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22C:188 HW3 ①
Spring 2014

HW3

#1 Formalization in FOL

1) $R(x)$: x is a red thing
$B(x)$: x is in the box

a) All red things are in the box.
$\forall x (R(x) \rightarrow B(x))$

b) Only red things are in the box
$\forall x (B(x) \rightarrow R(x))$

c) No animal is both a cat and a dog.
$A(x)$: x is an animal
$C(x)$: x is a cat
$D(x)$: x is a dog

$\forall x (A(x) \rightarrow \neg(C(x) \wedge D(x)))$
$\forall x (A(x) \rightarrow (\neg C(x) \vee \neg D(x)))$

d) Every prize was won by a boy.
$P(y)$: y is a prize
$B(x)$: x is a boy
$W(x,y)$: x won y (Adam won the lottery)

$\forall y (P(y) \rightarrow \exists x (B(x) \wedge W(x,y)))$

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#1 Formalization in FOL

1) e) A boy won every prize.

$P(y)$: y is a prize
$B(x)$: x is a boy
$W(x,y)$: x won y

$$\exists x \forall y \left(B(x) \wedge P(y) \rightarrow W(x,y) \right)$$

---

2) a) $\forall x \forall y (x = y)$ (where all quantifiers have the same domain)

- Every pair of elements consists of two equal elements.

b) $\exists x \exists y (x \neq y)$ (where all quantifiers have the same nonempty domain)

- There exists at least one pair of elements such that the elements are not equal.

c) $\exists x \exists y \left(x \neq y \wedge \forall z (z \neq x \rightarrow z = y) \right)$ (where all quantifiers have the same nonempty domain)

- There exists a model with only two distinct elements:

	Note:	$z \neq x$	$z = y$	$z \neq x \rightarrow z = y$	$x \neq y \wedge \forall z (z \neq x \rightarrow z = y)$
2 distinct elements	$(x, y = z, y = z)$	T	T	T	T
3 distinct elements	(x, y, z)	T	F	F	F
1 distinct element	$(x = z, y = z)$	F	T	T	F
2 distinct elements	$(x = z, y)$	F	F	T	T

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#1 Formalization in FOL

3) If there are any tax payers, then all politicians are tax payers. If there are any philanthropists, then all tax payers are philanthropists. Therefore, if there are any tax-paying philanthropists, then all politicians are philanthropists.

$T(x)$: x is a tax payer
$P(x)$: x is a politician
$H(x)$: x is a philanthropist

$$\exists x\Big(T(x) \rightarrow \big(\forall x\,(P(x) \rightarrow T(x))\big)\Big),$$

$$\exists x\big(H(x)\big) \rightarrow \forall x\,(T(x) \rightarrow H(x)),$$

$$\vdash \quad \exists x\big(T(x) \wedge H(x)\big) \rightarrow \forall x\big(P(x) \rightarrow H(x)\big)$$

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#1 Formalizing in FOL

4) It is possible to specify a formula for models with a specified concrete number of elements; for example a model with exactly three distinct elements or even at most three distinct elements. When not specifying a concrete bound as in the previous two examples, it is possible to specify an upper bound with the "$\bigwedge_{i=1}^{k}$" operator. Within this operator, the upper bound is 'k', where 'k' is an arbitrary natural number. A subset bounded by k is a subset of infinitely many elements. A subset of an infinite set is still an infinite set. Therefore, there can be no formula in first-order logic that is true only for models with finitely many distinct elements.

