{\pi} \approx 2r$$

Multidimensional case



- Local approximation to multidimensional minimum by multidimensional Taylor series:

$$f(\mathbf{x}+\mathbf{h}) = f + \nabla f^{T}\mathbf{h} + o\left(\|\mathbf{h}\|^{2}\right)$$

Gradient $\nabla f = \begin{Bmatrix} \dfrac{\partial f}{\partial x_1} \\ \dfrac{\partial f}{\partial x_2} \\ \vdots \\ \dfrac{\partial f}{\partial x_n} \end{Bmatrix}$

$$f(\mathbf{x}+\mathbf{h}) - f = \nabla f^{T}\mathbf{h} + o\left(\|\mathbf{h}\|^{2}\right) \geq 0$$

$$f(\mathbf{x}-\mathbf{h}) - f = -\nabla f^{T}\mathbf{h} + o\left(\|\mathbf{h}\|^{2}\right) \geq 0$$

(For any $\mathbf{h}$)

Condition for minimum: $\nabla f = \mathbf{0}$

Multidimensional case (2)

- For minimum, consider second-order approximation:

$$f(\mathbf{x}+\mathbf{h}) = f + \nabla f^T \mathbf{h} + \frac{1}{2}\mathbf{h}^T \mathbf{H}\mathbf{h} + O\left(\|\mathbf{h}\|^3\right)$$

Hessian $$\mathbf{H} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^{\ 2}} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^{\ 2}} & & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^{\ 2}} \end{bmatrix}$$

Multidimensional case (3)

- Second order approximation:

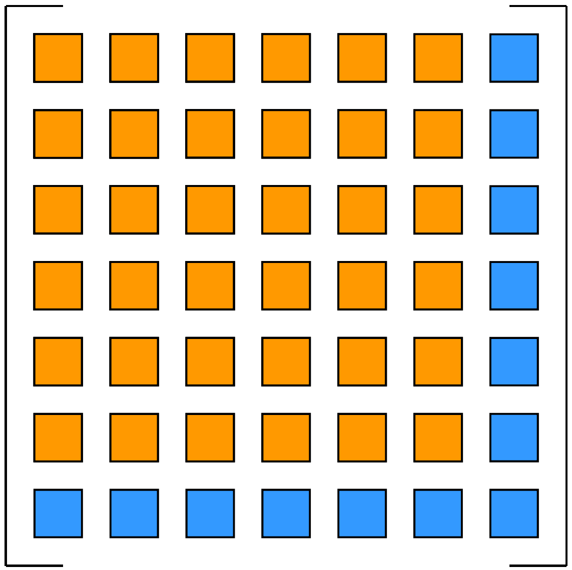
$$\left.\begin{aligned} f(\mathbf{x}+\mathbf{y})-f &= \frac{1}{2}\mathbf{y}^T\mathbf{H}\mathbf{y}+o\left(\|\mathbf{y}\|^3\right)>0 \\ f(\mathbf{x}+\mathbf{z})-f &= \frac{1}{2}\mathbf{z}^T\mathbf{H}\mathbf{z}+o\left(\|\mathbf{z}\|^3\right)>0 \\ f(\mathbf{x}+\mathbf{q})-f &= \frac{1}{2}\mathbf{q}^T\mathbf{H}\mathbf{q}+o\left(\|\mathbf{q}\|^3\right)>0 \end{aligned}\right\} \quad \mathbf{y}^T\mathbf{H}\mathbf{y}>0 \quad \forall\mathbf{y}\neq\mathbf{0}$$

- Local minimum: ***Optimality Conditions***:
  - First Order Necessity Condition: $\nabla f=\mathbf{0}$
  - Second Order Sufficiency Condition: $\mathbf{y}^T\mathbf{H}\mathbf{y}>0 \quad \forall\mathbf{y}$

Positive definiteness

- $\mathbf{y}^T\mathbf{H}\mathbf{y} > 0 \quad \forall \mathbf{y} \neq \mathbf{0}$ $\Leftrightarrow$ Hessian *positive definite*

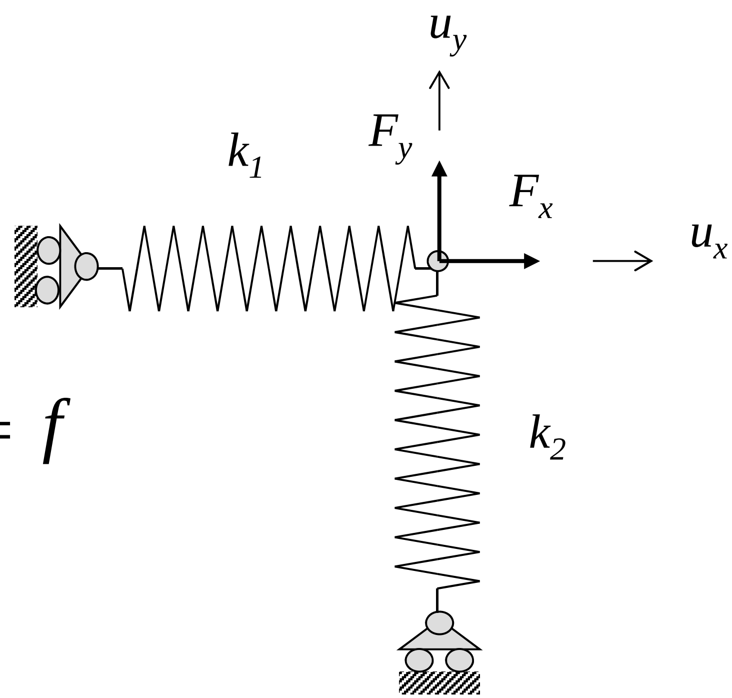
- Tests for positive definiteness:
  - Evaluate $\mathbf{y}^T\mathbf{H}\mathbf{y}$ for all $\mathbf{y}$ (impractical)
  - All *eigenvalues* λ_i of $\mathbf{H}$ positive
  - Sylvester's rule: all determinants of $\mathbf{H}$ and its *principal submatrices* are positive



Example

- Another loaded structure (small displacements):

$$\Pi = \frac{1}{2}k_1 u_x^{\,2} + \frac{1}{2}k_2 u_y^{\,2} - F_x u_x - F_y u_y = f$$



- Equilibrium: $\min\limits_{u_x, u_y} f$

- First order necessity condition:

$$\nabla f = \begin{Bmatrix} \dfrac{\partial f}{\partial u_x} \\ \dfrac{\partial f}{\partial u_y} \end{Bmatrix} = \begin{Bmatrix} k_1 u_x - F_x \\ k_2 u_y - F_y \end{Bmatrix} = 0 \quad \Rightarrow \quad \begin{Bmatrix} u_x \\ u_y \end{Bmatrix} = \begin{Bmatrix} \dfrac{F_x}{k_1} \\ \dfrac{F_y}{k_2} \end{Bmatrix}$$