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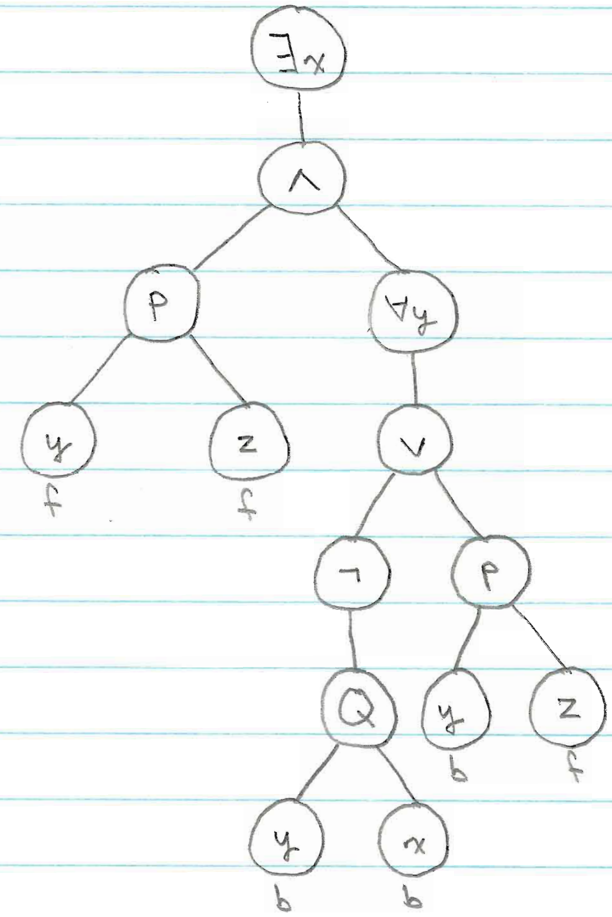
#2 Parse Trees, Variables and Substitutions

Let the formula Φ be

$$\exists x \left(P(y,z) \wedge \left(\forall y \left(\neg Q(y,x) \vee P(y,z)\right)\right)\right)$$

1)



2) For each variable occurrence in Φ identify if it is free or bound.

$$\exists x \left(P(\underset{f}{y},\underset{f}{z}) \wedge \left(\forall y \left(\neg Q(\underset{b}{y},\underset{b}{x}) \vee P(\underset{b}{y},\underset{f}{z})\right)\right)\right)$$

f ≡ free

b ≡ bound

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#2 Parse Trees, Variables and Substitutions

3) Consider the variable w, the unary function symbol f and the binary function symbol g.

(a) Which of w, $f(x)$, and $g(x,y)$ are free for x in Φ?
The variable w, $f(x)$, and $g(x,y)$ are free for x in Φ since there are no free occurences of x.

(b) Which of w, $f(x)$, and $g(x,y)$ are free for y in Φ?
Only the variable w is free for y in Φ.
Both $f(x)$ and $g(x,y)$ would be captured by '$\exists x$'.

(c) Compute $\Phi[w/x]$, $\Phi[f(w)/y]$, and $\Phi[g(w,z)/z]$.

$$\Phi[w/x] \equiv \exists x \left(P(y,z) \wedge \left(\forall y \left(\neg Q(y,x) \vee P(y,z)\right)\right)\right)$$

$$\Phi[f(w)/y] \equiv \exists x \left(P(f(w),z) \wedge \left(\forall y \left(\neg Q(y,x) \vee P(y,z)\right)\right)\right)$$

$$\Phi[g(w,z)/z] \equiv \exists x \left(P(y,g(w,z)) \wedge \left(\forall y \left(\neg Q(y,x) \vee P(y,g(w,z))\right)\right)\right)$$

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3 Natural deduction in FOL

1) $(y=\emptyset) \wedge (y=x) \vdash \emptyset = x$

1	$(y=\emptyset) \wedge (y=x) \vdash \emptyset = x$	premise
2	$y=\emptyset$	$\wedge e_1$ 1
3	$y=y$	$=i$
4	$\emptyset = y$	$=e$ 2,3
5	$y=x$	$\wedge e_2$ 1
6	$\emptyset = x$	$=e$ 5,4

$$\frac{y=\emptyset \qquad \phi[y/z]}{\phi[\emptyset/z]} =e$$
(where ϕ is $z=y$)

$$\frac{\emptyset=y \qquad \phi[y/z]}{\phi[x/z]} =e$$
where ϕ is $\emptyset=z$

2) $t_1 = t_2 \vdash (t+t_2)=(t+t_1)$

1	$t_1 = t_2$	premise
2	$(t+t_1)=(t+t_1)$	$=i$
3	$(t+t_2)=(t+t_1)$	$=e$ 1,2 where ϕ is $(t+x)=(t+t_1)$