Example (2)

- Second order sufficiency: $\mathbf{H}$ positive definite?

$$f = \frac{1}{2}k_1 u_x{}^2 + \frac{1}{2}k_2 u_y{}^2 - F_x u_x - F_y u_y \qquad \nabla f = \begin{Bmatrix} k_1 u_x - F_x \\ k_2 u_y - F_y \end{Bmatrix}$$

$$\mathbf{H} = \begin{bmatrix} \dfrac{\partial^2 f}{\partial u_x{}^2} & \dfrac{\partial^2 f}{\partial u_x \partial u_y} \\ \dfrac{\partial^2 f}{\partial u_y \partial u_x} & \dfrac{\partial^2 f}{\partial u_y{}^2} \end{bmatrix} = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \qquad \Rightarrow \qquad \begin{cases} k_1 > 0 \\ k_2 > 0 \end{cases}$$

Multidimensional quadratic functions

(for geometrical interpretation of optimality conditions)

- Polynomial terms up to 2nd order:

$$f(x_1, x_2, x_3) = 3 + 4x_2 + 2x_1x_2 - x_1^2 + x_3^2$$

- General form:

$$f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{b}^T\mathbf{x} + c$$

$$\frac{1}{2}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}^T \begin{bmatrix} -2 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 4 \\ 0 \end{Bmatrix}^T \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} + 3$$

- Note: 2nd order Taylor series is *exact*

Quadratic functions (2)

$$f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{b}^T\mathbf{x} + c$$

- Optimality conditions:
  - Gradient: $\nabla f = \mathbf{A}\mathbf{x} + \mathbf{b} = 0$
  - Hessian: $\mathbf{H} = \mathbf{A}$
- Stationary points: $\mathbf{x} = -\mathbf{A}^{-1}\mathbf{b}$

1D: compare

$$f(x) = ax^2 + bx + c$$

$$x = -\frac{b}{2a} \qquad \left(a = \frac{1}{2}\mathbf{A}\right)$$

Example

- Consider $f(\mathbf{x}) = \frac{1}{2}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}^T \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} -1 \\ 4 \end{Bmatrix}^T \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + 2$

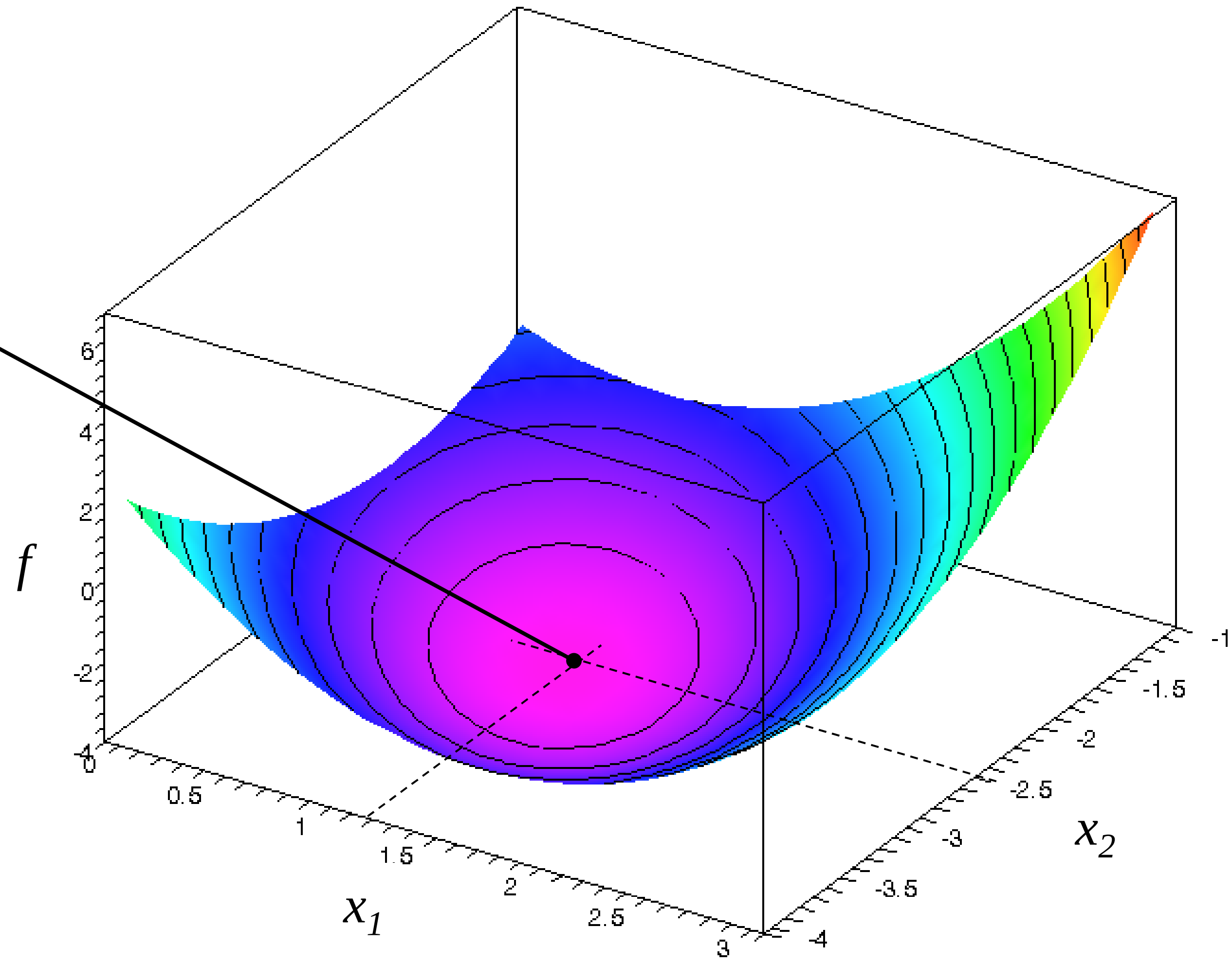
$$\Rightarrow \nabla f = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} -1 \\ 4 \end{Bmatrix} = \mathbf{0}$$

$$\Rightarrow \mathbf{x}^* = -\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{Bmatrix} -1 \\ 4 \end{Bmatrix} = \begin{Bmatrix} 1.2 \\ -2.6 \end{Bmatrix}$$ Stationary point

$$\mathbf{H} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \Rightarrow \begin{cases} 3 > 0 \\ |\mathbf{H}| = 6 - 1 > 0 \end{cases}$$ Hessian positive definite

Example (2)

- Result:
  minimum
  at (1.2, -2.6):



Example: least squares (Lect. 4)

- Least squares fitting: unconstrained optimization problem

$$\min_{\mathbf{a}} L = \boldsymbol{\varepsilon}^T \boldsymbol{\varepsilon} = \left(\tilde{\mathbf{f}} - \mathbf{M}\mathbf{a}\right)^T \left(\tilde{\mathbf{f}} - \mathbf{M}\mathbf{a}\right)$$

$$\Rightarrow f(\mathbf{a}) = \frac{1}{2}\mathbf{a}^T \underbrace{\left(2\mathbf{M}^T\mathbf{M}\right)}_{\mathbf{A}} \mathbf{a} - \underbrace{2\tilde{\mathbf{f}}^T\mathbf{M}}_{\mathbf{b}^T}\mathbf{a} + \underbrace{\tilde{\mathbf{f}}^T\tilde{\mathbf{f}}}_{c}$$

- Stationary point:

$$\mathbf{x} = -\mathbf{A}^{-1}\mathbf{b}$$

$$\Rightarrow \mathbf{a} = -\left(2\mathbf{M}^T\mathbf{M}\right)^{-1} \cdot -2\mathbf{M}^T\tilde{\mathbf{f}}$$

$$= \left(\mathbf{M}^T\mathbf{M}\right)^{-1}\mathbf{M}^T\tilde{\mathbf{f}} \quad ✓$$

- Hessian:

$$\mathbf{y}^T\mathbf{A}\mathbf{y} = \mathbf{y}^T\left(2\mathbf{M}^T\mathbf{M}\right)\mathbf{y}$$

$$= 2\left(\mathbf{M}\mathbf{y}\right)^T\left(\mathbf{M}\mathbf{y}\right) = 2\mathbf{z}^T\mathbf{z} > 0 \quad ✓$$

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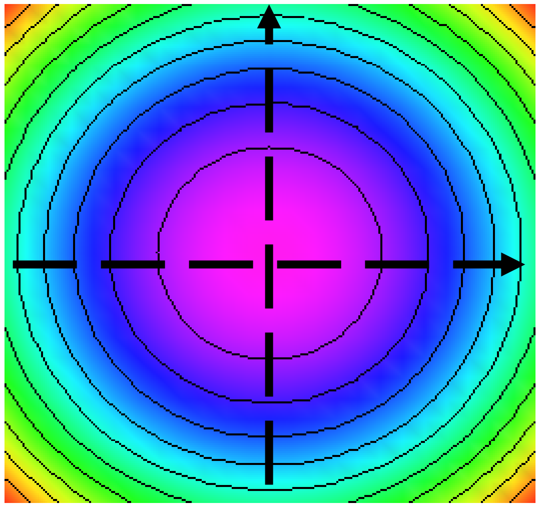
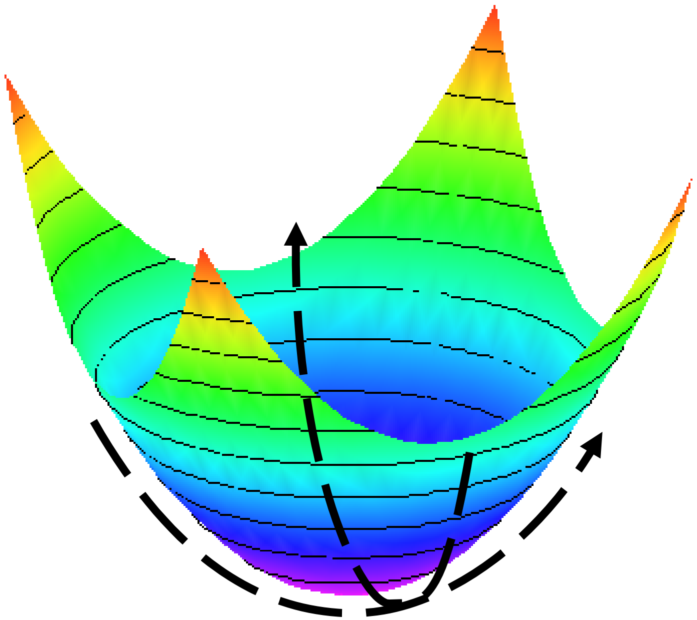
Nature of stationary points

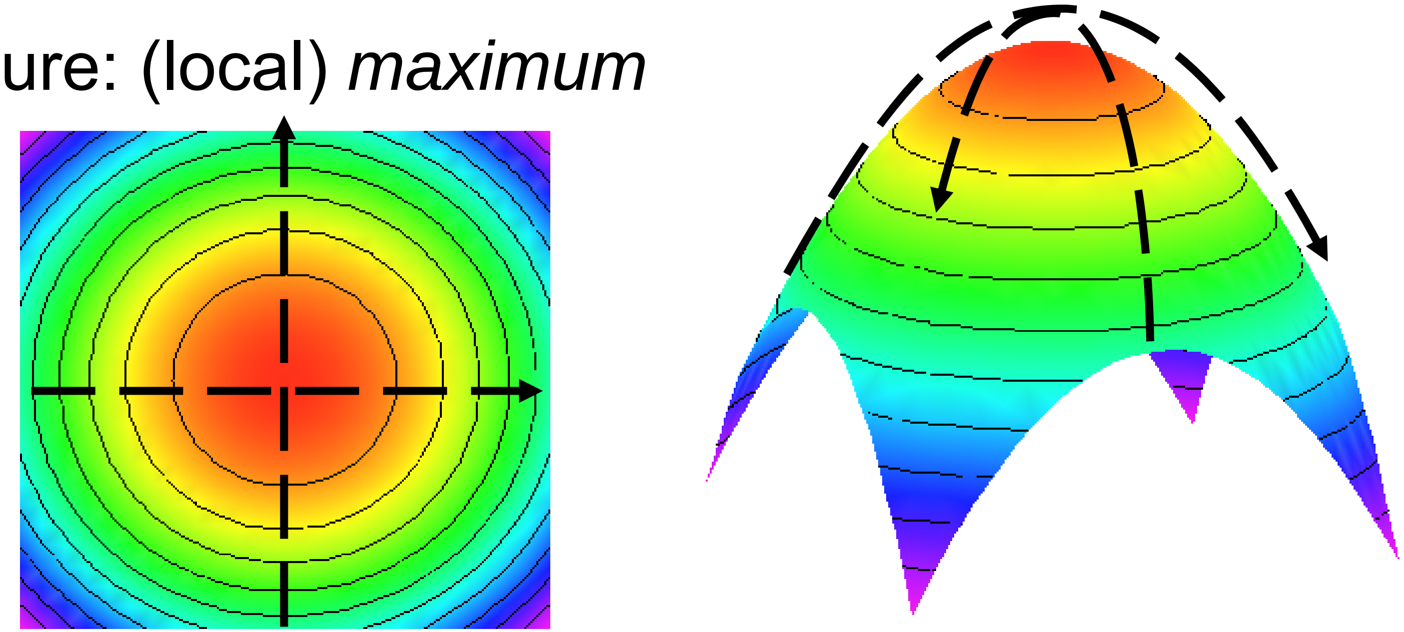
- Hessian $\mathbf{H}$ *positive definite*:
  - Quadratic form $\mathbf{y}^T\mathbf{H}\mathbf{y} > 0$
  - Eigenvalues $\lambda_i > 0$ (≈ curvature)
- Local nature: (local) *minimum*