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3 Natural Deduction in FOL

3) $(\forall x\, P(x)) \rightarrow (\forall x\, Q(x)) \vdash \forall x((\forall x\, P(x)) \rightarrow Q(x))$

1	$(\forall x\, P(x)) \rightarrow (\forall x\, Q(x))$	premise
2	x_0	
3	$\forall x\, P(x)$	assumption
4	$\forall x\, Q(x)$	$\rightarrow e$ 1, 3
5	$Q(x_0)$	$\forall x\, e$ 4
6	$(\forall x\, P(x)) \rightarrow Q(x_0)$	$\rightarrow i$ 3-5
7	$\forall x((\forall x\, P(x)) \rightarrow Q(x))$	$\forall x\, i$ 2-6

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3 Deductions in FOL

4) $\forall x\, P(x,z) \vdash \forall y\, P(y,z)$

1	$\forall x\, P(x,z)$	premise
2	x_0 $P(x_0,z)$	$\forall x$ e 1
3	$\forall y\, P(y,z)$	$\forall y$ 2

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#N Natural deduction in FOL

5) $\forall x(P(x) \rightarrow \neg Q(x)) \vdash \neg(\exists x(P(x) \wedge Q(x)))$

1	$\forall x(P(x) \rightarrow \neg Q(x))$	premise
2	$\exists x(P(x) \wedge Q(x))$	assumption
3	x_0	
4	$P(x_0) \wedge Q(x_0)$	assumption
5	$P(x_0)$	$\wedge e_1$ 4
6	$Q(x_0)$	$\wedge e_2$ 4
7	$P(x_0) \rightarrow \neg Q(x_0)$	$\forall x e$ 1
8	$\neg Q(x_0)$	$\rightarrow e$ 7, 5
9	$\perp$	$\neg e$ 6, 8
10	$\perp$	$\exists x e$ 2, 3-9
11	$\neg(\exists x(P(x) \wedge Q(x)))$	

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3 Natural Deduction in FOL

6) $\exists x \forall y\, P(x,y) \vdash \forall y\, \exists x\, P(x,y)$

see example pg 116 lines 3-6

1		$\exists x \forall y\, P(x,y)$	premise
2	y_0		
3	x_0	$\forall y\, P(x_0,y)$	assumption
4		$P(x_0,y_0)$	$\forall y\, e$ 3
5		$\exists x\, P(x,y_0)$	$\exists x\, i$ 4
6		$\exists x\, P(x,y_0)$	$\exists x\, e$ 1, 3-6
		$\forall y\, \exists x\, P(x,y)$	$\forall y$ 2-6

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#3 Natural Deduction in FOL

7) $P(a) \vdash \forall x \, (a = x \rightarrow P(x))$

1	$P(a)$	premise
2	x_0	
3	$a = x_0$	assume
4	$P(x_0)$	$=e$ 3,1
5	$a = x_0 \rightarrow P(x_0)$	$\rightarrow i$ 3-4
6	$\forall x((a = x) \rightarrow P(x))$	$\forall x\, i$ 2-5

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#4 Semantics of FOL

1) $\Phi \stackrel{def}{=} \forall x \forall y\, Q(g(x,y), g(y,y), z)$

We want g^m to perform some operation on 2 elements such that $g(x,y)$ represents some operation on unlike elements and $g(y,y)$ represents some operation on like elements. Then the result of $g(x,y)$ and $g(y,y)$ provide "input" to Q^m. However, let it be noted that while $g(y,y)$ obviously performs an operation on two like elements, $g(x,y)$ may also perform the same operation on two elements, that is to say that it is not required that $x \neq y$ but in fact it is allowable that $x = y$. Considering that the truth value of Φ is then determined by its only free variable, z, $l(z)$ and $l'(z)$ thus also determine the truth value of Φ.

$+, -, *, \div$

$$Q^m \stackrel{def}{=} \{ g_1^m, g_2^m, z \mid (g_1^m)(g_2^m) = z \}$$
$$g^m \stackrel{def}{=} \{ (x,y) \mid x - y \}$$

Such that any number multiplied by zero is zero.
Let the universe be M such that
$M \stackrel{def}{=} \{0,1,2\}$

Therefore, every product must equal 0.