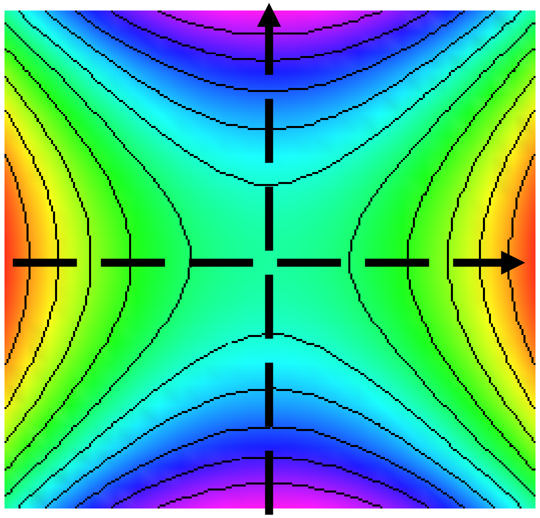
Nature of stationary points (2)

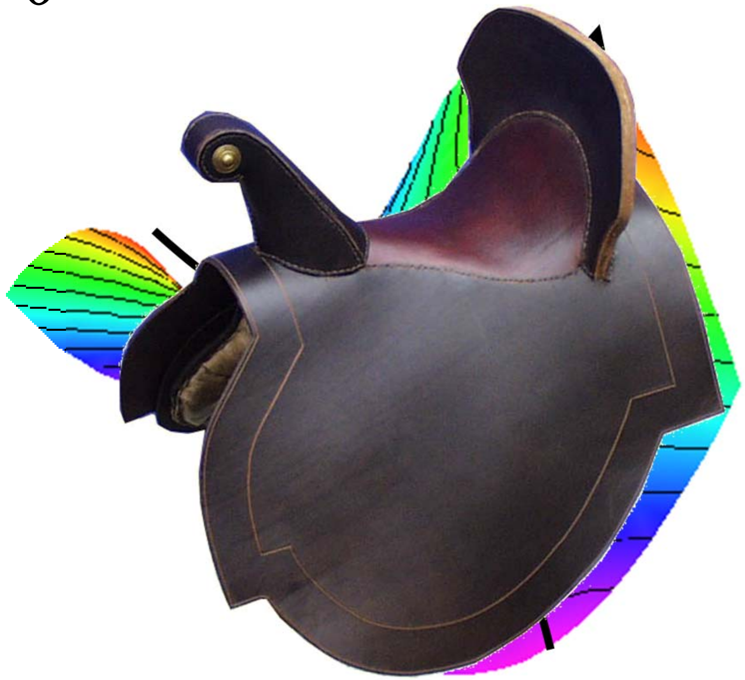
- Hessian $\mathbf{H}$ *negative definite*:
  - Quadratic form $\quad \mathbf{y}^T\mathbf{H}\mathbf{y} < 0$
  - Eigenvalues $\quad \lambda_i < 0$
- Local nature: (local) *maximum*



Nature of stationary points (3)

- Hessian $\mathbf{H}$ *indefinite*:
  - Quadratic form $\mathbf{y}^T\mathbf{H}\mathbf{y} \neq 0$
  - Eigenvalues $\lambda_i \neq 0$
- Local nature: *saddle point*



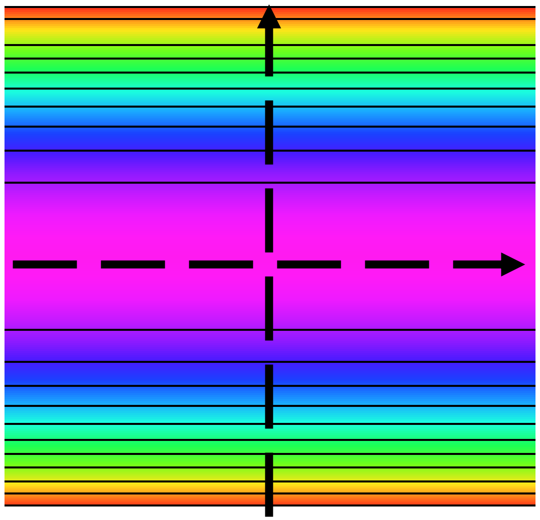


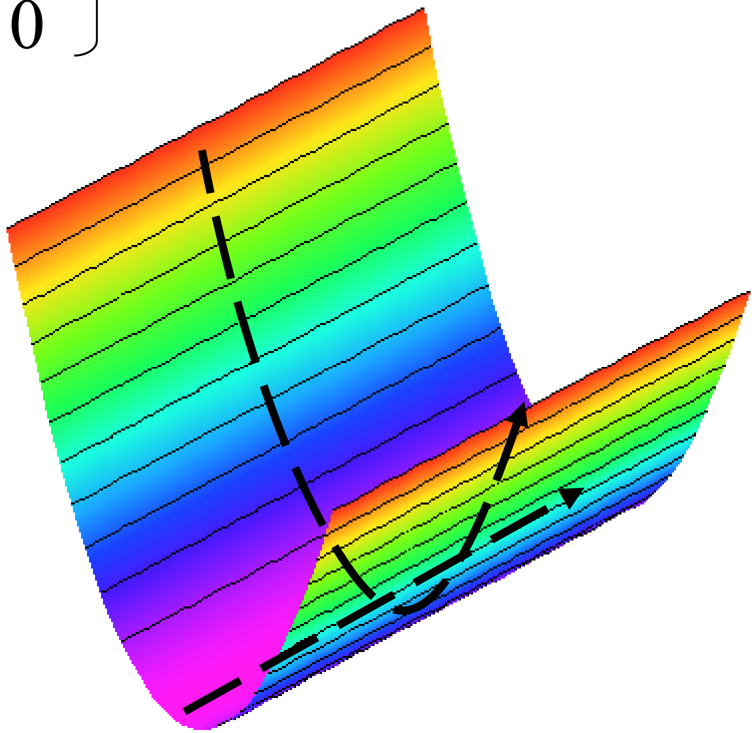
Nature of stationary points (4)

- Hessian $\mathbf{H}$ *positive semi-definite*:
  - Quadratic form $\mathbf{y}^T\mathbf{H}\mathbf{y} \geq 0$
  - Eigenvalues $\lambda_i \geq 0$

$$\left.\begin{array}{r} \mathbf{y}^T\mathbf{H}\mathbf{y} \geq 0 \\ \lambda_i \geq 0 \end{array}\right\} \longrightarrow \mathbf{H} \text{ singular}$$

- Local nature: *valley*



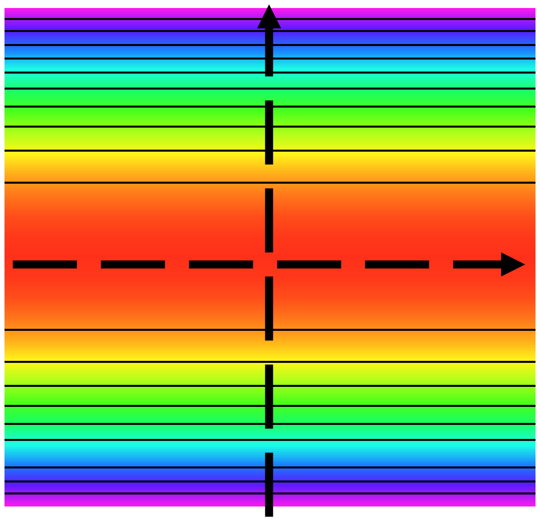


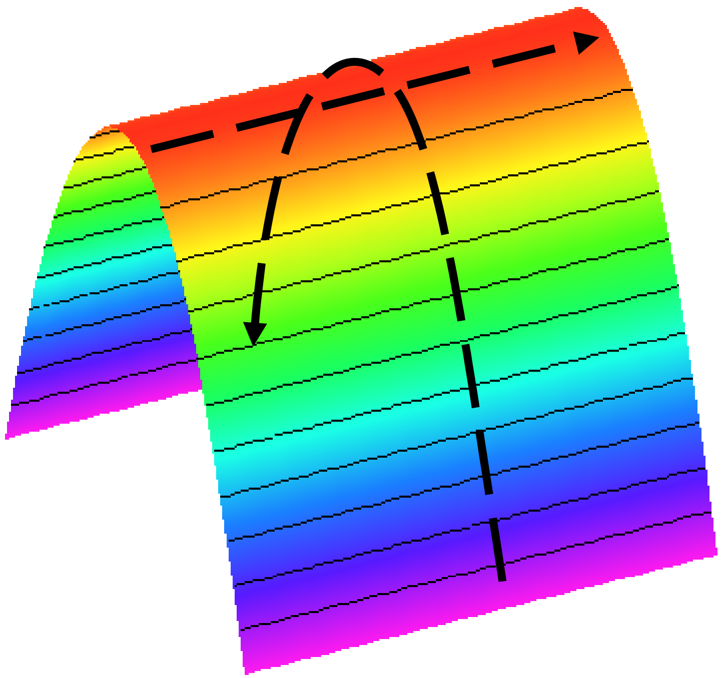
Nature of stationary points (5)

- Hessian $\mathbf{H}$ *negative semi-definite*:
  - Quadratic form $\quad \mathbf{y}^T\mathbf{H}\mathbf{y} \le 0$
  - Eigenvalues $\quad \lambda_i \le 0$

$\left.\begin{array}{l}\mathbf{y}^T\mathbf{H}\mathbf{y} \le 0 \\ \lambda_i \le 0\end{array}\right\} \longrightarrow$ $\mathbf{H}$ singular

- Local nature: *ridge*





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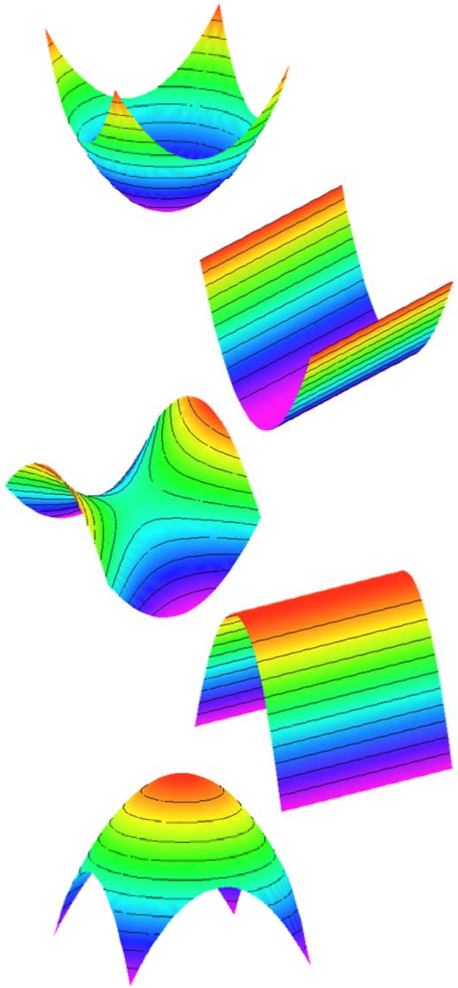
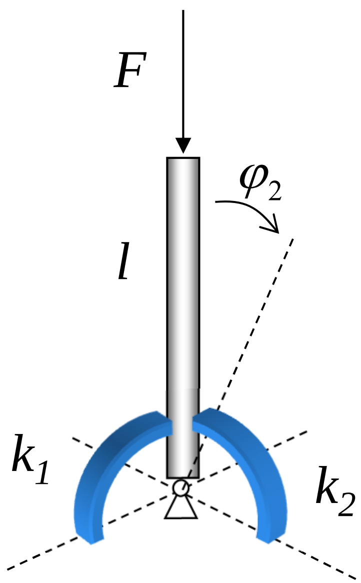
Stationary point nature summary

$\underbrace{\mathbf{y}^T\mathbf{H}\mathbf{y},\quad \lambda_i}$	**Definiteness H**	**Nature x***
>0	Positive d.	Minimum
≥ 0	Positive semi-d.	Valley
$\neq 0$	Indefinite	Saddlepoint
≤ 0	Negative semi-d.	Ridge
<0	Negative d.	Maximum



Structural example: pin-jointed bar with rotational springs

- Nature of initial position depends on load (buckling):