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#4 Semantics of FOL

1) $\Phi \stackrel{def}{=} \forall x \forall y \, Q(g(x,y), g(y,y), z)$

We want g^m to perform some operation on 2 elements such that $g(x,y)$ represents some operation on unlike elements and $g(y,y)$ represents some operation on like elements. Then the result of $g(x,y)$ and $g(y,y)$ provide "input" to Q^m. However, let it be noted that while $g(y,y)$ obviously performs an operation on two like elements, $g(x,y)$ may also perform the same operation on two elements, that is to say that it is not required that $x \neq y$ but in fact it is allowable that $x = y$. Considering that the truth value of Φ is then determined by its only free variable, z, $\ell(z)$ and $\ell'(z)$ thus also determine the truth value of Φ.

$+, -, *, \div$

$$Q^m \stackrel{def}{=} \{ g_1^m, g_2^m, z \mid (g_1^m)(g_2^m) = z \}$$
$$g^m \stackrel{def}{=} \{ (x,y) \mid x - y \}$$

Such that any number multiplied by zero is zero.
Let the universe be $\mathcal{M}$ such that
$\mathcal{M} \stackrel{def}{=} \{0, 1, 2\}$

Therefore, every product must equal $\emptyset$.

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#4 Semantics of FOL

2) $\Phi \stackrel{def}{=} \forall x \exists y \exists z ((P(x,y) \wedge P(z,y) \wedge (P(x,z) \rightarrow P(z,x)))$

Let the universe A be the set of natural numbers $\mathbb{N}$.

a) $P^m \stackrel{def}{=} \{(m,n) \mid m < n\}$; satisfies Φ

Let's first look at it as $\forall x \exists y \exists z (\overset{①}{(x<y)} \wedge \overset{②}{(z<y)} \wedge \overset{③}{((x<z) \rightarrow (z<x))})$
where all 3 terms must evaluate to True for Φ to be satisfied.

$x<z$	$z<x$	$((x<z) \rightarrow (z<x))$	$\forall x \in \mathbb{N}$ and $\exists z \in \mathbb{N}$
T	T	T	invalid
T	F	F	valid but false
F	T	T	$\forall x \in \mathbb{N}$, $x<z$ is true $\Rightarrow$ invalid
F	F	T	T when $x = z \in \mathbb{N}$

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#4 Semantics of FOL

2) b) $p^{m} \overset{def}{=} \{(m, 2m) \mid m \in A\}$; satisfies Φ

The set of natural numbers, $\mathbb{N}$, is the universe A, as given. Every natural number multiplied by 2 is still in the universe of natural numbers.

c) $p^{m} \overset{def}{=} \{(m,n) \mid m < n+1\}$

$$\forall x \exists y \exists z ((x < y+1) \wedge (z < y+1) \wedge ((x < z+1) \rightarrow (z < n+1)))$$

$x < z+1$	$z < x+1$	$(x < z+1) \rightarrow (z < x+1)$	$\forall x \in \mathbb{N}$ and $\exists z \in \mathbb{N}$
T	T	T (if $x=z$)	T when $x=z \in \mathbb{N}$
T	F	F	
F	T	T (if $z < x$)	but $z < x$ invalid $\forall x \in \mathbb{N}$
F	F	T	invalid $\forall x \in \mathbb{N}$

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#4 Semantics

3) $\Phi \overset{def}{=} \forall x \forall y \exists z \, (R(x,y) \rightarrow R(y,z))$

a) $A \overset{def}{=} \{a, b, c, d\}$ and $R^m \overset{def}{=} \{(b,c), (b,b), (b,a)\}$

No. For example, let's look at the element (b,c). Then an element (c,a) or (c,b) or even (c,d) should belong to R^m and no such element exists in R^m.

b) $A \overset{def}{=} \{a, b, c\}$ $R^m \overset{def}{=} \{(b,c), (a,b), (c,b)\}$

Yes.

$(b,c) \rightarrow (c,b)$ here $x=2$. ok

$(a,b) \rightarrow (b,c)$ ok

$(c,b) \rightarrow (b,c)$ here $x=2$ again. ok