$k_1 = 10,\ k_2 = 9.5,\ l = 2$

$$dz = l - l\cos\varphi_1\cos\varphi_2$$

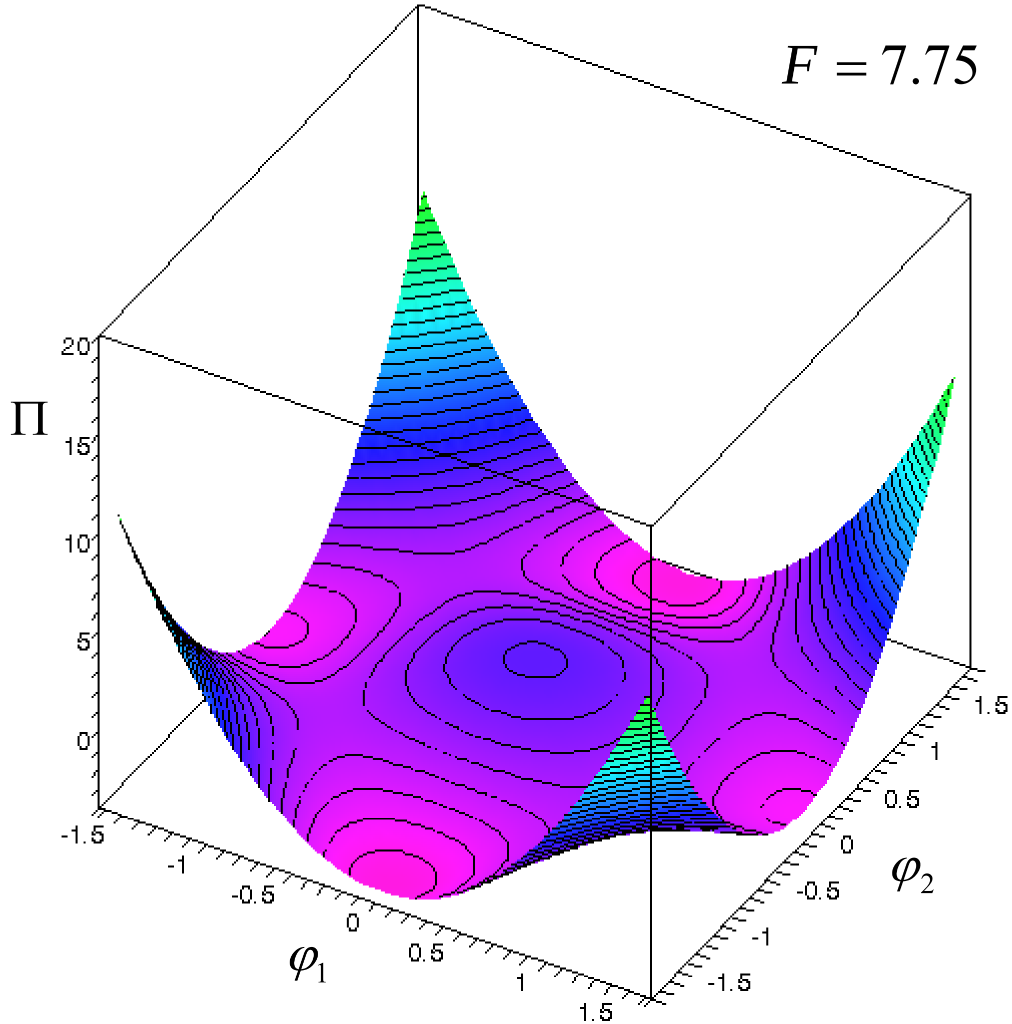
$$\Pi = \frac{1}{2}k_1\varphi_1^{\ 2} + \frac{1}{2}k_2\varphi_2^{\ 2} - Fdz$$

$$\nabla\Pi = \begin{Bmatrix} k_1\varphi_1 - Fl\sin\varphi_1\cos\varphi_2 \\ k_2\varphi_2 - Fl\sin\varphi_2\cos\varphi_1 \end{Bmatrix} = \mathbf{0} \quad \Rightarrow \varphi = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\Delta\Pi = \begin{bmatrix} k_1 - Fl & 0 \\ 0 & k_2 - Fl \end{bmatrix} \quad \Rightarrow F_{crit} = \min\left(\frac{k_1}{l}, \frac{k_2}{l}\right)$$

$$\Rightarrow F_{crit} = 4.75$$

Bar example: nature of stationary points



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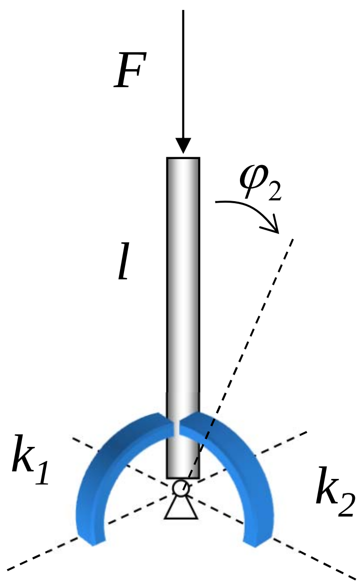
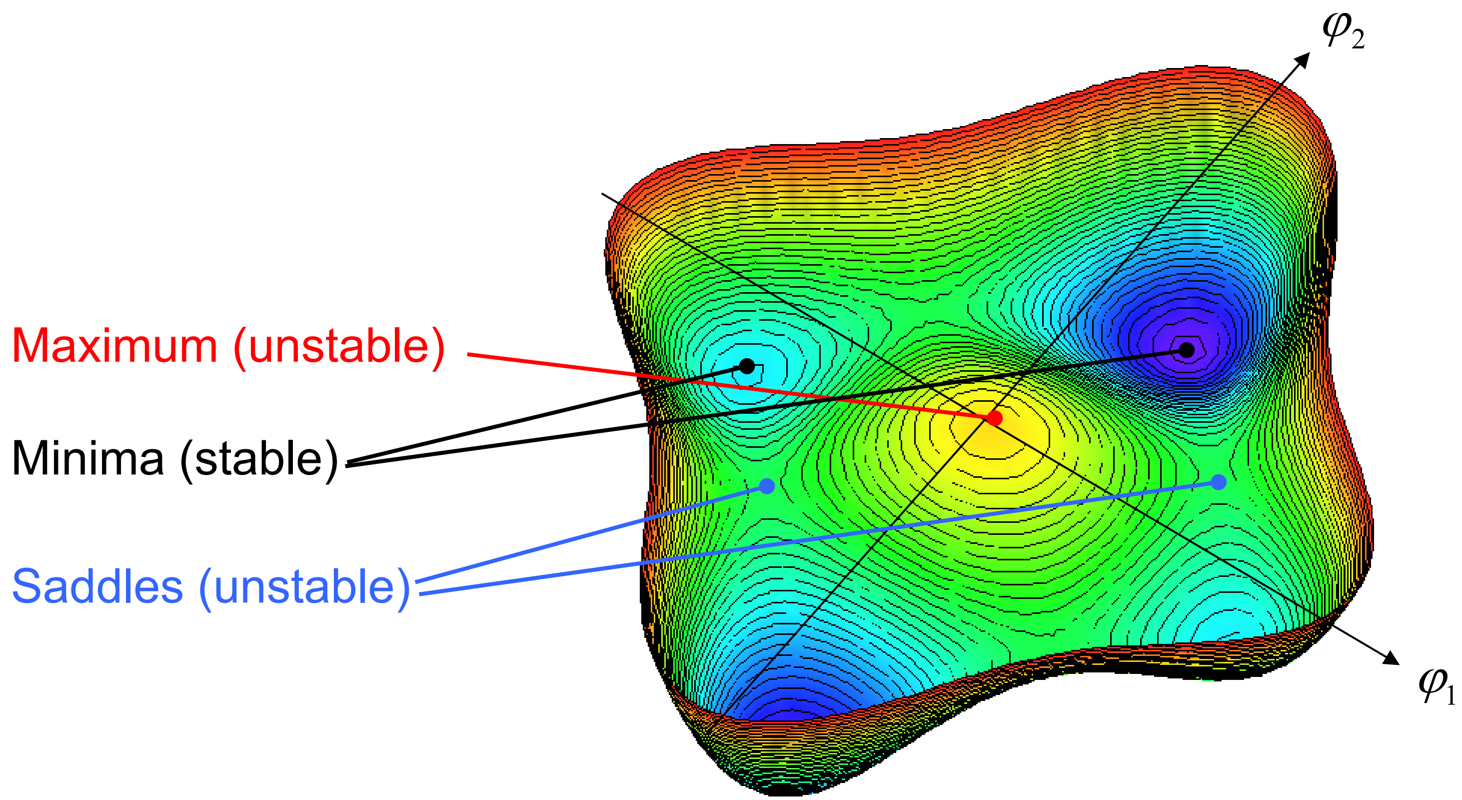
Bar example: nature of stationary points (2)

- Nature of stationary points at $F = 6$:



$k_1 = 10,\ k_2 = 9.5,\ l = 2$



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Contents

- Outline of remaining lectures
- Unconstrained problems
  - Transformation methods
  - Existence of solutions, optimality conditions
  - Nature of stationary points
  - Global optimality

Global optimality

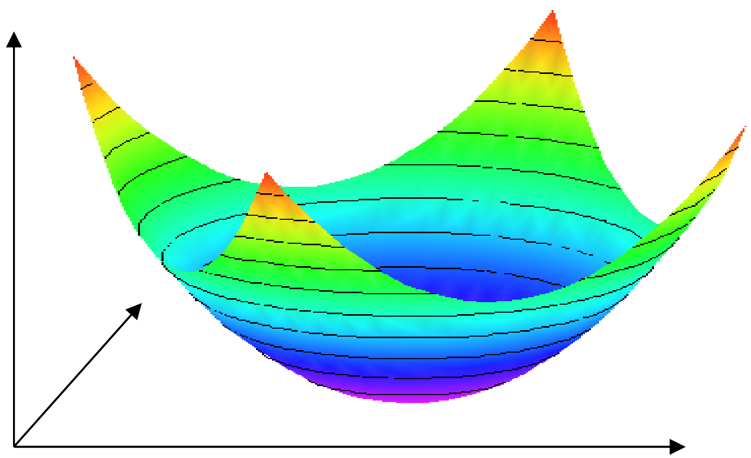
- Optimality conditions for unconstrained problem:
  - First order necessity: $\nabla f(\mathbf{x}^*) = 0$ (stationary point)
  - Second order sufficiency: $\mathbf{H}$ positive definite at $\mathbf{x}^*$
- Optimality conditions only valid locally:

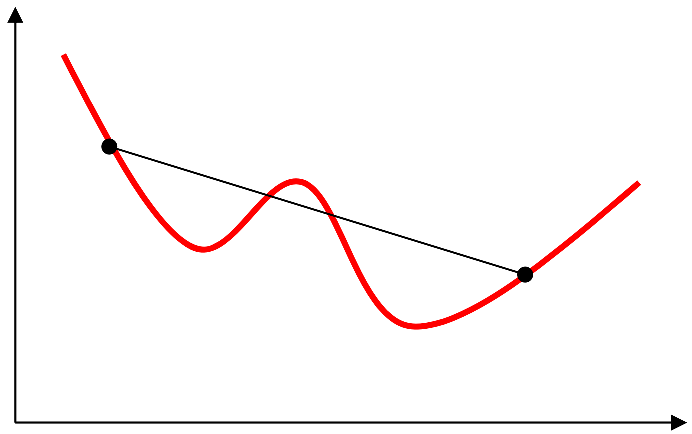
$\Rightarrow \quad \mathbf{x}^*$ *local* minimum

- When can we be sure $\mathbf{x}^*$ is a *global* minimum?

Convex functions

- Convex function: *any line connecting any 2 points on the graph lies above it (or on it)*:



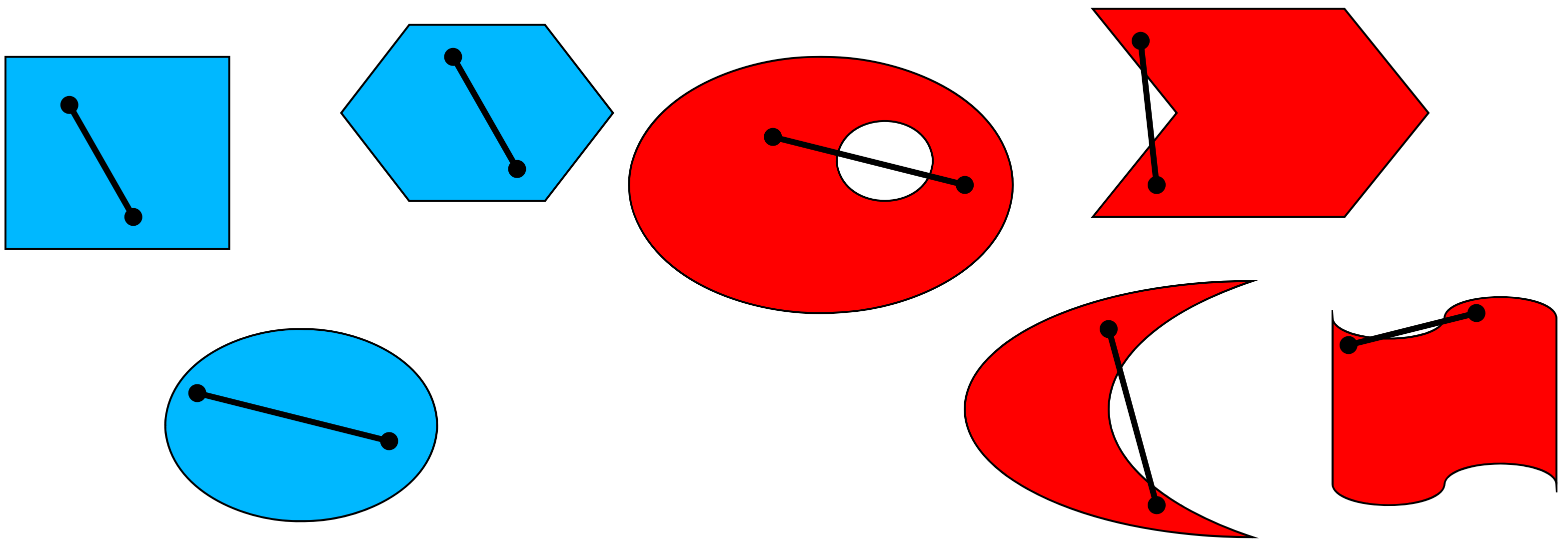


- $\mathbf{H}$ positive (semi-)definite $\Leftrightarrow f$ locally convex (proof by Taylor approximation)

Convex domains

- Convex set:

  "A set S is convex if for every two points $\mathbf{x}_1$, $\mathbf{x}_2$ in S, the connecting line also lies completely inside S"



Convexity and global optimality (constrained or unconstrained)

If:

- **Objective f = *(strictly)* convex function**
- **Feasible domain = convex set** (Ok for unconstrained optimization)

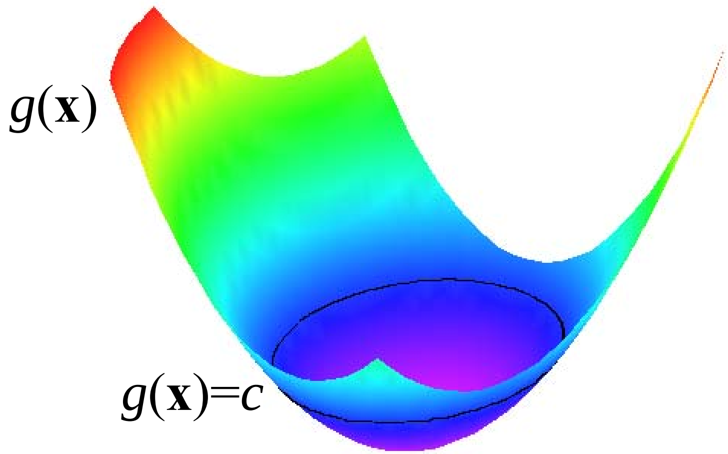
$\Rightarrow$ *Stationary point = (unique) global minimum*

- Special case: f, g, h all linear $\Rightarrow$ *linear programming*
- More general class: *convex optimization*

Convex set properties

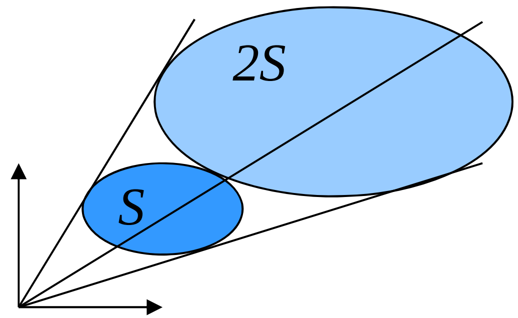
- Level sets (isocontours) of convex functions are convex sets:

$$S = \{\mathbf{x} \mid g(\mathbf{x}) \le c\}$$



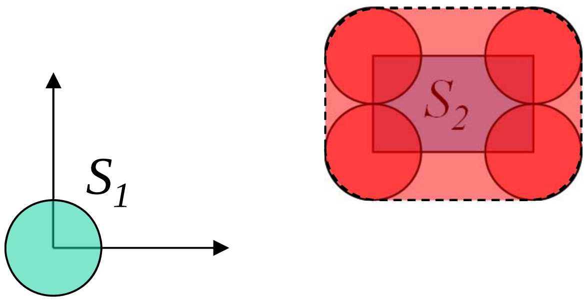
- Scaling:

S convex $\Rightarrow \alpha S$ convex



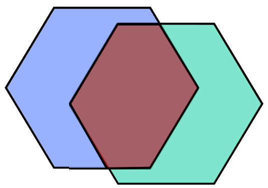
Convex set properties (2)

- Summation: S_1, S_2 convex $\Rightarrow S_1 + S_2$ convex



$$S_1 + S_2 = \{\mathbf{y} \mid \mathbf{y} = \mathbf{x}_1 + \mathbf{x}_2, \;\; \mathbf{x}_1 \in S_1, \; \mathbf{x}_2 \in S_2\}$$

- Intersection: S_1, S_2 convex $\Rightarrow S_1 \cap S_2$ convex



Applies to feasible domain defined by multiple constraints

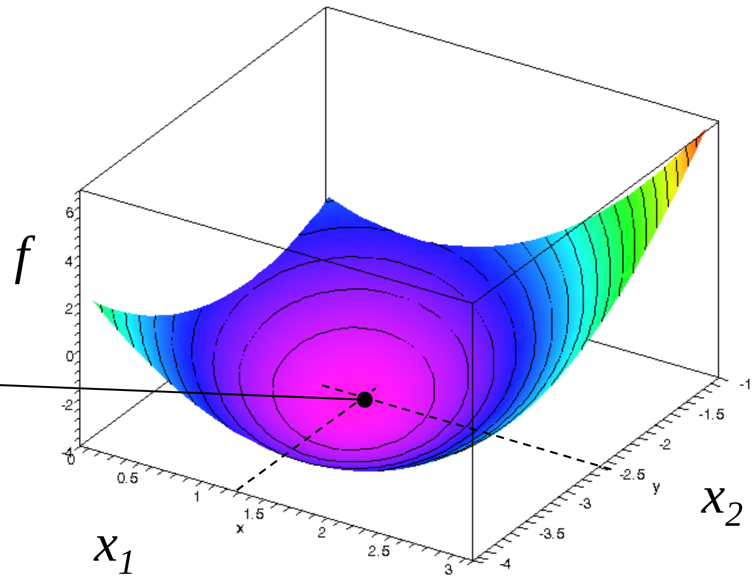
- Union generally nonconvex!

Example

- Quadratic functions with $\mathbf{A}$ positive definite are strictly convex:

$$f(\mathbf{x}) = \frac{1}{2}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}^T \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} -1 \\ 4 \end{Bmatrix}^T \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + 2$$

$\Rightarrow$ Stationary point (1.2, -2.6) must be *unique global optimum*



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Summary optimality conditions

- Conditions for *local* minimum of unconstrained problem:
  - First Order Necessity Condition: $\nabla f = \mathbf{0}$
  - Second Order Sufficiency Condition: $\mathbf{H}$ positive definite

- For *convex* f in *convex* feasible domain:
  condition for *global* minimum:
  - Sufficiency Condition: $\nabla f = \mathbf{0}$

Summary (2)

- More information: Papalambros 4.1 – 4.4

- Next lecture: **methods** for unconstrained optimization for **single-variable problems**

Sunday May 10: project proposals!

Project proposal [~ 1 A4]

- Context, what is the problem
- Optimization problem formulation (equations not needed):
  - Objective
  - Constraints
  - Design variables (≥ 2): continuous?
- Modeling: how to compute responses?
  Which approximations to make?
  What model do you already have?
- Cases, variations

Project ideas:
see handouts
Lecture 3

Project info:
see (Course info + Assessment – Report Guidelines)

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